DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY MADRAS
CHENNAI – 600036

Multipartite Operator Entanglement

&

Absolutely, Maximally Entangled States

A Thesis

Submitted by

ROHAN NARAYAN

For the award of the degree

Of

BACHELOR OF TECHNOLOGY

IN

ENGINEERING PHYSICS

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Do you guys just put the word 'Quantum' in front of everything?

– Scott Lang

THESIS CERTIFICATE

This is to undertake that the Thesis titled **Multipartite Operator Entanglement & Absolutely, Maximally Entangled States**, submitted by me to the Indian Institute of Technology Madras, for the award of **Bachelor of Technology in Engineering Physics**, is a bona fide record of the research work done by me under the supervision of **Dr. Arul Lakshminarayan**. The contents of this Thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

Chennai 600036 **Rohan Narayan**

Date: 23 May 2022

Prof. Arul Lakshminarayan
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ABSTRACT

KEYWORDS Quantum; Information; Operators; Multipartite; Absolutely, Maximally Entangled States; Entanglement; Unitaries; Entropy; Nonlinear Maps;

The importance of entanglement in quantum states for the design of quantum algorithms and the study of the dynamics of quantum systems has been known since the conception of the field. The study of the role played by quantum evolutions in generating such resources and shaping the nature of Hamiltonian dynamics has been garnering more attention in recent years. In particular, there has been increased focus on the study of operator entanglement as a measure of the non-locality of the operator within the Hilbert-Schmidt space. This increased focus has been motivated by the connection of the entanglement structure of the operator with its entangling power, its utility as a measure to characterize delocalization or information scrambling in an evolution and the importance of maximally entangled **bipartite** operators in quantum circuits and many body physics.

However, the problem of entanglement in multipartite operators has not been studied rigorously —partly due to the absence of a truly multipartite entanglement measure.

In this work, we seek to address this gap by considering entanglement in operator bipartitions —analogous to popular techniques used in quantifying entanglement in multipartite *states*. This allows us to generalize the notion of maximally entangled operators from the bipartite case to a multipartite setting.

Taking cognizance of the fact that the operator entanglement entropies alone do not characterize its non-local features, we design a construction scheme to generate LU-invariants for any (N, d).

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NOTATION

S' If $S \subset \{A_1 \dots A_N\}, S' = \{A'_r : A_r \in S\}$

CHAPTER 1

INTRODUCTION

1.1 MOTIVATION

Entanglement is a striking feature of quantum mechanics bearing no classical equivalent or description. The study of entanglement of states has been a major focus since the dawn of research into quantum information. While there exists a theory of bipartite entanglement and its quantification, its extension to multipartite cases has proven difficult with measures such as the **n-tangle** being formulated as recently, relatively, as 2000. In the absence of a well-studied measure of entanglement in multipartite systems, the extension of the study of bipartite entanglement to multipartite systems was through *bipartite cuts* made in the multipartite system and then quantifying the entropy of entanglement for the bipartition.

If $\mathcal{H}^d$ is a d-dimensional Hilbert space, the space of all linear operators acting on states in $\mathcal{H}^d$ is known as the Hilbert-Schmidt space, $\mathcal{H}^d_{HS}$ and it is a d^2-dimensional space, endowed with the Hilbert-Schmidt product as the inner product :—$\langle A|B\rangle := \mathrm{Tr}(A^\dagger B)$. Thus, operators can themselves be entangled in the Hilbert-Schmidt space and have all the structures associated with the entanglement of states.

Thus, analogous to the definition of entanglement in states, it is possible to define the entanglement of operators —with connections to the Schmidt coefficients of the operator Schmidt decomposition of the operator, U. Similar to the manner in which the lack of a truly multipartite measure of entanglement in states —barring the 'newly' designed n-tangle —implies that even multipartite state entanglement is studied via bipartite cuts, even multipartite operator entanglement is studied using bipartite cuts in the operator set of parties —for the moment.

While there exist perfectly valid mathematical reasons to study the entanglement structure of operator spaces, the study of operator entanglement is further motivated by the application and correlation of operator entanglement entropy measures to a bevy of problems in quantum information.

The primary quantity of interest when considering the entanglement features of quantum evolutions is the amount of entanglement created when they act on quantum states. In certain applications, one might seek operators that generally produce 'higher amounts of entanglement' and in other settings, operators that produce *no* entanglement. The quantity that encapsulates this idea and quantifies the notion of 'entanglement creation' is the entangling power of the operator —denoted as e_p for bipartite operators.

$e_p(U)$ is defined as the average entanglement produced by U due to its action on pure product states distributed according to the uniform Haar measure,

$$e_p(U) = C_d \mathcal{E}\overline{(U|\phi_1\rangle \otimes |\phi_2\rangle)}^{|\phi_1\rangle, |\phi_2\rangle},$$

for bipartite operators, where $\mathcal{E}(.)$ may be any entanglement measure, and C_d is a suitable scaling factor. The linear entropy is often used for the easy of integration over Haar Measure (7). We scale it so that, $0 \leq e_p(U) \leq 1$.

This quantity is correlated with the operator entanglement of the operator for bipartite operators. For a general multipartite operator, the definition of entangling power is in itself in murky waters given that the entanglement of multipartite states is not well defined apart from the entanglement across bipartite cuts. However, when the definition of $e_p(U)$ is extended to multipartite settings, we will be able to establish a relationship between it and the operator entanglement entropies.

Bipartite unitaries that maximize operator entanglement, i.e. unitaries that are maximally entangled, are known as 'dual-unitary' gates. These gates have applications in many-body physics and quantum chaotic systems. However, maximally entangled operators have **not** been studied in the multipartite setting to the extent that its applications in many-body physics are known. However, a theory of multipartite operator entanglement and checks for **maximal** multipartite operator entanglement could motivate further explorations to this extent since the very crux of many body physics lies in the description of joint evolutions. Also note that the operator entanglement entropies of a multipartite operator do **not** in themselves characterize the entire no$N$-local features of the operator. This can be shown as :—

$$\text{If} \quad U' = \left(u^1_{A_1} \otimes \ldots \otimes u^1_{A_N}\right) U \left(u^2_{A_1} \otimes \ldots \otimes u^2_{A_N}\right),$$

U' & U are N-partite locally connected operators. Thus, since connections through local unitaries does **not** alter the non-local structure of unitaries, **any** non-local characteristic of U' & U must correspond to each other. Thus, the number of **independent** non-local characteristics of an N-partite operator will be equal to the 'degrees of freedom' of the N-partite operator, reduced by the 'degrees of freedom' captured by the local unitaries —viz.

$$\text{LU-Invariants}(N, d) = d^{2N} - 1 - (2N)(d^2 - 1). \tag{1.1}$$

Thus, for any unitary operator acting on N parties, each of local dimension d, a total of $d^{2N} - 1 - (2N)(d^2 - 1)$ *independent* local unitary (LU)-invariants must be provided to encapture the non-local features of the operator.

The study of the operator entanglement of multipartite operators and other entanglement features of unitary operators finds application in the construction of Absolutely, Maximally Entangled (AME) states, viz. quantum states for which **any** bipartition is maximally entangled. The importance of AME states as a quantum resource of entanglement have been elaborated in detail in Ch. 3 and Ch. 5.

Moreover, the relationship between AME states and the entanglement features has been analyzed and well defined for bipartite operators with two entanglement features —$E(U)$ & $E(US)$ being used in the construction of AME states of 4 parties.

The Bipartite Case : Motivation

We shall now define the two entanglement features, $E(U)$ & $E(US)$, state their connection to matrix reshapings and express the entangling power of unitaries in terms of these, so as to motivate the connection between all of these concepts —at least for the bipartite case.

$E(U)$ is the standard operator entanglement entropy of the bipartite operator, U. Thus, if:—

$$U_{AB} := \sum_{i=0}^{d^2} \sqrt{\lambda_i} m_A^i \otimes m_B^i,$$

with $\sum_{i=0}^{d^2} \lambda_i = d^2$.

Then, analogous to the definition of entropic functions of entanglement defined for states, the operator entanglement entropy can also be defined as entropic functions on these $\{\lambda_i/d^2\}$. We pick the Tsallis-2 entropy and define it as the *linear* operator entanglement entropy of U. We pick the Tsallis-2 entropy and term it as a 'linear' entropy as the entropic function is a polynomic function of the elements of U.

Thus,

$$E(U) = 1 - \frac{1}{d^4} \sum_{i=1}^{d^2} \lambda^2 = 1 - \frac{1}{d^4} \operatorname{Tr}\left((U^{\mathcal{R}} U^{\mathcal{R}\dagger})^2\right), \tag{1.2}$$

where $U^{\mathcal{R}}$ is defined as:—

$$\langle ik|U^{\mathcal{R}}|jl\rangle := \langle ij|U|kl\rangle, \tag{1.3}$$

a matrix reshaping of U.

Eq. 1.2 is proven in Ch. 2 —along with a statement on the equivalence between operators and states.

The complementary quantity, $E(US)$ is defined congruently —except for the use of the operator Schmidt decomposition of US, where S - Two qubit SWAP gate.

Thus,

$$E(US) = 1 - \frac{1}{d^4}\sum_{i=1}^{d^2}\mu^2 = 1 - \frac{1}{d^4}\operatorname{Tr}\left((U^\Gamma U^{\Gamma\dagger})^2\right), \tag{1.4}$$

where U^Γ is defined as:—

$$\langle kj|U^\Gamma|il\rangle := \langle ij|U|kl\rangle, \tag{1.5}$$

a matrix reshaping of U.

Eq. 1.4 is proven in Ch. 2 as well.

In (13), Zanardi showed that for bipartite operators,

$$e_p(U) = \frac{1}{E(S)}(E(U) + E(US) - E(S)). \tag{1.6}$$

Thus, motivating the definition of operator entanglement entropies of U *as well as* that of other LU-invariants in view of its connection to the entangling power.

$E(U)$ & $E(US)$ maximize at $1 - \frac{1}{d^2}$ for a given local dimension, d.

We can thus define these classes of operators on bipartite operators acting on systems with local dimension, d.

Dual Unitaries (Duals):—

- Bipartite operators that maximize $E(U)$.
- $\lambda_i = 1\ \forall i$.
- Maximally entangled bipartite operators.
- $U^{\mathcal{R}}$ is unitary.

- equivalent conditions for the definition of dual unitary operators.

T-dual Unitaries (T-duals):—

- Bipartite operators that maximize $E(US)$.
- $\mu_i = 1 \; \forall i$.
- U^Γ is unitary.

- equivalent conditions for the definition of T-dual unitary operators.

2-Unitary Operators (2-Unitaries)

Bipartite operators that are both dual-unitary **and** T-dual unitary are termed as 2-unitary operators. These are **perfect tensors** as any reshaping of them results in unitary operators.

There is an isomorphism between AME(4,d) states & 2-unitary operators. Thus, the study of $E(U)$ & $E(US)$ is instrumental in constructing AME states as well.

Thus, it is reasonable to expect that the generalization of $E(U)$ and the other LU-invariant quantities to multipartite settings has similar connections to AME(N, d) for $N > 4$ and this is one of the motivations behind our analysis of operator entanglement.

1.2 THESIS OUTLINE

In this section, we provide a brief outline of our work —supplementing the introductory passages of each chapter that expound on the work described in it as well —along with a set of final remarks and comments. In Ch. 1, we introduce operator entanglement and AME states —motivating research into these problems. We then explicitly link operator entanglement, LU-invariants and AME states for the bipartite case through a bevy of known results stated and summarized in (10).

Before heading into multipartite operator entanglement and its quantification, in Ch. 2 we develop a tool that allows us to link operators and states through an isomorphism, which we term as the multipartite operator-state isomorphism. The utility of such an isomorphism lies in its ability to convert problems on multipartite states to problems on multipartite operators —and vice-versa. We utilize this inter-conversion to prove several results and restate the k_D problem (worked on by Shrigyan Brahmachari) as a problem on quantum states. The utility of this tool is immense and this can be viewed from the **innumerous** results on operator entanglement that we prove using this tool. One of the significant results that we prove using the multipartite operator-state isomorphism is that **AME($2N-1, d$) exists if AME($2N, d$) exists**.

In Ch. 3, we express ideas, definitions and theorems developed by us that build the foundation for a general theory of multipartite operator entanglement. We define maximally entangled multipartite *operators* in a manner analogous to the definition of AME states for quantum states and further extend the analogy by proving that to check for maximal operator entanglement, only balanced bipartitions (or *nearly* balanced bipartitions, for odd N) need to be considered. We conclude this chapter by providing a construction recipe for the construction of a class of LU-invariants analogous to the definition of $E(US)$ for the bipartite setting.

Ch. 4 & Ch. 5 deal with our work related to the construction of AME states. Ch. 4 deals with the nonlinear maps defined in (10) to create 2-unitaries, and thus AME(4,d) and probes the efficiency of such maps. Our aim in probing the efficiency of these maps is to show that the utility of these maps is limited to small d. Thus, the construction of AME states of the form $(4, d)$ for higher d itself **cannot** be done using these maps —let alone the construction of AME(N, d) for $N > 4$. In Ch. 5, we describe the manner in which we extend the nonlinear maps defined in (10) to settings where $N > 4$. We then describe the manner in which these maps are used to generate *3-unitaries*[1] of size 8×8. We then describe our approach to constructing AME$(8, 4)$ —which is an **open problem** in quantum information. Our approaches to constructing AME(8,4) fail due to a lack of rigour —but succeed in motivating further studies in operator entanglement.

[1]Defined as an extension of 2-unitaries in Ch. 2

CHAPTER 2

OPERATOR-STATE ISOMORPHISMS

2.1 INTRODUCTION

The importance of operator-state isomorphisms lies in its utility in translating problems in the operator space to that in the state space —and vice-versa. Insights can be gained into the nature of the less studied operator space through observations made regarding the isomorphic state space.

In this chapter, we summarize well-known aspects of the correspondence between operators and states, viz. the Choi-Jamialkowski isomorphism and discuss the isomorphism stated in (10) as a modified version of the CJ isomorphism for bipartite operators. We then use this isomorphism to prove the relations between the bipartite operator entanglement entropies, entanglement entropies of the isomorphic state and the matrix reshapings defined in Ch. 1. We then leverage the modified CJ isomorphism to restate the k_D problem, worked on by Shrigyan Brahmachari, as a problem on the distance measure between states.

We then **generalise the operator-state isomorphism** defined in (10) **to multipartite operators**. The consequence of this result is that it provides a natural isomorphism between multipartite operators and states.

We then examine the implications of the multipartite operator-state isomorphism vis-a-vis the k-uniformity of states and AME states. In particular, the multipartite operator-state isomorphism permits the construction of AME$(2N, d)$ through the construction of a perfect tensor of the form $d^N \times d^N$. We also state a construction scheme for the construction of AME states of the order AME$(2N - 1, d)$ —thereby permitting the construction of AME states of both an even and an odd number of parties.

To motivate our focus on AME states, we provide a brief description of its utility. We then **coin a term**, '**k-uniform unitary operator**', to describe unitary operators which under the multipartite operator-state isomorphism defined by us, is isomorphic to a k-uniform state. We conjecture about the behaviour of such unitaries under *any* other operator-state isomorphism that can be defined. Should our conjecture be true, it strengthens our definition of k-uniformity among unitary operators as the definition becomes independent of the isomorphism chosen.

Utilising the operator-state isomorphism, we also state and prove a theorem on the existence of k-uniform states, stating that if a k-uniform state on 2N parties, each of local dimension d exists, there exists a (k-1)-uniform state on $(2N-1)$ parties, each of local dimension d. This theorem is of **immense consequence** in that it implies that **AME**$(2N-1, d)$ **exists if AME**$(2N, d)$ **exists**. The fact that we just need to prove the existence of AME states of even dimensions, or the non-existence of AME states of odd dimensions greatly simplifies the AME existence problem and is a result of great significance given the magnitude of the importance of AME states and their diverse applications in quantum information. To increase our emphasis on the utility of this result, we also provide a brief motivation for the construction of AME states —as a sort of precursor hinting at our work on AME states, described in Ch. 5, along with a detailed overview of the application of AME states.

2.2 THE CHOI-JAMIALKOWSKI ISOMORPHISM

The association between the entangling measures of unitary operators and the matrix reshapings as stated in Ch. 1 can be shown using the correspondence between operators and states (14), which essentially leverages the Choi-Jamialkowski isomorphism (1)(6).

Linear operators acting on states in a d dimensional Hilbert space, $\mathcal{H}^d$ themselves form a d^2 dimensional Hilbert space. Any such operator, X, can thus be mapped to a state

$|X\rangle \in \mathcal{H}_A^d \otimes \mathcal{H}_{A'}^d$, as:—

$$|X\rangle = \sum_{ij} X_{ij}|ij\rangle = \frac{1}{\sqrt{d}}\sum_{ij}\langle i|X|j\rangle|i\rangle|j\rangle = \frac{1}{\sqrt{d}}\sum_{j} X|jj\rangle$$
$$= (X \otimes \mathbb{I})\,|\Phi^+\rangle_{AA'}, \quad |\Phi^+\rangle \coloneqq \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle \;\;\&\;\; \langle i|X|j\rangle \coloneqq \sqrt{d}\cdot X_{ij}, \tag{2.1}$$

and $\{|i\rangle\}_1^d$ being an orthonormal basis in $\mathcal{H}^d$. This isomorphism corresponds to the lexicographical vectorization of the operator, X (See Appendix).

2.3 BIPARTITE OPERATOR ENTANGLEMENT & MATRIX RESHAPINGS

The results of this section are shown in Suhail et al. (10). But the proofs and the relations defined are of importance with respect to the generalization of operator entanglement to multipartite settings. Thus, we state these results in detail here along with the proof techniques for the same.

To bridge the gap between bipartite operator entanglements & the matrix reshapings and prove the relationships defined in Ch. 1 for 4 parties, we consider the operator-state isomorphism for a bipartite unitary, U_{AB} defined over a Hilbert space, $\mathcal{H}_A^d \otimes \mathcal{H}_B^d$. This isomorphism is an extension of the CJ isomorphism and was stated by Suhail et al. (10). Such an operator will be isomorphic to a 4-party state in $\mathcal{H}_A^d \otimes \mathcal{H}_{A'}^d \otimes \mathcal{H}_B^d \otimes \mathcal{H}_{B'}^d$, $|U\rangle_{AA'BB'}$, defined as:—

$$|U\rangle_{AA'BB'} \coloneqq (U_{AB} \otimes \mathbb{I}_{A'B'})|\Phi^+\rangle_{AA'}|\Phi^+\rangle_{BB'}. \tag{2.2}$$

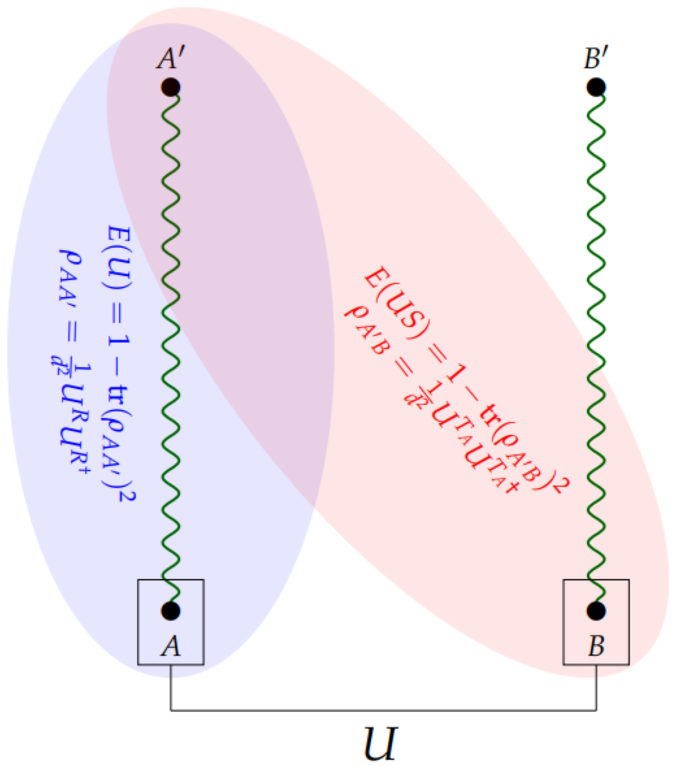

Figure 2.1: U acts on $\mathcal{H}_A \otimes \mathcal{H}_B$. It is isomorphic to the 4-party state, $|U\rangle_{AA'BB'}$. The 'squiggly' lines indicate maximum bipartite entanglement in the initial state —with A,A' & B,B' being maximally entangled. The relationship between the bipartite operator entanglements E(U) and E(US) with the corresponding reshapings is also shown. Figure from Suhail et al. (10)

Given such an operator-state equivalence between bipartite operators and 4-party states, we can define **3 unique** bipartitions of the state with equal number of parties on each side:—

1. $AB \mid A'B'$: The entanglement in this bipartition is determined by the reduced density matrix ρ_{AB}[1]$=\frac{1}{d^2}UU^\dagger = \frac{1}{d^2}\mathbb{I}_{d^2}$ —viz. the maximally mixed state. Thus, the bipartition $AB \mid A'B'$ is maximally entangled. This can be seen from Eq. 2.2 directly as $AB \mid A'B'$ is **already** maximally entangled and U is local to this bipartition and thus, does **not** alter the entanglement in this bipartition.

2. $AA' \mid BB'$: The entanglement in this bipartition is determined by the reduced density matrix $\rho_{AA'}$[2]$= \frac{1}{d^2}U^{\mathcal{R}}U^{\mathcal{R}\dagger}$, where $U^{\mathcal{R}}$ is the realignment of U. Thus, the entanglement in this bipartition is maximal iff. $U^{\mathcal{R}}$ is unitary —i.e. if U is

[1]And by $\rho_{A'B'}$ as for a pure state, any of the entanglement entropies calculated along a bipartite cut will be equal.

[2]And by $\rho_{BB'}$ (See footnote 1)

dual-unitary.

We can calculate the entanglement in this bipartition using the linear entropy ($S_2(\rho)$) as an entanglement measure of state. We pick the linear entropy as the entanglement measure as it directly relates to the operator entanglement entropies of U as :—

$$S_2(\rho_{AA'}) = 1 - \mathrm{Tr}(\rho_{AA'}^2) = 1 - \frac{1}{d^4}\mathrm{Tr}((U^{\mathcal{R}}U^{\mathcal{R}\dagger})^2). \tag{2.3}$$

Now, if $\{\sqrt{\lambda_i}\}_{i=1}^{d^2}$ are the Schmidt coefficients of U, i.e.

$$U = \sum_{i=1}^{d^2} \sqrt{\lambda_i}\, m_i^A \otimes m_i^B, with \sum_{i=1}^{d^2} \lambda_i = d^2$$
$$\implies U^{\mathcal{R}} = \sum_{i=1}^{d^2} \sqrt{\lambda_i} |m_i^A\rangle\langle m_i^B|,$$

where $|m_i^A\rangle$ & $|m_i^B\rangle$ are the lexicographical vectorizations of m_i^A & m_i^B.

Thus, Eq. 2.3 simplifies to:—

$$U^{\mathcal{R}}U^{\mathcal{R}\dagger} = \sum_{i=1}^{d^2} \lambda_i |m_i^A\rangle\langle m_i^A|$$
$$\implies (U^{\mathcal{R}}U^{\mathcal{R}\dagger})^2 = \sum_{i=1}^{d^2} \lambda_i^2 |m_i^A\rangle\langle m_i^A|$$
$$\therefore\ S_2(\rho_{AA'}) = 1 - \frac{1}{d^4}\sum_{i=1}^{d^2} \lambda_i^2 = E(U).$$

Thus, the operator entanglement measure, $E(U)$ is **equal** to the linear entropy of entanglement of the vectorized version of U, $|U\rangle_{AA'BB'}$ with respect to the bipartition $AA' \mid BB'$.

3. $A'B \mid AB'$: The entanglement in this bipartition is determined by the reduced density matrix $\rho_{A'B}$ [3]$= \frac{1}{d^2}U^{\Gamma}U^{\Gamma\dagger}$, where U^{Γ} is the partial transpose of U. Thus, the entanglement in this bipartition is maximal iff. U^{Γ} is unitary —i.e. if U is T-dual-unitary.

   Analogous to the bipartition $A'A \mid BB'$, we can calculate the entanglement in this bipartition using the linear entropy ($S_2(\rho)$) as an entanglement measure of state. We show that:—

$$S_2(\rho_{A'B}) = 1 - \mathrm{Tr}(\rho_{A'B}^2) = 1 - \frac{1}{d^4}\mathrm{Tr}((U^{\Gamma}U^{\Gamma\dagger})^2).$$

[3]And by $\rho_{AB'}$ (See footnote 1)

Now, if $\{\sqrt{\mu_i}\}_{i=1}^{d^2}$ are the Schmidt coefficients of US[4], i.e.

$$US = \sum_{i=1}^{d^2} \sqrt{\mu_i}\, m_i^A \otimes m_i^B.$$

By definition :—

$$\begin{aligned} \langle ij|US|kl\rangle &:= \langle ij|U|lk\rangle. \\ \langle ij|U|kl\rangle &:= \langle il|(U^\Gamma)^T|kj\rangle. \end{aligned}$$

Thus, the following steps follow :—

$$\begin{aligned} \langle ij|US|kl\rangle &= \langle ij|U|lk\rangle \\ &= \langle ik|(U^\Gamma)^T|lj\rangle \\ &= \langle ik|(U^\Gamma)^T|jl\rangle \\ &= \langle ij|(U^\Gamma)^T S)^{\mathcal{R}}|kl\rangle \end{aligned}$$

$$US = ((U^\Gamma)^T S)^{\mathcal{R}} \tag{2.4}$$

Thus, by analogy with the equivalence proven between E(U) and $S_2(\rho_{AA'})$, leveraging Eq. 2.4:—

$$\begin{aligned} S_2(\rho_{A'B}) &= 1 - \frac{1}{d^4}\,\mathrm{Tr}((U^\Gamma U^{\Gamma\dagger})^2) \\ &= 1 - \frac{1}{d^4}\,\mathrm{Tr}(((U^\Gamma)^T)S(U^\Gamma S)^\dagger)^2) \\ &= 1 - \frac{1}{d^4}\,\mathrm{Tr}(((US)^{\mathcal{R}}(US)^{\mathcal{R}\dagger})^2) \end{aligned}$$

$$S_2(\rho_{A'B}) = 1 - \frac{1}{d^4}\sum_{i=1}^{d^2}\mu_i^2 = E(US). \tag{2.5}$$

Thus, the operator entanglement measure, $E(US)$ is **equal** to the linear entropy of entanglement of the vectorized version of U, $|U\rangle_{AA'BB'}$ with respect to the bipartition $A'B \mid AB'$.

This connection between the operator entanglement entropies & the matrix reshapings has been formalised for bipartite operators. However, a generalisation of the operator

[4]Swap operator on 2 qudits

entanglement entropies to multipartite systems is, at the time of writing, unknown. In the rest of the chapter, we attempt at a generalisation of the concept of operator entanglement entropies to multipartite operators.

2.4 MULTIPARTITE OPERATOR-STATE ISOMORPHISMS

[5] In this section, we define an operator-state isomorphism for multipartite operators, extending the bipartite operator-state isomorphism stated in Sec. 2 to multipartite operators, show its equivalence to a vectorization of the operator and state a few of its general implications. The extension of the operator-state isomorphism finds its utility in AME state construction —as will be shown in Ch. 4 —and as a general mathematical tool.

2.4.1 Definition

Consider an N-partite unitary, $U_{A_1A_2\cdots A_N}$, defined over a Hilbert space, $\mathcal{H}^d_{A_1} \otimes \mathcal{H}^d_{A_2} \otimes \cdots \otimes \mathcal{H}^d_{A_N}$. Such an operator will be isomorphic to a 2N-party state in $\mathcal{H}^d_{A_1} \otimes \mathcal{H}^d_{A'_1} \cdots \otimes \mathcal{H}^d_{A_N} \otimes \mathcal{H}^d_{A'_N}$, $|U\rangle_{A_1A'_1\ldots A_NA'_N}$, defined as:—

$$|U\rangle_{A_1A'_1\ldots A_NA'_N} := U_{A_1A_2\ldots A_N} \otimes \mathbb{I}_{A'_1A'_2\ldots A'_N}\, |\Phi^+\rangle_{A_1A'_1}|\Phi^+\rangle_{A_2A'_2}\cdots|\Phi^+\rangle_{A_NA'_N} \tag{2.6}$$

2.4.2 The Isomorphism As A Vectorization

In this section, we re-frame the operator-state isomorphism as a vectorization of the unitary operator —in a manner similar to the lexicographical vectorization (See Appendix) of the unitary to produce the isomorphic state in the case of the CJ isomorphism. This view of the multipartite operator-state isomorphism and the understanding of the process of vectorization as a natural rearrangement of indices is essential mathematical background for the remainder of this chapter and is a useful restatement or interpretation of our

[5]This symbol means that the results stated within **the section** were proven, stated or derived by us.

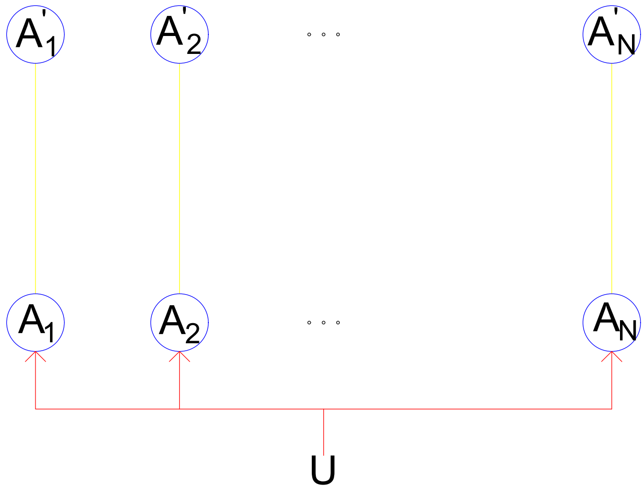

Figure 2.2: U acts on $A_1A_2\ldots A_N$. It is isomorphic to the 2N-party state, $|U\rangle_{A_1A_1'\ldots A_NA_N'}$. The lines in yellow indicate maximum bipartite entanglement in the initial state —with A_iA_i' being maximally entangled $\forall i \in [1:N]$.

isomorphism.

