Pi Day Project

Symbolic and Numerical Approximations for π

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Contents

1 Introduction

A simple celebration of Pi Day 2022 that utilises some things I find interesting about Lisp like HOFs and its ability to generate symbolic expressions (valid Lisp code) rather than just values.

The code generates symbolic (Lisp) expressions which represent particular infinite expressions for the value of π, computed to an arbitrary depth. It then uses a library to convert that to a valid LaTeX form, as well as evaluating the expression to get a numeric approximation for π.

Since this is written as a literate document, it's interspersed with some light rambling about the mathematics behind these approximations, which somebody hopefully finds interesting.

2 Preamble

Our first task is to create a generalisation for expressing infinite series as symbolic expressions, given a method of generating the i^{th} term of the series.

```
(require 'litex-mode)
(cl-defmacro series (expr &optional (var i) (lower 0) (upper n) (op ''+)) "Given a
↪  symbolic expression in terms of a variable like n or i, return a func which returns
↪  the long form as a symbolic expression"
          `(lambda (lnum unum)
             (cons ,op (-map (lambda (,var) ,expr) (number-sequence lnum unum)))
             )
          )
```

After that, we define some utilities and define our constant `LIM` which decides how far we should generate most series.

```
(defun odds (n)
  (-take n (-filter (lambda (x) (not (math-evenp x))) (number-sequence 0 (* 2 n))))
  )
(defun square (n)
  (expt n 2))
(defun sqrt (n)
  (expt n 0.5))

(cl-defun fib (n &optional (a 0) (b 1))
  (if (<= n 1)
      b
    (fib (- n 1) b (+ a b)))
  )
(setq LIM 50)
```

3 Approximations

3.1 Riemann Sum

It turns out that the sum $\sum_{i=1}^{\infty} \frac{1}{i^2}$ converges to $\frac{\pi^2}{6}$.

$sin(x) = 0$ when $x = n\pi$ (including $x = 0$), so we can approximate sin as a product of $1 - \frac{x}{n\pi}$, so $sin(x) \approx \ldots \times (1 + \frac{x}{n\pi}) \times x \times (1 - \frac{x}{n\pi})$. Since $(1-a)(1+a) = (1-a^2)$ this becomes $sin(x) = x \prod_{n=1}^{\infty} 1 - \frac{x^2}{n^2\pi^2}$.

The coefficient of x^3 in the Taylor series for sin is $-\frac{1}{6}$.

Multiplying out the infinite product for sin gets us $\frac{1}{6} = \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$.

```
(let* (
       (riemann (series `(/ ,(float 1) (square ,i)) i 1 n))
       (wrapped `(* (sqrt 6) (sqrt ,(funcall riemann 1 LIM))))
       (tex (s-wrap (litex-lisp2latex-all wrapped ) "$" "$"))
       (evalled (eval wrapped))
       )
  (list tex evalled))
```

- $(\sqrt{6}) \left(\sqrt{\left(\frac{1.0}{((1^2))} + \frac{1.0}{((2^2))} + \frac{1.0}{((3^2))} + \frac{1.0}{((4^2))} + \frac{1.0}{((5^2))} + \frac{1.0}{((6^2))} + \frac{1.0}{((7^2))} + \frac{1.0}{((8^2))} + \frac{1.0}{((9^2))} + \frac{1.0}{((10^2))} + \frac{1.0}{((11^2))} + \frac{1.0}{((12^2))} \right.} \right.$
- 3.122626522933726

3.2 Wallis Integral

The Wallis product formula is one of the oldest expressions for π, and it hinges on the expansion of an integral $\int_0^1 (1-x^2)^{\frac{1}{2}}\, dx$, which describes one fourth the area of the unit circle, i.e. $\frac{\pi}{4}$.

We can substitute $x = \cos(\theta)$, so this becomes $I_n = \int_0^{\pi} sin^n(\theta)\, d\theta$. Integrating this by parts gives us a recurrence relation, i.e a ratio between I_n and I_{n+2}. Expanding this recurrence relation gives us the product $\pi = 2 \prod_{k=1}^{\infty} \frac{2k}{2k-1} \frac{2k}{2k+1}$.

```
(let* (
       (wallis (series `(* (/ ,(* 2.0 i) ,(- (* 2.0 i) 1)) (/ ,(* 2.0 i) ,(+ (* 2.0 i)
       ↪  1))) i 1 n '*) )
```

```
       (wrapped `(* 2 ,(funcall wallis 1 LIM)))
       (tex (s-wrap (litex-lisp2latex-all wrapped) "$" "$"))
       (evalled (eval wrapped))
       )
  (list tex evalled)
  )
```

