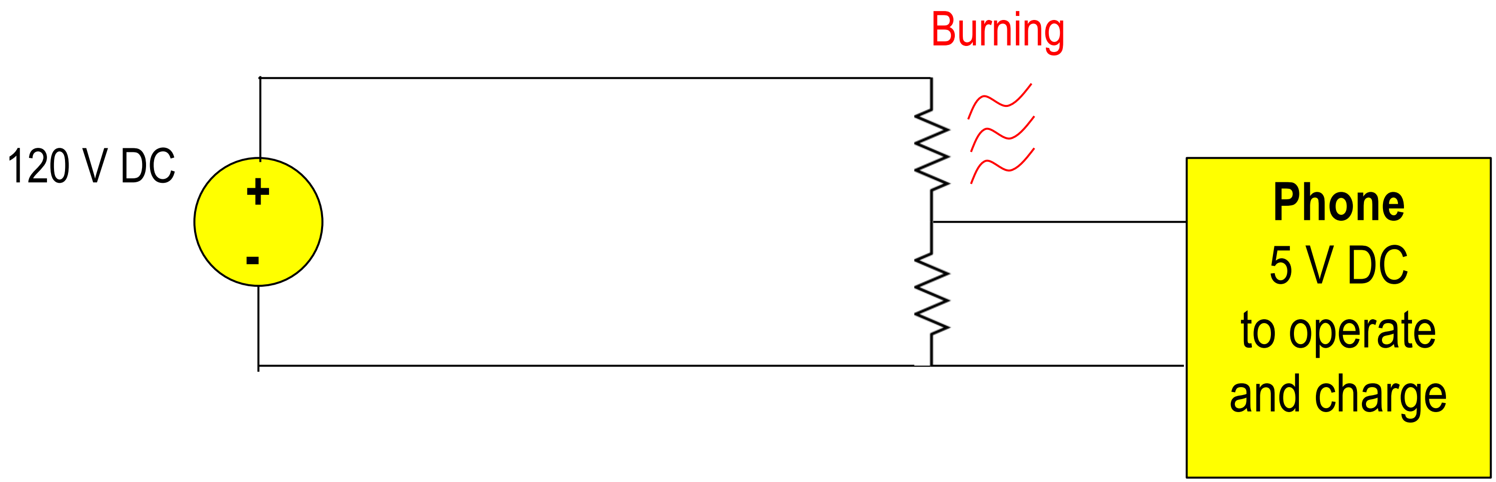
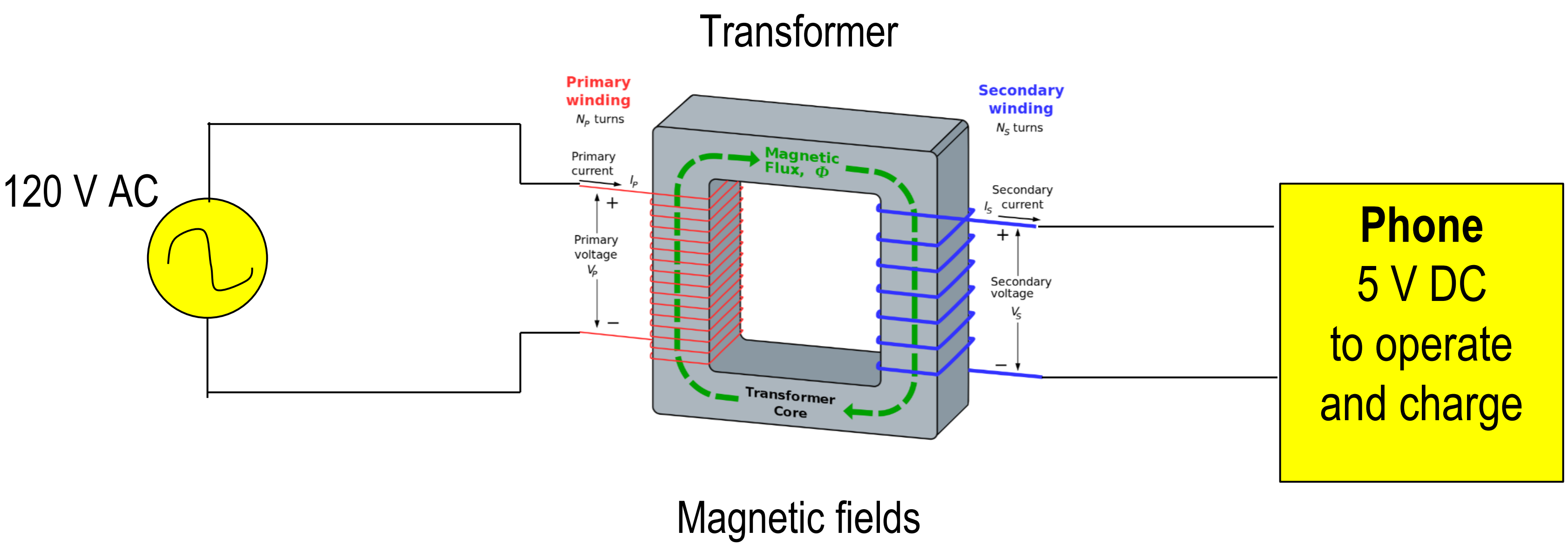
Chapter 10

Sinusoidal Steady-State Analysis

Direct Current vs. Alternating Current (Loss)





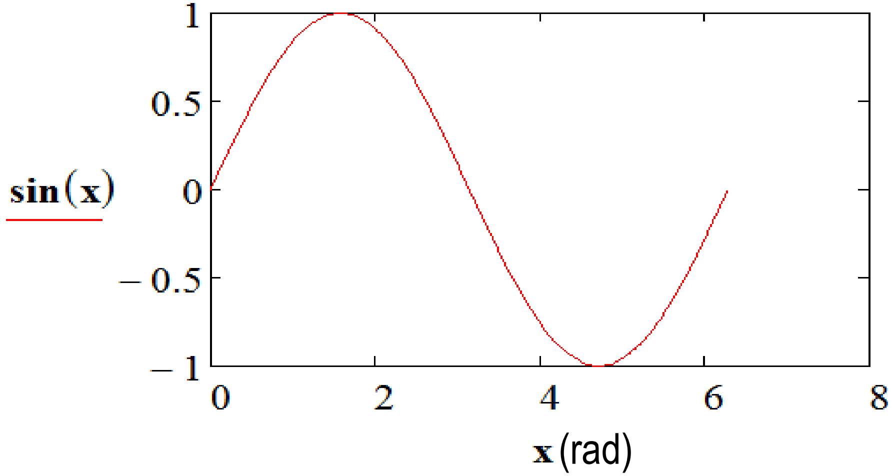
Sinusoidal Sources

sin (x)

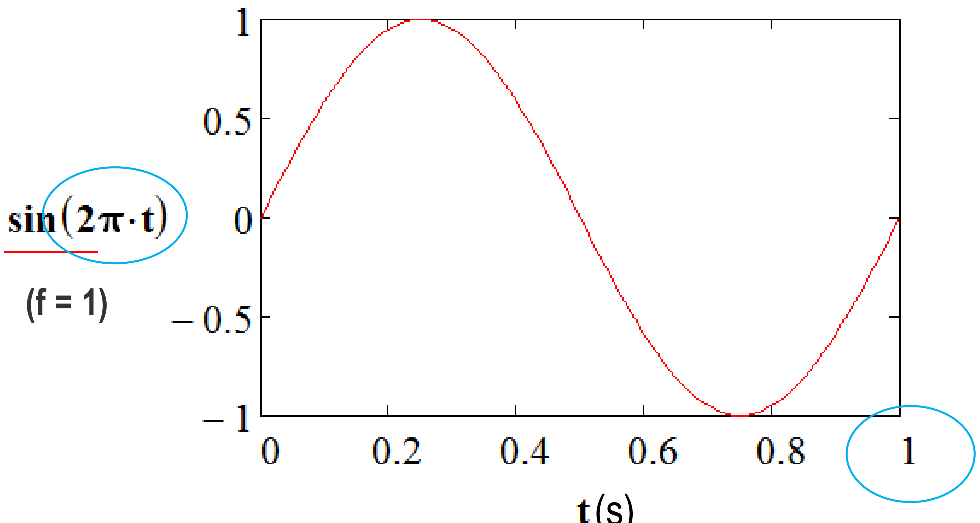
↓

$$\boxed{v_s = V_m \sin(\omega t + \phi)}$$

$$where\ \omega = 2\pi f$$



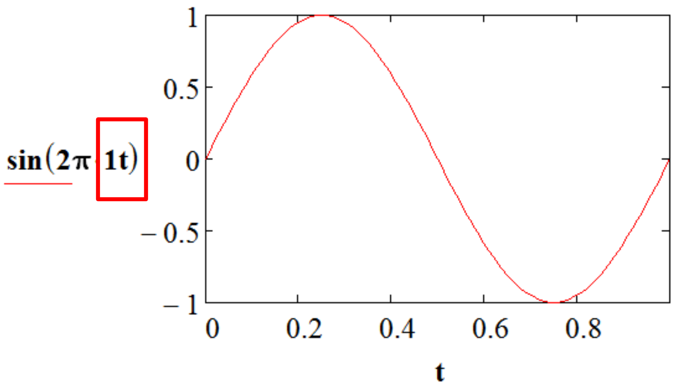
Convert sin(x) vs. x (rad) to sin(t) vs. time (s)
Plot 1 sinewave per second

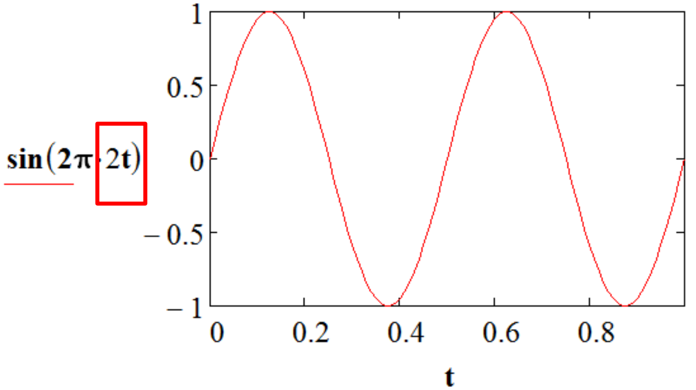


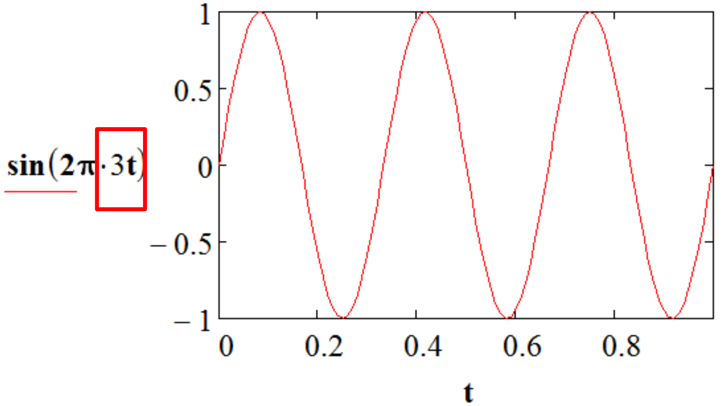
Sinusoidal Sources – Frequency

$$\sin(2\cdot\pi\cdot f\cdot t)$$

Frequency: Number of cycles per second
Unit: Hz



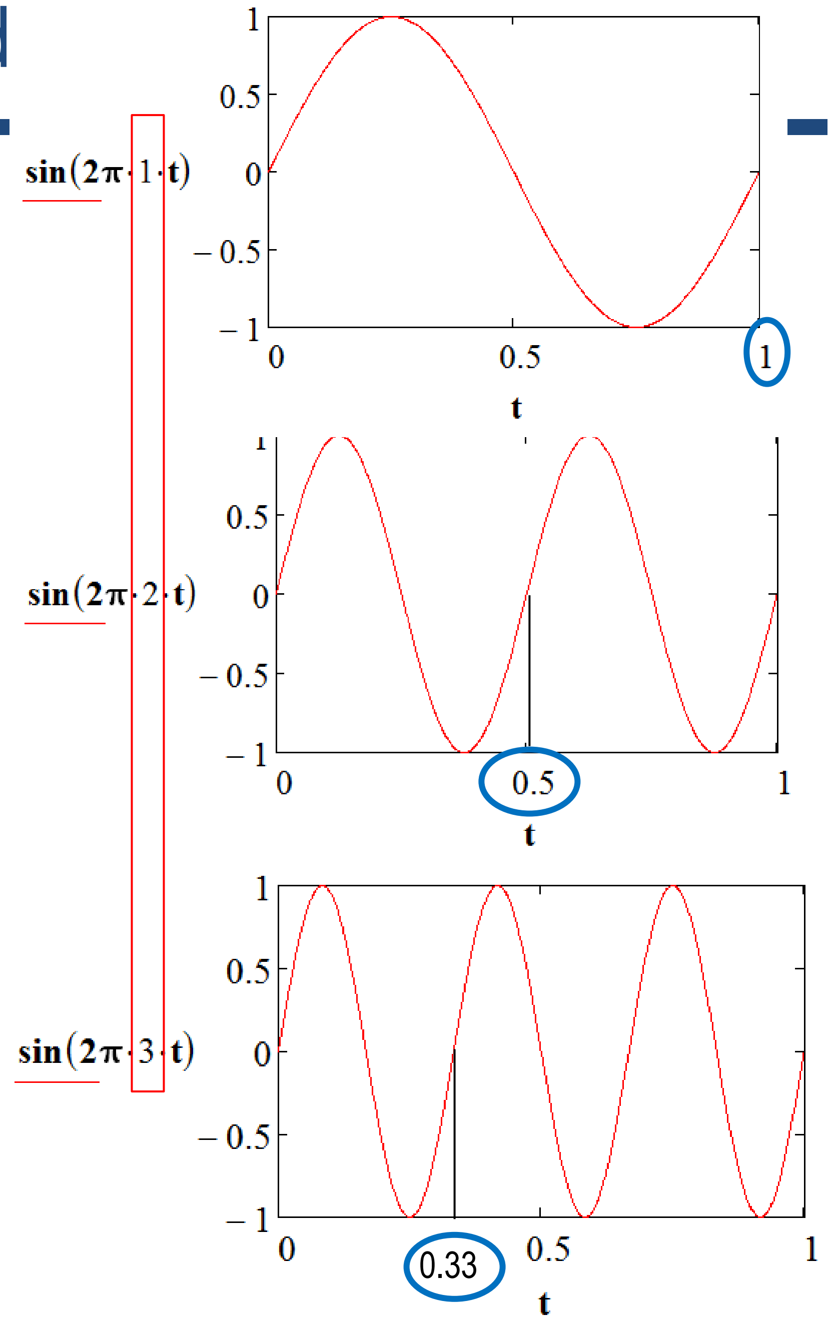




Sinusoidal Sources – Period

$$\sin\left(2 \cdot \pi \cdot \boxed{1/T} \cdot t\right)$$

Period: Duration of one cycle in a repeating event
T = 1/f
Unit: s (time)



