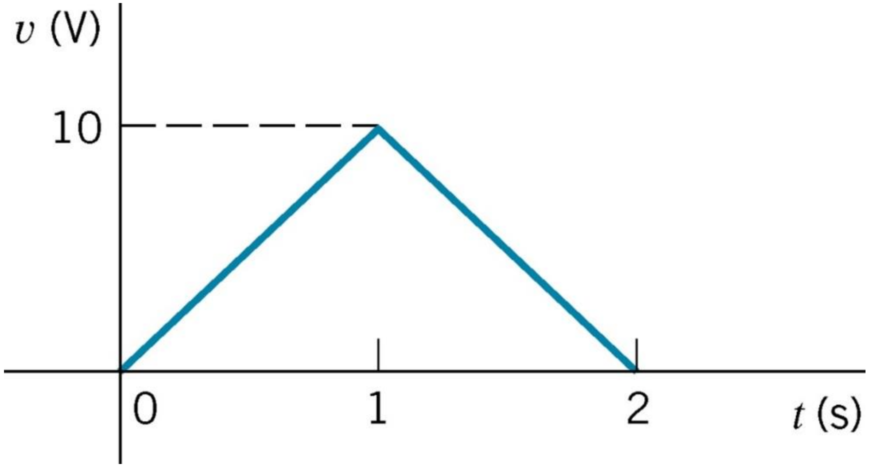
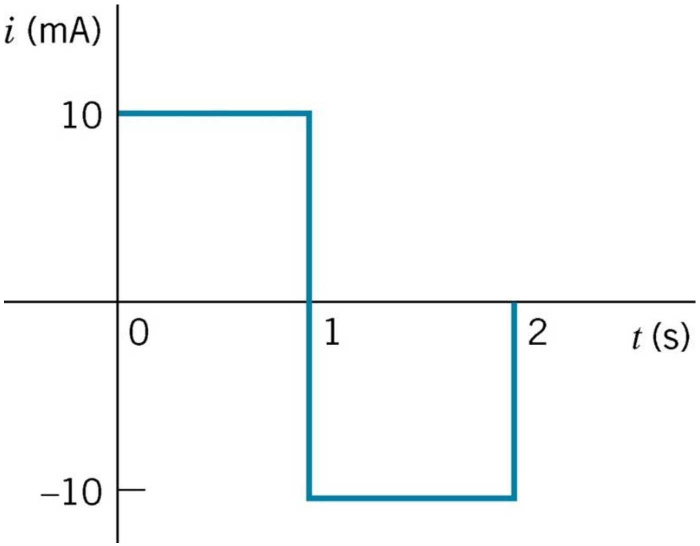
Chapter 7

Energy Storage Elements (Problems)

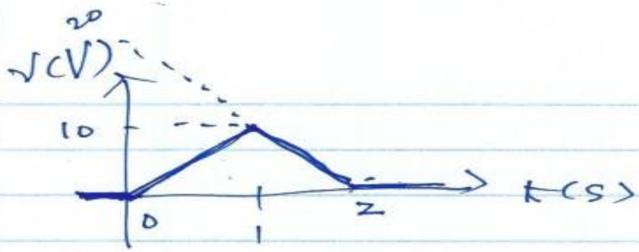
Example 7.2-1

- C = 1 mF and the voltage across the capacitor is given below. Calculate the current i(t) through the capacitor.





Example 7.2-1 Solution



$C = 1\,mF.$

$$v(t) = \begin{cases} 0, & t \le 0 \\ 10 \cdot t & 0 \le t < 1 \\ 20 - 10t & 1 \le t \le 2 \\ 0 & t \ge 2 \end{cases}$$

$i(t)$

$\rightarrow 1 \times 10^{-3} \cdot 0 = 0$

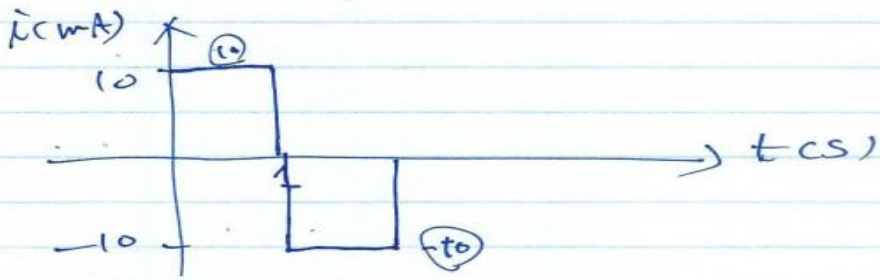
$\rightarrow 1 \times 10^{-3} \cdot \frac{d(10t)}{dt} = 10^{-2}$

$\rightarrow 1 \times 10^{-3}\, d(20 - 10t)/dt = -10^{-2}$

$\rightarrow 1 \times 10^{-3} \cdot 0 = 0$

$$i = C \cdot \frac{dv}{dt}$$

$$i(t) = \begin{cases} 0 & t \le 0 \\ 10^{-2} & 0 \le t < 1 \\ -10^{-2} & 1 \le t \le 2 \\ 0 & t \ge 2 \end{cases}$$



Example 7.2-1 Solution

Find the current for a capacitor $C = 1$ mF when the voltage across the capacitor is represented by the signal shown in Figure 7.2-6.

Solution

The voltage (with units of volts) is given by

$$v(t) = \begin{cases} 0 & t \le 0 \\ 10t & 0 \le t \le 1 \\ 20 - 10t & 1 \le t \le 2 \\ 0 & t \ge 2 \end{cases}$$

Then, because $i = C\,dv/dt$, where $C = 10^{-3}$ F, we obtain

$$i(t) = \begin{cases} 0 & t < 0 \\ 10^{-2} & 0 < t < 1 \\ -10^{-2} & 1 < t < 2 \\ 0 & t > 2 \end{cases}$$

Therefore, the resulting current is a series of two pulses of magnitudes 10^{-2} A and -10^{-2} A, respectively, as shown in Figure 7.2-7.

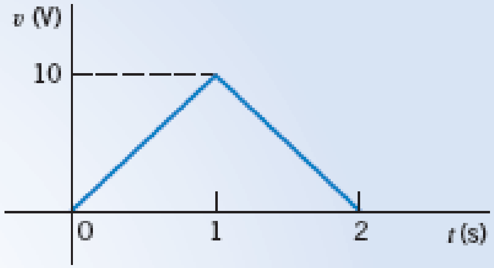


FIGURE 7.2-6 Waveform of the voltage across a capacitor for Example 7.2-1. The units are volts and seconds.

