22 SPRING CSCE 629 600: ANALYSIS OF ALGORITHMS - Homework 9

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Question (1) Textbook page 1100, Exercise 34.5-1.

The subgraph-isomorphism problem takes two undirected graphs G1 and G2, and it asks whether G1 is isomorphic to a subgraph of G2. Show that the subgraph-isomorphism problem is NP-complete.

ANSWER:

Given two graphs G1 and G2, the subgraph-isomorphism problem is to check whether G1 is isomorphic to a subgraph of G2.

Given two graphs X and Y are isomorphic to each other if:

1. They have the same number of vertices
2. They have the same number of edges
3. Their edge connectivity is retained
4. There is a bijective function f from the vertices of graph X to graph Y with the property that vertices a and b are adjacent in X if and only if f(a) and f(b) are adjacent in Y, for all vertices a, b belonging to X.

In order to prove that the given problem is NP-Complete, we need to show that it belongs to both:

1. NP class
2. NP-hard class

Now, if the given problem belongs to NP class then it must be verifiable in polynomial time i.e., with a provided certificate we should be able to verify if it is a solution to the problem in polynomial time.

Let us now prove that the subgraph-isomorphism problem belongs to NP:

1. **The certificate in order to prove that the subgraph-isomorphism problem belongs to NP:** Let G be a subgraph of G2. The mapping between the vertices of G1 and G is known.
2. **In order to verify the certificate:** We have to check if the given graph G1 is isomorphic to G or not. Now, checking if the mapping from G1 to G is a bijection can be done in polynomial time and verifying whether, for every edge (a, b) in G1, there

is an edge (f(a), f(b)) (where f is the bijective function) present in G can also be done in polynomial time.

Therefore, the subgraph-isomorphism problem has polynomial time verifiability and hence it belongs to the NP class.

Now we have to show that the subgraph-isomorphism problem also belongs to the NP-hard class. A given problem T belongs to NP-Hard if every NP problem is reducible to T in polynomial time. In order to prove that the subgraph-isomorphism problem (L) is NP-Hard, we will choose a known NP-complete problem (A) and show that:

$$A \leq_P L.$$

That is we will show that: a reduction from a known NP-Complete problem (we will choose the Clique Decision problem as the known NP-complete problem, all problems in NP can be reduced to a problem in NPC in polynomial time) to the subgraph-isomorphism problem is possible in polynomial time which effectively shows that all problems in NP can be reduced to the subgraph-isomorphism problem in polynomial time thereby showing it to be NP-Hard.

Proof for that:

We aim to prove that the Clique Decision Problem can be reduced to the subgraph-isomorphism problem in polynomial time.

If the input to the Clique Decision Problem is (G, k) then the output is true if the graph G has a clique of size k. Given a graph G=(V, E), a clique in G is a subgraph of G that is a complete graph.

Now, we assume G1 to be a complete graph of k vertices and G2 to be G.

Here, G1 and G2 are the given inputs to the subgraph-isomorphism problem.

If n is the number of vertices in G (which is G2) then k must be less than or equal to n (k<=n), otherwise, it is not possible to have a clique of size k as a subgraph of G.

Now, the time taken to create G1 is $O(k^2)$ as the number of edges in a complete graph of size k is k*(k-1)/2. And, $O(k^2)$ is equal to $O(n^2)$ as k<=n, where $O(n^2)$ is the time complexity required for creating G. So it only takes polynomial time to create G1 and G2.

Now, G (=G2) has a clique of size k, if and only if G1 is a subgraph of G2 (=G). And, if G1 is a subgraph of G2, then the result of the subgraph-isomorphism problem with G1 and G2 as inputs is also true since every graph is isomorphic to itself, so G1 is isomorphic to a subgraph of G2. So, if the Clique Decision problem is true, then the subgraph-isomorphism problem is also true and vice-versa. Therefore, the Clique Decision Problem can be reduced to the subgraph-isomorphism problem in polynomial time for a particular instance. Thus, the subgraph-isomorphism problem is an NP-Hard problem. Now, we have proved that the subgraph-isomorphism problem is both NP and NP-Hard. Hence, the subgraph-isomorphism problem is NP-Complete, which is the required proof.