COMPLEX NUMBERS

Consider the quadratic equation;

$$x^2 + 1 = 0$$

It has no solutions in the **real number system** since

$$x^2 = -1 \quad \text{or} \quad x = \pm\sqrt{1} = \pm j$$

$$j = \sqrt{-1} \quad \text{ie.} \quad j^2 = -1$$

Similarly $x^2 + 16 = 0$ gives $x = \pm\sqrt{16}$

or $x = \pm\sqrt{16}\,\sqrt{-1} = \pm 4\,j$

Powers of j

$$j = \sqrt{-1}$$

$$j^2 = -1$$

$$j^3 = j^2 .\, j = -j$$

$$j^4 = j.j^3 = j\,(-j) = -j^2 = 1$$

$$j^5 = j^4\, j = j$$

$$j^6 = j^4 .\, j^2 = j^2$$

$$j^7 = j^4 .\, j^3 = j^3$$

Clearly all the powers of j can be described by $j,\ j^2,\ j^3,\ j^4.$

$$\begin{aligned} \text{e.g. } j^{2307} &= j^{2304} .\, j^3 \\ &= (j^4)^{576} .\, (-j) \\ &= -j \end{aligned}$$

General form of complex numbers

Consider now the quadratic equation

$$x^2 + 2x + 10 = 0 \qquad (b^2 < 4ac)$$

$$x = \frac{-2 \pm \sqrt{4 - 40}}{2}$$

$$= \frac{-2 \pm \sqrt{-36}}{2}$$

$$= -\frac{2}{2} \pm \frac{6}{2} j$$

$$= -1 \pm 3j$$

This leads us t oa general form for complex numbers

$$z = a + bj \qquad a \text{ and } b \text{ are real}$$

a = real part of z = Re {z}

b = imaginary part of z = Im {z}

In the previous example;

$$\text{roots} = z$$
$$= -1 \pm 3j,$$

hence $a = -1, \quad b = \pm 3$

For complex numbers :

$3 + 2j, \quad -4, \quad 2 - 7j, \quad 16j$ are all special cases

$a + bj$ is **purely real** if $b = 0$, $(= a)$.
$a + bj$ is **purely imaginary** if $a = 0$, $(= bj)$.

Basic Rules of algebra:

Consider: $z_1 = a_1 + b_1 j,$
$z_2 = a_2 + b_2 j$

Sum: $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2) j$

Difference: $z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2) j$

Example:

for : $z_1 = 3 - 4 j$ and : $z_2 = -4 + 7 j$

$$z_1 + z_2 = (3 + (-4)) + ((-4) + 7) j = -1 + 3 j$$
$$z_1 - z_2 = (3 - (-4)) + ((-4) - 7) j = 7 - 11 j$$

Product: $z_1 . z_2 = (a_1 + b_1 j)(a_2 + b_2 j)$
$= a_1 a_2 + a_1 b_2 j + b_1 a_2 j + b_1 b_2 j^2$
$= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) j$

Example:

if $z_1 = 2 - 3 j$ and $z_2 = -1 + j$ find $z_1 (2 z_2 + 1)$

$$z_1 (2 z_2 + 1) = 2 z_1 z_2 + z_1$$
$$= 2[-2 + 3 + (2 + 3) j] + 2 - 3 j$$
$$= 2[1 + 5 j] + 2 - 3 j$$
$$= 4 + 7 j$$

Complex Conjugate :

For any complex number $a + bj$ there corresponds a complex number

$$a - bj$$

obtained by changing the sign of the **"imarigary part".**

$a - bj$ is the **Complex Conjugate** of $a + bj$.

Notation :

If $z = a + bj$ then $\bar{z} = a - bj$

Clearly $z + \bar{z} = 2a$
and $z - \bar{z} = 2bj$,

hence $$a = \frac{1}{2}(z + \bar{z}) = \text{Re}\{z\}$$

$$b = \frac{1}{2}(z - \bar{z}) = \text{Im}\{z\}$$

Also $$z\bar{z} = (a + bj)(a - bj)$$
$$= a^2 - (j^2)b^2,$$

hence $$z\bar{z} = a^2 + b^2,$$

Quotient :

