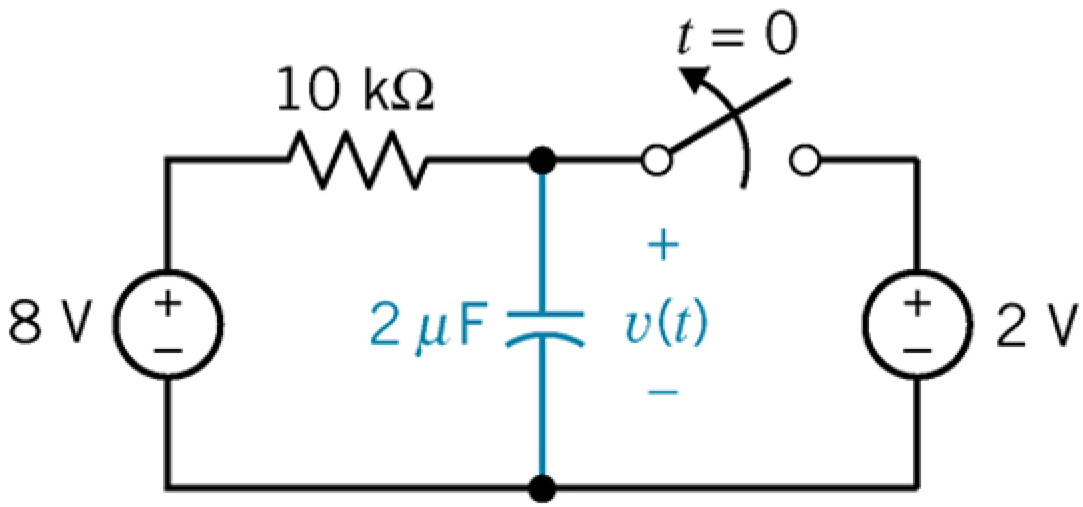
Chapter 8

The Complete Response of RL and RC Circuits (Problems)

Example 8.3-1

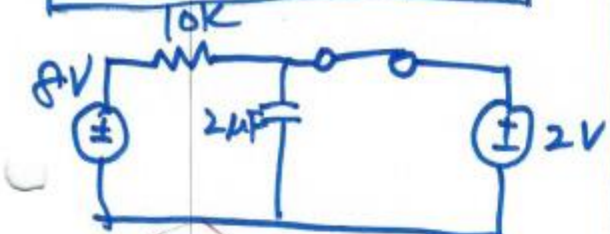
- What is the value of the ***capacitor voltage*** 50 ms after the switch opens? $t \geq 0$

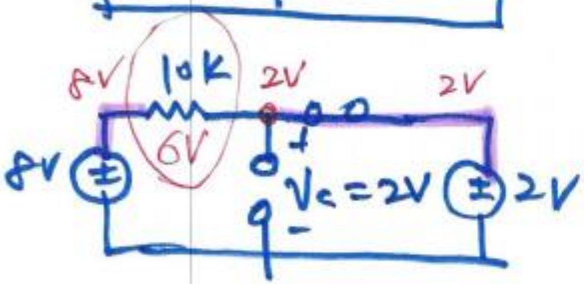


Example 8.3-1 Solution

$$v(t) = V + (v(0) - V)e^{-t/RC}.$$

$\boxed{t<0 : \text{obtain } v(0)}$

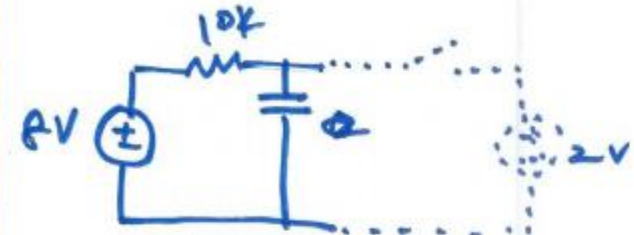


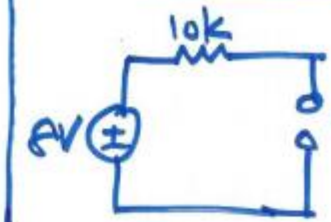


V across cap

$v_c = 2V = v(0)$

$\boxed{t \ge 0 : \text{obtain } V}$ $t=\infty$





$v_c = 8V = V$

$\tau = R\cdot C = (10\times10^{3})(2\times10^{-6}) = 20ms$

$$\boxed{v(t) = 8 + (2-8)e^{-t/20m} \quad V}$$

$$v(50ms) = 8 - 6e^{-50ms/20ms} = 7.51V$$

Example 8.3-1 Solution

Find the capacitor voltage after the switch opens in the circuit shown in Figure 8.3-4*a*. What is the value of the capacitor voltage 50 ms after the switch opens?

Solution

The 2-volt voltage source forces the capacitor voltage to be 2 volts until the switch opens. Because the capacitor voltage cannot change instantaneously, the capacitor voltage will be 2 volts immediately after the switch opens. Therefore, the initial condition is

$$v(0) = 2\ \text{V}$$

Figure 8.3-4*b* shows the circuit after the switch opens. Comparing this circuit to the *RC* circuit in Figure 8.3-1*b*, we see that

$$R_t = 10\ \text{k}\Omega \quad \text{and} \quad V_{oc} = 8\ \text{V}$$

The time constant for this first-order circuit containing a capacitor is

$$\tau = R_t C = \left(10 \times 10^3\right)\left(2 \times 10^{-6}\right) = 20 \times 10^{-3} = 20\ \text{ms}$$

Substituting these values into Eq. 8.3-6 gives

$$v(t) = 8 - 6e^{-t/20}\ \text{V} \tag{8.3-8}$$

where t has units of ms. To find the voltage 50 ms after the switch opens, let $t = 50$. Then,

$$v(50) = 8 - 6e^{-50/20} = 7.51\ \text{V}$$

Example 8.3-1 Solution

Figure 8.3-4*c* shows a plot of the capacitor voltage as a function of time.

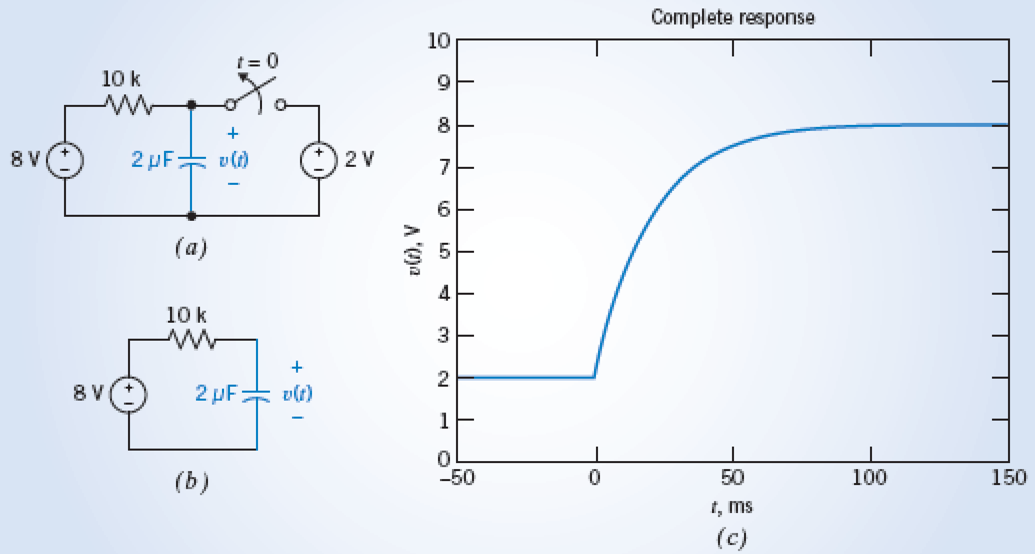
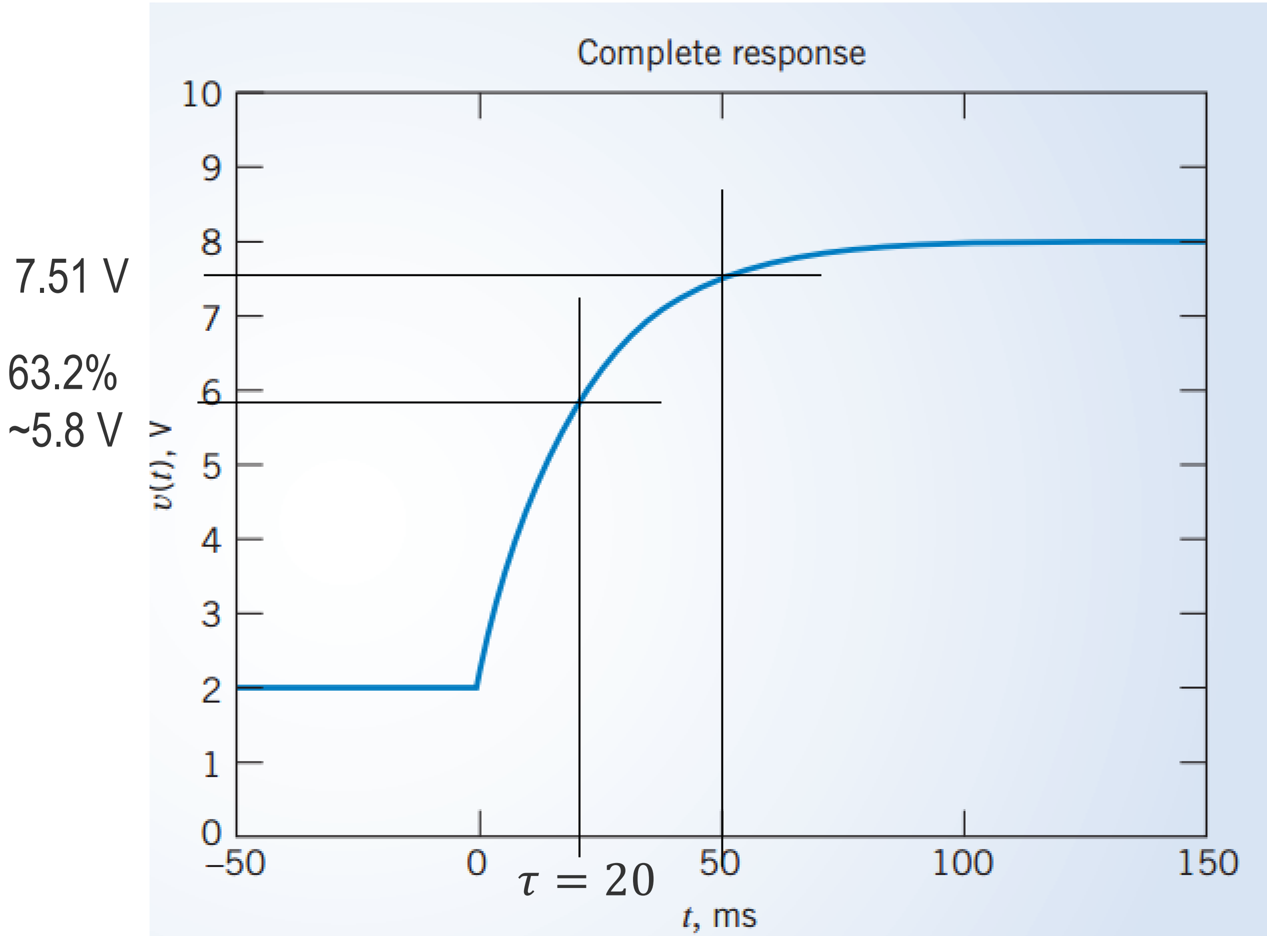


