LECTURE 2 – SIMPLICIAL SETS

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1. Notation

Categories will typically be sans serif (*i.e.* Top, sSet, etc.) or caligraphric (*i.e.* $\mathcal{C}$, $\mathcal{S}$, etc.). If $\mathcal{C}$ is an arbitrary category then we denote the the set of $\mathcal{C}$-morphisms from objects $A, B \in \mathcal{C}$ by $\mathcal{C}(A, B)$.

2. Simplices

Recall the definition of the *standard topological n-simplex* as the set

$$\Delta^n_{\mathsf{Top}} := \left\{ (t_0, \cdots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum t_i = 1 \text{ and } t_i \geqslant 0 \text{ for all } i \right\}$$

Alternatively we may think of Δ^n_{Top} as the convex hull of vertices $v_i = (0, \ldots, 1, \ldots, 0)$. We then have *codegeneracy maps* $s^i : \Delta^{n+1}_{\mathsf{Top}} \to \Delta^n_{\mathsf{Top}}$, and *coface maps* $d^i : \Delta^{n-1}_{\mathsf{Top}} \to \Delta^n_{\mathsf{Top}}$ defined by

$$s^i(t_0, \ldots, t_{n+1}) = (t_0, \ldots, t_i + t_{i+1}, \ldots, t_{n+1})$$
$$d^i(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_i, 0, t_{i+1}, \ldots, t_n)$$

Clearly d^i is just the map embedding $\Delta^{n-1}_{\mathsf{Top}}$ as the i^{th} face of Δ^n_{Top}, and s^i is a retration of $\Delta^{n+1}_{\mathsf{Top}}$ minus the i^{th} vertice v_i onto the face opposite v_i.

Given any topological space $X \in \mathsf{Top}$ we define the *singular n-simplices of X* to be the set of maps

$$\mathrm{Sing}(X)_n := \mathsf{Top}(\Delta^n_{\mathsf{Top}}, X)$$

This turns out to be an important example of something called a simplicial object. Before we can define what a simplicial object is, we must first define the *simplex category* Δ. The objects of Δ are the ordered sets $[n] = \{0, 1, \ldots, n\}$, and the morphisms $f : [m] \to [n]$ are the weakly-order-preserving (*i.e.* non-decreasing) functions. Similarly to above we have maps $s^i : [n+1] \to [n]$ and $d^i : [n-1] \to [n]$ given by repeating the i, and skipping i respectively. The following lemma says that these maps are the only maps we care about.

Lemma 2.1. *Any map f in Δ is the composition of some d^i and s^j.*

Proof sketch. If you have a map $f : [m] \to [n]$ then you have the inequality $f(0) \leqslant f(1) \leqslant \cdots \leqslant f(m)$. We obtain a unique chain of elements of the form $g_0 < \cdots < g_k$ for $k \leqslant \min\{m, n\}$. By composing d^0 with itself g_0 times we get a map that sends $[m]$ to $[g_0, g_0 + 1, \ldots, g_0 + m]$. We repeat g_0 with s^0 as many times as it occurs in the sequence of $f(i)$s. We then simply repeat this process inductively on the g_i for $0 < i \leqslant k$. □

Definition 2.2. Let $\mathcal{C}$ be a category. A *simplicial object in $\mathcal{C}$* is a functor $X : \Delta^{\mathrm{op}} \to \mathcal{C}$. Similarly a *cosimplicial object in $\mathcal{C}$* is a functor $Y : \Delta \to \mathcal{C}$.

We write X_n and Y^n for $X([n])$ and $Y([n])$ respectively, and hence we will often use the notation $X_\bullet$ and $Y^\bullet$ for X and Y respectively. Similarly we write d_i, s_i and d^i, s^i for the obvious maps in $\mathcal{C}$.

Example 2.3. The standard simplices $\Delta^\bullet_{\mathsf{Top}}$ is a cosimplicial space.

Example 2.4. Given any space $X \in \mathsf{Top}$ we therefore have that $\mathrm{Sing}(X)_\bullet$ is a simplicial set acting on objects by $[n] \mapsto \mathsf{Top}(\Delta^n_{\mathsf{Top}}, X)$, and on morphisms by $(f : [m] \to [n]) \mapsto f_*$ where f_* is precomposition with f.

