LECTURE 2 – SIMPLICIAL SETS

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1. Notation

Categories will typically be sans serif (*i.e.* Top, sSet, etc.). If $\mathcal{C}$ is an arbitrary category then we denote the the set of $\mathcal{C}$-morphisms from objects $A, B \in \mathcal{C}$ by $\mathcal{C}(A, B)$.

2. Garbage

Recall the definition of the *standard topological n-simplex* as the set

$$\Delta^n_{\mathsf{Top}} := \left\{ (t_0, \cdots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i \right\}$$

Alternatively we may think of Δ^n_{Top} as the convex hull of vertices $v_i = (0, \dots, 1, \dots, 0)$. We then have maps codegeneracy maps $s^i : \Delta^{n+1}_{\mathsf{Top}} \to \Delta^n_{\mathsf{Top}}$, and coface maps $d^i : \Delta^{n-1}_{\mathsf{Top}} \to \Delta^n_{\mathsf{Top}}$ defined by

$$s^i(t_0, \dots, t_{n+1}) = (t_0, \dots, t_i + t_{i+1}, \dots, t_{n+1})$$
$$d^i(t_0, \dots, t_{n-1}) = (t_0, \dots, t_i, 0, t_{i+1}, \dots, t_n)$$

Clearly d^i is just the map embedding $\Delta^{n-1}_{\mathsf{Top}}$ as the i^{th} face of Δ^n_{Top}, and s^i is a retration of $\Delta^{n+1}_{\mathsf{Top}}$ minus the i^{th} vertice v_i onto the face opposite v_i.

Given any topological space $X \in \mathsf{Top}$ we define the *singular n-simplices of X* to be the maps

$$\mathrm{Sing}(X)_n := \mathsf{Top}(\Delta^n_{\mathsf{Top}}, X)$$

This turns out to be an important example of something called a simplicial object. Before we can define what a simplicial object is, we must first define the *simplex category* Δ. The objects of Δ are the ordered sets $[n] = \{0, 1, \dots, n\}$, and the morphisms $f : [m] \to [n]$ are the weakly-order-preserving (*i.e.* non-decreasing) functions. Similarly to above we have maps $s^i : [n+1] \to [n]$ and $d^i : [n-1] \to [n]$ given by repeating the i, and skipping i respectively. The following lemma says that these maps are the only maps we care about.

Lemma 2.0.1. *Any map f in Δ is the composition of a series of d^i and s^j.*

Proof sketch. If you have a map $f : [m] \to [n]$ then you have the inequality $f(0) \leq f(1) \leq \cdots \leq f(m)$. We get unique elements of the form $g_0 < \cdots < g_k$ for $k \leq \min\{m, n\}$. By composing d^0 with itself g_0 times we get a map that sends $[m]$ to $[g_0, g_0 + 1, \dots, g_0 + m]$. We repeat g_0 with s^0 as many times as it occurs in the sequence of $f(i)$s. We then simply repeat this process inductively on the g_i for $0 < i \leq k$. $\square$

Definition 2.0.2. Let $\mathcal{C}$ be a category. A *simplicial object in* $\mathcal{C}$ is a functor $X : \Delta^{\mathrm{op}} \to \mathcal{C}$. Similarly a *cosimplicial object in* $\mathcal{C}$ is a functor $Y : \Delta \to \mathcal{C}$.

We write X_n and Y^n for $X([n])$ and $Y([n])$ respectively, and hence we will often use the notation $X_\bullet$ and $Y^\bullet$ for X and Y respectively. Similarly we write d_i, s_i and d^i, s^i for the obvious maps in $\mathcal{C}$.

Example 2.0.3. The standard simplices $\Delta^\bullet_{\mathsf{Top}}$ is a cosimplicial space.

Example 2.0.4. Given any space $X \in \mathsf{Top}$ we therefore have that $\mathrm{Sing}(X)_\bullet$ is a simplicial set acting on objects by $[n] \mapsto \mathsf{Top}(\Delta^n_{\mathsf{Top}}, X)$, and on morphisms by $(f : [m] \to [n]) \mapsto f_*$ where f_* is precomposition with f.

Date: November 26, 2019.

Properties 2.0.5. *The degeneracy, and face maps satisfy the simplicial properties:*

(1) $d_i \circ d_j = d_{j-1} \circ d_i$ *for* $i < j$
(2) $s_i \circ s_j = s_{j+1} \circ s_i$ *for* $i \leq j$
(3) *Lastly,*

$$d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & \text{if } i < j \\ \mathrm{id} & \text{if } i = j, j+1 \\ s_j \circ d_{i-1} & \text{if } i > j+1 \end{cases}$$

A dual statement holds for cosimplicial objects. These (dual) properties are obvious in the case of Δ. We next construct a functor which is adjoint to $\mathrm{Sing}(-)_\bullet$.

