22 SPRING CSCE 629 600: ANALYSIS OF ALGORITHMS

HOMEWORK-7

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(1) Textbook page 893, Exercise 29.5-5

Solve the following linear program using SIMPLEX:

maximize $x_1 + 3x_2$

subject to

$$x_1 - x_2 \le 8$$
$$-x_1 - x_2 \le -3$$
$$-x_1 + 4x_2 \le 2$$
$$x_1, x_2 \ge 0$$

ANSWER:

The given linear program is in the Standard Form and in order to solve we first convert this linear program into its Slack form. The slack form is given by:

$$z = x_1 + 3x_2$$
$$x_3 = 8 + x_2 - x_1$$
$$x_4 = -3 + x_1 + x_2$$
$$x_5 = 2 + x_1 - 4x_2$$

In order to obtain the basic solution, we set all the non-basic variables to 0.

Non basic variables in this equation are x_1 and x_2

setting them to 0 we obtain the basic solution as:

$$x_1 = 0, x_2 = 0, x_3 = 8, x_4 = -3, x_5 = 2$$

As we can see from the value of x_4 in the ~~[illegible]~~ initial basic solution, this basic solution is not feasible as value of x_4 is negative.

So, in order to ~~[illegible]~~ determine whether our LP is

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feasible or not, we will use a helper x_0 to form an Auxiliary LP.

The Auxiliary LP is given as:

maximize $-x_0$

subject to

$$x_1 - x_2 - x_0 \leq 8$$
$$-x_1 - x_2 - x_0 \leq -3$$
$$-x_1 + 4x_2 - x_0 \leq 2$$
$$x_1, x_2, x_0 \geq 0$$

Now, we convert this Auxiliary LP into its slack form. The slack form LP is as follows:

$$z = -x_0 \qquad \text{— (i)}$$
$$x_3 = 8 - x_1 + x_2 + x_0 \qquad \text{— (ii)}$$
$$x_4 = -3 + x_1 + x_2 + x_0 \qquad \text{— (iii)}$$
$$x_5 = 2 + x_1 - 4x_2 + x_0 \qquad \text{— (iv)}$$
$$x_0, x_1, x_2, x_3, x_4, x_5 \geq 0$$

In order to proceed, we choose x_0 and move it to left hand side and move the basic variable (whose constant term is the most negative) to the right hand side. So taking a pivot and rewriting equation (iii) as $x_0 = 3 - x_1 - x_2 + x_4$ we get:

$$z = -(3 - x_1 - x_2 + x_4) = -3 + x_1 + x_2 - x_4 \qquad \text{— (v)}$$
$$x_3 = 8 - x_1 + x_2 + (3 - x_1 - x_2 + x_4)$$
$$\Rightarrow x_3 = 11 - 2x_1 + x_4 \qquad \text{— (vi)}$$
$$x_0 = 3 - x_1 - x_2 + x_4 \qquad \text{— (vii)}$$
$$x_5 = 2 + x_1 - 4x_2 + (3 - x_1 - x_2 + x_4)$$
$$\Rightarrow x_5 = 5 - 5x_2 + x_4 \qquad \text{— (viii)}$$

Now the basic solution is:

$x_1 = 0, x_2 = 0, x_4 = 0, x_0 = 3, x_3 = 11, x_5 = 5$

So, we can see that the basic solution to Auxiliary LP is feasible, So, now we keep running the SIMPLEX algorithm on this LP to find the maximum value of $-x_0$ or the objective function. Since x_1 has a positive coefficient in the above equation number (v), we will now pivot on x_1 using x_0 in equation (vii). We now get:

$$z = -x_0$$

$$x_1 = 3 - x_0 - x_2 + x_4$$

$$x_3 = 5 + 2x_0 + 2x_2 - x_4$$

$$x_5 = 5 - 5x_2 + x_4$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Now the basic solution is:

$x_1 = 3, x_2 = 0,$ $x_0 = 0$, $x_3 = 5$, $x_4 = 0$, $x_5 = 5$

and the objective value is $z = 0$

The above basic solution is optimal and the optimal objective value that we obtain is 0.

