1 Notation

- **Sets**:
  I — elective surgeries (index i); B — OR–day blocks (index t).

- **Parameters (scalars)**:
  b_i (booked time), d_i (nominal duration),
  c^{OT}, c^{IT}, c^{REJ} (unit costs), $\text{Cap} = 480$ minutes (regular capacity).

- **Decision variables**:
  $Z_{it} \in \{0, 1\}$, $R_i \in \{0, 1\}$ (first stage); $OT_t, IT_t \geq 0$ (second stage).

- **Uncertain durations**:
  $\mathbf{D} = \big(D_i\big)_{i \in I} \in \mathbb{R}_+^{|I|}$ is a *random* vector with distribution $\mathbb{P}_{\mathbf{D}}$;
  realizations are denoted as $\mathbf{d}$.

- **Covariates for CSO**:
  $X \in \mathbb{R}^{d_x}$ is a random covariate vector revealed before scheduling,
  with realization x and joint law $\mathbb{P}_{X,\mathbf{D}}$.

2 Deterministic Baseline MILP

The baseline assumes actual durations d_i are known *a priori*.

$$
\begin{aligned}
\min_{Z,R,OT,IT} \quad & \sum_{t \in B} \big(c^{OT} OT_t + c^{IT} IT_t\big) + c^{REJ} \sum_{i \in I} b_i R_i && && (1) \\
\text{s.t.} \quad & \sum_{t \in B} Z_{it} + R_i = 1 && \forall i \in I, && (2) \\
& \sum_{i \in I} d_i Z_{it} - \text{Cap} = OT_t - IT_t && \forall t \in B, && (3) \\
& Z_{it}, R_i \in \{0, 1\},\ OT_t, IT_t \geq 0.
\end{aligned}
$$

3 Introducing Uncertainty

At allocation time the true durations are unknown; we therefore model $\mathbf{D} \sim \mathbb{P}_{\mathbf{D}}$ and observe $\mathbf{d}$ *after* choosing (Z, R). All expectations below are taken with respect to this distribution.

4 Two-Stage Stochastic Program

We now select (Z, R) to minimize the expected total cost, recognizing that OT and IT are adjusted in a second stage after the duration is realized.

First stage.

$$\min_{Z,R} \quad c^{REJ}\sum_{i\in I} b_i R_i \;+\; \mathbb{E}_{\mathbb{P}_{\mathbf{D}}}\big[Q(Z,R;\mathbf{D})\big] \tag{4}$$

$$\text{s.t.} \quad \sum_{t\in B} Z_{it} + R_i = 1 \qquad \forall i \in I, \tag{5}$$

$$Z_{it}, R_i \in \{0,1\}.$$

Second stage. After having a realization $\mathbf{d} = (d_i)_{i\in I}$ we solve:

$$Q(Z,R;\mathbf{d}) = \min_{OT,IT\geq 0} \sum_{t\in B}\big(c^{OT} OT_t + c^{IT} IT_t\big) \tag{6}$$

$$\text{s.t.} \sum_{i\in I} d_i Z_{it} - \text{Cap} = OT_t - IT_t,\ \forall t \in B. \tag{7}$$

5 Sample-Average Approximation (SAA)

When $\mathbb{P}_{\mathbf{D}}$ is unknown, we estimate the expectation in (4) using N i.i.d. samples of vectors $\mathbf{d}^{(1)}, \ldots, \mathbf{d}^{(N)} \sim \mathbb{P}_{\mathbf{D}}$. With second–stage variables $OT_t^{(n)}, IT_t^{(n)}$ for each sample n:

$$\min_{Z,R,OT^{(n)},IT^{(n)}} \quad c^{REJ}\sum_{i\in I} b_i R_i + \frac{1}{N}\sum_{n=1}^{N}\sum_{t\in B}\big(c^{OT} OT_t^{(n)} + c^{IT} IT_t^{(n)}\big) \tag{8}$$

$$\text{s.t.} \quad \sum_{t\in B} Z_{it} + R_i = 1 \qquad \forall i \in I, \tag{9}$$

$$\sum_{i\in I} d_i^{(n)} Z_{it} - \text{Cap} = OT_t^{(n)} - IT_t^{(n)} \qquad \forall t \in B,\ n = 1,\ldots,N, \tag{10}$$

$$Z_{it}, R_i \in \{0,1\},\ OT_t^{(n)}, IT_t^{(n)} \geq 0.$$

6 Contextual Stochastic Optimization

Covariate information x—for example day-of-week, surgeon, or patient status—often helps anticipate durations. Let $\mathbb{P}_{\mathbf{D}\,|\,X=x}$ denote the *true* conditional distribution of $\mathbf{D}$ given $X = x$. For each observed x we solve

$$z^\star(x) = \arg\min_{Z,R} \quad c^{REJ}\sum_{i\in I} b_i R_i + \mathbb{E}_{\mathbb{P}_{\mathbf{D}\,|\,X=x}}\big[Q(Z,R;\mathbf{D})\big] \tag{11}$$

$$\text{s.t.} \quad \sum_{t\in B} Z_{it} + R_i = 1 \qquad \forall i \in I, \qquad Z_{it}, R_i \in \{0,1\}.$$

Problem (11) is structurally identical to (4): only the expectation is now *conditional on the covariate*. Data-driven approximations (e.g. contextual SAA or predict-then-optimize) replace $\mathbb{P}_{\mathbf{D}\,|\,X=x}$ with an empirical or learned model; these will be developed in later sections.