Let X be a simplicial object in a category $\mathcal{C}$.

Definition 2.0.6. An n-simplex $x \in X_n$ is *degenerate* if there exists some $y \in X_{n-1}$ such that $x = s_i(y)$ for some $0 \leq i \leq n-1$.

Definition 2.0.7. The *geometric realisation of* X is the space given by

$$|X_\bullet| = \coprod_{n \geq 0} X_n \times \Delta^n \Big/ \sim$$

where the equivalence relation is given by identifying $(f^* x, u) \sim (y, f_* u)$ for any $f \in \Delta([m],[n])$, and $x \in X_n$, and $u \in \Delta^m$.

It turns out that it is enough to consider when f is a degeneracy, or face map. This is an obvious result from 2.0.1 and the following lemma,

Lemma 2.0.8. *Every object* $x \in X_n$ *may be written uniquely as* $s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_l}(y)$ *such that* $i_1 > \cdots > i_n$ *and* $y \in X_\bullet$ *is non-degenerate.*

Proof. By (2) of 2.0.5 it is easy to see that we can find an increasing sequence i_j. Furthermore this sequence must terminate since $x \in X_n$ and each degeneracy map increases degree by 1. Thus we obtain a y such that $x = s_{i_1} \circ \cdots \circ s_{i_n}(y)$. All that is left to check is that these are unique. This is immediate by starting at x and working backwards. ☐

Corollary 2.0.9. *Given a simplicial object* X *in* $\mathcal{C}$ *then its geometric realisation* $|X|$ *is a CW-complex with an* n*-cell for every non-degenerate* n*-simplex* $x \in X_n$.

For a proof of this see page 56 of [1].

Proposition 2.0.10. *The functors* $|-| : \mathsf{sSet} \rightleftarrows \mathsf{Top} : \mathrm{Sing}$ *are adjoint.*

Proof. We wish to show that, given any $X_\bullet \in \mathsf{sSet}$ and any $Y \in \mathsf{Top}$, the sets $\mathsf{sSet}(X_\bullet, \mathrm{Sing}(Y)_\bullet)$ and $\mathsf{Top}(|X_\bullet|, Y)$ are in bijection. This proof relies on the fact that any map $f_n \in \mathsf{sSet}(X_n, \mathrm{Sing}(Y)_n)$ corresponds to a map $\widetilde{f_n} : X_n \times \Delta^n_{\mathsf{Top}} \to Y$ where $\widetilde{f_n}(x,u) = f_n(x)(u)$. If we now consider any map $g \in \Delta([m],[n])$ then we get the following commutative diagram by definition

$$\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d, "X(g)"] & \mathsf{sSet}(\Delta^{\mathsf{n}}_{\mathsf{Top}}, \mathsf{Y}) \arrow[d, "(\Delta^\bullet_{\mathsf{Top}}(g))^*"] \\ X_m \arrow[r, "f_m"] & \mathsf{sSet}(\Delta^{\mathsf{m}}_{\mathsf{Top}}, \mathsf{Y}) \end{tikzcd}$$

By the existence of $\widetilde{f_n}$ in relation to f_n this gives us the following commutative diagram

$$\begin{tikzcd} X_n \times \Delta^m_{Top} \arrow[r, "\mathrm{id} \times \Delta^\bullet_{\mathsf{Top}}(g)"] \arrow[d, "X(g)\times \mathrm{id}"] & X_n \times \Delta^n_{\mathsf{Top}} \arrow[d, "\widetilde{f_n}"] \\ X_m \times \Delta^m_{\mathsf{Top}} \arrow[r, "\widetilde{f_m}"] & \mathsf{sSet}(\Delta^{\mathsf{m}}_{\mathsf{Top}}, \mathsf{Y}) \end{tikzcd}$$

Taking g to be (a composition of the) s^i gives us that $\widetilde{f_n}$ respects the equivalence relation defining $|X_\bullet|$ and hence is a map $\widetilde{f_n} \in \mathsf{Top}(|X_\bullet|, Y)$. We can go backwards in a similar fashion. $\square$

References

[1] Jon Peter May, *Simplicial Objects in Algebraic Topology*, University of Chicago Press, 1982