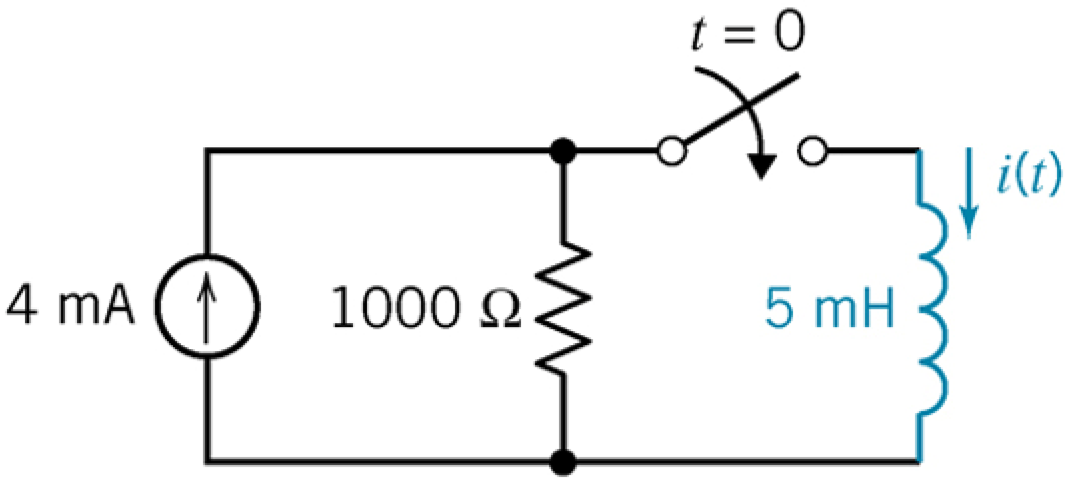
FIGURE 8.3-4 (*a*) A first-order circuit and (*b*) an equivalent circuit that is valid after the switch opens. (*c*) A plot of the complete response, $v(t)$, given in Eq. 8.3-8.

Example 8.3-1 Solution



Example 8.3-2

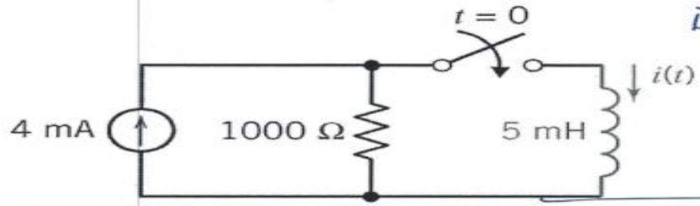
- Find the inductor current after the switch closes. How long will it take for the ***inductor current*** to reach **2 mA**? $t \geq 0$



Example 8.3-2 Solution

Example 8.3 – 2

Find the inductor current after the switch closes. How long will it take for the ***inductor current*** to reach **2 mA**?



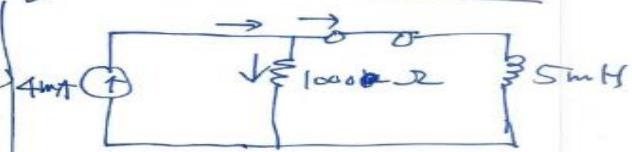
$i(t) = I + (i(0) - I)e^{-t/\tau}$, $I = i(\infty)$

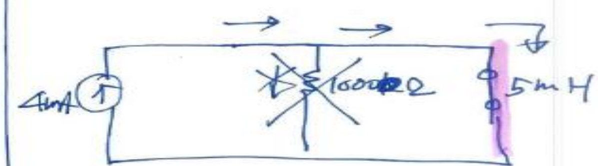
$\tau = \frac{L}{R}$

$\boxed{t < 0 ;\ \text{obtain } i(0)}$

$\boxed{i(0) = 0\text{mA}}$

$\boxed{t \geq 0 ;\ \text{obtain } I}$





$I = 4\text{mA}$

$\tau = \frac{L}{R} = 5\mu s$.

$i(t) = I + (i(0) - I)e^{-t/\tau}$

$2\text{mA} = i(t) = 4 + (0 - 4)e^{-t/5}$ (unit : μs).

$\left(\frac{2-4}{-4}\right) = e^{-t/5}$

$t = -5 \cdot \ln\left(\frac{2-4}{-4}\right) = \underline{3.47\ (\mu s)}$

Example 8.3-2 Solution

Find the inductor current after the switch closes in the circuit shown in Figure 8.3-5*a*. How long will it take for the inductor current to reach 2 mA?

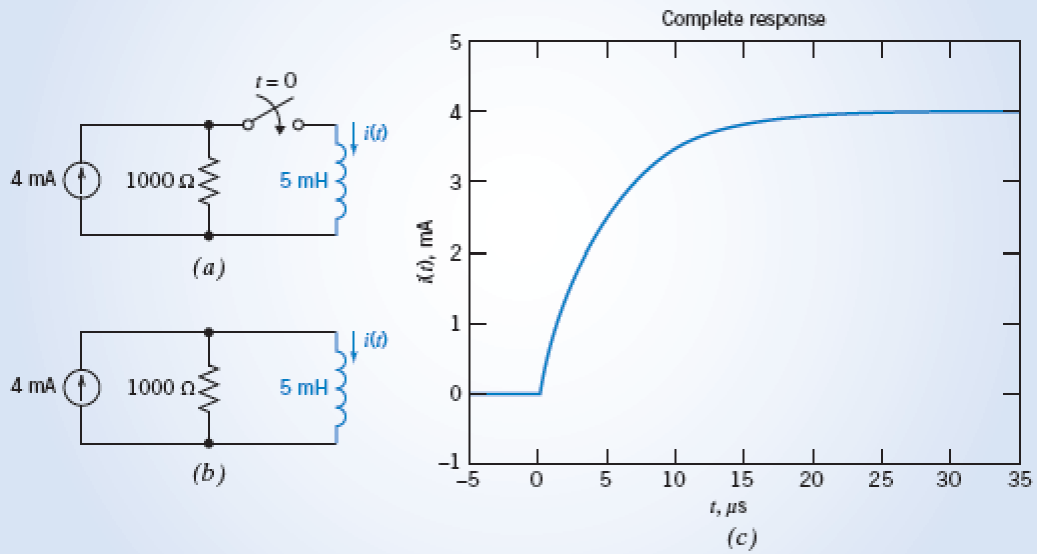
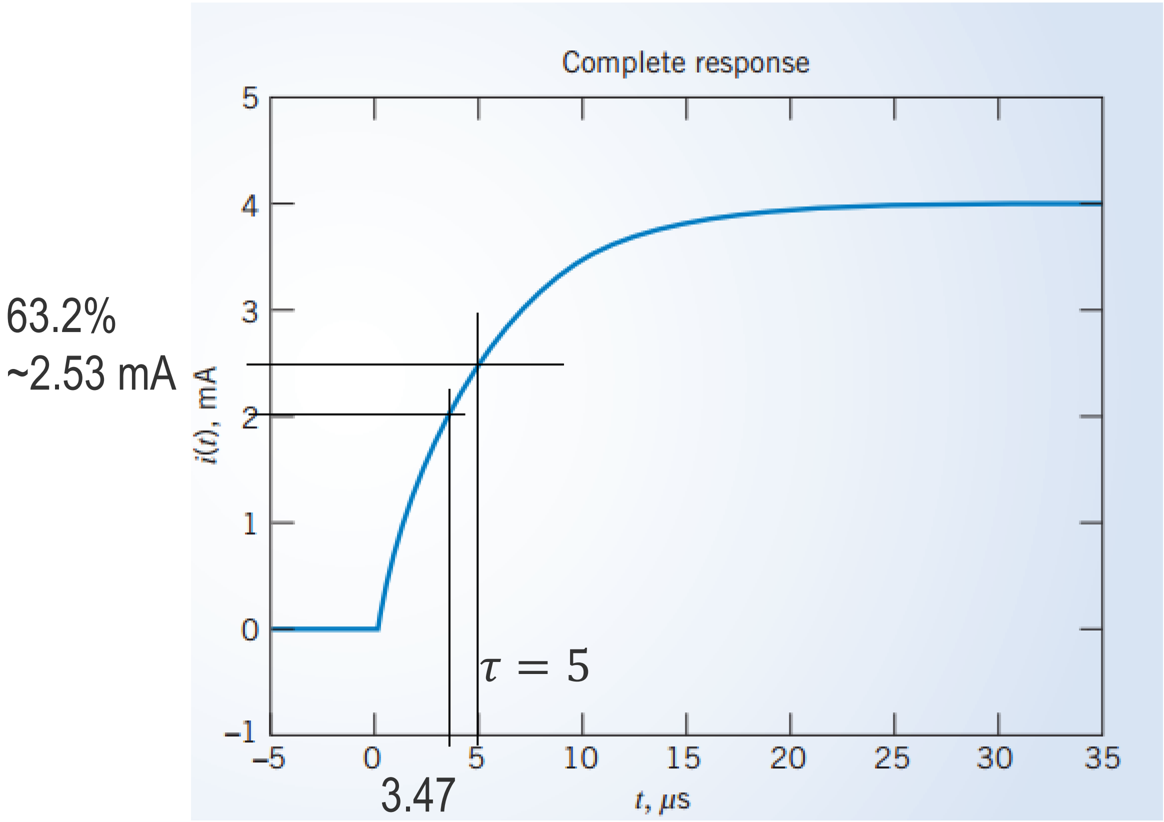


