Chapter 7

Energy Storage Elements

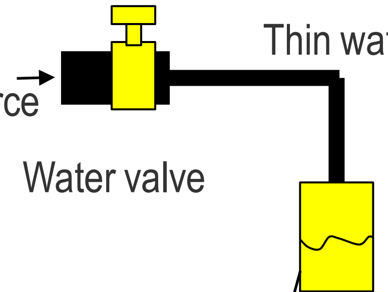
Introduction

- This chapter introduces two more circuit elements, the capacitor (C) and the inductor (L). The constitutive equations for the devices involve either integration or differentiation.
  - Electric circuits that contain capacitors and/or inductors are represented by differential equations. We say that circuits containing capacitors and/or inductors are dynamic circuits.
  - Circuits that contain capacitors and/or inductors are able to store energy.
  - Capacitor voltages and inductor currents are **continuous** functions of time.
  - Series or parallel capacitors can be reduced to an equivalent capacitor. Series or parallel inductors can be reduced to an equivalent inductor.

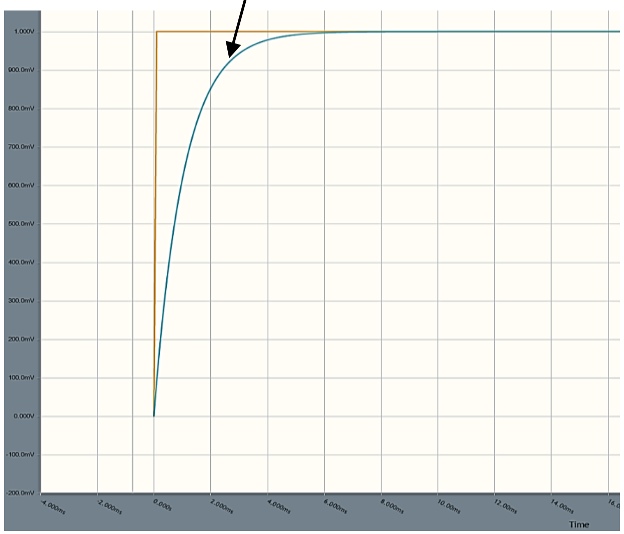
The Impact of “Time”

- Up to this point we have assumed that everything happens **instantaneously**
- We will now discuss elements that have input-output relationships depending on <u>time</u>
- These elements can “**store**” energy
- At different times they may either produce or absorb energy
- These elements are called “**energy storage elements**”

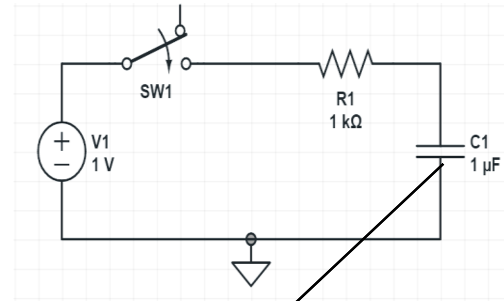
Example of Energy Storing Systems



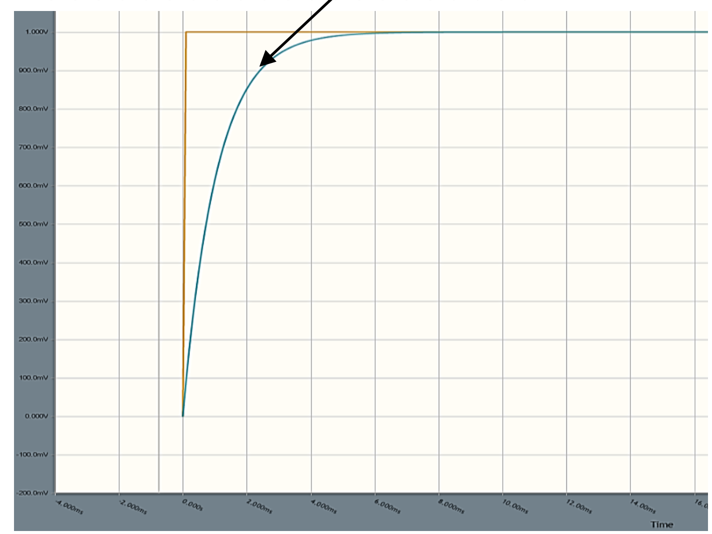
Amount of Water in the Tank



Time



Amount of Charge in the Capacitor



Time

Energy Storage Elements

- ***Capacitors*** store energy in an electric field
- ***Inductors*** store energy in a magnetic field
- Capacitors and inductors are passive elements:
  - Can store energy supplied by a circuit
  - Can return stored energy to a circuit
  - Cannot supply more energy to a circuit than the energy it stored

Power Generation and Distribution

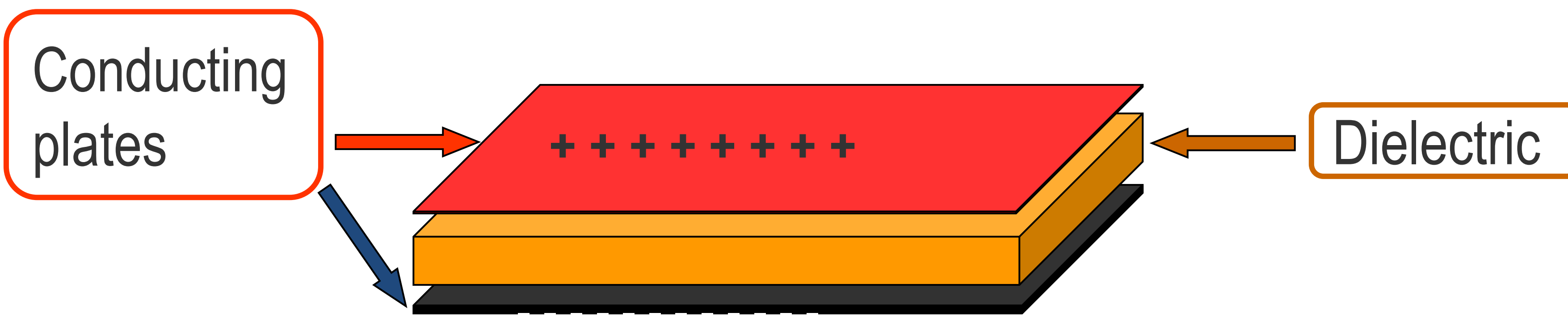
– ***Capacitors*** and ***Inductors*** are used to model electrical power transmission lines along with resistors
– Energy storage elements are used to model electrical loads:
  - ***Capacitors*** model computers and other electronics (power supplies)
  - ***Inductors*** model (large/small) motors

CAPACITORS

https://www.youtube.com/watch?v=X4EUwTwZ110

Capacitance

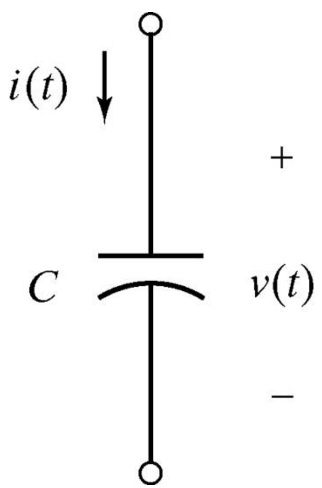
- Capacitors are represented by a parameter called *capacitance*.
- Capacitance occurs when two ***conductors*** (plates) are separated by a ***dielectric*** (insulator)
- Capacitance is a measure of the ability of a device to store energy in the form of a separated charge or an electric field

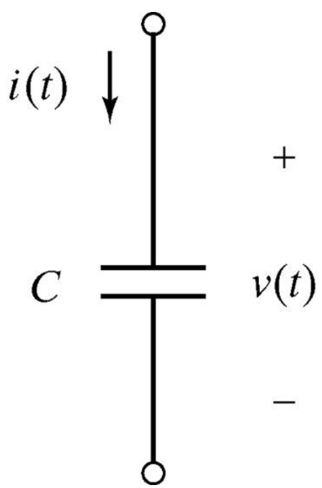


Capacitance

- Charge builds up on each of the plates when a voltage is applied
- Charge on the two conductors creates an electric field
- A capacitor is a circuit element that stores energy in an electric field

