CS170–Fall 2012 — Solutions to Homework 1

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1. For question 1, I assume the following basic relations:
   $n^c = \Omega(\log(n))$, $0 < c < 1$
   $n = \Omega(n^c)$, $0 < c < 1$
   $n = O(n^c)$, $c > 1$

   (a) $n - 100 = \Theta(n - 200)$, in the limits as $n \to \infty$, both act like n, $\lim_{n\to\infty} \frac{f}{g} = 1$

   (b)

$$\lim_{n\to\infty} \frac{n^{1/2}}{n^{2/3}}$$
$$\lim_{n\to\infty} n^{-1/3} = 0$$
$$n^{1/2} = O(n^{2/3})$$

   (c)

$$\frac{100n + \log(n)}{n + \log(n)^2}$$
$$\lim_{n\to\infty} \approx \frac{100n}{n} = 100$$
$$100n + \log(n) = \Theta(n + \log(n)^2)$$

(d)

$$\begin{aligned}
&\frac{n\log(n)}{10n\log(10n)}\\
&=\frac{n\log(n)}{10n(\log(10)+\log(n))}\\
&=\frac{n\log(n)}{10n\log(n)+10\log(10)n} \qquad\qquad \text{(shows } \log(cn)=\Theta(\log(n)))\\
\lim_{n\to\infty} &\approx\frac{n\log(n)}{10n\log(n)}\\
&=10\\
n\log(n) &=\Theta(10n\log(10n))
\end{aligned}$$

(e) $\log(2n) = \Theta(\log(3n))$, by part (d)

(f) $\log(n^2) = 2 * \log(n)$
$10\log(n) = \Theta(\log(n^2))$

(g)

$$\begin{aligned}
\lim_{n\to\infty}\frac{n^{1.01}}{n\log^2(n)}\\
\lim_{n\to\infty}\frac{n^{.01}}{\log^2(n)}\\
\lim_{n\to\infty}\frac{n^{.005}}{\log(n)} &= \infty\\
n^{1.01} &= \Omega(n\log^2(n))
\end{aligned}$$

(h)

$$\begin{aligned}
\lim_{n\to\infty}\frac{\frac{n^2}{\log(n)}}{n\log^2(n)}\\
\lim_{n\to\infty}\frac{n}{\log^3(n)} &= \infty \text{ using L'Hopital's rule}\\
\frac{n^2}{\log(n)} &= \Omega(n\log^2(n))
\end{aligned}$$

(i)

$$\lim_{n\to\infty}\frac{n^{.1}}{\log(n)^{10}} = \infty \text{ same as h, keep using L'Hopital's rule}$$

$$n^{.1} = \Omega(\log(n)^{10})$$

(j)

$$\lim_{n\to\infty} \frac{\log(n)^{\log(n)}}{\frac{n}{\log(n)}}$$
$$\lim_{n\to\infty} \frac{\log(n)^{\log(n)+1}}{n}$$
$$\text{let } m = \log(n) \lim_{m\to\infty} \frac{m^{m+1}}{2^m} = \infty$$
$$\log(n)^{\log(n)} = \Omega(\frac{n}{\log(n)})$$

(k)

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\log^3(n)} = \infty \text{ using L'Hopital's rule}$$
$$\sqrt{n} = \Omega(\log^3(n)$$

(l)

$$\lim_{n\to\infty} \frac{\sqrt{n}}{5^{\log_2(n)}}$$
$$\text{let } x = \log_2(n) \lim_{x\to\infty} \frac{2^{x/2}}{5^x} = 0$$
$$\sqrt{n} = O(5^{\log_2(n)})$$

(m)

$$\lim_{n\to\infty} \frac{n2^n}{3^n}$$
$$\lim_{n\to\infty} n * \left(\frac{2}{3}\right)^n = 0$$
$$n2^n = \Omega(3^n)$$

(n)

$$\lim_{n\to\infty} \frac{2^n}{2^{n+1}} = \frac{1}{2}$$
$$2^n = \Theta(2^{n+1})$$

(o)

$$\lim_{n\to\infty} \frac{n!}{2^n}$$
$$\frac{1}{2} * \frac{2}{2} * \frac{3}{2} * * \frac{n-1}{2} * \frac{n}{2}$$

as n approaches infinity, this series must approach infinity because each successive term is increasing and all except the first are at least one (I don't know if this is a formal proof but I forget 1B, the following conclusion seems rather intuitive)
$n! = \Omega(2^n)$

(p)

$$\lim_{n\to\infty} \frac{\log(n)^{\log(n)}}{2^{\log^2(n)}}$$
$$\lim_{x\to\infty} \frac{x^x}{2^{x^2}} = 0 \text{ courtesy of wolframalpha}$$
$$\log(n)^{\log(n)} = O(2^{\log^2(n)})$$

(q)

$$\lim_{x\to\infty} \frac{\sum_{i=1}^n i^k}{n^{k+1}}$$
$$\lim_{x\to\infty} 1^k + 2^k + 3^k + \; + (n-2)^k + (n-1)^k + \frac{n^k}{n^{k+1}} = \infty$$
$$\sum_{i=1}^n i^k = \Omega(n^{k+1})$$

2. (a) $g(n) = 1+c+c^2++c^n$, $c<1$ This is a geometric series with $c<1$ (it converges). $\lim_{\to\infty} \frac{g(n)}{1} = \frac{1+c+c^2++c^n}{1} = \frac{1}{1-c}$, a constant so $g(n) = \Theta(1)$

   (b) If $c = 1$ then the series turns into $\sum_{i=1}^{n} 1$, which is just n, $\frac{n}{n} = 1$, so $g(n) = \Theta(n)$

   (c) If $c > 1$ then we can determine bounds by doing $\lim_{n\to\infty} \frac{g(n)}{c^n}$:

$$\lim_{n\to\infty} \frac{1+c+c^2+\ +c^n}{c^n}$$
$$\lim_{n\to\infty} \frac{1}{c^n} + \frac{c}{c^n} + ... + \frac{c^{n-1}}{c^n} + \frac{c^n}{c^n}$$
$$\lim_{n\to\infty} c^{-n} + c^{-n+1} + c^{-n+2} +\ + c^{-1} + 1 = \frac{1}{1-\frac{1}{c}}$$

   This is a constant so we know $g(n) = \Theta(c^n)$

3. (a) PUT SOMETHING HERE
   (b) PUT SOMETHING HERE
   (c) PUT SOMETHING HERE
   (d) PUT SOMETHING HERE
   (e) PUT SOMETHING HERE

4. PUT SOMETHING HERE

5. (a) PUT SOMETHING HERE
   (b) PUT SOMETHING HERE
   (c) PUT SOMETHING HERE