Consider an N-partite unitary, $U_{A_1A_2\cdots A_N}$, defined over a Hilbert space, $\mathcal{H}^d_{A_1} \otimes \mathcal{H}^d_{A_2} \otimes \cdots \otimes \mathcal{H}^d_{A_N}$, with its isomorphic 2N-party state in $\mathcal{H}^d_{A_1} \otimes \mathcal{H}^d_{A_1'} \cdots \otimes \mathcal{H}^d_{A_N} \otimes \mathcal{H}^d_{A_N'}$ as $|U\rangle_{A_1A_1'\ldots A_NA_N'}$.

Let $\{i_\alpha\}_{i=0}^{d-1}$ be an orthonormal basis in $\mathcal{H}^d_{A_\alpha}$ and $\{j_\alpha\}_{j=0}^{d-1}$ be an orthonormal basis in $\mathcal{H}^d_{A_\alpha'}$.

Then,

$$
\begin{aligned}
\langle i_1 j_1 \ldots i_N j_N | U\rangle_{A_1A_1'\ldots A_NA_N'} &= \langle i_1 j_1 \ldots i_N j_N | U_{A_1A_2\ldots A_N} \otimes \mathbb{I}_{A_1'A_2'\ldots A_N'} \, |\Phi^+\rangle_{A_1A_1'} |\Phi^+\rangle_{A_2A_2'} \cdots |\Phi^+\rangle_{A_NA_N'} \\
&= \sum_{a_1\ldots a_N}^{d} \frac{1}{d^{\frac{N}{2}}} \langle i_1 \ldots i_N | U | a_1 \ldots a_N\rangle \langle j_1 | a_1\rangle \ldots \langle j_N | a_N\rangle \\
&= \frac{1}{d^{\frac{N}{2}}} \langle i_1 \ldots i_N | U | j_1 \ldots j_N\rangle.
\end{aligned}
$$

i.e.

$$\langle i_1 j_1 i_2 j_2 \dots i_N j_N | U \rangle_{A_1 A'_1 \cdots A_N A'_N} = \frac{1}{d^{N/2}} \langle i_1 i_2 \dots i_N | U_{A_1 \cdots A_N} | j_1 j_2 \dots j_N \rangle \tag{2.7}$$

2.4.3 Implications

The implication of the multipartite operator-state isomorphism is that it permits the study of operator partitions in terms of partitions in the state. It allows us to translate problems in the lesser studied operator spaces to equivalent problems in the corresponding state spaces. This is especially of utility in the study of operator entanglement and in the extension of the ideas of operator entanglement entropies and the special classes of duals, T-duals & 2-unitaries to multipartite operators. In the rest of this chapter, we utilize the operator-state isomorphism to define and derive these extensions.

2.4.4 Reduced State Density Operators & Operator Matrix Reshapings

In this subsection, we define a procedure to arrive at the reduced density operator of a state in terms of reshapings of the matrix form of the isomorphic operator. This is a neat trick in that it allows us to directly construct the reduced density operator with respect to **any** bipartition in terms of reshapings of the matrix form of the operator.

Theorem 2.4.1. *Consider an N-partite unitary, $U_{A_1 A_2 \cdots A_N}$, defined over a Hilbert space, $\mathcal{H}^d_{A_1} \otimes \mathcal{H}^d_{A_2} \otimes \cdots \otimes \mathcal{H}^d_{A_N}$ that is isomorphic to the 2N-party state, $|U\rangle_{A_1 A'_1 \dots A_N A'_N}$ in $\mathcal{H}^d_{A_1} \otimes \mathcal{H}^d_{A'_1} \cdots \otimes \mathcal{H}^d_{A_N} \otimes \mathcal{H}^d_{A'_N}$. Without any loss of generality, we take the bipartition $A_1 A_2 \dots A_p A'_{a_1} A'_{a_2} \dots A'_{a_q} | A_{p+1} \dots A_N A'_{a_{q+1}} \dots A'_{a_N}$, where $\{a_r\}_{r=1}^N$ is a permutation of $\{r\}_{r=1}^N$. Then, the reduced density matrix obtained on tracing out $A_{p+1} \dots A_N A'_{a_{q+1}} \dots A'_{a_N}$, ρ' is given by:—*

$$\rho' := \mathrm{Tr}_{A_{p+1} \dots A_N A'_{a_{q+1}} \dots A'_{a_N}} [\rho_{A_1 A'_1 \dots A_N A'_N}] = \frac{1}{d^N} U^{(x)} \cdot U^{(x)\dagger}, \tag{2.8}$$

where $U^{(x)}$ is a matrix reshaping of U, defined as:—

$$\langle i_1 \dots i_p j_{a_1} \dots j_{a_q} | U^{(x)} | i_{p+1} \dots i_N j_{a_{q+1}} \dots j_{a_N} \rangle = \langle i_1 \dots i_N | U | j_1 \dots j_N \rangle. \tag{2.9}$$

Note:— $U^{(x)}$ is a $d^{p+q} \times d^{2N-(p+q)}$ operator. If the bipartition is maximally entangled, $U^{(x)}$ will be proportional to an isometry, i.e.

$$U^{(x)} \cdot U^{(x)\dagger} = d^{N-(p+q)} \mathbb{I}_{d^{(p+q)}}. \tag{2.10}$$

We could have removed this proportionality factor by adding this in the definition of $U^{(x)}$ as well.

Proof. In this proof, we adopt the Einstein summation convention.

$$\begin{aligned}
\rho' &= |i_1 \dots i_p j_{a_1} \dots j_{a_q}\rangle \langle i_1 j_1 \dots i_N j_N | U \rangle_{A_1 A_1' \dots A_N A_N'} \\
&\qquad {}_{A_1 A_1' \dots A_N A_N'}\langle U | i_1' \dots i_p' i_{p+1} \dots i_N j_{a_1}' \dots j_{a_q}' j_{a_{q+1}} \dots j_{a_N} \rangle \langle i_1' \dots i_p' j_{a_1}' \dots j_{a_q}' | \\
&= \frac{1}{d^N} |i_1 \dots i_p j_{a_1} \dots j_{a_q}\rangle \langle i_1 \dots i_N | U | \alpha_1 \dots \alpha_N \rangle \langle j_1 | \alpha_1 \rangle \dots \langle j_N | \alpha_N \rangle \langle \beta_1 \dots \beta_N | U^\dagger | i_1' \dots i_p' i_{p+1} \dots i_N \rangle \\
&\qquad \langle \beta_{a_1} | j_{a_1}' \rangle \dots \langle \beta_{a_q} | j_{a_q}' \rangle \langle \beta_{a_{q+1}} | j_{a_{q+1}} \rangle \dots \langle \beta_{a_N} | j_{a_N} \rangle \langle i_1' \dots i_p' j_{a_1}' \dots j_{a_q}' | \\
&= \frac{1}{d^N} |i_1 \dots i_p j_{a_1} \dots j_{a_q}\rangle \langle i_1 \dots i_N | U | j_1 \dots j_N \rangle \langle j_{a_1}' \dots j_{a_q}' j_{a_{q+1}} \dots j_{a_N} | U^\dagger | i_1' \dots i_p' i_{p+1} \dots i_N \rangle \\
&\qquad \langle i_1' \dots i_p' j_{a_1}' \dots j_{a_q}' | \\
&= \frac{1}{d^N} |i_1 \dots i_p j_{a_1} \dots j_{a_q}\rangle \langle i_1 \dots i_p j_{a_1} \dots j_{a_q} | U^{(x)} | i_{p+1} \dots i_N j_{a_{q+1}} \dots j_{a_N} \rangle \\
&\qquad \langle i_{p+1} \dots i_N j_{a_{q+1}} \dots j_{a_N} | U^{(x)\dagger} | i_1' \dots i_p' j_{a_1}' \dots j_{a_q}' \rangle \langle i_1' \dots i_p' j_{a_1}' \dots j_{a_q}' | \\
&= \frac{1}{d^N} U^{(x)} \cdot U^{(x)\dagger}
\end{aligned}$$

∎

Thus, to construct the operator matrix reshaping which corresponds to a certain bipartition of the isomorphic state, we associate each of the 2N parties of the state with an index in $\{i_r, j_r\}_{r=1}^{N}$. The mapping is as follows :—

- A_r corresponds to the index i_r
- A'_r corresponds to the index j_r.

Let $f : A_r \to i_r$ & $f : A'_r \to j_r$. Then, the reduced density operator corresponding to the bipartition, $S|S^c$ where $S \subset \{A_1 A'_1 \ldots A_N A'_N\}$ is given by :—

$$\rho'_S = \frac{U^{(x)} \cdot U^{(x)\dagger}}{d^N}, \tag{2.11}$$

where,

$$\langle f^{\otimes|S|}(S)|U^{(x)}|f^{\otimes|S^c|}(S^c)\rangle = \langle i_1 \ldots i_N|U|j_1 \ldots j_N\rangle. \tag{2.12}$$

2.5 REINTERPRETING THE K_D PROBLEM USING OPERATOR-STATE ISOMORPHISMS

[This section was developed in conjunction with Shrigyan Brahmachari]

In this section, we re-imagine the k_D problem in an attempt to convert it to a form that is better studied and thus illustrate the utility of the operator-state isomorphisms.

We can view the k_D problem through the lens of the operator-state isomorphism to view it as a distance problem on restricted classes of quantum states. Our motivation to convert the k_D problem to a problem on quantum states stemmed from an attempt to convert the problem on operator distances to one on the distance between states —which is more commonly studied and better understood.

To convert the k_D problem to one on states, we use the modified version of the CJ isomorphism. Note that the modification in the CJ isomorphism and its use here is natural given that the k_D problem, as defined in Ch. 1, deals with bipartite unitary operators.

So, we vectorize the unitary operator matrix using the bipartite operator-state isomorphism, U_{AB} defined on $\mathcal{H}_A^d \otimes \mathcal{H}_B^d$ to obtain a state, $|U\rangle_{AA'BB'}$ in $\otimes_{i=1}^{4}\mathcal{H}_i^d$,

$$|U\rangle_{AA'BB'} = (U_{AB} \otimes \mathbb{I}_{A'B'})|\phi^+\rangle_{AA'}|\phi^+\rangle_{BB'}.$$

Similarly, the local unitaries are vectorized. Then, the equivalent distance measure is given by:—

$$k_D(U) = \min_{p,q} \|U - p \otimes q\|^2 = \min_{|p\rangle,|q\rangle} \||U\rangle - |p\rangle_{AA'}|q\rangle_{BB'}\|^2, \tag{2.13}$$

where $|p\rangle$ & $|q\rangle$ are maximally entangled states & $|U\rangle$ is maximally entangled with respect to the bipartition $AB|A'B'$.

Generalization to a multipartite distance measure:— The generalization of the k_D measure to multipartite operators is speculated to be a projection problem of a general unitary on N parties to a product of local unitaries. The conversion of this 'multipartite k_D' to the state picture would necessitate the usage of our multipartite operator-state isomorphism.

2.6 MULTI-UNIFORMITY & MULTI-UNITARITY

In this section, we leverage the multipartite operator-state isomorphism defined in Sec. 4 to define multi-uniform unitary operators and describe multi-unitary operators as a special class of multi-uniform unitary operators. We state certain well-studied properties of k-uniform and AME states, and expound on their implications on the corresponding multi-uniform unitaries. We then state and prove novel theorems regarding the existence of k-uniform and AME states that have far-reaching implications with respect to the problem of their construction.

2.6.1 k-Uniform States (11)

In this subsection, we state the definition of k-uniform states and some common properties —including a theorem on the separability of k-uniform states that we state and prove.

2.6.1.1 Definition

A k-uniform state $|\psi\rangle$ of N parties, each of local dimension d in $\mathcal{H}^d_{A_1} \otimes \cdots \otimes \mathcal{H}^d_{A_N}$ is one for which **any reductions** to k parties is maximally mixed. i.e. —

$$\rho_S := \mathrm{Tr}_{S^c} |\psi\rangle\langle\psi| = \frac{1}{d^k}\mathbb{I}_S, \tag{2.14}$$

where $S = \{A_{j_1}, A_{j_2}, \ldots, A_{j_k}\} \subset \{A_1, A_2, \ldots, A_N\}$ & $S^c = \{A_1, A_2, \ldots A_N\} \setminus S$

2.6.1.2 Properties

Theorem 2.6.1. *(11) A k-uniform state $|\psi\rangle$ of N parties is k'-uniform $\forall\ k' \leq k$.*

Theorem 2.6.2. *(11) Local unitary products do **not** alter k-uniformity. If $|\psi\rangle_{A_1A_2\ldots A_N}$ is a k-uniform state of N parties:*

$$|\varphi\rangle = u_{A_1} \otimes u_{A_2} \otimes \ldots \otimes u_{A_N} |\psi\rangle_{A_1A_2\ldots A_N},$$

is a k-uniform state of N parties.

Theorem 2.6.3. *(11) For N parties, $\exists$ **no** state, $|\psi\rangle$ that is k-uniform for any $k > \lfloor \frac{N}{2} \rfloor$.*

Theorem 2.6.4. *(11)Let $|\psi\rangle_{A_1A_2\ldots A_N}$ and $|\phi\rangle_{B_1B_2\ldots B_N}$ be k-uniform states in $(C^{d_1})^{\otimes N}$ and $(C^{d_2})^{\otimes N}$ respectively. Then, their tensor product, i.e.*

$$|\varphi\rangle_{(A_1B_1),\ldots,(A_NB_N)} = |\psi\rangle_{A_1A_2\ldots A_N} \otimes |\phi\rangle_{B_1B_2\ldots B_N},$$

is a k-uniform state of N parties on $C^{d_1d_2}$.

Theorem 2.6.5. *(12) For a local dimension, d that is a prime power, there exists a k-uniform state in $(C^d)^{\otimes N}$ for the following conditions:—*

1. $d \geq 2k - 1$ & $N \in [2k, d + 1]$

2. $d \geq 4k - 2$ & $N \geq 2k$

Theorem 2.6.6 (The Separability of k-Uniform States)**.** *The density matrix of a k-uniform state, ρ of N parties can be written as the tensor product of 2 density operators* ***corresponding to pure states*** *of N_1 & N_2 parties respectively, with $N_1 + N_2 = N$,* ***if:***

$$k \leq \lfloor \frac{\lfloor \frac{N}{2} \rfloor}{2} \rfloor \tag{2.15}$$

Proof. Let us assume that $\rho_N = \rho 1_{N_1} \otimes \rho 2_{N_2}$. Since ρ_N is k-uniform, any reduction to k parties must be maximally mixed. For a general reduction to k parties, let us assume that k_1 & k_2 parties from $\rho 1_{N_1}$ and $\rho 2_{N_2}$ are left : $k_1 + k_2 = k$ with the reduced partial density matrices being ρ_1' & ρ_2' respectively. Then,

$$\rho_1' \otimes \rho_2' = \frac{1}{d^k} \mathbb{I}_{d^k}$$

Now, this must be true for **any** k_1, k_2 such that $k_1 + k_2 = k$. Since k-uniformity implies k'-uniformity for $k' \leq k$, we just need to consider the extreme cases —$k_1 = k, k_2 = 0$ & $k_1 = 0, k_2 = k$.

$\therefore$ ρ_1 & ρ_2 must be, *at the least*, **k-uniform** states themselves.

To derive the necessary conditions, we assume that ρ_1 & ρ_2 are absolutely, maximally entangled (AME) states[6] of N_1 & N_2 parties respectively. Then,

$$k \leq \min\{\lfloor \frac{N_1}{2} \rfloor, \lfloor \frac{N_2}{2} \rfloor\}$$

We need this to be true for *at least* **one** set of N_1, N_2. Thus, the optimal case that allows us to impose a condition of necessity is when $N_1 = \lfloor \frac{N}{2} \rfloor$ & thus, $N_2 = \lceil \frac{N}{2} \rceil$. Thus:—

$$k \leq \min\{\lfloor \frac{\lfloor \frac{N}{2} \rfloor}{2} \rfloor, \lfloor \frac{\lceil \frac{N}{2} \rceil}{2} \rfloor\} = \lfloor \frac{\lfloor \frac{N}{2} \rfloor}{2} \rfloor.$$

∎

[6]defined in Sec. 5.3

2.6.2 k-Uniform Unitary Operators

In this subsection, we introduce and define the term ‘k-uniform unitary operator’ and state and prove some of its properties.

2.6.2.1 Definition

We define a unitary operator, U, over N parties, each of local dimension d, i.e. $\mathcal{H}^d_{A_1} \otimes \cdots \otimes \mathcal{H}^d_{A_N}$ to be a k-uniform unitary operator if its vectorization, $|U\rangle_{A_1A'_1\ldots A_NA'_N}$:—

$$|U\rangle_{A_1A'_1\ldots A_NA'_N} = U_{A_1A_2\ldots A_N} \otimes \mathbb{I}_{A'_1A'_2\ldots A'_N}\ |\Phi^+\rangle_{A_1A'_1}|\Phi^+\rangle_{A_2A'_2}\cdots|\Phi^+\rangle_{A_NA'_N},$$

is a k-uniform state on 2N parties.

Note:—We defined the notion of a k-uniform unitary operator with respect to the operator-state isomorphism used predominantly in our work. We do note that other operator-state isomorphisms may be constructed which lead to states that are **not** k-uniform as the k-uniformity of states is **not** preserved under isomorphisms. However, we conjecture that the *maximum* bipartite entanglement that can be present in the isomorphic state is that in the state isomorphic to the unitary operator under the multipartite operator-state isomorphism defined —thus motivating and strengthening our definition of k-uniform unitary operators.

2.6.2.2 Properties

Conjecture 2.6.7. *If $\exists$ an operator state isomorphism $f : \mathcal{H}^d \otimes \mathcal{H}^d \rightarrow \mathcal{H}^{d^2}$ and a unitary operator, U_0 in $\mathcal{H}^d \otimes \mathcal{H}^d$ is a k-uniform unitary operator under the multipartite operator-state isomorphism defined in Ch. 3 Sec. 3 such that $f(U_0) = |U_0^f\rangle$, then, if $|U_0^f\rangle$ is k'-uniform, then $k' \leq k$.*

Our conjecture is motivated by the fact that under our isomorphism, a significant number of reductions to k-parties is maximally mixed for any $k \leq \lfloor N/2 \rfloor$ as $A_1 \ldots A_N|A'_1 \ldots A'_N$ is maximally mixed.

Note:— If k-uniformity was preserved under isomorphisms, then $k' = k$ for all isomorphisms.

Theorem 2.6.8. *For N parties, $\nexists$ any k-uniform unitary operator for $k > \lfloor \frac{N}{2} \rfloor$.*

Proof. By definition, a k-uniform unitary operator defined on N parties is isomorphic to a k-uniform state of 2N parties. However, $\nexists$ *any* k-uniform state of 2N parties for $k > \lfloor \frac{N}{2} \rfloor$.Thus, if $\exists$ a k-uniform unitary operator on N parties, $k \leq \lfloor \frac{N}{2} \rfloor$. ∎

Theorem 2.6.9. *If U is a k-uniform unitary operator defined on N parties, each of local dimension d i.e. $\mathcal{H}^d_{A_1} \otimes \cdots \otimes \mathcal{H}^d_{A_N}$, then $|U\rangle_{A_1A'_1\ldots A_{N-1}A'_{N-1}A_N}$:—*

$$|U\rangle_{A_1A'_1\ldots A_{N-1}A_{N'-1}A_N} = U_{A_1A_2\ldots A_N} \otimes \mathbb{I}_{A'_1A'_2\ldots A_{N'-1}} |\Phi^+\rangle_{A_1A'_1} |\Phi^+\rangle_{A_2A'_2} \cdots |\Phi^+\rangle_{A_{N-1}A'_{N-1}} |0\rangle_{A_N},$$

is a (k-1)-uniform state on 2N-1 parties.

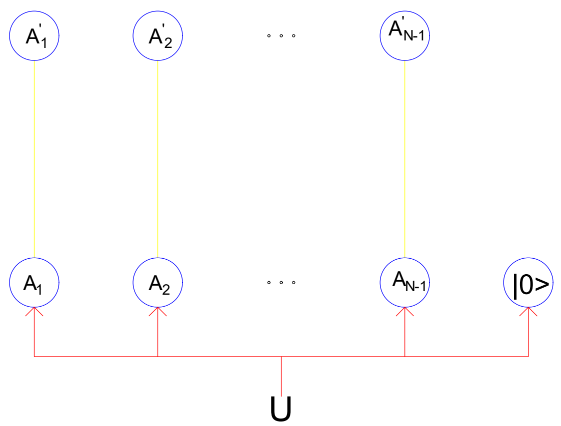

Figure 2.3: U acts on $A_1A_2\ldots A_N$. It is mapped to a (2N-1)-party state, $|U\rangle_{A_1A'_1\ldots A_{N-1}A'_{N-1}A_N}$. The lines in yellow indicate maximum bipartite entanglement in the initial state —with $A_iA'_i$ being maximally entangled $\forall i \in [1 : N-1]$. The A_N party is in the $|0\rangle$ state. Generalization to A_N being $|\psi\rangle_N$ is possible due to the fact that local unitaries do **not** affect entanglement.

Proof. Once again, we adopt the Einstein summation convention for this proof.

(k-1)-uniformity means that any reduction **to** k-1 parties should be maximally mixed.

We consider two cases :—

1. Without any loss of generality, tracing out all parties apart from $\{A_1 \dots A_{k_1}, A'_{a_1} \dots A_{a_{k_2}}\}'$ such that $k_1 + k_2 = k - 1$ where $\{a_r\}_{r=1}^{N-1}$ is a permutation of $\{1, 2 \dots N - 1\}$.

2. Without any loss of generality, tracing out all parties apart from $\{A_1 \dots A_{k_1}, A_N, A'_{a_1} \dots A'_{a_{k_2}}\}$ such that $k_1 + k_2 = k - 2$ where $\{a_r\}_{r=1}^{N-1}$ is a permutation of $\{1, 2 \dots N\}$.

Case 1:—

$$
\begin{aligned}
\rho' &:= \mathrm{Tr}_{A_{k_1+1}\dots A_N A'_{a_{k_2+1}}\dots A_{a'_{N-1}}}[\rho_{A_1 A'_1 \dots A_{N-1} A'_{N-1} A_N}] \\
&= |i_1 \dots i_{k_1} j_{a_1} \dots j_{a_{k_2}}\rangle\langle i_1 j_1 \dots i_{N-1} j_{N-1} i_N | U\rangle_{A_1 A'_1 \dots A_{N-1} A_{N-1}' A_N} \\
&\qquad {}_{A_1 A'_1 \dots A_{N-1} A'_{N-1} A_N}\langle U | i'_1 \dots i'_{k_1} i_{k_1+1} \dots i_N j'_{a_1} \dots j'_{a_{k_2}} j_{a_{k_2+1}} \dots j_{a_{N-1}}\rangle\langle i'_1 \dots i'_{k_1} j'_{a_1} \dots j'_{a_{k_2}}| \\
&= |i_1 \dots i_{k_1} j_{a_1} \dots j_{a_{k_2}}\rangle\langle i_1 \dots i_n j_{a_1} \dots j_{a_{N-1}} | U_{A_1 \dots A_N} \otimes \mathbb{I}_{A'_1 \dots A'_{N-1}} |\phi^+\rangle_{A_1 A'_1} \dots |\phi^+\rangle_{A_{N-1} A'_{N-1}} |0\rangle_{A_N} \\
&\quad {}_{A_1 A'_1}\langle\phi^+| \dots {}_{A_{N-1} A'_{N-1}}\langle\phi^+| \, {}_{A_N}\langle 0 | U^\dagger_{A_1 \dots A_N} \otimes \mathbb{I}_{A'_1 \dots A'_{N-1}} | i'_1 \dots i'_{k_1} i_{k_1+1} \dots i_N j'_{a_1} \dots j'_{a_{k_2}} j_{a_{k_2+1}} \dots j_{a_{N-1}}\rangle \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \langle i'_1 \dots i'_{k_1} j'_{a_1} \dots j'_{a_{k_2}}| \\
&= \frac{1}{d^N} |i_1 \dots i_{k_1} j_{a_1} \dots j_{a_{k_2}}\rangle\langle i_1 \dots i_N | U | j_1 \dots j_{N-1} 0\rangle\langle j'_{a_1} \dots j'_{a_{k_2}} j_{a_{k_2+1}} \dots j_{a_{N-1}} 0 | U^\dagger | i'_1 \dots i'_{k_1} i_{k_1+1} \dots i_N\rangle \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \langle i'_1 \dots i'_{k_1} j'_{a_1} \dots j'_{a_{k_2}}| \\
&= \frac{1}{d^N} |i_1 \dots i_{k_1} j_{a_1} \dots j_{a_{k_2}}\rangle\langle i_1 \dots i_{k_1} j_{a_1} \dots j_{a_{k_2}} 0 | U^{(x)} | i_{k_1+1} \dots i_N j_{a_{k_2}+1} \dots j_{a_{N-1}}\rangle \\
&\quad \langle i_{k_1+1} \dots i_N j_{a_{k_2}+1} \dots j_{a_{N-1}} | U^{(x)\dagger} | i'_1 \dots i'_{k_1} j'_{a_1} \dots j'_{a_{k_2}}\rangle\langle i'_1 \dots i'_{k_1} j'_{a_1} \dots j'_{a_{k_2}}| \\
&= \frac{1}{d^N} \, {}_{A_N}\langle 0 | U^{(x)} U^{(x)\dagger} | 0\rangle_{A_N},
\end{aligned}
$$

where :—

$$
\langle i_1 \dots i_{k_1} j_{a_1} \dots j_{a_{k_2}} j_N | U^{(x)} | i_{k_1+1} \dots i_N j_{a_{k_2}+1} \dots j_{a_{N-1}}\rangle = \langle i_1 \dots i_N | U | j_1 \dots j_N\rangle. \tag{2.16}
$$

But, this matrix reshaping corresponds to the 2N-party state bipartition —$A_1 \dots A_{k_1} A'_{a_1} \dots A'_{a_{k_2}} A'_N | A_{k_1+1} \dots A_N A'_{a_{k_2+1}} \dots A'_{a_{N-1}}$ and the size of the bipartition is $k|2N - k$.

Since U is a k-uniform unitary operator, the corresponding state of 2N parties will be k-uniform. Thus, $U^{(x)}$ will be proportional to an isometry as the reduced density operator of k states should be the maximally mixed state. Then,

$$\begin{aligned}
\rho' &= \frac{1}{d^N}\,{}_{A_N}\langle 0|U^{(x)} \cdot U^{(x)\dagger}|0\rangle_{A_N} \\
&= \frac{1}{d^{k-1}}\,{}_{A_N}\langle 0|\mathbb{I}_{d^k}|0\rangle_{A_N} \\
&= \frac{1}{d^{k-1}}\mathbb{I}_{d^{k-1}} \rightarrow \text{Maximally mixed state.}
\end{aligned}$$

Case 2:—

$$\begin{aligned}
\rho' &:= \mathrm{Tr}_{A_{k_1+1}\ldots A_{N-1}A'_{a_{k_2+1}}\ldots A_{a'_{N-1}}}[\rho_{A_1A'_1\ldots A_{N-1}A'_{N-1}A_N}] \\
&= |i_1\ldots i_{k_1}i_N j_{a_1}\ldots j_{a_{k_2}}\rangle\langle i_1\ldots i_N j_{a_1}\ldots j_{a_{N-1}}|U_{A_1\ldots A_N}\otimes \mathbb{I}_{A'_1\ldots A'_{N-1}}|\phi^+\rangle_{A_1A'_1}\ldots|\phi^+\rangle_{A_{N-1}A'_{N-1}}|0\rangle_{A_N} \\
&\quad {}_{A_1A'_1}\langle\phi^+|\ldots{}_{A_{N-1}A'_{N-1}}\langle\phi^+|\,{}_{A_N}\langle 0|U^\dagger_{A_1\ldots A_N}\otimes\mathbb{I}_{A'_1\ldots A'_{N-1}}|i'_1\ldots i'_{k_1}i_{k_1+1}\ldots i_{N-1}i'_N j'_{a_1}\ldots j'_{a_{k_2}}j_{a_{k_2+1}}\ldots j_{a_{N-1}}\rangle \\
&\qquad\qquad \langle i'_1\ldots i'_{k_1}i'_N j'_{a_1}\ldots j'_{a_{k_2}}| \\
&= \frac{1}{d^N}|i_1\ldots i_{k_1}i_N j_{a_1}\ldots j_{a_{k_2}}\rangle\langle i_1\ldots i_N|U|j_1\ldots j_{N-1}0\rangle\langle j'_{a_1}\ldots j'_{a_{k_2}}j_{a_{k_2+1}}\ldots j_{a_{N-1}}0|U^\dagger \\
&\qquad\qquad |i'_1\ldots i'_{k_1}i_{k_1+1}\ldots i_{N-1}i'_N\rangle\langle i'_1\ldots i'_{k_1}i'_N j'_{a_1}\ldots j'_{a_{k_2}}| \\
&= \frac{1}{d^N}|i_1\ldots i_{k_1}i_N j_{a_1}\ldots j_{a_{k_2}}\rangle\langle i_1\ldots i_{k_1}i_N j_{a_1}\ldots j_{a_{k_2}}0|U^{(x)}|i_{k_1+1}\ldots i_{N-1}j_{a_{k_2}+1}\ldots j_{a_{N-1}}\rangle \\
&\qquad\qquad \langle i_{k_1+1}\ldots i_{N-1}j_{a_{k_2}+1}\ldots j_{a_{N-1}}|U^{(x)\dagger}|i'_1\ldots i'_{k_1}i'_N j'_{a_1}\ldots j'_{a_{k_2}}\rangle\langle i'_1\ldots i'_{k_1}i'_N j'_{a_1}\ldots j'_{a_{k_2}}| \\
&= \frac{1}{d^N}\,{}_{A_N}\langle 0|U^{(x)(x)\dagger}|0\rangle_{A_N},
\end{aligned}$$

where :—

$$\langle i_1\ldots i_{k_1}i_N j_{a_1}\ldots j_{a_{k_2}}0|U^{(x)}|i_{k_1+1}\ldots i_{N-1}j_{a_{k_2}+1}\ldots j_{a_{N-1}}\rangle = \langle i_1\ldots i_N|U|j_1\ldots j_N\rangle. \tag{2.17}$$

But, this matrix reshaping corresponds to the $2N$-party state bipartition —$A_1 \ldots A_{k_1} A_N A'_{a_1} \ldots A'_{a_{k_2}} A'_N | A_{k_1+1} \ldots A_{N-1} A'_{a_{k_2+1}} \ldots A'_{a_{N-1}}$ and the size of the bipartition is $k|2N-k$.