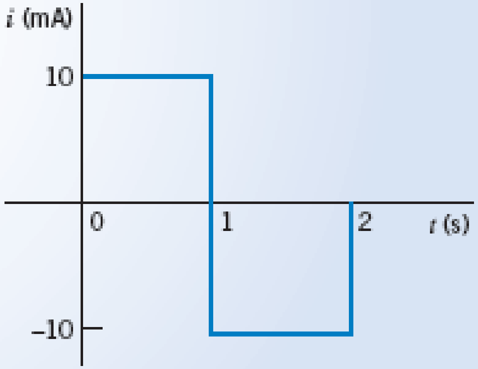


FIGURE 7.2-7 Current for Example 7.2-1.

Example 7.2-5

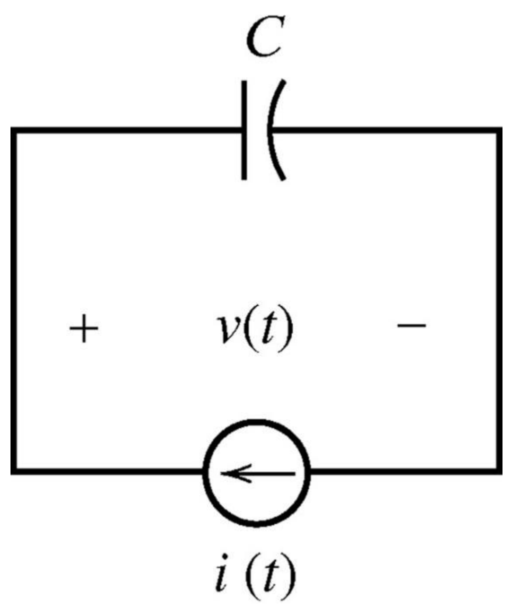
- The input current is:

$$i(t) = 3.75e^{-1.2t} \ A \quad for \ t > 0$$

- The output capacitor voltage is:

$$v(t) = 4 - 1.25e^{-1.2t} \ V \quad for \ t > 0$$

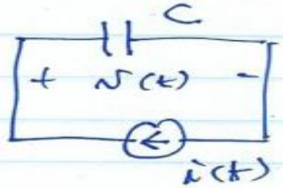
- Find the value of the capacitance, ***C***.



Example 7.2-5 Solution

$i(t) = 3.75\, e^{-1.2t}$ A for $t > 0$

$v(t) = 4 - 1.25\, e^{-1.2t}$ V for $t > 0$.



$$\boxed{v(t) = \frac{1}{C}\int_0^t i(\tau)\, d\tau + v(0)}$$

$$4 - 1.25\, e^{-1.2t} = \frac{1}{C}\int_0^t 3.75\, e^{-1.2\tau}\, d\tau + v(0)$$

$$= \frac{3.75}{C(-1.2)}\, e^{-1.2\tau}\Big|_0^t + v(0)$$

$$= -\frac{3.125}{C}\left(e^{-1.2t} - 1\right) + v(0)$$

$$= \frac{-3.125}{C}\, e^{-1.2t} + \frac{3.125}{C} + v(0)$$

$$-1.25\, \cancel{e^{-1.2t}} = \frac{-3.125}{C}\, \cancel{e^{-1.2t}} \qquad -1.25 = \frac{-3.125}{C},$$

$$4 = \frac{3.125}{C} + v(0) \qquad C = \frac{3.125}{1.25} = \boxed{2.5\ \text{F}}$$

Example 7.2-5 Solution

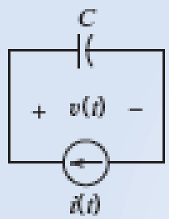


FIGURE 7.2-12 The circuit considered in Example 7.2-5.

The input to the circuit shown in Figure 7.2-12 is the current

$$i(t) = 3.75e^{-1.2t} \text{ A} \quad \text{for } t > 0$$

The output is the capacitor voltage

$$v(t) = 4 - 1.25e^{-1.2t} \text{ V} \quad \text{for } t > 0$$

Find the value of the capacitance C.

Solution

The capacitor voltage is related to the capacitor current by

$$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau + v(0)$$

That is,

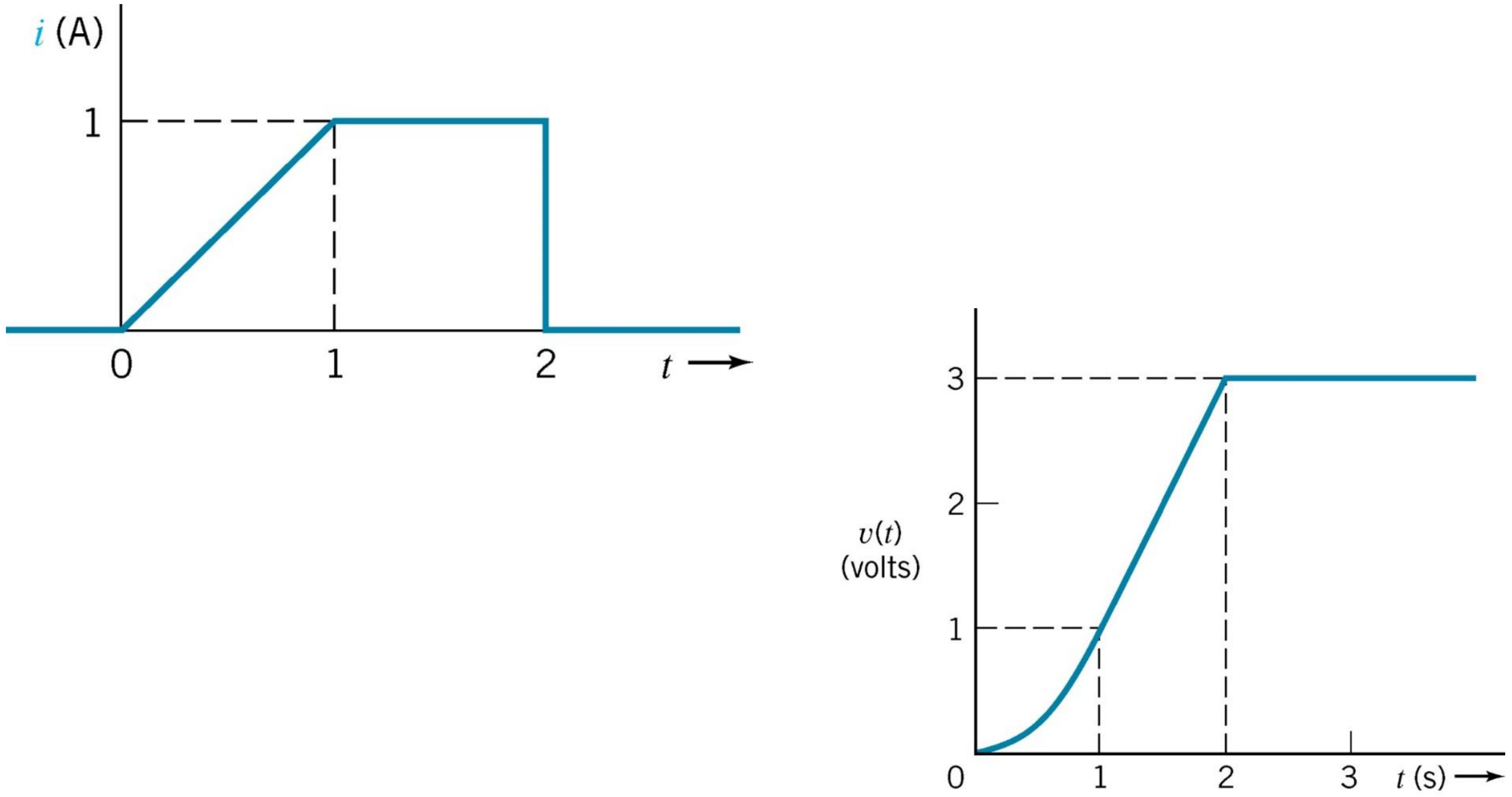
$$4 - 1.25e^{-1.2t} = \frac{1}{C}\int_0^t 3.75e^{-1.2\tau}d\tau + v(0) = \frac{3.75}{C(-1.2)}e^{-1.2\tau}\Big|_0^t + v(0) = \frac{-3.125}{C}\left(e^{-1.2t} - 1\right) + v(0)$$

