

## LECTURE 2 – SIMPLICIAL SETS

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### 1. NOTATION

Categories will typically be sans serif (*i.e.* **Top**, **sSet**, etc.) or caligraphic (*i.e.*  $\mathcal{C}$ ,  $\mathcal{S}$ , etc.). If  $\mathcal{C}$  is an arbitrary category then we denote the the set of  $\mathcal{C}$ -morphisms from objects  $A, B \in \mathcal{C}$  by  $\mathcal{C}(A, B)$ .

### 2. SIMPLICES

Recall the definition of the *standard topological  $n$ -simplex* as the set

$$\Delta_{\text{Top}}^n := \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i \right\}$$

Alternatively we may think of  $\Delta_{\text{Top}}^n$  as the convex hull of vertices  $v_i = (0, \dots, 1, \dots, 0)$ . We then have maps codegeneracy maps  $s^i : \Delta_{\text{Top}}^{n+1} \rightarrow \Delta_{\text{Top}}^n$ , and coface maps  $d^i : \Delta_{\text{Top}}^{n-1} \rightarrow \Delta_{\text{Top}}^n$  defined by

$$s^i(t_0, \dots, t_{n+1}) = (t_0, \dots, t_i + t_{i+1}, \dots, t_{n+1})$$

$$d^i(t_0, \dots, t_{n-1}) = (t_0, \dots, t_i, 0, t_{i+1}, \dots, t_n)$$

Clearly  $d^i$  is just the map embedding  $\Delta_{\text{Top}}^{n-1}$  as the  $i^{\text{th}}$  face of  $\Delta_{\text{Top}}^n$ , and  $s^i$  is a retraction of  $\Delta_{\text{Top}}^{n+1}$  minus the  $i^{\text{th}}$  vertex  $v_i$  onto the face opposite  $v_i$ .

Given any topological space  $X \in \text{Top}$  we define the *singular  $n$ -simplices of  $X$*  to be the maps

$$\text{Sing}(X)_n := \text{Top}(\Delta_{\text{Top}}^n, X)$$

This turns out to be an important example of something called a simplicial object. Before we can define what a simplicial object is, we must first define the *simplex category*  $\Delta$ . The objects of  $\Delta$  are the ordered sets  $[n] = \{0, 1, \dots, n\}$ , and the morphisms  $f : [m] \rightarrow [n]$  are the weakly-order-preserving (*i.e.* non-decreasing) functions. Similarly to above we have maps  $s^i : [n+1] \rightarrow [n]$  and  $d^i : [n-1] \rightarrow [n]$  given by repeating the  $i$ , and skipping  $i$  respectively. The following lemma says that these maps are the only maps we care about.

**Lemma 2.0.1.** *Any map  $f$  in  $\Delta$  is the composition of a series of  $d^i$  and  $s^j$ .*

*Proof sketch.* If you have a map  $f : [m] \rightarrow [n]$  then you have the inequality  $f(0) \leq f(1) \leq \dots \leq f(m)$ . We get unique elements of the form  $g_0 < \dots < g_k$  for  $k \leq \min\{m, n\}$ . By composing  $d^0$  with itself  $g_0$  times we get a map that sends  $[m]$  to  $[g_0, g_0 + 1, \dots, g_0 + m]$ . We repeat  $g_0$  with  $s^0$  as many times as it occurs in the sequence of  $f(i)$ s. We then simply repeat this process inductively on the  $g_i$  for  $0 < i \leq k$ .  $\square$

**Definition 2.0.2.** Let  $\mathcal{C}$  be a category. A *simplicial object in  $\mathcal{C}$*  is a functor  $X : \Delta^{\text{op}} \rightarrow \mathcal{C}$ . Similarly a *cosimplicial object in  $\mathcal{C}$*  is a functor  $Y : \Delta \rightarrow \mathcal{C}$ .

We write  $X_n$  and  $Y^n$  for  $X([n])$  and  $Y([n])$  respectively, and hence we will often use the notation  $X_\bullet$  and  $Y^\bullet$  for  $X$  and  $Y$  respectively. Similarly we write  $d_i, s_i$  and  $d^i, s^i$  for the obvious maps in  $\mathcal{C}$ .

**Example 2.0.3.** The standard simplices  $\Delta_{\text{Top}}^\bullet$  is a cosimplicial space.

**Example 2.0.4.** Given any space  $X \in \text{Top}$  we therefore have that  $\text{Sing}(X)_\bullet$  is a simplicial set acting on objects by  $[n] \mapsto \text{Top}(\Delta_{\text{Top}}^n, X)$ , and on morphisms by  $(f : [m] \rightarrow [n]) \mapsto f_*$  where  $f_*$  is precomposition with  $f$ .