Sinusoidal Sources – Summary

- A sinusoid is a periodic function defined by the property:

$$x(t+T) = x(t)$$

T = period

f = 1/T = frequency (number of cycles per second) [Hz]

ω = angular frequency (radian frequency) = $2 \times \pi \times f$ [rad/s]

$$v_s = V_m \sin(\omega t)$$

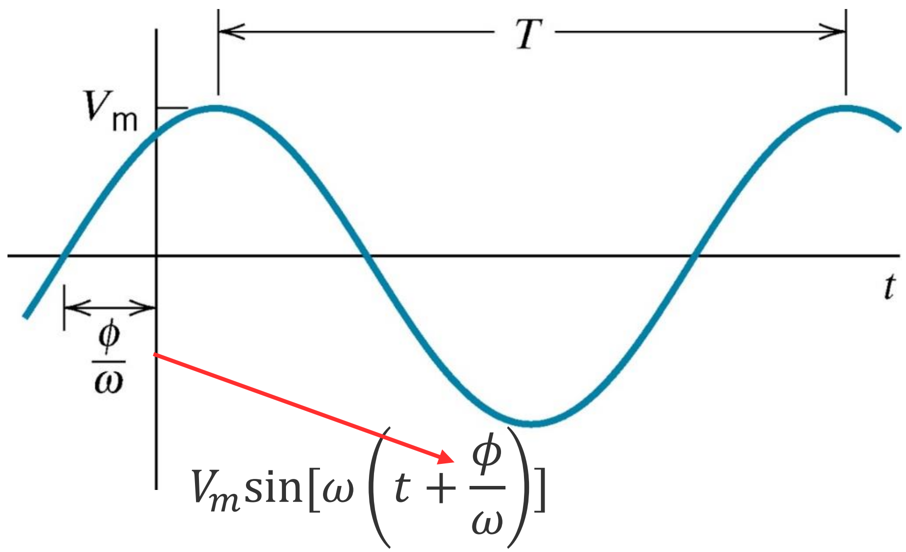
$$i_s = I_m \sin(\omega t)$$

Sinusoidal Sources

$$v_s = V_m \sin(\omega t + \phi)$$

Maximum magnitude

Phase angle (phase shift) in rad or degrees



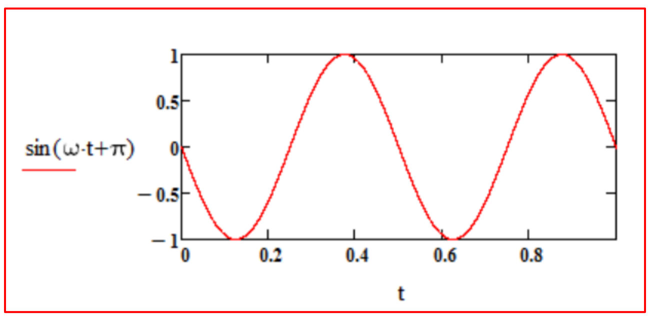
$$\omega = 2 \cdot \pi \cdot f$$
$$T = 1/f$$

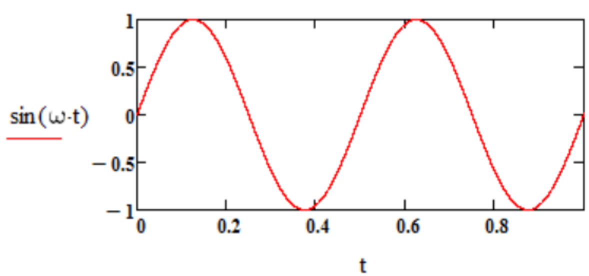
Review – Important Formulas

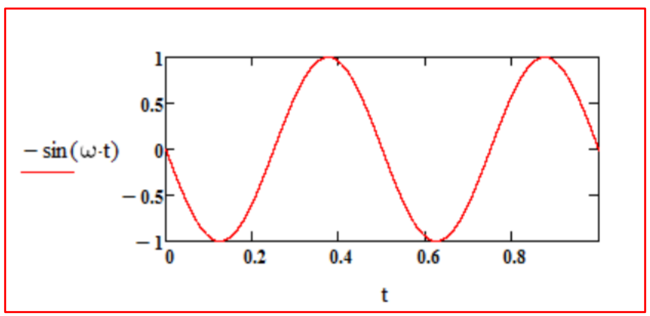
$$\sin(\omega t) = \cos(\omega t - 90°) = \cos\left(\omega t - \frac{\pi}{2}\right)$$

$$\cos(\omega t) = \sin(\omega t + 90°) = \sin\left(\frac{\pi}{2} + \omega t\right)$$

$$\sin(\omega t + 180°) = -\sin(\omega t)$$





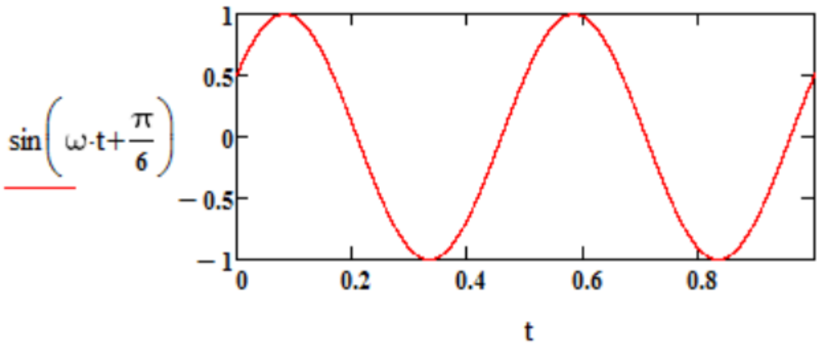


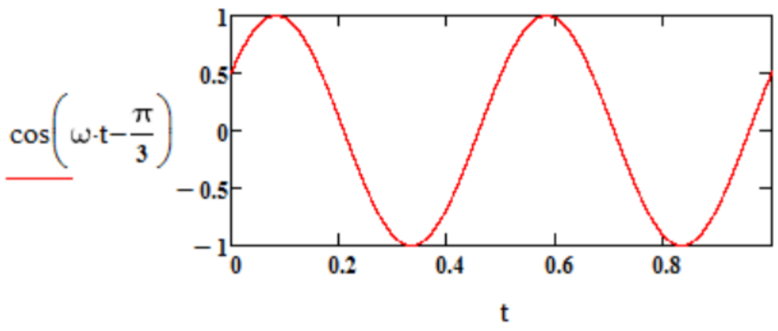
Example 1

$$v_s = V_m \sin(\omega t + 30^\circ) = V_m \cos(?)$$

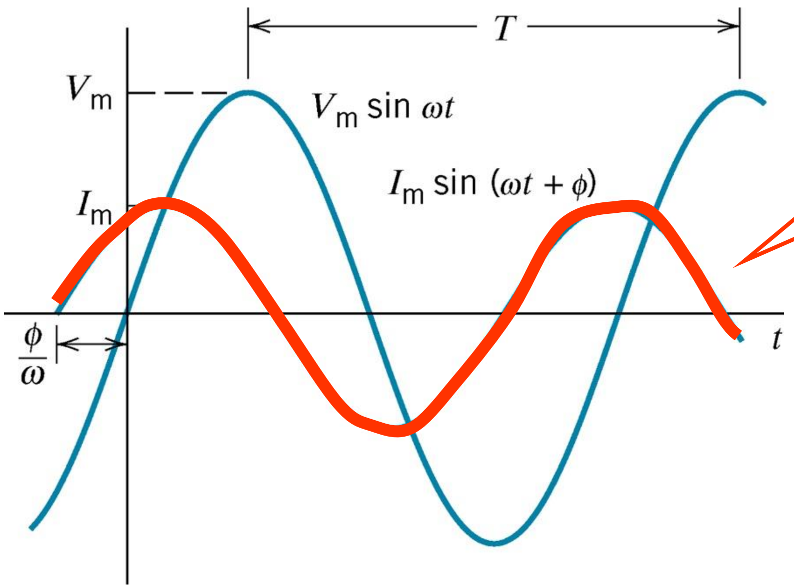
$$= V_m \cos(\omega t + 30^\circ - 90^\circ)$$

$$= V_m \cos(\omega t - 60^\circ)$$





Sinusoidal Sources



The current reaches its peak before the voltage: Current ***leads*** the voltage OR Voltage ***lags*** the current

Example 2

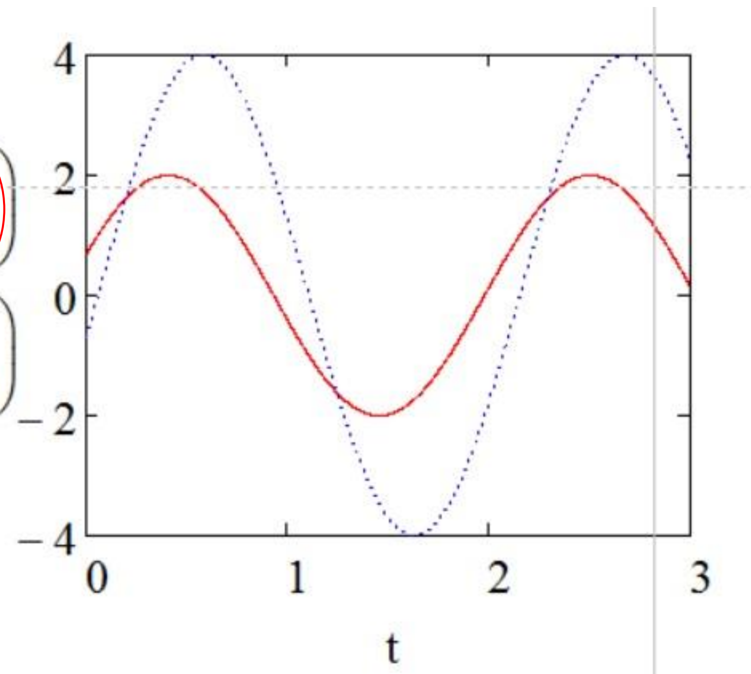
$$v = 2\sin(3t + 20^\circ)$$
$$i = 4\sin(3t - 10^\circ)$$

Voltage, *v*, leads (or advances) the current, *i*, by 20-(-10) = +30°

Degree to Radian

$$v = 2\sin\left(3 \cdot t + 20 \cdot \frac{\pi}{180}\right)$$

$$i = 4\sin\left(3 \cdot t - 10 \cdot \frac{\pi}{180}\right)$$



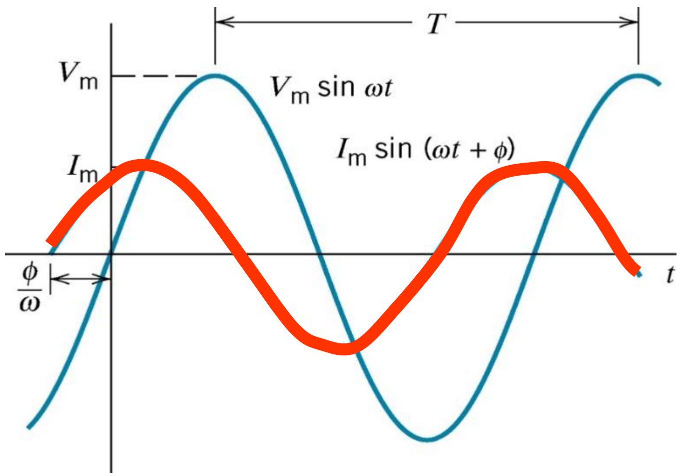
Example 10.2-1

- Consider the voltages $v_1 = 10\cos(200t+45^o)$ V and $v_2 = 8\sin(200t+15°)$ A. Determine the time by which $v_2(t)$ is advanced or delayed with respect to $v_1(t)$.

$$\boxed{\sin(\omega t) = \cos(\omega t - 90°) = \cos\left(\omega t - \frac{\pi}{2}\right)}$$

$$\cos(\omega t) = \sin(90° - \omega t) = \sin\left(\frac{\pi}{2} - \omega t\right)$$

$$\sin(\omega t + 180°) = -\sin(\omega t)$$



Current ***leads*** the voltage OR Voltage ***lags*** the current

Example 10.2-1 Solution

$$v_2 = \sin(200t{+}15^o)$$

$$v_2 = 8\cos(200t + 15^o - 90^o) = 8\cos(200t - 75^o)\,V$$

$$\theta_2 - \theta_1 = -75^o - 45^o = -120^o = -\frac{\pi}{3} rad$$

$$\frac{\phi}{\omega} = -\frac{\frac{\pi}{3}}{200} = -5.2\,ms$$

$$\theta_2 - \theta_1 < 0$$

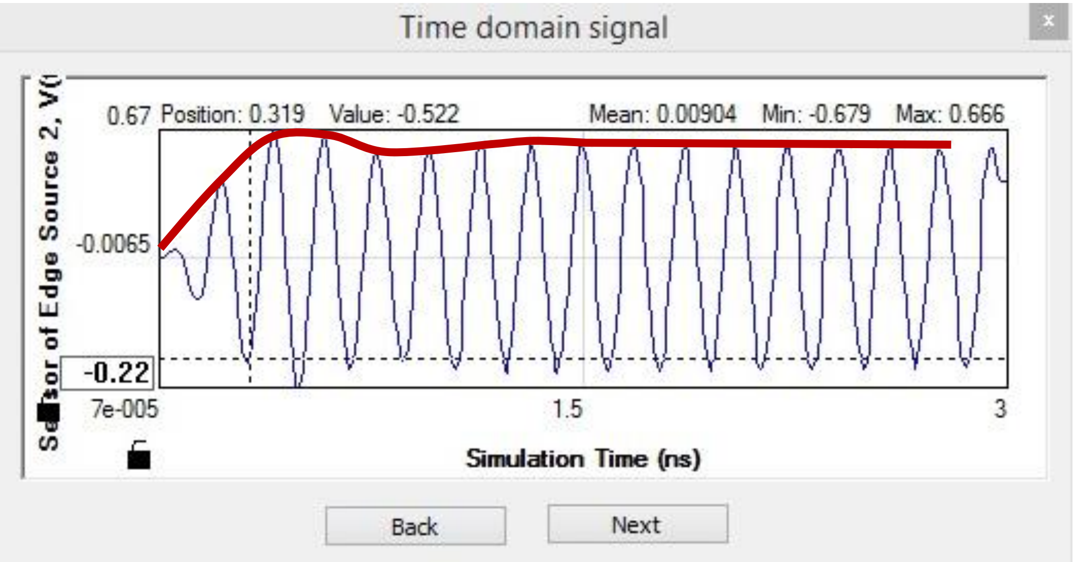
This indicates a delay (lag)

PHASORS

Phasors

- Phasors may be used when:
  - the circuit is ***linear***
  - the ***steady-state response*** is sought and
  - all independent sources are **sinusoidal** and have the **same frequency**

Example of a Steady State Response



Phasor

- Challenge in circuit analysis → sin or cos functions exist
  - Most of the realistic circuits include capacitors and/or inductors
  - The analytical equations will include:

$$C\frac{dv}{dt}\ \&\ L\frac{di}{dt}$$

Can we make the analysis simpler?