Equating the coefficients of $e^{-1.2t}$ gives

$$1.25 = \frac{3.125}{C} \quad \Rightarrow \quad C = \frac{3.125}{1.25} = 2.5 \text{ F}$$

Example 7.2-2

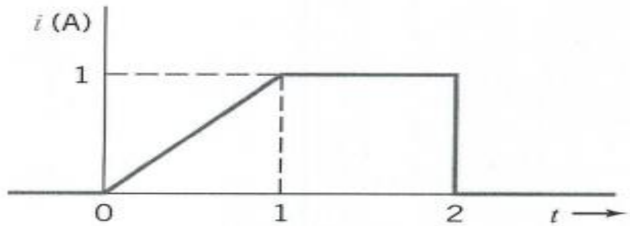
- C = ½ F, the current through the capacitor is given below. Calculate the voltage ***v***(t) across the capacitor.



Example 7.2-2 Solution

Example 7.2 – 2

- C = ½ F, current through the capacitor is given, voltage across the capacitor, $\mathbf{v}$(t) = ?



$$i(t) = \begin{cases} 0 & t \le 0 \\ t & 0 \le t \le 1 \\ 1 & 1 \le t \le 2 \\ 0 & 2 < t \end{cases}$$

$$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau + \cancel{v(0)}^{\,0} = \frac{1}{C}\int_0^t i(\tau)d\tau$$

$C = \frac{1}{2}$

$$v(t) = \begin{cases} 0 & t \le 0 & \\ 2\int_0^t \tau d\tau & 0 \le t \le 1, & v(t) = t^2, \quad v(1) = 1V \\ 2\int_1^t (1) d\tau + v(1) & 1 \le t \le 2, & 2(t-1)+1, \quad v(2) = 3V \\ v(2) & 2 \le t & 3V. \end{cases}$$

Example 7.2-2 Solution

Find the voltage $v(t)$ for a capacitor $C = 1/2$ F when the current is as shown in Figure 7.2-8 and $v(t) = 0$ for $t \leq 0$.

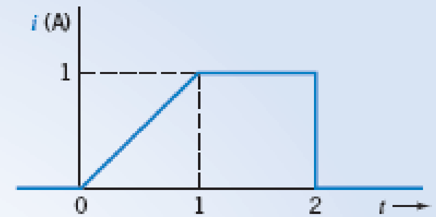


FIGURE 7.2-8 Circuit waveform for Example 7.2-2. The units are in amperes and seconds.

Solution

First, we write the equation for $i(t)$ as

$$i(t) = \begin{cases} 0 & t \leq 0 \\ t & 0 \leq t \leq 1 \\ 1 & 1 \leq t \leq 2 \\ 0 & 2 < t \end{cases}$$

Then, because $v(0) = 0$

$$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau + v(0) = \frac{1}{C}\int_0^t i(\tau)d\tau$$

and $C = 1/2$, we have

$$v(t) = \begin{cases} 0 & t \leq 0 \\ 2\displaystyle\int_0^t \tau d\tau & 0 \leq t \leq 1 \\ 2\displaystyle\int_1^t (1)d\tau + v(1) & 1 \leq t \leq 2 \\ v(2) & 2 \leq t \end{cases}$$

with units of volts. Therefore, for $0 < t \leq 1$, we have

$$v(t) = t^2$$

For the period $1 \leq t \leq 2$, we note that $v(1) = 1$ and, therefore, we have

$$v(t) = 2(t-1) + 1 = (2t - 1)\text{ V}$$

The resulting voltage waveform is shown in Figure 7.2-9. The voltage changes with t^2 during the first 1 s, changes linearly with t during the period from 1 to 2 s, and stays constant equal to 3 V after $t = 2$ s.

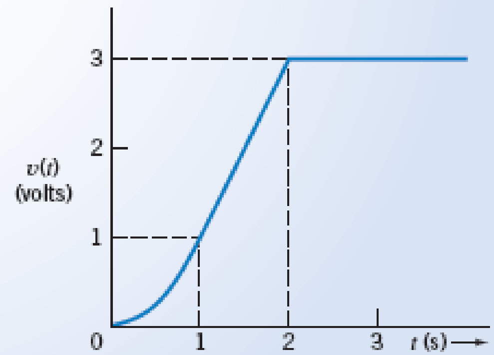
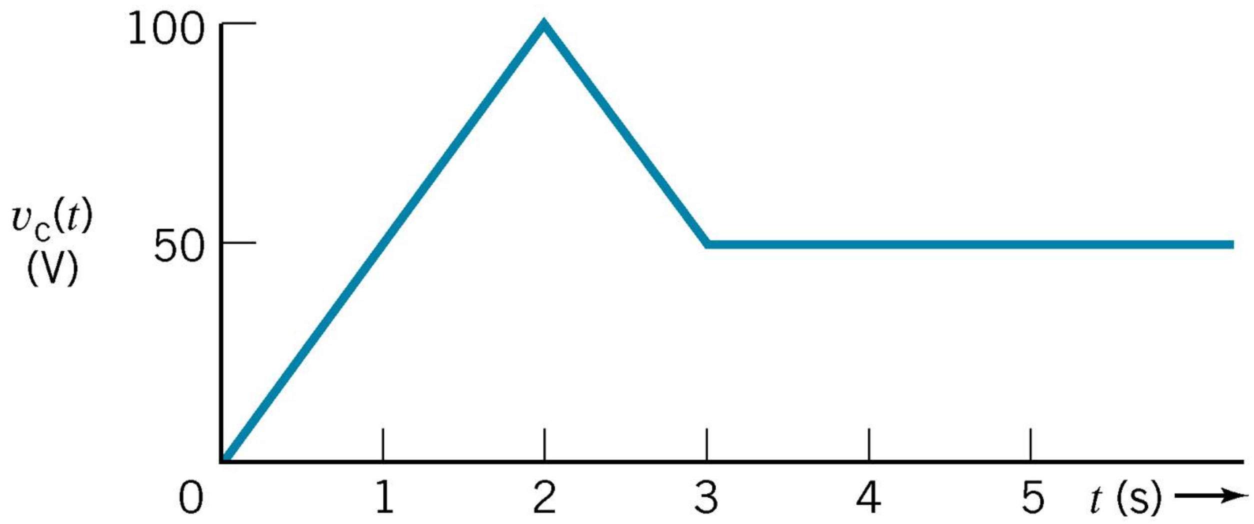


