LECTURE 2 – SIMPLICIAL SETS

NOTES ANOTATED BY ROGER MURRAY

1. Notation

Categories will typically be sans serif (*i.e.* Top, sSet, etc.) or caligraphric (*i.e.* $\mathcal{C}$, $\mathcal{S}$, etc.). If $\mathcal{C}$ is an arbitrary category then we denote the the set of $\mathcal{C}$-morphisms from objects $A, B \in \mathcal{C}$ by $\mathcal{C}(A, B)$.

2. comments for readers

This is a set of notes based on lectures given by Vigleik Angeltveit at the Australian National University in the (AUS) summer of 2019. For quite a few people who are taking this course this is their first early-graduate level course in algebraic topology, and hence I have tried to be very explicit (read: overly explicit) in much of the exposition. Furthermore, due to the pace of the course, many of the proofs are simplified or omitted. Where this has occured I have either attempted to generate my own proof, or I have replicated arguments from places such as [Hov], [LURIE...] or nLab. Finally, since most people attending the class have complete board notes from the lectures, in order for these notes to be somewhat useful (for myself as writing practise, and for the students taking this course) I have inserted my own exposition and explanation throughout, and I may slightly deviate from Vigleik's own notes in certain places based on my own personal preferences.

These notes are constantly being updated as the course won't be finished until at least the end of January, and I encourage you to check in regularly.

3. model categories and some classical examples

3.1. Model categories.

Definition 3.1.1. A *model category* is a category $\mathcal{C}$ together with 3 distinct classes of maps - fibrations, cofirbrations and weak equivalences - satisfying 5 axioms. If we denote any fibration by $A \twoheadrightarrow B$, any cofibration by $A \rightarrowtail B$, and any weak equivalence by $A \xrightarrow{\sim} B$, then these axioms may be stated as follows:

(M1) $\mathcal{C}$ is bicomplete

(M2) Weak equivalences satisfy the 2/3 axiom, *i.e.* given any diagram in $\mathcal{C}$ of the form

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & B \\ & C \arrow[ur] & \end{tikzcd}$$

such that any two of the maps are weak equivalences, then the third must be one also.

(M3) Any map $f \in \mathcal{C}(A, B)$ can be factored for some $C \in \mathcal{C}$ as either

$$A \overset{\sim}{\underset{g}{\rightarrowtail}} C \underset{h}{\twoheadrightarrow} B \qquad \text{or} \qquad A \overset{g}{\rightarrowtail} C \overset{\sim}{\underset{h}{\twoheadrightarrow}} B$$

(M4) Given either of the following commutative squares,

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "g"', "\wr"] & X \arrow[d, two heads, "f"] \\ B \arrow[r] & Y \end{tikzcd} \qquad \text{or} \qquad \begin{tikzcd} A \arrow[r] \arrow[d, tail, "g"'] & X \arrow[d, two heads, "f", "\wr"'] \\ B \arrow[r] & Y \end{tikzcd}$$

then there is a (not necessarily unique) lift $B \to X$ which commutes with the given diagram.

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(M5) If f is a retract of g, and if g is a fibration/cofibration/weak-equivalence then f is also.

Recall that we say $f : A \to B$ is a retract of $g : C \to D$ if their exist maps i, r, j, s making the following diagram commute,

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "f"] \arrow[rr, bend left, "\mathrm{id}"] & C \arrow[r, "r"] \arrow[d, "g"] & A \arrow[d, "f"] \\
B \arrow[r, "j"] \arrow[rr, bend right, "\mathrm{id}"'] & B \arrow[r, "s"] & B
\end{tikzcd}$$

Example 3.1.2. The proto-typical example is Top with fibrations the Serre fibrations, and cofibrations and weak-equivalences decided accordingly (see chapter 4 of [Hat]).