Since U is a k-uniform unitary operator, the corresponding state of 2N parties will be k-uniform. Thus, $U^{(x)}$ will be proportional to an isometry as the reduced density operator of k states should be the maximally mixed state. Then,

$$
\begin{aligned}
\rho' &= \frac{1}{d^N} {}_{A_N}\langle 0| U^{(x)(x)\dagger} |0\rangle_{A_N} \\
&= \frac{1}{d^{k-1}} {}_{A_N}\langle 0| \mathbb{I}_{d^k} |0\rangle_{A_N} \\
&= \frac{1}{d^{k-1}} \mathbb{I}_{d^{k-1}} \rightarrow \text{Maximally mixed state.}
\end{aligned}
$$

The two cases together encompass *any* reduction to $(k-1)$ parties and thus, this proof shows that any such reduction leads to the maximally mixed state, i.e. the $2N-1$-party state, $|U\rangle_{A_1 A'_1 \ldots A_{N-1} A'_{N-1} A_N}$ is $(k-1)$-uniform. ■

Corollary 2.6.9.1. *The existence of a k-uniform state on $2N$-parties —each of local dimension d —implies the existence of a $(k-1)$-uniform state on $2N-1$ parties —each of local dimension d.*

Corollary 2.6.9.2. *If $U_{A_1 A_2 \ldots A_N}$ is a k-uniform unitary operator,*

$$|U\rangle_{A_1 A'_1 \ldots A_{N-1} A_{N'-1} A_N} = U_{A_1 A_2 \ldots A_N} \otimes \mathbb{I}_{A'_1 A'_2 \ldots A_{N'-1}} \, |\Phi^+\rangle_{A_1 A'_1} |\Phi^+\rangle_{A_2 A'_2} \cdots |\Phi^+\rangle_{A_{N-1} A'_{N-1}} |\psi\rangle_{A_N},$$

is a (k-1)-uniform state on (2N-1) parties and $|\psi\rangle$ is a pure state in $\mathcal{H}^d_{A_N}$.

Proof. This generalization is a direct extension of Thm. 2.6.2 where multiplication by local unitaries does **not** alter entanglement characteristics such as uniformity. ■

This theorem implies that vectorizing certain columns of a k-uniform unitary preserves correlations and gives further insight into the structure of k-uniform unitary operators in general.

This form of vectorization, coupled with the vectorization implemented in the definition of k-uniform unitary operators, permits the creation of k-uniform states of **any dimension**.

2.6.3 Absolutely, Maximally Entangled (AME) States

In this subsection, we state the definition of a special class of k-uniform states known as **AME states** —in a slightly different manner compared to the definition in Ch. 1 —and hint at the motivation for their construction (since we invested a **lot** of time in constructing[7] AME states!). An analysis of the various quantum protocols that AME states serve as useful quantum resources for as well as a detailed account of our explorations in the construction of AME(N, d) states for $N > 4$ is presented in Ch. 5. We also present certain well-known properties of AME states for completion.

2.6.3.1 Definition

An $\lfloor \frac{N}{2} \rfloor$-uniform state $|\psi\rangle$ of N parties, each of local dimension d is said to be AME(N, d).

It is said to be absolutely, **maximally** entangled as it maximizes entanglement between **all** of its bipartitions, i.e. $\exists$ no state which has 'more' bipartite entanglement.

2.6.3.2 Significance & Utility

The fact that any reduction of AME states to subsystems of size N'$\leq \lfloor \frac{N}{2} \rfloor$ is maximally mixed is utilised in quantum secret sharing protocols. The intuitive reasoning behind the utility of AME states in such protocols is due to the fact that even in cases where *up to 50%* of the qudits used in the protocol are intercepted by an adversary, **no** information about the overall state of the system is gained.

AME states also have connections with combinatorial designs and structures (3) in

[7] and attempting to construct them, if we are being honest.

mathematics. The connection of AME states to classical error correcting codes and quantum error correcting codes further motivates the construction of these states. We further expound on the applications of AME states in Ch. 5 as we further motivate our attempts at constructing such states.

2.6.3.3 Properties

Theorem 2.6.10. *(4) $|\Phi\rangle$ must be maximally entangled for any bipartition.*
Thus, if we divide the set P into two disjoint sets A and $B : A \cup B = P$, and $|B| = m$ such that without loss of generality, we assume that $|B| < |A|$, then the state $|\Phi\rangle$ can be expressed as:—

$$|\Phi\rangle = \frac{1}{\sqrt{d^m}} \sum_{k \in \mathbb{Z}_d^m} |k_1\rangle_{B_1} \ldots m|k_m\rangle_{B_m} |\phi(k)\rangle_A,$$

with $\phi(k)|\phi(k') = \delta_{kk'}$.

Theorem 2.6.11. *(4) The reduced density matrix of every subset of parties $A \subset P$ with $|A| \leq \lfloor \frac{n}{2} \rfloor$ is totally mixed, $\rho_A = d^{-|A|} 1_{d^{|A|}}$.*

Theorem 2.6.12. *(4) The von Neumann entropy of every subset of parties $A \subset P$ with $|A| \leq \lfloor \frac{n}{2} \rfloor$ is maximal, $S(A) = |A| \log(d)$.*

Theorem 2.6.13. *(11) If $|\psi\rangle_{A_1 A_2 \ldots A_N}$ and $|\phi\rangle_{B_1 B_2 \ldots B_N}$ are AME(N,d_1) and AME(N,d_2) states in $(C^{d_1})^{\otimes N}$ and $(C^{d_2})^{\otimes N}$ respectively, then, their tensor product, i.e.*

$$|\varphi\rangle_{(A_1 B_1), \ldots, (A_N B_N)} = |\psi\rangle_{A_1 A_2 \ldots A_N} \otimes |\phi\rangle_{B_1 B_2 \ldots B_N}, \tag{2.18}$$

is an AME(N,$d_1 d_2$) state in $C^{d_1 d_2}$.

2.6.4 Multi-unitary Operators

In this subsection, we state the definition of multi-unitary operators and express it as a special class of k-uniform unitary operators. The property of multi-unitary operators shown is a consequence of the multipartite-operator state isomorphism defined in Sec. 4.

2.6.4.1 Definition

A unitary operator, U, defined over N parties, each of local dimension d in $\mathcal{H}^d_{A_1} \otimes \cdots \otimes \mathcal{H}^d_{A_N}$ is defined to be a multi-unitary operator if its vectorization, $|U\rangle_{A_1A'_1\ldots A_NA'_N}$:—

$$|U\rangle_{A_1A'_1\ldots A_NA'_N} = U_{A_1A_2\ldots A_N} \otimes \mathbb{I}_{A'_1A'_2\ldots A'_N} |\Phi^+\rangle_{A_1A'_1}|\Phi^+\rangle_{A_2A'_2}\cdots|\Phi^+\rangle_{A_NA'_N},$$

is an AME(N, d) state.

Multi-unitary operators are also known as *perfect tensors* as **any** reordering of their indices leads to a unitary operator.

2.6.4.2 Properties

Theorem 2.6.14. *A multi-unitary operator, U, defined over N parties is an N-uniform unitary operator.*

Proof. On applying the multipartite operator-state isomorphism of Sec. 2.4 to a multi-unitary operator, $U_{A_1A_2\ldots A_N}$, the resultant state of $2N$ parties, $|U_{A_1A'_1A_2A'_2\ldots A_NA'_N}\rangle$, must be N-uniform by the definition of multi-unitarity and AME states.

Thus, by the definition of k-uniform unitary operators, U is an N-uniform unitary operator. ∎

Theorem 2.6.15. *If U is a multi-unitary operator defined to act on N parties —i.e. an N-uniform unitary operator —each of local dimension d i.e. $\mathcal{H}^d_{A_1} \otimes \cdots \otimes \mathcal{H}^d_{A_N}$, then $|U\rangle_{A_1A'_1\ldots A_{N-1}A'_{N-1}A_N}$ constructed as:—*

$$|U\rangle_{A_1A'_1\ldots A_NA'_N} = U_{A_1A_2\ldots A_N} \otimes \mathbb{I}_{A'_1A'_2\ldots A'_N} |\Phi^+\rangle_{A_1A'_1}|\Phi^+\rangle_{A_2A'_2}\cdots|\Phi^+\rangle_{A_{N-1}A'_{N-1}}|0\rangle_{A_N}, \tag{2.19}$$

is an AME(2N-1,d) state.

Proof. A multi-unitary operator on N parties is an N-uniform unitary operator. Then, $|U\rangle_{A_1A_1'\ldots A_{N-1}A_{N'-1}A_N}$:—

$$|U\rangle_{A_1A_1'\ldots A_{N-1}A_{N'-1}A_N} = U_{A_1A_2\ldots A_N}\otimes\mathbb{I}_{A_1'A_2'\ldots A_{N'-1}}\, |\Phi^+\rangle_{A_1A_1'}|\Phi^+\rangle_{A_2A_2'}\cdots|\Phi^+\rangle_{A_{N-1}A_{N-1}'}|0\rangle_{A_N}$$

is an $(N-1)$-uniform state on $2N-1$ parties.

An $(N-1)$-uniform state on $2N-1$ parties, each of local dimension d, is an AME($2N-1$, d) state. ∎

Corollary 2.6.15.1. *The existence of AME($2N$, d) implies the existence of AME($2N-1$, d).*

Corollary 2.6.15.2. *The non-existence of AME(2N-1,d) implies the non-existence of AME(2N,d).*

The two corollaries stated above can be used to determine if the construction of certain AME states is possible. The AME Table (5) maintained by Felix Huber can be used to perform a basic check of this result.

2.7 REMARKS

Thus, in this chapter we generalized the bipartite operator-state isomorphism to a multipartite setting and utilized it to coin some novel terms such as k-uniform unitary operators and prove certain results on k-uniform states as well as AME states —including existence proofs that bear immense significance to the problem of the existence of a general AME(N, d) state.

We also speculate that the theorem on the separability of k-uniform states is of some significance in quantum secret sharing protocols —although we are, at the time of writing, unable to precisely determine the manner in which the result would come into the picture.

However, the result of maximal import stated in this chapter —for the purposes of the work described in the rest of the thesis, is that of the multipartite operator-state isomorphism and the connection between the reduced density operators of the state and the operator matrix reshapings.

These two mathematical tools couple together to allow us to alternate between the operator picture and the state picture with relative ease. It allows us to restate the entanglement features of multipartite operators in terms of the entanglement features of the state equivalent to the operator under our isomorphism. Thus, we can probe the entanglement features of multipartite operators in terms of the entanglement features of multipartite states —converting a less studied problem to one that is greater understood.

Let us now investigate multipartite entanglement in operators using the tools defined in this chapter.

CHAPTER 3

OPERATOR ENTANGLEMENT & MAXIMALLY ENTANGLED *OPERATORS*

3.1 INTRODUCTION

It is a fundamental advantage to be able to break down the state space of a composite system into simpler state-systems associated with its parts and thus describe the physical world in terms of its associated subsystems. Quantum theory provides an axiomatic advantage to this effect in that the state-space associated with the combination of N systems —$S_1 \ldots S_N$ —can be expressed as the tensor product of the individual state-spaces.

To this extent, the question of when a state belonging to this composite state-space could be expressed as the tensor product of states belonging to the state-spaces of the individual subsystems became important. This question led to the discovery of entanglement as a quantum resource that produces correlations markedly different from those produced by the standard product states. The utility of entanglement as a quantum resource motivated the construction of such states and thus, the study of the entangling capabilities of quantum evolutions was thrust into the limelight.

In this regard, it was shown that for **bipartite quantum operators**, the *entangling* power —viz. the average entanglement created by the quantum operator when acting on product states with the average taken with respect to the Haar measure on states, is correlated with the **entanglement of the operator** (13)—thus motivating the study of operator entanglement.

The sequential application of quantum gates results in the 'thermalization' of the entanglement properties of the resultant gate (7). The 'thermalization' of a property in this setting means that within a given number of iterations or 'interaction times' n, the quantity studied reaches values corresponding to the Haar average over the unitary group, i.e. the value of that particular property *for any random seed*, within n interactions, would be equal to the average of that property over CUE-sampled unitaries. This observation can be used for the modelling of generic Hamiltonian dynamics.

With respect to non-thermalizing phases, it has also been shown (8) that operator entanglement characterizes information scrambling/delocalization. Thus, the study of operator entanglement can be used to characterize localisation in many body systems.

Moreover, the notion of operator entanglement and that of an operator itself being entangled with another arises naturally when we consider the fact that operators themselves belong to a multipartite state-space —as shown by the multipartite operator-state isomorphism defined in Ch. 2. Thus, all the ideas and intuitions developed for the entanglement of states can be adopted for the entanglement of operators as well. The most important of these is the idea of the Schmidt vector.

The Schmidt decomposition of vectors in the tensor product of two subsystems, as stated in the Appendix for the unacquainted, into vectors belonging to its subsystems, and its connection to the entanglement of quantum states across bipartitions, is one of the important ideas rooted in the study of entanglement of states that can be adopted for the study of entanglement of operators.

Similar to the definition of entanglement in quantum states, we define an operator to be unentangled *with respect to* a bipartition **iff**. it has only one Schmidt coefficient. Thus, in this manner, we can perform a binary test to determine if an operator is entangled with respect to a given bipartition. But just as in the state-space, there is a requirement for a *measure* of entanglement capable of quantifiying the notion of entanglement.

This problem was addressed by the work of Zanardi in (13). It defines operator entanglement entropies for bipartite operators that have the requisite capabilities to serve as a measure of entanglement for such bipartite operator spaces. It also expresses the entangling power of these bipartite operators in terms of these operator entanglement entropies —thus, cementing the correlation between operator entanglement entropies and the entangling power of bipartite quantum evolutions.

The notion of the operator entanglement of a general N-partite operator is, at its core, a measure of the non-locality of the operator or a measure of its 'far-awayness' from local unitary products. Thus, the operator entanglement of an N-partite operator can be considered as a means to quantify the extent to which the evolution dynamics brought about by the operator will be different from that brought about by a local unitary product.

The operator entanglement of an operator can, in principle, be linked to a cost function required to simulate the action of the operator using a universal set of gates. This is once again due to the fact that the operator entanglement of U captures features regarding the non-locality of U. The fact that k_D is maximized by dual unitaries for bipartite operators is a consequence of this fact. The links between k_D and quantum complexity of U have been explored in Shrigyan Bramachari's Masters thesis. It is also speculated that a generalization of k_D to multipartite settings would result in it being maximized by maximally entangled operators —the generalization of duals to multipartite settings. Thus, it is important to have a theory for their quantification and one that could lead to algorithms aimed at creating ensembles of the same —like the M_R map.

While it is known that dual-unitary circuits are solvable many-body non-integrable quantum chaotic systems and are also connected to Bernoulli circuits, at present, the many-body interpretation or utility of multipartite maximally entangled unitaries is unknown and waits to be unearthed.

Thus, from serving as measures correlated with entangling power and measures of non-locality to permitting the definition of cost functions for the quantum complexity of the unitary, operator entanglement is a very fundamental property of operators and thus, operator entanglement entropies are quantities of great interest.

In this chapter, we define the **operator entanglement entropies of** U by first studying tripartite operators and then providing a generalised scheme to define and construct such entropic quantities for multipartite operators. Even for multipartite operators, we consider the Schmidt decomposition into bipartite cuts given the known non-existence of the Schmidt decomposition for the decomposition of tripartite operator(& state) systems into sums of the tensor products of operators (states) on the individual subsystems.

By our extension of the theory of operator entanglement entropies, we define a class of operators known as **maximally entangled *operators***, which generalize the notion of duality in bipartite unitary operators, to the multipartite case. In our definition of such unitary operators, we prove that it is sufficient to **consider balanced partitions** —analogous to the AME case for quantum states where maximal entanglement across balanced bipartitions implies maximal entanglement across **all** bipartitions. We leverage the operator-state isomorphism to this end.

We then plot the distribution of these entropic quantities for the CUE cloud of tripartite unitary operators acting on quantum states with local dimension 2. The analytic calculation of the entanglement entropies for certain unitaries is added to provide the reader with a sense of the manner in which the entanglement entropies serve as a measure of non-locality in the operator space.

However, the operator entanglement entropies of U alone do **not** capture the essential entanglement features or non-local of U. A way to affirm this statement is to note that operator entanglement entropies of U only consider bipartite cuts in the operator picture and that none of these entropic quantities are truly multipartite in definition. Moreover, the operator entanglement entropies of U alone do not suffice to distinguish between a general maximally entangled multipartite unitary operator and multi-unitary operators —despite the fact that multi-unitary operators are **perfect tensors**. Thus, to get closer to capturing the entire set of non-local features of U, we must consider the generation of LU-invariants.

LU-invariants characterize non-local features of operators. For bipartite two-qubit operators, it is known that 3 LU-invariants characterize the entire entanglement features of the operator —these are $E(U)$, $E(US)$ and another **independent** LU-invariant. For a general (N, d), the number of **independent** LU-invariants needed to characterize the non-local features of U is given by Eq. 1.1. In this chapter, we design a construction scheme for the generation of a class of LU-invariants that are **all independent**. The significance of this result is that these LU-invariants can be chosen as the first elements towards a set of independent LU-invariants of cardinality specified in Eq. 1.1 that completely characterize the non-local features of U.

3.2 ENTANGLEMENT IN TRIPARTITE UNITARY OPERATORS

We describe the $N = 3$ case separately and then extend the theory of multipartite operator entanglement to any arbitrary N as the tripartite case marks the transition from bipartite entanglement features —where there is only one bipartition possible —to the entanglement features seen in a general multipartite case. Thus, the intuitive understanding of the tripartite case generalises easily to the other multipartite cases. To further probe multipartite entanglement features, we study tripartite unitary operators defined on Hilbert spaces of local dimension $d = 2$ and provide some illustrative plots and sample cases.

3.2.1 Tripatite Operator Entanglement Entropies

In this subsection, we define the operator entanglement entropies for a general tripartite unitary and like in the case of bipartite operator entanglement, we state and prove the connection of these operator entanglements to the entanglement in bipartitions of the isomorphic 6-party state.

3.2.1.1 Definition

For a bipartite unitary operator, U_{AB}, the operator entanglement entropy that leads to the definition of duals (E(U)) is defined on the distribution of the Schmidt coefficients ($\sqrt{\lambda_i}$) of U_{AB}. As proven in Ch. 2, the Schmidt coefficients are the singular values of the matrix obtained on performing the realignment operation on U, defined as $^{\mathcal{R}}$.

Thus, the natural extension to the operator entanglement entropy of a tripartite unitary operator, U_{ABC}, that leads to the definition of a 'maximally entangled' operator would be based on the Schmidt decomposition of U_{ABC} for all possible bipartitions —$A|BC, B|AC$ & $C|BC$. i.e. if,

$$U_{A|BC} = \sum_{i=1}^{d^2} \sqrt{\lambda_i} m_i^A \otimes m_i^{BC}, \tag{3.1}$$

$$U_{B|AC} = \sum_{i=1}^{d^2} \sqrt{\mu_i} m_i^B \otimes m_i^{AC}, \tag{3.2}$$

$$U_{C|AB} = \sum_{i=1}^{d^2} \sqrt{\nu_i} m_i^C \otimes m_i^{AB}, \tag{3.3}$$

are the Schmidt decompositions of U_{ABC} with respect to the 3 bipartitions mentioned.

Then, the normalizations are :—

$$\sum_{i=1}^{d^2} \lambda_i = d^3, \tag{3.4}$$

$$\sum_{i=1}^{d^2} \mu_i = d^3, \tag{3.5}$$

$$\sum_{i=1}^{d^2} \nu_i = d^3, \tag{3.6}$$

to account for the normalization that $\mathrm{Tr}(UU^\dagger) = d^3$ and $\mathrm{Tr}(m_i^A * m_{i'}^{A\dagger}) = \delta_{ii'}$

Then, using the Tsallis-2 entropy of the distributions $\{\lambda_i/d^3\}$, $\{\mu_i/d^3\}$ & $\{\nu_i/d^3\}$, we can define the 3 operator entanglement entropies as linear entropy functions as:—

$$E_1(U_{ABC}) := \frac{1}{1 - \frac{1}{d^2}} \left(1 - \frac{1}{d^6} * \sum_{i=1}^{d^2} \lambda_i^2 \right), \tag{3.7}$$

$$E_2(U_{ABC}) \coloneqq \frac{1}{1-\frac{1}{d^2}}\left(1 - \frac{1}{d^6} * \sum_{i=1}^{d^2} \mu_i^2\right), \tag{3.8}$$

$$E_3(U_{ABC}) \coloneqq \frac{1}{1-\frac{1}{d^2}}\left(1 - \frac{1}{d^6} * \sum_{i=1}^{d^2} \nu_i^2\right). \tag{3.9}$$

Note:— The linear operator entanglement entropies are normalized such that their maximum value is 1 —when $\lambda_i = d$, $\mu_i = d$ & $\nu_i = d \forall i$.

3.2.1.2 Connection to 6-Party State Bipartitions

Analogous to the connection between the operator entanglement entropies of bipartite unitary operators and the entropy of entanglement in balanced bipartitions of the corresponding 4-party state obtained through the modified CJ isomorphism, in this subsection, we derive the connection between the operator entanglement entropies of tripartite unitary operators and the entanglement entropies of bipartitions of the corresponding 6-party state obtained via the multipartite operator-state isomorphism, i.e. for a tripartite unitary, U_{ABC}, the operator entanglement entropies **should** correspond to the entanglement entropies of some bipartitions in the 6-party state, $|U\rangle_{AA'BB'CC'}$ where,

$$|U\rangle_{AA'BB'CC'} \coloneqq U_{ABC} \otimes \mathbb{I}_{A'B'C'} \, |\Phi^+\rangle_{AA'}|\Phi^+\rangle_{BB'}|\Phi^+\rangle_{CC'}.$$

To this end, we must show that the Schmidt coefficients of the decomposition of U_{ABC} for the 3 bipartitions, as shown in the previous subsection, are the singular values of some reshaped version of U_{ABC}. Intuitively, the bipartitions whose entanglement entropies should correspond to the operator entanglement entropies for a tripartite unitary operator should be $AA'|BB'CC'$, $BB'|AA'CC'$ & $CC'|AA'BB'$. This is due to the fact that the Schmidt coefficients chosen to define the operator entanglement entropies are based on the operator bipartitions —$A|BC$, $B|AC$ & $C|AB$, which correspond to these state bipartitions.

Using the connection between the reduced density operators of $|U\rangle_{AA'BB'CC'}$ and the reshapings of U_{ABC} proven in Ch. 2, the corresponding reshapings we must consider are,

$$\langle i_1 j_1|U^{\mathcal{R}_1}|i_2 j_2 i_3 j_3\rangle = \langle i_1 i_2 i_3|U|j_1 j_2 j_3\rangle, \tag{3.10}$$

$$\langle i_2 j_2|U^{\mathcal{R}_2}|i_1 j_1 i_3 j_3\rangle = \langle i_1 i_2 i_3|U|j_1 j_2 j_3\rangle, \tag{3.11}$$

$$\langle i_3 j_3|U^{\mathcal{R}_3}|i_1 j_1 i_2 j_2\rangle = \langle i_1 i_2 i_3|U|j_1 j_2 j_3\rangle. \tag{3.12}$$

These reshapings correspond to the reduced density operators of the 6-party state as:—

$$\rho'_{AA'} = \mathrm{Tr}_{\mathrm{BB'CC'}}(\rho) = \frac{1}{d^3}U^{\mathcal{R}_1}U^{\mathcal{R}_1\dagger}, \tag{3.13}$$

$$\rho'_{BB'} = \mathrm{Tr}_{\mathrm{AA'CC'}}(\rho) = \frac{1}{d^3}U^{\mathcal{R}_2}U^{\mathcal{R}_2\dagger}, \tag{3.14}$$

$$\rho'_{CC'} = \mathrm{Tr}_{\mathrm{AA'BB'}}(\rho) = \frac{1}{d^3}U^{\mathcal{R}_3}U^{\mathcal{R}_3\dagger}, \tag{3.15}$$

$$\tag{3.16}$$

by the results of Ch. 2, with the normalization remaining d^3 as Tr$(A^\dagger B)$=Tr$(A^{(x)\dagger}B^{(x)})$ (See Appendix). Thus, if these bipartitions are maximally entangled, the corresponding reshaped matrix will be *proportional* to an isometry.

We anoint these reshapings of the unitary operator as $\mathcal{R}_1, \mathcal{R}_2$ & $\mathcal{R}_3$ so as to correspond to the realignment operation ($\mathcal{R}$) defined over bipartite unitary operators. This correspondence is maintained to further emphasize on the correspondence between these reshapings and the extension of the definition of the maximal operator entanglement to multipartite systems.

Now, we need to prove that the singular values of these reshapings are the Schmidt coefficients of the corresponding decomposition of U_{ABC}. To that end, we utilize the following lemmas.

Lemma 3.2.1. *Using the lexicographic vectorization (See Appendix) defined via the standard CJ isomorphism in Ch. 2, i.e. $A \to |A\rangle$,*

$$(A \otimes B \otimes C)^{\mathcal{R}_1} = |A\rangle\langle B^* \otimes C^*|. \tag{3.17}$$

Proof.

$$\begin{aligned}
\langle i_1 i_2 i_3 | A \otimes B \otimes C | j_1 j_2 j_3 \rangle &= \langle i_1 | A | j_1 \rangle \langle i_2 | B | j_2 \rangle \langle i_3 | C | j_3 \rangle \\
&= \langle i_1 j_1 | A \rangle \langle i_2 j_2 | B \rangle \langle i_3 j_3 | C \rangle \\
&= \langle i_1 j_1 | A \rangle \langle B^* | i_2 j_2 \rangle \langle C^* | i_3 j_3 \rangle \\
&= \langle i_1 j_1 | A \rangle \langle B^* \otimes C^* | i_2 j_2 i_3 j_3 \rangle
\end{aligned}$$

But,

$$\begin{aligned}
\langle i_1 i_2 i_3 | A \otimes B \otimes C | j_1 j_2 j_3 \rangle &= \langle i_1 j_1 | (A \otimes B \otimes C)^{\mathcal{R}_1} | i_2 j_2 i_3 j_3 \rangle \\
\implies \langle i_1 j_1 | (A \otimes B \otimes C)^{\mathcal{R}_1} | i_2 j_2 i_3 j_3 \rangle &= \langle i_1 j_1 | A \rangle \langle B^* \otimes C^* | i_2 j_2 i_3 j_3 \rangle \\
\therefore \quad (A \otimes B \otimes C)^{\mathcal{R}_1} &= |A\rangle\langle B^* \otimes C^*|
\end{aligned}$$

■

By symmetry, the following corollaries hold true :—

Corollary 3.2.1.1. $(A \otimes B \otimes C)^{\mathcal{R}_2} = |B\rangle\langle A^* \otimes C^*|$.

Corollary 3.2.1.2. $(A \otimes B \otimes C)^{\mathcal{R}_3} = |C\rangle\langle A^* \otimes B^*|$.

Lemma 3.2.2. *With the vectorization used being lexicographic (See Appendix) and U_{BC} being a unitary operator on systems B and C,*

$$(A \otimes U_{BC})^{\mathcal{R}_1} = |A\rangle\langle U^{\mathcal{R}*}|, \tag{3.18}$$

where $U^{\mathcal{R}}$ is the realigned matrix of the bipartite operator U.