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Properties 2.5. *The degeneracy, and face maps belonging to any simplicial object X in $\mathcal{C}$ satisfy the following,*

(1) $d_i \circ d_j = d_{j-1} \circ d_i$ *for* $i < j$
(2) $s_i \circ s_j = s_{j+1} \circ s_i$ *for* $i \leqslant j$
(3) *Lastly,*

$$d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & \text{if } i < j \\ \mathrm{id} & \text{if } i = j, j+1 \\ s_j \circ d_{i-1} & \text{if } i > j+1 \end{cases}$$

We refer to these properties as the *simplicial properties.* A dual statement holds for cosimplicial objects. These (dual) properties are obvious in the case of Δ. We next construct a functor which is adjoint to $\mathrm{Sing}(-)_\bullet$.

Let X be a simplicial object in a category $\mathcal{C}$.

Definition 2.6. An n-simplex $x \in X_n$ is *degenerate* if there exists some $y \in X_{n-1}$ such that $x = s_i(y)$ for some $0 \leqslant i \leqslant n-1$.

Definition 2.7. The *geometric realisation of* X is the space given by

$$|X_\bullet| = \coprod_{n \geqslant 0} X_n \times \Delta^n_{\mathsf{Top}} \Big/ \sim$$

where the equivalence relation is given by identifying $(f^*x, u) \sim (y, f_* u)$ for any $f \in \Delta([m],[n])$, and $x \in X_n$, and $u \in \Delta^m_{\mathsf{Top}}$.

By 2.1 it is enough to consider when f is a degeneracy, or face map.

We have the following useful lemma,

Lemma 2.8. *Every object $x \in X_n$ may be written uniquely as $s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_l}(y)$ such that $i_1 > \cdots > i_n$ and $y \in X_\bullet$ is non-degenerate.*

Proof. By (2) of 2.5 it is easy to see that we can find an increasing sequence i_j. Furthermore this sequence must terminate since $x \in X_n$ and each degeneracy map increases degree by 1. Thus we obtain a y such that $x = s_{i_1} \circ \cdots \circ s_{i_n}(y)$. All that is left to check is that these are unique. This is immediate by starting at x and working backwards. □

Given two simplicial sets X and Y we obtain another simplicial set $X \times Y$ defined by $(X \times Y)_n = X_n \times Y_n$. The above lemma implies that we may combine two degenerate simplices $x \in X_n$ and $y \in Y_n$ to get a non-degenerate simplex (x, y). This will be important below.

Lemma 2.9. *Given a simplicial set X its geometric realisation $|X|$ is a CW-complex with an n-cell for every non-degenerate n-simplex $x \in X_n$.*

For a classical proof of this see page 56 of [May].

Proposition 2.10. *The functors* $|-| : \mathsf{sSet} \xleftrightarrow{} \mathsf{Top} : \mathrm{Sing}$ *are adjoint.*

Proof. We wish to show that, given any $X_\bullet \in \mathsf{sSet}$ and any $Y \in \mathsf{Top}$, the sets $\mathsf{sSet}(X_\bullet, \mathrm{Sing}(Y)_\bullet)$ and $\mathsf{Top}(|X_\bullet|, Y)$ are in bijection. This proof relies on the fact that any map $f_n \in \mathsf{sSet}(X_n, \mathrm{Sing}(Y)_n)$ corresponds to a map $\widetilde{f_n} : X_n \times \Delta^n_{\mathsf{Top}} \to Y$ where $\widetilde{f_n}(x, u) = f_n(x)(u)$. If we now consider any map $g \in \Delta([m],[n])$ then we get the following commutative diagram by definition

$$\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d, "X(g)"] & \mathsf{sSet}(\Delta^{\mathsf{n}}_{\mathsf{Top}}, \mathsf{Y}) \arrow[d, "(\Delta^\bullet_{\mathsf{Top}}(g))^*"] \\ X_m \arrow[r, "f_m"] & \mathsf{sSet}(\Delta^{\mathsf{m}}_{\mathsf{Top}}, \mathsf{Y}) \end{tikzcd}$$