3.2. **Simplicies and simlpicial sets.** Recall the definition of the *standard topological n-simplex* as the set

$$\Delta^n_{\mathsf{Top}} := \left\{ (t_0, \cdots, t_n) \in \mathbb{R}^{n+1} \,\middle|\, \sum t_i = 1 \text{ and } t_i \geqslant 0 \text{ for all } i \right\}$$

Alternatively we may think of Δ^n_{Top} as the convex hull of vertices $v_i = (0, \ldots, 1, \ldots, 0)$. We then have *codegeneracy maps* $s^i : \Delta^{n+1}_{\mathsf{Top}} \to \Delta^n_{\mathsf{Top}}$, and *coface maps* $d^i : \Delta^{n-1}_{\mathsf{Top}} \to \Delta^n_{\mathsf{Top}}$ defined by

$$s^i(t_0, \ldots, t_{n+1}) = (t_0, \ldots, t_i + t_{i+1}, \ldots, t_{n+1})$$
$$d^i(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_i, 0, t_{i+1}, \ldots, t_n)$$

Clearly d^i is just the map embedding $\Delta^{n-1}_{\mathsf{Top}}$ as the i^{th} face of Δ^n_{Top}, and s^i is a retration of $\Delta^{n+1}_{\mathsf{Top}}$ minus the i^{th} vertice v_i onto the face opposite v_i.

Given any topological space $X \in \mathsf{Top}$ we define the *singular n-simplices of X* to be the set of maps

$$\mathrm{Sing}(X)_n := \mathsf{Top}(\Delta^n_{\mathsf{Top}}, X)$$

This turns out to be an important example of something called a simplicial object. Before we can define what a simplicial object is, we must first define the *simplex category* Δ. The objects of Δ are the ordered sets $[n] = \{0, 1, \ldots, n\}$, and the morphisms $f : [m] \to [n]$ are the weakly-order-preserving (*i.e.* non-decreasing) functions. Similarly to above we have maps $s^i : [n+1] \to [n]$ and $d^i : [n-1] \to [n]$ given by repeating the i, and skipping i respectively. The following lemma says that these maps are the only maps we care about.

Lemma 3.2.1. *Any map f in Δ is the composition of some d^i and s^j.*

Proof sketch. If you have a map $f : [m] \to [n]$ then you have the inequality $f(0) \leqslant f(1) \leqslant \cdots \leqslant f(m)$. We obtain a unique chain of elements of the form $g_0 < \cdots < g_k$ for $k \leqslant \min\{m, n\}$. By composing d^0 with itself g_0 times we get a map that sends $[m]$ to $[g_0, g_0 + 1, \ldots, g_0 + m]$. We repeat g_0 with s^0 as many times as it occurs in the sequence of $f(i)$s. We then simply repeat this process inductively on the g_i for $0 < i \leqslant k$. $\square$

Definition 3.2.2. Let $\mathcal{C}$ be a category. A *simplicial object in $\mathcal{C}$* is a functor $X : \Delta^{\mathrm{op}} \to \mathcal{C}$. Similarly a *cosimplicial object in $\mathcal{C}$* is a functor $Y : \Delta \to \mathcal{C}$.

We write X_n and Y^n for $X([n])$ and $Y([n])$ respectively, and hence we will often use the notation $X_\bullet$ and $Y^\bullet$ for X and Y respectively. Similarly we write d_i, s_i and d^i, s^i for the obvious maps in $\mathcal{C}$.

Example 3.2.3. The standard simplices $\Delta^\bullet_{\mathsf{Top}}$ is a cosimplicial space.

Example 3.2.4. Given any space $X \in \mathsf{Top}$ we therefore have that $\mathrm{Sing}(X)_\bullet$ is a simplicial set acting on objects by $[n] \mapsto \mathsf{Top}(\Delta^n_{\mathsf{Top}}, X)$, and on morphisms by $(f : [m] \to [n]) \mapsto f_*$ where f_* is precomposition with f.

Properties 3.2.5. *The degeneracy, and face maps belonging to any simplicial object X in $\mathcal{C}$ satisfy the following,*

(1) $d_i \circ d_j = d_{j-1} \circ d_i$ *for* $i < j$
(2) $s_i \circ s_j = s_{j+1} \circ s_i$ *for* $i \leqslant j$
(3) *Lastly,*

$$d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & \text{if } i < j \\ \mathrm{id} & \text{if } i = j, j+1 \\ s_j \circ d_{i-1} & \text{if } i > j+1 \end{cases}$$

We refer to these properties as the *simplicial properties.* A dual statement holds for cosimplicial objects. These (dual) properties are obvious in the case of Δ. We next construct a functor which is adjoint to $\mathrm{Sing}(-)_\bullet$.

Let X be a simplicial object in a category $\mathcal{C}$.