Symbol





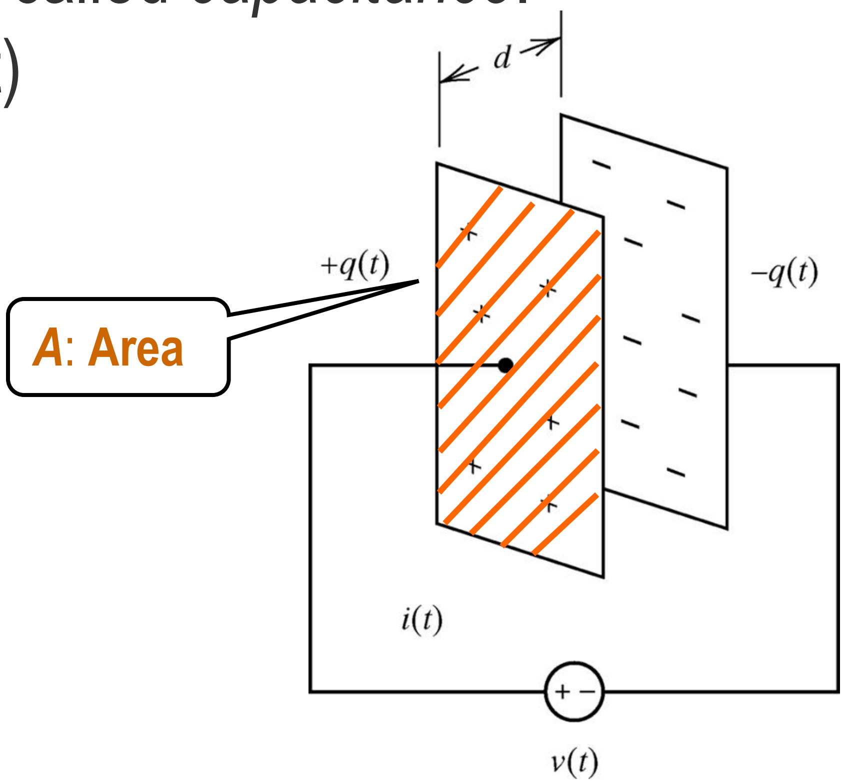
Capacitance

- The voltage difference between the two conductors is proportional to the charge:

$$\boxed{q = C \cdot v}$$

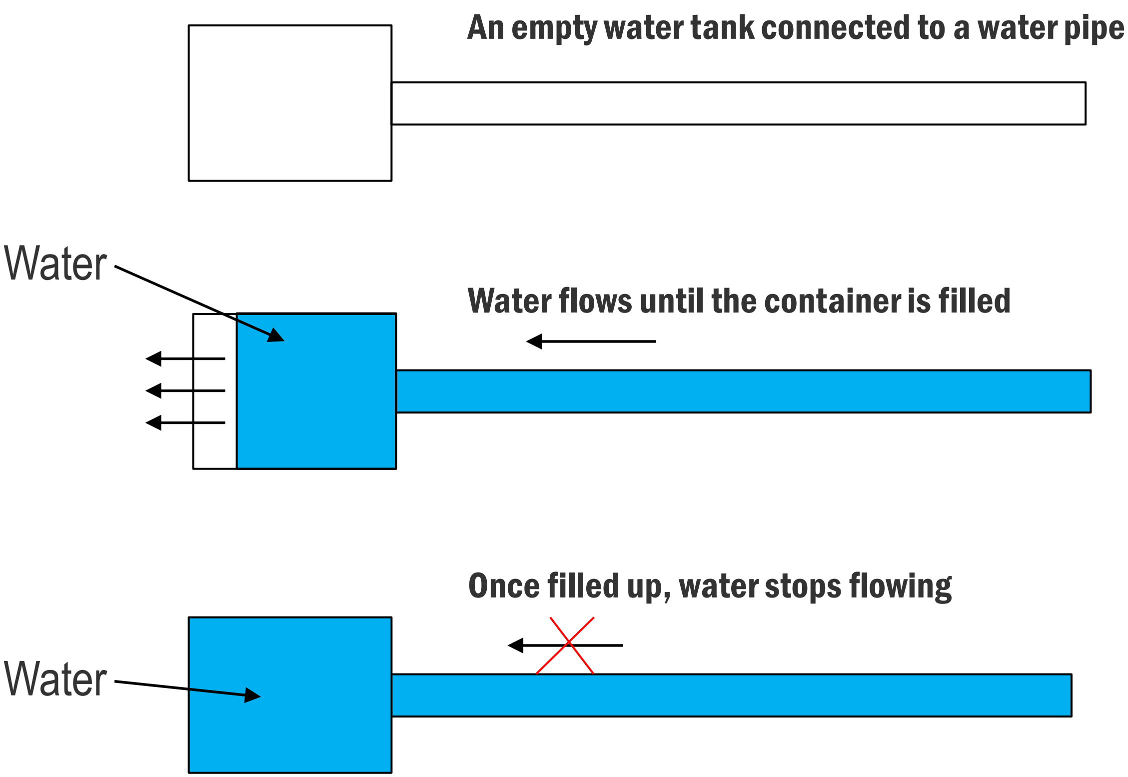
- The proportionality constant C is called *capacitance*.
- Units: **Farads** (F = Coulomb/Volt)

$$\boxed{\begin{aligned} C &= \frac{\varepsilon \cdot A}{d} \\ q &= \frac{\varepsilon \cdot A}{d} \cdot v \end{aligned}}$$



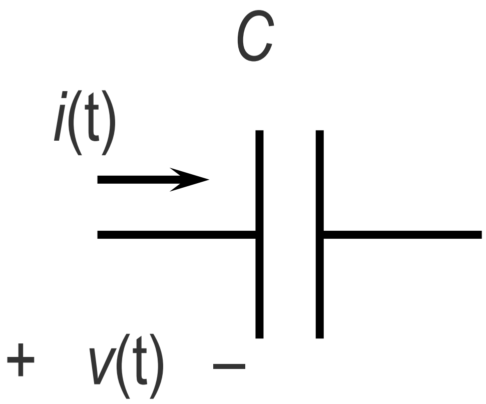
C: capacitance in Farads (*F*)
ε: permittivity in *F*/*m* $(8.854 \cdot 10^{-12} \cdot \varepsilon_r)$
A: area in m^2
d: length in *m*

Capacitor (DC)



DC current flowing into the water will be eventually stopped
Likewise, DC current flowing into a capacitor will be eventually stopped
A capacitor is a DC blocking element

I-V Relationship



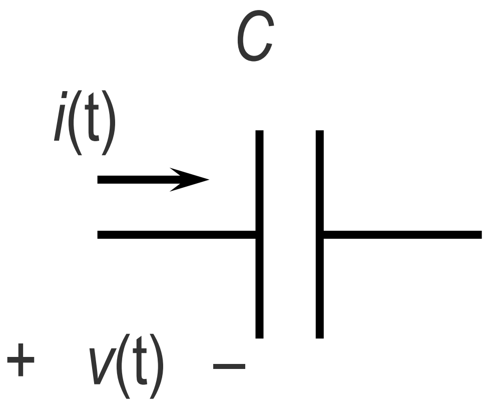
$$q(t) = C \cdot v(t)$$

$$\frac{d}{dt}[q(t)] = \frac{d}{dt}[C \cdot v(t)]$$

$$\frac{dq(t)}{dt} = C\frac{dv(t)}{dt}$$

$$\boxed{i(t) = C\frac{dv(t)}{dt}}$$

I-V Relationship



$$i(t) = C\frac{dv(t)}{dt}$$

$$\Rightarrow i(\tau)\cdot d\tau = C\cdot dv(\tau)$$

$$\Rightarrow \int_{t_0}^{t} i(\tau)\cdot d\tau = \int_{v(t_0)}^{v(t)} C\cdot dv(\tau)$$

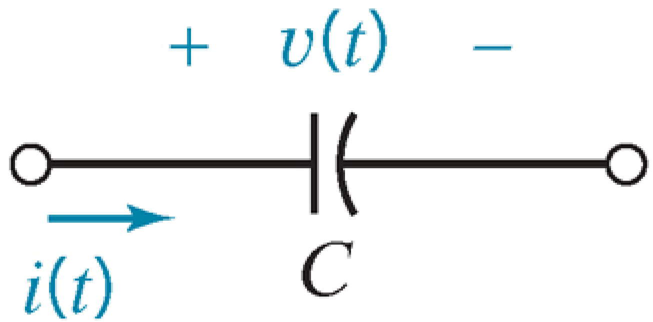
Initial time

$$\Rightarrow \int_{t_0}^{t} i(\tau)\cdot d\tau = C\left[v(t) - v(t_0)\right]$$

$$\Rightarrow \boxed{v(t) = v(t_0) + \frac{1}{C}\int_{t_0}^{t} i(\tau)\cdot d\tau}$$