To find $\frac{z_1}{z_2}$ where $z_1 = a_1 + b_1 j$,

$z_2 = a_2 + b_2 j$

$$\frac{z_1}{z_2} = \frac{z_1}{z_2}\,\frac{\overline{z_2}}{\overline{z_2}} = \frac{z_1\,\overline{z_2}}{z_2\,\overline{z_2}} = \frac{(a_1 + b_1 j)\ (a_2 - b_2 j)}{(a_2 + b_2 j)\ (a_2 - b_2 j)}$$

$$= \frac{(a_1 a_2 + b_1 b_2) + (b_1 a_2 - a_1 b_2) j}{{a_2}^2 - {b_2}^2 j^2}$$

$$= \frac{a_1 a_2 + b_1 b_2}{{a_2}^2 + {b_2}^2} + \frac{(a_2 b_1 - a_1 b_2) j}{{a_2}^2 + {b_2}^2}$$

Complex number in general form

NB : to find the quotient $\frac{z_1}{z_2}$ in general $a + bj$ form we multiply above and below by the complex conjugate of the denominator

ie. by $\overline{z_2}$

Equality

Let $z_1 = a_1 + b_1 j$
and $z_2 = a_2 + b_2 j$

then $z_1 = z_2$ when $a_1 = a_2$ and $b_1 = b_2$

ie $\text{Re}\{z_1\} = \text{Re}\{z_2\}$
and $\text{Im}\{z_1\} = \text{Im}\{z_2\}$

Complex Number zero :

$0 + 0j$ $(a = 0 \text{ and } b = 0)$ ie. both real part and imaginary part are zero.

Example :

find the real numbers α and β such that $z_1 + z_2 = 0$,
given that $z_1 = 3 + \alpha j$ and $z_2 = \beta - 5 j$.

$z_1 + z_2 = 3 + \beta + (\alpha - 5) j = 0$ if $3 + \beta = 0$ or $\beta = -3$
and $\alpha - 5 = 0$ or $\alpha = 5$

For complex numbers, z_1, z_2 **and** z_3 **:**

Commutative property:

$z_1 + z_2 = z_2 + z_1$ and $z_1 z_2 = z_2 z_1$

Associative property:

$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ and
$(z_1 z_2) z_3 = z_1 (z_2 z_3)$

Distributive property:

$z_1 (z_2 + z_3) = z_1 z_2 + z_2 z_3$

Geometrical Representation:

A complex number **a + b j** can be represented either as a **point (a, b)** or as a **position vector of (a, b)** in the **x y – plane** (complex plane) or **z – plane.**

x axis is called **real axis**
y axis is called **imaginary axis**.

This geographical representation is called **Argand Diagram**

Example:

Modulus of a complex number

Let **z = a + bj, t**he modulus of **z** is denoted by **| z |** and

$$|z| = \sqrt{(a^2 + b^2)}$$

Example: For **z = 3 – 4 j**

$$\begin{aligned} |z| &= \sqrt{3^2 + (-4)^2} \\ &= \sqrt{9 + 16} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

| z | = length of the vector for z and is always ≥ 0

Also $$\begin{aligned} z\bar{z} &= (a + bj)(a - bj) \\ &= a^2 + b^2 \\ &= |z|^2 \end{aligned}$$

Polar form of z

Point (x, y) represents $z = x + yj$, where $x = r\cos\theta$, $y = r\sin\theta$

thus $z = r\cos\theta + j\,r\sin\theta$
$= r\,(\cos\theta + j\sin\theta)$ This is called the **polar form** of **z**

Here $r = \sqrt{(x^2 + y^2)} = |z|$

θ is called the **"argument of z"** denoted by $\theta = \arg z$

The $\theta = \arg z$ is not unique since $\theta + 2k\pi$ (k an integer) produces another value of arg z.

However $-\pi < \theta \le \pi$ gives the principle value of **arg z.**

Clearly $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{x}{y}\right)$

Also $\cos\theta = \frac{x}{r} = \frac{x}{\sqrt{(x^2 + y^2)}}$

$\sin\theta = \frac{y}{r} = \frac{y}{\sqrt{(x^2 + y^2)}}$

Determination of pricipal

Argument of z: $-\pi < \theta \leq \pi$

1. $z = x + jy$ in first quadrant $(x > 0,\ Y > 0)$

$$\theta = \arg(z) == \tan^{-1}\left(\frac{x}{y}\right) \quad 0 < \theta \leq \frac{\pi}{2}$$