By the existence of $\widetilde{f_n}$ in relation to f_n this gives us the following commutative diagram

$$\begin{tikzcd} X_n \times \Delta^m_{\mathsf{Top}} \arrow[r, "\mathrm{id} \times \Delta^\bullet_{\mathsf{Top}}(g)"] \arrow[d, "X(g)\times \mathrm{id}"] & X_n \times \Delta^n_{\mathsf{Top}} \arrow[d, "\widetilde{f_n}"] \\ X_m \times \Delta^m_{\mathsf{Top}} \arrow[r, "\widetilde{f_m}"] & \mathsf{sSet}(\Delta^{\mathsf{m}}_{\mathsf{Top}}, \mathsf{Y}) \end{tikzcd}$$

Taking g to be (a composition of the) s^i gives us that $\widetilde{f_n}$ respects the equivalence relation defining $|X_\bullet|$ and hence is a map $\widetilde{f_n} \in \mathsf{Top}(|X_\bullet|, Y)$. We can go backwards in a similar fashion. □

Example 2.11. We define the *standard n-simplex* Δ^n as the simplicial set given by letting its k-simplices be the set of maps $\Delta([k],[n])$.

Lemma 2.12. *A k-simplex $x \in \Delta([k],[n])$ is non-degenerate if and only if x is injective.*

Proof. This is immediate once one considers the effect of s_i (which is precomposition by s^i). □

Lemma 2.13. *The geometric realisation of Δ^n is Δ^n_{Top}. Similarly the geometric realisation of $\Delta^n \times \Delta^m$ is $\Delta^n_{\mathsf{Top}} \times \Delta^m_{\mathsf{Top}}$.*

Proof. This first statement is a result of 2.9. The second statement also requires 2.8 □

Example 2.14. We then define a simplicial set called *the boundary $\partial\Delta^n$ of Δ^n*. We define this by specifying that $(\partial\Delta^n)_k$ is the set of non-surjective maps $x \in \mathsf{sSet}([n],[k])$. Indeed $\partial\Delta^n$ is a subsimplex of Δ^n, and it matches our intuition from Δ^n_{Top} since any element of $(\Delta^n)_k$ which is the image of some d^i is clearly not surjective.

Lemma 2.15 (The Yoneda Lemma)**.** *Let A be an object in a category $\mathcal{C}$ and let $F : \mathcal{C} \to \mathsf{Set}$ be a functor. There is an isomorphism of sets*

$$\mathrm{Natl}\left(\mathcal{C}(A, \_), F\right) \cong F(A)$$

It is not hard to find a proof of this, and even though the lemma is ubiquitous throughout category theory the proof is very straightforward.

An immediate consequence of the Yoneda lemma is,

Lemma 2.16. *Given a simplicial set $X_\bullet$ we have that*

$$X_n \cong \mathsf{sSet}(\Delta^n, X_\bullet)$$

3. Horns and Nerves

Definition 3.1. The horn Λ^n_k is the simplicial subset of $\partial\Delta^n$ defined by

$$(\Lambda^n_k)_i = \left\{x \in \Delta^n_k \,\middle|\, x([k]) \cup \{i\} \supsetneq [n]\right\}$$

Intuitively one thinks of this as the boundary of Δ^n excluding the i^{th} face.

Definition 3.2. Let $\mathcal{C}$ be a small category. The *nerve of $\mathcal{C}$* is the simplicial set $N\mathcal{C}$ defined by

$$N\mathcal{C}_n = \left\{C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} C_n\right\}$$

The maps s_i and d_i are defined as follows. Let $(f_1, \ldots, f_n)$ be an ordered n-tuple of composable maps in $\mathcal{C}$. We define s_i to simply be the map $(f_1, \ldots, \mathrm{id}, \ldots, f_n) \in N\mathcal{C}_{n+1}$. The map d_i is slightly more complicated.

$$d_i(f_1, \ldots, f_n) = \begin{cases} (f_2, \ldots, f_n) & \text{if } i = 0, \\ (f_1, \ldots, f_i \circ f_{i+1}, \ldots, f_n) & \text{if } 0 < i < n, \\ (f_1, \ldots, f_{n-1}) & \text{if } i = n \end{cases}$$

An important example of such a construction is the nerve of the category G associated to a group G. In this case $N\mathsf{G}_n = G^n$ and thus the geometric realisation of $N\mathcal{C}$ is

$$\coprod G^n \times \Delta^n_{\mathsf{Top}} / \sim$$

This is isomorphic to the Eilenberg Maclane space $K(G,1)$.