Initial condition

I-V Relationship



$$i(t) = C\frac{d}{dt}v(t)$$

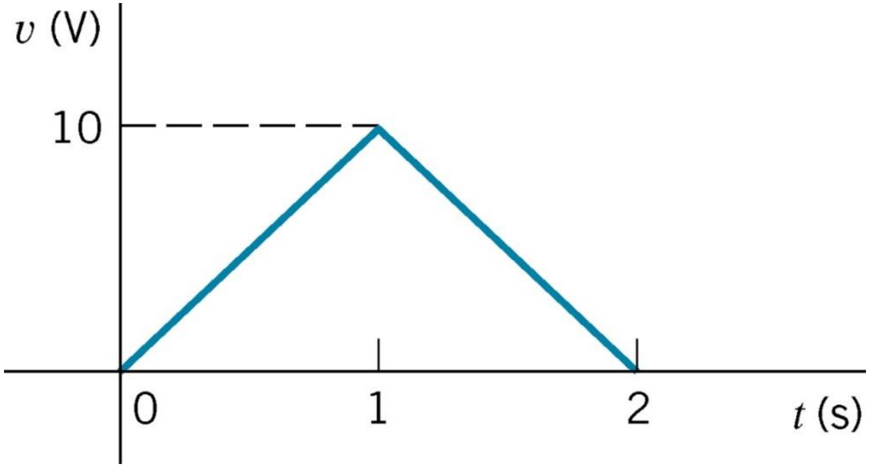
$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau + v(t_0)$$

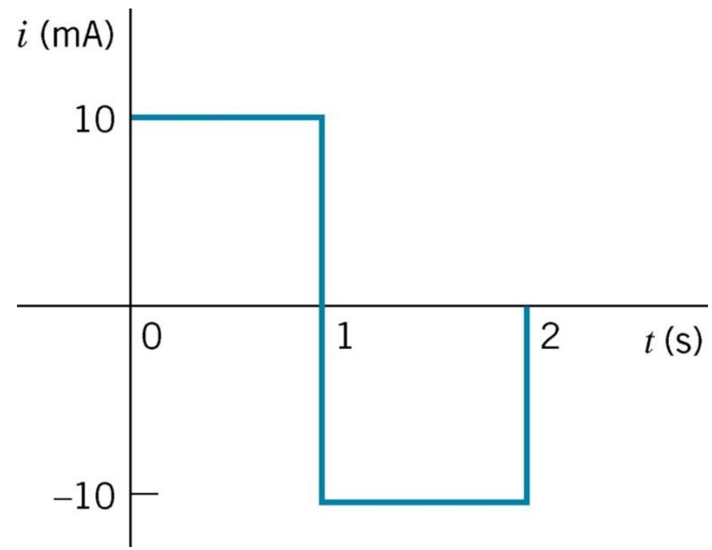
Voltage *across a capacitor* ***CANNOT*** *change instantaneously*

These equations define the behavior of the ***capacitors***

Example 7.2-1

- C = 1 mF and the voltage across the capacitor is given below. Calculate the current i(t) through the capacitor.





Example 7.2-5

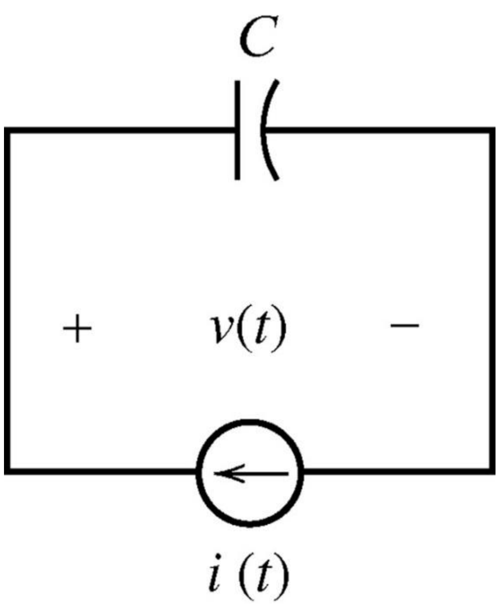
- The input current is:

$$i(t) = 3.75e^{-1.2t} \ A \quad for \ t > 0$$

- The output capacitor voltage is:

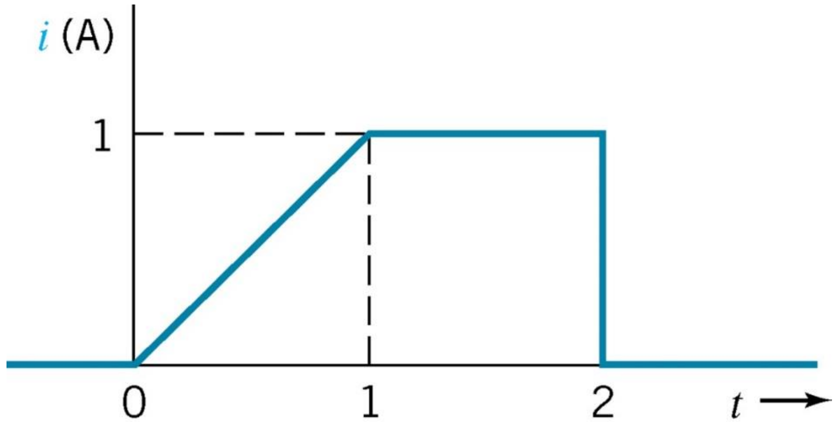
$$v(t) = 4 - 1.25e^{-1.2t} \ V \quad for \ t > 0$$

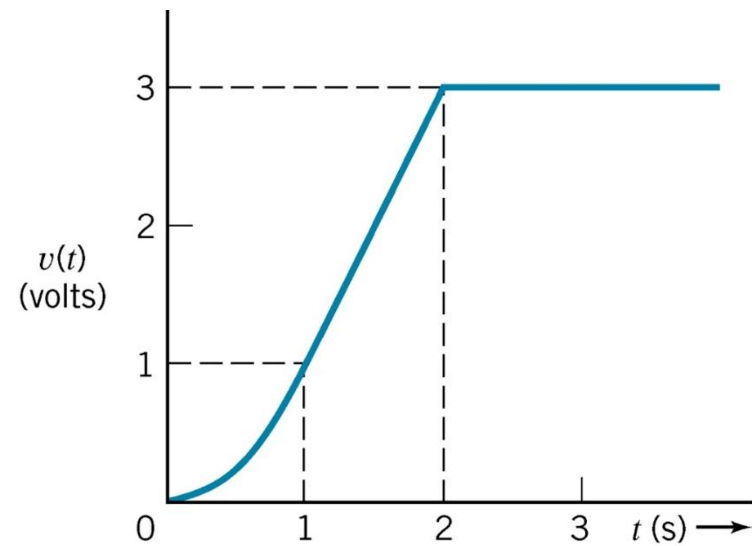
- Find the value of the capacitance, ***C***.



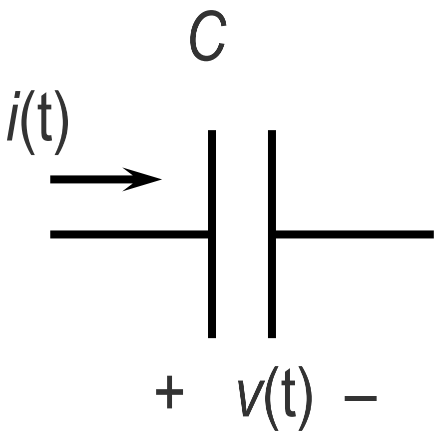
Example 7.2-2

- C = ½ F, the current through the capacitor is given below. Calculate the voltage $\boldsymbol{v}(t)$ across the capacitor.