Exponential function

Widely used in physics, chemistry, engineering, mathematical biology, economics, mathematics, etc.

$\ln e^x = x \quad \text{and} \quad e^{\ln x} = x.$

$\ln e = 1.$

$\frac{d}{dx} e^x = e^x$

$\int e^x \, dx = e^x + C$

Euler's Identity

$$e^{jx} = cos(x) + j\,sin(x)$$

Phasor

$$e^{jx} = cos(x) + j\,sin(x)$$

$$v_s(t) = V_m \cos(\omega t) = \mathrm{Re}\{V_m e^{j\omega t}\}$$

$$v_s(t) = V_m \cos(\omega t + \phi) = \mathrm{Re}\{V_m e^{j(\omega t + \phi)}\} = \mathrm{Re}\{V_m e^{j\omega t} e^{j\phi}\}$$

Phasor

$$i(t) = I_m \cos(\omega t) = \cancel{\mathrm{Re}}\left\{I_m \cancel{e^{j\omega t}}\right\}$$

$$i(t) = I_m \cos(\omega t + \phi) = \cancel{\mathrm{Re}}\left\{I_m e^{j(\cancel{\omega t}+\phi)}\right\} = \cancel{\mathrm{Re}}\left\{I_m \cancel{e^{j\omega t}} e^{j\phi}\right\}$$

1. We can drop the Re{}, since we know that it is the real part
2. We can also drop the part with radian frequency, because we know that we are solving the circuit for *one specific frequency*
3. Then ***i*(*t*)** information can be represented by the **phasor** of *i*(*t*), $\mathbf{I}$

$$\boxed{\mathbf{I} = I_m e^{j\phi} = I_m \angle \phi}$$

Phasor Notation

<u>Time domain</u>

$$i(t) = I_m \cos(\omega t + \phi)$$

$$= \mathrm{Re}\left\{ I_m e^{j(\omega t + \phi)} \right\}$$

$$= \mathrm{Re}\left\{ I_m e^{j\omega t} e^{j\phi} \right\}$$

<u>Frequency domain</u>

$$I = I_m e^{j\phi} = I_m \angle \phi$$

Phasor Notation

- Phasor notation:

$$i(t) = I_m \cos(\omega t + \phi)$$

$$\mathbf{I} = I_m e^{j\phi} = I_m \angle \phi$$

- **Cosine** function is usually chosen as the **standard** for phasor notation
- Phasor quantities are complex (complex exponential function)
- Although we dropped the $e^{j\omega t}$, note that we are performing the calculations in the *frequency domain*, instead of in the *time domain*

Phasor Notation

- Phasor notation:

$$\mathbf{I} = I_m e^{j\phi} = I_m \angle \phi$$

- This way, we avoid the complete solution of the circuits (with energy storage elements) in time domain that requires differential equation solutions

$$\boxed{C\frac{dv}{dt} \; \& \; L\frac{di}{dt}}$$

- Remember that ***frequency domain analysis*** only provides the ***steady-state solution***
- If you need to know the **transient behavior** of the circuit, you **CANNOT** use frequency domain solution (phasors)

Phasor Summary

Table 10.5-1 Transformation from the Time Domain to the Frequency Domain

1. Write the function in the time domain, $y(t)$, as a cosine waveform with a phase angle ϕ as
$$y(t) = Y_{\mathrm{m}} \cos(\omega t + \phi)$$
2. Express the cosine waveform as the real part of a complex quantity by using Euler's identity so that
$$y(t) = \mathrm{Re}\{Y_{\mathrm{m}} e^{j(\omega t + \theta)}\}$$
3. Drop the real part notation.
4. Suppress the $e^{j\omega t}$ while noting the value of ω for later use, obtaining the phasor
$$\mathbf{Y} = Y_{\mathrm{m}} e^{j\phi} = Y_{\mathrm{m}} \underline{/\phi}$$

Table 10.5-2 Transformation from the Frequency Domain to the Time Domain

1. Write the phasor in exponential form as
$$\mathbf{Y} = Y_{\mathrm{m}} e^{j\beta}$$
2. Reinsert the factor $e^{j\omega t}$ so that you have
$$Y_{\mathrm{m}} e^{j\beta} e^{j\omega t}$$
3. Reinsert the real part operator Re as
$$\mathrm{Re}\{Y_{\mathrm{m}} e^{j\beta} e^{j\omega t}\}$$
4. Use Euler's identity to obtain the time function
$$y(t) = \mathrm{Re}\{Y_{\mathrm{m}} e^{j(\omega t + \beta)}\} = Y_{\mathrm{m}} \cos(\omega t + \beta)$$

Example 3

- Transform the current expression in time domain $i(t) = 5\sin(100t+120°)$ to the frequency domain.

$$\sin(\omega t) = \cos(\omega t - 90°) = \cos\left(\omega t - \frac{\pi}{2}\right)$$

$$\cos(\omega t) = \sin(90° - \omega t) = \sin\left(\frac{\pi}{2} - \omega t\right)$$

$$\sin(\omega t + 180°) = -\sin(\omega t)$$

Example 4

- Transform the voltage expression in frequency domain **V = 24 ∠125°** to time domain.

$$\sin(\omega t) = \cos(\omega t - 90^\circ) = \cos\left(\omega t - \frac{\pi}{2}\right)$$

$$\cos(\omega t) = \sin(90^\circ - \omega t) = \sin\left(\frac{\pi}{2} - \omega t\right)$$

$$\sin(\omega t + 180^\circ) = -\sin(\omega t)$$

Review: Complex Numbers

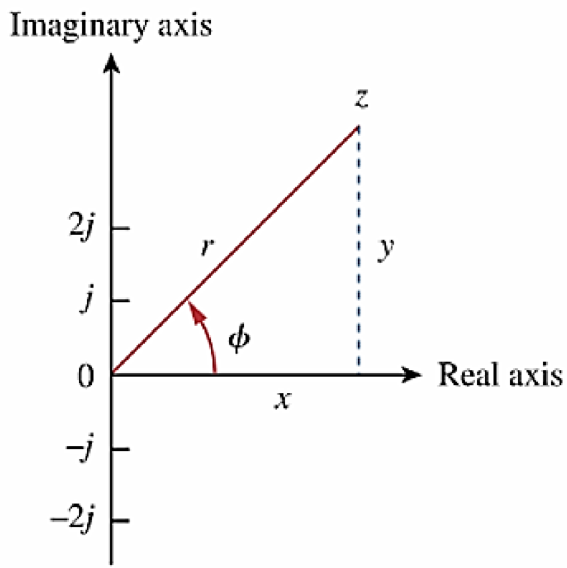


Figure 9.6
Representation of a complex number $z = x + jy = r\angle\phi$.

- $z = x + jy$ (rectangular form) $+$ and $-$
- $z = r\angle\phi$ (polar form) $\times$ and $\div$
- $z = re^{j\phi}$ (exponential form) $\times$ and $\div$

Useful for the calculator

Rectangular to polar

Inverse tan.

$r = \sqrt{x^2 + y^2}$ $\phi = \tan^{-1}\frac{y}{x}$ and

$\phi = 180^o - \tan^{-1}\frac{y}{x}$, when x<0

$$z = \sqrt{x^2 + y^2}\angle\tan^{-1}\frac{y}{x}$$

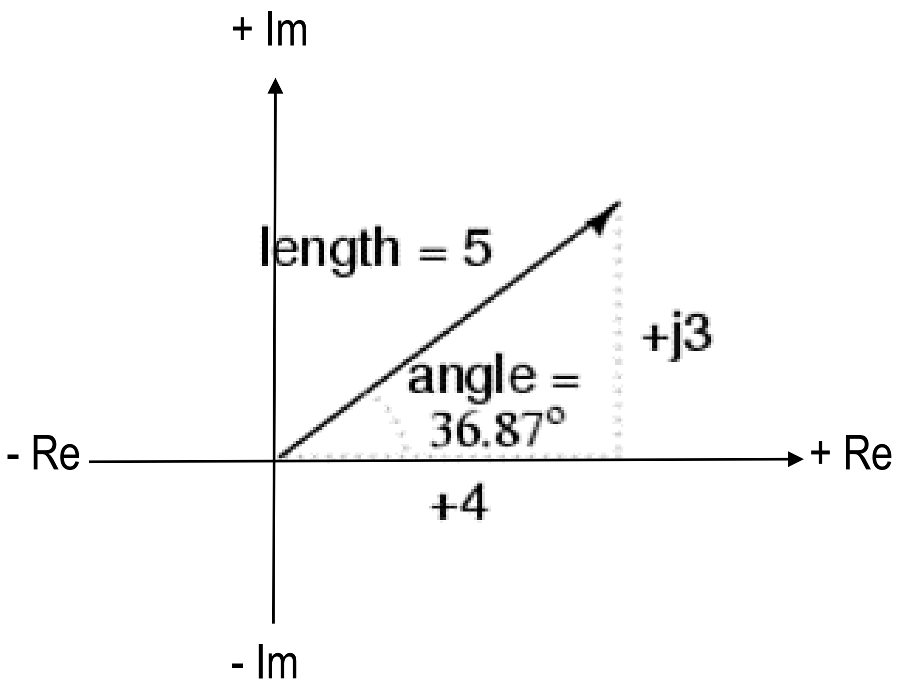
Polar to rectangular

$x = rcos\phi$

$y = rsin\phi$

$z = r(cos\phi + jsin\phi)$

Review: Complex Numbers – Example



Convert from rectangular to polar and vice versa

$$\boxed{5\angle 36.87^o \longleftrightarrow 4 + j3}$$

Polar to rectangular

$5 \angle 36.87^\circ$	**(polar form)**
$(5)(\cos 36.87^\circ) = 4$	(real component)
$(5)(\sin 36.87^\circ) = 3$	(imaginary component)
$4 + j3$	**(rectangular form)**

Rectangular to polar

$4 + j3$	**(rectangular form)**
$c = \sqrt{a^2 + b^2}$	(pythagorean theorem)

polar magnitude = $\sqrt{4^2 + 3^2}$

polar magnitude = 5

polar angle = $\arctan \dfrac{3}{4}$

polar angle = 36.87°

$5 \angle 36.87^\circ$ **(polar form)**

Review: Complex Numbers

Given:

$$Z_1 = x_1 + jy_1 = r_1 \angle \phi_1$$

$$Z_2 = x_2 + jy_2 = r_2 \angle \phi_2$$

Addition and Subtraction of Complex Number (<u>rectangular</u>)

$$Z_1 + Z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

$$Z_1 - Z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

Multiplication/Division of Complex Number (<u>polar</u>)

$$Z_1 \cdot Z_2 = r_1 \cdot r_2 \angle (\phi_1 + \phi_2)$$

$$r_1 e^{j\phi_1} \cdot r_2 e^{j\phi_2}$$

$$= r_1 \cdot r_2 e^{j(\phi_1 + \phi_2)}$$

$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2} \angle (\phi_1 - \phi_2)$$

Example 10.3-2

Consider the phasors

$$\mathbf{V}_1 = 4.25\angle 115^\circ \text{ and } \mathbf{V}_2 = -4 + j3$$

Convert $\mathbf{V}_1$ to rectangular form and $\mathbf{V}_2$ to polar form.

Polar to rectangular

$$x = r\cos\phi$$

$$y = r\sin\phi$$

$$z = r(\cos\phi + j\sin\phi)$$

Rectangular to polar

$$r = \sqrt{x^2 + y^2} \quad \phi = \tan^{-1}\frac{y}{x}$$

$$z = \sqrt{x^2 + y^2}\angle\tan^{-1}\frac{y}{x}$$

Example 10.3-3

Consider the phasors

$$\mathbf{V}_1 = -1.796 + j3.852 = 4.25\,\underline{/115^\circ} \text{ and } \mathbf{V}_2 = -4 + j3 = 5\,\underline{/143^\circ}$$

Determine $\mathbf{V}_1 + \mathbf{V}_2$, $\mathbf{V}_1 \cdot \mathbf{V}_2$ and $\frac{\mathbf{V}_1}{\mathbf{V}_2}$.

Addition and Subtraction of Complex Number (rectangular)

$$Z_1 + Z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

$$Z_1 - Z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

Multiplication/Division of Complex Number (polar)

$$Z_1 \cdot Z_2 = r_1 \cdot r_2 \angle(\phi_1 + \phi_2)$$

$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2} \angle(\phi_1 - \phi_2)$$

Example 10.3-4 – Kirchhoff's Law for AC Circuits

The input to the circuit shown in Figure 10.3-3 is the voltage source voltage,

$$v_s(t) = 25\cos(100t + 15^\circ)\ \text{V}$$

The output is the voltage across the capacitor,

$$v_C(t) = 20\cos(100t - 22^\circ)\ \text{V}$$

Determine the resistor voltage $v_R(t)$.

