22 SPRING CSCE 629 600: ANALYSIS OF ALGORITHMS - Homework 10

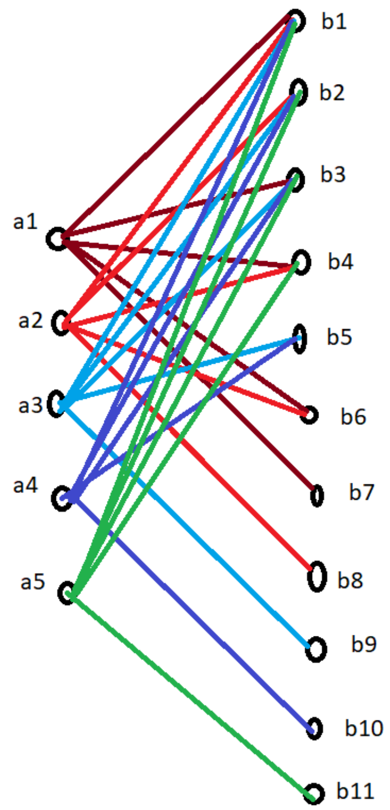
Name: Rohan Chaudhury

UIN: 432001358

Question (1) Textbook page 1111, Exercise 35.1-3.
Professor Bundchen proposes the following heuristic to solve the vertex-cover problem. Repeatedly select a vertex of highest degree, and remove all of its incident edges. Give an example to show that the professor's heuristic does not have an approximation ratio of 2. (*Hint:* Try a bipartite graph with vertices of uniform degree on the left and vertices of varying degree on the right.)

ANSWER:

Following the given hint let's consider the bipartite graph shown below with vertices a1, a2, a3, a4, a5, b1, b2, b3, b4, b5, b6, b7, b8, b9, b10, and b11 and the following mentioned connections.



a1 is connected to b1, b3, b4, b6, and b7
a2 is connected to b1, b2, b4, b6, and b8
a3 is connected to b1, b2, b3, b5, and b9
a4 is connected to b1, b2, b3, b5, and b10
a5 is connected to b1, b2, b3, b4, and b11

b1 is connected to a1, a2, a3, a4, and a5
b2 is connected to a2, a3, a4, and a5
b3 is connected to a1, a3, a4, and a5
b4 is connected to a1, a2, and a5
b5 is connected to a3 and a4
b6 is connected to a1 and a2
b7 is connected to a1
b8 is connected to a2
b9 is connected to a3
b10 is connected to a4
b11 is connected to a5

So, there are 16 vertices in the graph with 25 undirected edges in total. Let's consider the left part of the bipartite graph containing vertices (a1, a2, a3, a4, a5) as 'A' and the right part containing the vertices (b1, b2, b3, b4, b5, b6, b7, b8, b9, b10, b11) as 'B'. The vertices on 'A' side (a1, a2, a3, a4, a5) have degrees (5, 5, 5, 5, 5) and the vertices on ‘B’ side (b1, b2, b3, b4, b5, b6, b7, b8, b9, b10, b11) have degrees (5, 4, 4, 3, 2, 2, 1, 1, 1, 1, 1) as can be seen from the graph shown above.

So, in this graph, we can see that there exists a vertex-cover of size 5 (considering only the vertices a1, a2, a3, a4, a5). We need to prove that the algorithm proposed by the professor is not a 2-approximate algorithm.

Now, following Professor Bundchen’s proposed heuristic we can take the following steps:

1. Vertex b1 on the ‘B’ side has degree 5 (degree 5 is the highest currently), so we select that first and remove all of its incident edges. This reduces the degrees of the vertices on the ‘A’ side to (4, 4, 4, 4, 4) and the highest degree value to 4. Degrees of the vertices on the ‘B’ side are currently (4, 4, 3, 2, 2, 1, 1, 1, 1, 1).
2. Vertex b2 on the ‘B’ side has degree 4 (degree 4 is the highest currently), so we select that now and remove all of its incident edges. This reduces the degrees of the vertices on the ‘A’ side to (4, 3, 3, 3, 3) and the highest degree value to 4. Degrees of the vertices on the ‘B’ side are currently (4, 3, 2, 2, 1, 1, 1, 1, 1).
3. Vertex b3 in the ‘B’ side has degree 4 (degree 4 is the highest currently), so we select that now and remove all of its incident edges. This reduces the degrees of the vertices on the ‘A’ side to (3, 3, 2, 2, 2) and the highest degree value to 3. Degrees of the vertices on the B side are currently (3, 2, 2, 1, 1, 1, 1, 1).
4. Vertex b4 in the ‘B’ side has degree 3 (degree 3 is the highest currently), so we select that now and remove all of its incident edges. This reduces the degrees of the vertices on the ‘A’ side to (2, 2, 2, 2, 1) and the highest degree value to 2. Degrees of the vertices on the ‘B’ side are currently (2, 2, 1, 1, 1, 1, 1).
5. Vertex b5 in the ‘B’ side has degree 2 (degree 2 is the highest currently), so we select that now and remove all of its incident edges. This reduces the degrees of the vertices on the ‘A’ side to (2, 2, 1, 1, 1) and the highest degree value to 2. Degrees of the vertices on the ‘B’ side are currently (2, 1, 1, 1, 1, 1).
6. Vertex b6 in the ‘B’ side has degree 2 (degree 2 is the highest currently), so we select that now and remove all of its incident edges. This reduces the degrees of the vertices on the ‘A’ side to (1, 1, 1, 1, 1) and the highest degree value to 1. Degrees of the vertices on the ‘B’ side are currently (1, 1, 1, 1, 1).
7. Now our algorithm has to choose 5 more vertices from (b7, b8, b9, b10, b11) or (a1, a2, a3, a4, a5) (all of them have the same degrees) to complete the vertex-cover.

So in total, this algorithm chooses 11 vertices for the vertex-cover problem however we know that a vertex cover of 5 exists for this graph. Hence the approximation ratio is given by:

Approximation ratio = C/C^{*}= 11/5 = 2.2 which is greater than 2.

Hence this example shows that the professor’s heuristic has an approximation ratio greater than 2, thus it is not a 2-approximate algorithm. (Proved)