Power Stored in a Capacitor



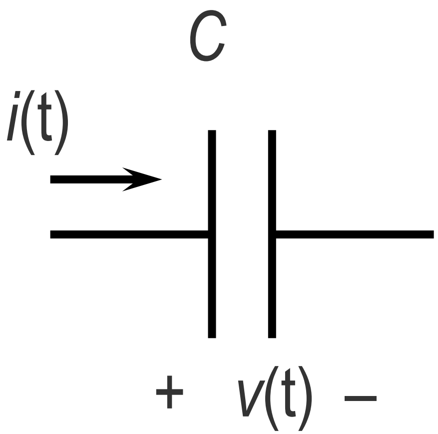
$$i(t) = C\frac{dv(t)}{dt}$$

$$p(t) = v(t) \cdot i(t)$$

$$= v(t) \cdot C\frac{dv(t)}{dt}$$

$$\boxed{p(t) = C \cdot v(t)\frac{dv(t)}{dt}}$$

Energy Stored in a Capacitor



$$w(t) = \int_{-\infty}^{t} p(t)dt = \int_{-\infty}^{t} C \cdot v(t)\frac{dv(t)}{dt}dt$$

$$= \int_{-\infty}^{v(t)} C \cdot v \cdot dv = C \int_{-\infty}^{(t)} v \cdot dv$$

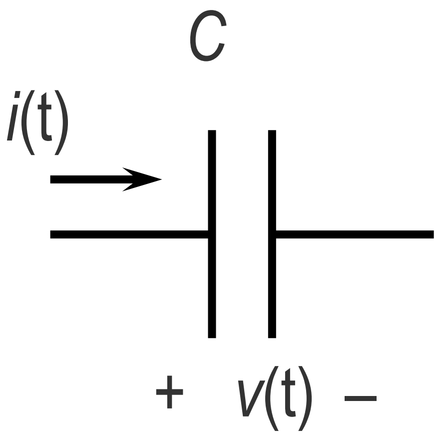
$$= \frac{1}{2} C \cdot v^2 \Big|_{v(-\infty)}^{v(t)}$$

$$w(t) = \frac{1}{2} C \cdot \left[v(t)^2 - v(-\infty)^2 \right]$$

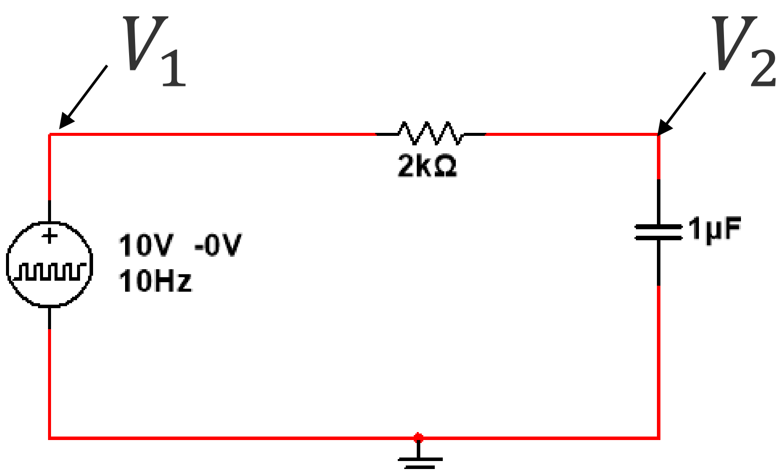
Because the capacitor is uncharged at t = $-\infty$

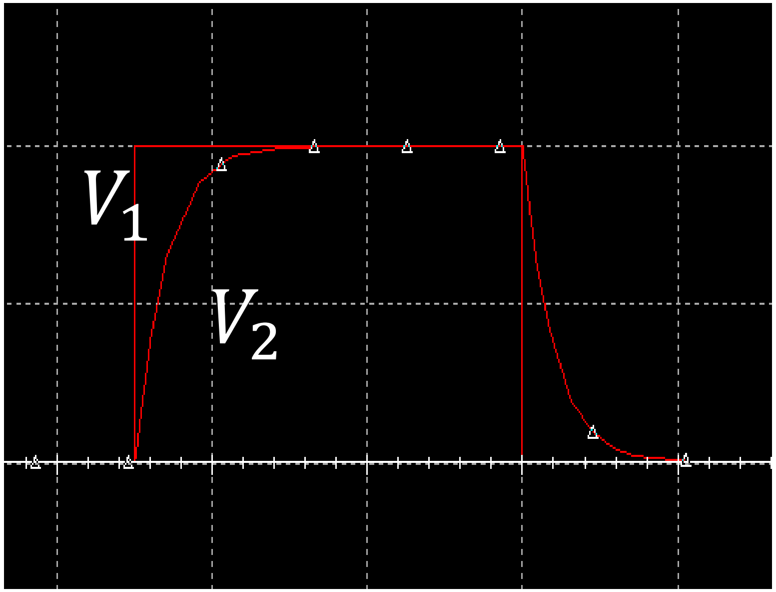
$$\boxed{w(t) = \frac{1}{2} C \cdot v(t)^2}$$

Energy Stored in a Capacitor



***Voltage** & **Charge** on a capacitor **CANNOT** change instantaneously. It takes time to charge the capacitor.*

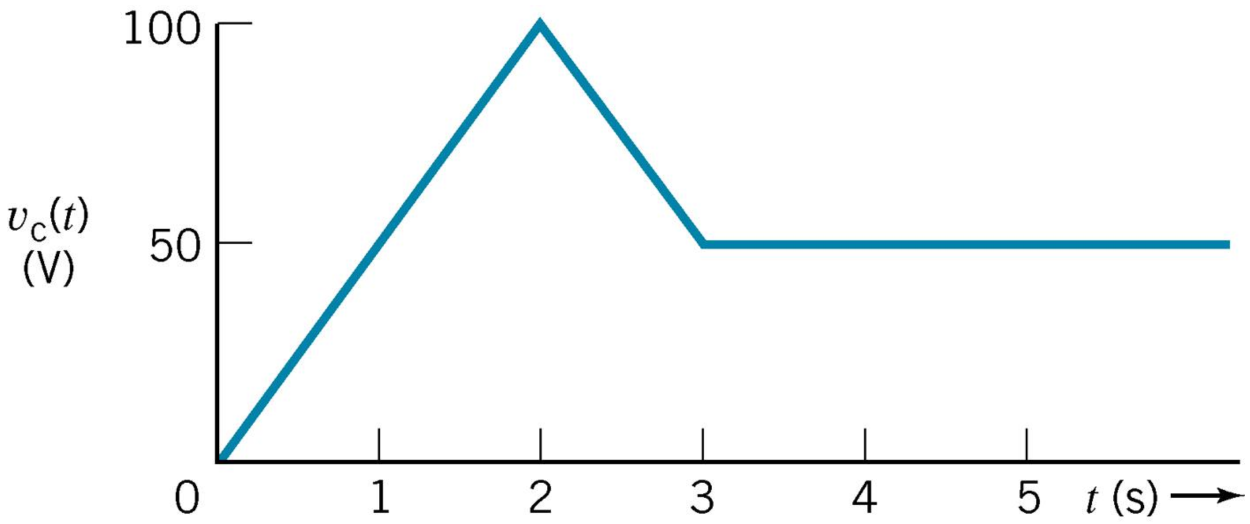




Multisim simulation result

Example 7.3-2

- The voltage across a 5-mF capacitor is shown below. Determine and plot the capacitor **current**, **power** and **energy**.