FIGURE 7.2-9 Voltage waveform for Example 7.2-2.

Example 7.3-2

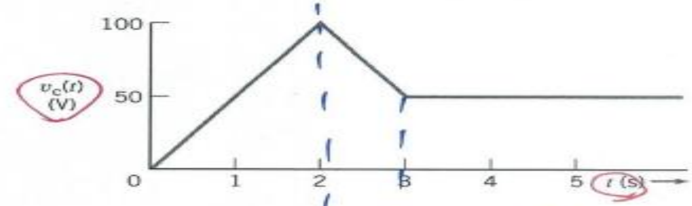
- The voltage across a 5-mF capacitor is shown below. Determine and plot the capacitor **current**, **power** and **energy**.



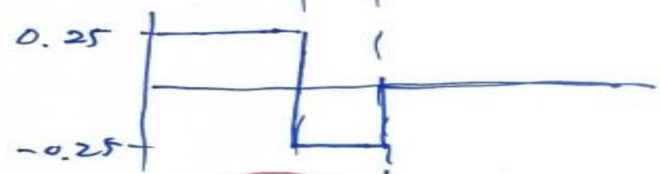
Example 7.3-2 Solution

Example 7.3 – 2

The voltage across a 5-mF capacitor is shown below. Determine and plot the capacitor ① **current**, ② **power** and ③ **energy**.

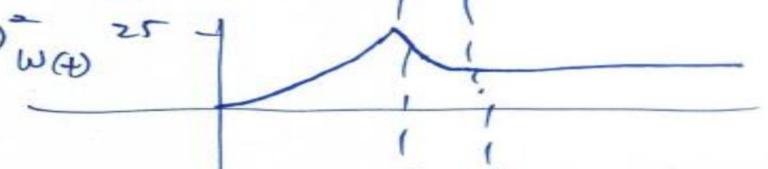


$i(t) = C\,\frac{dv}{dt} \rightarrow$ slope of $v(t)$ graph.



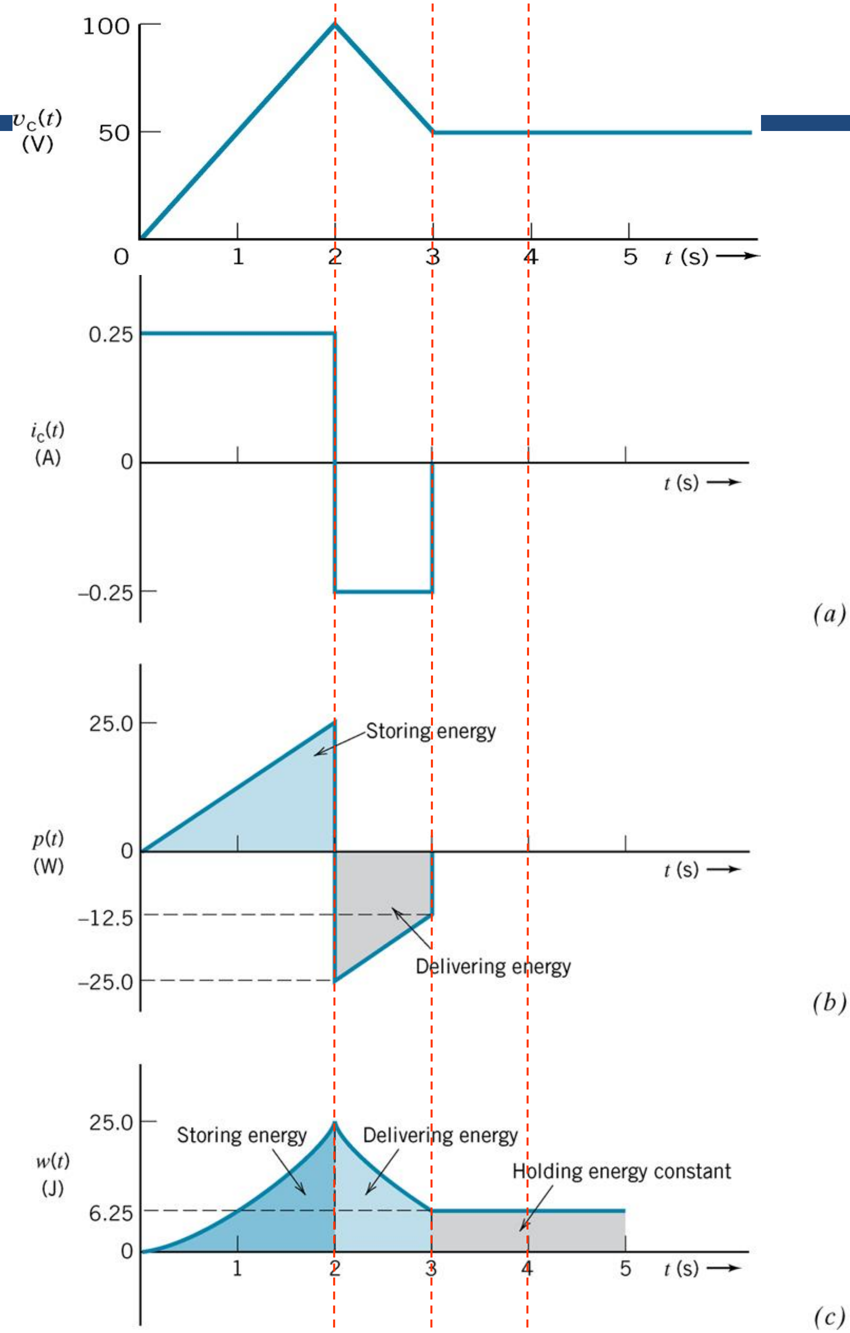
$p(t) = C \cdot v(t)\,\frac{dv(t)}{dt} = v(t) \cdot C\,\frac{dv(t)}{dt}$

$w(t) = \frac{1}{2} C\, v(t)^2$



Example 7.3-2 Solution

C = 5 mF



Example 7.3-2 Solution

The voltage across a 5-mF capacitor varies as shown in Figure 7.3-3. Determine and plot the capacitor current, power, and energy.