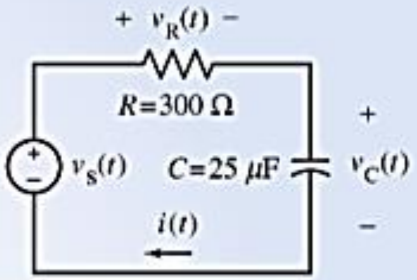


FIGURE 10.3-3 The circuit in Example 10.3-4

Addition and Subtraction of Complex Number (rectangular)

$$Z_1 + Z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

$$Z_1 - Z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

Multiplication/Division of Complex Number (polar)

$$Z_1 \cdot Z_2 = r_1 \cdot r_2 \angle(\phi_1 + \phi_2)$$

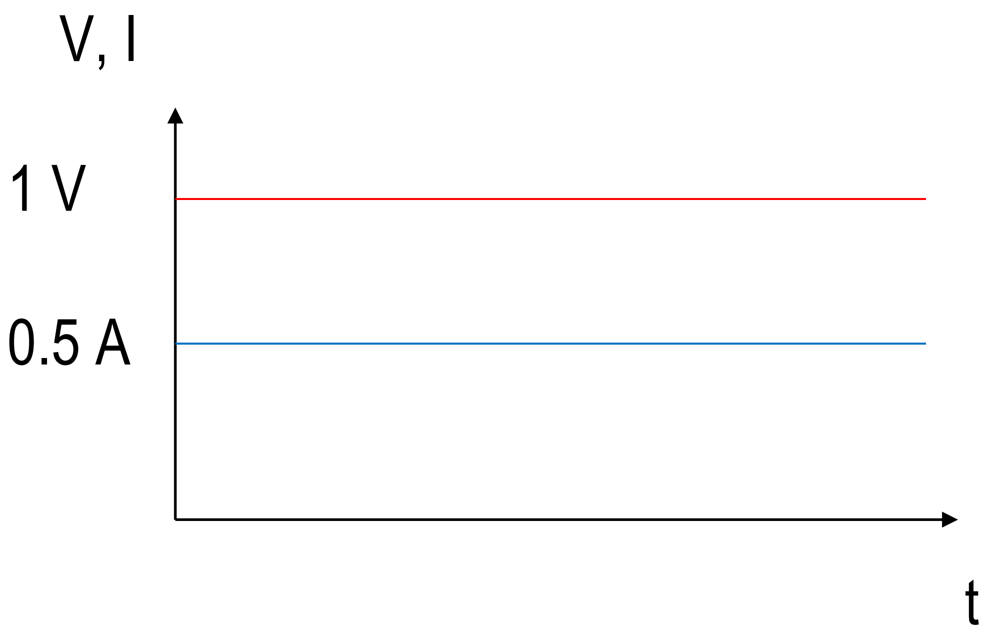
$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2}\angle(\phi_1 - \phi_2)$$

IMPEDANCE

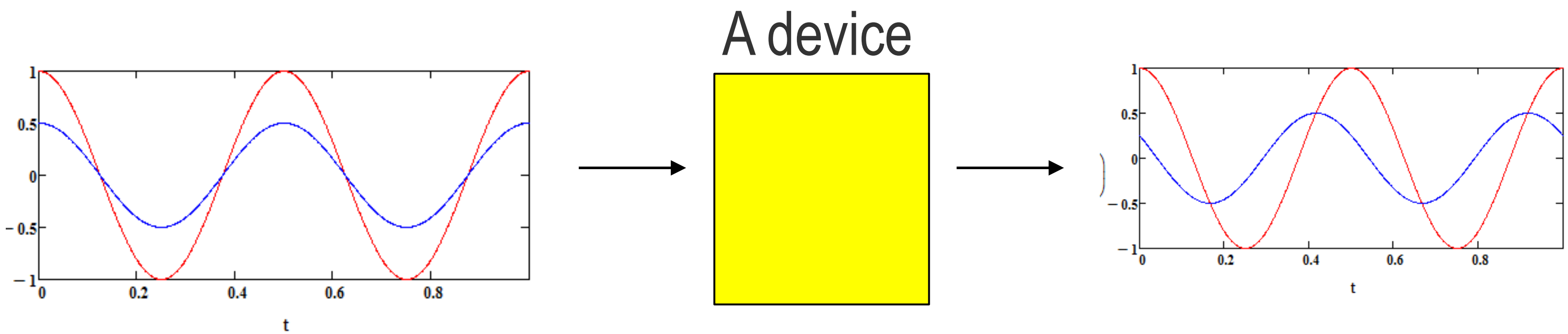
In DC Analysis

if $\begin{cases} V = 1\text{ V} \\ I = 0.5\ A \end{cases}$

$$R = \frac{V}{I} = \frac{1}{0.5} = 2\ \Omega$$



Concept of Impedance in AC Circuits



Does "R" describe this situation?

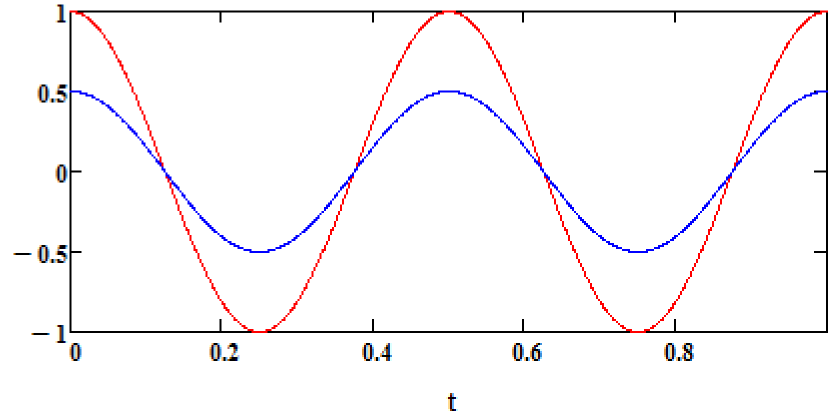
Magnitude can be calculated by R

How about the phase difference?

The phase information is missing!

Impedance

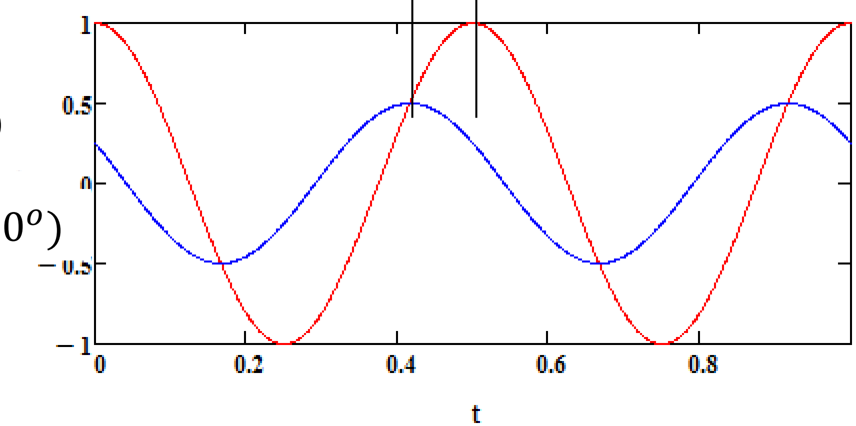
Freq.	Time
$V = 1\angle 0^o$	$v(t) = 1 \cdot \cos(\omega t)$
$I = 0.5\angle 0^o$	$i(t) = 0.5 \cdot \cos(\omega t)$



$$R = \frac{V}{I} = \frac{1\angle 0^o}{0.5\angle 0^o} = 2\angle 0^o \Omega$$ (2 Ω resistor)

Phase

$V = 1\angle 0^o$	$v(t) = 1 \cdot \cos(\omega t)$
$I = 0.5\angle 60^o$	$i(t) = 0.5 \cdot \cos(\omega t + 60^o)$

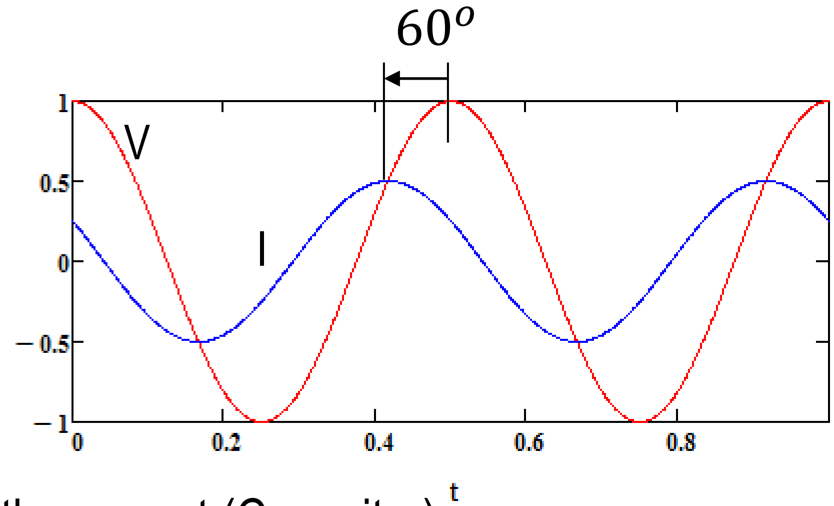


$$Z\ \cancel{R} = \frac{V}{I} = \frac{1\angle 0^o}{0.5\angle 60^o} = 2\angle(-60^o)\ \Omega$$

I leads V

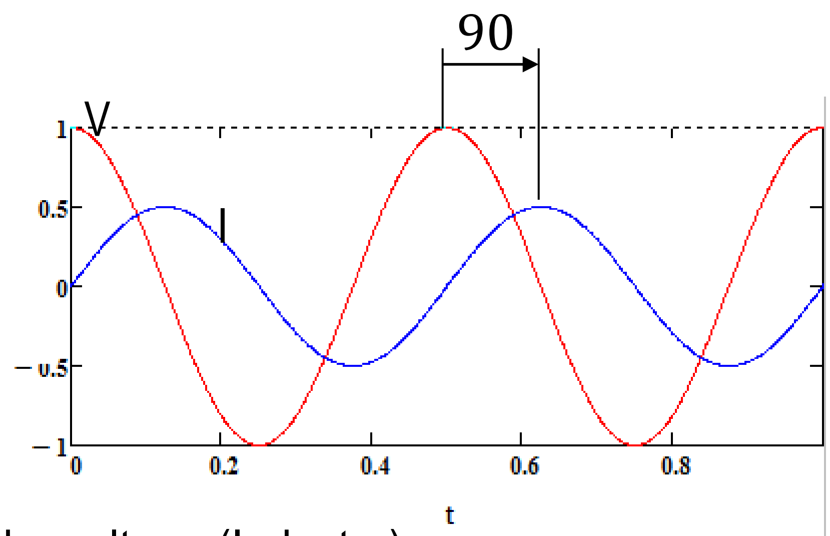
Example of Impedance

$$Z = \frac{V}{I} = 2\angle(-60^o)\ \Omega$$



Current *leads* the voltage OR Voltage *lags* the current (Capacitor)

$$Z = \frac{V}{I} = 2\angle(90^o)\ \Omega$$



Voltage *leads* the current OR Current *lags* the voltage (Inductor)

Impedance

- The ratio of the phasor voltage to the phasor current is defined as ***impedance*** and denoted by **Z**
- Impedance in AC circuits has a similar role to the role of resistance in DC circuits

$$\mathrm{V} = V_m e^{j\phi}, \mathrm{I} = I_m e^{j\beta}$$

$$\Rightarrow \mathrm{Z} = \frac{\mathrm{V}}{\mathrm{I}} = \frac{V_m e^{j\phi}}{I_m e^{j\beta}} = \frac{V_m}{I_m} e^{j(\phi-\beta)} = \frac{V_m}{I_m} \angle(\phi-\beta)$$

Magnitude, |**Z**|

Phase angle

Impedance

$$Z = |Z|\angle\theta \quad \rightarrow \text{polar form}$$

$$= |Z|e^{j\theta} \quad \rightarrow \text{exponential form}$$

$$= R + jX \quad \rightarrow \text{rectangular form}$$

$$|Z| = \sqrt{R^2 + X^2}, \quad \theta = \tan^{-1}\left(\frac{X}{R}\right)$$

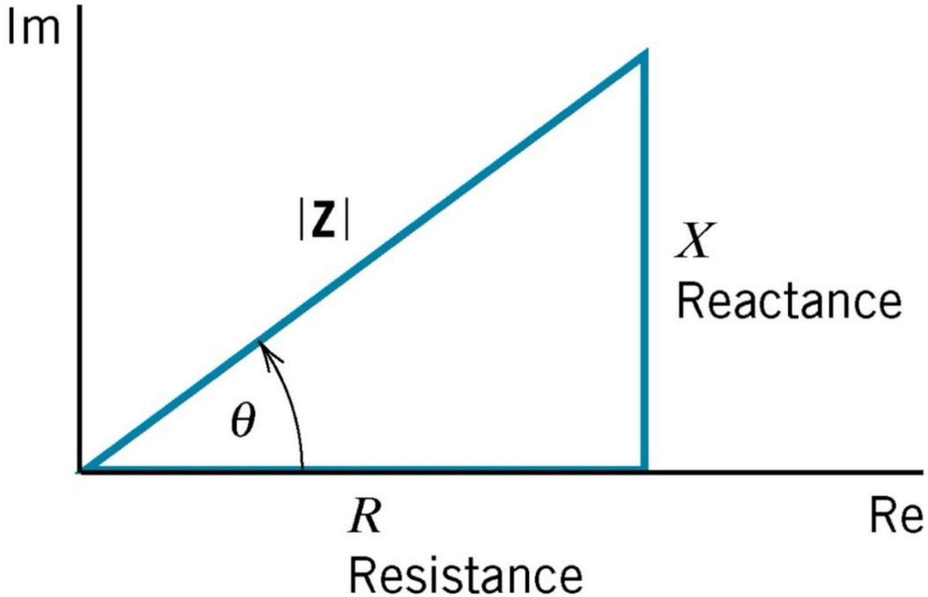
Real part:
Resistance [Ω]

Imaginary part: **Reactance** [Ω]
+ *jX* (Positive) → Inductor
– *jX* (Negative) → Capacitor

Impedance

$$Z = |Z|\angle\theta = |Z|e^{j\theta} = R + jX$$

$$|Z| = \sqrt{R^2 + X^2},\ \theta = \tan^{-1}\left(\frac{X}{R}\right)$$



Admittance

$$\mathrm{Y} = \frac{1}{\mathrm{Z}} = \frac{1}{|\mathrm{Z}|\angle\theta} = \frac{1}{|\mathrm{Z}|}\angle -\theta$$

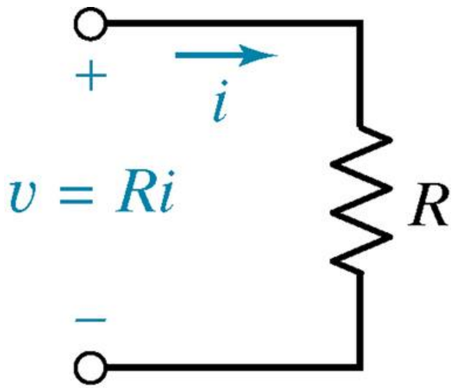
$$\mathrm{Y} = \frac{1}{R + jX} = \frac{R - jX}{R^2 + X^2} = \frac{R}{R^2 + X^2} - j\frac{X}{R^2 + X^2}$$

$$= G + jB$$

Real part: **Conductance**
[Siemens]

Imaginary part: **Susceptance**
[Siemens]
+ *jB* (Positive) → Capacitor
– *jB* (Negative) → Inductor