Series & Parallel Capacitors

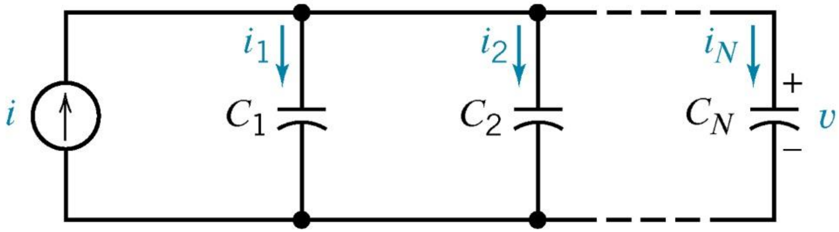
- **Series capacitors** combine like **parallel resistors**

$$1/C_{eq\text{-}series} = 1/C_1 + 1/C_2 + 1/C_3 + \cdots$$

- **Parallel capacitors** combine like **series resistors**

$$C_{eq\text{-}parallel} = C_1 + C_2 + C_3 + \cdots$$

Series & Parallel Capacitors

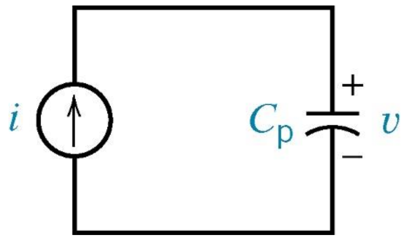


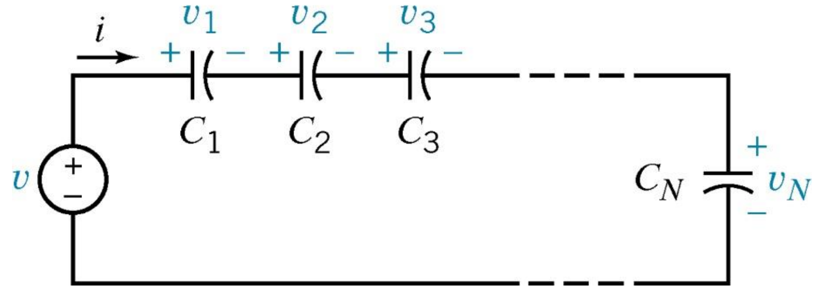
$$i = i_1 + i_2 + i_3 + \cdots + i_N$$

$$= C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + \cdots + C_N \frac{dv}{dt}$$

$$= (C_1 + C_2 + \cdots + C_N) \frac{dv}{dt}$$

$$i = C_p \frac{dv}{dt}$$



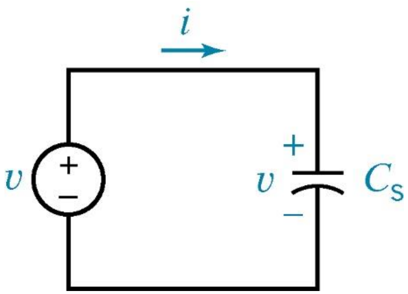


$$v = v_1 + v_2 + v_3 + \cdots + v_N$$

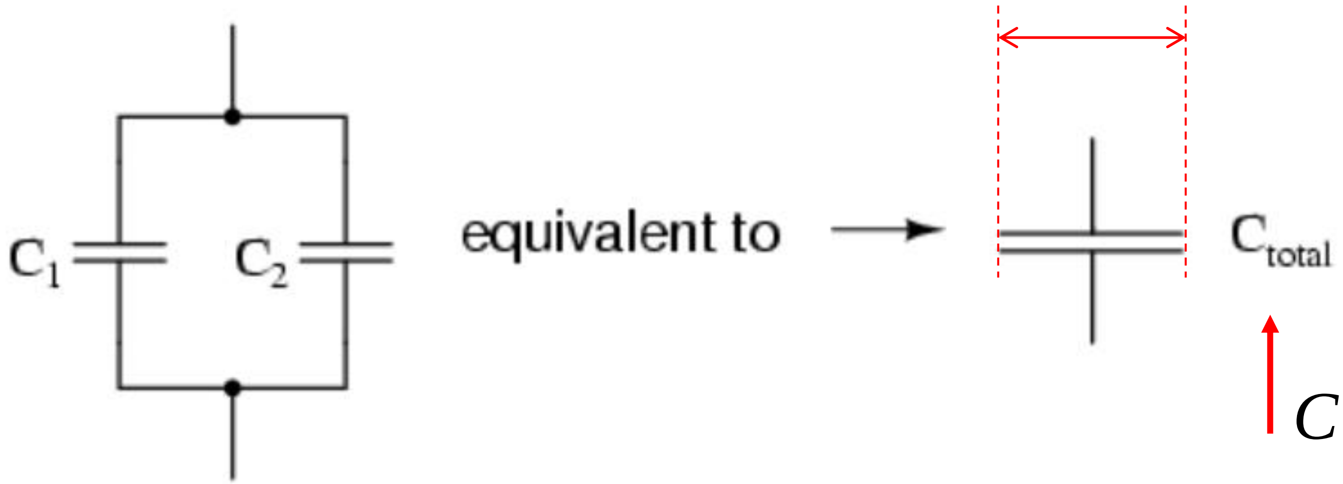
$$= \frac{1}{C_1} \int_{t_0}^{t} i \cdot d\tau + \frac{1}{C_2} \int_{t_0}^{t} i \cdot d\tau + \cdots + \frac{1}{C_N} \int_{t_0}^{t} i \cdot d\tau$$

$$= \left(\frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N} \right) \int_{t_0}^{t} i \cdot d\tau$$

$$v = \frac{1}{C_s} \int_{t_0}^{t} i \cdot d\tau$$



Capacitors in Parallel



$$C = \frac{\varepsilon \cdot A}{d}$$

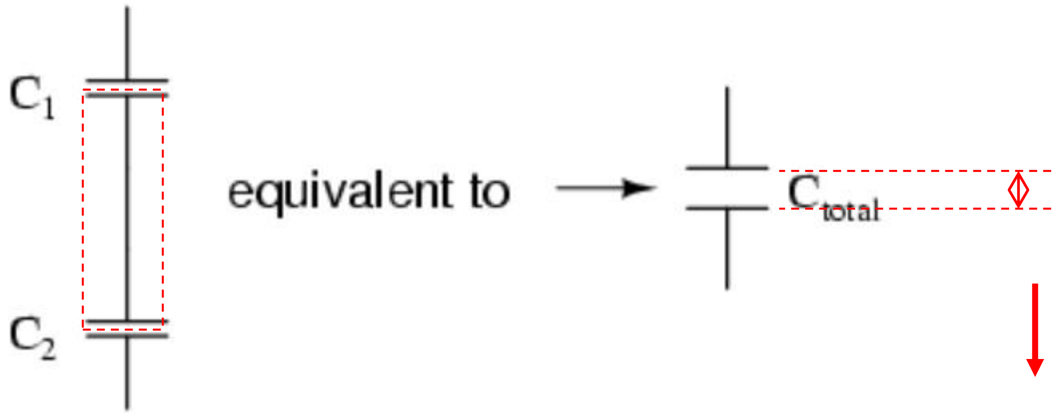
The overall effect is that of a single equivalent capacitor having the sum total of the plate areas of the individual capacitors.

The total capacitance is the sum of the individual capacitors' capacitances.

Parallel Capacitances

$$C_{eq} = C_1 + C_2 + \ldots\ C_n$$

Capacitors in Series



$$C = \frac{\varepsilon \cdot A}{d}$$

The overall effect is that of a single (equivalent) capacitor having the sum total of the plate spacings of the individual capacitors.

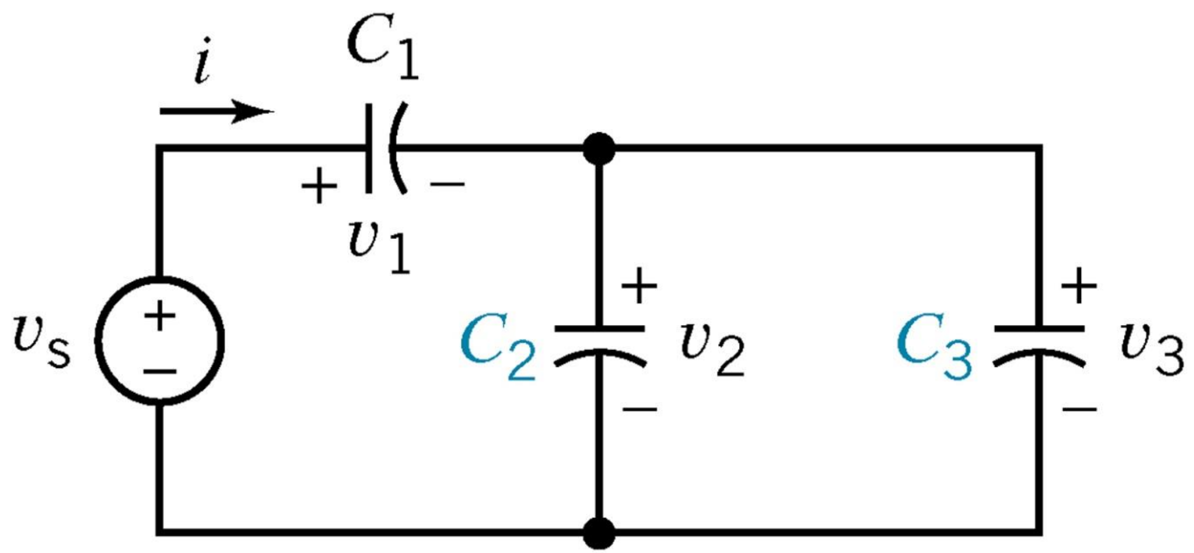
The total capacitance is less than any of the individual capacitors' capacitance.