Let X be a simplicial set. We define $M : \mathsf{sSet} \to \mathsf{Ch}_{\geqslant 0}$ by specifying $M(X)_n$ to the free abelian group generated by the non-degenerate $n-$*simplices* of X. Our boundary maps are then the maps $d(x) = \sum_i (-1)^i d_i(x)$. Another way of stating this is that, by 2.9, $M(X)$ is the singular chain complex of $|X|$.

Theorem 3.3. *Let X and Y be simplicial sets. Then $M(X \times Y) \cong M(X) \otimes M(Y)$*

This follows from the following theorem,

Theorem 3.4 (The Eilenberg-Zilber theorem)**.** *Given topological spaces X, Y, and their product $X \times Y$, Then*

$$C_\bullet(X) \otimes C_\bullet(Y) \cong C_\bullet(X \times Y)$$

in the category of chain complexes.

Proof. See [EZ]. □

Proof of 3.3 (IGNORE FOR NOW – NOT COMPLETE). $(M(X) \otimes M(Y))_n = \oplus_{i+j=n} M(X)_i \otimes M(Y)_j$. By 2.1 we have that (x, y) is non-degenerate when either x or y is non-degenerate, or when they are incompatibly degenerate. By incompatibly degenerate I mean that they decompose as $x = s_{i_l} \circ \cdots \circ s_{i_1} x'$ and $y = s_{j_k} \circ \cdots \circ s_{j_1} y'$ such that $s_{i_r} \neq s_{j_s}$ for all r, s. Let us refer to l and k from the decomposition above as the order of degeneracy for arbitrary degenerate elements x and y respectively. A non-degenerate element has order of degeneracy 0. To see that the inequality holds note that if (x, y) is non-degenerate in $(X \times Y)_n$ then we must have that the total order of degeneracy $l + k$ is less than or equal to n. This follows since if $l + k > n$ then x and y could not possibly be incompatibly degenerate by the pigeonhole principle. We may restate this succinctly as (x, y) non-degenerate in $X_n \times Y_n$ is a pair $(s_{i_l} \circ \cdots s_{i_1} x', s_{j_k} \circ \cdots \circ s_{j_1} y')$ in $X_n \times Y_n$ where $j_r \neq i_s$ and where x' is non-degenerate in X_{n-l} and y' is non-degenerate in Y_{n-k}, where $l + k \leqslant n$. Consider the pair (x', y') in $X_{n-l} \times Y_{n-k}$. We therefore have that $|X \times Y| = |X| \times |Y|$. This allows us to use the Eilenberg-Zilber theorem, which gives us the result (by the discussion preceeding 3.3). □

4. sSet as a Model Category

We can put a model category structure on the category of simplicial sets. We first define two sets $I = \bigcup_{n \geqslant 0} \mathsf{sSet}(\partial\Delta^n, \Delta^n)$ and $J = \left\{ \mathsf{sSet}(\Lambda^n_k, \Delta^n) \,\middle|\, n > 0 \text{ and } 0 \leqslant k \leqslant n \right\}$.

Definition 4.1. Given a map $f \in \mathsf{sSet}(X, Y)$, we say f is

- a fibration if $J \boxslash f$, which is to say that if we have any map $p \in J$, and any $\Lambda^n_k \to X$ and $\Delta^n \to Y$, making the following diagram commute,

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, "p"] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] & Y \end{tikzcd}$$

  then we must have a lift $h : \Delta^n \to X$ which commutes with the diagram,
- a weak equivalence if $|f| : |X| \to |Y|$ is a weak equivalence in Top,
- a cofibration if it is injective.

Lemma 4.2. *This defines a model structure on* sSet*.*

Proof. (I think we're showing this next lecture). □

I'm not entirey sure why we introduced the set I. I thought it would be used to define cofibrations but apparently not. Perhaps this will become apparent next week.

References

[May] Jon Peter May, *Simplicial Objects in Algebraic Topology*, University of Chicago Press, 1982

[EZ] Eilenberg, S., & Zilber, J. (1953). *On Products of Complexes*. American Journal of Mathematics, 75(1), 200-204. doi:10.2307/2372629

[BK] Bousfield, A.K., & Kan, D.M. (1972). *Homotopy Limits, Completions and Localizations*. Springer-Verlag Berlin Heidelberg.