Example : determine the modulus and argument of

$z = 3\,2\,j$

$r = |z| = |3 + 2j|$

$= \sqrt{3^2 + 2^2}$

$= \sqrt{13}$

$\theta = \arg(z) = \tan^{-1}\left(\frac{2}{3}\right) = 0.588$ radians

2. $z = x + zy$ in second quadrant $(x < 0,\ y > 0)$

$$\theta = \arg(z) = \pi \tan^{-1}\left(\frac{x}{y}\right), \qquad \frac{\pi}{2} < \theta < \pi$$

Example : Determine $|z|$ and $\arg(z)$ for $z = -1 + j$

$r = |z|$

$= |-1 + j|$

$= \sqrt{((1^2) + 1^2}$

$= \sqrt{2}$

Since $\tan^{-1}\left|\left(\frac{1}{-1}\right)\right| = \tan^{-1} 1 = \frac{\pi}{4}$

thus $\theta = \arg(z) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ rad.

3. $z = x + zy$ in third quadrant $(x < 0,\ y < 0)$:

$$\theta = \arg(z) = \tan^{-1}\left(\frac{x}{y}\right) - \pi, \qquad -\pi < \theta < -\frac{\pi}{2}$$

Example

Determine the polar form of complex number

$$z = \sqrt{-6} - \sqrt{2}\, j$$

$$r = |z|$$

$$= \left| -\sqrt{6} - \sqrt{2}\, j \right|$$

$$= \sqrt{((-6^2) + (-\sqrt{2})^2}$$

$$= \sqrt{8}$$

Also $\tan^{-1}\left(\frac{x}{y}\right) = \tan^{-1}\left(\frac{-\sqrt{2}}{-\sqrt{6}}\right)$

$$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{\pi}{6}\ \text{rad.}$$

Example : Verify the inequailities

$| z_1 + z_2 | \leq | z_1 | + (| z)_2 |$ (Triangle inequality)

$$= 5 + \sqrt{2}$$

Also $| z_1 - z_2 | = | 2 + 5 j |$

$$= \sqrt{29}$$

$$= 5 - \sqrt{2} < (| z)_1 - z_2 |$$

$$= \sqrt{29}$$

Polar form of z :

$$z = \sqrt{8} \left[\cos\left(-\frac{5\pi}{6}\right) + j \sin\left(-\frac{5\pi}{6}\right)\right]$$

$$= \sqrt{8} \left[\cos\left(\frac{5\pi}{6}\right) - j \sin\left(\frac{5\pi}{6}\right)\right]$$

4. $z = x + jy$ in fourth quadrant $(x > -, \; y < 0)$:

$$\theta = \arg(z) = -\tan^{-1} \left| \left(\frac{x}{y}\right) \right| \qquad -\frac{\pi}{2} < \theta \leq 0$$

Example : find the priciple argument of $z = 1 - j$.

$$\theta = \arg(z) = -\tan^{-1} \left| \left(\frac{x}{y}\right) \right|$$

$$= -\tan^{-1} \left| \left(\frac{-1}{1}\right) \right|$$

$$= -\tan^{-1} 1$$

$$= \frac{\pi}{4} \text{ rad.}$$

Exponential form of z :

Recall (refer to "Tables of mathmatical formulas")

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$$

Let $x = j\theta$ $\left(j = \sqrt{-1}\right)$

$$e^{j\theta} = 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + ...$$

$$= 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + ... + j\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + ...\right)$$

$$= \cos\theta + j\sin\theta$$

Thus $z = r\,(\cos\theta + j\sin\theta)$ has a new form

$z = r\,e^{j\theta}$ the exponential form

Important Points :

1. For the correct value in exponential form θ must be in radians not in degrees.

2. Negative θ is measured in the clockwise direction. Replacing θ by $-\theta$ in

$$e^{j\theta} = \cos\theta + j\sin\theta$$

we get $e^{-j\theta} = \cos(-\theta) + \sin(-\theta)$,

or $e^{j\theta} = \cos\theta - j\sin\theta$

3. $e^{j\theta}$ and $e^{-j\theta}$ are complex cojugates hence

$z = re^{j\theta}$ and $\bar{z} = re^{-j\theta}$

Consider $e^{j\theta} = \cos(\theta) + \sin(\theta)$ and
$e^{-j\theta} = \cos(\theta) - \sin(\theta)$

Adding and substracting

$$e^{j\theta} - e^{-j\theta} = 2\,j\sin\theta$$

gives two important results :