Phasor of R



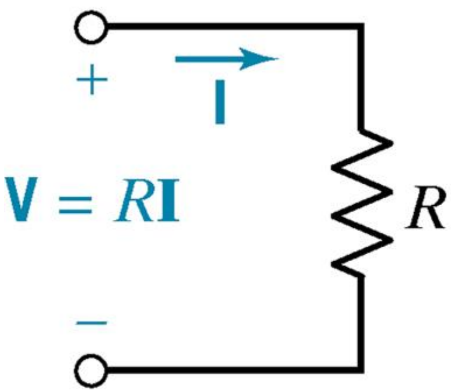
(a)

$$v(t) = Ri(t)$$

<u>Ohm's law</u>

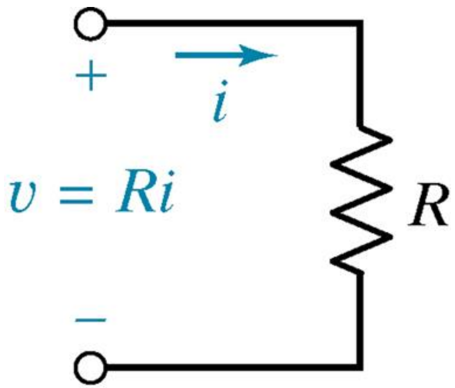
$$v = V_m \cos(\omega t + \phi) = \mathrm{Re}\left\{V_m e^{j(\omega t + \phi)}\right\}$$

$$i = I_m \cos(\omega t + \beta) = \mathrm{Re}\left\{I_m e^{j(\omega t + \beta)}\right\}$$

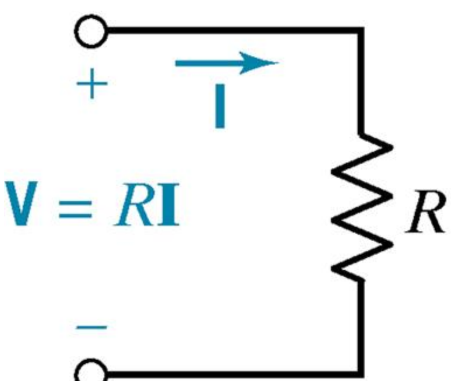


(b)

Phasor of *R*



(a)



(b)

$$v(t) = Ri(t)$$ Ohm's law

$$v = V_m \cos(\omega t + \phi) = \mathrm{Re}\left\{V_m e^{j(\omega t + \phi)}\right\}$$

$$i = I_m \cos(\omega t + \beta) = \mathrm{Re}\left\{I_m e^{j(\omega t + \beta)}\right\}$$

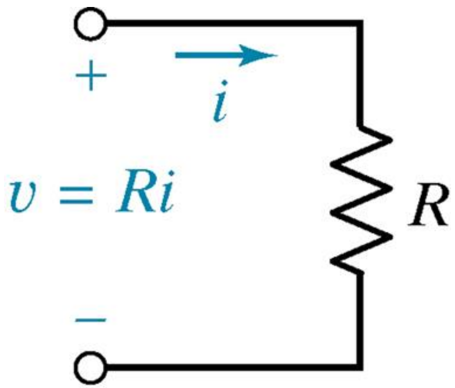
$$V_m e^{j(\omega t + \phi)} = R \times I_m e^{j(\omega t + \beta)}$$

$$V_m e^{j\omega t} e^{j\phi} = R \times I_m e^{j\omega t} e^{j\beta}$$

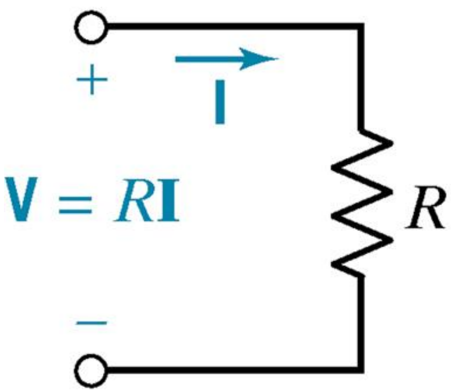
$$V_m e^{j\phi} = R \times I_m e^{j\beta} \Rightarrow \phi = \beta$$

$$\boxed{\mathrm{V} = R \times \mathrm{I}}$$

Phasor of R



(a)



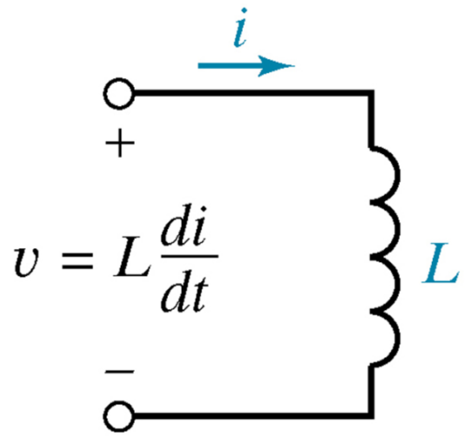
(b)

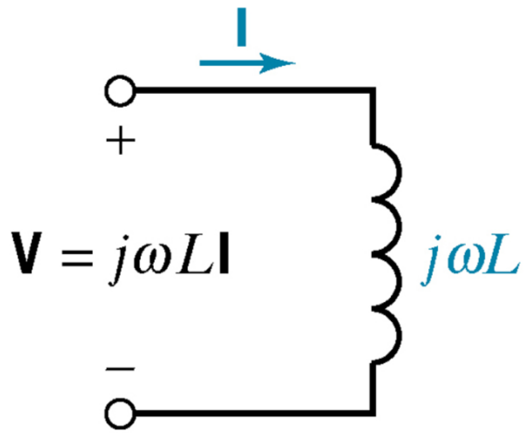
$$v(t) = 10\cos(10t);\, i(t) = ?$$

$$\mathrm{V} = R \times \mathrm{I} \Rightarrow \mathrm{I} = \frac{\mathrm{V}}{\mathrm{R}} = \frac{10\angle 0^\circ}{R}$$

$$\Rightarrow \quad i(t) = \frac{10\cos(10t)}{R}$$

Phasor of L



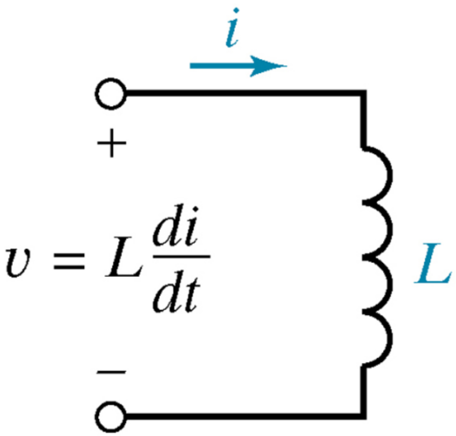


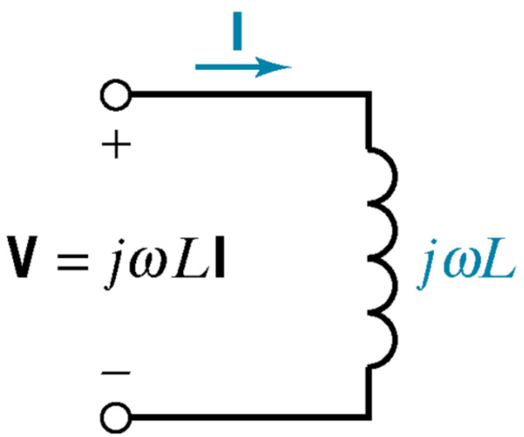
$$v(t) = L\frac{di(t)}{dt}$$

$$v = V_m\cos(\omega t + \phi) = \mathrm{Re}\left\{V_m e^{j(\omega t + \phi)}\right\}$$

$$i = I_m\cos(\omega t + \beta) = \mathrm{Re}\left\{I_m e^{j(\omega t + \beta)}\right\}$$

Phasor of *L*





$$v(t) = L\frac{di(t)}{dt}$$

$$v = V_m \cos(\omega t + \phi) = \mathrm{Re}\left\{V_m e^{j(\omega t+\phi)}\right\}$$

$$i = I_m \cos(\omega t + \beta) = \mathrm{Re}\left\{I_m e^{j(\omega t+\beta)}\right\}$$

$$V_m e^{j(\omega t+\phi)} = L \times \frac{dI_m e^{j(\omega t+\beta)}}{dt}$$

$$V_m e^{j\omega t} e^{j\phi} = L \times \frac{dI_m e^{j(\omega t+\beta)}}{dt}$$

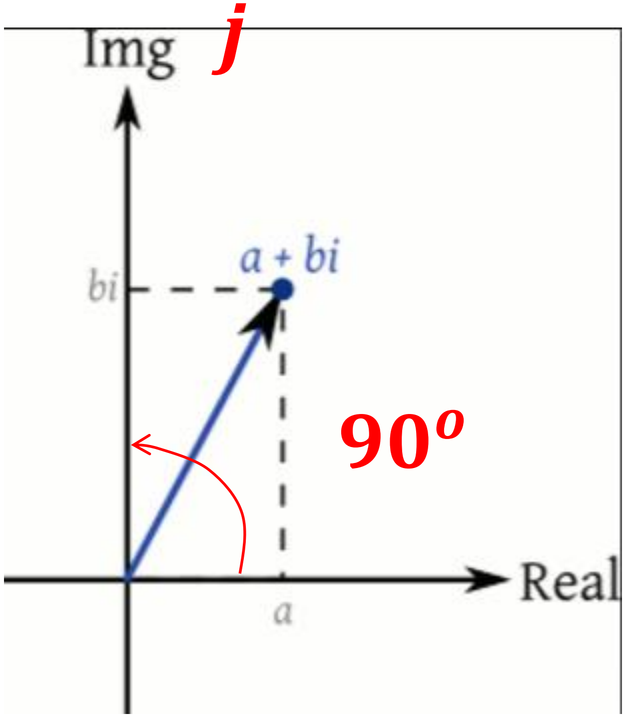
$$V_m e^{j\omega t} e^{j\phi} = L \times j\omega \times I_m e^{j(\omega t+\beta)}$$

$$V_m e^{j\omega t} e^{j\phi} = L \times j\omega \times I_m e^{j\omega t} e^{j\beta}$$

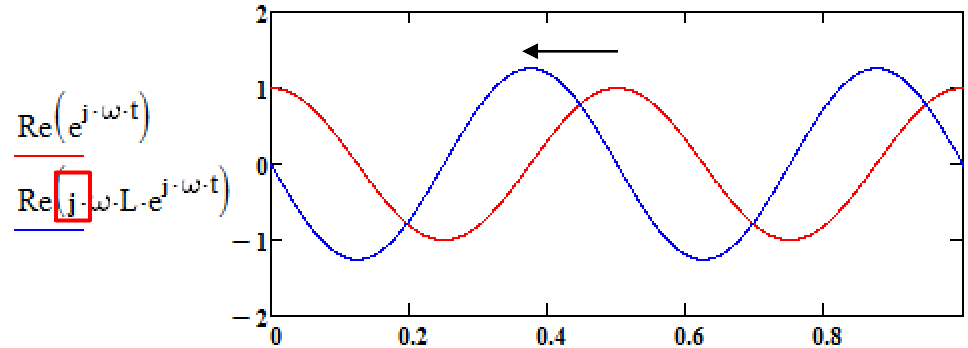
$$V_m e^{j\phi} = L \times j\omega \times I_m e^{j\beta}$$

$$\boxed{\mathrm{V} = j\omega L \times \mathrm{I}}$$

Complex Number *j*

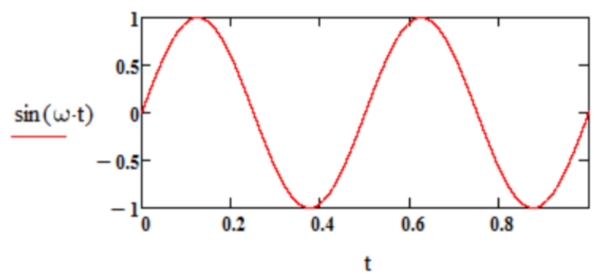


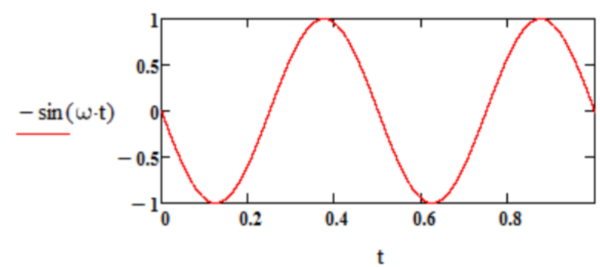
$j\ causes\ 90^o phase\ shift$: leading



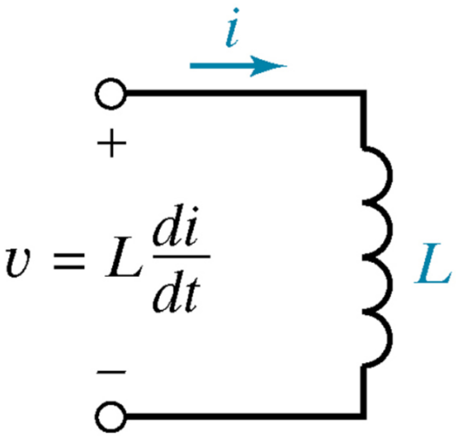
$$e^{j90^o} = j$$

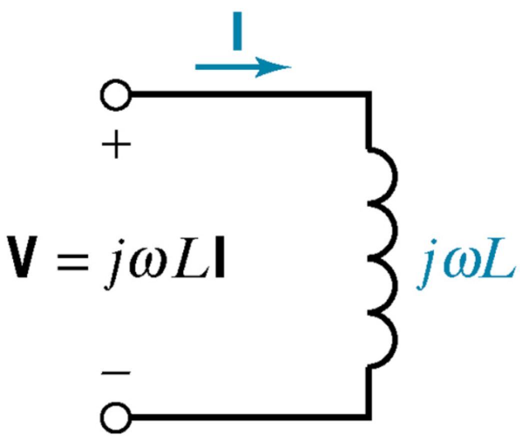
$$1 \cdot e^{j180^o} = -1$$





Phasor of L



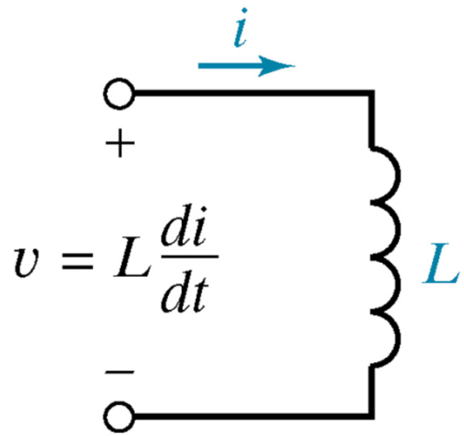


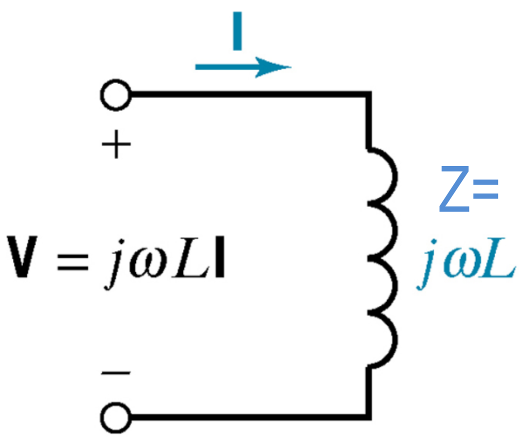
$$\mathrm{V} = j\omega L \times \mathrm{I}$$

$$j = e^{j90^\circ} \Rightarrow V_m e^{j\phi} = L \times \omega \times I_m e^{j90^\circ} e^{j\beta}$$