Proof.

$$\begin{aligned}
\langle i_1 i_2 i_3|A \otimes U_{BC}|j_1 j_2 j_3\rangle &= \langle i_1|A|j_1\rangle\langle i_2 i_3|U_{BC}|j_2 j_3\rangle \\
&= \langle i_1|A|j_1\rangle\langle i_2 j_2|U_{BC}^{\mathcal{R}}|i_3 j_3\rangle \\
&= \langle i_1 j_1|A\rangle\langle i_2 j_2 i_3 j_3|U_{BC}^{\mathcal{R}}\rangle \\
&= \langle i_1 j_1|A\rangle\langle U_{BC}^{\mathcal{R}*}|i_2 j_2 i_3 j_3\rangle
\end{aligned}$$

But,

$$\begin{aligned}
\langle i_1 i_2 i_3|A \otimes U_{BC}|j_1 j_2 j_3\rangle &= \langle i_1 j_1|(A \otimes U_{BC})^{\mathcal{R}_1}|i_2 j_2 i_3 j_3\rangle \\
\implies \langle i_1 j_1|(A \otimes U_{BC})^{\mathcal{R}_1}|i_2 j_2 i_3 j_3\rangle &= \langle i_1 j_1|A\rangle\langle U_{BC}^{\mathcal{R}*}|i_2 j_2 i_3 j_3\rangle \\
\implies (A \otimes U_{BC})^{\mathcal{R}_1} &= |A\rangle\langle U^{\mathcal{R}*}|.
\end{aligned}$$

■

Then, leveraging this set of results, if U_{ABC} has a Schmidt decomposition with respect to the bipartition $A|BC$ as :—

$$U_{A|BC} = \sum_{i=1}^{d^2} \sqrt{\lambda_i} m_i^A \otimes m_i^{BC},$$

with $\sum_{i=1}^{d^2} \lambda_i = d^3$: as in Ch. 4 Eq. 1.

Then,

$$
\begin{aligned}
U^{\mathcal{R}_1} &= \sum_{i=1}^{d^2} \sqrt{\lambda_i} (m_i^A \otimes m_i^{BC})^{\mathcal{R}_1}, \\
&= \sum_{i=1}^{d^2} \sqrt{\lambda_i} |m_i^A\rangle\langle (m_i^{BC})^{\mathcal{R}*}|.
\end{aligned}
$$

But, we know that $\mathrm{Tr}(A^\dagger B) = \mathrm{Tr}(A^{(x)\dagger} B^{(x)})$ for *any* reshaping operation,(x) by Lemma. SIP(See Appendix). Thus, $\{(m_i^{BC})^{\mathcal{R}}\}$ also forms an orthogonal set $\implies |(m_i^{BC})^{\mathcal{R}}\rangle$ also forms an orthogonal set in $\mathcal{H}_B^d \otimes \mathcal{H}_C^d$.

Thus:—

$$
\begin{aligned}
(U^{\mathcal{R}_1} * U^{\mathcal{R}_1 \dagger})^2 &= \sum_{i=1}^{d^2} \lambda_i^2 |m_i^A\rangle\langle m_i^A| \\
\implies E_1(U_{ABC}) &= \frac{1}{1-\frac{1}{d^2}} \Big(1 - \frac{1}{d^6} \mathrm{Tr}((U^{\mathcal{R}_1} U^{\mathcal{R}_1 \dagger})^2)\Big).
\end{aligned}
$$

By symmetry, we can derive similar relations for E_2 & E_3 as well:—

$$
E_1(U_{ABC}) = \frac{1}{1-\frac{1}{d^2}} \Big(1 - \frac{1}{d^6} \mathrm{Tr}((U^{\mathcal{R}_1} U^{\mathcal{R}_1 \dagger})^2)\Big), \tag{3.19}
$$

$$
E_2(U_{ABC}) = \frac{1}{1-\frac{1}{d^2}} \Big(1 - \frac{1}{d^6} \mathrm{Tr}((U^{\mathcal{R}_2} U^{\mathcal{R}_2 \dagger})^2)\Big), \tag{3.20}
$$

$$
E_3(U_{ABC}) = \frac{1}{1-\frac{1}{d^2}} \Big(1 - \frac{1}{d^6} \mathrm{Tr}((U^{\mathcal{R}_3} U^{\mathcal{R}_3 \dagger})^2)\Big). \tag{3.21}
$$

Note:— In this definition of operator entanglement, we renormalize the operator entanglements so that $E_i(U_{ABC}) \in [0, 1]$.

3.2.2 Maximally Entangled Tripartite Operators

The extension of the definition of maximally entangled bipartite unitary operators to tripartite unitary operators is natural as we move from there being **only** one bipartition with respect to which a bipartite unitary operator can be Schmidt-decomposed into to there now being **3 bipartitions** with respect to which the tripartite unitary operator can be Schmidt-decomposed into —$A|BC$, $B|AC$ & $C|AB$.

This is analogous to the extension of bipartite *state* entanglement to multipartite systems where the entanglement in different bipartite cuts are studied for the lack of a measure of true multipartite entanglement. Thus, even in the case of operator entanglement, we are restricted to the study of the entanglement in bipartitions and define the notion of 'maximal entanglement' with respect to bipartitions —analogous to the definition adopted for AME states.

3.2.2.1 Definition

A tripartite unitary operator, U_{ABC} is defined as a maximally entangled tripartite operator iff. the Schmidt coefficients of the decomposition of U with respect to $A|BC$, $B|AC$ & $C|AB$ are **all equal** to d, i.e. by Ch. 3 Eq. 1, if:—

$$U_{A|BC} = \sum_{i=1}^{d^2} \sqrt{\lambda_i} m_i^A \otimes m_i^{BC},$$

$$U_{B|AC} = \sum_{i=1}^{d^2} \sqrt{\mu_i} m_i^B \otimes m_i^{AC},$$

$$U_{C|AB} = \sum_{i=1}^{d^2} \sqrt{\nu_i} m_i^C \otimes m_i^{AB},$$

then, U_{ABC} is a maximally entangled tripartite unitary operator iff. $\forall i \in [1 : d^2]$,

$$\lambda_i = d, \tag{3.22}$$

$$\mu_i = d, \tag{3.23}$$

$$\nu_i = d. \tag{3.24}$$

$$\tag{3.25}$$

An equivalent set of conditions, based on the linear operator entanglements defined in Ch. 4 Sec. 2.1 are :—

$$E_1(U_{ABC}) = 1, \tag{3.26}$$

$$E_2(U_{ABC}) = 1, \tag{3.27}$$

$$E_3(U_{ABC}) = 1. \tag{3.28}$$

Utilizing the connection to 6-Party State Bipartitions shown in Ch. 4 Sec. 2.2, another equivalent set of conditions can be derived as:—

$$U^{\mathcal{R}_1} U^{\mathcal{R}_1 \dagger} = \mathrm{d}\mathbb{I}_{d^2}, \tag{3.29}$$

$$U^{\mathcal{R}_2} U^{\mathcal{R}_2 \dagger} = \mathrm{d}\mathbb{I}_{d^2}, \tag{3.30}$$

$$U^{\mathcal{R}_3} U^{\mathcal{R}_3 \dagger} = \mathrm{d}\mathbb{I}_{d^2}. \tag{3.31}$$

If any subset of the 3 conditions for maximal tripartite operator entanglement is satisfied, the tripartite unitary operator is said to be maximally entangled with respect to those particular bipartitions.

3.2.2.2 Connection to 6-Party Bipartitions

By the results stated in Ch. 4 Sec. 2.2, for the 6-party state, $|U\rangle_{AA'BB'CC'}$, isomorphic to U_{ABC} defined as:—

$$|U\rangle_{AA'BB'CC'} \coloneqq U_{ABC} \otimes \mathbb{I}_{A'B'C'}\ |\Phi^+\rangle_{AA'}|\Phi^+\rangle_{BB'}|\Phi^+\rangle_{CC'},$$

the reduced density operators for the 6-party bipartitions $AA'|BB'CC'$, $BB'|AA'CC'$ & $CC'|AA'BB'$ are given by:—

$$\begin{aligned}
\rho'_{AA'} &= \mathrm{Tr}_{\mathrm{BB'CC'}}(\rho) = \frac{1}{d^3}U^{\mathcal{R}_1}U^{\mathcal{R}_1\dagger},\\
\rho'_{BB'} &= \mathrm{Tr}_{\mathrm{AA'CC'}}(\rho) = \frac{1}{d^3}U^{\mathcal{R}_2}U^{\mathcal{R}_2\dagger},\\
\rho'_{CC'} &= \mathrm{Tr}_{\mathrm{AA'BB'}}(\rho) = \frac{1}{d^3}U^{\mathcal{R}_3}U^{\mathcal{R}_3\dagger},
\end{aligned}$$

Utilizing Eq. 3.30-3.32, for a maximally entangled operator, all of these 6-party bipartitions —$AA'|BB'CC'$, $BB'|AA'CC'$ & $CC'|AA'BB'$ will be **maximally entangled** as the corresponding reduced density operators will be **maximally mixed**.

Analogously, if only a subset of the operator entanglement entropies are maximum, only the bipartitions of the isomorphic state corresponding to *those* operator entanglement entropies will be maximally entangled.

This connection between a bipartition of the operator being maximally entangled and the corresponding bipartitions of the isomorphic state being maximally entangled is utilized in proving the **Bipartition Thoerem of Operator Entanglement**.

3.3 NUMERICAL STUDIES OF OPERATOR ENTANGLEMENT ON TRIPARTITE OPERATORS OF ORDER 8

In this section, we present **some** of our numerics based results on the distribution of the operator entanglement entropies —$E_1(U)$,$E_2(U)$ & $E_3(U)$ —over sets of tripartite operators with $d = 2$.

In particular, we show the distribution of the 3 operator entanglement entropies over CUE(8) and an ensemble we coin 'AME62-Perturbed', obtained by perturbing an AME(6,2) obtained through our work in Ch. 5. The motivation behind constructing an ensemble of unitary operators by perturbing an AME(6,2) was to obtain a better idea of the distribution of the tripartite entanglement entropies for unitaries close to the perfect tensor. We have taken 10^6 CUE samples and 10^4 samples of 'AME62-Perturbed'.

We then show explicit constructions of permutations of order 8 that are maximally entangled with respect to one or more of the 3 possible operator bipartitions in tripartite unitary operators.

3.3.1 Distribution of Tripartite Operator Entanglement Entropies over Ensembles

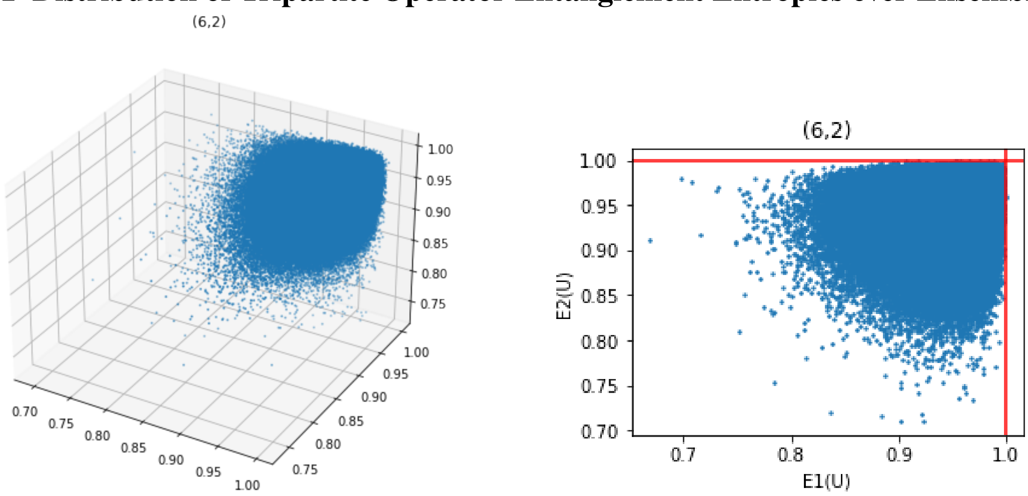

Figure 3.1: On the left is the distribution of the tripartite operator entanglement entropies over the CUE(8) ensemble.
On the right is the distribution of $E_2(U)$ v.s. $E_1(U)$ for CUE(8). The tripartite operator entanglement entropies will all behave in the **same manner**.

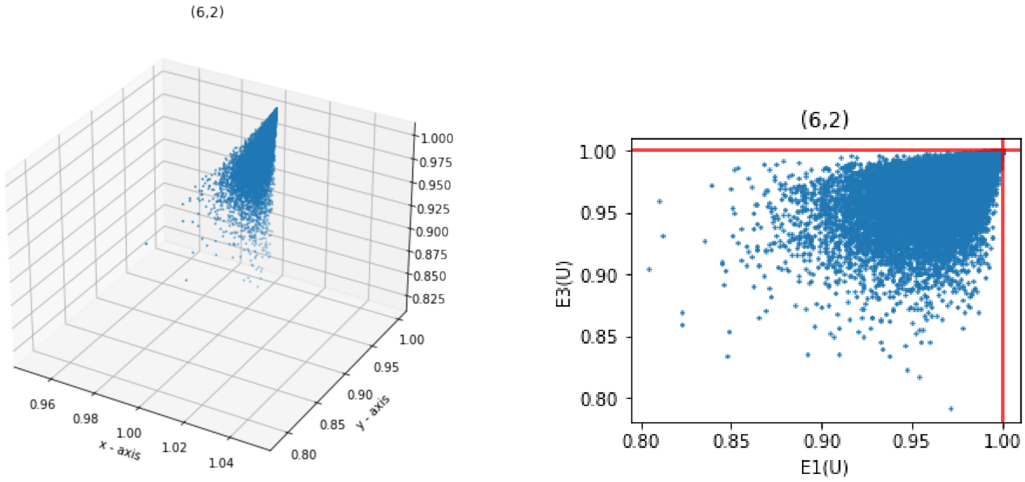

Figure 3.2: On top is the distribution of the tripartite operator entanglement entropies over the so-called ‘AME62-Perturbed’ ensemble.
On the bottom is the distribution of $E_3(U)$ v.s. $E_1(U)$ for ‘AME62-Perturbed’. The symmetric perturbation scheme implies that the distribution of the operator entanglements will be same for all 3 operator entanglement entropies.

3.3.2 The Tripartite Operator Entanglement Entropies of Permutations

We iterate over all the 8 factorial permutation matrices of order 8 and study the values of the tripartite entanglement entropies for such well structured matrices. We pick permutations as they have a good structure and have ‘well-distributed’ values of operator entanglements in general —ranging from minimal to maximal.

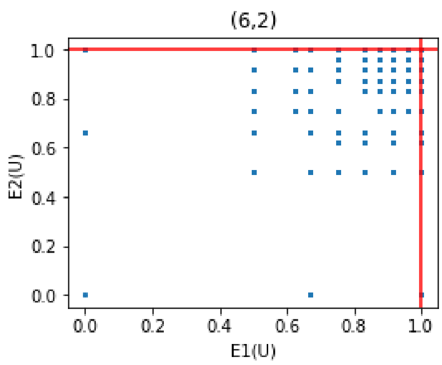

Figure 3.3: The distribution of $E_1(U)$ v.s. $E_2(U)$ for permutations of order 8.

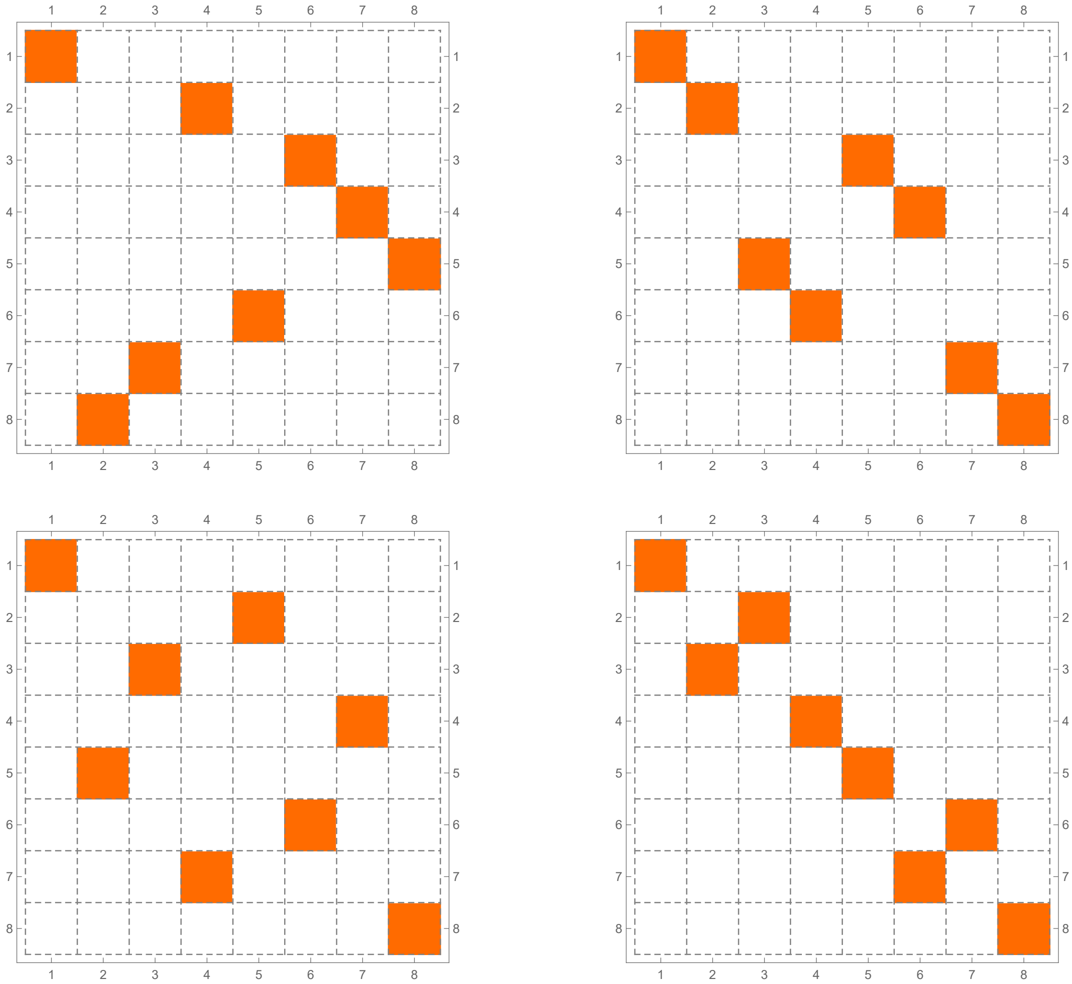

Figure 3.4: (In clockwise from top-left) These represent the matrix structures of permutation matrices of order 8 that are maximally entangled with respect to :—all 3 bipartitions of order 2|4, $AA'|BB'CC'$ & $BB'|AA'CC'$,$AA'|BB'CC'$ & $CC'|AA'BB'$ and $BB'|AA'CC'$ & $CC'|AA'BB'$ respectively.

3.3.3 Inferences

We draw the following results from the distributions of the tripartite entanglement entropies over CUE(8), 'AME62-Perturbed' and over the set of permutation matrices of order 8.

- The measure of maximally entangled tripartite unitary operators with $d = 2$ in CUE(8) tends to 0.

- This motivates the construction of maps equivalent to the M_R and M_{TR} maps, with the caveat that these maps would deal with rectangular matrices of the order (in general) $d^2 \times d^4$.

- The plot depicting the distribution of $E_1(U)$ and $E_2(U)$ for CUE(8) shows that there are unitaries such that two of the three entropies are maximized.

- The plots depicting the 'AME62-Perturbed' ensemble indicate behaviour in the neighbourhood of a maximally entangled operator.

- There are no lines of approach for AME(6,2) along the paths $E_1(U) = 1$ or $E_3(U) = 1$. The same holds for the line $E_2(U) = 1$. This behaviour is similar to that seen for P_9[1], which is another indicator of the fact that all the AME(6,2)s may be LU-connected.

- The permutation matrices take discrete values of operator entanglement at discrete intervals.

- There are **no** permutation matrices which are maximally entangled for **only** one of the three bipartitions.

- The permutations that are maximally entangled for the three operator bipartitions are **not** perfect tensors (3-unitaries) of order 8.

- Thus, there exists **no** permutation matrix that is 3-unitary.

- The above statement is equivalent to saying that there exists **no** minimal support AME state for the tuple (6,2)

[1]A 2-unitary that is conjectured to be LU-connected to any other 2-unitary of order 9.

3.4 A THEORY OF MULTIPARTITE OPERATOR ENTANGLEMENT

In this section, we extend the study of operator entanglement from the tripartite case to a general multipartite case. There is a natural extension of the definition of 'maximally entangled' operators from the tripartite to the general multipartite case —albeit with a more involved proof to show the connection between operator entanglement & state entanglement.

3.4.1 Multipartite Operator Entanglement Entropies

In this subsection, we define the operator entanglement entropies for a general multipartite unitary operator and like in the case of bipartite and tripartite operator entanglement, we state and prove the connection of these operator entanglements to the entanglement in bipartitions of the isomorphic $2N$-party state.

3.3.1.1 Definition

For a general multipartite operator $U_{A_1 \dots A_N}$, an operator entanglement entropy can be defined based on the Schmidt decomposition of $U_{A_1 \dots A_N}$ with respect to *any* bipartition of its N parties. An exhaustive list of these operator entanglement entropies would have cardinality $(2^N - 2)/2$ —as each party can join one half of the bipartition independent of the other parties **but** the cases where there are 0 or N parties in one half must be discarded. The factor of 1/2 arises when one considers the symmetry between the two halves of a bipartition implying that $S|S^c$ is the same as $S^c|S$.

Consider an exhaustive superset of **all** unique bipartitions of the N parties, $\mathcal{P}$ with each set, P in $\mathcal{P}$ corresponding to a particular bipartition of the N parties. By the arguments made in the previous paragraph, $|\mathcal{P}| = (2^N - 2)/2$.

Now, for each bipartition, $P \subset \mathcal{P}$, let S correspond to the set of parties with smaller cardinality, i.e. if $P = S|S^c$, $|S| \le |S^c|$ or equivalently, $|S| \le \lfloor N/2 \rfloor$.

Then, if the Schmidt decomposition of $U_{A_1 \dots A_N}$ with respect to the bipartition $S|S^c$ is given by:—

$$U_{A_1\ldots A_N} = \sum_{i=1}^{d^{2|S|}} \lambda_i^S m_i^S \otimes m_i^{S^c}, \tag{3.32}$$

with the normalization as:—

$$\sum_{i=1}^{d^{2|S|}} \lambda_i^S = d^N, \tag{3.33}$$

to account for the fact that $\mathrm{Tr}(UU^\dagger) = d^N$ & $\mathrm{Tr}(m_i^S m_{i'}^{S\dagger}) = \delta_{ii'}$.

Then, using the Tsallis-2 entropy of the distributions $\{\lambda_i^S/d^N\}$, we can define the operator entanglement entropy corresponding to the bipartition $S|S^c$ as linear entropy functions as:—

$$E_S(U_{A_1\ldots A_N}) := \frac{1}{1-d^{2|S|}}\left(1 - \frac{1}{d^{2N}}\sum_{i=1}^{d^{2|S|}} (\lambda_i^S)^2\right). \tag{3.34}$$

Note:—The linear operator entanglement entropies are normalized such that their maximum value is 1.

3.3.1.2 Connection to 2N-Party State Bipartitions

Analogous to the connection between the operator entanglement entropies of bipartite & tripartite unitary operators and the entropy of entanglement in bipartitions of the corresponding 4-party & 6-party states respectively, obtained via the operator-state isomorphisms, in this subsection, we establish the relationship between the operator entanglement entropies of a general multipartite unitary operator and the entanglement entropies of bipartitions of the corresponding 2N-Party state obtained via the multipartite operator-state isomorphism, i.e. for a general multipartite unitary operator defined on N parties —$A_1 \ldots A_N$, $U_{A_1\ldots A_N}$, the operator entanglement entropies **should** correspond to the entanglement entropies of some bipartitions in the 2N-party state, $|U\rangle_{A_1 A_1' \ldots A_N A_N'}$ where:—

$$|U\rangle_{A_1A_1'\dots A_NA_N'} := U_{A_1A_2\dots A_N} \otimes \mathbb{I}_{A_1'A_2'\dots A_N'} \; |\Phi^+\rangle_{A_1A_1'}|\Phi^+\rangle_{A_2A_2'} \cdots |\Phi^+\rangle_{A_NA_N'}.$$

To establish this connection, we must show that the Schmidt coefficients of the decomposition of $U_{A_1\dots A_N}$ into any bipartition, $P \subset \mathcal{P}$ - $S|S^c$, are the singular values of some reshaped version of $U_{A_1\dots A_N}$, which can then be connected to particular bipartitions in the 2N-party state by the results in Ch. 3.

Let P be the bipartition we consider : $P = S|S^c$ with $S = A_1 \dots A_k$ without loss of generality. Then, intuitively, by the operator-state isomorphism, the bipartitions whose entanglement entropies should correspond to this operator entanglement entropy for a general multipartite unitary operator should be $S \cup S'|S^c(S')^c$, where $S' = A_1' \dots A_k'$.

Using the connection between the reduced density operators of $|U\rangle_{A_1\dots A_N}$ and the reshapings of $U_{A_1\dots A_N}$ proven in Ch. 3 Sec. 4.4, the corresponding reshaping we must consider is:—

$$\langle i_1 j_1 \dots i_k j_k | U^{\mathcal{R}_S} | i_{k+1} j_{k+1} \dots i_N j_N \rangle = \langle i_1 \dots i_N | U | j_1 \dots j_N \rangle, \qquad (3.35)$$

where $\mathcal{R}_S$ is a generalised realignment operation.

These reshapings correspond to the reduced density operators of the 2N-party state as:—

$$\rho_{S \cup S'} = \mathrm{Tr}_{S^c(S')^c}(\rho_{A_1A_1'\dots A_NA_N'}) = \frac{1}{d^N} U^{\mathcal{R}_S} U^{\mathcal{R}_S \dagger}, \qquad (3.36)$$

by Sec. 3.4 with the normalization remaining d^N as $\mathrm{Tr}(A^\dagger B) = \mathrm{Tr}(A^{\mathcal{R}_S \dagger} B^{\mathcal{R}_S})$.

Thus, we need to prove that the singular values of this reshaped matrix are the Schmidt coefficients of the decomposition of $U_{A_1\dots A_N}$ with respect to the bipartition $S|S^c$. To that end, we utilize the following lemma.

Lemma 3.4.1. *Consider the vectorization used here as the vectorization resulting from the multipartite operator-state isomorphism defined in Ch. 3 Sec. 3 —**not** the lexicographical vectorization.*

Thus, $M \mapsto |M\rangle$ *as:—*

$$\langle i_1 j_1 i_2 j_2 \ldots i_N j_N | M\rangle = \langle i_1 i_2 \ldots i_N | M | j_1 j_2 \ldots j_N\rangle.$$

Note:— *For a **Hilbert-Schmidt (HS) norm-1** matrix* M*,* $|M\rangle$ *is also normalized.*

Then, for $S \subset \{1, \ldots, N\}$*, if* m^S *is a* $d^{|S|} \times d^{|S|}$ *with HS norm* $= 1$ *& if* m^{S^c} *is a* $d^{|S^c|} \times d^{|S^c|}$ *with HS norm* $= 1$,

$$(m^S \otimes m^{S^c})^{\mathcal{R}_S} = |m^S\rangle\langle (m^{S^c})^*|, \tag{3.37}$$

where $\mathcal{R}_S$ *is defined as:—*

$$\langle i_1 j_1 \ldots i_k j_k | U^{\mathcal{R}_S}_{A_1 \ldots A_N} | i_{k+1} j_{k+1} \ldots i_N j_N\rangle = \langle i_1 \ldots i_N | U_{A_1 \ldots A_N} | j_1 \ldots j_N\rangle$$

Proof.

$$\begin{aligned}
\langle i_1 \ldots i_N | m^S m^{S^c} | j_1 \ldots j_N\rangle &= \langle i_1 \ldots i_k | m^S | j_1 \ldots j_k\rangle\langle i_{k+1} \ldots i_N | m^{S^c} | j_{k+1} \ldots j_N\rangle \\
&= \langle i_1 j_1 \ldots i_k j_k | m^S\rangle\langle i_{k+1} j_{k+1} \ldots i_N j_N | m^{S^c}\rangle \\
&= \langle i_1 j_1 \ldots i_k j_k | m^S\rangle\langle (m^{S^c})^* | i_{k+1} j_{k+1} \ldots i_N j_N\rangle
\end{aligned}$$

But,

$$\begin{aligned}
\langle i_1 \ldots i_N | m^S m^{S^c} | j_1 \ldots j_N\rangle &= \langle i_1 j_1 \ldots i_k j_k | (m^S \otimes m^{S^c})^{\mathcal{R}_S} | i_{k+1} j_{k+1} \ldots i_N j_N\rangle \\
\therefore \quad (m^S \otimes m^{S^c})^{\mathcal{R}_S} &= |m^S\rangle\langle (m^{S^c})^*|.
\end{aligned}$$

■

Leveraging this result, if $U_{A_1 \dots A_N}$ has a Schmidt decomposition with respect to the bipartition $S|S^c$ as:—

$$U_{A_1 \dots A_N} = \sum_{i=1}^{d^{2|S|}} \lambda_i^S m_i^S \otimes m_i^{S^c},$$

with $\sum_{i=1}^{d^{2|S|}} \lambda_i^S = d^N$.

Then,

$$\begin{aligned} U_{A_1 \dots A_N}^{\mathcal{R}_S} &= \sum_{i=1}^{d^{2|S|}} \lambda_i^S (m_i^S \otimes m_i^{S^c})^{\mathcal{R}_S} \\ &= \sum_{i=1}^{d^{2|S|}} \lambda_i^S |m_i^S\rangle\langle (m_i^{S^c})^*|. \end{aligned}$$

But, we know that $\mathrm{Tr}(A^\dagger B) = \mathrm{Tr}(A^{(x)\dagger} B^{(x)})$ for *any* reshaping operation, (x) by Lemma. SIP(See Appendix). Thus, any vectorization also preserves orthogonality $\implies \{|m_i^S\rangle\}$ & $\{|m_i^{S^c}\rangle\}$ form an orthogonal set.