$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$

$$\sin\theta = \frac{1}{2}\left(e^{j\theta} - e^{-j\theta}\right)$$

Log of complex numbers :

Let $z = re^{j\,(\theta + 2k\pi)}$ $(k = 0, 1, 2, \ldots)$

$$\begin{aligned} \ln z &= \ln\left[re^{j\,(\theta+2k\pi)}\right] \\ &= \ln r + j\,(\theta + 2k\pi)\ \ln e \\ &= \ln r + j\,(\theta + 2k\pi) \end{aligned}$$

The principal value of $\ln z$ is $\ln z = \ln r + j\theta$ $(k = 0)$

Example :

Principal value of $\ln\left[2\,e^{j\,(\frac{\pi}{4} + 2k\pi)}\right]$

is $\ln 2 + j\,\frac{\pi}{4} = 0.693 + 0.785\,j$

Multiplication in polar form

Let $z_1 = r_1 (\cos\theta_1 + j \sin\theta_1)$
$z_2 = r_2 (\cos\theta_2 + j \sin\theta_2)$

then

$$z_1 z_2 = r_1 r_2 (\cos\theta_1 + j \sin\theta_1)(\cos\theta_2 + j \sin\theta_2)$$

$$= r_1 r_2 [\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 + j (\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2]$$

$$= r_1 r_2 [\cos(\theta_1 + \theta_2) + j\sin(\theta_1 + \theta_2)]$$

$$= r_1 r_2\, e^{(j\theta_1 + \theta_2)}$$

Similarly

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + j\sin(\theta_1 - \theta_2)]$$

$$= \frac{r_1}{r_2} \quad e^{(j\theta_1 + \theta_2)}$$

Clearly

$$| z_1 z_2 | = r_1 r_2 = (| z)_1 || z_2 \Big|$$

$$\frac{(| z)_1}{z_2} \Big| = \frac{r_1}{r_2} = \frac{(| z)_1 |}{(| z)_2 |}$$

and $\arg z_1 z_2 = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$

$$\arg \frac{z_1}{z_2} = \theta_1 - \theta_2 = \arg z_1 - \arg z_2$$

Example :

for $z_1 = 2 \left[\cos \frac{\pi}{4} + j \sin \frac{\pi}{4} \right]$

and $z_2 = 3 \left[\cos \frac{\pi}{3} + j \sin \frac{\pi}{3} \right]$

$$z_1 z_2 = 2 \times 3 \left[\cos\left(\frac{\pi}{4} + \frac{\pi}{3}\right) + j \sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right)\right]$$

$$= 3 \left[\cos\left(\frac{7\pi}{12}\right) + j \sin\left(\frac{7\pi}{12}\right)\right.$$

and $$\frac{z_1}{z_2} = \frac{2}{3} \left[\cos\left(\frac{\pi}{12}\right) - \right.$$

$$j \sin \left(\frac{\pi}{\ } \right)]$$

De Moivre's Theorem :

If $z_1 = r_1\, e^{j\theta_1}$

$z_2 = r_2\, e^{j\theta_2}$

$z_n = r_n\, e^{j\theta_n}$

then

$$z_1\, z_2\, z_n = r_1\, r_2\, r_n\, e^{j\,(\theta_1+\theta_2+..+\theta_n)}$$
$$= r_1\, r_2\, r_n\, [\cos\,(\theta_1+\theta_2+.+\theta_n)$$
$$+\, j \sin\,(\theta_1+\theta_2+.+\theta_n)\,]$$

Set $z_1 = z_2 = .\;.. = z_n = z = r\, e^{j\theta} = r\,(\cos\theta + j\sin\theta)$

then, $z^n = r^n\,(\cos\theta + j\sin\theta)^n$

$= r^n\, e^{jn\theta}$

$= r^n\,(\cos n\theta + j\sin n\theta)$

Set $r = 1$ to give **De Moivre's Theorem**

$$(\cos\theta + j\sin\theta)^n = (\cos n\theta + j\sin n\theta)$$

Application :

