LECTURE 2 – SIMPLICIAL SETS

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1. Notation

Categories will typically be sans serif (*i.e.* Top, sSet, etc.). If $\mathcal{C}$ is an arbitrary category then we denote the the set of $\mathcal{C}$-morphisms from objects $A, B \in \mathcal{C}$ by $\mathcal{C}(A, B)$.

2. Garbage

Recall the definition of the *standard topological n-simplex* as the set

$$\Delta^n_{\mathsf{Top}} := \left\{ (t_0, \cdots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i \right\}$$

An alternative way of thinking of Δ^n_{Top} is as the convex hull of vertices $v_i = (0, \ldots, 1, \ldots, 0)$. We then have maps codegeneracy maps $s^i : \Delta^{n+1}_{\mathsf{Top}} \to \Delta^n_{\mathsf{Top}}$, and coface maps $d^i : \Delta^{n-1}_{\mathsf{Top}} \to \Delta^n_{\mathsf{Top}}$ defined by

$$s^i(t_0, \ldots, t_{n+1}) = (t_0, \ldots, t_i + t_{i+1}, \ldots, t_{n+1})$$
$$d^i(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_i, 0, t_{i+1}, \ldots, t_n)$$

Clearly d^i is just the map embedding $\Delta^{n-1}_{\mathsf{Top}}$ as the i^{th} face of Δ^n_{Top}. Similarly s^i is a retration of $\Delta^{n+1}_{\mathsf{Top}}$ minus the i^{th} vertice v_i onto the face opposite v_i.

Given any topological space $X \in \mathsf{Top}$ we define the *singular n-simplices of X* to be the maps

$$\mathrm{Sing}(X)_n := \mathsf{Top}(\Delta^n_{\mathsf{Top}}, X)$$

This turns out to be an important example of something called a simplicial object. Before we can define what a simplicial object is, we must first define the *simplex category* Δ. The objects of Δ are the ordered sets $[n] = \{0, 1, \ldots, n\}$, and the morphisms $f : [m] \to [n]$ are the weakly-order-preserving (*i.e.* non-decreasing) functions. Similarly to above we have maps $s^i : [n+1] \to [n]$ and $d^i : [n-1] \to [n]$ given by repeating the i, and skipping i respectively. The following lemma says that these maps are the only maps we care about.

Lemma 2.0.1. *Any map f in Δ is the composition of a series of d^i and s^j.*

Proof sketch. If you have a map $f : [m] \to [n]$ then you have the inequality $f(0) \leq f(1) \leq \cdots \leq f(m)$. We get unique elements of the form $g_0 < \cdots < g_k$ for $k \leq \min\{m, n\}$. By composing d^0 with itself g_0 times we get a map that sends $[m]$ to $[g_0, g_0 + 1, \ldots, g_0 + m]$. We repeat g_0 with s^0 as many times as it occurs in the sequence of $f(i)$s. We then simply repeat this process inductively on the g_i for $0 < i \leq k$. □

Definition 2.0.2. A *simplicial object* is a functor $X : \Delta^{\mathrm{op}} \to \mathsf{Set}$. Similarly a *cosimplicial object* is a functor $X : \Delta \to \mathsf{Set}$.

References

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