FIGURE 8.3-5 (*a*) A first-order circuit and (*b*) an equivalent circuit that is valid after the switch closes. (*c*) A plot of the complete response, $i(t)$, given by Eq. 8.3-9.

Example 8.3-2 Solution



Example 8.3-2 Solution

Solution

The inductor current will be 0 A until the switch closes. Because the inductor current cannot change instantaneously, it will be 0 A immediately after the switch closes. Therefore, the initial condition is

$$i(0) = 0$$

Figure 8.3-5*b* shows the circuit after the switch closes. Comparing this circuit to the *RL* circuit in Figure 8.3-2*b*, we see that

$$R_t = 1000\ \Omega \quad \text{and} \quad I_{sc} = 4\ \text{mA}$$

The time constant for this first-order circuit containing an inductor is

$$\tau = \frac{L}{R_t} = \frac{5 \times 10^{-3}}{1000} = 5 \times 10^{-6} = 5\ \mu\text{s}$$

Substituting these values into Eq. 8.3-7 gives

$$i(t) = 4 - 4e^{-t/5}\ \text{mA} \qquad (8.3\text{-}9)$$

where t has units of microseconds. To find the time when the current reaches 2 mA, substitute $i(t) = 2$ mA. Then

$$2 = 4 - 4e^{-t/5}\ \text{mA}$$

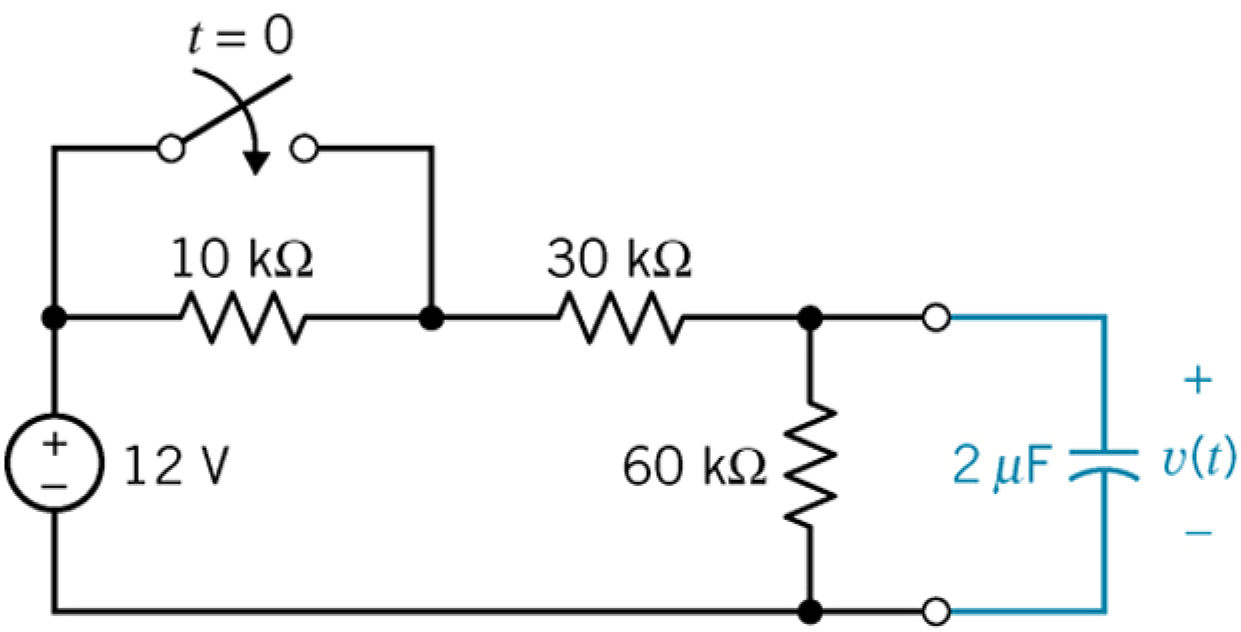
Solving for t gives

$$t = -5 \times \ln\left(\frac{2-4}{-4}\right) = 3.47\ \mu\text{s}$$

Figure 8.3-5*c* shows a plot of the inductor current as a function of time.

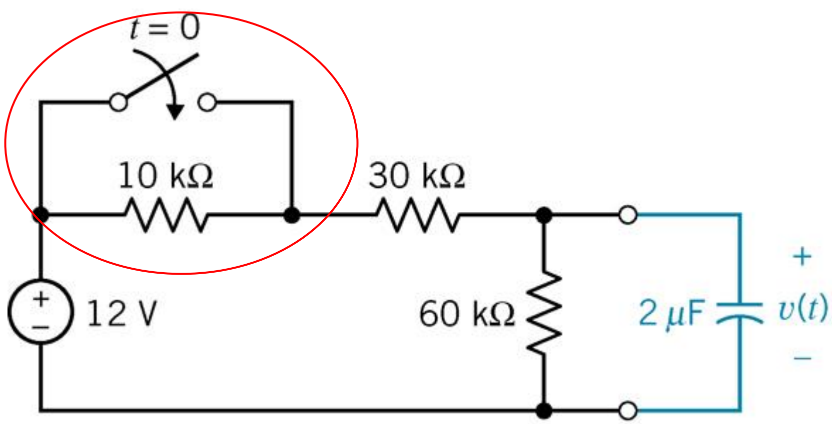
Example 8.3-3

- The switch has been open for a long time and the circuit has reached steady state before the switch closes at time t = 0. Find the ***capacitor voltage*** for $t \geq 0$.

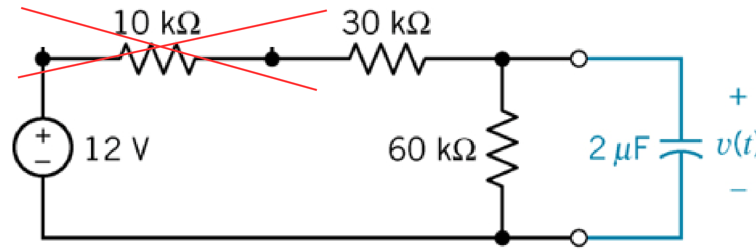


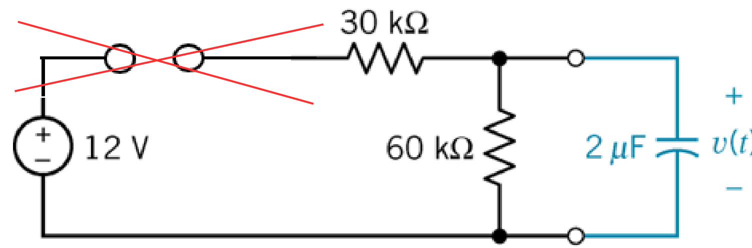
Example 8.3-3

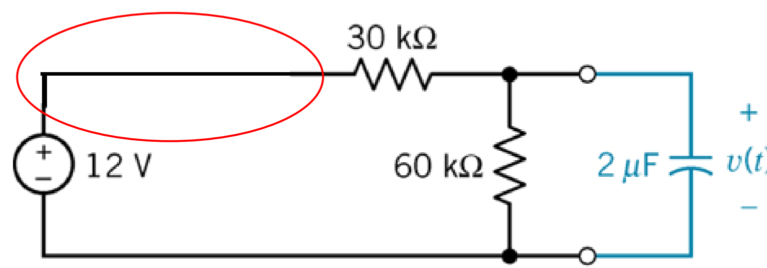
What is the equivalent circuit at t = ∞ of the circuit shown in the circle below?