Series Capacitances

$$C_{eq} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \ldots \frac{1}{C_n}}$$

Example 7.4-1

- Find the equivalent capacitance C_{eq} when $C_1 = C_2 = C_3 = 2$ mF, $v_1(0) = 10$ V and $v_2(0) = v_3(0) = 20$ V.



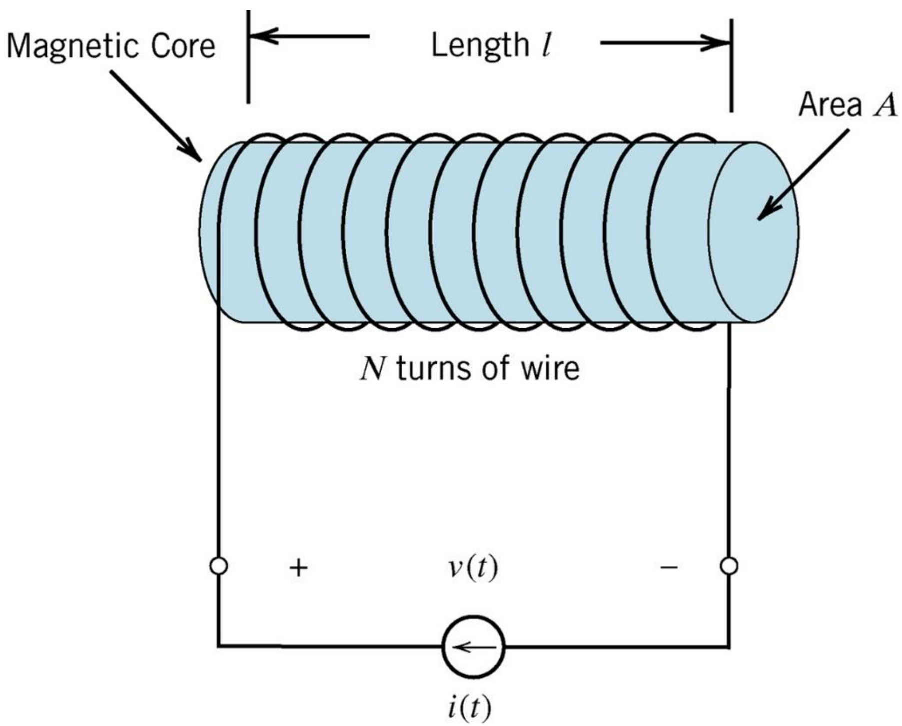
INDUCTANCE

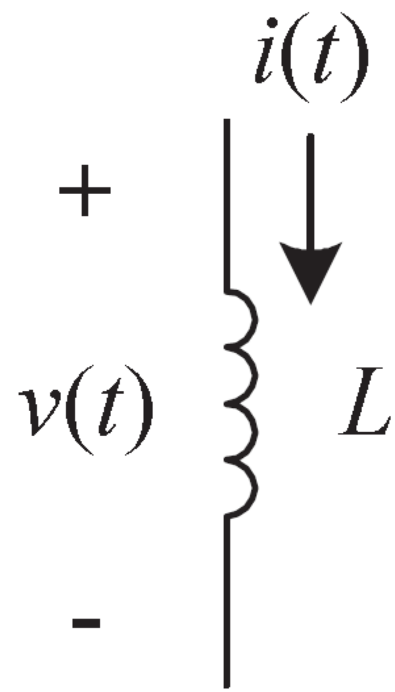
https://www.youtube.com/watch?v=KSylo01n5FY

Inductance is a measure of the ability of a device to store energy the form of a magnetic field

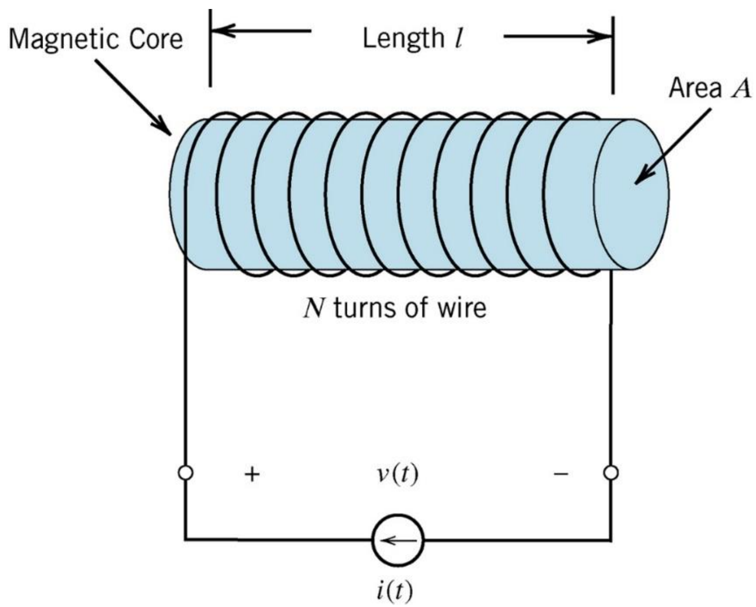
Inductance

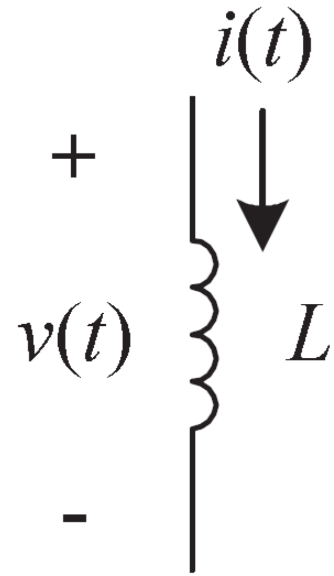
- An inductor can be constructed by winding a coil of wire around a magnetic core (any magnetic material)
- Energy stored in the magnetic field created by current flowing through the wire





Inductance





$$\boxed{L = \frac{\mu N^2 A}{l}}$$

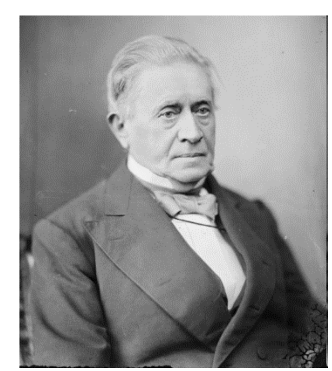
$L =$ Inductance in Henrys (H)

$N =$ Number of turns

$\mu =$ Magnetic permeability of the core

$A =$ Cross-sectional area (m^2)

$l =$ Length (m)



I-V Relationship

Faraday's Law:

$$e = -\frac{d\psi}{dt}$$

$$\psi = N \cdot \phi$$

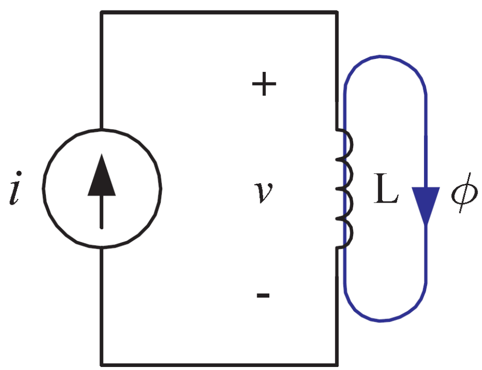
$$\phi = N \cdot \frac{\mu \cdot A}{l} \cdot i$$

$$\psi = N \cdot N \cdot \frac{\mu \cdot A}{l} \cdot i$$

$$= N^2 \frac{\mu \cdot A}{l} \cdot i$$

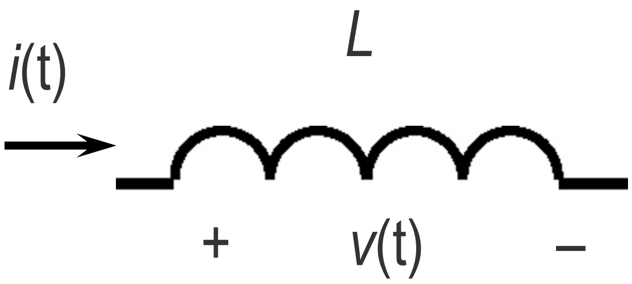
$$e = -\frac{d\left(N^2 \frac{\mu \cdot A}{l} \cdot i\right)}{dt}$$

$$= -N^2 \frac{\mu \cdot A}{l} \frac{di}{dt} = -L \frac{di}{dt}$$



$$\boxed{v = L \frac{di}{dt}}$$

I-V Relationship



$$v(t) = L\frac{di(t)}{dt}$$

$$\Rightarrow v(\tau)\cdot d\tau = L\cdot di(\tau)$$

$$\Rightarrow \int_{t_0}^{t} v(\tau)\cdot d\tau = \int_{i(t_0)}^{i(t)} L\cdot di(\tau)$$

