awbacks of Bisection Method

- The convergence of bisection method is slow as it is simply based on halving the interval.
- If one of the initial guesses is closer to the root, it will take larger number of iterations to reach the root.
- If a function $f(x)$ is such that it just touches the x-axis it will be unable to find the lower guess, x_ℓ, and upper guess, x_u, such that $f(x_\ell)f(x_u)<0$

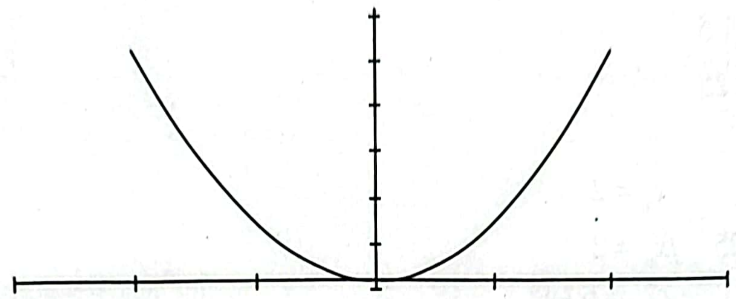

Figure 2.6: Function has a single root that cannot be bracketed.

lgorithm

1. Start
2. Choose x_ℓ and x_u as two guesses for the root such that $f(x_\ell)f(x_u)<0$ and stopping criterion E
3. Compute f(x_ℓ) and $f(x_u)$
4. Estimate the root, x_m of the equation $f(x)=0$ as the mid-point between x_ℓ and x_u as

$$x_m = \frac{x_\ell + x_u}{2}$$

5. Now check the following
   a. If $f(x_\ell)f(x_m)=0$; then the root is x_m. Go to step 8.
   b. Else If $f(x_\ell)f(x_m)<0$, the root lies between x_ℓ and x_m. Set $x_u = x_m$.
   c. Else if $f(x_\ell)f(x_m)>0$, the root lies between x_m and x_u. Set $x_\ell = x_m$.
6. Find the absolute approximate relative error as

$$|E_{ra}| = \left|\frac{x_l - x_u}{x_u}\right|$$

7. If $|E_{ra}| > E$, then go to Step 3, else go to step 8.
8. Stop

Example

Solve the Equation f(x) = $3x^2$-6x+2=0

$$\Rightarrow \quad I = \int_{x_0}^{x_2} f(x)dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] \qquad (2)$$

This equation (2) is called Simpson's 1/3 rule.

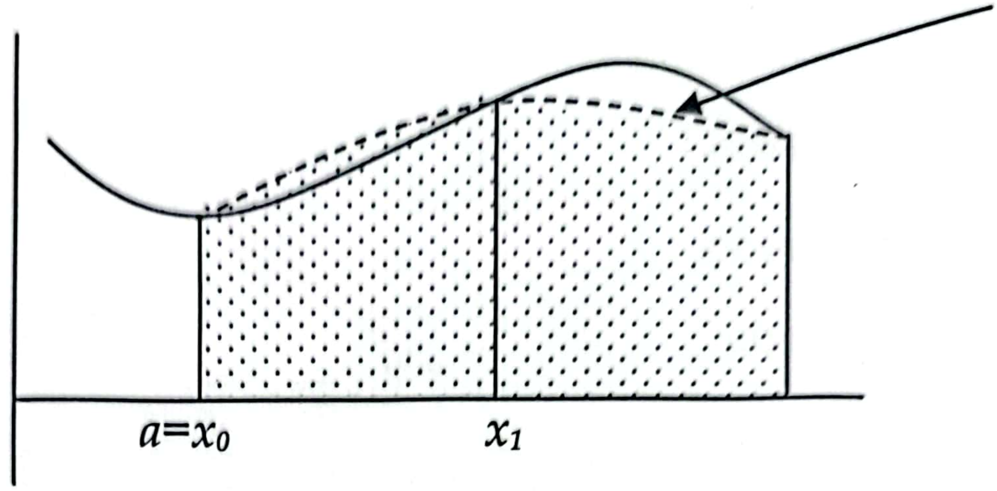

Figure: Geometrical Interpretation of Simpsons 1/3 rule

Algorithm

1. Start
2. Read value of lower & upper limit, say x_0 & $x2$
3. Set n=2
4. $h=(x_2-x_0)/n$
5. $x_1=x_0+h$
6. Calculate values $f(x_0)$, $f(x_1)$ and $f(x_2)$
7. Calculate the value of integration by using formula