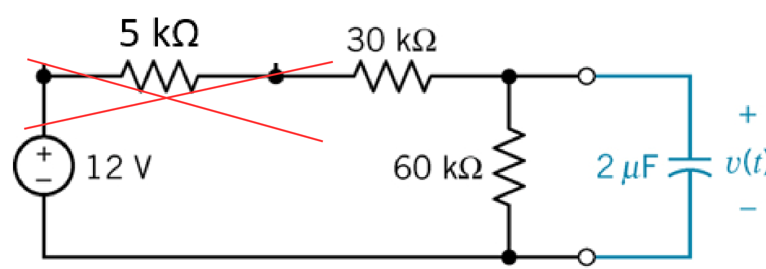


(a)









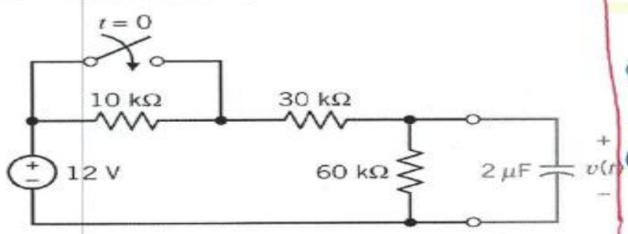
Example 8.3-3 Solution

Example 8.3 – 3

The switch has been open for a long time and the circuit has reached steady state before the switch closes at time t = 0. Find the ***capacitor voltage*** for $t \ge 0$.

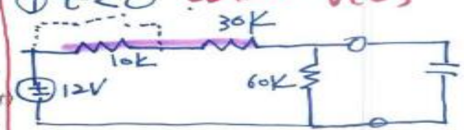
$v(t) = V + (v(0) - V)e^{-t/\tau}$

$\tau = R \cdot C$

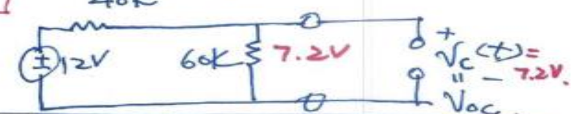


(a)

① $t<0$ obtain $v(0)$



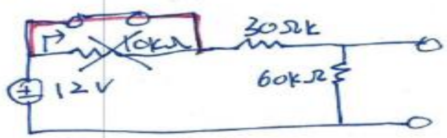
⇓

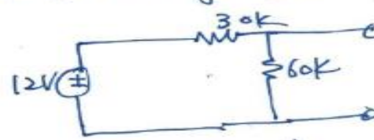


$v_c(t) = 7.2V$ = V_{oc}

$$v(0) = \frac{60K}{40K+60K} \times 12V = 7.2V.$$

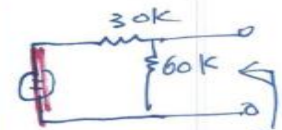
② $t \ge 0$; obtain V; Thevenin eq. ckt.





$v(t) = 8V$ = V_{oc} = $v_c(t)$

$$V_{oc} = \frac{60K}{30K+60K} \cdot 12V = 8V.$$



$R_T =$

$$R_T = \frac{30K \cdot 60K}{30K+60K} = 20K\Omega$$

$v(t) = V + (v(0) - V)e^{-t/\tau}$ R·C

$\tau = R \cdot C = 20K \cdot 2\mu = 40ms.$

$v(t) = 8 + (7.2 - 8)e^{-t/40}$

(unit. ms)

Example 8.3-3 Solution

The switch in Figure 8.3-6*a* has been open for a long time, and the circuit has reached steady state before the switch closes at time $t = 0$. Find the capacitor voltage for $t \geq 0$.

Solution

The switch has been open for a long time before it closes at time $t = 0$. The circuit will have reached steady state before the switch closes. Because the input to this circuit is a constant, all the element currents and voltages will be constant when the circuit is at steady state. In particular, the capacitor voltage will be constant. The capacitor current will be

$$i(t) = C\frac{d}{dt}v(t) = C\frac{d}{dt}(\text{a constant}) = 0$$

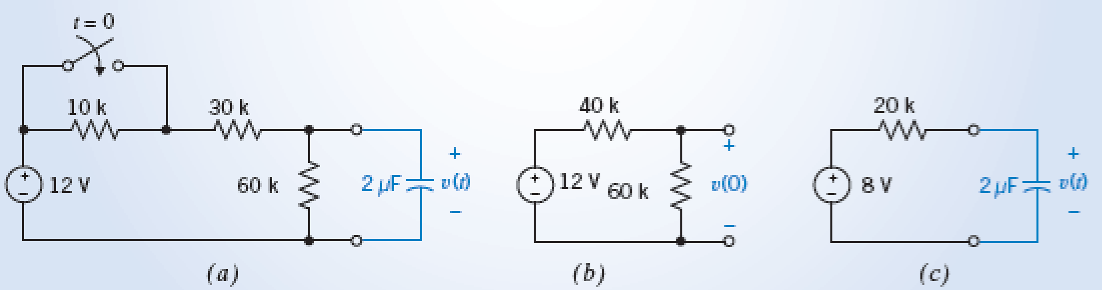


FIGURE 8.3-6 (*a*) A first-order circuit. The equivalent circuit for (*b*) $t < 0$ and (*c*) $t > 0$.

Example 8.3-3 Solution

The capacitor voltage is unknown, but the capacitor current is zero. In other words, the capacitor acts like an open circuit when the input is constant and the circuit is at steady state. (By a similar argument, inductors act like short circuits when the input is constant and the circuit is at steady state.)

Figure 8.3-6*b* shows the appropriate equivalent circuit while the switch is open. An open switch acts like an open circuit; thus, the 10-kΩ and 30-kΩ resistors are in series. They have been replaced by an equivalent 40-kΩ resistor. The input to the circuit is a constant (12 volts), and the circuit is at steady state; therefore, the capacitor acts like an open circuit. The voltage across this open circuit is the capacitor voltage. Because we are interested in the initial condition, the capacitor voltage has been labeled as $v(0)$. Analyzing the circuit in Figure 8.3-6*b* using voltage division gives

$$v(0) = \frac{60 \times 10^3}{40 \times 10^3 + 60 \times 10^3} 12 = 7.2 \text{ V}$$

Figure 8.3-6*c* shows the appropriate equivalent circuit after the switch closes. Closing the switch shorts out the 10-kΩ resistor, removing it from the circuit. (A short circuit in parallel with any resistor is equivalent to a short circuit.) The part of the circuit that is connected to the capacitor has been replaced by its Thévenin equivalent circuit. After the switch is closed,

$$V_{oc} = \frac{60 \times 10^3}{30 \times 10^3 + 60 \times 10^3} 12 = 8 \text{ V}$$

and

$$R_t = \frac{30 \times 10^3 \times 60 \times 10^3}{30 \times 10^3 + 60 \times 10^3} = 20 \times 10^3 = 20 \text{ k}\Omega$$

and the time constant is

$$\tau = R_t \times C = \left(20 \times 10^3\right) \times \left(2 \times 10^{-6}\right) = 40 \times 10^{-3} = 40 \text{ ms}$$

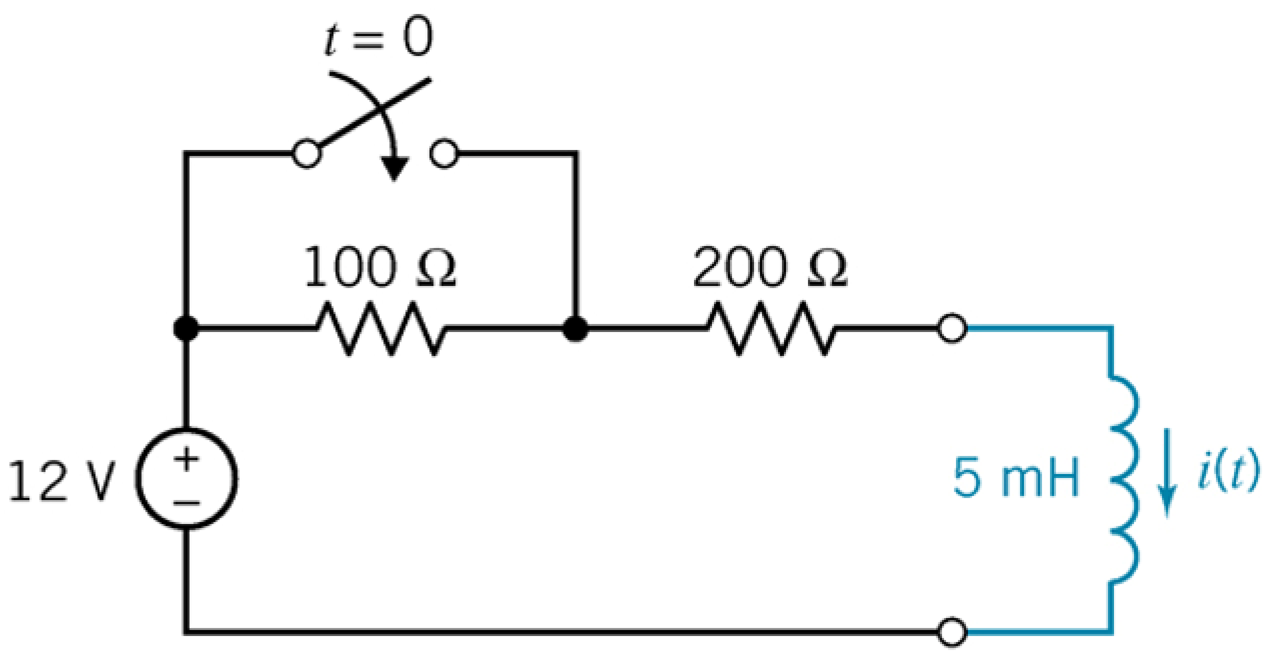
Substituting these values into Eq. 8.3-6 gives

$$v(t) = 8 - 0.8e^{-t/40} \text{ V}$$

where t has units of ms.