**Properties 2.0.5.** *The degeneracy, and face maps satisfy the simplicial properties:*

- (1)  $d_i \circ d_j = d_{j-1} \circ d_i$  for  $i < j$
- (2)  $s_i \circ s_j = s_{j+1} \circ s_i$  for  $i \leq j$
- (3) *Lastly,*

$$d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & \text{if } i < j \\ \text{id} & \text{if } i = j, j+1 \\ s_j \circ d_{i-1} & \text{if } i > j+1 \end{cases}$$

A dual statement holds for cosimplicial objects. These (dual) properties are obvious in the case of  $\Delta$ . We next construct a functor which is adjoint to  $\text{Sing}(-)_\bullet$ .

Let  $X$  be a simplicial object in a category  $\mathcal{C}$ .

**Definition 2.0.6.** An  $n$ -simplex  $x \in X_n$  is *degenerate* if there exists some  $y \in X_{n-1}$  such that  $x = s_i(y)$  for some  $0 \leq i \leq n-1$ .

**Definition 2.0.7.** The *geometric realisation* of  $X$  is the space given by

$$|X_\bullet| = \coprod_{n \geq 0} X_n \times \Delta_{\text{Top}}^n / \sim$$

where the equivalence relation is given by identifying  $(f^*x, u) \sim (y, f_*u)$  for any  $f \in \Delta([m], [n])$ , and  $x \in X_n$ , and  $u \in \Delta_{\text{Top}}^m$ .

By 2.0.1 it is enough to consider when  $f$  is a degeneracy, or face map.

We have the following useful lemma,

**Lemma 2.0.8.** *Every object  $x \in X_n$  may be written uniquely as  $s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_l}(y)$  such that  $i_1 > \cdots > i_l$  and  $y \in X_\bullet$  is non-degenerate.*

*Proof.* By (2) of 2.0.5 it is easy to see that we can find an increasing sequence  $i_j$ . Furthermore this sequence must terminate since  $x \in X_n$  and each degeneracy map increases degree by 1. Thus we obtain a  $y$  such that  $x = s_{i_1} \circ \cdots \circ s_{i_l}(y)$ . All that is left to check is that these are unique. This is immediate by starting at  $x$  and working backwards.  $\square$

Given two simplicial sets  $X$  and  $Y$  we obtain another simplicial set  $X \times Y$  defined by  $(X \times Y)_n = X_n \times Y_n$ . The above lemma implies that we may combine two degenerate simplices  $x \in X_n$  and  $y \in Y_n$  to get a non-degenerate simplex  $(x, y)$ . This will be important below.

**Lemma 2.0.9.** *Given a simplicial set  $X$  its geometric realisation  $|X|$  is a CW-complex with an  $n$ -cell for every non-degenerate  $n$ -simplex  $x \in X_n$ .*

For a classical proof of this see page 56 of [1].

**Proposition 2.0.10.** *The functors  $|-| : \mathbf{sSet} \rightleftarrows \mathbf{Top} : \text{Sing}$  are adjoint.*

*Proof.* We wish to show that, given any  $X_\bullet \in \mathbf{sSet}$  and any  $Y \in \mathbf{Top}$ , the sets  $\mathbf{sSet}(X_\bullet, \text{Sing}(Y)_\bullet)$  and  $\mathbf{Top}(|X_\bullet|, Y)$  are in bijection. This proof relies on the fact that any map  $f_n \in \mathbf{sSet}(X_n, \text{Sing}(Y)_n)$  corresponds to a map  $\widetilde{f}_n : X_n \times \Delta_{\text{Top}}^n \rightarrow Y$  where  $\widetilde{f}_n(x, u) = f_n(x)(u)$ . If we now consider any map  $g \in \Delta([m], [n])$  then we get the following commutative diagram by definition

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & \mathbf{sSet}(\Delta_{\text{Top}}^n, Y) \\ \downarrow X(g) & & \downarrow (\Delta_{\text{Top}}^\bullet(g))^* \\ X_m & \xrightarrow{f_m} & \mathbf{sSet}(\Delta_{\text{Top}}^m, Y) \end{array}$$

By the existence of  $\widetilde{f}_n$  in relation to  $f_n$  this gives us the following commutative diagram

$$\begin{array}{ccc} X_n \times \Delta_{\text{Top}}^m & \xrightarrow{\text{id} \times \Delta_{\text{Top}}^{\bullet}(g)} & X_n \times \Delta_{\text{Top}}^n \\ \downarrow X(g) \times \text{id} & & \downarrow \widetilde{f}_n \\ X_m \times \Delta_{\text{Top}}^m & \xrightarrow{\widetilde{f}_m} & \text{sSet}(\Delta_{\text{Top}}^m, Y) \end{array}$$