Definition 3.2.6. An n-simplex $x \in X_n$ is *degenerate* if there exists some $y \in X_{n-1}$ such that $x = s_i(y)$ for some $0 \leqslant i \leqslant n-1$.

Definition 3.2.7. The *geometric realisation of* X is the space given by

$$|X_\bullet| = \coprod_{n \geqslant 0} X_n \times \Delta^n_{\mathsf{Top}} \Big/ \sim$$

where the equivalence relation is given by identifying $(f^*x, u) \sim (y, f_*u)$ for any $f \in \Delta([m],[n])$, and $x \in X_n$, and $u \in \Delta^m_{\mathsf{Top}}$.

By 3.2.1 it is enough to consider when f is a degeneracy, or face map.

We have the following useful lemma,

Lemma 3.2.8. *Every object* $x \in X_n$ *may be written uniquely as* $s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_l}(y)$ *such that* $i_1 > \cdots > i_n$ *and* $y \in X_\bullet$ *is non-degenerate.*

Proof. By (2) of 3.2.5 it is easy to see that we can find an increasing sequence i_j. Furthermore this sequence must terminate since $x \in X_n$ and each degeneracy map increases degree by 1. Thus we obtain a y such that $x = s_{i_1} \circ \cdots \circ s_{i_n}(y)$. All that is left to check is that these are unique. This is immediate by starting at x and working backwards. □

Given two simplicial sets X and Y we obtain another simplicial set $X \times Y$ defined by $(X \times Y)_n = X_n \times Y_n$. The above lemma implies that we may combine two degenerate simplices $x \in X_n$ and $y \in Y_n$ to get a non-degenerate simplex (x, y). This will be important below.

Lemma 3.2.9. *Given a simplicial set* X *its geometric realisation* $|X|$ *is a CW-complex with an* n*-cell for every non-degenerate* n*-simplex* $x \in X_n$.

For a classical proof of this see page 56 of [May].

Proposition 3.2.10. *The functors* $|-| : \mathsf{sSet} \overset{\longleftarrow}{\longrightarrow} \mathsf{Top} : \mathrm{Sing}$ *are adjoint.*

Proof. We wish to show that, given any $X_\bullet \in \mathsf{sSet}$ and any $Y \in \mathsf{Top}$, the sets $\mathsf{sSet}(X_\bullet, \mathrm{Sing}(Y)_\bullet)$ and $\mathsf{Top}(|X_\bullet|, Y)$ are in bijection. This proof relies on the fact that any map $f_n \in \mathsf{sSet}(X_n, \mathrm{Sing}(Y)_n)$ corresponds to a map $\widetilde{f_n} : X_n \times \Delta^n_{\mathsf{Top}} \to Y$ where $\widetilde{f_n}(x, u) = f_n(x)(u)$. If we now consider any map $g \in \Delta([m],[n])$ then we get the following commutative diagram by definition

$$\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d, "X(g)"] & \mathsf{sSet}(\Delta^n_{\mathsf{Top}}, \mathsf{Y}) \arrow[d, "(\Delta^\bullet_{\mathsf{Top}}(g))^*"] \\ X_m \arrow[r, "f_m"] & \mathsf{sSet}(\Delta^m_{\mathsf{Top}}, \mathsf{Y}) \end{tikzcd}$$

By the existence of $\widetilde{f_n}$ in relation to f_n this gives us the following commutative diagram

$$\begin{tikzcd} X_n \times \Delta^m_{\mathsf{Top}} \arrow[r, "\mathrm{id} \times \Delta^\bullet_{\mathsf{Top}}(g)"] \arrow[d, "X(g)\times \mathrm{id}"] & X_n \times \Delta^n_{\mathsf{Top}} \arrow[d, "\widetilde{f_n}"] \\ X_m \times \Delta^m_{\mathsf{Top}} \arrow[r, "\widetilde{f_m}"] & \mathsf{sSet}(\Delta^m_{\mathsf{Top}}, \mathsf{Y}) \end{tikzcd}$$

Taking g to be (a composition of the) s^i gives us that $\widetilde{f_n}$ respects the equivalence relation defining $|X_\bullet|$ and hence is a map $\widetilde{f_n} \in \mathsf{Top}(|X_\bullet|, Y)$. We can go backwards in a similar fashion. □

Example 3.2.11. We define the *standard n-simplex* Δ^n as the simplicial set given by letting its k-simplices be the set of maps $\Delta([k], [n])$.