   $$v=\int_{x_0}^{x_2} f(x)dx=\frac{h}{3}[f(x_0)+4f(x_1)+f(x_2)]$$

8. Display the value of integration "v"
9. Terminate

$$+\frac{h}{3}[f(x_{n-4})+4f(x_{n-3})+f(x_{n-2})]+\frac{h}{3}[f(x_{n-2})+4f(x_{n-1})+f(x_n)]$$

$$=\frac{h}{3}[f(x_0)+4\{f(x_1)+f(x_3)+\ldots+f(x_{n-1})\}+2\{f(x_2)+f(x_4)+\ldots+f(x_{n-2})\}+f(x_n)]$$

$$\Rightarrow \int_{x_0}^{x_n} f(x)dx=\frac{h}{3}\left[f(x_0)+4\sum_{\substack{i=1\\ i=0dd}}^{k-1} f(x_i)+2\sum_{\substack{i=2\\ i=even}}^{k-2} f(xi)+f(x_n)\right] \qquad (3)$$

s equation (3) is called composite Simpson's 1/3 rule

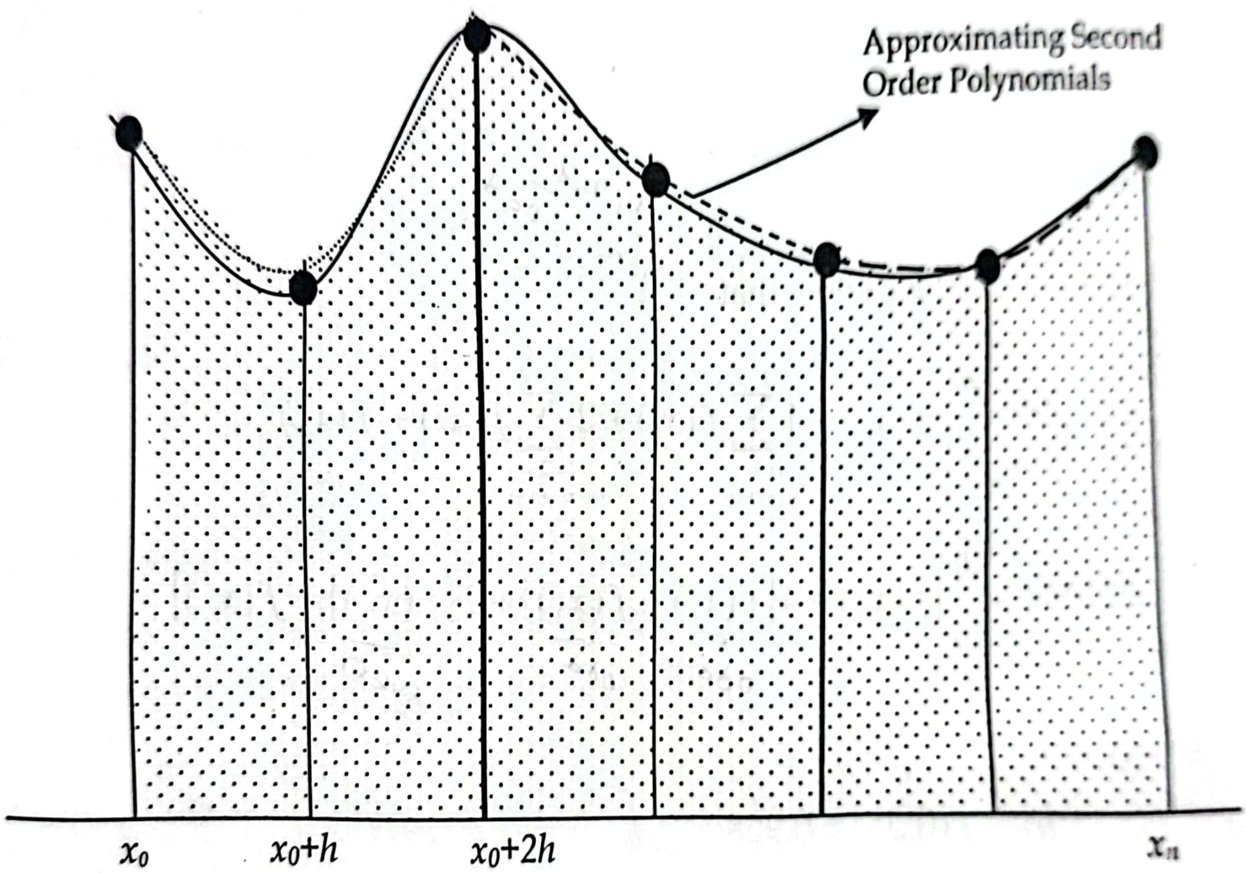

Figure: geometrical Interpretation of composite Simpson's 1/3 rule

lgorithm

1. Start
2. Read value of lower & upper limit, say x_0 & x_n
3. Read number of segments, say k
4. Calculate $h=(x_n - x_0)/k$
5. Set term=$f(x_0)+f(x_n)$
6. For i=1 to k-1

    term=term +4*$f(x_0+i*h)$

    $i=i+2$

7. End for
8. For i=2 to k-2

    term=term +2*$f(x_0+i*h)$

    $i=i+2$
9. End for
10. Calculate the value of integration by using formula $v=\frac{h}{3}*term$
11. Display the value of integration "v"
12. Terminate

Example

Apply Simpson's 1/3 rule to calculate $\int_0^1 \sqrt{1-x^2}\,dx$ by using 4 segments (i.e k=4) a

segments (i.e. k=8)

Solution

Here, x_0=0 and x_n=1

<u>For k=4</u>

h=(x_n -x_0)/k=(1-0)/4=0.25

From composite Simpson's rule, we know that

1, 2, 3 odd odd

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{3}\left[f(x_0)+4\sum_{\substack{i=1\\ i=0dd}}^{k-1} f(x_i)+2\sum_{i=2}^{k-2} f(xi)+f(x_n)\right]$$