Initial time

$$\Rightarrow \int_{t_0}^{t} v(\tau)\cdot d\tau = L\left[i(t) - i(t_0)\right]$$

$$\Rightarrow \boxed{i(t) = i(t_0) + \frac{1}{L}\int_{t_0}^{t} v(\tau)\cdot d\tau}$$

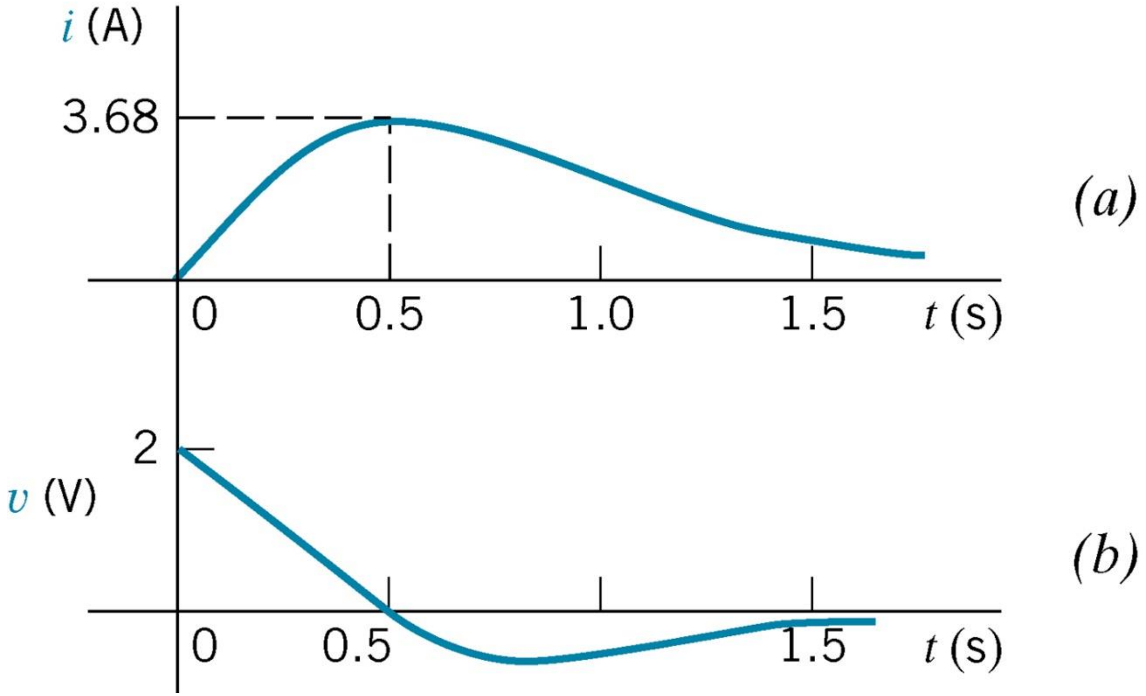
Initial condition

***Current** in an inductor **CANNOT** change instantaneously*

Example 7.5-1

- Find the voltage across an inductor, ***L***=0.1 H, when the current in the inductor is:

$$i(t) = 20 \cdot t \cdot e^{-2t} \ A, \qquad t > 0, i(0) = 0$$



Example 7.5-3

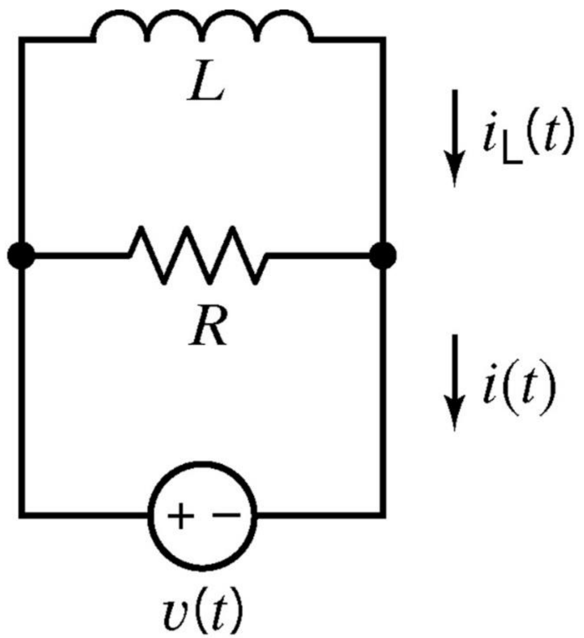
- Calculate ***R*** and ***L*** if $i_L(0) = -3.5$ A,

the input to the circuit is the voltage:

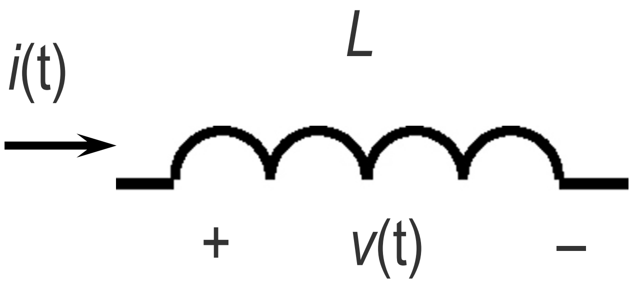
$$v(t) = 4 \cdot e^{-20t} \text{ V}, \qquad t > 0$$

and the output is the current:

$$i(t) = -1.2 \cdot e^{-20t} - 1.5 \text{ A}, \qquad t > 0$$



Power Stored in an Inductor



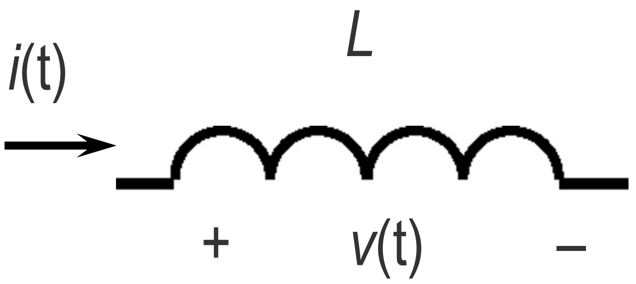
$$v(t) = L\frac{di(t)}{dt}$$

$$p(t) = v(t)\cdot i(t)$$

$$= L\frac{di(t)}{dt}\cdot i(t)$$

$$\boxed{p(t) = L\cdot i(t)\frac{di(t)}{dt}}$$

Energy Stored in an Inductor



$$w(t) = \int_{-\infty}^{t} p(t)dt = \int_{-\infty}^{t} L \cdot i(t)\frac{di(t)}{dt}dt$$

$$= \int_{-\infty}^{i(t)} L \cdot i \cdot di = L\int_{-\infty}^{i(t)} i \cdot di$$