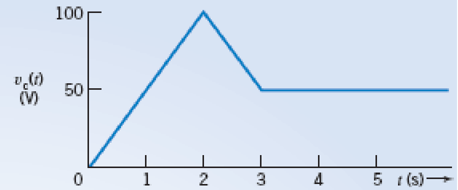


FIGURE 7.3-3 The voltage across a capacitor.

Solution

The current is determined from $i_c = C\,dv/dt$ and is shown in Figure 7.3-4*a*. The power is $v(t)i(t)$—the product of the current plot (Figure 7.3-4*a*) and the voltage plot (Figure 7.3-3)—and is shown in Figure 7.3-4*b*. The capacitor receives energy during the first two seconds and then delivers energy for the period $2 < t < 3$.

The energy is $\omega = \int p\,dt$ and can be found as the area under the $p(t)$ plot. The plot for the energy is shown in Figure 7.3-4*c*. Note that the capacitor increasingly stores energy from $t = 0$ s to $t = 2$ s, reaching a maximum energy of 25 J. Then the capacitor delivers a total energy of 18.75 J to the external circuit from $t = 2$ s to $t = 3$ s. Finally, the capacitor holds a constant energy of 6.25 J after $t = 3$ s.

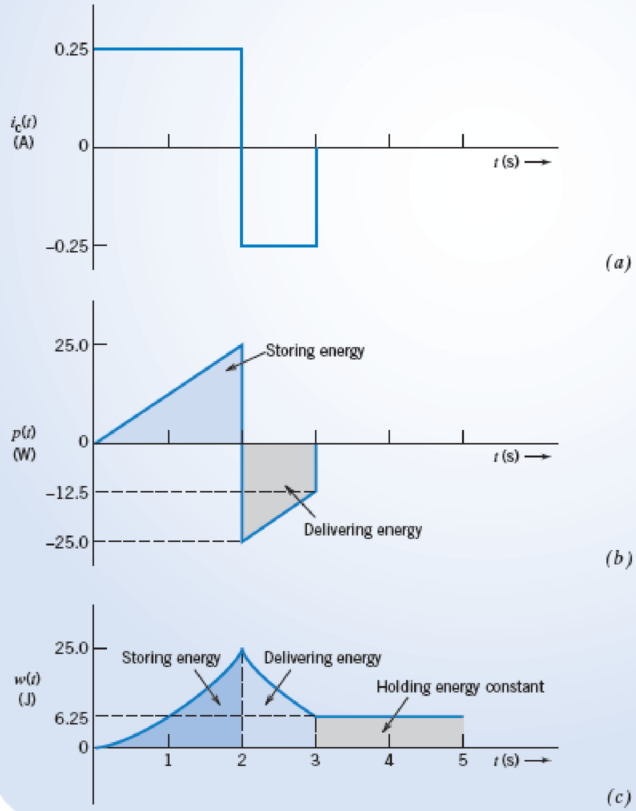
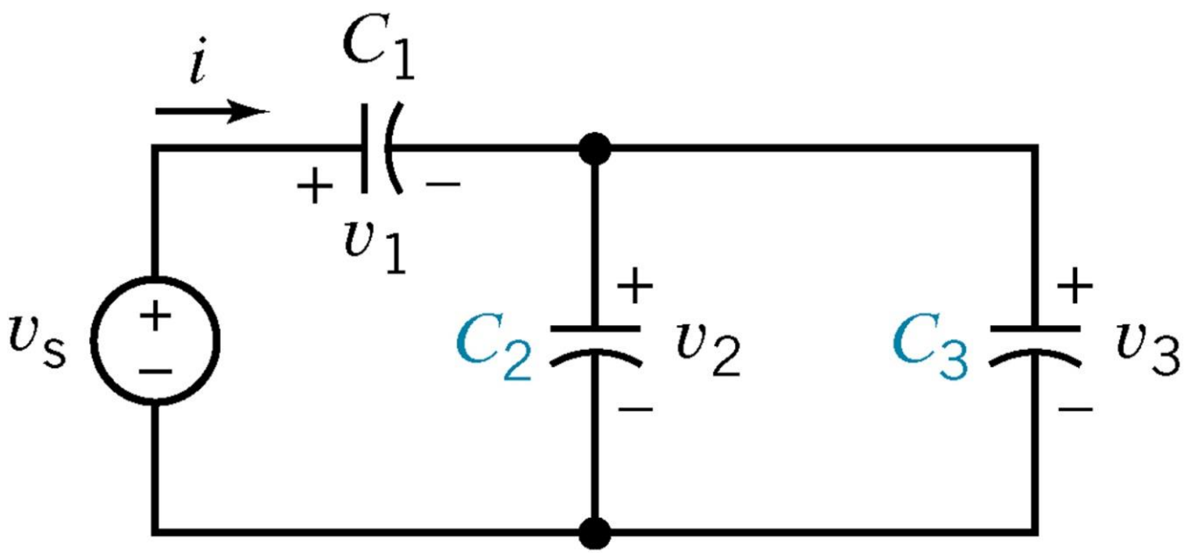


FIGURE 7.3-4 The current, power, and energy of the capacitor of Example 7.3-2.

Example 7.4-1

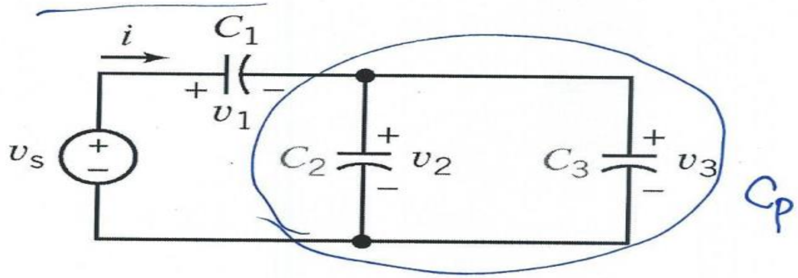
- Find the equivalent capacitance $\boldsymbol{C}_{eq}$ when $\boldsymbol{C}_1 = \boldsymbol{C}_2 = \boldsymbol{C}_3 = \mathbf{2}$ mF, $\boldsymbol{v}_1(0) = \mathbf{10}$ V and $\boldsymbol{v}_2(0) = \boldsymbol{v}_3(0) = \mathbf{20}$ V.