Inductor voltage leads the inductor current by exactly 90°

Example 5





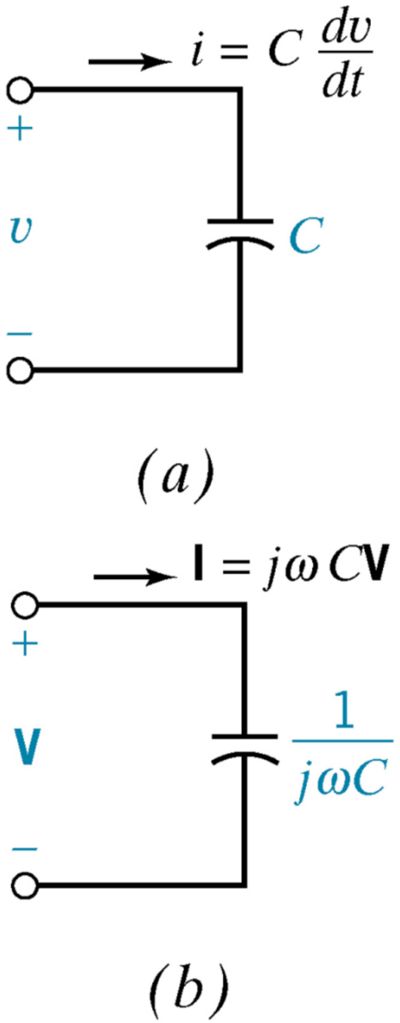
$$L = 2\,H,\ \omega = 100\,rad/s,\ v(t) = 10\cos(\omega t + 50^\circ)$$

$$\mathrm{V} = j\omega L \times \mathrm{I} \Rightarrow \mathrm{I} = \frac{\mathrm{V}}{j\omega L} = \frac{10\angle 50^\circ}{j \times 100 \times 2} = \frac{10\angle 50^\circ}{j200}$$

$$= \frac{10\angle 50^\circ}{200\angle 90^\circ} = 0.05\angle -40^\circ$$

$$\Rightarrow i(t) = 0.05\cos(100t - 40^\circ)$$

Phasor of C

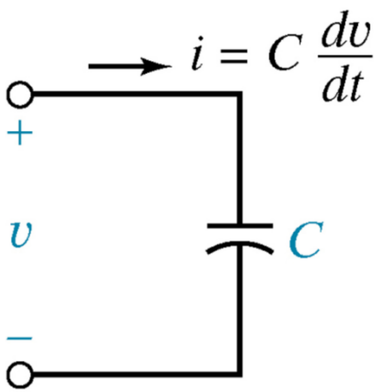


$$i(t) = C\frac{dv(t)}{dt}$$

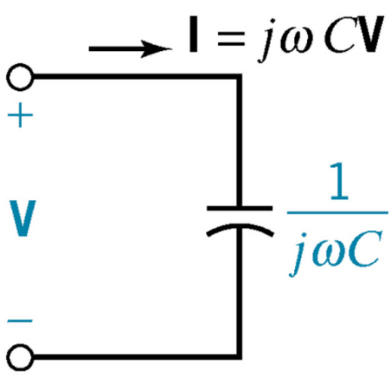
$$v = V_m \cos(\omega t + \phi) = \mathrm{Re}\left\{V_m e^{j(\omega t + \phi)}\right\}$$

$$i = I_m \cos(\omega t + \beta) = \mathrm{Re}\left\{I_m e^{j(\omega t + \beta)}\right\}$$

Phasor of C



(a)



(b)

$$i(t) = C\frac{dv(t)}{dt}$$

$$v = V_m\cos(\omega t + \phi) = \mathrm{Re}\left\{V_m e^{j(\omega t+\phi)}\right\}$$

$$i = I_m\cos(\omega t + \beta) = \mathrm{Re}\left\{I_m e^{j(\omega t+\beta)}\right\}$$

$$I_m e^{j(\omega t+\beta)} = C\times\frac{dV_m e^{j(\omega t+\phi)}}{dt}$$

$$I_m e^{j\omega t}e^{j\beta} = C\times\frac{dV_m e^{j(\omega t+\phi)}}{dt}$$

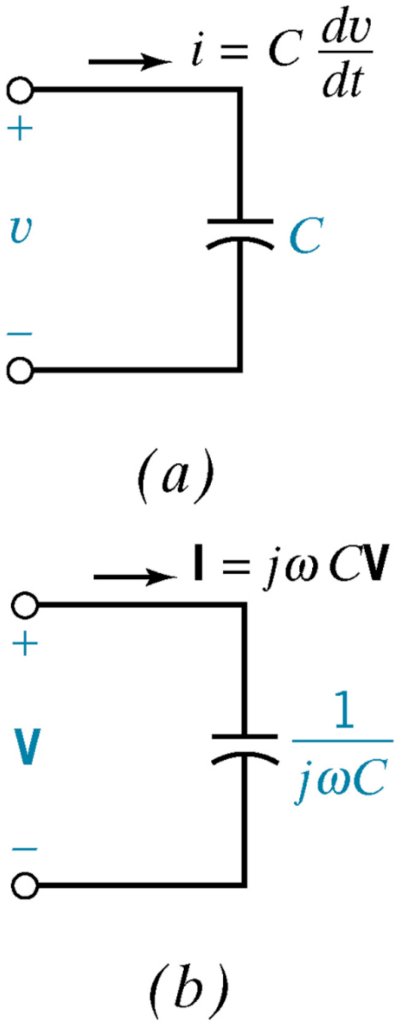
$$I_m e^{j\omega t}e^{j\beta} = C\times j\omega\times V_m e^{j(\omega t+\phi)}$$

$$I_m e^{j\omega t}e^{j\beta} = C\times j\omega\times V_m e^{j\omega t}e^{j\phi}$$

$$I_m e^{j\beta} = C\times j\omega\times V_m e^{j\phi}$$

$$\boxed{\mathrm{I} = j\omega C\times \mathrm{V}} \longrightarrow \boxed{V = \frac{1}{j\omega C}\cdot \mathrm{I}}$$

Phasor of C

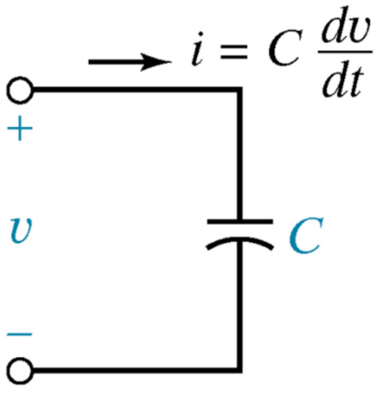


$$\mathrm{I} = j\omega C \times \mathrm{V}$$

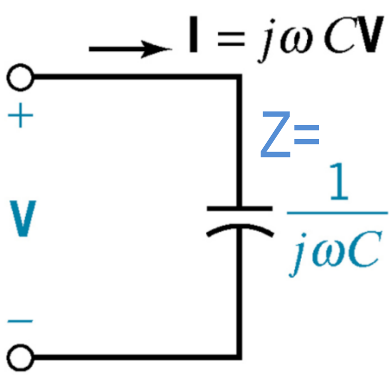
$$j = e^{j90^\circ} \Rightarrow I_m e^{j\beta} = L \times \omega \times V_m e^{j\phi}{}_m e^{j90^\circ}$$

Capacitor current leads the capacitor voltage by exactly 90°

Example 6



(a)



(b)

$$C = 1mF, v(t) = 100\cos(1000t)V$$

$$\mathrm{I} = j\omega C \times \mathrm{V} = j \times 1000 \times 1m \times 100\angle 0^\circ$$

$$= j100\angle 0^\circ = 100\angle 90^\circ$$

$$\Rightarrow i(t) = 100\cos(1000t + 90^\circ)$$

Impedance Summary

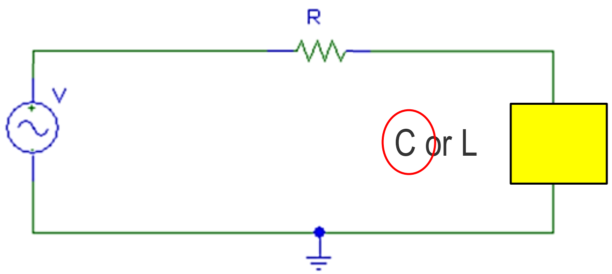
$$\text{For a resistor } R;\ Z = \mathrm{R}$$

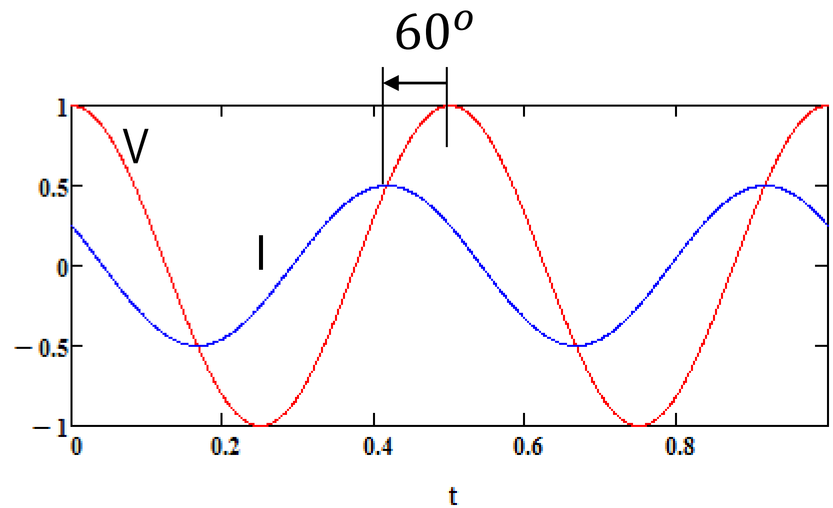
$$\text{For an inductor } L; Z = j\omega L$$

$$\text{For a capacitor } C; Z = \frac{1}{j\omega C} = \frac{-j}{\omega C}$$

$1 = 1\angle 0^o, j = 1\angle 90^o$, -1 $= 1\angle \pm 180^o$,
$-\mathrm{j} = 1\angle -90^o\ or\ 1\angle 270^o$

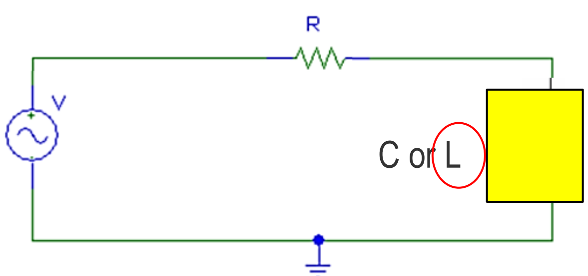
Example of Impedance

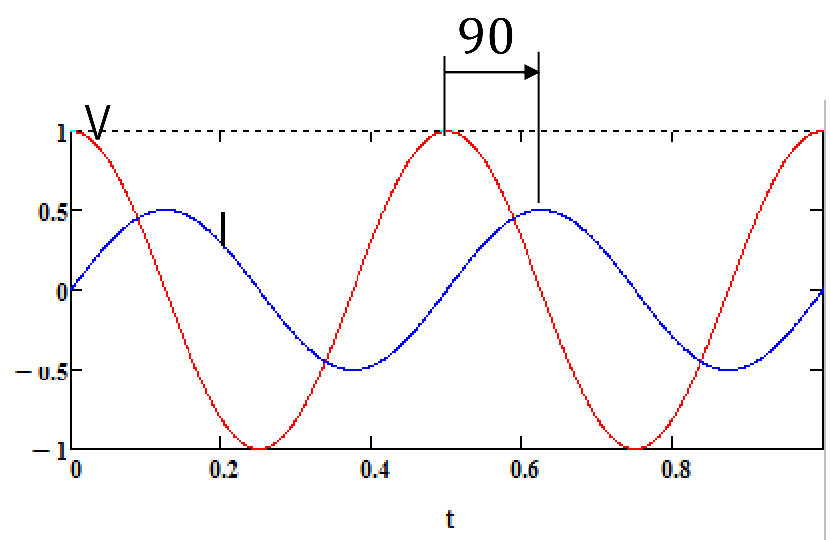




$$Z = \frac{V}{I} = 2\angle(-60^{o})\ \Omega = 1 - 1.732j\ \Omega$$

Polar form Rectangular form





$$Z = \frac{V}{I} = 2\angle(90^{o})\ \Omega = 0 + 2j\ \Omega$$

Polar form Rectangular form

Summary

- DC circuit analysis techniques compute only magnitudes of V and I

  (Parallel/Series, Volt/Current, division, KVL, KCL, Mesh, Nodal, Thévenin/Norton Equivalent, Dependent sources, OpAmps, etc.)