Lemma 3.2.12. *A k-simplex $x \in \Delta([k], [n])$ is non-degenerate if and only if x is injective.*

Proof. This is immediate once one considers the effect of s_i (which is precomposition by s^i). □

Lemma 3.2.13. *The geometric realisation of Δ^n is Δ^n_{Top}. Similarly the geometric realisation of $\Delta^n \times \Delta^m$ is $\Delta^n_{\mathsf{Top}} \times \Delta^m_{\mathsf{Top}}$.*

Proof. This first statement is a result of 3.2.9. The second statement also requires 3.2.8 □

Example 3.2.14. We then define a simplicial set called *the boundary $\partial\Delta^n$ of Δ^n*. We define this by specifying that $(\partial\Delta^n)_k$ is the set of non-surjective maps $x \in \mathsf{sSet}([n], [k])$. Indeed $\partial\Delta^n$ is a subsimplex of Δ^n, and it matches our intuition from Δ^n_{Top} since any element of $(\Delta^n)_k$ which is the image of some d^i is clearly not surjective.

Lemma 3.2.15 (The Yoneda Lemma)**.** *Let A be an object in a category $\mathcal{C}$ and let $F : \mathcal{C} \to \mathsf{Set}$ be a functor. There is an isomorphism of sets*

$$\mathrm{Natl}\left(\mathcal{C}(A, \_), F\right) \cong F(A)$$

It is not hard to find a proof of this, and even though the lemma is ubiquitous throughout category theory the proof is very straightforward.

An immediate consequence of the Yoneda lemma is,

Lemma 3.2.16. *Given a simplicial set $X_\bullet$ we have that*

$$X_n \cong \mathsf{sSet}(\Delta^n, X_\bullet)$$

4. HORNS AND NERVES

Definition 4.0.1. The horn Λ^n_k is the simplicial subset of $\partial\Delta^n$ defined by

$$(\Lambda^n_k)_i = \left\{x \in \Delta^n_k \,\middle|\, x([k]) \cup \{i\} \supsetneq [n]\right\}$$

Intuitively one thinks of this as the boundary of Δ^n excluding the i^{th} face.

Definition 4.0.2. Let $\mathcal{C}$ be a small category. The *nerve of $\mathcal{C}$* is the simplicial set $N\mathcal{C}$ defined by

$$N\mathcal{C}_n = \left\{C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} C_n\right\}$$

The maps s_i and d_i are defined as follows. Let $(f_1, \ldots, f_n)$ be an ordered n-tuple of composable maps in $\mathcal{C}$. We define s_i to simply be the map $(f_1, \ldots, \mathrm{id}, \ldots, f_n) \in N\mathcal{C}_{n+1}$. The map d_i is slightly more complicated.

$$d_i(f_1, \ldots, f_n) = \begin{cases} (f_2, \ldots, f_n) & \text{if } i = 0, \\ (f_1, \ldots, f_i \circ f_{i+1}, \ldots, f_n) & \text{if } 0 < i < n, \\ (f_1, \ldots, f_{n-1}) & \text{if } i = n \end{cases}$$

An important example of such a construction is the nerve of the category G associated to a group G. In this case $N\mathsf{G}_n = G^n$ and thus the geometric realisation of $N\mathcal{C}$ is

$$\coprod G^n \times \Delta^n_{\mathsf{Top}} / \sim$$

This is isomorphic to the Eilenberg Maclane space $K(G,1)$.

Let X be a simplicial set. We define $M : \mathsf{sSet} \to \mathsf{Ch}_{\geqslant 0}$ by specifying $M(X)_n$ to the free abelian group generated by the non-degenerate $n-$*simplices* of X. Our boundary maps are then the maps $d(x) = \sum_i (-1)^i d_i(x)$. Another way of stating this is that, by 3.2.9, $M(X)$ is the singular chain complex of $|X|$.

Theorem 4.0.3. *Let X and Y be simplicial sets. Then $M(X \times Y) \cong M(X) \otimes M(Y)$*

This follows from the following theorem,

Theorem 4.0.4 (The Eilenberg-Zilber theorem)**.** *Given topological spaces X, Y, and their product $X \times Y$, Then*

$$C_\bullet(X) \otimes C_\bullet(Y) \cong C_\bullet(X \times Y)$$

in the category of chain complexes.