Then,

$$\begin{aligned} (U^{\mathcal{R}_S} U^{\mathcal{R}_S \dagger})^2 &= \sum_{i=1}^{d^{2|S|}} \lambda_i^2 |m_i^S\rangle\langle m_i^S| \\ \implies E_S(U_{A_1 \dots A_N}) &= \frac{1}{1 - \frac{1}{d^{2|S|}}} \left(1 - \frac{1}{d^{2N}} \mathrm{Tr}((U^{\mathcal{R}_S} U^{\mathcal{R}_S \dagger})^2) \right). \end{aligned}$$

In this set of definitions of operator entanglement entropies, we renormalize the operator entanglements so that $E_S(U_{A_1 \dots A_N}) \in [0, 1]$.

Summarizing the result of this subsection:—

Theorem 3.4.2. *If $U_{A_1 \dots A_N}$ is a N-partite unitary operator isomorphic to $|U\rangle_{A_1 A_1' \dots A_N A_N'}$ under the multipartite operator-state isomorphism defined by us in Sec. 3.1, then the linear operator entanglement entropy of $U_{A_1 \dots A_N}$ with respect to a bipartition $S|S^c$ (where*

$|S| \leq |S^c|$) is related to the **Tsallis-2** *entropy of entanglement of $|U\rangle_{A_1A'_1\ldots A_NA'_N}$ in the bipartition $S \cup S'|S^c \cup (S^{'})^c$ where $S' = \{A'_i : A_i \in S\}$.*

$$E_S(U_{A_1\ldots A_N}) = S_2(\rho_{S\cup S'}) = \frac{1}{1-\frac{1}{d^{2|S|}}}\left(1-\frac{1}{d^{2N}}\operatorname{Tr}((U^{\mathcal{R}_S}U^{\mathcal{R}_S\dagger})^2)\right) = \frac{1}{1-\frac{1}{d^{2|S|}}}\left(1-\frac{1}{d^{2N}}\sum_{i=1}^{d^{2|S|}}\lambda_i^2\right), \tag{3.38}$$

where

$$\langle i_1 j_1 \ldots i_k j_k | U^{\mathcal{R}_S} | i_{k+1} j_{k+1} \ldots i_N j_N\rangle = \langle i_1 \ldots i_N | U | j_1 \ldots j_N\rangle, \tag{3.39}$$

& $\{\lambda_i\}$ are the Schmidt coefficients of the decomposition of $U_{A_1\ldots A_N}$ with respect to the bipartition $S \cup S'|S^c \cup (S^{'})^c$: $\sum_{i=1}^{d^{2|S|}} \lambda_i = d^N$.

3.3.1.3 Maximal Operator Entanglement

If $U_{A_1\ldots A_N}$ is a N-partite unitary operator isomorphic to the 2N-partite state $|U\rangle_{A_1A'_1\ldots A_NA'_N}$ with $S \subset \{1, \ldots, N\}$, the following conditions are equivalent and sufficient to state that $U_{A_1\ldots A_N}$ is **maximally entangled** with respect to the bipartition $S|S^c$:—

- $E_S(U_{A_1\ldots A_N}) = 1$.
- $S_2(\rho_{S\cup S'}) = 1$, where $S' = \{A'_i : A_i \in S\}$.
- $\lambda_i = d^{N-2|S|}$ $\forall i \in [1 : d^{2|S|}]$, where $\{\lambda_i\}$ are the Schmidt coefficients of the decomposition of $U_{A_1\ldots A_N}$ into $S|S^c$.
- $U^{\mathcal{R}_S}$ is **proportional** to an isometry, where $\langle S_1|U^{\mathcal{R}_S}|S_2\rangle = \langle i_1 \ldots i_N|U|j_1 \ldots j_N\rangle$, $\{S_1 = \otimes i_r j_r : r \in S\}$ & $\{S_2 = \otimes i_r j_r : r \notin S\}$.

3.4.2 The Bipartition Theorem for Maximal Operator Entanglement

The bipartite theorem for maximal operator entanglement is analogous to, and based on the following (well-known) property of multipartite quantum states:—

Property 3.4.3. *If a N-partite state $|\psi\rangle_{A_1\ldots A_N}$ is maximally entangled with respect to the bipartition $S|S^c$, where $S \subset 1, \ldots, N$ and $|S| \leq |S^c|$ without loss of generality, then $|\psi\rangle_{A_1\ldots A_N}$ will be maximally entangled with respect to $S'|(S^{'})^c$ for any $S' \subset S$.*

Theorem 3.4.4. *If $U_{A_1 \ldots A_N}$ is maximally entangled for a bipartition $S|S^c$, with $|S| \le |S^c|$, then U is maximally entangled for the bipartition - $S_1|S_1^c$, for any $S_1 \subset S$.*

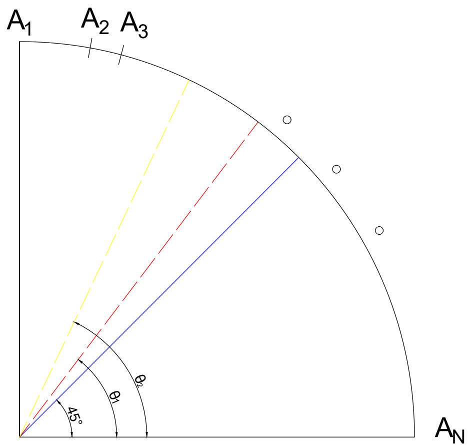

Figure 3.5: The 'notches' or markings on the quarter circle for every $\theta = 90/(N-1)$ represent the N parties numbered from 1 to N without loss of generality. The line in blue, drawn at 45 marks the middle of the set of parties. A bipartition of these N parties in the given order is represented by a radius at an angle θ on the quarter circle with parties on either side of the radius forming the set of parties on either side of the bipartition. Without loss of generality, the S is the set including A_1 and $|S| \le |S^c|$ implies that $\theta \ge 45$. Then, the bipartition theorem states that if $U_{A_1 \ldots A_N}$ is maximally entangled for a bipartition θ_1, it will be maximally entangled for *any* bipartition $\theta_2 : \theta_2 \ge \theta_1 \ge 45$.

Proof. The theorem can be proven by a sequence of arguments connecting the operator picture and the state picture.

1. By Thm. 3.3.2, $U_{A_1 \ldots A_N}$ will be maximally entangled with respect to the bipartition $S|S^c$ iff. the state $|U\rangle_{A_1 A_1' \ldots A_N A_N'}$ related to $U_{A_1 \ldots A_N}$ by the operator-state isomorphism defined by us in Sec. 2.4 is maximally entangled with respect to the bipartition $S \cup S' vert S^c \cup S'^c$ where $S' = \{A_i' : A_i \in S\}$ & $S'^c = A_i' : A_i \notin S$.

2. $U_{A_1 \ldots A_N}$ being maximally entangled with respect to $S|S^c$ implies that $|U\rangle_{A_1 A_1' \ldots A_N A_N'}$ is maximally entangled with respect to the bipartition $S \cup S' vert S^c \cup S'^c$.

3. If $|U\rangle_{A_1A_1'\ldots A_NA_N'}$ is maximally entangled with respect to $S \cup S'|S^c \cup S'^c$, it will be maximally entangled with respect to $S_1 \cup S_1'|S_1^c \cup S_1'^c$ for *any* $S_1 \subset S$, with S_1' & $S_1'^c$ being defined analogously to S'& S'^c.

4. By Thm. 3.3.2, since $|U\rangle_{A_1A_1'\ldots A_NA_N'}$ is maximally entangled with respect to $S_1 \cup S_1'|S_1^c \cup S_1'^c$, $U_{A_1\ldots A_N}$ will be maximally entangled with respect to the bipartition $S_1|S_1^c$.

■

3.4.3 Maximally Entangled Multipartite Operators

Similar to the definition of maximally entangled tripartite operators, we term a multipartite operator as maximally entangled if the operator is maximally entangled with respect to *every* possible bipartition $S|S^c$ i.e. $E_S(U_{A_1\ldots A_N}) = 1$ $\forall S \subset \{1, \ldots, N\} : S \neq \{\}$.

The fact that the bipartition $S|S^c$ is equivalent to $S^c|S$ implies that we only need to check if $E_S(U_{A_1\ldots A_N}) = 1$ $\forall S : |S| \leq \lfloor N/2 \rfloor$. This reduces the complexity of the problem to $O\big((2^N - 2)/2\big)$.

However, the Bipartition Theorem for Maximal Operator Entanglement of Sec. 3.3.2 implies that if $U_{A_1\ldots A_N}$ is maximally entangled $\forall S : |S| = \lfloor N/2 \rfloor$. This, along with the symmetric nature of bipartitions implies that we only have to calculate the entanglement entropy across $\frac{1}{2}\binom{N}{N/2}$ & $\binom{N}{\lfloor N/2 \rfloor}$ bipartitions for even & odd N, respectively.

Thus, the Bipartition Theorem for Maximal Operator Entanglement allows us to greatly reduce the number of bipartitions corresponding to which the entanglement entropy of the operator, $U_{A_1\ldots A_N}$ must be calculated to ascertain if the operator is maximally entangled.

Theorem 3.4.5. *Due to the Bipartition Theorem for Maximal Operator Entanglement, the complexity of the problem of determining if a given multipartite operator,* $U_{A_1\ldots A_N}$ *is maximally entangled reduces from* $O\big((2^N\big)$ *to* $O(2^N/\sqrt{N})$*, where* $O(1)$ *represents the* *complexity of calculating the entanglement entropy of $U_{A_1 \ldots A_N}$ with respect to a constant number of bipartitions.*

Proof. Since the 'Big O' notation of time complexity analysis only considers the limiting behaviour of the computation, we need only consider cases where N is even. Thus, we need to calculate the entanglement entropy corresponding to $\frac{1}{2}\binom{N}{N/2}$ bipartitions.

$$\frac{1}{2}\binom{N}{N/2} = \frac{1}{2}\frac{N\,!}{\left((N/2)\,!\right)^2}$$

Using Stirling's approximation,

$$N\,! \sim \sqrt{2\pi N}\frac{N^N}{N}$$
$$(N/2)\,! \sim \sqrt{\pi N}\frac{(N/2)^{(N/2)}}{N/2}$$

Thus,

$$\frac{1}{2}\frac{N\,!}{\left((N/2)\,!\right)^2} \sim \frac{1}{2}\frac{2^N}{\sqrt{\pi(N/2)}}.$$

If we neglect the scaling factors and only take the terms that scale with N, this corresponds to a complexity of $O(\frac{2^N}{\sqrt{N}})$. ■

3.5 CONSTRUCTION OF LU-INVARIANTS FOR TRIPARTITE OPERATORS

The moments and entropies of each set of LU-invariants associated with an operator, U, characterize the 'extent' of non-locality of the operator. Thus, any attempt to generalize the construction of LU-invariants for increasing number of parties(N) **or** increasing local dimension(d) is of importance. In this section, we provide a scheme for the construction of a class of sets of LU-invariants for increasing N an define entropies using them. The number of such sets of invariants constructed via this construction scheme scales **only** with N and thus, is **not** exhaustive. We also leverage the operator-state isomorphism yet again and prove the connection between the Tsallis-2 entropies of entanglement of the isomorphic state & the corresponding entropies defined using the sets of LU-invariants.

We study the $N = 3$ case separately as in the study of operator entanglement as it marks a transition from the bipartite case where there exists only **one** bipartition with respect to which the operator can be decomposed to a case where multiple decompositions are possible. Based on the trends seen in the tripartite case, we will be better positioned to generalize the construction to the multipartite case.

3.5.1 A scheme for the Construction of a Class of LU-Invariants

In the bipartite case, the Schmidt decomposition of US, where S is the two qudit generalized SWAP gate, led to a new set of LU-invariants but SU leads to the same set of LU-invariants as that created by US. However, the rationale behind the LU-invariance of the Schmidt coefficients of the decomposition of US is :—

$$\begin{aligned} \text{If} \quad U' &:= (u_{A_1} \otimes u_{B_1})U(u_{A_2} \otimes u_{B_2}), \\ U'S &= (u_{A_1} \otimes u_{B_1})U(u_{A_2} \otimes u_{B_2})S \\ &= (u_{A_1} \otimes u_{B_1})US(u_{B_2} \otimes u_{A_2}). \end{aligned}$$

Hence, the Schmidt eigenvalues of U'S are the same as the Schmidt eigenvalues of US.

Thus, the reason that US leads to LU-invariant Schmidt coefficients is that $S^2 = \mathbb{I}$ & the fact that it leaves the local product unitaries as local product unitaries itself when 'pulled' through them.

By a similar analogy, for a tripartite unitary operator, UP will leave the local product unitaries as local product unitaries itself when 'pulled' through them **iff.** P is a permutation matrix of order $d^3 \times d^3$ where d - local dimension under consideration, i.e.

$$\begin{aligned}
\text{If} \quad U' &:= (u_{A_1} \otimes u_{B_1} \otimes u_{C_1}) U (u_{A_2} \otimes u_{B_2} \otimes u_{C_3}), \\
U'P &= (u_{A_1} \otimes u_{B_1} \otimes u_{C_1}) U (u_{A_2} \otimes u_{B_2} \otimes u_{A_3}) P \\
&= (u_{A_1} \otimes u_{B_1} \otimes u_{C_1}) UP \Big(P(u_{A_2} \otimes u_{B_2} \otimes u_{C_2}) P \Big),
\end{aligned}$$

where $\Big(P(u_{A_2} \otimes u_{B_2} \otimes u_{C_2}) P \Big)$ will just be a permutation of these local unitary products.

However, this also requires $P^2 = \mathbb{I}$. This is in general a very non-trivial restriction on permutation matrices. For permutation matrices on 3 qudit states, the only permutation matrices that satisfy these criteria are the **2 qudit SWAP gates** —S_{AB}, S_{BC} & S_{AC}.

However, unlike the case of bipartite operators, there are multiple bipartitions with respect to which the resultant operator can be decomposed in the case of tripartite operators.

Consider S_{AB}. If US_{AB} is decomposed with respect to $A|BC$ or $B|AC$, a new set of LU-invariants will be created. However, the decomposition of US_{AB} with respect to $C|AB$ will **not** produce new LU-invariants. We can show this as follows.

If U_{ABC} has a Schmidt decomposition with respect to $C|AB$ as:—

$$U_{C|AB} = \sum_{i=1}^{d^2} \sqrt{\nu_i} m_i^C \otimes m_i^{AB},$$

then, the decomposition of US_{AB} with respect to this bipartition will be;—

$$U_{C|AB} S_{AB} = \sum_{i=1}^{d^2} \sqrt{\nu_i} m_i^C \otimes (m_i^{AB} S_{AB}).$$

Thus, the orthonormal basis $\{m_i^{AB}\}$ is transformed into $\{m_i^{AB} S_{AB}\}$ but the Schmidt coefficients remain the same as that resulting from the decomposition of U with respect to the bipartition $C|AB$.

Thus, in the case of US' where S' are the generalized two-qudit SWAP operators, only decompositions with respect to bipartitions for which the particular SWAP gate is **not** local will lead to new sets of LU-invariants.

In the case of bipartitions, US produced the same invariants as SU. The equivalent question for tripartite unitary operators is if the sets of LU-invariants corresponding to US_{AB} is different from the sets of LU-invariants corresponding to $S_{AB}U$. We find that the **Schmidt coefficients** of the two operators US_{AB} **&** $S_{AB}U$ **are related** and establish this relationship after analyzing the state equivalent description of the entropic measures derived from these sets of LU-invariants.

Thus, for tripartite operators, the sets of LU-invariants that can be constructed by the scheme described above are the Schmidt coefficients of:—

- US_{AB} decomposed with respect to $A|BC$ & $B|AC$
- US_{AC} decomposed with respect to $A|BC$ & $C|AB$
- US_{BC} decomposed with respect to $B|AC$ & $C|AB$

The notation we adopt for each of these decomposition will be as follows :—

$$US_{AB} = \sum_{i=1}^{d^2} \sqrt{\lambda^{A|BC}_{S_{AB},i}} \;\; m^{A}_{S_{AB},i} \otimes m^{BC}_{S_{AB},i}, \tag{3.40}$$

where $\sqrt{\lambda^{A|BC}_{S_{AB},i}}$ are the Schmidt coefficients of the decomposition of US_{AB} with respect to $A|BC$ & $\sum_{i=1}^{d^2} \lambda^{A|BC}_{S_{AB},i} = d^3$.
Similar notation is followed for the other sets of LU-invariants as well.

Thus, following this scheme, we get a class of **6 sets** of LU-invariants with each Schmidt coefficient within a set of LU-invariants being LU-invariant in itself. However, it is easier to define an entropy on the probability distribution $\{\lambda^{A|BC}_{S_{AB},i}/d^3\}$ and study those. We define the Tsallis-2 entropy of these distributions in the next section —which are essentially the linear operator entanglement entropies of US (S - corresponding two-qudit SWAP) with respect to the corresponding bipartitions.

3.5.2 Tsallis-2 Entropy of the Distribution of the LU-Invariants

If a tripartite unitary operator U_{ABC} is decomposed with respect to the bipartition $A|BC$:

$$US_{AB} = \sum_{i=1}^{d^2} \sqrt{\lambda_{S_{AB},i}^{A|BC}} \; m_{S_{AB},i}^{A} \otimes m_{S_{AB},i}^{BC},$$

with $\sum_{i=1}^{d^2} \lambda_{S_{AB},i}^{A|BC} = d^3$, then $\{\lambda_{S_{AB},i}^{A|BC}/d^3\}$ forms a probability distribution and we denote the Tsallis-2 entropy of this distribution as:—

$$E_{S_{AB}}^{A|BC}(U) \equiv \frac{1}{1-\frac{1}{d^2}}\left(1-\frac{1}{d^6}\sum_{i=1}^{d^2}(\lambda_{S_{AB},i}^{A|BC})^2\right) = E_1(US_{AB}), \tag{3.41}$$

where $E_1(U)$ is the linear operator entanglement entropy of U with respect to the bipartition $A|BC$.

Thus, we get a series of 6 entanglement entropies related to U_{ABC}.

3.5.3 Connection to 2N-Party State Bipartitions

Analogous to the connection between $E(US)$ for bipartite unitary operators and the entropy of entanglement in bipartitions of the corresponding 4-party state as well as the connections between the linear operator entanglement entropies of multipartite unitary operators and the entanglement entropy of bipartitions in the isomorphic state, we establish a connection between the Tsallis-2 entropies defined in the previous subsection and the entanglement entropies in bipartitions of the 6-party state.

We leverage this connection to prove the redundancy of the operator entanglement entropy of $S_\alpha U$, for *any* two-qudit SWAP gate S_α, with respect to *any* bipartition of $\{A, B, C\}$ given the operator entanglement entropy of US_α with respect to all possible bipartitions. Moreover, this connection forms the foundation for our generalization of the construction of LU-invariants to arbitrary N.

The connection between the Tsallis-2 entropies of the class of LU-invariants constructed using the scheme described and the entanglement entropies of the corresponding 6-party state —constructed using the multipartite operator-state isomorphism defined by us —is intuitive. Establishing the connection for the Tsallis-2 entropy of one particular set of LU-invariants facilitates its generalization for the Tsallis-2 entropy of any arbitrary set of LU-invariants.

Here we show the connection between $E_{S_{AB}}^{A|BC}$ $(= E_1(US_{AB}))$ and the entanglement entropy of $|U\rangle_{AA'BB'CC'}$ with respect to the bipartition $AB'|BA'CC'$.

We state:—

$$E_{S_{AB}}^{A|BC} = E_1(US_{AB}) = S_2(\rho_{AB'}). \tag{3.42}$$

Proof. We have proven in Ch. 3 Sec. 4.4 the connection between the matrix reshapings and the reduced density operators. Thus, to prove the above result, we just need to show that $(US_{AB})^{\mathcal{R}_1}$ is equal to a matrix reshaping of U_{ABC} that corresponds to the bipartition $AB'|BA'CC'$ of the 6-party state.

Let $(U_{ABC}S_{AB})^{\mathcal{R}_1} = U^{(x)}$. Then,

$$\begin{aligned}
&\langle i_1 i_2 i_3 | U S_{AB} | j_1 j_2 j_3 \rangle = \langle i_1 j_1 | U^{(x)} | i_2 j_2 i_3 j_3 \rangle \\
\implies &\langle i_1 i_2 i_3 | U | j_2 j_1 j_3 \rangle = \langle i_1 j_1 | U^{(x)} | i_2 j_2 i_3 j_3 \rangle \\
\implies &\langle i_1 i_2 i_3 | U | j_1 j_2 j_3 \rangle = \langle i_1 j_2 | U^{(x)} | i_2 j_1 i_3 j_3 \rangle
\end{aligned}$$

.

$\langle i_1 i_2 i_3 | U_{ABC} | j_1 j_2 j_3 \rangle = \langle i_1 j_2 | U^{(x)} | i_2 j_1 i_3 j_3 \rangle$ is in the 'canonical' form —which allows us to 'read-off' the bipartition which the matrix reshaping corresponds to.

$$\therefore\ U^{(x)} = US_{AB} \text{ corresponds to the bipartition } AB'|BA'CC'.$$

■

The manner in which this can be extended to an arbitrary set of LU-invariants is by noting that $E_1(U) = S_2(\rho_{AA'})$, i.e. the $\mathcal{R}_1$ reshaped unitary corresponds to the bipartition $AA'|BB'CC'$ of the 6-party state. Then $E_1(US_{AB})$ merely swaps A' & B' with respect to the bipartition that $\mathcal{R}_1$ corresponded to, to get $AB'|BA'CC'$. The rationale behind this operation can be explained as:—

- We know that the bipartition that US_{AB}, decomposed with respect to $A|BC$, must correspond to will be related to the bipartition that U decomposed with respect to $A|BC$ is related to. Thus, we start off with the bipartition $AA'|BB'CC'$.

- The effect of S_{AB} is in swapping A' & B' as $\langle i_1 i_2 i_3|US_{AB}|j_1 j_2 j_3\rangle = \langle i_1 i_2 i_3|U|j_2 j_1 j_3\rangle$ and thus, to bring it back to the 'canonical' for where we can 'read-off' the bipartition it corresponds to, we must once again exchange $j_1 \& j_2$. When this operation is performed, we are essentially swapping A'& B' from the bipartition that $U^{\mathcal{R}_1}$ corresponds to —recollect that i_r corresponds to the party A_r & j_r corresponds to A'_r when we connect the matrix reshapings with the state bipartitions.

From the second point, it follows that $S_{AB}U$ would result in a swapping of the A & B parties as it results in the swapping of i_1 & i_2. This algorithm to determine to which bipartition of the 6-party state, a particular decomposition of a particular US_α or $S_\alpha U$ corresponds, can be summarized as:—

1. Check with respect to which operator bipartition is the decomposition being performed.

2. Write down the corresponding state bipartition.

3. For the decomposition of US_α, swap the 'primed' version of the parties that S_α acts on.
   For the decomposition of $S_\alpha U$, swap the 'unprimed' version of the parties that S_α acts on.

Using this algorithm, we determine to which bipartition of the 6-party state each of the 6 sets of LU-invariants correspond and indicate them in the table below. As a reminder, by 'correspond', we mean that the Tsallis-2 entropy of the distribution of the set of LU-invariants will be equal to the Tsallis-2 entropy of entanglement of the 6-party state with respect to the corresponding bipartition.

Operator DecomposedOperator Bipartition	$A\|BC$	$B\|AC$	$C\|AB$
U	$AA' \mid BB'CC'$	$BB' \mid AA'CC'$	$CC' \mid AA'BB'$
US_{AB}	$AB' \mid BA'CC'$	$BA' \mid AB'CC'$	$CC' \mid AA'BB'$
$S_{AB}U$	$BA' \mid AB'CC'$	$AB' \mid BA'CC'$	$CC' \mid AA'BB'$
US_{BC}	$AA' \mid BB'CC'$	$BC' \mid AA'CB'$	$CB' \mid AA'BC'$
$S_{BC}U$	$AA' \mid BB'CC'$	$CB' \mid AA'BC'$	$BC' \mid AA'CB'$
US_{AC}	$AC' \mid BB'CA'$	$BB' \mid AA'CC'$	$CA' \mid AC'BB'$
$S_{AC}U$	$CA' \mid BB'AC'$	$BB' \mid AA'CC'$	$AC' \mid CA'BB'$

Table 3.1: An exhaustive list of the operator decompositions whose Schmidt coefficients form the 6 sets of the class of LU-invariants constructed by the scheme described. The correspondence of these decompositions with bipartitions of the 6-party state is shown.

Inferences:—

- As expected, the decompositions all correspond to state bipartitions of the form $2|4$.
- The Schmidt decompositions of $S_\alpha U$ does **not** produce any *new* set of LU-invariants when compared to the LU-invariants produced by the Schmidt decompositions of US_α.
- The class of LU-invariants produced by this scheme correspond to 9 bipartitions of the 6-party state of the form $2|4$. In total, there are 15 possible bipartitions of the form $2|4$ of a set of 6 parties. The remaining 6 bipartitions are trivially maximally entangled given that by the isomorphism used by us in constructing the 6-party state, $ABC|A'B'C'$ is *already* maximally entangled. Thus, the entanglement entropy in all of the non-trivial bipartitions of the 6-party state of the form $2|4$ corresponds to the Tsallis-2 entropy of a set of LU-invariants constructed by the Schmidt decomposition of US_α with respect to one of the 3 bipartitions —$A|BC$, $B|AC$ or $C|AB$, where S_α is a 2-qudit SWAP gate.

3.6 CONSTRUCTION OF LU-INVARIANTS FOR MULTIPARTITE OPERATORS

Note that even though the Bipartition Theorem of Maximal Operator Entanglement stated in Sec. 3.2 allows us to only consider balanced bipartitions (or *nearly* balanced bipartitions for odd N) when testing for maximal operator entanglement, for a general multipartite operator, $U_{A_1 \ldots A_N}$ the entropy information conveyed by the Schmidt coefficients obtained on decomposing the operator with respect to bipartitions of uneven sizes is **not** contained in that conveyed by the Schmidt coefficients obtained on decomposing the operator with respect to balanced bipartitions.

Thus, we can consider *all* possible Schmidt decompositions of a multipartite operator and the product of the multipartite operator with various generalized SWAP gates, with each one generating a different set of LU-invariants.

However, even for quadripartite operators, this scheme for the construction of LU-invariants becomes increasingly complicated when viewed in the operator picture as the Schmidt decomposition of a quadripartite operator, U_{ABCD} can be performed with respect to bipartitions of the order 1|3 and 2|2 —leading to a total of 7 possible bipartitions with respect to which the operator can be decomposed. Moreover, the number of generalized SWAP gates that we would need to consider also increases greatly. Even **products of SWAP gates** will need to be considered since a subset of such gates do leave the entanglement properties of U unchanged.

In contrast to the increasing complexity of the construction scheme in the operator picture with increasing N, the construction scheme in the state picture remains fairly uncomplicated. Thus, we state the generalization of the LU-invariants construction scheme utilizing the state picture.

3.6.1 A Generalized Scheme for the Construction of LU-Invariants

Consider an N-partite unitary operator, $U_{A_1\ldots A_N}$ with its state equivalent, $|U\rangle_{A_1A_1'\ldots A_NA_N'}$. A set of LU-invariants can be defined in the operator picture by a tuple $(S_{\alpha_1}U_{A_1\ldots A_N}S_{\alpha_2}\ ,\ B_{op} = S_1|S_1^c)$ such that the LU-invariants are the Schmidt coefficients of the decomposition of $S_{\alpha_1}U_{A_1\ldots A_N}S_{\alpha_2}$ with respect to the bipartition, B_{op}. In the state picture, a set of LU-invariants can be defined using a bipartition $B_s = S_2 \cup S_3'|S_2^c \cup (S_3')^c$ such that the LU-invariants are the *square roots* of the singular values of the reduced density operator, $\rho_{S_2\cup S_2'} = \mathrm{Tr}(\rho)$.

Note that any tuple in the operator picture will correspond to a bipartition in the state picture such that $|S_2| + |S_3'|\%2 = 0$.

Moreover, in the definition of a set of LU-invariants in the state picture, even though $S_2 \neq S_3$ in general, $|S_2| = |S_3|$ as for *any* general $S_{\alpha_1}U_{A_1\ldots A_N}S_{\alpha_2}$, the SWAP gates either swap the elements $\{A_1, \ldots, A_N\}$ and displace them two at a time or swap the elements $\{A_1', \ldots, A_N'\}$ and displace them two at a time. This condition implies the former and thus, supersedes it.

Thus, not all bipartitions in the state picture correspond to a set of LU-invariants. We state that any bipartition, $B_s = (S_2 \cup S_3'|S_2^c \cup (S_3')^c)$ in the state picture corresponds to a set of LU-invariants constructed by the generalization of the scheme described in the previous section iff. it corresponds to a tuple $(S_{\alpha_1}U_{A_1\ldots A_N}S_{\alpha_2}\ ,\ B_{op} = S_1|S_1^c)$ in the operator picture, i.e. iff. $|S_2| = |S_3|$.

We do not, at the time of writing, have a cognizance of the equivalents of the other bipartitions of the state picture, in the operator picture. But, we do believe that the entanglement entropy of those bipartitions do represent, in some form, the entanglement features of $U_{A_1\ldots A_N}$.

3.6.2 Mapping of State Bipartitions to Tuples in the Operator Picture

For the bipartitions in the state picture that do correspond to a set of LU-invariants, the mapping back from the state picture to the operator picture can be done. However, note that this mapping is **not** one-to-one. To arrive at **a** (non-unique in general) tuple $(S_{\alpha_1}U_{A_1\ldots A_N}S_{\alpha_2}\ ,\ B_{op} = S_1|S_1^c)$, we construct the following algorithm:—

1. Convert B_s to the following form —$S \cup S'|S^c \cup S'^c$ using **two-party swap operations**.
2. Note down the swap operations performed and their **ordering** and convert them to the equivalent two-qudit SWAP gates.
3. If a swap operation was performed on the parties belonging to $\{A_1, \ldots, A_N\}$, it acts on U from the left. If a swap operation was performed on the parties belonging to $\{A'_1, \ldots, A'_N\}$, it acts from the right.
4. Keeping in mind the order in which the swap operations were performed, the operator-SWAP product —$S_{\alpha_1}U_{A_1\ldots A_N}S_{\alpha_2}$ can be constructed.
5. If the final state bipartition, $S \cup S'|S^c \cup S'^c$, $B_{op} = S|S^c$.