Example 8.3-4

- The switch has been open for a long time and the circuit has reached steady state before the switch closes at time $t = 0$. Find the ***inductor current*** for $t \geq 0$.

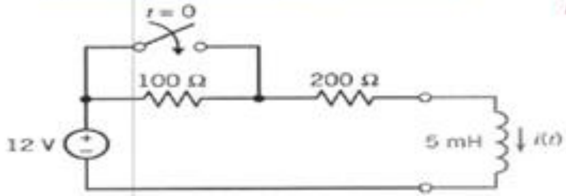


Example 8.3-4 Solution

8.3-4: The switch has been open for a long time and the circuit has reached steady state before the switch closes at time t = 0. Find the ***inductor current*** for t ≥ 0.

$i(t) = I + (i(0) - I)e^{-t/\tau}$ for $t \geq 0$

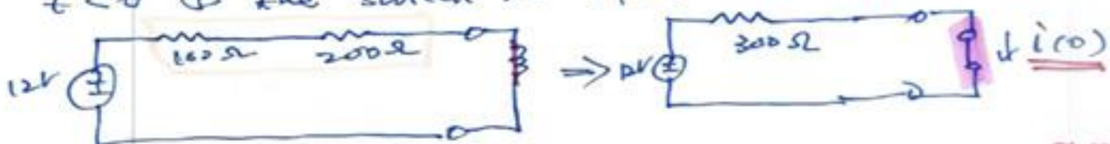
from Norton eq. ckt.



(a)

$L/R = 5\,mH / 200\,\Omega = 25\,\mu s$.

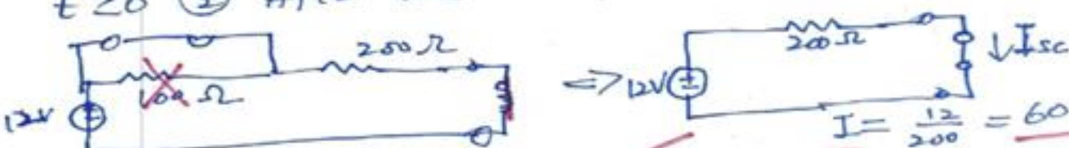
$t < 0$ ① the switch is open → obtain $i(0)$



$i(0) = 12/300 = 40\,mA$.

obtain ② I

$t \geq 0$ ② After the sw is closed → Norton →



$I = \frac{12}{200} = 60\,mA$.

$i(t) = I + (i(0) - I)e^{-t/\tau}$

$\tau = L/R = 25\,\mu s$

$i(t) = 60 + (40 - 60)e^{-t/\tau}$ mA

Example 8.3-4 Solution

The switch in Figure 8.3-7a has been open for a long time, and the circuit has reached steady state before the switch closes at time $t = 0$. Find the inductor current for $t \geq 0$.

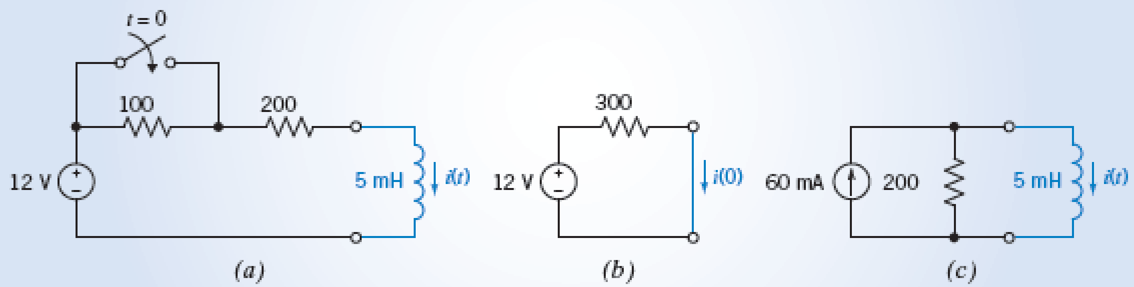


FIGURE 8.3-7 (a) A first-order circuit. The equivalent circuit for (b) $t < 0$ and (c) $t > 0$.

Example 8.3-4 Solution

Solution

Figure 8.3-7*b* shows the appropriate equivalent circuit while the switch is open. The 100-Ω and 200-Ω resistors are in series and have been replaced by an equivalent 300-Ω resistor. The input to the circuit is a constant (12 volts), and the circuit is at steady state; therefore, the inductor acts like a short circuit. The current in this short circuit is the inductor current. Because we are interested in the initial condition, the initial inductor current has been labeled as $i(0)$. This current can be calculated using Ohm's law:

$$i(0) = \frac{12}{300} = 40\text{ mA}$$

Figure 8.3-7*c* shows the appropriate equivalent circuit after the switch closes. Closing the switch shorts out the 100-Ω resistor, removing it from the circuit. The part of the circuit that is connected to the inductor has been replaced by its Norton equivalent circuit. After the switch is closed,

$$I_{sc} = \frac{12}{200} = 60\text{ mA} \quad \text{and} \quad R_t = 200\ \Omega$$

and the time constant is

$$\tau = \frac{L}{R_t} = \frac{5 \times 10^{-3}}{200} = 25 \times 10^{-6} = 25\ \mu\text{s}$$

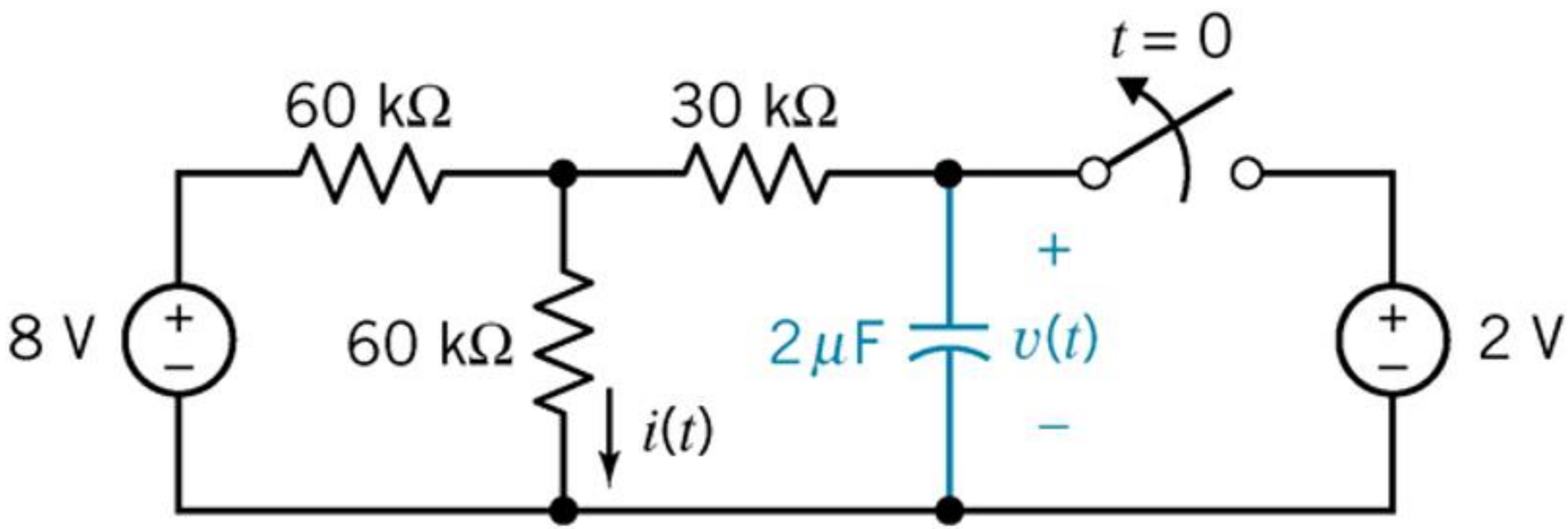
Substituting these values into Eq. 8.3-7 gives

$$i(t) = 60 - 20e^{-t/25}\text{ mA}$$

where t has units of microseconds.

Example 8.3-5

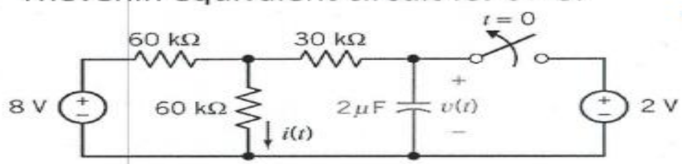
- The circuit is at steady state before the switch opens. Find the current ***i*(t)** for t > 0. What is the voltage $v(t)$ at t = 60 ms?



Example 8.3-5 Solution

Example 8.3 – 5

- The circuit is at steady state before the switch opens. Find the Thevenin equivalent circuit for t > 0.