Proof. See [EZ]. □

Proof of 4.0.3. $(M(X) \otimes M(Y))_n = \oplus_{i+j=n} M(X)_i \otimes M(Y)_j$. By 3.2.1 we have that (x, y) is non-degenerate when either x or y is non-degenerate, or when they are incompatibly degenerate. By incompatibly degenerate I mean that they decompose as $x = s_{i_l} \circ \cdots \circ s_{i_1} x'$ and $y = s_{j_k} \circ \cdots \circ s_{j_1} y'$ such that $s_{i_r} \neq s_{j_s}$ for all r, s. Let us refer to l and k from the decomposition above as the order of degeneracy for arbitrary degenerate elements x and y respectively. A non-degenerate element has order of degeneracy 0. To see that the inequality holds note that if (x, y) is non-degenerate in $(X \times Y)_n$ then we must have that the total order of degeneracy $l + k$ is less than or equal to n. This follows since if $l + k > n$ then x and y could not possibly be incompatibly degenerate by the pigeonhole principle. We may restate this succinctly as (x, y) non-degenerate in $X_n \times Y_n$ is a pair $(s_{i_l} \circ \cdots s_{i_1} x', s_{j_k} \circ \cdots \circ s_{j_1} y')$ in $X_n \times Y_n$ where $j_r \neq i_s$ and where x' is non-degenerate in X_{n-l} and y' is non-degenerate in Y_{n-k}, where $l + k \leqslant n$. Consider the pair (x', y') in $X_{n-l} \times Y_{n-k}$. We therefore have that $|X \times Y| = |X| \times |Y|$. This allows us to use the Eilenberg-Zilber theorem, which gives us the result (by the discussion preceeding 4.0.3). □

5. sSet as a model category

We can put a model category structure on the category of simplicial sets. We first define two sets $I = \bigcup_{n \geqslant 0} \mathsf{sSet}(\partial\Delta^n, \Delta^n)$ and $J = \{\mathsf{sSet}(\Lambda^n_k, \Delta^n) \mid n > 0 \text{ and } 0 \leqslant k \leqslant n\}$.

Definition 5.0.1. Given a map $f \in \mathsf{sSet}(X, Y)$, we say f is

- a fibration if $J \boxslash f$, which is to say that if we have any map $p \in J$, and any $\Lambda^n_k \to X$ and $\Delta^n \to Y$, making the following diagram commute,

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, "p"] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] & Y \end{tikzcd}$$

  then we must have a lift $h : \Delta^n \to X$ which commutes with the diagram,
- a weak equivalence if $|f| : |X| \to |Y|$ is a weak equivalence in Top,
- a cofibration if it is injective.

Lemma 5.0.2. *This defines a model structure on* sSet.

Proof. (I think we may be proving this later but if not I'm sure it's somewhere in [Hov].) □

You may be wondering why we went to the effort of defining the set I when it appears we haven't used it in defining a model structure on sSet. Our reason for including it will become apparent in the following section, where we define weak factorisation systems.

6. Weak factorisation systems and the small object argument

Sadly, given an arbitrary model category $\mathcal{M}$, we can't expect the factorisation of a map (axiom M3) to be functorial. Indeed if you take (bounded) chain complexs of abelian groups then the factorisation of the composite of two maps is not necessarily the same as the composite of the factorisation of the individual maps. This occurs because of an extension problem. One way to rectify this is to simply require your factorisations to be functorial in M3 as in [Hov]. Another way, however is to consider a nice subclass of model categories called *cofibrantly generated model categories*. Finally we give a useful method for constructing cofibrantly generated model categories from the small object argument due to Quillen [REFFF].

6.1. **Weak factorisation systems.**

Let us first observe some properties of model categories as we have defined them.

From now on, given an arbitrary model category $\mathcal{M}$, let us denote the classes of fibrations by Fib, cofibrations by $coFib$ and weak-equivalences by $\mathcal{W}$.