- AC analysis is a lot more complex than DC analysis
  - In addition to magnitudes, phase should also be considered
  - Essentially, sine and cosine function(s) are to be used, but then
    - Analytical computations are too complicated to handle by human (or computers)
  - It is necessary to adapt simpler ways to compute AC circuits
  - It is also desired to utilize the known DC analysis techniques (listed above) for AC circuits

Summary

Solution for AC circuit analysis (Advantages of phasors)

- Phasor is adapted to remove cosine functions from analytical equations
- By using phasors, the same DC circuit analysis techniques (that we have studied so far) can be used to analyze AC circuits

Disadvantages of Phasor

- Phasors are complex numbers (still better than having COS and SIN functions)
- Students should be familiar with the complex mathematics (still better than using COS and SIN functions)
- Steady-state only (transient response cannot be obtained by phasor)

COS and SIN: time domain

Phasors: frequency domain

Element	Impedance	Admittance
R	$\mathbf{Z} = R$	$\mathbf{Y} = \frac{1}{R}$
L	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \frac{1}{j\omega L}$
C	$\mathbf{Z} = \frac{1}{j\omega C}$	$\mathbf{Y} = j\omega C$

Methods we have learned

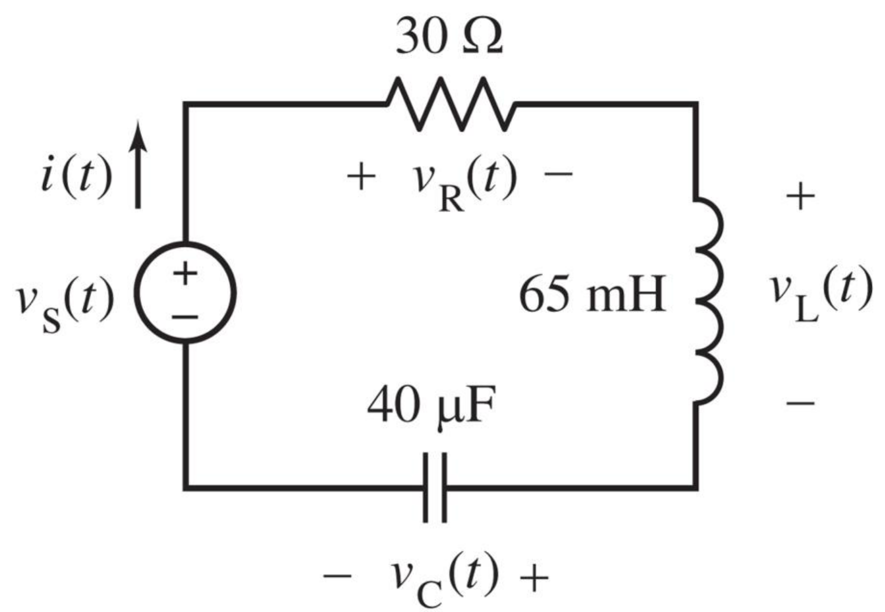
- **Kirchhoff's Laws**
  - Hold true in the frequency domain
  - The sum of sinusoidal currents in/out of a node equals zero
    - $I_1 + I_2 + I_3 + \ldots + I_N = 0$
  - The sum of sinusoidal voltages around a loop equals zero
    - $V_1 + V_2 + \ldots + V_N = 0$
- **Nodal Analysis** is valid in the frequency domain
  - First, transform a time-domain circuit to the frequency domain
  - Use nodal analysis, as you have done before
  - Finally, transform the answer to the time domain
- **Mesh Analysis** is also valid in the frequency domain
  - Follow the same steps as for Nodal analysis

Example 10.4-1

The input to the AC circuit shown below is the source voltage

$$v_s(t) = 12\cos(1000t + 15^o)\,V$$

Determine (a) the impedances of the capacitor, inductor, and resistance and (b) the current $i(t)$



Reminder

Polar to rectangular

$$x = rcos\phi$$

$$y = rsin\phi$$

$$z = r(cos\phi + jsin\phi)$$

Rectangular to polar

$$r = \sqrt{x^2 + y^2} \quad \phi = \tan^{-1}\frac{y}{x}$$

$$z = \sqrt{x^2 + y^2}\angle \tan^{-1}\frac{y}{x}$$

Addition and Subtraction of Complex Number (rectangular)

$$Z_1 + Z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

$$Z_1 - Z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

Multiplication/Division of Complex Number (polar)

$$Z_1 \cdot Z_2 = r_1 \cdot r_2\angle(\phi_1 + \phi_2)$$

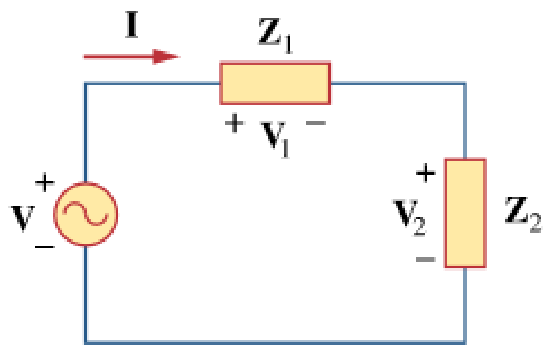
$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2}\angle(\phi_1 - \phi_2)$$

Element	Impedance	Admittance
R	$\mathbf{Z} = R$	$\mathbf{Y} = \frac{1}{R}$
L	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \frac{1}{j\omega L}$
C	$\mathbf{Z} = \frac{1}{j\omega C}$	$\mathbf{Y} = j\omega C$

Impedance Combination

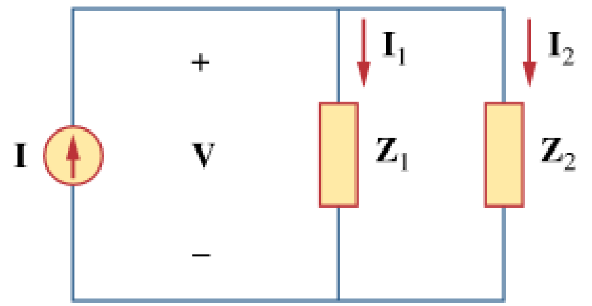
Series elements

$$Z_{eq} = Z_1 + Z_2 + \ldots + Z_N$$



Parallel elements

$$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \ldots \frac{1}{Z_N}$$



Voltage Division

$$V_1 = V \frac{Z_1}{(Z_1 + Z_2)}$$

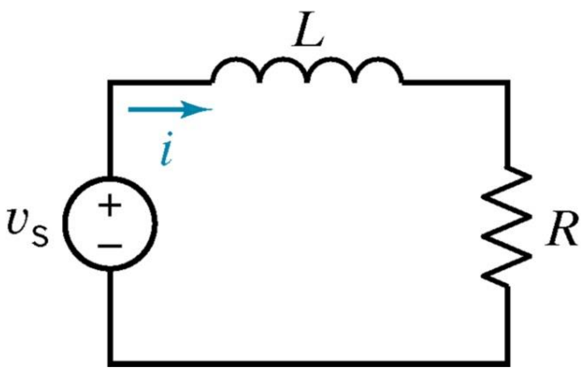
$$V_2 = V \frac{Z_2}{(Z_1 + Z_2)}$$

Current Division

$$I_1 = I \frac{Z_2}{Z_1 + Z_2}$$

$$I_2 = I \frac{Z_1}{Z_1 + Z_2}$$

Example 7



$$\omega = 100\ rad/s,\ R = 200\ \Omega,\ L = 2\,H,\ v(t) = V_m \cos\omega t\ \text{V}$$

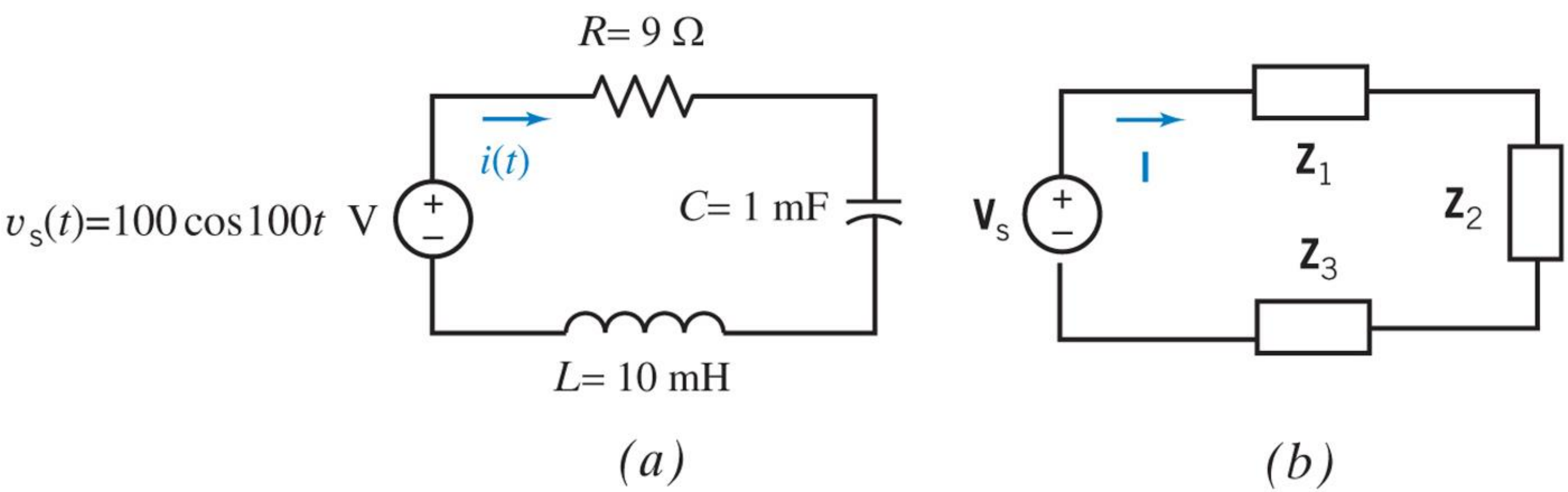
$$\boldsymbol{i} = ?$$

$$\mathrm{I} = \frac{\mathrm{V}}{j\omega L + R} = \frac{V_m\angle 0^\circ}{j\times 100\times 2 + 200} = \frac{V_m\angle 0^\circ}{j\times 200 + 200} = \frac{V_m\angle 0^\circ}{283\angle 45^\circ} = \frac{V_m\angle -45^\circ}{283}$$

$$\Rightarrow i(t) = \frac{V_m}{283}\cos\left(100t - 45^\circ\right)$$

Example 10.5-1 – KVL

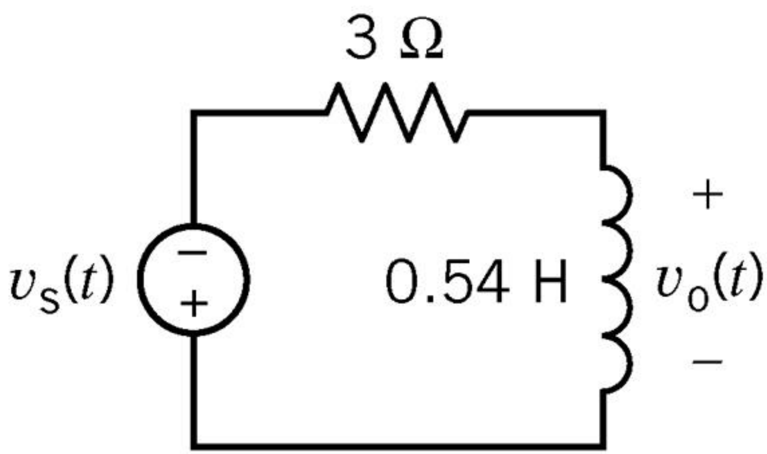
- Determine the steady state current $i(t)$ using phasor and impedances



Example 10.5-2 – Voltage Division

- Determine the steady-state output voltage, $v_o(t)$ if the source voltage is:

$$v_s(t) = 7.28\cos(4t + 77°)\ V$$



(a)

Example 10.5-3 – KVL

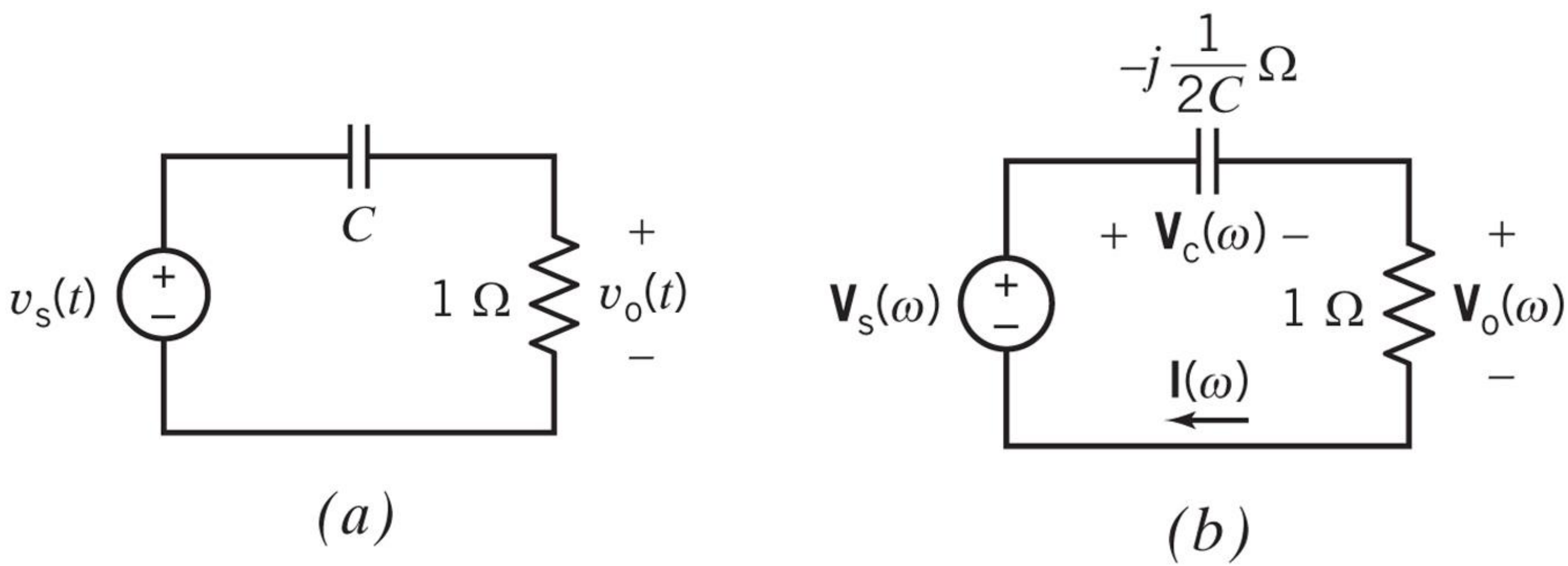
The input to the circuit is the voltage of the voltage source:

$$v_s(t) = 7.68\cos(2t + 47^o)\text{ V}$$

The output is the voltage across the resistor:

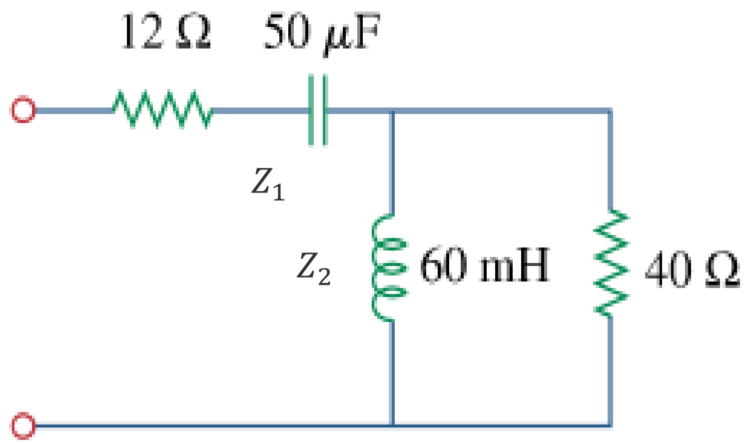
$$v_o(t) = 1.59\cos(2t + 125^o)$$

Determine capacitance C of the capacitor.