Hence our original LP is feasible.

Now, our original LP becomes:

$$z = (3 - x_2 + x_4) + 3x_2 = 3 + 2x_2 + x_4 \quad \text{—— (ix)}$$

$$x_1 = 3 - x_2 + x_4 \quad \text{—— (x)}$$

$$x_3 = 5 + 2x_2 - x_4 \quad \text{—— (xi)}$$

$$x_5 = 5 - 5x_2 + x_4 \quad \text{—— (xii)}$$

Now in this set of equations, we can see that x_2 and x_4 are non-basic variables with positive co-efficients in the objective function. So, we pick x_2 and try to increase the value of x_2 as much as possible given the other equations as constraints.

Equation (xii) will limit the value of x_2, so we ~~take~~ take a pivot and obtain $x_2 = 1 + \frac{x_4}{5} - \frac{x_5}{5}$ and the set of equations become:

$$Z = 3 + 2\left(1 + \frac{x_4}{5} - \frac{x_5}{5}\right) + x_4 = 5 + \frac{7x_4}{5} - \frac{2x_5}{5}$$

$$x_2 = 1 + \frac{x_4}{5} - \frac{x_5}{5}$$

$$x_3 = 5 + 2\left(1 + \frac{x_4}{5} - \frac{x_5}{5}\right) - x_4 = 7 - \frac{3x_4}{5} - \frac{2x_5}{5} \quad \text{(xiii)}$$

$$x_1 = 3 - \left(1 + \frac{x_4}{5} - \frac{x_5}{5}\right) + x_4$$

$$\Rightarrow x_1 = 2 + \frac{4x_4}{5} + \frac{x_5}{5} \quad \text{——— (xiv)}$$

Now the basic solution is:

$x_4 = 0,\ x_5 = 0,\ x_1 = 2,\ x_2 = 1,\ x_3 = 7$

and the objective value is $= 5$

Now, in the above LP, x_4 is the only non-basic variable which has a positive co-efficient in the objective function. So, now we pick x_4 and increase its value as much as possible. Equation (xiii) will limit the value of x_4. So we re-write equation (xiii) from $x_3 = 7 - \frac{3x_4}{5} - \frac{2x_5}{5}$ to $x_4 = \frac{35}{3} - \frac{5x_3}{3} - \frac{2x_5}{3}$. Doing this pivot operation we obtain the new set of equations as:

$$Z = 5 + \frac{7}{5}\left(\frac{35}{3} - \frac{5x_3}{3} - \frac{2x_5}{3}\right) - \frac{2x_5}{5} = \frac{64}{3} - \frac{7x_3}{3} - \frac{4x_5}{3}$$

$$x_4 = \frac{35}{3} - \frac{5x_3}{3} - \frac{2x_5}{3}$$

$$x_2 = 1 + \frac{1}{5}\left(\frac{35}{3} - \frac{5x_3}{3} - \frac{2x_5}{3}\right) - \frac{x_5}{5} = \frac{10}{3} - \frac{x_3}{3} - \frac{x_5}{3}$$

$$x_1 = 2 + \frac{4}{5}\left(\frac{35}{3} - \frac{5x_3}{3} - \frac{2x_5}{3}\right) + \frac{x_5}{5} = \frac{34}{3} - \frac{4x_3}{3} - \frac{x_5}{3}$$

So now the basic solution that we get by setting all non-basic variable x_3 and x_5 to 0 is :

<u>Basic solution</u>: $x_3 = 0, x_5 = 0$, $x_1 = \frac{34}{3}$, $x_2 = \frac{10}{3}$, $x_4 = \frac{35}{3}$

and the objective value is $= \frac{64}{3}$

Since in the objective function z written above, the coefficients of all the non-basic variables are negative, so this is our final solution (optimal solution). So our final solution is the basic solution :

$$x_1 = \frac{34}{3}, x_2 = \frac{10}{3}, x_3 = 0, x_4 = \frac{35}{3}, x_5 = 0$$

and the objective value with this solution is $= \frac{64}{3}$