(a)

$v(t) = V + (v(0) - V)e^{-t/\tau}$

$\tau = R\cdot C$

- $t<0$: obtain $v(0)$.

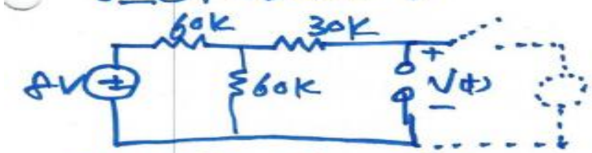
$\boxed{v(0) = 2V}$.

$\tau = R\cdot C = 60K \cdot 2\mu F = 120 mS$.

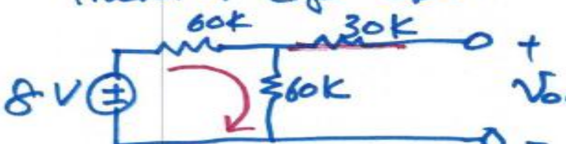
$v(t) = 4 - 2e^{-t/120}$ V

$v(60) = 4 - 2e^{-60/120} = 2.787$ V.

- $t \geq 0$: obtain V



Thevenin eq ckt.



$v_{oc} = V_{Th} = \frac{60K}{60K+60K}\cdot 8 = \boxed{4V}$

Req

Example 8.3-5 Solution

The circuit in Figure 8.3-8*a* is at steady state before the switch opens. Find the current $i(t)$ for $t > 0$.

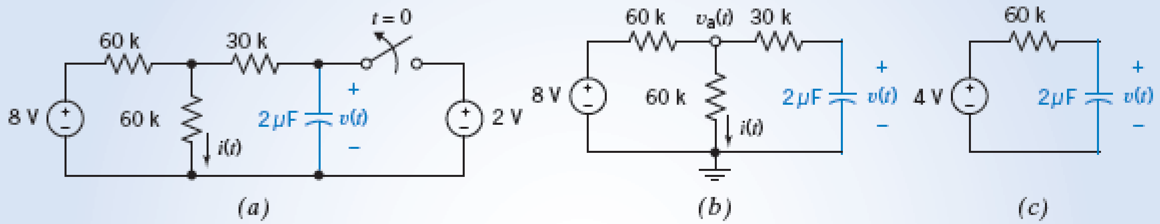


FIGURE 8.3-8 (*a*) A first-order circuit, (*b*) the circuit after the switch opens, and (*c*) the equivalent circuit after the switch opens.

Solution

The response or output of a circuit can be any element current or voltage. Frequently, the response is not the capacitor voltage or inductor current. In Figure 8.3-8*a*, the response is the current $i(t)$ in a resistor rather than the capacitor voltage. In this case, two steps are required to solve the problem. First, find the capacitor voltage using the methods already described in this chapter. Once the capacitor voltage is known, write node or mesh equations to express the response in terms of the input and the capacitor voltage.

First we find the capacitor voltage. Before the switch opens, the capacitor voltage is equal to the voltage of the 2-volt source. The initial condition is

$$v(0) = 2\text{ V}$$

Figure 8.3-8*b* shows the circuit as it will be after the switch is opened. The part of the circuit connected to the capacitor has been replaced by its Thévenin equivalent circuit in Figure 8.3-8*c*. The parameters of the Thévenin

Example 8.3-5 Solution

equivalent circuit are

$$V_{oc} = \frac{60 \times 10^3}{60 \times 10^3 + 60 \times 10^3} 8 = 4\,\text{V}$$

and
$$R_t = 30 \times 10^3 + \frac{60 \times 10^3 \times 60 \times 10^3}{60 \times 10^3 + 60 \times 10^3} = 60 \times 10^3 = 60\,\text{k}\Omega$$

The time constant is

$$\tau = R_t \times C = \left(60 \times 10^3\right) \times \left(2 \times 10^{-6}\right) = 120 \times 10^{-3} = 120\,\text{ms}$$

Substituting these values into Eq. 8.3-6 gives

$$v(t) = 4 - 2e^{-t/120}\,\text{V}$$

where t has units of ms.

Now that the capacitor voltage is known, we return to the circuit in Figure 8.3-8*b*. Notice that the node voltage at the middle node at the top of the circuit has been labeled as $v_a(t)$. The node equation corresponding to this node is

$$\frac{v_a(t) - 8}{60 \times 10^3} + \frac{v_a(t)}{60 \times 10^3} + \frac{v_a(t) - v(t)}{30 \times 10^3} = 0$$

Substituting the expression for the capacitor voltage gives

$$\frac{v_a(t) - 8}{60 \times 10^3} + \frac{v_a(t)}{60 \times 10^3} + \frac{v_a(t) - \left(4 - 2e^{-t/120}\right)}{30 \times 10^3} = 0$$

or
$$v_a(t) - 8 + v_a(t) + 2\left[v_a(t) - \left(4 - 2e^{-t/120}\right)\right] = 0$$

Solving for $v_a(t)$, we get

$$v_a(t) = \frac{8 + 2\left(4 - 2e^{-t/120}\right)}{4} = 4 - e^{-t/120}\,\text{V}$$

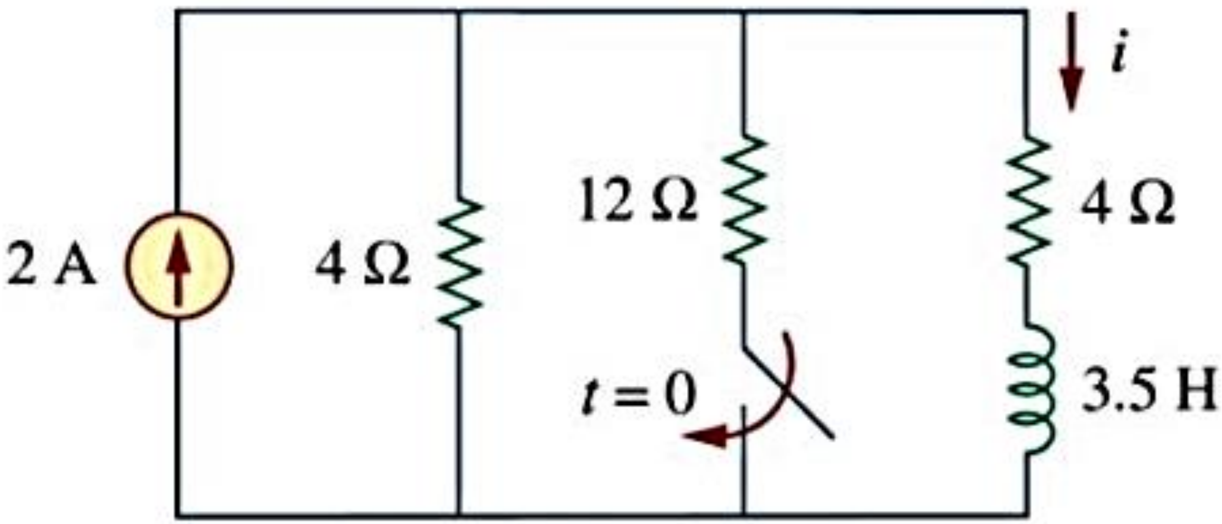
Finally, we calculate $i(t)$ using Ohm's law:

$$i(t) = \frac{v_a(t)}{60 \times 10^3} = \frac{4 - e^{-t/120}}{60 \times 10^3} = 66.7 - 16.7e^{-t/120}\,\mu\text{A}$$

where t has units of ms.

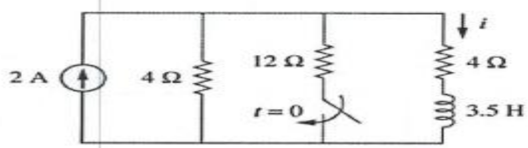
Example 2

Obtain the inductor current for both $t < 0$ and $t > 0$ in each of the circuits in Fig. 7.120.

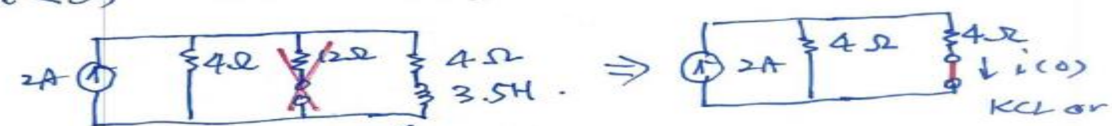


Example 2 Solution

Obtain the inductor current for both $t < 0$ and $t > 0$ in each of the circuits in Fig. 7.120.



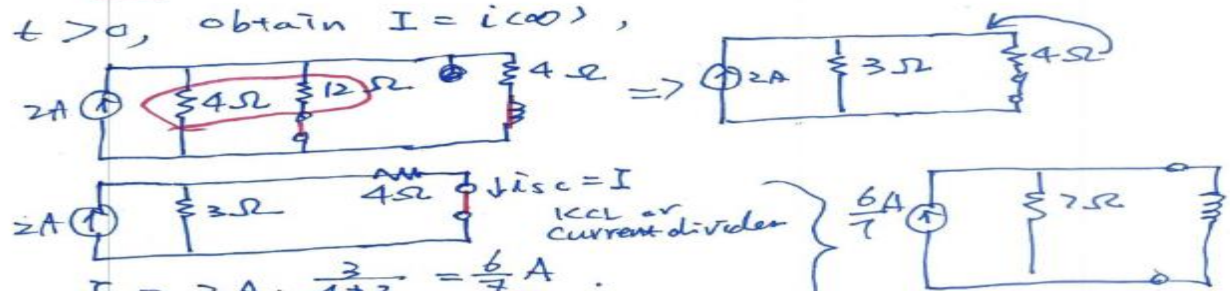
$t<0$, obtain $i(0)$ the init. current.