- $2\frac{2.0}{(1.0)}\frac{2.0}{(3.0)}\frac{4.0}{(3.0)}\frac{4.0}{(5.0)}\frac{6.0}{(5.0)}\frac{6.0}{(7.0)}\frac{8.0}{(7.0)}\frac{8.0}{(9.0)}\frac{10.0}{(9.0)}\frac{10.0}{(11.0)}\frac{12.0}{(11.0)}\frac{12.0}{(13.0)}\frac{14.0}{(13.0)}\frac{14.0}{(15.0)}\frac{16.0}{(15.0)}\frac{16.0}{(17.0)}\frac{18.0}{(17.0)}\frac{18.0}{(19.0)}\frac{20.0}{(19.0)}\frac{20.0}{(21.0)}\frac{22.0}{(21.0)}\frac{22.0}{(23.0)}\frac{24.0}{(23.0)}$
- 3.126078900215408

3.3 Brouncker's Continued Fraction

This is basically equivalent to Wallis' formulation. It's just interestingly recursive, and has a nice pattern.

```
(cl-defun brouncker (n &optional (i 0))
  (if (>= i n) n
    `(/ ,(square (float (+ (* 2 i) 1))) (+ 2 ,(brouncker n (+ 1 i))))))
(let* (
       (br (brouncker 5))
       (wrapped `(/ 4 (+ 1 ,br)))
       (tex (s-wrap (litex-lisp2latex-all wrapped ) "$" "$"))
       (evalled (eval wrapped))
       )
  (list tex evalled))
```

- $\frac{4}{\left(1+\frac{1.0}{\left(2+\frac{9.0}{\left(2+\frac{25.0}{\left(2+\frac{49.0}{\left(2+\frac{81.0}{(2+5)}\right)}\right)}\right)}\right)}\right)}$
- 3.0896825396825394

3.4 Leibniz Expansion

The derivative of $\arctan x$ is $\frac{1}{1+x^2}$, which has power series expansion $\sum_{i=0} -x^{2n}$ since it represents the infinite sum of a geometric series. Integrating each term gives us $x - \frac{x^3}{3} + \frac{x^5}{5} \ldots$ as an expansion for $\arctan x$. Since $\arctan 1 = \frac{\pi}{4}$, $\pi = 4 \arctan 1$, which expands to:

```
(let* (
       (leibniz (series `(,(if (math-evenp i) '+ '-) (/ ,(float 4) ,(+ (* 2 i) 1))) i 0
       ↪  n))
       (wrapped (funcall leibniz 0 LIM))
       (tex (s-wrap (litex-lisp2latex-all wrapped ) "$" "$"))
       (evalled (eval wrapped))
       )
  (list tex evalled))
```

- $\frac{4.0}{(1)}+-\frac{4.0}{(3)}+\frac{4.0}{(5)}+-\frac{4.0}{(7)}+\frac{4.0}{(9)}+-\frac{4.0}{(11)}+\frac{4.0}{(13)}+-\frac{4.0}{(15)}+\frac{4.0}{(17)}+-\frac{4.0}{(19)}+\frac{4.0}{(21)}+-\frac{4.0}{(23)}+\frac{4.0}{(25)}+-\frac{4.0}{(27)}+\frac{4.0}{(29)}+-\frac{4.0}{(31)}+\frac{4.0}{(33)}+-\frac{4.0}{(35)}+\frac{4.0}{(37)}+-\frac{4.0}{(39)}+\frac{4.0}{(41)}+-\frac{4.0}{(43)}+\frac{4.0}{(45)}+-\frac{4.0}{(47)}+\frac{4.0}{(49)}+-\frac{4.0}{(51)}+\frac{4.0}{(53)}+-\frac{4.0}{(55)}+\frac{4.0}{(57)}+-\frac{4.0}{(59)}+\frac{4.0}{(61)}+-\frac{4.0}{(63)}+\frac{4.0}{(65)}+-\frac{4.0}{(67)}+\frac{4.0}{(69)}+-\frac{4.0}{(71)}+\frac{4.0}{(73)}+-\frac{4.0}{(75)}+\frac{4.0}{(77)}+-\frac{4.0}{(79)}+\frac{4.0}{(81)}+-\frac{4.0}{(83)}+\frac{4.0}{(85)}+-\frac{4.0}{(87)}+\frac{4.0}{(89)}+-\frac{4.0}{(91)}+\frac{4.0}{(93)}+-\frac{4.0}{(95)}+\frac{4.0}{(97)}+-\frac{4.0}{(99)}+\frac{4.0}{(101)}$
- 3.1611986129870506

4 Sources

- `https://github.com/Atreyagaurav/litex-mode` for converting Lisp expressions to valid LaTeX.
- `http://www.geom.uiuc.edu/~huberty/math5337/groupe/expresspi.html` for the initial ideas.
- `https://math.stackexchange.com/questions/8337/different-methods-to-compute-sum-limits-k-1-inf` for the proof of the Riemann sum expression.
- `https://mindyourdecisions.com/blog/2016/10/12/the-wallis-product-formula-for-pi-and-its-proof` for proof and history of the Wallis integral expansion.
- `https://proofwiki.org/wiki/Power_Series_Expansion_for_Real_Arctangent_Function` for part of the proof behind Leibniz' formula.