If $|\,z\,| = 1$ then $z = \cos\theta + j\sin\theta = e^{j\theta}$

$$\frac{1}{z} = \frac{1}{\cos\theta + j\sin\theta}\,\frac{\cos\theta - j\sin\theta}{\cos\theta - j\sin\theta}$$

$$= \frac{\cos\theta - j\sin\theta}{\cos^2\theta + \sin^2\theta}$$

$$= \frac{\cos\theta - j\sin\theta}{1}$$

$$= e^{-j\theta}$$

Hence $z^n = e^{jn\theta} = \cos n\theta + j\sin n\theta$

and $z^{-n} = e^{-jn\theta} = \cos n\theta - j\sin n\theta$

Adding and subtracting gives

$$2\cos n\theta = z^n + \frac{1}{z^n}$$

and $2\; j\sin n\theta = z^n - \dfrac{1}{z^n}$

Example : Evaluate

(i) $\left(\cos\frac{\pi}{6} + j\sin\frac{\pi}{6}\right)^3$

(ii) $(1+j)^{25}$,

(iii) $\frac{1}{(1+j)^8}$.

Solution : By **De Moivre's Theorem**

(i) $\left(\cos\frac{\pi}{6} + j\sin\frac{\pi}{6}\right)^3 = \cos 3\left(\frac{\pi}{6}\right) + j\sin 3\left(\frac{\pi}{6}\right)$

$$= \cos\left(\frac{\pi}{2}\right) + j\sin\left(\frac{\pi}{2}\right)$$

$$= 0 + j$$

$$= j$$

(ii) Let $z = 1 + j$, then $r = |z| = \sqrt{2}$

and $\theta = \arg z = \tan^{-1} = \frac{\pi}{4}$

Thus $z^{25} = \left(\sqrt{2}\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right)\right)^{25}$

$$= \left(\sqrt{2}\right)^{25}\left(\cos\frac{25\pi}{4} + j\sin\frac{25\pi}{4}\right)$$

$$= 2^{\frac{25}{2}}\left(\cos\left(6\pi + \frac{\pi}{4}\right) + j\sin\left(6\pi + \frac{\pi}{4}\right)\right)$$

$$= 2^{12}\sqrt{2}\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right)$$

$$= 2^{12}\sqrt{2}\left(\frac{1}{\sqrt{2}} + j\left(\frac{1}{\sqrt{2}}\right)\right)$$

$$= 2^{12}(1+j)$$

(iii) Set $z = 1 + j$ thus from (ii) $r = |z| = \sqrt{2}$

and $\theta = \arg z = \tanh^{-1} = \frac{\pi}{4}$

Thus $\frac{1}{z^8} = z^{-8} = \left(\sqrt{2}\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right)\right)^{-8}$

$$= \left(\sqrt{2}\right)^{-8} \left(\cos\left(-\frac{8\pi}{4}\right) + j \sin\left(-\frac{8\pi}{4}\right)\right)$$

$$= \frac{1}{2^4} (\cos 2\pi - j \sin 2\pi)$$

$$= \frac{1}{16}$$

Powers of cos θ and sin θ :

Example : $\cos^6\theta = \left[\left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right]^6$

Binomial expansion gives

$$\cos^6\theta = \frac{1}{2^6}[z^6 + 6z^4 + 15z^2 + 20 + 15z^{-2} + 6z^{-4} + z^{-6}]$$

$$= \frac{1}{64}\left[\left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20\right]$$

$$= \frac{1}{64}[2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20]$$

$$= \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$$

Example: Express $\cos 6\theta$ and $\sin 6\theta$ in terms of $\cos\theta$ and $\sin\theta$

Example: Sketch $z = \frac{1}{2}\left(\sqrt{3} - j\right)$ on an **Argand Diagram**
and evaluate z^{-24}

$$|z| = \sqrt{\frac{3}{4} + \frac{1}{4}} = \sqrt{1} = 1$$

$$\arg z = -\frac{\pi}{6}$$

Polar form of z is

$$z = 1\left(\cos\left(-\frac{\pi}{6}\right) + j\sin\left(-\frac{\pi}{6}\right)\right)$$

$$= \cos\frac{\pi}{6} - j\sin\frac{\pi}{6}$$

$$= e^{-j\frac{\pi}{6}}$$

Hence $z^{-24} = \left(e^{-j\frac{\pi}{6}}\right)^{-24} = e^{j4\pi} = \cos 4\pi + j\sin 4\pi = 1$

Roots of complex numbers:

Recall de Moivre's theorem;

For $z = r(\cos\theta + j\sin\theta)$

$$z^n = r^n(\cos n\theta + j\sin n\theta)$$

fFor any value of $\mathbf{n}$ (integer or fraction, positive or negative)
This is a very important result.