Taking  $g$  to be (a composition of the)  $s^i$  gives us that  $\widetilde{f}_n$  respects the equivalence relation defining  $|X_{\bullet}|$  and hence is a map  $\widetilde{f}_n \in \text{Top}(|X_{\bullet}|, Y)$ . We can go backwards in a similar fashion.  $\square$

**Example 2.0.11.** We define the *standard  $n$ -simplex*  $\Delta^n$  as the simplicial set given by letting its  $k$ -simplices be the set of maps  $\Delta([k], [n])$ .

**Lemma 2.0.12.** A  $k$ -simplex  $x \in \Delta([k], [n])$  is non-degenerate if and only if  $x$  is injective.

*Proof.* This is immediate once one considers the effect of  $s_i$  (which is precomposition by  $s^i$ ).  $\square$

**Lemma 2.0.13.** The geometric realisation of  $\Delta^n$  is  $\Delta_{\text{Top}}^n$ . Similarly the geometric realisation of  $\Delta^n \times \Delta^m$  is  $\Delta_{\text{Top}}^n \times \Delta_{\text{Top}}^m$ .

*Proof.* This first statement is a result of 2.0.9. The second statement also requires 2.0.8  $\square$

**Example 2.0.14.** We then define a simplicial set called *the boundary*  $\partial\Delta^n$  of  $\Delta^n$ . We define this by specifying that  $(\partial\Delta^n)_k$  is the set of non-surjective maps  $x \in \text{sSet}([n], [k])$ . Indeed  $\partial\Delta^n$  is a subsimplex of  $\Delta^n$ , and it matches our intuition from  $\Delta_{\text{Top}}^n$  since any element of  $(\Delta^n)_k$  which is the image of some  $d^i$  is clearly not surjective.

**Lemma 2.0.15** (The Yoneda Lemma). Let  $A$  be an object in a category  $\mathcal{C}$  and let  $F : \mathcal{C} \rightarrow \text{Set}$  be a functor. There is an isomorphism of sets

$$\text{Nat}(\mathcal{C}(A, -), F) \cong F(A)$$

It is not hard to find a proof of this, and even though the lemma is ubiquitous throughout category theory the proof is very straightforward.

An immediate consequence of the Yoneda lemma is,

**Lemma 2.0.16.** Given a simplicial set  $X_{\bullet}$  we have that

$$X_n \cong \text{sSet}(\Delta^n, X_{\bullet})$$

### 3. HORNS AND NERVES

**Definition 3.0.1.** The horn  $\Lambda_k^n$  is the simplicial subset of  $\partial\Delta^n$  defined by

$$(\Lambda_k^n)_i = \{x \in \Delta_k^n \mid x([k]) \cup \{i\} \supseteq [n]\}$$

Intuitively one thinks of this as the boundary of  $\Delta^n$  excluding the  $i^{\text{th}}$  face.

**Definition 3.0.2.** Let  $\mathcal{C}$  be a small category. The *nerve* of  $\mathcal{C}$  is the simplicial set  $N\mathcal{C}$  defined by

$$N\mathcal{C}_n = \left\{ C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} C_n \right\}$$

The maps  $s_i$  and  $d_i$  are defined as follows. Let  $(f_1, \dots, f_n)$  be an ordered  $n$ -tuple of composable maps in  $\mathcal{C}$ . We define  $s_i$  to simply be the map  $(f_1, \dots, \text{id}, \dots, f_n) \in N\mathcal{C}_{n+1}$ . The map  $d_i$  is slightly more complicated.

$$d_i(f_1, \dots, f_n) = \begin{cases} (f_2, \dots, f_n) & \text{if } i = 0, \\ (f_1, \dots, f_i \circ f_{i+1}, \dots, f_n) & \text{if } 0 < i < n, \\ (f_1, \dots, f_{n-1}) & \text{if } i = n \end{cases}$$

An important example of such a construction is the nerve of the category  $\mathbf{G}$  associated to a group  $G$ . In this case  $NG_n = G^n$  and thus the geometric realisation of  $N\mathcal{C}$  is

$$\coprod G^n \times \Delta_{\text{Top}}^n / \sim$$

This is isomorphic to the Eilenberg MacLane space  $K(G, 1)$ .