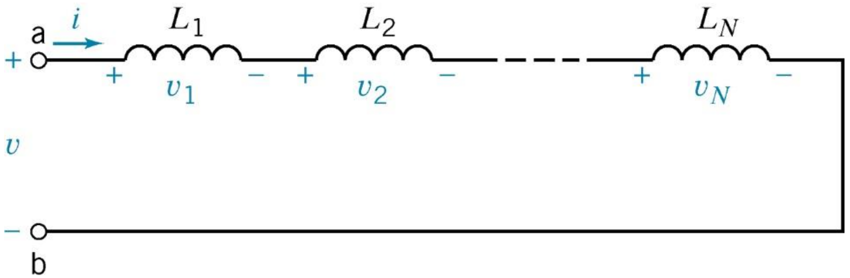
$$= \frac{1}{2}L \cdot i^2 \Big|_{i(-\infty)}^{i(t)}$$

$$= \frac{1}{2}L \cdot \left[i(t)^2 - i(-\infty)^2\right]$$

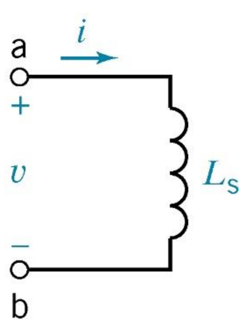
If the inductor is initially not magnetized: $i(-\infty) = 0$

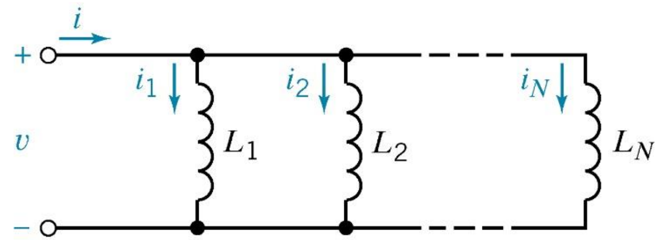
$$\boxed{w(t) = \frac{1}{2}L \cdot i(t)^2}$$

Series & Parallel Inductors

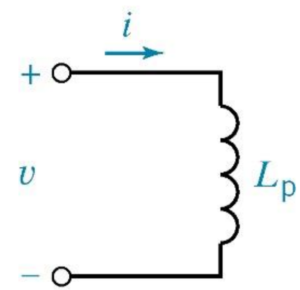


$$v = v_1 + v_2 + v_3 + \cdots + v_N$$
$$= L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + \cdots + L_N \frac{di}{dt}$$
$$= (L_1 + L_2 + \cdots + L_N) \frac{di}{dt}$$
$$v = L_s \frac{di}{dt}$$



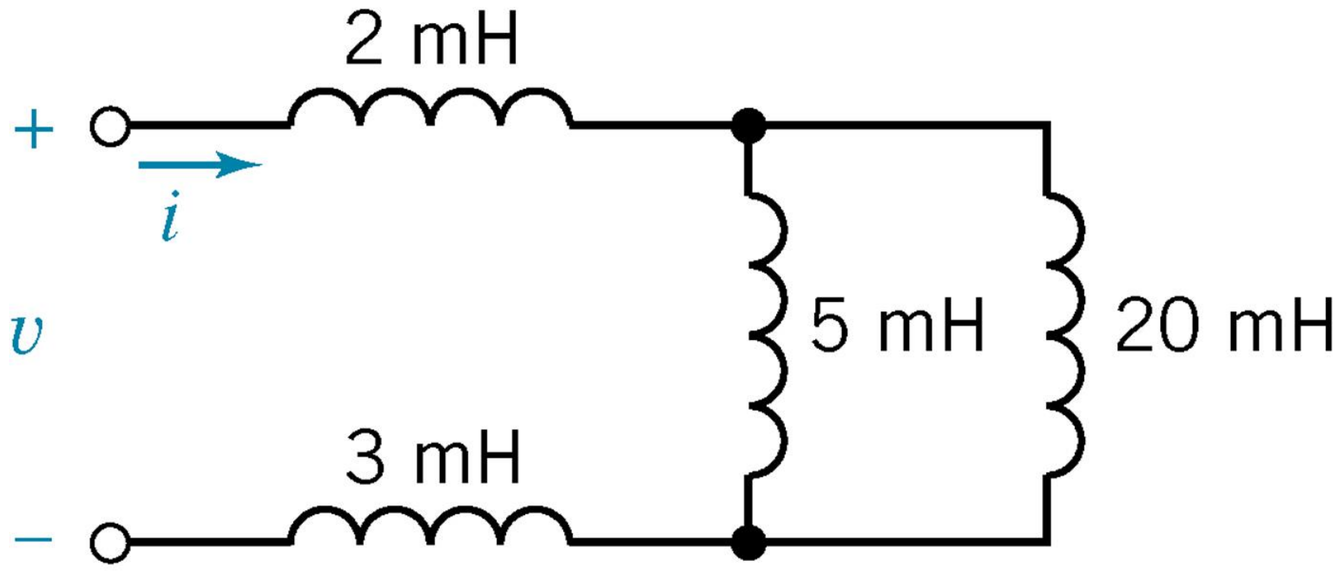


$$i = i_1 + i_2 + i_3 + \cdots + i_N$$
$$= \frac{1}{L_1} \int_{t_0}^{t} v \cdot d\tau + \frac{1}{L_2} \int_{t_0}^{t} v \cdot d\tau + \cdots + \frac{1}{L_N} \int_{t_0}^{t} v \cdot d\tau$$
$$= \left(\frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N} \right) \int_{t_0}^{t} v \cdot d\tau$$
$$i = \frac{1}{L_p} \int_{t_0}^{t} v \cdot d\tau$$



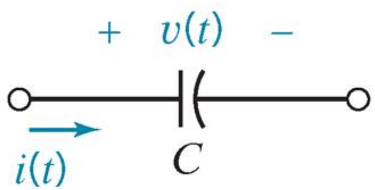
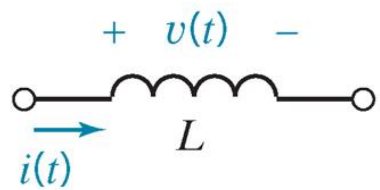
Example 7.7-1

- Find $\boldsymbol{L}_{\text{eq}}$ assuming that $\boldsymbol{i}(0) = 0$ A.



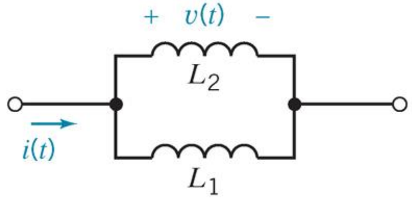
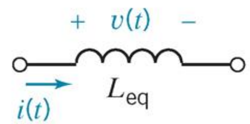
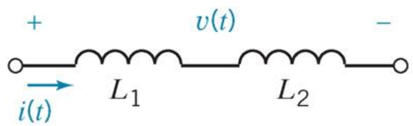
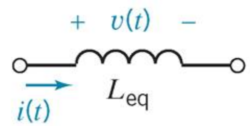
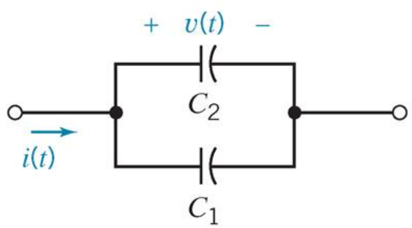
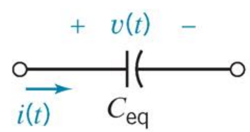
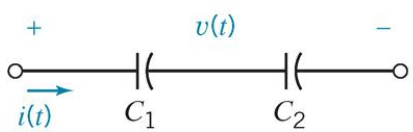
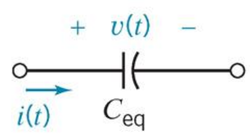
Summary

Table 7.13-1 Element Equations for Capacitors and Inductors

CAPACITOR	INDUCTOR
	
$i(t) = C\dfrac{d}{dt}v(t)$	$i(t) = \dfrac{1}{L}\displaystyle\int_{t_0}^{t} v(\tau)d\tau + i(t_0)$
$v(t) = \dfrac{1}{C}\displaystyle\int_{t_0}^{t} i(\tau)d\tau + v(t_0)$	$v(t) = L\dfrac{d}{dt}i(t)$

Summary

Table 7.13-2 Parallel and Series Capacitors and Inductors

SERIES OR PARALLEL CIRCUIT	EQUIVALENT CIRCUIT	EQUATION
		$L_{eq} = \dfrac{1}{\dfrac{1}{L_1} + \dfrac{1}{L_2}}$
		$L_{eq} = L_1 + L_2$
		$C_{eq} = C_1 + C_2$
		$C_{eq} = \dfrac{1}{\dfrac{1}{C_1} + \dfrac{1}{C_2}}$