22 SPRING CSCE 629 600: ANALYSIS OF ALGORITHMS - Homework 4

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Question (1) Textbook page 630, Exercise 23.1-6.

Show that a graph has a unique minimum spanning tree if, for every cut of the graph, there is a unique light edge crossing the cut. Show that the converse is not true by giving a counterexample.

Answer:

Let us try to show the proof with a contradiction.

For that, we will assume that there exists 2 Minimum Spanning Trees T and T' which are distinct from each other for a graph G=(V, E) which has a unique light edge crossing the cut for every cut of the graph. Let an edge e1 be in T and not in T'.

Now, if we remove the edge e1 from the tree T, it will cut the tree into 2 different components/trees with vertices V1 and V2 which contained the edge e1. Now, let us consider this cut between V1 and V2 as (V1, V2). Let the unique light edge crossing the cut (V1, V2) be x.

Now, if e1≠x, then, the edge weight of x is less than the edge weight of e1 since x is the unique light edge crossing the cut (V1, V2), and the spanning tree T∪(x)-(e1) (i.e., the spanning tree T with edge e1 removed between vertices V1 and V2 and edge x added between the vertices V1 and V2) has lesser total edge weight than T which contradicts our assumption that T is a minimum spanning tree of G.

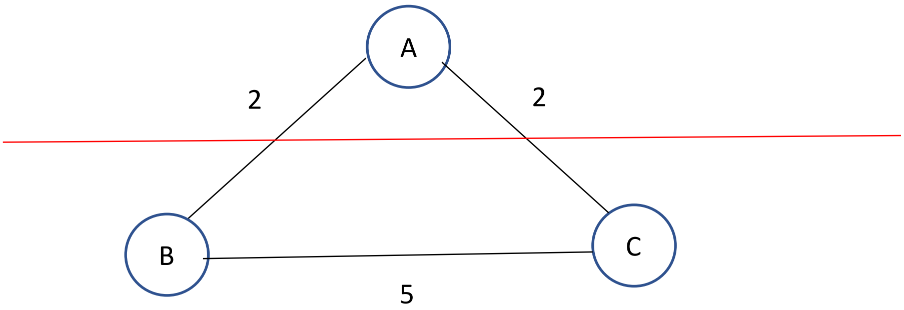
So, we have to assume that e1=x. That means that the unique light edge x crossing the cut (V1, V2) is not a part of the minimum spanning tree T' as we had initially assumed that edge e1(=x) is not in T'. Now, let us consider the path between vertices V1 and V2 for T' which must be connected by an edge e2 in T' crossing the cut (V1, V2). Now, as established, since e1 is a unique light edge crossing the cut between V1 and V2, the edge weight of e1 must be less than the edge weight of e2. So, if we remove e2 from T' and add e1 between the vertices V1 and V2 in T' (i.e, the spanning tree T'∪(e1)-(e2)) then the resulting spanning tree must have a lesser total edge weight that T' which contradicts our assumption that T' is a minimum spanning tree of G.

Hence, e2=x=e1, so our initial assumption that edge e1 is not in MST T' is not valid. Since e1 is chosen at random, so this is a general case for all the edges in T and T'. Hence, both the minimum spanning trees are the same and the graph has a unique minimum spanning tree if, for every cut of the graph, there is a unique light edge crossing the cut.

A counterexample for the above:

We have to show that a graph can contain a unique minimum spanning tree even if it doesn't have a unique light edge crossing the cut for every cut of the graph.

Let us consider the following graph and the following cut given in the color red:



The graph has vertices A, B, C, and edge weights of edges (A, B), (B, C), and (A, C) is 2, 5, 2 respectively as shown. This graph has a unique minimum spanning tree containing the edges (A, B) and (A, C) however the cut shown in red between the vertex A and edge (B, C) doesn't have a unique light edge crossing the cut (both edges (A, C) and (A, B) crossing the cut have edge weight equal to 2).

Hence this serves as a counterexample that shows that a graph can contain a unique minimum spanning tree even if it doesn't have a unique light edge crossing the cut for every cut of the graph.

In fact, this holds true for any set of edge weights in the above graph such that:

w(A, B)=w(A, C), w(B, C)>w(A, B), and w(B, C)>w(A, C),

where w(X, Y) denotes the edge weight of edge (X, Y)