We provide the following illustrative example in the case of a quadripartite operator, U_{ABCD}.

Consider the state bipartition $AB'C'D|A'BCD'$.

Then:—

$$
\begin{aligned}
U &\rightarrow AB'C'D|A'BCD' \\
US_{AB} &\rightarrow AA'C'D|BB'CD' \\
S_{CD}US_{AB} &\rightarrow AA'CC'|BB'DD'.
\end{aligned}
$$

Thus, *one of the* tuples in the operator picture that corresponds to $AB'C'D|A'BCD'$ is $(S_{CD}U_{ABCD}S_{AB}, AC|BD)$.

We can show that this tuple is **not** unique as follows:—

$$
\begin{aligned}
U &\rightarrow AB'C'D|A'BCD' \\
US_{AB} &\rightarrow AA'C'D|BB'CD'
\end{aligned}
$$

$$US_{AB}S_{CD} \to AA'DD'|BB'CC'.$$

Thus, another tuple in the operator picture that corresponds to $AB'C'D|A'BCD'$ is $(U_{ABCD}S_{AB}S_{CD},\ AD|BC)$. Also note that since S_{AB} & S_{CD} commute, even the tuple $(U_{ABCD}S_{CD}S_{AB},\ AD|BC)$ corresponds to the same set of LU-invariants.

Thus, the many-to-one mapping from the state bipartitions to tuples in the operator picture is another motivation for carrying out the generalization of the LU-invariant construction scheme in the state picture.

3.6.3 Salient Features of the Construction Scheme

- The LU-invariants constructed **all** correspond to a bipartition in the state picture. This **implies their independence** as the entanglement entropy across any bipartition, while constrained by the entanglement entropy across other bipartitions, is **not** fixed by them, i.e. specifying the entanglement entropies of $\alpha - 1$ bipartitions does **not** fix the entanglement entropy across the α^{th} bipartition.

- The LU-invariants constructed all correspond to the singular values of some reshaping of U (by the connection with the state bipartitions). Thus, these LU-invariants will be polynomial functions of the entries of the operator matrix.

The only drawback of this construction is that the number of LU-invariants generated does **not** scale with d —only with N. But the number of LU-invariants needed to represent the non-local properties of U scales with **both** N & d.

Thus, for any given (N, d), as d increases, the fraction of the total number of LU-invariants generated by this construction scheme reduces.

3.7 REMARKS

Thus, in this section, we have extended the definition of dual unitaries to multipartite settings by defining maximally entangled operators that maximize **all** the multipartite operator entanglement entropies of U.

Using this notion, we can create non-linear maps analogous to the M_R map that generate such maximally entangled operators.

Note that **not** all maximally entangled operators correspond to AME states as multi-unitarity is a stricter condition to enforce as opposed to maximal operator entanglements.

However, we hope to use the idea of maximal operator entanglement and the class of LU-invariants defined to construct an efficient version of the M_{TR} map that works in the multipartite case —effectively for at least small N and small d. At the time of writing, though we came up with an extension of the M_{TR} map for multipartite settings that works effectively to generate 3-unitaries of order 8, the extension scheme is **not** very rigorous and fails to generate 4-unitaries of order 256. Whether this is due to the fact that we are unable to pick 'good' reshapings to define our non-linear maps or if for higher N, the initial seed must possess certain characteristics is unknown.

Nonetheless, in the next two chapters, we examine the efficiency of the nonlinear maps so as to give an indication as to the difficulty of the problem we are faced with. We then state our extension of the M_{TR} map for multipartite settings, its efficiency when generating 3-unitaries of order 8 —the smallest (N,d) of interest beyond N=4[2] and the failure of the generalized 'M_{TR}' when dealing with the construction of 4-unitaries of order 256. The findings of the next two chapters further motivate the necessity of a rigorous extension of the M_{TR} map based on the study of operator entanglement.

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[2] N=5 is not considered as by the theorem proven in Ch. 2, only AME states of even parties need to be considered given that they exist—and we know that AME(6,2) exists.

CHAPTER 4

CREATING ENSEMBLES OF DUALS, T-DUALS AND 2-UNITARY GATES

4.1 INTRODUCTION

The aim of this chapter is to define the nonlinear maps used in (10) and probe their efficiency with respect to convergence as well as convergence rates with respect to the number of iterations, n.

By probing the efficiency of the maps, we aim to understand the reason for their effectiveness and their shortcomings. These learnings will be used to construct a scheme for the extension of the M_{TR} map to arbitrary N, to generate multi-unitary operators.

During the course of the studied conducted, we also noted that all the **period cycles** of the M_R map are of period-2 —a result that allows us to better comprehend the space of these operators and the space of these operators under influence of the nonlinear maps.

4.2 THE NONLINEAR MAPS

The previous chapter motivated and formalised the notion of duals, T-duals and 2-unitary gates. Though the existence of these classes of unitaries has been long known (9) —with duals being referred to as *bi-unitaries* in mathematical parlance since (2), there existed no subroutine capable of generating these classes of unitaries until the work of Suhail et al. (10) which introduced nonlinear maps capable of generating ensembles of these classes of unitaries.

Here, we introduce the nonlinear maps formalised and outline arguments for their efficiency. A numerical analysis of their efficiency is provided in the next section where tests on the convergence and the convergence rates are performed on these maps.

4.2.1 The Realignment-Nearest-Unitary Map (M_R)

The $M_R : U(d^2) \mapsto U(d^2)$ map, as defined in (10) is in two stages:—

1. The linear stage: $U \mapsto U^{\mathcal{R}}$, i.e. the realignment operation is performed on the input unitary.

2. The non-linear stage: $U^{\mathcal{R}} \mapsto V$, the nearest unitary to $U^{\mathcal{R}}$ —where the distance measure used is the Hilbert-Schmidt norm[1]. It is also known that if $U^{\mathcal{R}} = V\sqrt{U^{\mathcal{R}\dagger}\mathcal{U}^{\mathcal{R}}}$, the nearest unitary to $U^{\mathcal{R}}$ is V.

Then, given a seed unitary, U_0, we construct $U_n = M_R^n(U_0)$. It is known that the distance of $U_n^{\mathcal{R}}$ to its nearest unitary is *non-increasing* with iteration n. Thus, with each iteration, the distance of $U_n^{\mathcal{R}}$ from its nearest unitary is non-increasing[2]. Moreover, duals are **period-2 fixed points** of the map. If these fixed points are attracting, U_n tends to become dual.

4.2.2 The Partial Transpose-Nearest-Unitary Map (M_T)

The $M_T : U(d^2) \mapsto U(d^2)$ map is defined in (10) in a manner analogous to the definition of the M_R map with the realignment operation in the linear stage of the M_R map being replaced by the partial transpose operation.

Arguments regarding the convergence of the map to T-dual are analogous to those made for the convergence of the M_R map to duals with T-duals being **period-2 fixed points** of the M_T map.

4.2.3 The Realignment-Partial Transpose-Nearest Unitary Map (M_{TR})

The $M_{TR} : U(d^2) \mapsto U(d^2)$ map is defined in (10) in two stages:—

1. The linear stage: $U \mapsto U^{\mathcal{R}} \mapsto U^{\mathcal{R}\Gamma}$, i.e. the realignment operation followed by the partial transpose operation is performed on the input unitary.

2. The non-linear stage: $U^{\mathcal{R}\Gamma} \mapsto V$, the nearest unitary to $U^{\mathcal{R}\Gamma}$ —where the distance measure used is the Hilbert-Schmidt norm [3]. It is also known that if $U^{\mathcal{R}\Gamma} =$

[1] If the distance measures used are unitarily invariant, the definition of the *closest unitary* is norm invariant.

[2] Note that if this distance were **decreasing**, convergence of the map to dual unitaries could be easily concluded. The property of non-increasing is not conclusive to that effect.

[3] See footnote 1

$V\sqrt{U^{\mathcal{R}\Gamma\dagger}U^{\mathcal{R}\Gamma}}$, the nearest unitary to $U^{\mathcal{R}\Gamma}$ is V.

Then, given a seed unitary, U_0, we construct $U_n = M_{TR}^n(U_0)$. Similar to the M_R map, it is known that the distance of $U^{\mathcal{R}\Gamma}$ to its nearest unitary is *non-increasing* with iteration n. Thus, with each iteration, the distance of $U^{\mathcal{R}\Gamma}$ from its nearest unitary is non-increasing[4]. Moreover, 2-unitaries are **period-3 fixed points** of the M_{TR} map. If these fixed points are attracting, $U^{\mathcal{R}\Gamma}$ tends to become 2-unitary.

4.2.4 The $M_T - M_R$ Map ($M_T M_R$)

The $M_T M_R$ map is constructed by successively performing the M_R & the M_T maps. Thus, it is similar to the M_{TR} map but for an additional nearest-unitary projection performed *after* the realignment operation. Analogous to the M_{TR} map, 2-unitaries are **period 3 fixed points** of the $M_T M_R$ map as well, since the only point of difference between the two maps lies in the additional projection subroutine that is supplemented in the case of the $M_T M_R$ map. However, 2-unitaries are perfect tensors with **every** reshaping of them being unitary. Thus, the additional projection operation will not affect the dynamics of 2-unitaries, which will thus undergo the same evolution under the $M_T M_R$ and the M_{TR} maps.

4.3 ENSEMBLES OF THESE SPECIAL CLASSES OF UNITARIES

Given the constructions of non-linear maps that can generate these special classes of unitaries, we can now define ensembles of these classes of unitaries.

The ensembles termed as ‘dual-CUE(n)’ & ‘T-dual-CUE(n)’ are defined in (10). We define, in this work, the ‘M_{TR}^{∞} (CUE(n))’ and $M_T M_R^{\infty}$(CUE(n)) ensembles in an analogous manner —albeit with an important caveat that these ensembles are **not** ensembles of 2-unitaries in general —given the non-convergence of these maps for $d > 3$.

[4]See footnote 2

One particular category of ensembles is the set of ensembles generated on iteratively running the non-linear maps on CUE(n) sampled random unitaries to obtain asymptotic behaviour. Symbolically, these ensemble of unitaries are $M_R^\infty(CUE(n))$ and $M_T^\infty(CUE(n))$ —termed as 'dual-CUE(n)' and 'T-dual-CUE(n)' —with n being the size of the unitaries. However, for practical considerations, only a finite number of iterations are performed.

The 'M_{TR}^∞ (CUE(n))' and the $M_T M_R^\infty$(CUE(n)) ensembles are the ensembles of unitaries obtained on iteratively running the M_{TR} and the $M_T M_R$ maps on CUE(n). As described earlier, these are **not** ensembles of 2-unitaries.

4.4 THE EFFICIENCY OF THE NONLINEAR MAPS

In this section, we examine the non-linear maps with regard to their efficiency in producing duals, T-duals and 2-unitaries for local dimensions, d, ranging from 2 to 8. A subset of this analysis has been performed in (10). We extend their analysis to dimensions of interest to us —and add the corresponding analysis for the $M_T M_R$ map.

We also highlight the $d = 2$ case in which whilst there are **no** 2-unitaries, serves as an interesting case study for the same reason. Moreover, the efficiency of the M_R map in generating duals for this case is leveraged in the numerical study of the k_D problem described in Shrigyan Brahmachari's thesis —a project *entangled* with me.

4.4.1 The M_R map

The efficiency of the M_R map lies in its proclivity to generate duals and to do so in fewer iterations, n. Thus, there are two aspects of the efficiency of the M_R map to be probed—its 'asymptotic' efficiency and its convergence rate with regard to n.

To examine the 'asymptotic' convergence, which is mathematically as $n \to \infty$, we take $n = 500$ and examine the distribution of the linear operator entanglement entropy —E(U_n) for unitaries obtained on evolving CUE distributed unitaries for n iterations under the M_R map. We claim that $n = 500$ gives an accurate characterization of the distribution for $n\infty$, we verify that the means and the variances for $n = 500$ & $n = 1000$ are approximately equal. These tests have been performed for all the convergence tests.

The prospect of examining the convergence rate for a given local dimension d arises only if the M_R map asymptotically converges to duals for *most* seeds of local dimension d. We probe convergence rate by taking initial seeds, U, and plotting their linear operator entanglement entropy E(U_n) on evolving them under M_R.

2.3.1.1 Asymptotic convergence

In each of the graphs below, **in pink**[5] is the distribution of E(U) on the CUE and **in red** is the distribution of E(U) for the ensemble obtained by iterating the M_R map 500 times on the CUE —viz. called the 'dual CUE'[6]The range of E(U) values of the samples is on the horizontal axis and the number of samples, out of a total of 10^4, is on the vertical axis. The distribution of E(U) for the CUE serves as a contrast against the distribution of E(U) for the 'dual CUE' to further highlight the efficiency of the M_R map.

The parameters of interest are :—

1. $\overline{E(U)}$ —Average value of E(U) for the given distribution.

2. η —The percentage of duals generated using the given distribution.

3. η_{app} —The percentage of unitaries for which E(U) is within 10^{-4}% of the maximum value. These can be considered as duals, upto a certain tolerance, accounting for numerical errors in the implementation of the algorithm.

[5]To account for red-green colour blindness

[6]In actuality, the moniker 'dual CUE' is used to refer to the ensemble obtained on acting the M_R map on CUE n times —as $n \rightarrow \infty$. However, for $n = 500$, the ensemble begins to assume its asymptotic behaviour up to a certain tolerance (a claim supported by studies in the Appendix) and this thus justifies our usage of the term *as well as* our study of the asymptotic convergence of the map with 500 iterations.

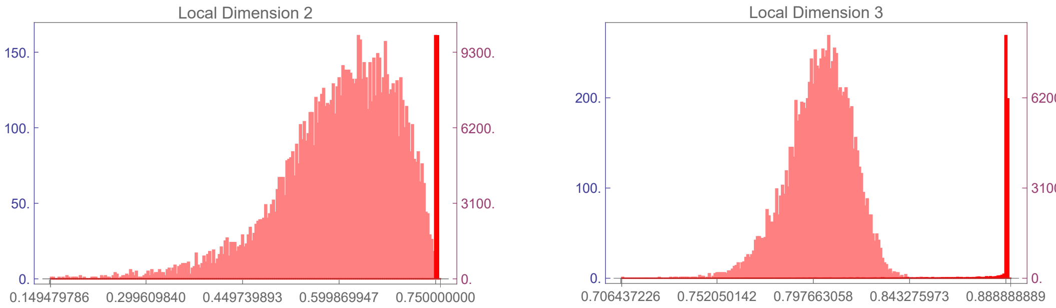


Figure 4.1: For CUE(4), $\overline{E(U)} = 0.5984$, $\eta, \eta_{app} = 0$. For 'dual-CUE(4)', $\overline{E(U)} = 0.75$(maximum) with $\eta = 99.94\%$ & $\eta_{app} = 100\%$.
For CUE(9), $\overline{E(U)} = 0.8$, $\eta, \eta_{app} = 0$. For 'dual-CUE(9)', $\overline{E(U)} = 8/9$(maximum) with $\eta = 61.17\%$ & $\eta_{app} = 95.39\%$.

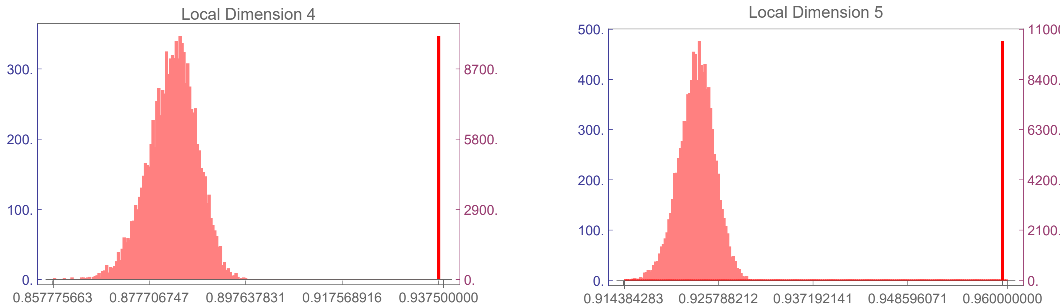


Figure 4.2: For CUE(16), $\overline{E(U)} = 0.8823$, $\eta, \eta_{app} = 0$. For 'dual-CUE(16)', $\overline{E(U)} = 0.937486$(E(U) for duals = 0.9375) with $\eta = 0.03\%$ & $\eta_{app} = 5.11\%$. **All** the 'dual-CUE' unitaries fall within $10^{-2}\%$ of the maximum value.
For CUE(25), $\overline{E(U)} = 0.9230$, $\eta, \eta_{app} = 0$. For 'dual-CUE(25)', $\overline{E(U)} = 0.959976$(maximum = 0.959996, E(U) for duals = 0.96) with $\eta = 0\%$ & $\eta_{app} = 0\%$. However, **all** the 'dual-CUE' unitaries fall within $10^{-2}\%$ of the maximum value $\Rightarrow$ they are practically duals.

2.3.1.2 Convergence rate

Since for all the local dimensions probed in the previous section, the M_R map generates duals, *up to a reasonable tolerance*, for all seeds, here we contrast the convergence rate of the map for different local dimensions. To ensure uniformity of initial conditions across local dimensions, each convergence plot begins with a product unitary. While we examined the convergence starting with various initial seeds, here is provided the evolution of only one seed per local dimension —for representative purposes.

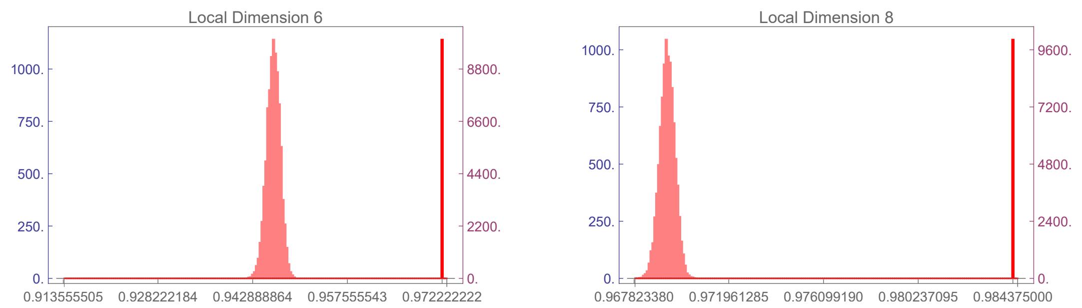
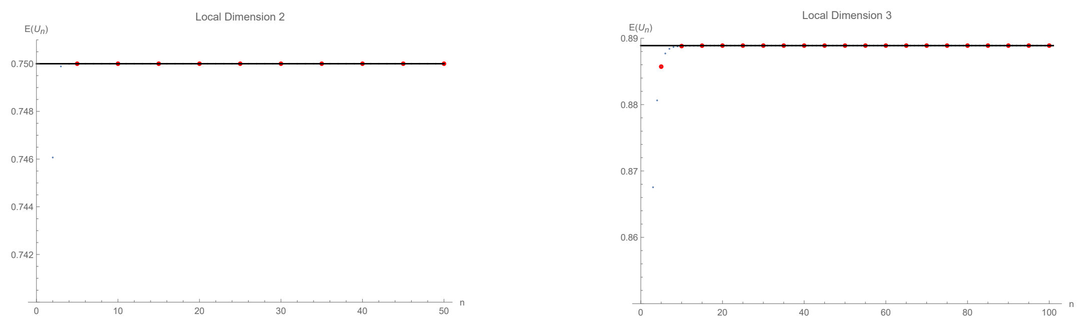


Figure 4.3: For CUE(36), $\overline{E(U)} = 0.9459$, $\eta, \eta_{app} = 0$. For 'dual-CUE(36)', $\overline{E(U)} = 0.972205$(E(U) for duals = $0.97\overline{2}$) with $\eta = 0\%$ & $\eta_{app} = 0\%$. However, once again, **all** the 'dual-CUE' unitaries fall within $10^{-2}\%$ of the maximum value.
For CUE(64), $\overline{E(U)} = 0.96923$, $\eta, \eta_{app} = 0$. For 'dual-CUE(64)', $\overline{E(U)} = 0.98436$(maximum = 0.984368, E(U) for duals = 0.984375) with $\eta = 0\%$ & $\eta_{app} = 0\%$. But once again, **all** the 'dual-CUE' unitaries fall within $10^{-2}\%$ of the maximum value.



From the explorations made of the asymptotic convergence & the convergence rate of the M_R map, we draw the following inferences :—

1. There is numerical evidence to suggest that the set of duals is a set with measure zero with respect to the Haar measure for the local dimensions studied. This motivates the definition of the M_R map and shows its power.

2. The average value of E(U) with respect to the Haar measure increases as a fraction of the maximum value as the local dimension increases.

3. There is numerical evidence to suggest that the M_R map asymptotically converges to duals with a probability tending to 1 for CUE-sampled unitaries.

4. The convergence rate of the M_R map in producing duals reduces as the local dimension increases.

5. The efficiency of the M_R map is also a statement on the efficiency of the M_T **map** as well. This fact is extrapolated from the symmetry between the definitions of the reshuffling and the partial transpose operations —which consequently translates

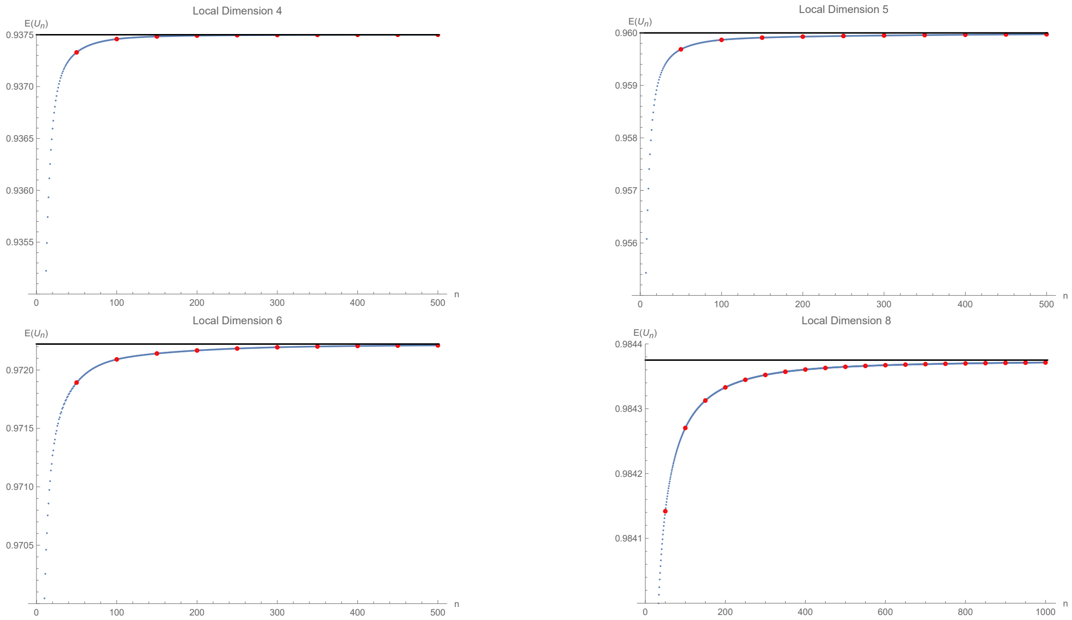


Figure 4.4: Growth of $E(U_n)$ for d = 2-6 and 8. For d=2 and d=3, $E(U_n)$ converges to the maximum possible value of $(d^2-1)/d^2$ for $n \approx 5$ & $n \approx 20$ respectively. Beyond d=3, the convergence rate drops. However, for sufficiently large n, for all the local dimensions considered, $E(U_n)$ converges to that of duals.

into a symmetry between the two maps. Thus, the efficiency of the M_T map need not be studied independently of the efficiency of the M_R map.

2.3.1.3 Generation of 2-unitaries using the M_R map

In this section, we study the tendency of the M_R map to produce 2-unitaries in certain local dimensions.

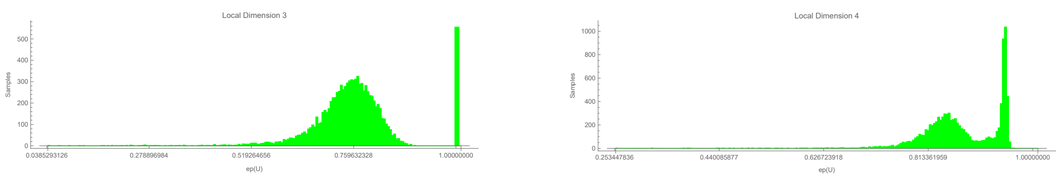


Figure 4.5: For d=3, the M_R map generates 2-unitaries for $\approx 6\%$ of input seeds, the seeds being sampled with the Haar measure. However, the same is **not** seen for d=4 with the maximum entangling power generated being 0.963074.

Thus, we find that only for $d = 3$, does the M_R map result in 2-unitaries with an appreciable, non-zero measure in the 'dual-CUE' space. The percentage of unitary seeds sampled via the Haar measure that end up as 2-unitaries for d=3 is approximately 5.5-6%[7].

However, in $d = 4$, we do not find 2-unitaries in the 'dual-CUE' space with the entangling power attaining a maximum value of ≈ 0.96. For the local dimensions greater than 4 analysed, we was unable to generate 2-unitaries using the M_R map. Thus, there is strong numerical evidence that suggests that the M_R map only produces 2-unitaries for $d = 3$. Venturing to conjecture at a possible explanation for this observation, we argue that the fact that the $d = 3$ case is a transition case between the $d = 2$ space where there are **no** 2-unitaries and the $d = 4$ space influences this behaviour.

Thus, the inability of the M_R map to generate 2-unitaries for local dimensions greater than 3 motivates the need for the M_{TR} map.

2.3.1.4 Dynamics of the M_R Map

[Joint result with Shrigyan Brahmachari]

Theorem 4.4.1. *If M_R map's eventually runs into a period cycle, it must be a period-2 cycles.*

Proof. We know that the projective distances of the M_R map is non-increasing.Now if for some initially seed, if these distances were in fact simply *decreasing*,by using a Banach Fixed Point theorem like arguement,we can show that it would eventually converge to 0, which would imply that the map has found a Dual-unitary.
On the converse of this set of seeds, would at some point would have projective distances which stop decreasing; then two consecutive projective distances must be equal;i.e, for

[7]The test has been performed multiple times with the percentage of 2-unitaries being generates being in this range.

some n, $d_n = d_{n+1}$.Now,

$$d_n^2 = \min_{W \in \mathbb{U}(d^2)} \|U_n^R - W\|^2 = \|U_n^R - U_{n+1}\|^2$$

$$= \|U_{n+1}^R - U_n\|^2$$

Which follows from fact that the Hilbert-schmidt distance is invariant to reshapings.Equality holds due to our assumption about the seed.

$$d_n^2 = \min_{W \in \mathbb{U}(d^2)} \|U_{n+1}^R - W\|^2 = \|U_{n+1}^R - U_{n+2}\|^2 = d_{n+1}^2$$

However in the previous lemma we have shown that for operator,and therefore U_{n+1}^2 has an unique projection (i.e, only one such unitary can have the same minimum H-S distance from U_{n+1}^2).Therefore,

$$U_n = U_{n+2}$$

, Which would imply that $U_{n+1} = U_{n+3}$,which would in turn imply that this is a period-2 map. Note that dual-unitaries are a special case of this where $U_n^R = U_{n+1}$.
It is important to add that our numerical studies have, however,not resulted in finding any such non-dual fixed points at all.Whether this is due to the nature and vastness of the space we are exploring, or that the might perhaps be decreasing after all, isn't clear.

4.4.2 The M_{TR} map

The efficiency of the M_{TR} map lies in its ability to generate 2-unitaries and to do so in fewer iterations. In this section, we probe its asymptotic efficiency and the convergence rate. The parameters that characterize the efficiency of the M_{TR} map are similar to those used in the study of the efficiency of the M_R map but for the use of the entangling power, e_p, and its average value to characterise the efficiency of the M_{TR} map. Analogous to the maximisation of $E(U)$ by duals, 2-unitaries maximise $e_p(U)$ —thus motivating the use of the entangling power as a measure to characterise the efficiency o the M_{TR} map.

2.3.2.1 Asymptotic convergence

In each of the graphs below, the distribution of the entangling power, $e_p(U_n)$, under the M_{TR} map is shown, after $n = 500$ iterations. we dub the ensemble obtained by iterating the M_{TR} map 500 times on the corresponding CUE as the 'M_{TR} CUE(n)' ensemble with the hypothesis that the ensemble thus generated encapsulates the features of the ensemble obtained by running the M_{TR} map as $n \longrightarrow \infty$. The insets show the distributions of $e_p(U)$ for the corresponding CUE to further highlight the efficiency of the M_{TR} map.

∴ The parameters of interest are :—

1. $\overline{e_p(U)}$ —Average value of e_p for the given distribution.

2. η —The percentage of 2-unitaries generated using the given distribution.

3. η_{app} —The percentage of unitaries for which $e_p(U)$ is within 10^{-4}% of the maximum value[8]. These can be considered as 2-unitaries, up to a certain tolerance —accounting for numerical errors in the implementation of the algorithm.