$$\int_{x_0}^{x_0+nh} f(x)dx = \int_{x_0}^{x_0+3h} f(x)dx = 3h\left[f(x_0)+\frac{3}{2}\Delta f(x_0)+\frac{3}{4}\Delta^2 f(x_0)+\frac{1}{8}\Delta^3 f(x_0)\right]$$

$$=\frac{3}{8}h[8f(x_0)+12(f(x_1)-f(x_0))+6(f(x_0)-2f(x_1)+f(x_2))+(-f(x_0)+3f(x_1)-3f(x_2)+f(x_3))]$$

$$=\frac{3}{8}h[f(x_0)+3f(x_1)+3f(x_2)+f(x_3)]$$

$$\Rightarrow \quad I=\int_{x_0}^{x_3} f(x)dx=\frac{3}{8}h[f(x_0)+3f(x_1)+3f(x_2)+f(x_3)] \qquad (2)$$

This equation (2) is called Simpson's 3/8 rule.

Algorithm

1. Start
2. Read value of lower & upper limit, say x_0 & $x3$
3. Set n=3
4. Set $h=(x_3-x_0)/n$
5. Set $x_1=x_0+h$ $\quad x_2=x_0+2h$
6. Calculate values $f(x_0)$, $f(x_1)$, $f(x_2)$ *and* $f(x_3)$
7. Calculate the value of integration by using formula

   $$v=\int_{x_0}^{x_3} f(x)dx=\frac{3}{8}h[f(x_0)+3f(x_1)+3f(x_2)+f(x_3)]$$

8. Display the value of integration "v"
9. Terminate

Example

Apply Simpson's 3/8 rule to calculate $\int_0^1 \sqrt{1-x^2}\,dx$

Solution

Here, $h=\dfrac{b-a}{3}=\dfrac{1-0}{3}=0.33$

Simpson's 3/8 rule is given by

$$I=\int_{x_0}^{x_3} f(x)dx=\frac{3}{8}h[f(x_0)+3f(x_1)+3f(x_2)+f(x_3)]$$

}

Output

Enter Lower & Upper Limit
1 2
Value of Integration=4.750000

6.3.6 Composite Simpson's 3/8 Rule

It is also called multi-segment Simpson's 3/8 rule or multiple segment 3/8 Simpson's, divides the interval $[x_0, x_n]$ into n segments and apply Simpson's 3/8 rule repeated every three segments. Therefore n needs to be multiple of 3. Now, the segment width is by