Lemma 6.1.1. *Given a model category $\mathcal{M}$ and a map $f \in \mathcal{M}(A, B)$, then*

(1) *f is a cofibration if and only if $f \boxslash (Fib \cap \mathcal{W})$*
(2) *f is an acyclic cofibration if and only if $f \boxslash Fib$*

The obvious mirror statement is true for fibrations,

Lemma 6.1.2. *Given a model category $\mathcal{M}$ and a map $f \in \mathcal{M}(A, B)$, then*

(1) *f is a fibration if and only if $(coFib \cap \mathcal{W}) \boxslash f$*
(2) *f is an acyclic fibration if and only if $coFib \boxslash f$*

Proof. We begin by proving 6.1.1.1. The forward direction follows immediately by definition. For the reverse direction suppose we have a map $f : A \to B$ in $\mathcal{M}$ which lifts against all acyclic fibrations. Then by (M3) we may factor f as

$$\begin{tikzcd} A \arrow[r, tail, "i"'] & C \arrow[r, two heads, "\sim", "j"'] & B \end{tikzcd}$$

for some object $C \in \mathcal{M}$. This gives us the following commutative diagram,

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"'] & C \arrow[d, two heads, "j", "\wr"'] \\ B \arrow[r, equal] & B \end{tikzcd}$$

and since f lifts against all trivial fibrations we obtain a lift $h : B \to C$. It's straightforward to see that the following diagram commutes as a result,

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "f"'] & A \arrow[r, equal] \arrow[d, tail, "g"] & A \arrow[d, "f"] \\ B \arrow[r, "h"'] & C \arrow[r, "\sim", "j"'] & B \end{tikzcd}$$

Thus f is a retract of g which is a cofibration and therefore, by (M5), f is also a cofibration.

The proof of the other statements in lemmas 6.1.1 and 6.1.2 are essentially identical. $\square$

Corollary 6.1.3.

(1) *$coFib$ is closed under pushouts*
(2) *$coFib \cap \mathcal{W}$ is closed under pushouts*
(3) *similarly for Fib and $Fib \cap \mathcal{W}$.*

Proof. Suppose we have a pushout square

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, tail, "g"'] & C \arrow[d, "f"] \\ B \arrow[r, "j"'] & D \end{tikzcd}$$

We wish to show that f is a cofibration. By the above lemmas it is sufficient to show that f has the left lifting property in relation to $Fib \cap \mathcal{W}$. Let us consider the following diagram,

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, tail, "g"'] & C \arrow[r] \arrow[d, "f"] & X \arrow[d, two heads, "\wr"] \\ B \arrow[r, "j"'] & D \arrow[r] & Y \end{tikzcd}$$

If the righthand square (and hence the whole diagram) commutes then since g is a cofibration we get a lift of the outer square $h : B \to X$. Now by appealing to the universal property of a pushout square we obtain a map $h' : D \to X$ which is a lift for f, and hence f is a cofibration.

The proof for the other cases is similar. □

Notation 6.1.4. Given a set of maps S in a category $\mathcal{D}$ let us denote the set of maps which lift on the left of all maps in S by ${}^{\boxslash}S$, and all maps which lift on the right by $S^{\boxslash}$. We say a map $\sigma \in {}^{\boxslash}S$ has the left-lifting property with respect to S, and similarly for $\sigma \in S^{\boxslash}$.

We next present an important definition.

Definition 6.1.5. Given a category $\mathcal{C}$ a weak factorisation system (WFS) on $\mathcal{C}$ is a pair (L, R), where L and R are classes of maps satisfying the following conditions,

(1) any map $f \in \mathcal{C}(X, Y)$ factors as $X \xrightarrow{i} Z \xrightarrow{j} Y$ where $i \in L$ and $j \in R$.
(2) $L = {}^{\boxslash}R$ and $R = L^{\boxslash}$

Example 6.1.6. By lemma 6.1.1 and 6.1.2 any model category $\mathcal{M}$ has at least two weak factorisation systems on itself, namely $(coFib, Fib \cap \mathcal{W})$ and $(coFib \cap \mathcal{W}, Fib)$.

6.2. **Cofibrantly generated model categories.**

Definition 6.2.1. Given a set I of maps in $\mathcal{C}$, a *relative I-cell* is the pushout of transfinite-compositions of maps in I. We denote to the set of I-cells in $\mathcal{C}$ by $\mathrm{cell}(I)$.

Let us unpack this. First suppose we have some ordinal α, and some functor $X_\bullet : \alpha \to \mathcal{C}$ with the properties that

(1) the successor map is sent to a‘ map $X_\bullet(s) : X_\bullet \to X_{\bullet+1}$ with $X_i(s) \in I$ for all $i \in \alpha$
(2) given any ordinal $a \in \alpha$ we have that: $X_a = \operatorname{colim} X_\bullet$ where the colimit is taken over the set of ordinals less than a.