Example 8 – Z_{eq}

- At $\omega = 377\ rad/s$, find the input impedance of the circuit.



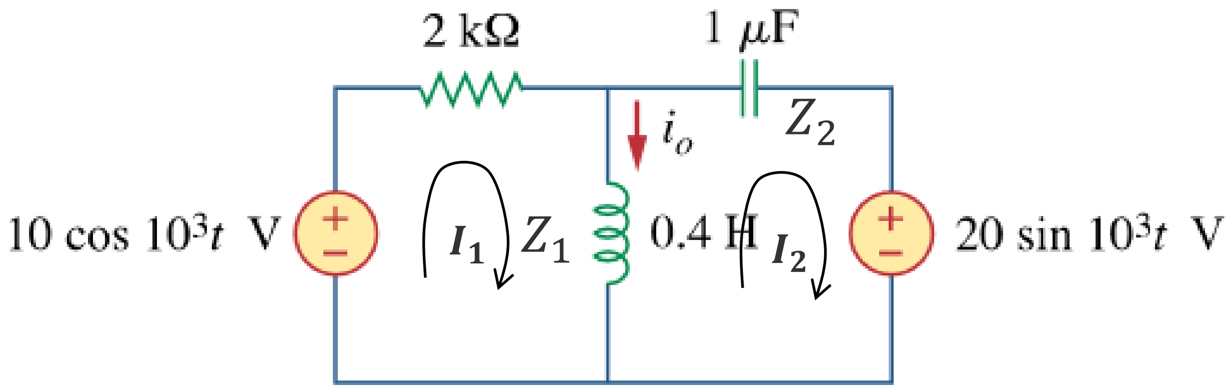
R: $Z = R$

L: $Z = j\omega L$

C: $Z = -j\frac{1}{\omega C}$

Example 9 – Mesh Analysis

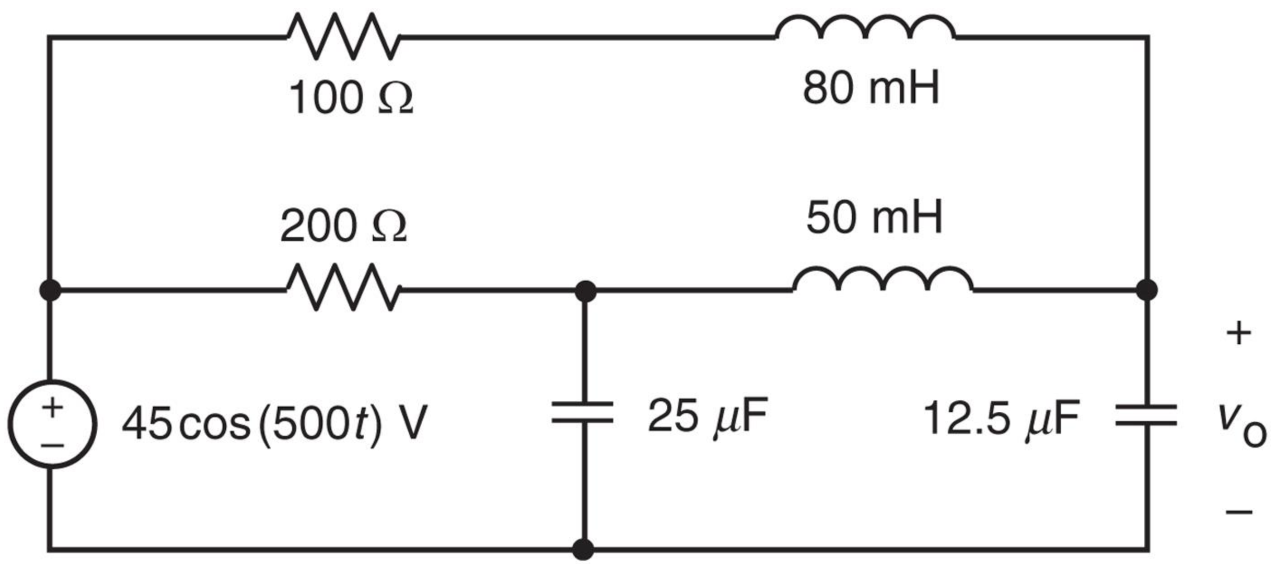
- Calculate i_o



$$\sin(\omega t) = \cos(\omega t - 90°) = 1\angle -90° = -1j$$

Example 10.6-2

- Determine the mesh currents for the circuit shown below.



Exercise 10.6-3 – Node Analysis

The input to the circuit shown in Figure 10.6-10 is the voltage source voltage

$$v_s(t) = 10 \cos(10t) \text{ V}$$

The output is the current $i(t)$ in resistor R_1. Determine $i(t)$.

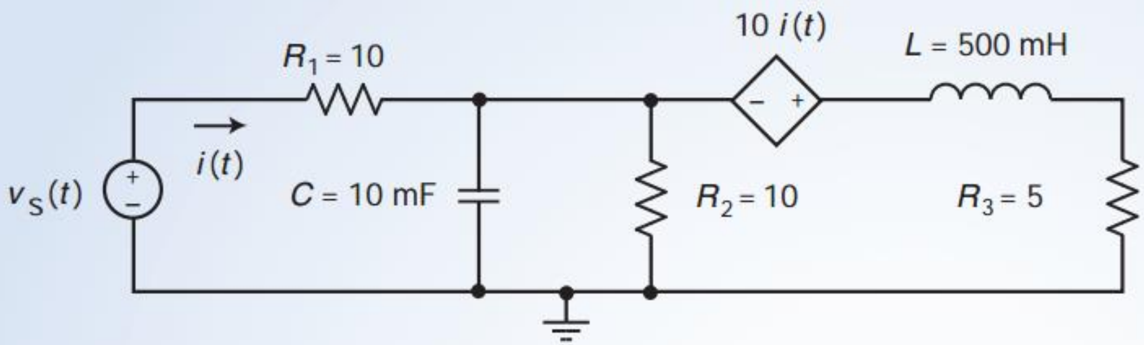
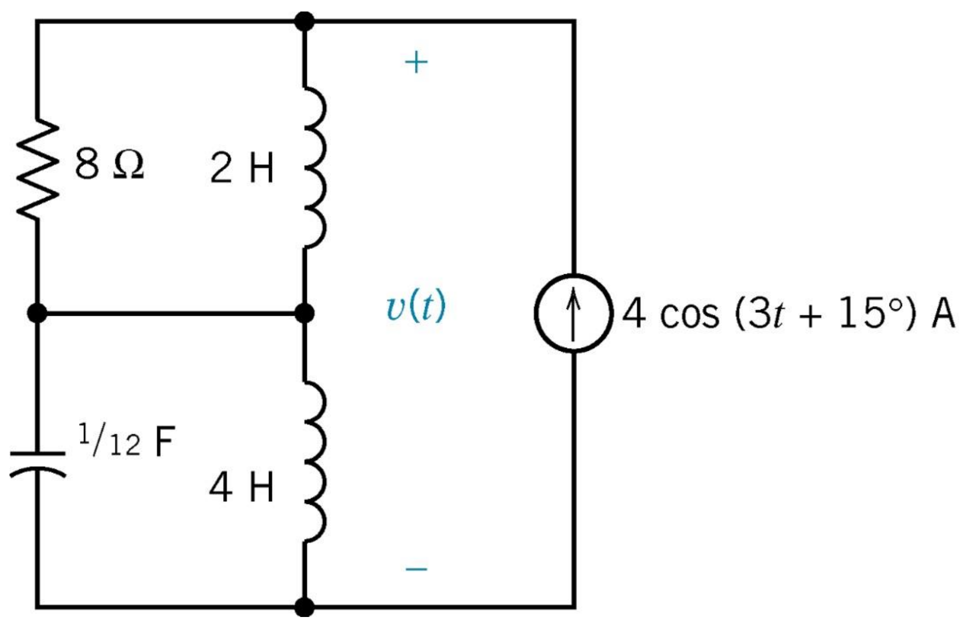


FIGURE
Example

Exercise 10.7-2

- Determine the phasor representation of each circuit element.



Methods we have learned

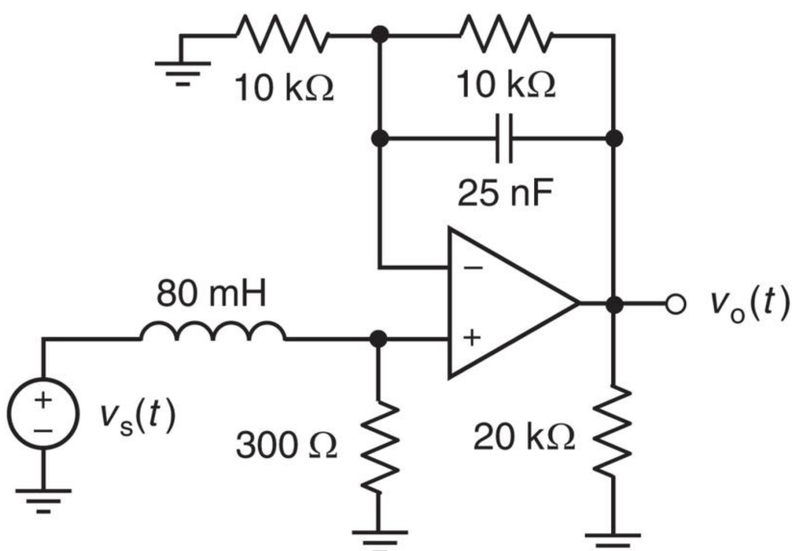
- **OpAmp**
- **Thévenin/Norton Equivalent**
  - No dependent sources: Just like in DC
  - Turn off independent sources
  - Determine the equivalent impedance
  - *Include impedance from capacitors and inductors*
  - Turn on independent sources
  - Determine phasor voltage (Thévenin) or current (Norton) at the terminals
- **Use source transformation to go from Norton to Thévenin**
- **With dependent sources**
  - Add a phasor source across the terminals
  - V_m = *1, angle = 0*
  - Determine the phasor current it provides to the circuit
  - Use Ohm's law to calculate the equivalent impedance

Example 10.6-4 – OpAmp

The input to the ac circuit shown in Figure 10.6-13 is the voltage source voltage

$$v_s(t) = 125\cos(500t + 15°)\ \text{mV}$$

Determine the output voltage $v_o(t)$.

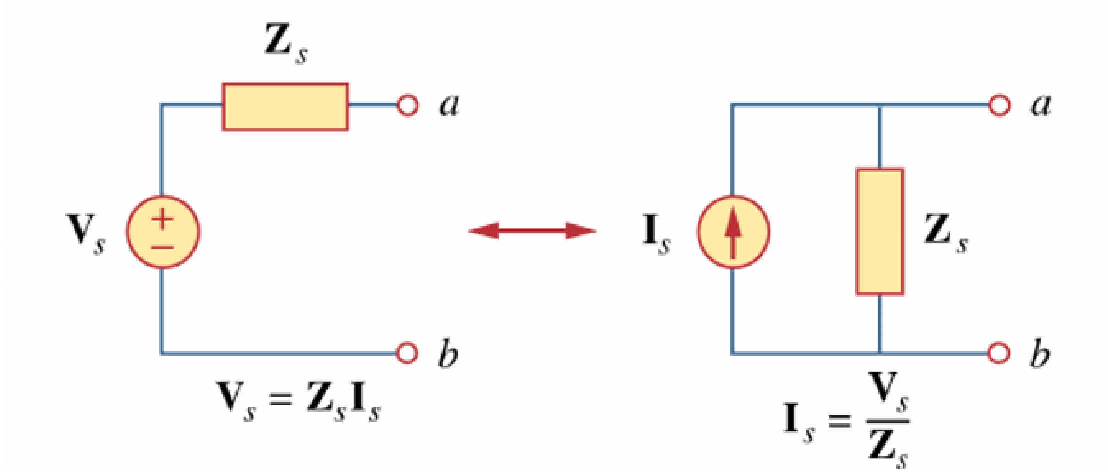


Source Transformation

- Source transformation in the frequency domain is just like in the time domain, except that we use impedance instead of resistance

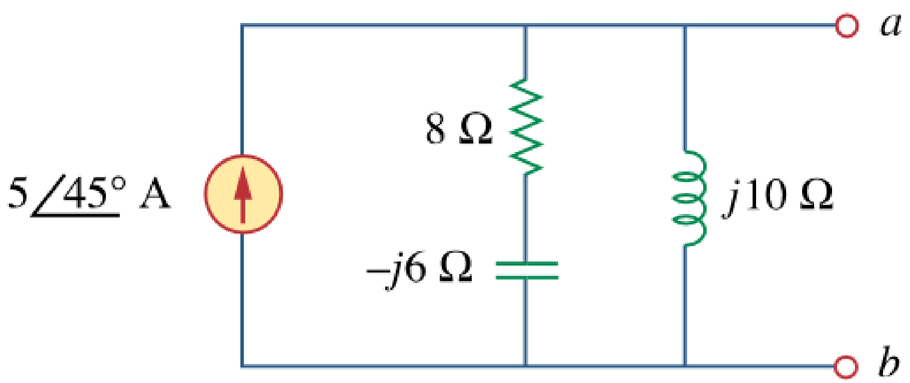
$$V_s = \frac{Z_s}{I_s}$$

$$I_s = \frac{V_s}{Z_s}$$



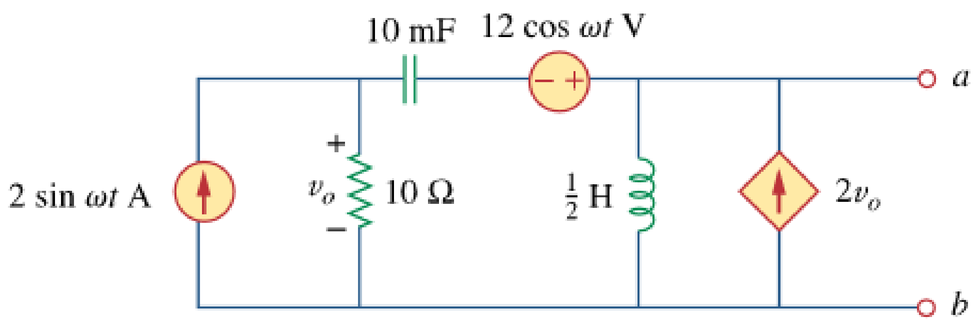
Example 10

For the circuit depicted in Fig. 10.101, find the Thevenin equivalent circuit at terminals *a*-*b*.



Example 11

10.66 At terminals a-b, obtain Thevenin and Norton equivalent circuits for the network depicted in Fig. 10.109. Take $\omega = 10$ rad/s.



Example 12