When $\mathbf{n}$ is a fraction we are finding roots of complex numbers

Consider $w^n = z$ (integer)

The $\mathbf{n}$ different solutions $w_0, w_1, w_2, \ldots, w_{n-1}$ of this equation are the '**n^{th} roots of z**' denoted by

$$\sqrt[n]{z} \quad \text{or} \quad z^{\frac{1}{n}}$$

Let $z = r(\cos\theta + j\sin\theta)$

and $w = \rho(\cos\phi + j\sin\phi)$

Equation $w^n = z$

gives $\rho^n(\cos n\phi + j\sin n\phi) = r(\cos\theta + j\sin\theta)$

Equality of two complex numbers in polar form means that;

(i) their modulus are equal

i.e. $\rho^n = r$ or $\rho = r^{\frac{1}{n}}$

(ii) their arguments, arg w^n and $\arg z$ may differ by a multiple of 2π, say $2k\pi$,

k = 0, 1, 2. ..

Thus $n\phi = \theta + 2k\pi$

$$\text{or} \quad \phi = \frac{\theta + 2k\pi}{n}, \qquad k = 0, 1, 2, \ldots, n-1$$

(all other choicesof k duplicate the values of ϕ)

Thus n^{th} roots of z, $w_0, w_1, w_2, \ldots, w_{n-1}$ are given by

$$w_k = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + j\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

$$= r^{\frac{1}{n}}\exp^{j\left(\frac{\theta+2k\pi}{n}\right)} \qquad k = 0, 1, 2. .., n-1$$

Example : Find all the cube roots of -8

Set $z = -8$,

then $r = |-8| = 8$,

$\arg(-8) = \pi$

Hence $-8 = 8(\cos\pi + j\sin\pi)$

and $(-8)^{\frac{1}{3}} = 8^{\frac{1}{3}} \exp^{j\frac{\pi+2k\pi}{3}}$

$$= \sqrt[3]{8}\left[\cos\left(\frac{\pi + 2k\pi}{3}\right) + j\sin\left(\frac{\pi + 2k\pi}{3}\right)\right]$$

$k = 0, 1, 2$ (since $n = 3$)

There are 3 cube roots of $w^3 = z = -8$

Thus the 3 cube roots of $z = -8$ **are**

$$w_0 = 2\left(\cos\frac{\pi}{3} + j\sin\frac{\pi}{3}\right)$$

$$= 2\left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$$

$$= 1 + \sqrt{3}\,j$$

$$w_1 = 2(\cos\pi + j\sin\pi)$$

$$= 2(-1 + 0)$$

$$= -2$$

$$w_2 = 2\left(\cos\frac{5\pi}{3} + j\sin\frac{5\pi}{3}\right)$$

$$= 2\left(\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)$$

$$= 1 - \sqrt{3}\,j$$

w_0, w_1, w_2 are equally spaced $\frac{2\pi}{3}$ radians $(120°)$ apart on circle of radius $2\left(=\sqrt[3]{8}\right)$, centered at $(0, 0)$

For $w^n = z$ roots are spaced by $\frac{2\pi}{n}$ rad and lie on a circle of radius $\sqrt[n]{r}$

Example : Find the values of $j^{\frac{2}{3}}$

There are two ways depending upon how you express $j^{\frac{2}{3}}$

(i) $j^{\frac{2}{3}} = (j^2)^{\frac{1}{3}} = (-1)^{\frac{1}{3}}$
i.e find 3 cube roots of -1

(ii) $j^{\frac{2}{3}} = \left(j^{\frac{1}{3}}\right)^2$ i.e find the 3 cube roots of j and square them.

Here it appears sensible to use method (i)

(i) polar form of -1 :

$r = |-1| = 1$ and $\theta = \arg(-1) = \pi$

$w_k = (-1)^{\frac{1}{3}}$

$$= \sqrt[3]{1}\left(\cos\left(\frac{\pi + 2k\pi}{3}\right) + j\sin\left(\frac{\pi + 2k\pi}{3}\right)\right]$$

$k = 0, 1, 2$

$$w_0 = \cos\left(\frac{\pi}{3}\right) + j\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}j$$

$$w_1 = \cos\pi + j\sin\pi = -1$$

$$w_2 = \cos\left(\frac{5\pi}{3}\right) + j\sin\left(\frac{5\pi}{3}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}j$$

Exercise : Solve the equation

$$w^{\frac{3}{2}} = j$$