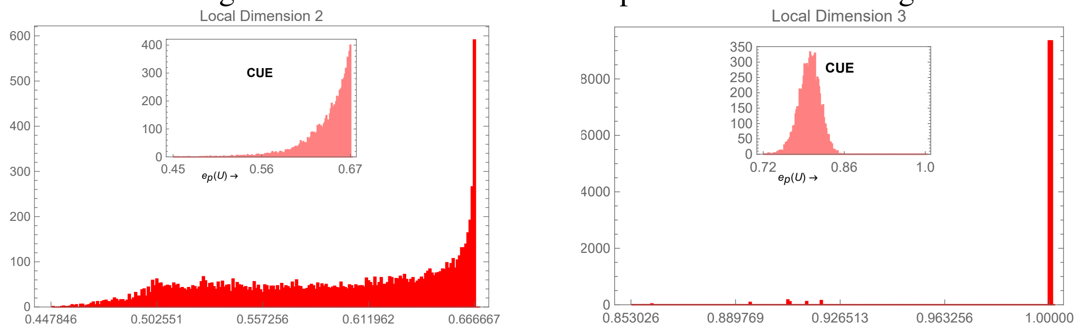


Figure 4.6: For CUE(4), $\overline{e_p(U)} = 0.601$, $\eta, \eta_{app} = 0\%$. For 'M_{TR} CUE(4)', $\overline{e_p(U)} = 0.610$ with $\eta \& \eta_{app} = 0\%$ as expected as there are **no** 2-unitaries of size 4×4.
For CUE(9), $\overline{e_p(U)} = 0.8$, $\eta, \eta_{app} = 0$. For 'M_{TR} CUE(9)', $\overline{e_p(U)} = 0.994$ with $\eta = 93.49\%$ & $\eta_{app} = 93.49\%$.

[8]Note that due to the re-normalisation of $e_p(U)$, its maximum value is 2/3 for $d = 22$-unitaries & 1 for $d \neq 2$

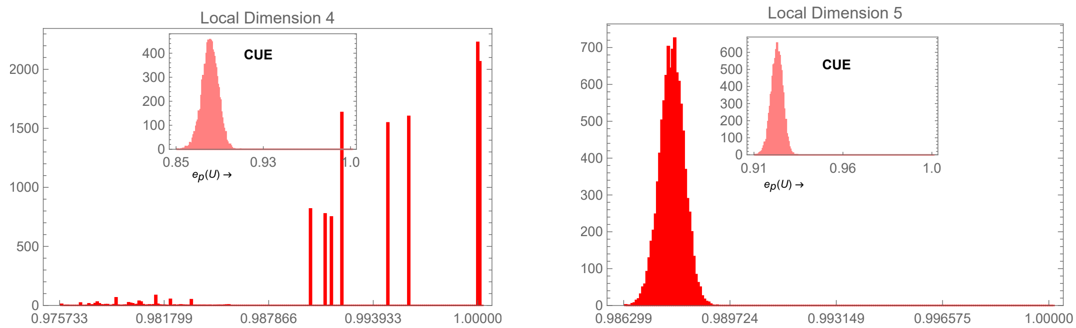


Figure 4.7: For CUE(16), $\overline{e_p(U)} = 0.8824$, $\eta, \eta_{app} = 0\%$. For 'M_{TR} CUE(16)', $\overline{e_p(U)} = 0.9938$ with $\eta = 20.65\%, \eta_{app} = 22.16\%$.
For CUE(25), $\overline{e_p(U)} = 0.9231$, $\eta, \eta_{app} = 0$. For 'M_{TR} CUE(25)', $\overline{e_p(U)} = 0.9878$ with η & $\eta_{app} = 0\%$.

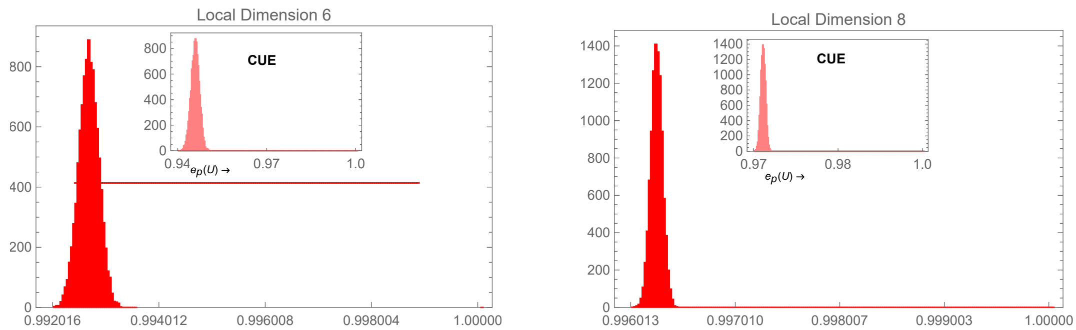


Figure 4.8: For CUE(36), $\overline{e_p(U)} = 0.9459$, $\eta, \eta_{app} = 0$. For 'M_{TR} CUE(36)', $\overline{e_p(U)} = 0.9927$ with η & $\eta_{app} = 0\%$.
For CUE(64), $\overline{e_p(U)} = 0.9692$, $\eta, \eta_{app} = 0$. For 'dual-CUE(64)', $\overline{e_p(U)} = 0.9965$ with $\eta = 0\%$ & $\eta_{app} = 0\%$.

2.3.2.2 Convergence rate

Only for $d = 3$ & $d = 4$, does the M_{TR} map produce 2-unitaries. Thus, studies on the convergence rate of the map can only be performed for these local dimensions. Moreover, for $d = 4$, the map does **not** converge to 2-unitaries for $\approx 80\%$ of all initial seeds sampled from CUE(16). Thus, when we contrast the convergence rates for $d = 3$ & $d = 4$ in this section, it is with the added caveat that the convergence rate is contingent on the map itself converging. From the explorations made of the asymptotic convergence & the convergence rate of the M_{TR} map, we draw the following inferences :—

1. There is numerical evidence to suggest that the set of 2-unitaries is a set with measure zero with respect to the Haar measure for the local dimensions studied.

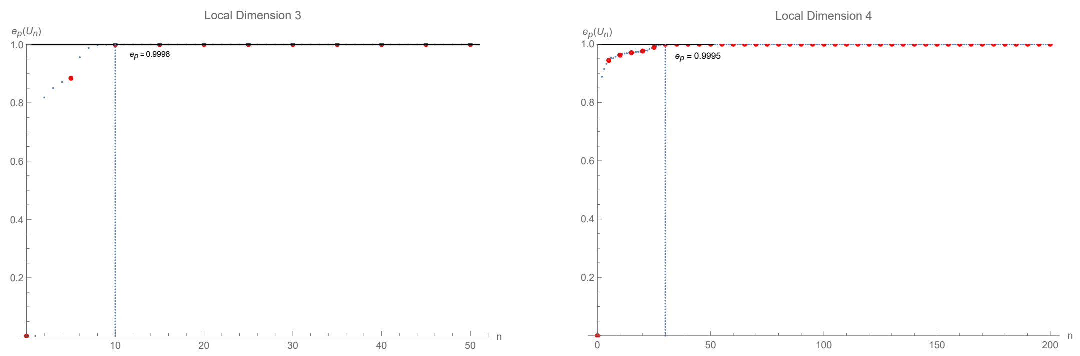


Figure 4.9: Growth of $e_p(U_n)$ for $d = 3$ & $d = 4$. For $d = 3$, at the end of 10 iterations, the $e_p(U_n)$ value is ~ 0.9998 while for $d = 4$, it takes ≈ 30 iterations for most initial seeds.

This motivates the definition of the M_{TR} map and shows its power.

2. The average of the entangling power with respect to the Haar measure increases as the local dimension increases.

3. The average of the entangling power with respect to the Haar measure is functionally equivalent to $(d^2 - 1)/(d^2 + 1)$ —as stated in [Bhargavi].

4. For $d = 3$, the M_{TR} map produces 2-unitaries with a very high probability.

5. The M_{TR} map can be used to produce 2-unitaries for $d = 4$ as we show in [Chapter: (4,4)]. However, it is necessary to work with a large number of initial seeds as the map converges to 2-unitaries at a rate of *only* $\sim 21\%$.

2.3.2.3 Comparison with the $M_T M_R$ Map

It is possible that the $M_T M_R$ map is more powerful than the M_{TR} map in terms of asymptotic convergence or convergence rates given the additional projection subroutine after each reshaping. Here, we contrast the effectiveness of the $M_T M_R$ map with that of the M_{TR} map.

Asymptotic Convergence of the $M_T M_R$ Map:—

Convergence Rate of the $M_T M_R$ map contrasted with that of the M_{TR} map:—

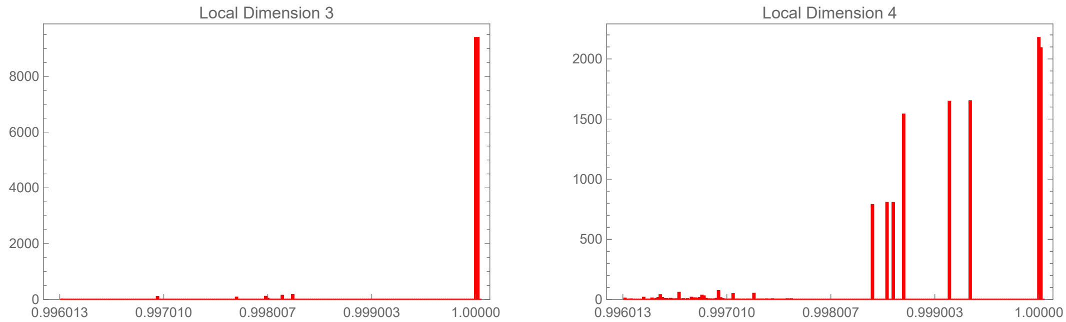


Figure 4.10: For the ensemble generated by the action of the $M_T M_R$ map on CUE(9), $\overline{e_p(U)} = 0.994$, $\eta, \eta_{app} = 93.87\%$ as opposed to $\overline{e_p(U)} = 0.994$, $\eta, \eta_{app} = 93.49\%$ On the other hand, the ensemble generated by the action of the $M_T M_R$ map on CUE(16), $\overline{e_p(U)} = 0.9939$, $\eta = 20.92\%, \eta_{app} = 21.77\%$. For '$M_{TR}$ CUE(16)', $\overline{e_p(U)} = 0.9938$ with $\eta = 20.65\%$ & $\eta_{app} = 22.16\%$.

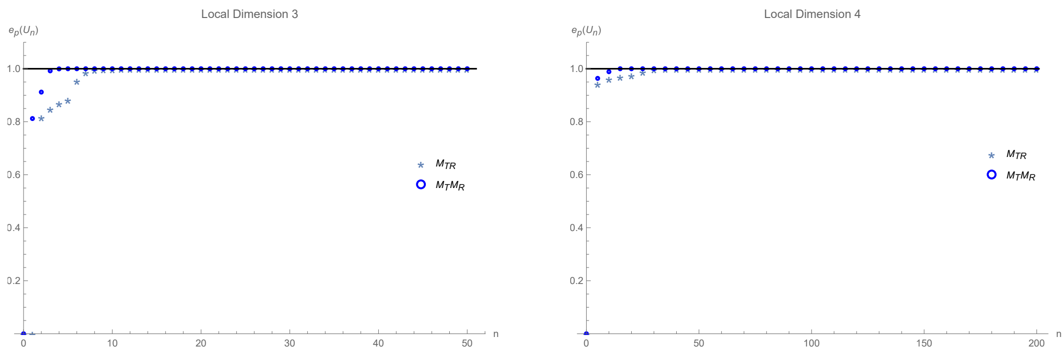


Figure 4.11: For both $d = 3$ & $d = 4$, the convergence rate of the generated seeds under the $M_T M_R$ map is superior to that of the M_{TR} map.

From the explorations made contrasting the $M_T M_R$ and the M_{TR} maps, we conclude that:—

1. The $M_T M_R$ map is as efficient as the M_{TR} map for considerations of asymptotic convergence.

2. Neither the $M_T M_R$ map nor the M_{TR} map converge to 2-unitaries for CUE sampled seeds with an appreciable probability.

3. The $M_T M_R$ map converges to 2-unitaries at a faster rate as compared to the M_{TR} map.

4. The time complexity of the projection subroutine is $O(n^3)$. Thus, the convergence rate of the $M_T M_R$ map outstripping than that of the M_{TR} map by the margins depicted in the studies shown does not necessarily lead to faster convergence *with*

*respect to **time***. Practically, we find that the M_{TR} map exhibits faster convergence with respect to time in comparison to the $M_T M_R$ map. Thus, for subsequent studies on bipartite unitaries, the M_{TR} map has been chosen.

4.5 IMPORTANT FAMILIES OF $\mathcal{U}(9)$ UNITARIES

- Joint result with Shrigyan Brahmachari

The lack of the existence of a simple form like the Cartan Decomposition for dimensions that are not powers of 2 implies that even lower dimensions like d=3 can be quite complicated to study. Moreover, d=3, is of special interest because it acts like a transitionary class from the relatively simply d=2 case to higher dimensions, while being numerically approachable. In this section, we study 3 families of $\mathcal{U}(9)$ motivated by the Cartan decompositon of 2-qubit gates.

Let $\vec{c} = c_1 x + c_2 y + c_3 z \in \mathbb{R}^3$

1. $e^{(i\cdot\vec{c}\cdot(S_x\otimes S_x x + S_y\otimes S_y y + S_z\otimes S_z z))}$
2. $e^{(i\cdot c_1 * S_x\otimes S_x)} * e^{(i\cdot c_2 * S_y\otimes S_y)} * e^{(i\cdot c_3 * S_z\otimes S_z)}$
3. $e^{(i\cdot\vec{c}\cdot((S_x\otimes S_x)^2 x + (S_y\otimes S_y)^2 y + (S_z\otimes S_z)^2 z))}$

Where S_x, S_y & S_z. are the Spin-1 matrices. These seem like natural extensions of the Cartan 2-qubit decomposition and have some useful properties.

Note:— While the Spin-1 operators do not have the same anti-commutation properties that Spin-$\frac{1}{2}$ operators enjoy, the —$(S_w e \otimes S_i)^2$,$(S_j \otimes S_j)^2$ commute $\forall$ i, j $\in$ $[1:3]$

$$\implies e^{(i\cdot\vec{c}\cdot((S_x\otimes S_x)^2 x + (S_y\otimes S_y)^2 y + (S_z\otimes S_z)^2 z))} \tag{4.1}$$

$$= e^{(i\cdot c_1 (S_x\otimes S_x)^2)} e^{(i\cdot c_2 (S_y\otimes S_y)^2)} e^{(i\cdot c_3 (S_z\otimes S_z)^2)} \tag{4.2}$$

It is instructive to study how these families populate the $e_p - g_t$ plots, but for the sake of brevity, we have added this in the Appendix.

Now we list the outstanding properties that make these families a subject of our study:

1. We have been able to show that the 3^{rd} family is entirely a subset of T-Duals. This gives us a large continuous family of T-dual operators, and by multiplying by the generalized 2 qutrit Swap gate, a family of duals.

2. In the past investigations in our group, while M_R and M_{TR} shows great success for $d \leq 3$, its efficiency to produce dual unitaries and 2-unitaries decreases for higher dimensions. This warrants a closer look at the dynamics of these maps in d=3, where this transition happens. This is a difficult task due to the fact that it is difficult to generate random matrices that do not converge to duals and 2-unitaries under the M_R and M_{TR} respectively-essentially indicating that the measure associated with such matrices is very small (w.r.t. the Haar measure). However, we find that random operators from both the 1^{st} and 2^{nd} families rarely converge to 2-unitary matrices under the influence of the M_{TR} maps, which makes them a precious resource for an analytical investigation of these maps.

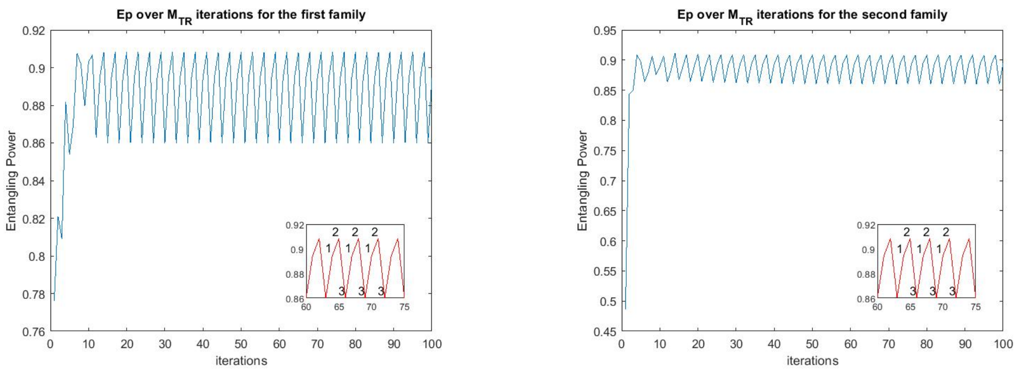


Figure 4.12: We observe that the entangling power values over iterations M_{TR} are repeating in a period-3 fashion, which follows from the resultant period-3 fixed orbit that we have observed.

Thus, the first 3 families of unitary operators in $\mathcal{U}(9)$ are a useful resource.

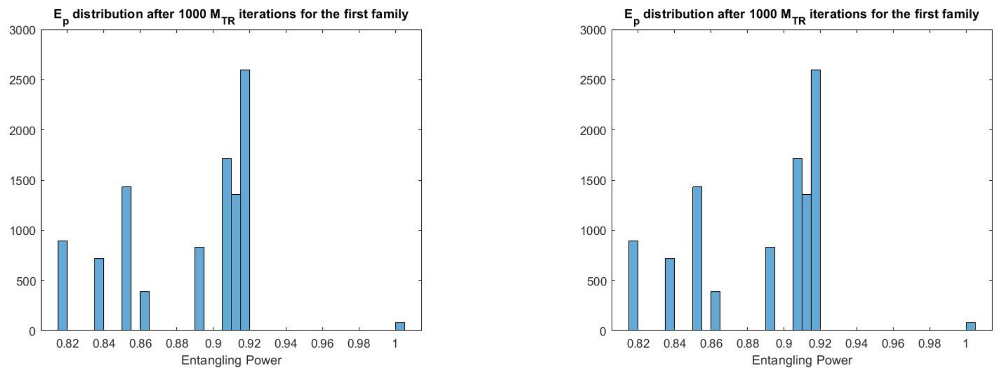


Figure 4.13: We see that only a small fraction $\sim 0.79\%$ become 2-unitaries,we have found that the rest gather into some set of period-3 fixed points. On the other hand, as shown earlier in this chapter, the efficiency of the M_{TR} map on $\mathcal{U}(9)$ is $\sim 94\%$.

4.6 REMARKS

As we observe through the non-convergence of the M_{TR} map to 2-unitaries for local dimensions $d > 3$ and as we observe via our work in the construction of multi-unitary operators, explained in Ch. 5, there is a need for a rigorous study of operator entanglement, to guide us in our aim to construct 2-unitary operators, and in general, multi-unitary operators of arbitrary size.

Moreover, the study of multipartite operator entanglement is very well motivated in itself —independent of the AME problem. We have stated some of the applications of the study of operator entanglement in the introductory section of Ch. 3.

This chapter also illustrated the manner of construction of 2-unitary operators by the M_{TR} map, which is linked to the construction of AME states of 4 parties by the bipartite operator-state isomorphism used in (10). This statement can be generalized to state that the construction of AME states of any arbitrary N is equivalent to the construction of multi-unitary operators —a generalization of the notion of 2-unitarity.

In the next chapter, we construct an algorithm that generalizes the working of the M_{TR} map for $N > 4$ and we give examples of its working in the generation of 3-unitaries of order 8. However, there are several drawbacks to this algorithm which we run into as we attempt to deal with higher order unitary operators such as operators of the order 256 which arise when we attempted to construct AME(8,4).

CHAPTER 5

AME STATES - MOTIVATION, CONSTRUCTION & ANALYSIS

5.1 INTRODUCTION

AME states are states for which **all** of their bipartitions are maximally entangled i.e. any reduction to N' parties ($N' < N$) results in the maximally mixed state for $N' \leq \lfloor \frac{N}{2} \rfloor$. This definition is, without much complication, extended to higher number of parties and larger local dimensions. Thus, AME states are natural extensions of the Bell states and the GHZ states. The fact that AME(2,d) & AME(3,d) are the generalised Bell and the generalised GHZ states, respectively, is indicative of the same.

However, AME(N,d) states do **not** exist for all tuples (N,d). The 'AME problem' refers to the problem of constructing AME states of a specific number of parties and a certain local dimension.

But why are we interested in constructing AME states?

The primary and most direct utility of AME states is in the study of quantum entanglement. Given that the class of AME states maximizes bipartite entanglement, distinctions between AME states that are **not** LU-connected[1] offer an insight for further classifications of entanglement.

In protocols ranging from multipartite quantum teleportation and quantum error correction to quantum secret sharing, AME states serve as a major entanglement resource. Thus, the existence and the construction of such states is an important problem in the field of quantum information theory with the primary question of interest being the discovery of the set of tuples (N,d) such that an AME state of N parties, each of local dimension d exists.

[1]Connected by local unitary transformations

In this section, we seek to motivate the construction of such states by sketching out some of the quantum protocols that utilize AME states.

1. Quantum teleportation schemes (4):- The standard quantum teleportation scheme involves the reproduction of an unknown quantum state in the possession of one of the communicators, Alice, at the location of the other communicator, Bob, using $log_2(d)$ ebits (Bell pair of local dimension d) of quantum resource & using only local operations and classical communication protocols (LOCC).

   While the generalized Bell state of local dimension d is an AME(2,d) state, generalizations of such teleportation schemes to multipartite settings involves the existence and construction of AME(N,d).

   There are two such protocols that generalize the notion of quantum teleportation to multipartite parties of arbitrary local dimension —parallel teleportation & open destination teleportation. These protocols leverage the fact that in AME states, all bipartitions are maximally entangled.

2. Entanglement swapping (4):- The bipartite entanglement swapping protocol involves 3 parties —Alice, Bob & Chelsea that are physically separated. Each of them has access to 2 physical quantum systems —A_1, A_2, B_1, B_2 & C_1, C_2 respectively. Alice & Bob and Bob & Chelsea share $log_2(d)$ ebits (Bell pair of local dimension d) of quantum resource each & using only LOCC, it is possible to deterministically generate a shared resource of $log_2(d)$ ebits between Alice & Chelsea.

   Similar to the case of quantum teleportation, the extension of the entanglement swapping protocol to multipartite settings requires the sharing of AME states. To this extent, the following corollary holds:—

   Corollary 5.1.0.1. *Consider (M+2)k parties denoted as $\{A_1 \dots A_{(M+2)k}$ —each of local dimension d, where $M, k \in \mathcal{Z}$. Then, if each set of 2k parties defined continuously at intervals of size k i.e. —$\{A_1, \dots, A_{2k}\}, \{A_{k+1}, \dots A_{3k}\}, \dots, \{A_{Mk+1}, \dots, A_{(M+2)k}\}$, share an AME(2k,d) state (subject to its existence) given as:—*

$$|U\rangle_{A_{mk+1}A_{mk+2}\dots A_{(m+2)k}} := U_{A_{mk+1}A_{mk+3}\dots A_{mk+2k-1}} \otimes \mathbb{I}_{A_{mk+2}\dots A_{(m+2)k}} |\Phi^+\rangle_{A_{mk+1}A_{mk+2}} \cdots |\Phi^+\rangle_{A_{mk+2k-1}A_{mk+2k}},$$

   where $U_{A_{mk+1}A_{mk+3}\dots A_{mk+2k-1}}$ is a 2k-unitary operator of local dimension d $\forall m \in [0 : M]$
   & U^M ***is LU-equivalent to*** *U.*

   Then, through LOCC and the use of auxiliary quantum states, it is possible to deterministically create a shared AME(2k,d) state between $\{A_1, \dots, A_k, A_{(M+1)k+1}, \dots A_{(M+2)k}\}$.

3. Quantum secret sharing :—Applications of AME states to quantum secret sharing protocols is based on the fact that even if an eavesdropper or a 'spy' gains access to k parties out of N, for $k \leq \lfloor N/2 \rfloor$, no information about the overall state of the system or the information encoded in it can be extracted as the reduced density operator is maximally mixed.

Thus, the question of the existence of AME(N,d) for any given tuple (N,d) and its construction is of great importance. Leveraging the operator-state isomorphism stated in Ch. 3, the problem of constructing AME states is now a problem of constructing multi-unitary operators with appropriate features.

In this chapter, we generalize the M_{TR} map to arbitrary N and state a general scheme for the construction of AME(N,d) subject to its existence. Given that in Ch. 3, we have proven that it suffices to study AME states of even N, the smallest multipartite k-unitary operators of interest are 3-unitary operators of local dimension d=2. We construct a general scheme to construct 3-unitaries of the order 8×8. The constructed scheme works with an unexpected, and currently inexplicable, efficiency for most cases.

Then, we attempt to utilize our understanding of the construction of k-unitary operators to construct a 4-unitary operator of order 256×256, which would correspond to AME(8,4) under the multipartite operator-state isomorphism defined by us in Ch. 3. However, we have failed to construct such a unitary operator and our inability to construct such an operator is perhaps a statement on the size of the space.

5.2 GENERALISING THE APPROACH TO FIND AME STATES

In this section, we extend the ideas that we introduced in the 4-party case to general dimensions. In effect, we introduce a general approach and algorithm to find such multi-unitaries, building on the ideas of maps such as M_{TR} and $M_T M_R$. It is important to note that as size of space grows very quickly with dimension, the performances of these maps can decline very steeply with dimension and thus, the number of random seeds needed to finally find an AME state is much greater. In this section we will simply address the states for even number of particles.

Let us take a state with $2N$ parties —$A_1 \dots A_N$. Since in practice we begin with a matrix which assumes a certain ordering (in our case, lexicographical), let us associate these parties with the following set of indices- $\mathbb{S} = i_1, i_2, .., i_N, j_1 \dots j_N$. Each bipartition divides $\mathbb{S}$ to two subsets S_1 and S_2. While in general, the ordering of parties within the bipartition (as well as sets) does **not** affect the entanglement, in writing a matrix representation we add some ordering. Since S_1 and S_2 have a cardinality of N each, let $f_1(k)$and$f_2(k) \forall k \in 1, ..., N$ reflect this ordering.

Explicitly, let the reshaping be such that the sets:—

$$S_1 : i_1, ..., i_N \rightarrow f_1(i_1), ..., f_1(i_N)$$
$$S_2 : j_1, ...j_N \rightarrow f_2(j_1), ..., f_2(j_N)$$

In the Dirac notation, the reshaping of an operator M, called $M^{(x)}$ is given by:—

$$\langle i_1, ..., i_N | M | j_1, ...j_N \rangle = \langle f_1(i_1), ..., f_1(i_N) | M^{(x)} | f_2(j_1), ...N(j_N) \rangle \tag{5.1}$$

Next, it is important to identify what is a *sufficient* set of bipartitions whose entanglement we need to maximise. While for an AME state every bipartition is maximally entangled, there is great deal of redundancy arising from two sources:—

- The ordering inside a bipartition is irrelevant
- For the same bipartition, we can have 2 possible choices of which subset represents rows and columns respectively, this too is redundant.

These redundancies can be circumvented by considering every possible *combination* of N parties from 2N —then arranging this set and its converse in lexicographical order (the choice of ordering is trivial as long as its convenient) and then selecting only those that include i_1 (without loss of generality) as the subset that corresponds to the rows and the converse subset as the columns. This construction avoids both the aforementioned have these redundancies.

Now that we have identified the bipartitions, we build a nonlinear map based algorithm along the lines of M_{TR}. Let $\mathbb{P}$ be the set of all bipartitions one intends to address. The underlying idea remains the same —we aim to construct a map whose fixed points (atleast *one* fixed point) is such an operator that all the bipartitions in $\mathbb{P}$ are maximally entangled.

We adopt a new notation, for any bipartition P in $\mathbb{P}$, we call the reshaped operator M, M^P.

Algorithm 1: Nonlinear Map to find $n - unitary$ operators

Data: Require U to be a random $\mathbb{U}(d^2)$

. **while** *iteration less than n* **do**

  while *every bipartition p in $\mathbb{P}$* **do**

    $U \leftarrow$ reshaping, U^p

  end

  $[w_1, D, w_2] \leftarrow SVD(U)$

  $U \leftarrow w_1 w_2^{\dagger}$- nearest unitary

end

In theory, this completes the algorithm **but** it does **not** guarantee that one would find a multi-unitary, but at least it any multi-unitary is a fixed point, and if one begins with a sufficiently close seed, in theory, it will find a multi-unitary. However, in practice, the cardinality grows very fast with d —for 6 and 8 parties, the number of bipartitions becomes 10 and 35 respectively.

In general, the cardinality grows as $\binom{2n-1}{n-1}$. This makes the algorithm very slow, considering the number of iterations before convergence also grows very quickly.

This, as it turns out can be avoided. Numerically, we discovered that even for several incomplete set of bipartitions, the algorithm tends to converge to such k-unitary operators —with 3/9 reshapings sufficing for the generation of 3-unitary operators of local dimension d=2.

The startling efficiency of picking a subset of all possible reshapings in $\mathbb{P}$ is that starting with *any* initial seed, most of the subsets of the superset of $\mathbb{P}$ picked of a given cardinality witnesses a high degree of convergence to N-unitaries for sufficiently low N & d.

A plausible explanation to express the sufficiency of picking a subset of reshapings from the set $\mathbb{P}$ as opposed to picking **all** of them could be that the measure of N-unitaries in the set of k-unitaries that the map is 'expected' to generate is very high under the measure induced by the map. In fact, we do see that if we take certain reshapings, we do **not** generate N-unitaries, which further supports our claim.