$$h = \frac{x_n - x_0}{n}$$

Apply Simpson's 3/8 Rule over each three interval

$$\int_{x_0}^{x_n} f(x)dx = \frac{3}{8}h[f(x_0)+3f(x_1)+3f(x_2)+f(x_3)]+\frac{3}{8}h[f(x_3)+3f(x_4)+3f(x_5)+f(x_6)]+\cdots$$

$$+\frac{3}{8}h[f(x_{n-6})+3f(x_{n-5})+3f(x_{n-4})+f(x_{n-3})]+\frac{3}{8}h[f(x_{n-3})+3f(x_{n-2})+3f(x_{n-1})+\ldots$$

$$\Rightarrow \quad \int_{x_0}^{x_n} f(x)dx = \frac{3}{8}h\left[f(x_0)+3\sum_{\substack{i=1 \\ i \bmod 3 \neq 0}}^{n-1} f(x_i)+2\sum_{\substack{i=2 \\ i \bmod 3 \equiv 0}}^{n-1} f(x_i)+f(x_n)\right]$$

This equation (3) is called composite Simpson's 3/8 rule

Algorithm

1. Start
2. Read value of lower & upper limit, say x_0 & x_n
3. Read number of segments, say k
4. Calculate $h=(x_n - x_0)/k$
5. Set term=$f(x_0)+f(x_n)$
6. For i=1 to k-1

    if(i mod 2 $\neq$ 0)

        term=term +3*$f(x_0+i*h)$

    else

        term=term +2*$f(x_0+i*h)$
7. End for
8. Calculate the value of integration by using formula $v=\frac{3}{8}*h*term$

9. Display the value of integration "v"
10. Terminate

mple

culate the integral value of following tabulated function from x=0 to x=1.6 using Simpson's rule.

x	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
$f(x)$	0	0.24	0.55	0.92	1.63	1.84	2.37	2.95	3.56

ution

Here, $h = 0.2$

Composite Simpson's 3/8 rule is given by

$$I = \int_{x_0}^{x_n} f(x)dx = \frac{3}{8}h\left[f(x_0) + 3\sum_{\substack{i=1 \\ i \bmod n \neq 0}}^{n-1} f(x_i) + 2\sum_{\substack{i=2 \\ i \bmod n = 0}}^{n-1} f(xi) + f(x_n) \right]$$

$$\Rightarrow I = \frac{3}{8}h[f(x_0) + \{3f(x_1) + 3f(x_2)\} + 2f(x_3) + \{3f(x_4) + 3f(x_5)\} + 2f(x_6) + \{3f(x_7) + 3f(x_8)\} + f(x_n)]$$

Thus,

$$I = \frac{3}{8}h[0 + 3*0.24 + 3*0.55 + 2*0.92 + 3*1.63 + 3*1.84 + 2*2.37 + 3*2.95 + 3.56]$$

$$= \frac{3}{8}*0.2[0 + 3*0.24 + 3*0.55 + 2*0.92 + 3*1.63 + 3*1.84 + 2*2.37 + 3*2.95 + 3.56] = 2.38$$

cond Example

alculat ... by using composite Simpson's 3/8 rule with ...

$$p_2(x) = \frac{f_0(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + \frac{f_1(x-x_2)(x-x_0)}{(x_1-x_2)(x_1-x_0)} + \frac{f_2(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

ove equation can be written as

$$p_2(x) = f_0 l_0 + f_1 l_1 + f_2 l_2 = \sum_{i=0}^{2} f_i l_i(x) \text{ ---------------------(2)}$$

Where,

$$l_i(x) = \prod_{j=0, j\neq i}^{2} \frac{(x-x_j)}{(x_i-x_j)}$$

eneralizing equation (2) for n+1 points, we get the relation

$$p_n(x) = \sum_{i=0}^{n} f_i l_i(x) \text{ -----------------------------(3)}$$

Where,

$$l_i(x) = \prod_{j=0, j\neq i}^{n} \frac{(x-x_j)}{(x_i-x_j)}$$

quation (3) is called Lagrange interpolation formula.

lgorithm

1. Start
2. Read number of points, say n
3. Read the value at which value is needed, say x
4. Read given data points
5. Calculate values of L_i as below:

```
for i=1 to n
for j=1 to n
        if (j!=i )
        L[i]=L[i]*((x-x[j])/(x[i]-x[j]))
        End if
End for
```

End for

6. Calculate interpolated value at x as below

   For i=1 to n

   v=v +fx[i]*Lx[i]

   End for

7. Print the interpolation value v at x
8. Terminate

Example

The upward velocity of a rocket is given as a function of time in Table below.