Let  $X$  be a simplicial set. We define  $M : \mathbf{sSet} \rightarrow \mathbf{Ch}_{\geq 0}$  by specifying  $M(X)_n$  to be the free abelian group generated by the non-degenerate  $n$ -simplices of  $X$ . Our boundary maps are then the maps  $d(x) = \sum_i (-1)^i d_i(x)$ . Another way of stating this is that, by 2.0.9,  $M(X)$  is the singular chain complex of  $|X|$ .

**Theorem 3.0.3.** *Let  $X$  and  $Y$  be simplicial sets. Then  $M(X \times Y) \cong M(X) \otimes M(Y)$*

*Proof Sketch (PROBABLY IGNORE THIS MESS).*  $(M(X) \otimes M(Y))_n = \oplus_{i+j=n} M(X)_i \otimes M(Y)_j$ . By 2.0.1 we have that  $(x, y)$  is non-degenerate when either  $x$  or  $y$  is non-degenerate, or when they are incompatibly degenerate. By incompatibly degenerate I mean that they decompose as  $x = s_{i_l} \circ \cdots \circ s_{i_1} x'$  and  $y = s_{j_k} \circ \cdots \circ s_{j_1} y'$  such that  $s_{i_r} \neq s_{j_s}$  for all  $r, s$ . Let us refer to  $l$  and  $k$  from the decomposition above as the order of degeneracy for arbitrary degenerate elements  $x$  and  $y$  respectively. A non-degenerate element has order of degeneracy 0. To see that the inequality holds note that if  $(x, y)$  is non-degenerate in  $(X \times Y)_n$  then we must have that the total order of degeneracy  $l + k$  is less than or equal to  $n$ . This follows since if  $l + k > n$  then  $x$  and  $y$  could not possibly be incompatibly degenerate by the pigeonhole principle. We may restate this succinctly as  $(x, y)$  non-degenerate in  $X_n \times Y_n$  is a pair  $(s_{i_l} \circ \cdots \circ s_{i_1} x', s_{j_k} \circ \cdots \circ s_{j_1} y')$  in  $X_n \times Y_n$  where  $j_r \neq i_s$  and where  $x'$  is non-degenerate in  $X_{n-l}$  and  $y'$  is non-degenerate in  $Y_{n-k}$ , where  $l + k \leq n$ . Consider the pair  $(x', y')$  in  $X_{n-l} \times Y_{n-k}$ .

I'm not exactly sure how the final part of this proof goes where we show that this corresponds to a unique pair with  $l + k$  actually equal to  $n$ . Maybe we can discuss this at the student meeting. If we managed to show that  $|X \times Y| = |X| \times |Y|$  then the Eilenberg-Zilber theorem [2] would give the result.  $\square$

#### 4. $\mathbf{sSet}$ AS A MODEL CATEGORY

We can put a model category structure on the category of simplicial sets. We first define two sets  $I = \bigcup_{n \geq 0} \mathbf{sSet}(\partial \Delta^n, \Delta^n)$  and  $J = \{\mathbf{sSet}(\Lambda_k^n, \Delta^n) \mid n > 0 \text{ and } 0 \leq k \leq n\}$ .

**Definition 4.0.1.** Given a map  $f \in \mathbf{sSet}(X, Y)$ , we say  $f$  is

- a fibration if  $J \Box f$ , which is to say that if we have any map  $p \in J$ , and any  $\Lambda_k^n \rightarrow X$  and  $\Delta^n \rightarrow Y$ , making the following diagram commute,

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow p & & \downarrow f \\ \Delta^n & \longrightarrow & Y \end{array}$$

then we must have a lift  $h : \Delta^n \rightarrow X$  which commutes with the diagram,

- a weak equivalence if  $|f| : |X| \rightarrow |Y|$  is a weak equivalence in  $\mathbf{Top}$ ,
- a cofibration if it is injective.

**Lemma 4.0.2.** *This defines a model structure on  $\mathbf{sSet}$ .*

*Proof.* (I think we're showing this next lecture).  $\square$

I'm not entirely sure why we introduced the set  $I$ . I thought it would be used to define cofibrations but apparently not. Perhaps this will become apparent next week.

#### REFERENCES

- [1] Jon Peter May, *Simplicial Objects in Algebraic Topology*, University of Chicago Press, 1982
- [2] Eilenberg, S., & Zilber, J. (1953). *On Products of Complexes*. American Journal of Mathematics, 75(1), 200-204. doi:10.2307/2372629
- [3] Bousfield, A.K., & Kan, D.M. (1972). *Homotopy Limits, Completions and Localizations*. Springer-Verlag Berlin Heidelberg.