5.3 GENERATION OF 3-UNITARIES OF ORDER 8×8 : AME(6,2)

In this section, we provide an example of the manner in which the algorithm that generates the sufficient set of reshapings works and highlight one particular case where constructing a non-linear map using a very small subset of the sufficient set of reshapings still results in the generation of the requisite 3-unitary. We then use the non-linear map to arrive at a general form a particular class of 3-unitaries of the order 8×8 that has a 'good' structure. We futher conjecture that **all** 3-unitaries of the order 8×8 are locally connected to this unitary —a fact supported by our studies in the LU-connectedness of unitary operators.

5.3.1 Set of Reshapings

The cardinality of $\mathbb{P}$ will be $\binom{6}{3}/2$, viz. 10. We exhaustively list these reshapings as $\langle i_1 i_2 i_3 | U | j_1 j_2 j_3 \rangle$ =:—

1. $\langle i_1 i_2 j_1 | U^{(1)} | i_3 j_2 j_3 \rangle$
2. $\langle i_1 i_2 j_2 | U^{(2)} | j_1 i_3 j_3 \rangle$
3. $\langle i_1 i_2 j_3 | U^{(3)} | j_1 j_2 i_3 \rangle$
4. $\langle i_1 j_1 i_3 | U^{(4)} | i_2 j_2 j_3 \rangle$
5. $\langle i_1 j_2 i_3 | U^{(5)} | j_1 i_2 j_3 \rangle$
6. $\langle i_1 j_3 i_3 | U^{(6)} | j_1 j_2 i_2 \rangle$
7. $\langle i_1 j_1 j_2 | U^{(7)} | i_2 i_3 j_3 \rangle$
8. $\langle i_1 j_1 j_3 | U^{(8)} | i_2 j_2 i_3 \rangle$

9. $\langle i_1 j_2 j_3 | U^{(9)} | j_1 i_2 i_3 \rangle$

with the corresponding entropies being defined as :—

$$E_i(U) := \frac{1}{1-\frac{1}{d^3}}\Big(1 - \frac{1}{d^6}\operatorname{Tr}\left((U^i U^{i\dagger})^2\right)\Big), \tag{5.2}$$

with the entropy definition being normalized to 1.

We find that taking 3 of these 9 reshapings (along with the trivial reshaping, $U^{(0)} = U$, they form the set of 10 reshapings) to define our non-linear map results in the generation of 3-unitaries for *most* of the cardinal-3 set considered. This extraordinary efficiency was not observed for 4-unitaries in N=8 as shown in the next section.

5.3.2 A ‘Good’ Structure of 3-Unitaries of Order 8×8

A ‘good’ structure of a general matrix is conventionally stated to be one as sparse as possible —implying an approach towards as minimal support as possible, with a well-defined block structure and possibly some pattern in its entries.

In the case of matrices of order 8, it is known that $\nexists$ any permutation matrix of order 8 that is 3-unitary. Thus, there exists no corresponding minimal support AME(6,2). The next best structure would be one in which one particular row has 2 values. However, by the requirement of unitarity, if one row has 2 values, another complementary row must also have 2 values. By this argument, we can consider only unitary matrices with 8,10 . . . number of entries.

If we start with our initial seed being a permutation matrix and try different combinations of 3 reshapings, trying to optimize over the number of non-zero values in the matrix representation, finally, the 3-unitary generated by the non-linear map was observed to be of a ‘good’ structure.

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

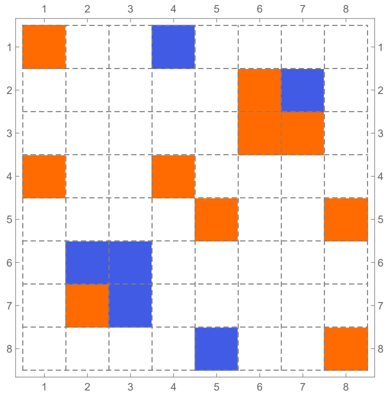


Figure 5.1: On the left is the 8×8 3-unitary that corresponds to AME(6,2). The matrix plot on the right shows the block structure of the constructed 3-unitary. The 3-unitary consists of 4 blocks of 2×2 unitaries.

5.3.3 LU-Connectedness of 3-Unitaries of Order 8×8

Two unitary operators defined on the same space —U_1 & U_2 are defined to be LU-connected iff. for some $u_A^{(1)}$, $u_B^{(1)}$ & $u_C^{(1)}$ and some $u_A^{(2)}$, $u_B^{(2)}$ & $u_C^{(2)}$,

$$(u_A^{(1)} \otimes u_B^{(1)} \otimes u_C^{(1)})U_1(u_A^{(2)} \otimes u_B^{(2)} \otimes u_C^{(2)}) = U_2 \tag{5.3}$$

Then,

Conjecture 5.3.1. *All 3-unitaries of order 8 are LU-connected.*

We are currently in the process of aggregating evidence to back this claim and have a few necessary conditions for LU-connectedness satisfied.

5.4 AN ATTEMPT AT A CONSTRUCTION OF AME(8,4)

While the approach of optimizing over non-linear maps constructed by picking arbitrary subsets of the superset of $\mathbb{P}$ for a given (N,d) and iterating the resultant non-linear map over a large set of random seeds has been quite successful for lower dimensions, it did not, at the time of writing, find an AME(8,4) state.

Thus, in place of utilizing random initial seeds and optimizing over the set of all possible non-linear maps —an extremely large set in itself with $|\mathcal{P}| = 35$ itself[2], we attempted to optimize over the space of all possible non-linear maps by picking a 'good' initial seed.

Now the question was how we could characterize a seed as 'good'? Should we take the average of all the entropic quantities corresponding to the 35 reshapings possible and optimize this average? Or should we pick an initial seed with many entropic quantities being maximized, i.e. with many of the 35 possible reshapings leading to unitaries, or equivalently, many of the 35 possible bipartitions being maximally entangled.

If we recall the study of the efficiency of the maps performed in Ch. 2, we realize that as the size of the matrices increases[3], the average values of these entropic quantities increases. Thus, for 256×256, the average values of each of these entropic values with respect to the Haar measure, will be relatively high. However, this **does not** imply that the given unitary is 'close' to being a perfect tensor. Rather, it is a commentary on the nature of the space.

Thus, it is easier to have high averages for each of the entropic quantities as opposed to having a few bipartitions of the system being maximally entangled. Thus, our notion of a 'good' unitary would be one which 'many' bipartitions maximally entangled. So it made intuitive sense to try a state with 'many' bipartitions entangled.

To this end, we decided to pick operators whose state equivalents under the multipartite operator-state isomorphism corresponded to AME(4,16), i.e. 2-unitaries of local dimension 16. However, the M_{TR} map does not produce 2-unitaries for d=16. Thus, we had to produce a 2-unitary of local dimension d=16 and then use it as a seed.

5.4.1 Generation of a 2-unitary of (4,16)

First we state a well known theorem:—

[2]Number of possible non-linear maps is then $2^{|\mathcal{P}|}$

[3]Which we saw as an increase in the local dimension

Theorem 5.4.1. *(11) Let $|\psi\rangle_{A_1A_2\ldots A_N}$ and $|\phi\rangle_{B_1B_2\ldots B_N}$ be k-uniform states in $(C^{d_1})^{\otimes N}$ and $(C^{d_2})^{\otimes N}$ respectively. Then, their tensor product, i.e.*

$$|\varphi\rangle_{(A_1B_1),\ldots,(A_NB_N)} = |\psi\rangle_{A_1A_2\ldots A_N} \otimes |\phi\rangle_{B_1B_2\ldots B_N},$$

is a k-uniform state of N parties on $C^{d_1d_2}$.

We consider this theorem in the context of two AME(4,4) states, $|\phi\rangle_{ABCD}$ and $|\phi\rangle_{A'B'C'D'}$. It follows that, if we pair the qudits into subsystems AA',BB',CC' and DD' subsystems, each of $d = 16$, taking the partially reduced density matrix by tracing any 2 parties will give a maximally mixed state. It follows that this is an AME(4,16) state and in the operator picture, it will be a 2-unitary with $d = 16$. We exploit this to create 2-unitaries with d=16.

The operator equivalent of AME(4,4)s can be easily generated by the M_{TR} algorithm. Then we tensor them, and do an appropriate reshaping to get the operator equivalent of AME(4,16).

We describe this method in the following algorithm:—

Algorithm 2: The Rearrangement algorithm

Data: Require Let U_A and U_B be operators corresponding to AME(4,d)'s and $U = U_A \otimes U_B$

for *indices* i_1, i_2, i_3, i_4 *and* j_1, j_2, j_3, j_4 *go from 0,....,(d-1)* **do**

$U(d^2(di_1 + j_1) + (di_2 + j_2), d^2(di_3 + j_3) + (di_4 + j_4) \leftarrow$
$U(d^7(i_1) + d^6(i_2 - 1) + d^5(i_3) + d^4(i_4) + d^3(j_1) + d^2(j_2) + d(j_3) + j_4)$

end

Note that this method only finds a particular subset of 2-unitaries of order 256×256.

Proof. Let us take some arbitrary 2-unitary of order 256×256 generated by this method —corresponding to the state $|\phi\rangle_{(AA')(BB')(CC')(DD')}$. Now let us pick some arbitrary unitary $V_{AA'}$ acting on particles AA' and apply it to the state, creating some entanglement between A and A'. The resultant state is also AME(4,16) w.r.t. the same subsystems, because of our choice of subsystem, A and A' are never separated (one can easily check that the relevant bipartitions remain max. entangled). This new AME state is such that particles A and A' have some entanglement; however it is impossible to create a state from our method where A and A' are not separable. ■

Note:— The 4-unitary corresponding to AME(8,4) is a 2-unitary in the (4,16) space. This follows from the converse of the argument made when pairing up parties. Thus, in theory, if we could iterate over all possible 2-unitaries of order 256×256, we will find one that is a 4-unitary of the same order, when viewed with N=8. However, as proven above, the 2-unitaries created by our method can **never** be a perfect tensor.

5.4.2 Optimization Over The Set of Non-Linear Maps

We considered several approaches to optimize the non-linear map picked in several ways —with neither of them having an advantage over the other that can be explained in a rigorous manner.

We also considered a dynamic approach to our non-linear map in which during each iteration, the reshaping and subsequent projection-to-the-nearest unitary that resulted in the minimum disruption to the bipartitions that were already maximally entangled, was picked.

We also attempted to dynamically pick the reshaping to be chosen at each step by optimizing with respect to a cost function defined as the norm mean square of the vector created as a measure of the deviation of each of the 35 entropic quantities from the maximum (All the entropic quantities were normalized to 1).

However, all of these approaches had their flaws in that they either destroyed the entanglement features of the 2-unitary seed picked or struck fixed points.

5.5 REMARKS

The complexity of the problem arises from two issues in that both the space of all possible non-linear maps is large and the number of fixed points other than 4-unitaries (in our case) also increases.

The study of operator entanglement and LU-invariants can offer insights as to the manner in which we can optimize over the set of all possible non-linear maps and thus, the study of the entanglement features of operators is central to the problem of generating AME states of arbitrary (N,d).

We end the discussion on AME(8,4) construction with the hope that some researcher somewhere either constructs the state or proves its non-existence. Should the proof of its non-existence arrive via the proof that AME(7,4) does not exist, thus leveraging the theorem we have stated in Ch. 3.

CHAPTER 6

SUMMARY & FUTURE DIRECTIONS

6.1 SUMMARY

The body of work described in the thesis can be broadly classified into results related to multipartite operator-state isomorphism, operator entanglement and results related to AME states.

6.1.1 Multipartite Operator-State Isomorphism : A Summary

The generalization of the bipartite operator-state isomorphism stated in (10) to multipartite operators in Ch. 3 led to the development of a very versatile mathematical tool that can be leveraged to restate problems in the operator picture as problems in the state picture —and vice-versa.

Herein lies the true utility of the operator-state isomorphism as certain proofs that are difficult in the operator picture become trivial in the state picture and vice-versa. For instance, in Ch. 4, the Bipartition Theorem for Maximal Operator Entanglement seems like a very complex result in the operator picture, but naturally follows when it is viewed in the state picture.

Moreover, the manner in which the isomorphism can be viewed as a vectorization of the multipartite operator into a state results in a *neat trick* to directly write down the reduced density matrix with respect to a particular bipartition in the state picture, in terms of a reshaped version of the matrix form of the operator by mapping each of the indices of the operator to one of the 2N parties of the state and reshaping the matrix as dictated by the bipartition in the state picture.

Also, it was due to the multipartite operator-state isomorphism that we arrived at the existence theorem related to k-uniform and AME states —a significant result when one considers the fact that the existence and the generation of such entanglement resources is an open problem in Quantum Information.

6.1.2 Operator Entanglement Results : A Summary

The concepts introduced and the results proven in the area of operator entanglement form the foundations of a general theory of multipartite entanglement with existing postulations and ideas regarding bipartite operators extended to the multipartite case and additional concepts that are not emergent in the bipartite case, described, analysed and discussed.

The extension of the notion of 'duality' or maximal operator entanglement of U in bipartite operators to multipartite operators is a major step towards a general theory of multipartite operator entanglement and it is analogous to the definition of AME states for multipartite entanglement in states.

The Bipartition Theorem for Maximal Entanglement completes the analogy of maximally entangled operators with AME states —proving that the operator entanglement of U in only balanced (or *nearly* balanced for odd N) bipartitions is of importance if we are to determine if U is maximally entangled.

The extension of the entropic quantity, E(US) for bipartite operators to multipartite settings by the definition and construction of a class of LU-invariants for any general multipartite operator means that a greater amount of information regarding the entanglement features of U can be determined.

The equivalence of the operator and the state pictures once again allowed us to extend the generalization of the construction scheme for the construction of LU-invariants to arbitrary N-partite operators.

6.1.3 AME States : A Summary

In Ch. 4, we explored the efficiency of the non-linear maps defined by Suhail et al. (10) and reproduced some of the results stated in their paper regarding asymptotic convergence & convergence rate —extending it to local dimensions of interest to us.

Then, as a continuation of our exploration of bipartite operators, we explore 3 classes of families in $\mathcal{U}(9)$. We find interesting properties regarding the 3 families that make them a useful resource to probe the space.

We then constructed a 3-unitary operator of order 8×8 that has a 'good' structure and conjecture that **all** 3-unitary operators of order 8×8 are LU-connected to this.

Though our attempts at constructing AME(8,4) - a 4-unitary of order 256×256, failed, through our work, we attain a better understanding of the space and get a greater cognizance of the requirement for a more rigorous approach to selecting particular reshapings in our construction of the nonlinear map.

In fact, this further motivates the requirement of understanding operator entanglement and the general entanglement features of operators —thereby resulting in a scenario where our failure to generate a 4-unitary of order 256×256 motivates **all** of our other results.

We have also performed tests on the LU-connectedness of unitaries and derived results that do numerically support the fact that there exists only one 3-unitary gate of order 8×8 —up to local transformations.

6.2 FUTURE DIRECTIONS

In this section, we state the future directions that research along these lines can take and gaps in this work that need to be addressed before publication as a manuscript[1].

1. Explicitly **define**, **motivate** and **derive** the entangling power $e_p(U)$ equivalents for higher N. State its connection in terms of the operator entanglement entropies

[1] Hopefully; subject to Prof. Arul's approval:')

and the entropies of the distributions of certain LU-invariants —if possible.

2. Explore the operator entanglement distributions for the tripartite case for different families of unitaries. Attempt to find a family for which there can be found a closed form expression for the same.

3. Find the operator entanglement entropies for the generalized Fourier transform.

4. Construct nonlinear maps that generalize the action of M_R that are capable of generating maximally entangled operators.

5. Obtain stronger numerical evidence for the fact that all 3-unitaries of size 8×8 are LU-connected.

6. Use the idea of operator entanglements and the LU-invariants to generalize the construction of AME states to any (N,d).

APPENDIX A

MATHEMATICAL FOUNDATIONS

In this chapter we review some important mathematical concepts from quantum information that will be used frequently in the thesis.This is by no way self-contained; we have not deemed it necessary to add those introductory ideas that would be taught in an introductory graduate level course, but for the most part the rest has been covered.

We discuss in some detail about the surprisingly powerful concept of vectorisation and operator reshapings in some detail here,then we move onto the schmidt-decomposition of operators-we show a simple derivation that uses the ideas of reshaping to connect it to singular value decomposition.

Finally,in the third section we move on to norms and their properties, we will use this extensively in our work in strength measures, and we discuss some important inequalities we will use.

A.0.1 Vectorisation

We begin by discussing some elementary transformations on matrices.Let A be a rectangular matrix on $\mathbb{C}^{M\times N}$ a rectangular matrix.We define vectorisation as the process of reshaping A by placing its elements in lexicographical order (i.e. row after row), into a vector, which we can $|A\rangle$ of dimension MN,

$$\langle (i-1)N + j|A\rangle = A_{ij}$$

Where $i = 1, ..., M$ and $j = 1, ..., N$.As a short hand, now onwards, we will represent $|(i-1)N + j\rangle$ simply as $|ij\rangle$ where the bases should be clear from context. Similarly,a vector of dimension MN can be reshaped into a matrix of dimensions $M \times N$.These ideas of vectorisation come from a powerful concept in quantum information known as the Choi-Jamialkowski isomorphism.While the Choi-Jamialkowski isomorphism is a very powerful statement which discusses vectorisation even for quantum channels, this text only requires some simpler ideas, and hence will only dwell on those. There can be several orderings in doing so, and have different significances in Quantum information. We illustrate the simplest, case, a converse of vectorization,which we will call Matrixification,throughout the text.Let,

$$A=\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \rightarrow |A\rangle = (A_{11}, A_{12}, A_{21}, A_{22}).$$

For simplicity, hereon we will deal with square matrices alone, since it covers the scope of the topics covered here. This interchange of representation is frequently used in quantum information, but this requires some operational equivalence.One such correspondance is with respect to inner products.Let A,B be two matrices of $N \times N$ dimensions,

$$Tr(A^{\dagger}B) = \langle A|B\rangle$$

While it is straightforward to see why this is true, it is a very powerful statement.This implies that the Hilbert-Schmidt norm of a matrix is equal to the norm of the corresponding vector,$||A||^2_{HS} = tr(A^{\dagger}A) = \langle A|A\rangle$.Moreover, for any reshaping of an operator, (denoted as A^s if A is reshaped),

$$\mathrm{tr}((A^s)^{\dagger}B^s) = \langle A^s|B^s\rangle$$

But the reordering of a vector does not affect the inner product of two vectors, as long both vectors are reordered symmetrically.So,

$$= \langle A|B\rangle = \mathrm{tr}(A^{\dagger}B)$$

In short for any such reshaping of an operator,

$$\mathrm{tr}((A^s)^{\dagger}B^s) = \mathrm{tr}(A^{\dagger}B) \tag{A.1}$$

Corollary A.0.0.1. *Let U be an unitary operator of $d \times d$ dimensions.Then the its vectorised state, $|U\rangle$ is maximally entangled 2-particle (generalised bell) state, where each particle is of d dimensions each.*

This can be proven by the following arguement;

$$|U\rangle = (U \otimes \mathbb{I})|\phi^+\rangle_{AA'}$$

Where we name the particles (qudits) A and A'.Now operationally, we are acting local unitary on the qudit named A. Because this is a local operation, it can't affect the entanglement of $|\phi^+\rangle_{AA'}$, so the resultant state $|U\rangle$ remains maximally entangled.
Next we consider what vectorisation (matrixification) does to separable states (operators). Now we establish an interesting property of this operation, Let X be an $d \times d$ matrix,

$$(X \otimes \mathbb{I})|\phi^+\rangle = \frac{1}{\sqrt{d}} \sum_i X \otimes \mathbb{I}|ii\rangle$$

$$= \frac{1}{\sqrt{d}} \sum_{ij} |j\rangle\langle j|X|i\rangle \otimes |i\rangle = \frac{1}{\sqrt{d}} \sum_{ij} \langle j|X|i\rangle |j\rangle \otimes |i\rangle$$

$$= \frac{1}{\sqrt{d}} \sum_{ij} \langle i|X^T|j\rangle |j\rangle \otimes |i\rangle$$

The last equality follows from the definition of matrix transpose,

$$= \frac{1}{\sqrt{d}} \sum_{ij} |j\rangle \otimes (|i\rangle\langle i|X^T|j\rangle) = \frac{1}{\sqrt{d}} \sum_{j} (\mathbb{I} \otimes X^T)|jj\rangle$$

$$= (\mathbb{I} \otimes X^T)|\phi^+\rangle$$

We want to end this section noting that here we have discussed a very specific, lexicographical vectorisation;there are other possible orderings, indeed we will discuss a different ordering , so to speak in the next chapter, for its relevance to the theory 4-partite entanglement. An interesting question is the following; say we have a product of operators, $A \otimes B$, will the vectorisation of this operator be seperable as well,that is, will it be $|A\rangle \otimes |B\rangle$? We find that this is not the case for this particular vectorisation, i.e,vectorisation in lexicographical order,further we will see alternate form that does satisfy this condition later on.

Finally we state, without proof, some other relevant identities regarding vectorisation. Let B be an operator on a 2- particle system (X and Y) and let A_0, A_1 be local operators to each subsystem/particle.

$$(A_0 \otimes A_1)|B\rangle = |(A_0 B A_1^T)\rangle.$$

$$tr_Y(\langle A|B\rangle) = AB^*.$$

The next two properties are regarding partial trace operations,

$$tr_X(\langle A|B\rangle) = A^T B.$$

A.0.2 Schmidt Decomposition of Operators

The Schmidt Decomposition of quantum states is a very common and well known concept in quantum information. In this section, we do not comment on it any further, instead we describe the schmidt decomposition for square matrices with dimensions that are square numbers,i.e, the operators act on 2 particles.The idea is very similar, and there are a several ways to prove this, we offer one such proof, using the idea of reshuffling. Let X be an operator of $d^2 \times d^2$ dimensions.Since the dimensions are square, the reshuffling operation is valid. We consider the singular value decomposition of X^R,

$$X^R = \sum_i^{d^2} |a_i\rangle\langle b_i|$$

By the definition of SVD, λ_i's are real and positive. Where $|a_i\rangle, |b_i\rangle$ are orthonormal bases. We reshuffling X^R once more, on both sides,

$$(X^R)^R = X = \sum_i^{d^2} \lambda_i(|a_i\rangle\langle b_i|)^R$$

$$= \sum_i^{d^2} \lambda_i([a_i] \otimes [b_i^*])$$

Where $[a_i], [b_i]$ are orthonormal operator bases (in the last section we showed that $tr(A^\dagger B) = \langle A|B\rangle$, so matrixification of an orthonormal bases of states, will result in an orthonormal bases on operators) This shows that any square operator acting on 2 particles of equal dimensions has has schmidt decomposition, it can be shown that such a decomposition is possible in 2-particles with different dimensions as well.

Corollary A.0.0.2. *The sum of the squares of the schmidt numbers of a unitary operator acting on 2 particles is equal to* d^2

Let the schmidt decomposition of U, a 2-particle unitary be,

$$U = \sum_i^{d^2} \lambda_i([a_i] \otimes [b_i^*])$$

Then,

$$U^\dagger U = \sum_{i,j}^{d^2} \lambda_j \lambda_i([a_j]^\dagger [a_i] \otimes [b_j^*]^\dagger [b_i^*])$$

Taking the trace,

$$tr(U^\dagger U) = \sum_{i,j}^{d^2} \lambda_j \lambda_i tr([a_j]^\dagger [a_i] \otimes tr([b_j^*]^\dagger [b_i^*]))(\lambda_j \lambda_i)$$

$$= \sum_{i,j} (\lambda_j \lambda_i)\delta_{i,j}$$

This equality follows from the definition of orthonormal bases.

$$= \sum_i (\lambda_i)^2$$

But we know that $tr(U^\dagger U) = d^2$ So

$$\sum_i (\lambda_i)^2 = d^2$$

Note that this is only a necessary and not sufficient conditions for unitarity.

A.0.3 Norms in Quantum Information

In this section we discuss some important norms used throughout the text,and their properties.

Definition A.1. A norm on linear operators is a map $||.||$:L(A) $\rightarrow$ [0,∞) which satisfies the following properties, for any L,K $\in L(A)$.

- Positive-Definitenes: $||L|| \geq 0$ with equality iff $L = 0$.
- absolute scalability: $||cL|| = |c|||L||$ for all $c \in \mathbb{C}$
- subadditivity:$||L + K|| \leq ||L|| + ||K||$

A norm $||.||$ is unitary invariant if it further satisfies

$$||ULV^{\dagger}|| = ||L||$$

Where U,V are unitary operators.

We can show that norms are convex by combining subadditivity and scalability,for $\lambda \in [0, 1]$:

$$||\lambda L + (1 - \lambda)K|| \leq \lambda||L|| + (1 - \lambda)||K||$$

We now discuss a popular class of norms, called the Schatten norms.These norms are functions of the singular values of the operator,i.e the eigenvalues of the positive semi-definite operator $|L| = \sqrt{L^{\dagger}L}$ (which is called the modulus of L).The Schatten p-norm is simply the p-norm of its singular values. So for any $L \in \mathbb{L}(A)$, the Schatten p-norm , where $p \geq 1$ is,

$$||L||_p := (tr(|L|^p))^{\frac{1}{p}}$$

It is important to note that for $p \in [0, 1)$,this function fails to satisfy the conditions necessary for a norm, in that, it doesn't satisfy the subadditivity property. Some important instances on Schatten norms are,

$$||L||_{\infty} = ||L||, ||L||_2 = \sqrt{tr(L^{\dagger}L)}, ||L||_1 = tr||L||$$

Where these norms are the operator norm or spectral norm, Frobenius or Hilbert-Schmidt

Norm and trace norm respectively.

- *The Frobenius Norm.*A d iscussed earlier , the Frobenius norm $\|.\|_2$ is given by

$$\|A\|_2 = (tr(A^*A)))^{\frac{1}{2}} = \sqrt{\langle A, A\rangle}$$

  Which makes it analogous to the the Euclidean norm for vectors.More explicitly,the frobenius norm of the oprator corresponds to the euclidean norm of an operator viewed as a vector:

$$\|A\|_2 = \|vec(A)\| = \sqrt{\sum_{i,j} |A_{i,j}|^2}$$

  where i, j serve as the indices of the matrix representation of A.

- *Trace Norm.*The trace norm is defined as

$$\|A\|_1 = tr\left(\sqrt{A^\dagger A}\right)$$

  which is simply equal to the sum of the singular values of A. For density operators, ρ andσ , $\|\rho - \sigma\|_1$ is typically referred to as the trace distance ρ andσ. Later on we will show that,

$$\|X\|_1 = \max_U |tr(XU)|$$

  Where U is an unitary operator. This is a profound property; for instance,using this we are able to show that the trace norm is non-increasing on partial tracing,let $X \in \mathbb{L}(A \otimes B)$ where A and B refer to the hilbert spaces of two sub-systems.

$$\|tr_B(X)\|_1 = \max |tr((U_A \otimes \mathbb{I}_B)X)|$$

Where U_A is a local unitary on sub-system A.

$$\leq \max |tr((V_{AB}X)| = \|X\|_1$$

Where V is an unitary acting on the whole system AB. Because the Schatten Norms are function of the singular values of the operator alone, they are unitary invariant as well. Finally we state without proof some powerful inequalities on the Schatten norm, known as Holder's Inequalities.

Lemma A.0.1. *Let $L, K \in \mathbb{L}(A)$ and $p, q \in \mathbb{R}$, such that $p \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then we have*

$$|tr(LK)| \leq tr|LK| \leq \|L\|_p \|K\|_q$$

Moreover, for every L there exists a K such that equality is achieved.

This is a very powerful inequality, used in many instances. One powerful application is in deriving norm duality of the Schatten Norm,

$$\|L\|_p = \max_{K\in\mathbb{L}(A),\|K\|_q\leq 1} |tr(L^\dagger K)|$$

for $\frac{1}{p} + \frac{1}{q} = 1$ and $p, q \geq 1$.

While Holder's equality is very powerful, and very general and frequently used, there is an even tighter inequality called the Von Neumann trace inequality,which bounds the hilbert-schmidt inner product in terms of the singular values of the the respective operators.We once more state this without proof.However we extend this inequality, so the bound can be written in terms of the Schmidt numbers of the operators instead.

Theorem A.0.2. ***Von Neumann Trace Inequality*** *For any two operators A,B in the $\mathbb{C}^{d\times d}$ space, with singular values*

$$\alpha_1 \geq \alpha_2 \geq \ldots \alpha_{d-1} \geq \alpha_d$$

$$\beta_1 \geq \beta_2 \geq \ldots \beta_{d-1} \geq \beta_d$$

respectively. Then

$$|tr(A^\dagger B)| \leq \sum_i^d \alpha_i \beta_i$$

that is the sum of the product of the the singular values of operators A and B.

Now, from Eq. A.1 we have,

$$|\mathrm{tr}(A^\dagger B)| = |\mathrm{tr}((A^R)^\dagger B^R)|$$

But now the singular values of A^R and B^R are in fact the schmidt numbers

$$\lambda_1^A \geq \lambda_2^A \geq \ldots \lambda_{d-1}^A \geq \lambda_d^A$$

$$\lambda_1^B \geq \lambda_2^B \geq \ldots \lambda_{d-1}^B \geq \lambda_d^B$$

of A and B respectively.So, using this in the last expression and Von Neumann inequality

$$|\mathrm{tr}(A^\dagger B)| = |\mathrm{tr}((A^R)^\dagger B^R)| \geq \sum_i \lambda_i^A \lambda_i^B$$

In short we show that the modulus of the Hilbert-Schmidt inner-product is greater than equal to the sum of the products of the schmidt numbers of those operators.

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