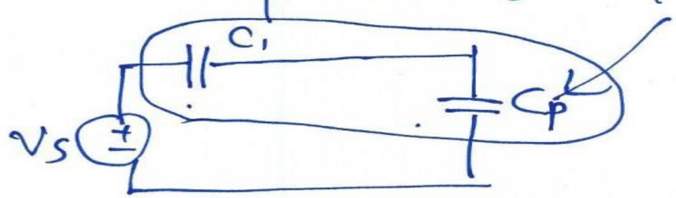
Example 7.4-1 Solution

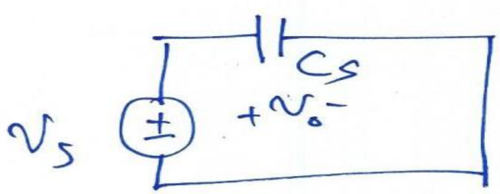
Example 7.4 – 1

- Find the C_{eq} when $C_1 = C_2 = C_3 = 2$ mF, $v_1(0) = 10$ V and $v_2(0) = v_3(0) = 20$ V



$C_p = C_2 + C_3 = 4\,mF$





C_s = C_1 & C_p in Series

$$C_s = \frac{C_1 C_p}{C_1 + C_p} = \frac{2\times10^{-3} \cdot 4\times10^{-3}}{2\times10^{-3} + 4\times10^{-3}} = \frac{8}{6}\,mF.$$

$V_o = ?$ $\quad V_o = 10 + 20 = 30V.$

Example 7.4-1 Solution

Find the equivalent capacitance for the circuit of Figure 7.4-5 when $C_1 = C_2 = C_3 = 2$ mF, $v_1(0) = 10$ V, and $v_2(0) = v_3(0) = 20$ V.

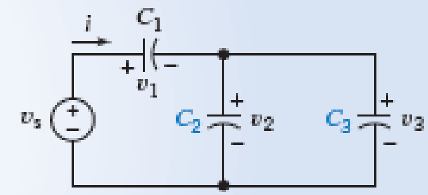


FIGURE 7.4-5 Circuit for Example 7.4-1.

Solution

Because C_2 and C_3 are in parallel, we replace them with C_p, where

$$C_p = C_2 + C_3 = 4\,\text{mF}$$

The voltage at $t = 0$ across the equivalent capacitance C_p is equal to the voltage across C_2 or C_3, which is $v_2(0) = v_3(0) = 20$ V. As a result of replacing C_2 and C_3 with C_p, we obtain the circuit shown in Figure 7.4-6.

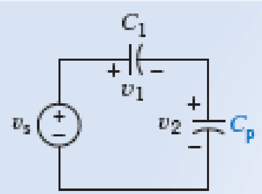


FIGURE 7.4-6 Circuit resulting from Figure 7.4-5 by replacing C_2 and C_3 with C_p.

We now want to replace the series of two capacitors C_1 and C_p with one equivalent capacitor. Using the relationship of Eq. 7.4-9, we obtain

$$C_s = \frac{C_1 C_p}{C_1 + C_p} = \frac{(2 \times 10^{-3})(4 \times 10^{-3})}{(2 \times 10^{-3}) + (4 \times 10^{-3})} = \frac{8}{6}\,\text{mF}$$

The voltage at $t = 0$ across C_s is

$$v(0) = v_1(0) + v_p(0)$$

where $v_p(0) = 20$ V, the voltage across the capacitance C_p at $t = 0$. Therefore, we obtain

$$v(0) = 10 + 20 = 30\,\text{V}$$

Thus, we obtain the equivalent circuit shown in Figure 7.4-7.

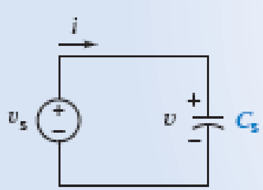
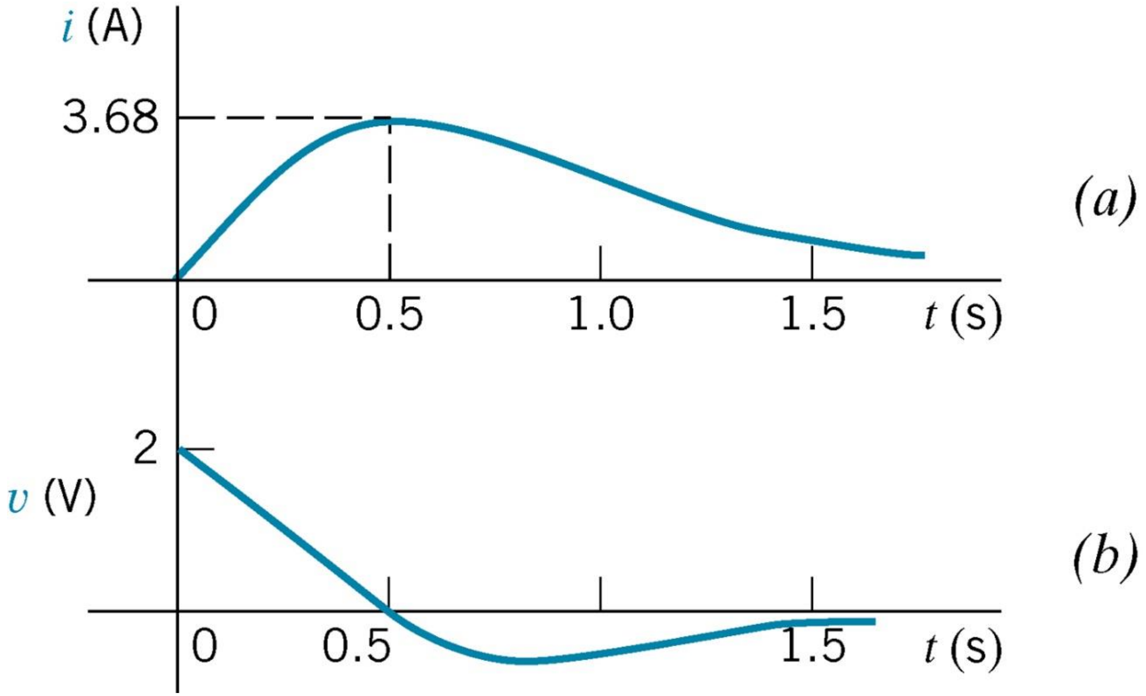


FIGURE 7.4-7 Equivalent circuit for the circuit of Example 7.4-1.