Then a relative I-cell is the map $X_0 \to X_\alpha := \operatorname{colim} X_\bullet$ induced by composing the successor maps,

$$X_0 \xrightarrow{X_0(s)} X_1 \xrightarrow{X_1(s)} X_2 \longrightarrow \cdots \longrightarrow X_\alpha$$

It's easy to see that the definition presented in class is this definition in the special case that $\alpha = \omega$.

Definition 6.2.2. An object $A \in \mathcal{C}$ is a *small object in $\mathcal{C}$* (or a κ-small object) if there exists some cardinal κ such that the functor $\hom_{\mathcal{C}}(A, \_)$ preserves κ-directed colimits.

The last property (which is refered to as being *κ-compact*) says that given some poset J with the property that any sub-poset J' of cardinality $< \kappa$ has an upper-bound in J, and some functor $F : \mathsf{J} \to \mathcal{C}$, then the canonical map

$$\operatorname{colim}_{\mathsf{j} \in \mathsf{J}} \hom_{\mathcal{C}}(A, F(\mathsf{j})) \longrightarrow \hom_{\mathcal{C}}(A, \operatorname{colim}_{\mathsf{j} \in \mathsf{J}} F(\mathsf{j}))$$

arising from the universal property of the lefthand side (after mapping each $\hom_{\mathcal{C}}(A, F(d)) \to \hom_{\mathcal{C}}(A, \operatorname{colim}_d F(d))$ by post-composition with the obvious inclusion $F(d) \to \operatorname{colim}_d F(d))$ is actually an isomorphism.

We are also interested in the case where we have some full-subcategory $\mathcal{D} \subset \mathcal{C}$. In this case our we simply replace the functor above by $F : \mathsf{J} \to \mathcal{D}$, and leave everything else the same. In this context we say A is *small relative to* $\mathcal{D}$.

Definition 6.2.3. A model category $\mathcal{M}$ is *cofibrantly generated* if there are sets of morphisms I, J such that

(1) The retracts of any I-cell are all the cofibrations
(2) The retracts of any J-cell are all the acyclic cofibrations
(3) The domains of maps in I are small relative to $\operatorname{cell}(I)$
(4) The domains of maps in J are small relative to $\operatorname{cell}(J)$

The idea of this definition is that we think of I as generating the set of cofibrations, and we think of J as generating the trivial cofibrations. We make this precise in the following proposition,

Proposition 6.2.4. *Suppose $\mathcal{M}$ is a cofibrantly generated model category. Then the retracts of* $\operatorname{cell}(I)$ *are exactly the elements of* ${}^{\boxslash}(I^{\boxslash})$*, and the retracts of* $\operatorname{cell}(J)$ *are exactly the elements of* ${}^{\boxslash}(J^{\boxslash})$

The proof will be delayed until later.

6.3. **The small object argument.**

Theorem 6.3.1 (Small Object Argument)**.** *Let I be a set of maps in a category $\mathcal{C}$. If $\mathcal{C}$ is cocomplete, and if the domain of any map in I is small, then all maps $f \in \mathcal{C}(X, Y)$ factor as*

$$X \xrightarrow{i} Z \xrightarrow{s} Y$$

where i is a relative I-cell, and $s \in I^{\boxslash}$

Proof. Given some morphism $f \in \mathcal{C}(X, Y)$ consider the set of all commutative squares of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd}$$

where $i \in I$. Call this set of squares S. Taking coproducts gives rise to the following diagram,

$$\begin{tikzcd} \coprod_{s \in S} A_s \arrow[r] \arrow[d, "\coprod i_s"'] & X \arrow[d, "f"] \\ \coprod_{s \in S} B_s \arrow[r] & Y \end{tikzcd}$$

We can take a pushout to get the following diagram,

$$\begin{tikzcd} \coprod_{s \in S} A_s \arrow[r] \arrow[d, "\coprod i_s"'] & X \arrow[d] \arrow[ddr, bend left, "f"] & \\ \coprod_{s \in S} B_s \arrow[r] \arrow[drr, bend right] & Z_1 \arrow[dr, dashed] & \\ & & Y \end{tikzcd}$$