KCL or current divider.

$$i(0) = 2A \cdot \frac{4}{4+4} = 1A.$$

$t>0$, obtain $I = i(\infty)$,

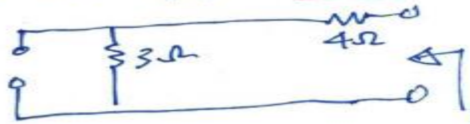


$i_{sc} = I$

KCL or current divider

$$I = 2A \cdot \frac{3}{4+3} = \frac{6}{7} A.$$

$R_{eq} = R_N$



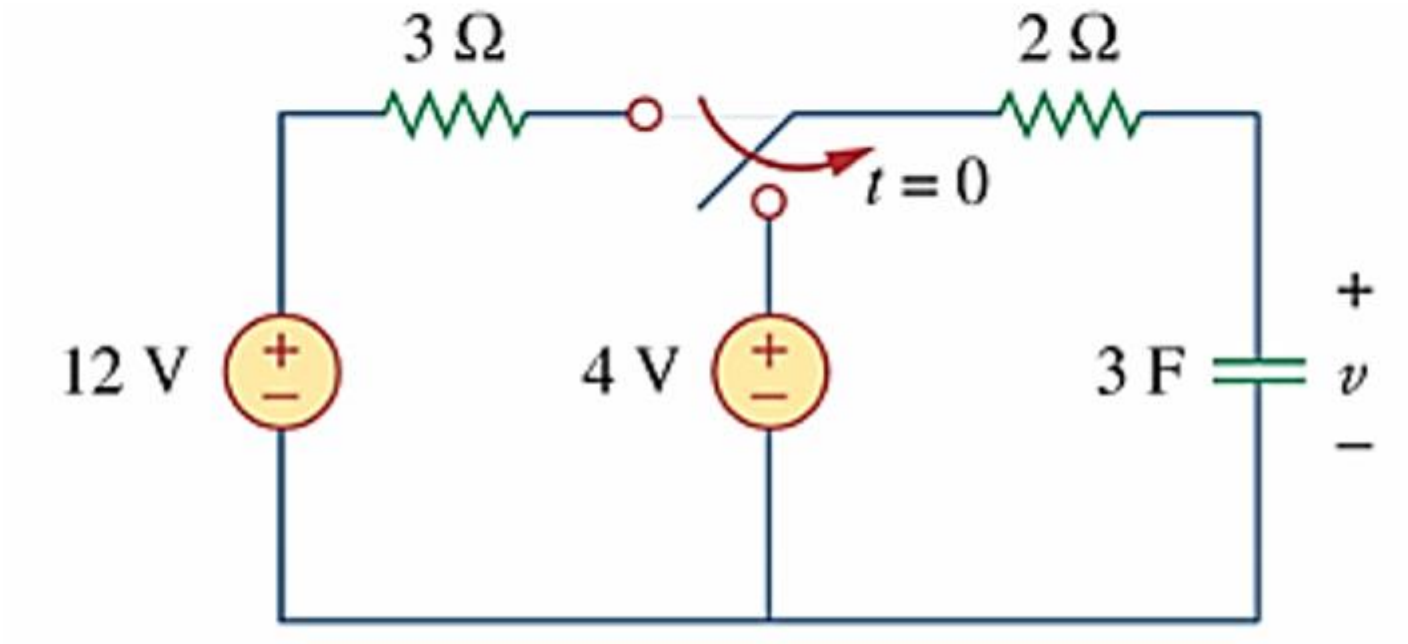
$R_{eq} = 3+4 = 7\Omega$

$$i(t) = I + (i(0) - I)\, e^{-t/\tau} \qquad \tau = L/R = 3.5/7 = 0.5$$

$$= \frac{6}{7} + \left(1 - \frac{6}{7}\right) e^{-t/0.5} = 0.857 + 0.143\, e^{-t/0.5}$$

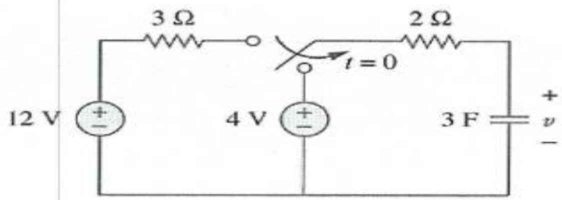
Example 3

Find the capacitor voltage for $t < 0$ and $t > 0$ for each of the circuits in the figure

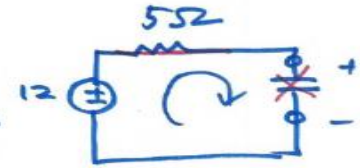


Example 3 Solution

Find the capacitor voltage for $t < 0$ and $t > 0$ for each of the circuits in the figure

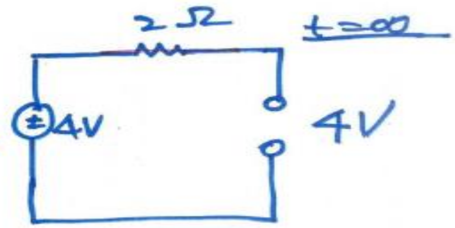


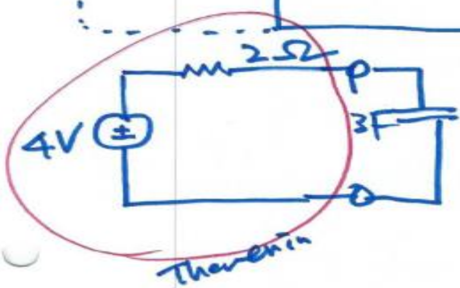
$t<0$, obtain $v(0)$



$v(t) = 12V.$

$t \geq 0$, obtain V .





What is R_{eq} (or R_t) $= 2\Omega$.

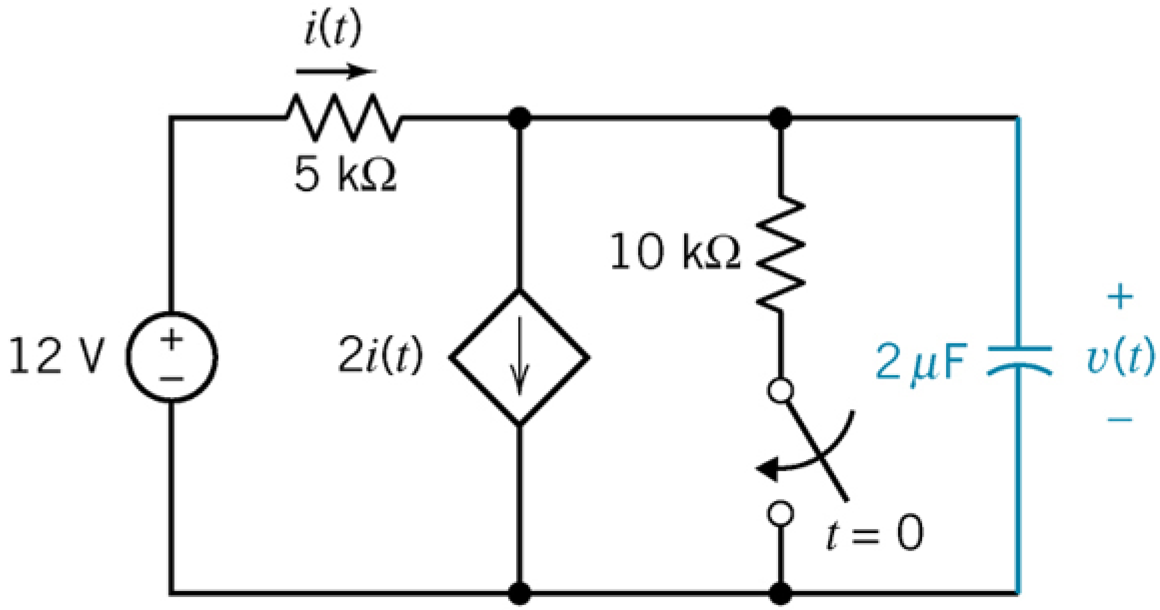
$\tau = R \cdot C = \frac{2}{\Omega} \cdot 3F = 6\,s.$

$v(t) = 4 + (12-4)\,e^{-t/6}$

$= 4 + (8)\,e^{-t/6}$

Example 8.5-1

- The first order switch is at steady state before the switch closes at $t = 0$. Find the capacitor voltage, $v(t)$, for $t > 0$.



Example 8.5-1 Solution

The first-order circuit shown in Figure 8.5-1*a* is at steady state before the switch closes at $t = 0$. This circuit contains a dependent source and so may be unstable. Find the capacitor voltage $v(t)$ for $t > 0$.