Example 7.5-1

- Find the voltage across an inductor, ***L***=0.1 H, when the current in the inductor is:

$$i(t) = 20 \cdot t \cdot e^{-2t} \;\; A, \qquad t > 0,\, i(0) = 0$$



Example 7.5-1 Solution

Ex 7.5-1 $\quad L = 0.1\,H$

$i(t) = 20 \cdot t e^{-2t}\ A, \quad t > 0, \quad i(0) = 0.$

for $t > 0.$

$$v(t) = L\frac{di}{dt} = (0.1)\frac{d}{dt}(20 \cdot t e^{-t})$$

$$= (0.1)\cdot(20)\,(t\,e^{-2t\,\prime} + t'\,e^{-2t})$$

$$= 2 \cdot (-2t\,e^{-2t} + e^{-2t})$$

$$= \underline{2 \cdot e^{-2t}(1-2t)\,V}.$$

Example 7.5-1 Solution

Find the voltage across an inductor, $L = 0.1$ H, when the current in the inductor is

$$i(t) = 20te^{-2t} \text{ A}$$

for $t > 0$ and $i(0) = 0$.

Solution

The voltage for $t < 0$ is

$$v(t) = L\frac{di}{dt} = (0.1)\frac{d}{dt}\left(20te^{-2t}\right) = 2\left(-2te^{-2t} + e^{-2t}\right) = 2e^{-2t}(1-2t) \text{ V}$$

The voltage is equal to 2 V when $t = 0$, as shown in Figure 7.5-6*b*. The current waveform is shown in Figure 7.5-6*a*.

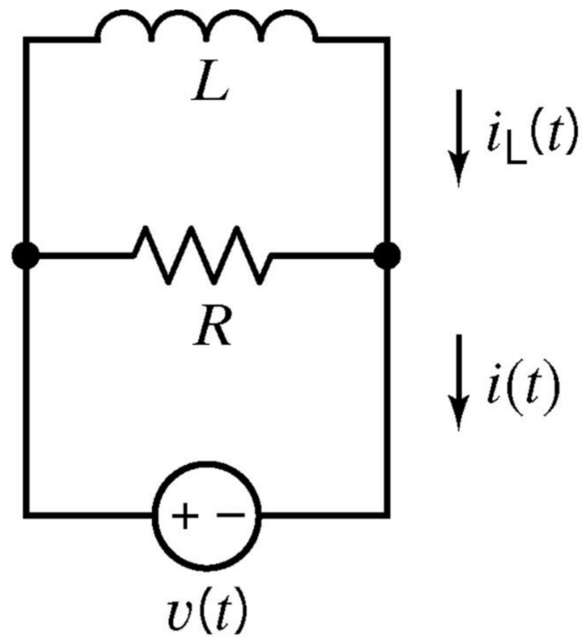
Example 7.5-3

- Calculate ***R*** and ***L*** if $i_L(0) = -3.5$ A, the input to the circuit is the voltage:

$$v(t) = 4 \cdot e^{-20t} \ V, \qquad t > 0$$

and the output is the current:

$$i(t) = -1.2 \cdot e^{-20t} - 1.5 \ A, \qquad t > 0$$



Example 7.5-3 Solution

- the input to the circuit is the voltage;

$$v(t) = 4 \cdot e^{-20t}\ V, \qquad t > 0 \quad \text{— (1)}$$

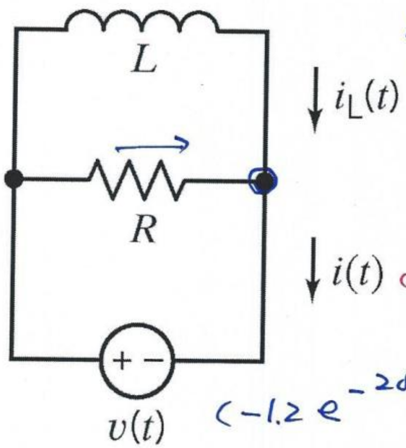
- the output is the current

$$i(t) = -1.2 \cdot e^{-20t} - 1.5\ A, \qquad t > 0 \quad \text{— (2)}$$

$$i_L(0) = -3.5A \quad \text{— (3)}$$

$$v(t) = L\frac{di(t)}{dt}$$

$$i(t) = i(t_0) + \frac{1}{L}\int_{t_0}^{t} v(\tau) \cdot d\tau$$



KCL

$$i(t) = \frac{v(t)}{R} + i_L(t)$$

$$\underset{(2)}{i(t)} = \underset{(1)}{\frac{v(t)}{R}} + \left[\underset{(3)}{i(t_0)} + \frac{1}{L}\int_{t_0}^{t} \underset{(1)}{v(\tau)}\,d\tau\right]$$

$$(-1.2e^{-20t} - 1.5) = \frac{4e^{-20t}}{R} - 3.5 + \frac{1}{L}\int_0^t 4e^{-20\tau}\,d\tau$$

$$'' \quad = \frac{4e^{-20t}}{R} - 3.5 + \frac{4}{L(-20)}(e^{-20t} - 1)$$

$$'' \quad = e^{-20t}\left(\frac{4}{R} - \frac{1}{5L}\right) - 3.5 + \frac{1}{5L}$$

$$-1.5 = -3.5 + \frac{1}{5L}, \qquad L = 0.1\,H$$

$$-1.2 = \frac{4}{R} - \frac{1}{5L}, \qquad R = 5\,\Omega.$$

Example 7.5-3 Solution

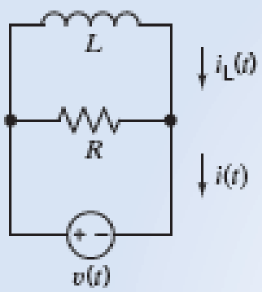


FIGURE 7.5-8 The circuit considered in Example 7.5-3.

The input to the circuit shown in Figure 7.5-8 is the voltage

$$v(t) = 4e^{-20t}\ \text{V} \quad \text{for } t > 0$$

The output is the current

$$i(t) = -1.2e^{-20t} - 1.5\ \text{A} \quad \text{for } t > 0$$

The initial inductor current is $i_L(0) = -3.5$ A. Determine the values of the inductance L and resistance R.