Time(t)	Velocity (v)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Determine the value of the velocity at $t = 16$ seconds using a first order Lagrang

Solution

For first order polynomial interpolation (also called linear interpolation), the ve
by

$$p_1(x) = \sum_{i=0}^{1} l_i(x) f_i \quad = l_0(x) f_0 + l_1(x) f_1$$

Since we want to find the velocity at $t = 16$, and we are using a first order polyno
to choose the two data points that are closest to $t = 16$ that also bracket $t = 1$

$$\frac{\partial E}{\partial a} = 2\sum_{i=1}^{n}(y_i - a - bx_i)(-1) = 0 \text{ and}$$

$$\frac{\partial E}{\partial b} = 2\sum_{i=1}^{n}(y_i - a - bx_i)(-x_i) = 0$$

nplifying above equations, we get

$$-\sum_{i=1}^{n} y_i + \sum_{i=1}^{n} a + \sum_{i=1}^{n} bx_i = 0 \quad \text{and}$$

$$-\sum_{i=1}^{n} y_i x_i + \sum_{i=1}^{n} ax_i + \sum_{i=1}^{n} bx_i^2 = 0$$

nce,

$$\sum_{i=1}^{n} a = a + a + \ldots + a = na$$

bove equations can be written as

$$na + b\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \qquad (1)$$

and

$$a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i \qquad (2)$$

olving the above Equations (1) and (2) gives

$$b = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$a = \frac{\sum_{i=1}^{n} y_i}{n} - b\frac{\sum_{i=1}^{n} x_i}{n} = \bar{y} - b\bar{x}$$

Where, $\bar{y} = \frac{\sum_{i=1}^{n} y_i}{n}$ *and* $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$

Algorithm

1. Start
2. Read number of points, say n
3. Read give data points, say x[i] and y[i]
4. Find summations of x, y, xy, and x^2 as below
   for i=0 to n-1

```
        sx=sx+x[i]
        sy=sy+y[i]
        sxy=sxy+x[i]*y[i],
        sx2=sx2+x[i]*x[i]
    End for
```

5. Calculate values of parameters as below:
   b=((n*sxy) - (sx*sy))/((n*sx2) - (sx*sx)) and
   a=(sy/n) - (b*sx/n)
6. Display the equation *ax+b*
7. Terminate

Example

Fit a straight line that best fits the following set of data by using linear regression.

x	1	2	3	4	5
$f(x)$	3	5	7	10	12

Solution

x_i	y_i	x_i^2	$x_i y_i$
1	3	1	3
2	5	4	10
3	7	9	21
4	10	16	40
5	12	25	60
$\sum x_i = 15$	$\sum y_i = 37$	$\sum x_i^2 = 55$	$\sum x_i y_i = 134$