Choose some cardinal κ such that all domains in I are κ-small. We may assume that κ is infinite since if $\alpha < \beta$ are cardinals then α-smallness implies β-smallness. We continue this inductively over the natural numbers. Given the limit ordinal ω, we then define Z_ω to be $\operatorname{colim}_{\alpha<\omega} Z_\alpha$. We repeat this process – taking pushout squares for all successor ordinals, and taking the colimit of all small successor ordinals for limit ordinals. This gives us a map $X \to Z$ (which is clearly a relative I-cell), and hence also the following commutative square,

$$\begin{tikzcd} A \arrow[r] \arrow[d] & Z \arrow[d, "s"] \\ B \arrow[r] & Y \end{tikzcd}$$

The final thing for us to show is that $s \in I^{\boxslash}$. To see that we wish to produce a lift $B \to Z$. Since we assumed that any possible choice of A is κ-small we know that the map $A \to Z$ factor through Z_α for some $\alpha < \kappa$. Taking the pushout of the obvious diagram gives us a map $B \to Z_{\alpha+1}$ which produces the lift we were looking for by composing with the inclusion $Z_{\alpha+1} \hookrightarrow Z$. Hence we may factor the map $X \to Y$ as

$$X \to Z \to Y$$

where the map $X \to Z$ is a relative I-cell, and where the map $Z \to Y$ has the right lifting property with respect to I. □

Remark 6.3.2. The set-theoretic reason why we ask for I to be a set and not a class is because we want to be guarenteed the existence of a single cardinal κ such that each of the domains of maps in I is κ-small. If we dropped this requirement, it is possible that we could find some model category, some I (perhaps a proper class), and some f such that no κ exists, and then our argument would fall apart.

We now return to the proof of 6.2.4.

Proof of 6.2.4. The proof is the same regardless of choice of I or J, so consider I. Clearly, any retract of an I-cell is in ${}^{\boxslash}(I^{\boxslash})$, so we only need to show the reverse inclusion.

Suppose we have an element $f : A \to C \in {}^{\boxslash}(I^{\boxslash})$. Then we may factor this (by the small-object argument) as the composition of maps

$$A \xrightarrow{i} B \xrightarrow{s} C$$

where i is a retract of an element of $\mathrm{cell}(I)$, and where s is in $I^{\boxslash}$. We then get the following commutative diagram,

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, "s"] \\ C \arrow[r, equal] & C \end{tikzcd}$$

and since $s \in I^{\boxslash}$, we obtain a lift $\ell : C \to B$. This fits into the following commutative diagram,

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "f"'] & A \arrow[r, equal] \arrow[d, "i"] & A \arrow[d, "f"] \\ C \arrow[r, "\ell"'] & B \arrow[r, "s"'] & C \end{tikzcd}$$

and thus f is a retract of an element of $\mathrm{cell}(I)$ as we wanted to show. □

6.4. **Constructing cofibrantly generated model categories.**

Theorem 6.4.1. *Suppose you have a bicomplete category $\mathcal{C}$, a subcategory $\mathcal{W} \subset \mathcal{C}$, and two sets of maps I and J. Then these produce a cofibrantly generated model structure if and only if*

(1) *$\mathcal{W}$ satisfies the 2/3 property*
(2) *The domains of maps in I are small relative to* $\mathrm{cell}(I)$
(3) *The domains of maps in J are small relative to* $\mathrm{cell}(J)$
(4) $\mathrm{cell}(J) \subset \mathcal{W} \cap {}^{\boxslash}(I^{\boxslash})$
(5) $I^{\boxslash} \subset \mathcal{W} \cap J^{\boxslash}$
(6) *Either* $\mathcal{W} \cap {}^{\boxslash}(I^{\boxslash}) \subset {}^{\boxslash}(J^{\boxslash})$ *or* $\mathcal{W} \cap J^{\boxslash} \subset I^{\boxslash}$

Proof. Clearly if $\mathcal{C}$ is a cofibrantly generated model category, then these conditions are satisfied. Let us consider the converse situation.

The axioms M1, M2, and M5 are satisfied trivially by definition. The factorisation axiom M3 follows from the small object argument. Hence all that's left to check is the lifting axiom M4.

Let us first assume that $\mathcal{W} \cap {}^{\boxslash}(I^{\boxslash}) \subset {}^{\boxslash}(J^{\boxslash})$. □

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