Solution

Apply KCL at either node to get

$$i(t) = \frac{v(t)}{R} + i_L(t) = \frac{v(t)}{R} + \left[\frac{1}{L}\int_0^t v(\tau)d\tau + i(0)\right]$$

That is

$$-1.2e^{-20t} - 1.5 = \frac{4e^{-20t}}{R} + \frac{1}{L}\int_0^t 4e^{-20t}d\tau - 3.5 = \frac{4e^{-20t}}{R} + \frac{4}{L(-20)}(e^{-20t} - 1) - 3.5$$

$$= \left(\frac{4}{R} - \frac{1}{5L}\right)e^{-20t} + \frac{1}{5L} - 3.5$$

Equating coefficients gives

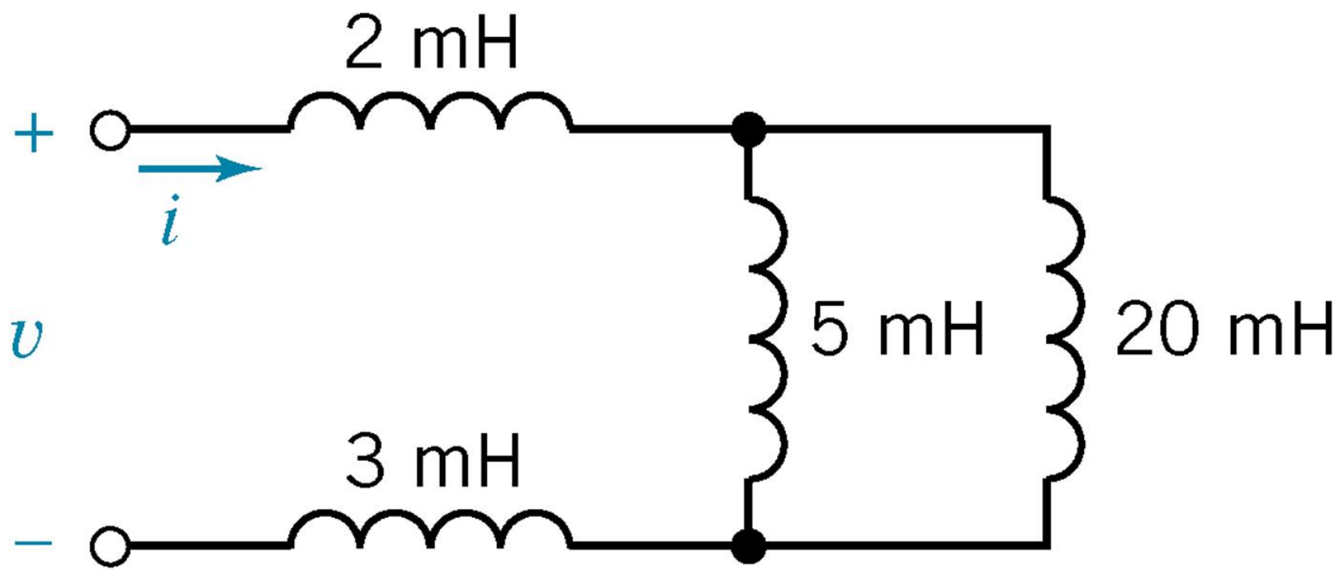
$$-1.5 = \frac{1}{5L} - 3.5 \quad \Rightarrow \quad L = 0.1\ \text{H}$$

and

$$-1.2 = \frac{4}{R} - \frac{1}{5L} = \frac{4}{R} - \frac{1}{5(0.1)} = \frac{4}{R} - 2 \quad \Rightarrow \quad R = 5\ \Omega$$

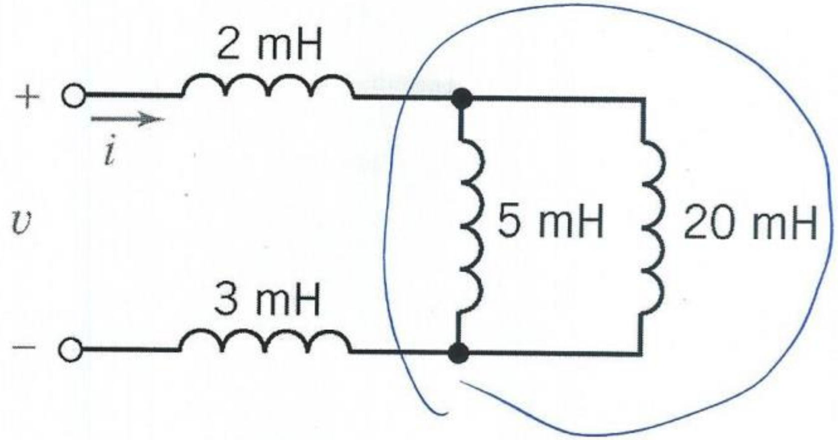
Example 7.7-1

- Find $\boldsymbol{L}_{eq}$ assuming that $\boldsymbol{i}(0) = 0$ A.



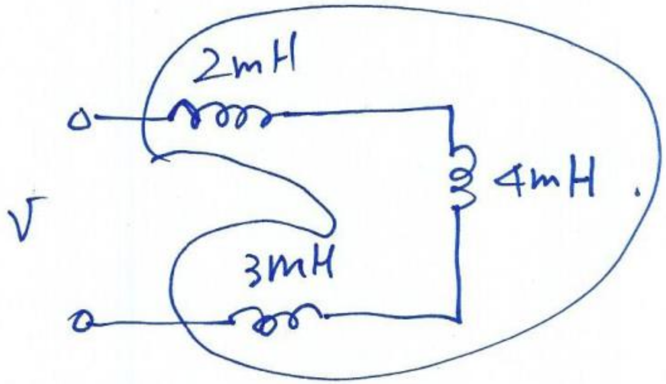
Example 7.7-1 Solution

Find the L_{eq} where $i(0) = 0$ A



$L_{eq} = \frac{5m \cdot 20m}{5m + 20m}$

$= 4mH.$



Series.

$L_{eq} = 2mH + 4mH + 3mH = \boxed{9mH}$

Example 7.7-1 Solution

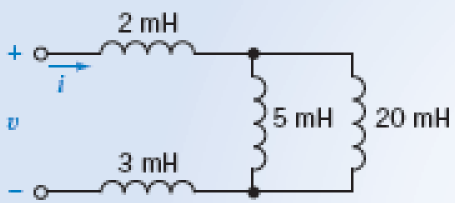


FIGURE 7.7-5 The circuit of Example 7.7-1.

Find the equivalent inductance for the circuit of Figure 7.7-5. All the inductor currents are zero at t_0.

Solution

First, we find the equivalent inductance for the 5-mH and 20-mH inductors in parallel.

From Eq. 7.7-4, we obtain

$$\frac{1}{L_p} = \frac{1}{L_1} + \frac{1}{L_2}$$

or

$$L_p = \frac{L_1 L_2}{L_1 + L_2} = \frac{5 \times 20}{5 + 20} = 4 \text{ mH}$$

This equivalent inductor is in series with the 2-mH and 3-mH inductors. Therefore, using Eq. 7.7-1, we obtain

$$L_{eq} = \sum_{n=1}^{N} L_n = 2 + 3 + 4 = 9 \text{ mH}$$