Now,

$n \quad n \quad n$

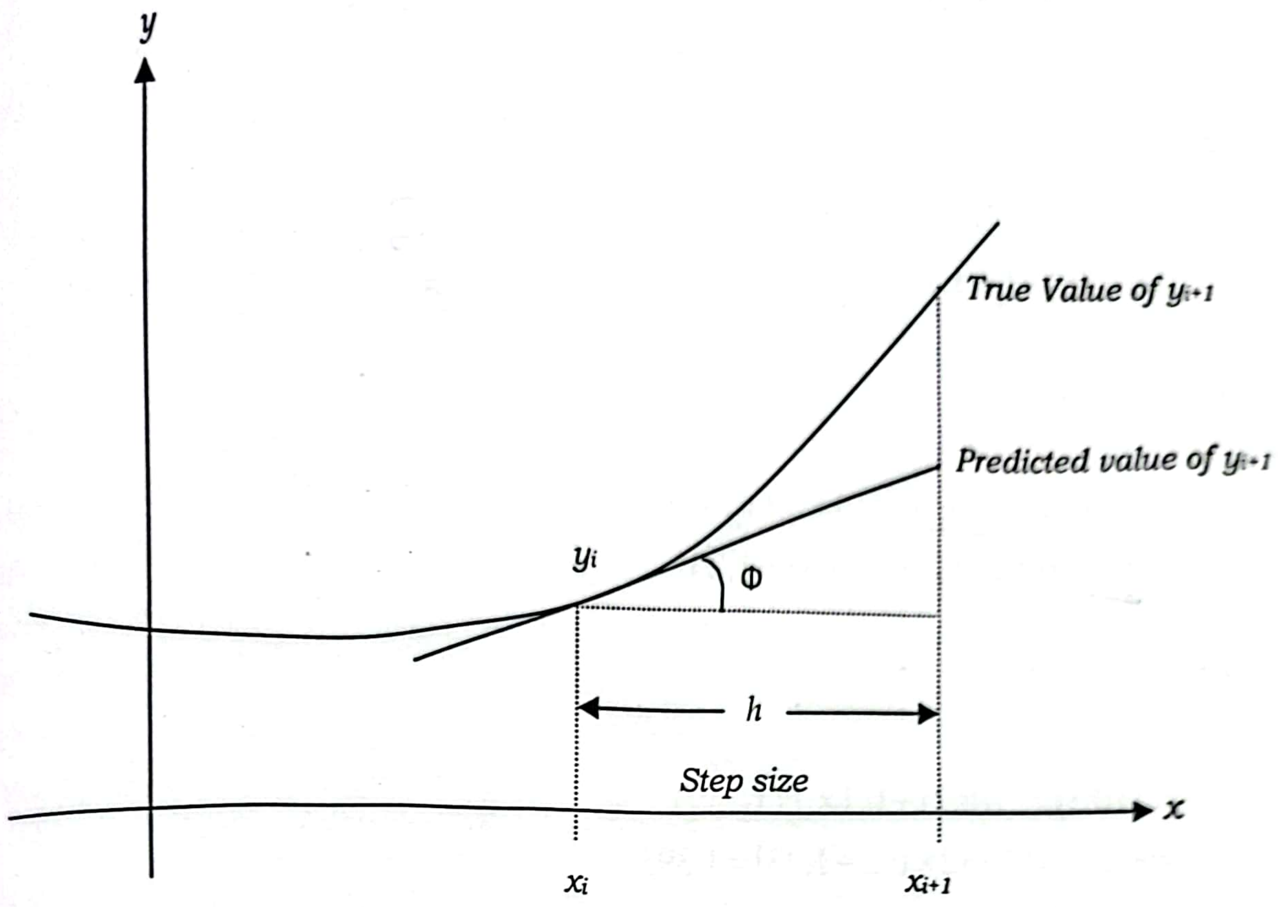

Figure: General graphical interpretation of Euler's method.

Algorithm

1. Start
2. Read initial values of x & y, say x_0 & y_0
3. Read the value at which functional value is required, say xp
4. Read step size, say h
5. Set $y=x_0y=y_0$
6. Compute value of y as below

   For $x=x_0$ to xp

   y=y+f(x,y)

   x=x+h

   End for
7. Display functional value, *y*
8. Terminate

Example

Approximate the solution of the initial-value problem $y'=2x + y$, $y(0) =1$ by using Euler method with step size of 0.1. Approximate the value of $y(0.4)$.

Solution

$$y(x_{i+1}) = y(x_i) + \frac{h}{2}(m_1 + m_1)$$

$$=> y(x_{i+1}) = y(x_i) + \frac{h}{2}(f(x_i, y_i) + f(x_{i+1}, y_{i+1})) \qquad (3)$$

Since $y(x_{i+1})$ appears on both side of equation (3), it cannot be evaluated until the value $y(x_{i+1})$ inside the function $f(x_{i+1}, y_{i+1}))$ is available. This value $y(x_{i+1})$ inside the function $f(x_{i+1}, y_{i+1}))$ can be predicted by using Euler formula as below:

$$y(x_{i+1}) = y(x_i) + hf(x_i, y_i)$$

Thus the Heun's formula given in (3) becomes

$$y(x_{i+1}) = y(x_i) + \frac{h}{2}(f(x_i, y_i) + f(x_{i+1}, y(x_i) + hf(x_i, y_i)) \qquad (4)$$

Algorithm

1. Start
2. Read initial values of x & y, say x_0 & y_0
3. Read the value at which functional value is required, say xp
4. Read step size, sya h
5. Set $y=x_0 y=y_0$
6. Compute values of $y(x_p)$ as below