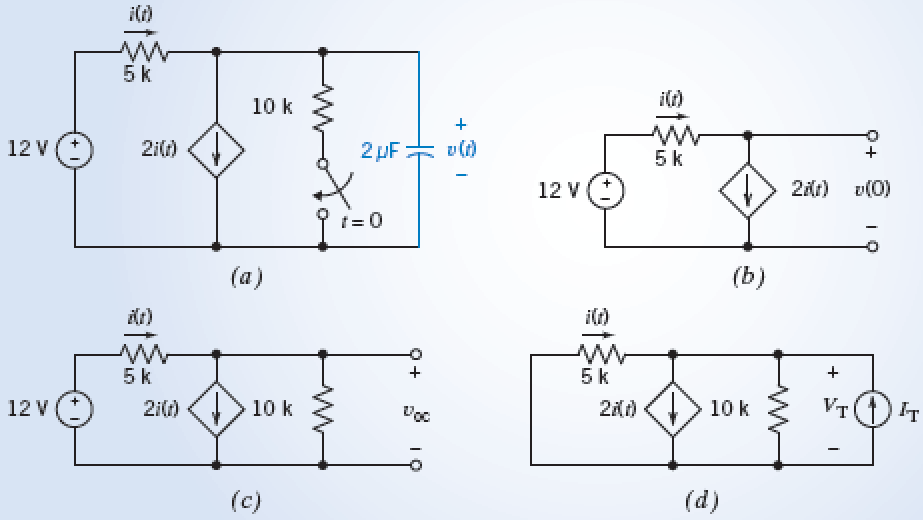


FIGURE 8.5-1 (*a*) A first-order circuit containing a dependent source. (*b*) The circuit used to calculate the initial condition. (*c*) The circuit used to calculate V_{oc}. (*d*) The circuit used to calculate R_t.

Example 8.5-1 Solution

Solution

The input to the circuit is a constant, so the capacitor acts like an open circuit at steady state. We calculate the initial condition from the circuit in Figure 8.5-1*b*. Applying KCL to the top node of the dependent current source, we get

$$-i + 2i = 0$$

Therefore, $i = 0$. Consequently, there is no voltage drop across the resistor, and

$$v(0) = 12\text{ V}$$

Next, we determine the Thévenin equivalent circuit for the part of the circuit connected to the capacitor. This requires two calculations. First, calculate the open-circuit voltage, using the circuit in Figure 8.5-1*c*. Writing a KVL equation for the loop consisting of the two resistors and the voltage source, we get

$$12 = \left(5 \times 10^3\right) \times i + \left(10 \times 10^3\right) \times (i - 2i)$$

Solving for the current, we find

$$i = -2.4\text{ mA}$$

Applying Ohm's law to the 10-kΩ resistor, we get

$$V_{oc} = \left(10 \times 10^3\right) \times (i - 2i) = 24\text{ V}$$

Now calculate the Thévenin resistance using the circuit shown in Figure 8.5-1*d*. Apply KVL to the loop consisting of the two resistors to get

$$0 = \left(5 \times 10^3\right) \times i + \left(10 \times 10^3\right) \times (I_T + i - 2i)$$

Solving for the current,

$$i = 2I_T$$

Applying Ohm's law to the 10-kΩ resistor, we get

$$V_T = 10 \times 10^3 \times (I_T + i - 2i) = -10 \times 10^3 \times I_T$$

The Thévenin resistance is given by

$$R_t = \frac{V_T}{I_T} = -10\text{ k}\Omega$$

The time constant is

$$\tau = R_t C = -20\text{ ms}$$

This circuit is unstable. The complete response is

$$v(t) = 24 - 12\, e^{t/20}$$

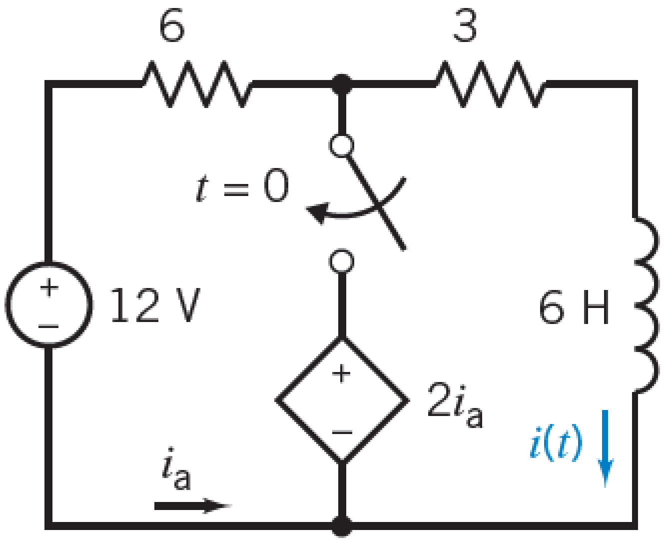
The capacitor voltage *decreases* from $v(0) = 12$ V rather than *increasing* toward $v_f = 24$ V. Notice that

$$v(\infty) = \lim_{t \to \infty} v(t) = -\infty$$

It's not appropriate to refer to the forced response as a steady-state response when the circuit is unstable.

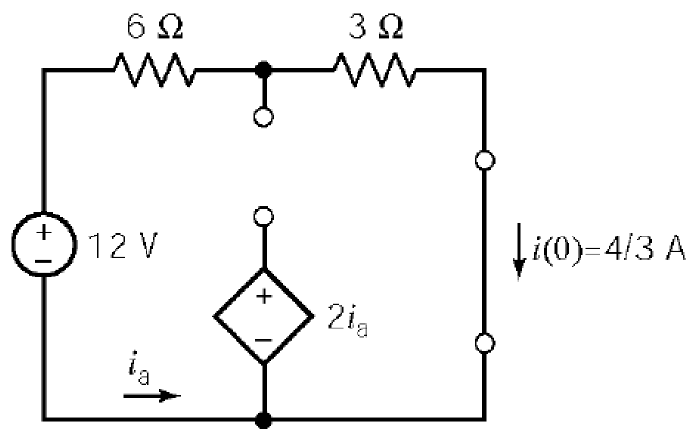
Problem 8.3-4

P 8.3-4 The circuit shown in Figure P 8.3-4 is at steady state before the switch closes at time $t = 0$. Determine the inductor current $i(t)$ for $t > 0$.



Problem 8.3-4 Solution

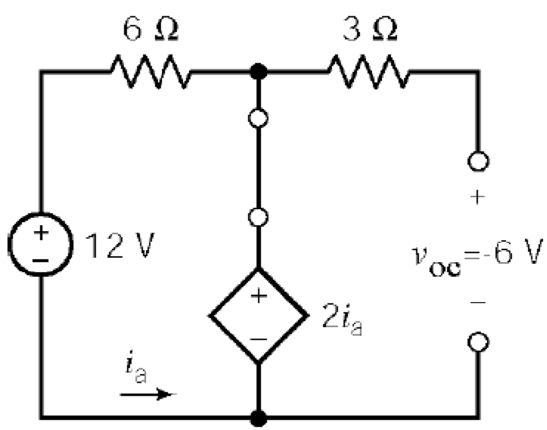
Solution: Before the switch closes:

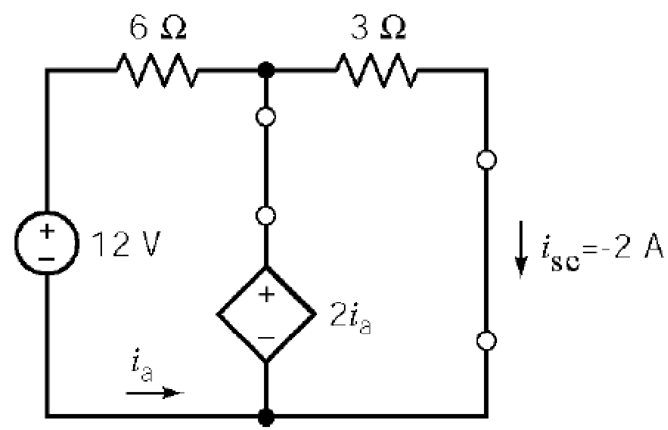


Ohm's law:

$i(0) = 12V/9\Omega$

After the switch closes:





Problem 8.3-4 Solution

$i_a = -i_1$

$i_{sc} = i_2$

* (3) $\begin{cases} v_{oc} = 2 i_a \\ -12 + 6 i_a + 2 i_a = 0 \Rightarrow i_a = \dfrac{12}{-4} = -3\,A \end{cases}$

$\boxed{v_{oc} = 2(-3) = -6\,V}$

* (2) $\begin{cases} -12 - 6 i_a + 2 i_a = 0 \Rightarrow i_a = -3\,A \\ -2 i_a + 3 i_{sc} = 0 \Rightarrow \boxed{i_{sc} = \dfrac{2(-3)}{3} = -2\,A} \end{cases}$

Therefore $R_t = \dfrac{\overset{v_{oc}}{-6}}{\underset{i_{sc}}{-2}} = 3\,\Omega$ so $\tau = \dfrac{6H}{3\Omega} = 2$ s. $(\tau = L/R_t)$

Finally, $i(t) = i_{sc} + (i(0) - i_{sc})\, e^{-\frac{t}{\tau}} = -2 + \dfrac{10}{3} e^{-0.5t}$ A for $t > 0$

$$i(t) = \underset{-2A}{I} + (\underset{4/3\,A}{i(0)} - \underset{-2A}{I})\, e^{-t/\tau} = -2 + \frac{10}{3} e^{-0.5t}$$

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Therefore $R_t = \dfrac{-6}{-2} = 3\,\Omega$ so $\tau = \dfrac{6}{3} = 2$ s.

Finally, $i(t) = i_{sc} + (i(0) - i_{sc})\, e^{-\frac{t}{\tau}} = -2 + \dfrac{10}{3} e^{-0.5t}$ A for $t > 0$