Pi Day Project

Symbolic and Numerical Approximations for π

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1 Introduction

A simple celebration of Pi Day 2022 that utilises some things I find interesting about Lisp like HOFs and its ability to generate symbolic expressions (valid Lisp code) rather than just values.

The code generates symbolic (Lisp) expressions which represent particular infinite expressions for the value of π , computed to an arbitrary depth. It then uses a library to convert that to a valid LATEX form, as well as evaluating the expression to get a numeric approximation for π .

Since this is written as a literate document, it's interspersed with some light rambling about the mathematics behind these approximations, which somebody hopefully finds interesting.

2 Preamble

Our first task is to create a generalisation for expressing infinite series as symbolic expressions, given a method of generating the i^{th} term of the series.

After that, we define some utilities and define our constant LIM which decides how far we should generate most series.

```
(defun odds (n)
    (-take n (-filter (lambda (x) (not (math-evenp x))) (number-sequence 0 (* 2 n))))
)
(defun square (n)
    (expt n 2))
(defun sqrt (n)
    (expt n 0.5))

(cl-defun fib (n &optional (a 0) (b 1))
    (if (<= n 1)
        b
        (fib (- n 1) b (+ a b)))
    )
(setq LIM 50)</pre>
```

3 Approximations

3.1 Riemann Sum

$$\bullet \left(\sqrt{6}\right) \left(\sqrt{\left(\frac{1.0}{\left((1^2)\right)} + \frac{1.0}{\left((2^2)\right)} + \frac{1.0}{\left((3^2)\right)} + \frac{1.0}{\left((4^2)\right)} + \frac{1.0}{\left((5^2)\right)} + \frac{1.0}{\left((6^2)\right)} + \frac{1.0}{\left((7^2)\right)} + \frac{1.0}{\left((8^2)\right)} + \frac{1.0}{\left((9^2)\right)} + \frac{1.0}{\left((10^2)\right)} + \frac{1.0}{\left((11^2)\right)} + \frac{1.0}{\left((12^2)\right)} +$$

3.122626522933726

3.2 Wallis Integral

The Wallis product formula is one of the oldest expressions for π , and it hinges on the expansion of an integral $\int_0^1 (1-x^2)^{\frac{1}{2}} dx$, which describes one fourth the area of the unit circle, i.e. $\frac{\pi}{4}$.

We can substitute $x = \cos(\theta)$, so this becomes $I_n = \int_0^{\pi} \sin^n(\theta) d\theta$. Integrating this by parts gives us a recurrence relation, i.e a ratio between I_n and I_{n+2} . Expanding this recurrence relation gives us the product $\pi = 2 \prod_{k=1}^{\infty} \frac{2k}{2k-1} \frac{2k}{2k+1}$.

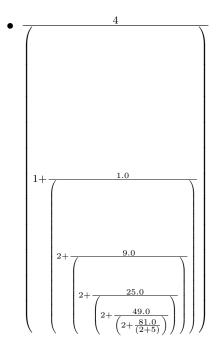
```
(wrapped `(* 2 ,(funcall wallis 1 LIM)))
  (tex (s-wrap (litex-lisp2latex-all wrapped) "$" "$"))
  (evalled (eval wrapped))
  )
(list tex evalled)
)
```

- $\bullet \quad 2\frac{2.0}{(1.0)}\frac{2.0}{(3.0)}\frac{4.0}{(3.0)}\frac{4.0}{(5.0)}\frac{6.0}{(5.0)}\frac{6.0}{(5.0)}\frac{8.0}{(7.0)}\frac{8.0}{(7.0)}\frac{8.0}{(9.0)}\frac{10.0}{(11.0)}\frac{12.0}{(11.0)}\frac{12.0}{(13.0)}\frac{14.0}{(13.0)}\frac{14.0}{(15.0)}\frac{16.0}{(15.0)}\frac{16.0}{(17.0)}\frac{18.0}{(17.0)}\frac{18.0}{(19.0)}\frac{20.0}{(21.0)}\frac{22.0}{(21.0)}\frac{22.0}{(23.0)}\frac{24.0}{(23.0)}\frac{22.0}{(23.0)}\frac{24.0}{(23.0)}\frac{22.0}{(23.0)}\frac{24.0}{(23.0)}\frac{22.0}{(23.0)}\frac{24.0}{(23.0)}\frac{$
- 3.126078900215408

3.3 Brouncker's Continued Fraction

This is basically equivalent to Wallis' formulation. It's just interestingly recursive, and has a nice pattern.

```
(cl-defun brouncker (n &optional (i 0))
  (if (>= i n) n
      `(/ ,(square (float (+ (* 2 i) 1))) (+ 2 ,(brouncker n (+ 1 i))))))
(let* (
          (br (brouncker 5))
          (wrapped `(/ 4 (+ 1 ,br)))
          (tex (s-wrap (litex-lisp2latex-all wrapped ) "$" "$"))
          (evalled (eval wrapped))
          )
          (list tex evalled))
```



• 3.0896825396825394

3.4 Leibniz Expansion

The derivative of $\arctan x$ is $\frac{1}{1+x^2}$, which has power series expansion $\sum_{i=0} -x^{2n}$ since it represents the infinite sum of a geometric series. Integrating each term gives us $x - \frac{x^3}{3} + \frac{x^5}{5} \dots$ as an expansion for $\arctan x$. Since $\arctan 1 = \frac{\pi}{4}$, $\pi = 4\arctan 1$, which expands to:

• 3.1611986129870506

4 Sources

- https://github.com/Atreyagaurav/litex-mode for converting Lisp expressions to valid LATEX.
- http://www.geom.uiuc.edu/~huberty/math5337/groupe/expresspi.html for the initial ideas.
- https://math.stackexchange.com/questions/8337/different-methods-to-compute-sum-limits-k-1-inf for the proof of the Riemann sum expression.
- https://mindyourdecisions.com/blog/2016/10/12/the-wallis-product-formula-for-pi-and-its-proof for proof and history of the Wallis integral expansion.
- https://proofwiki.org/wiki/Power_Series_Expansion_for_Real_Arctangent_Function for part of the proof behind Leibniz' formula.