   For $x=x_0$ to xp

   m1=f(x,y);
   *m2=f(x+h,y+h*m1);*
   y=y+h/2(m1+m2);*

   End for
7. Display functional value, *y*
8. Terminate

Example

Approximate the solution of the initial-value problem $y'=2x + y$, $y(0) =1$ by using Heun's method with step size of 0.1. Approximate the value of y(0.4)

Solution

Iteration 1

$$i = 0, \quad x_0 = 0, \quad y_0 = 1$$

$$m_1 = f(x_0, y_0) = 1$$

$$m_2 = f(x_1, y(x_i) + hf(x_0, y_0) = f(0.1, 1 + 0.1 * 1) = f(0.1, 1.1) = 1.3$$

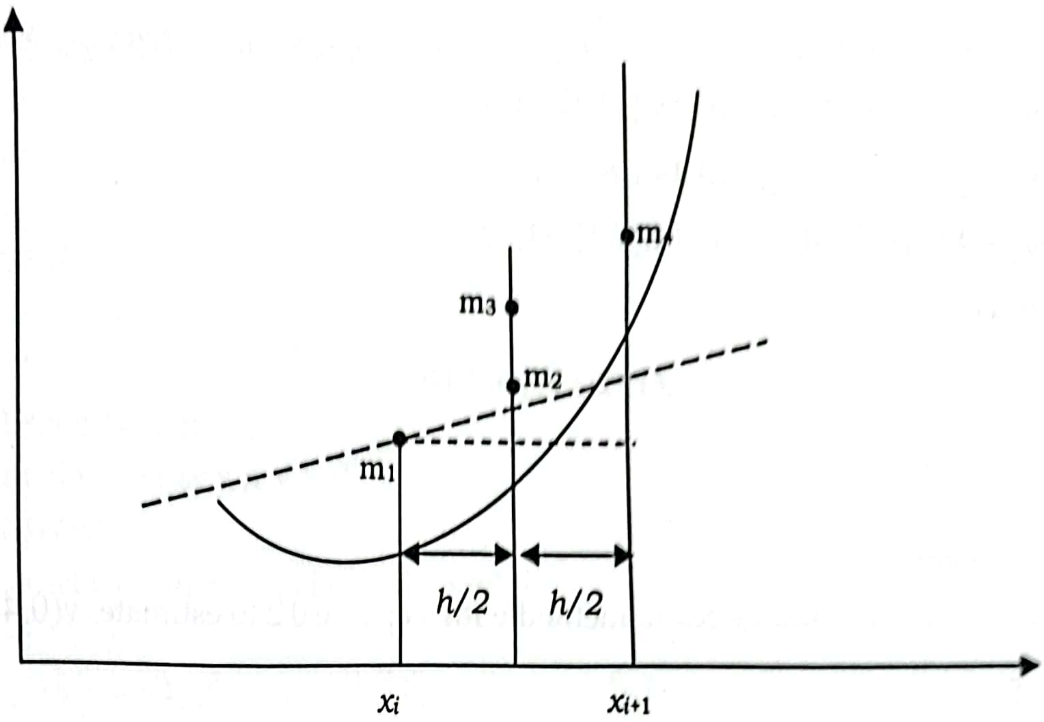

Figure: Geometrical Interpretation of 4th Order RK Method

Algorithm

1. Start
2. Read initial values of x & y, say x_0 & y_0
3. Read the value at which functional value is required, say xp
4. Read step size, say h
5. Set $y=x_0y=y_0$
6. Approximate value of y as below

   For $x=x_0$ to xp

   m1=f(x,y);
   *m2=f(x+h/2,y+h/2*m1);*
   *m3=f(x+h/2,y+h/2*m2);*
   *m4=f(x+h,y+h*m3);*
   y=y+h/6(m1+2*m2+2*m3+m4);*

   End for
7. Display functional value, *y*
8. Terminate

Example

Use the Runge-Kutta method to estimate $y(0.4)$ if $y' = 2x + y$, $y(0) = 1$

Solution

Here,

$$f(x, y) = 2x + y, \quad x_0 = 0, \text{ and } \quad y_0 = 1, h = 0.4$$

Now from Runge-Kutta method, we have