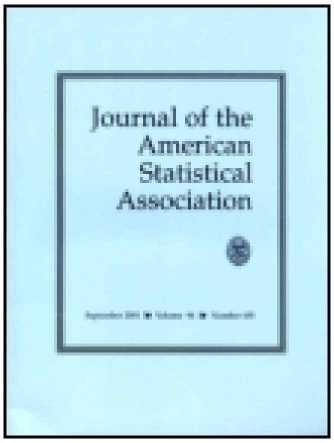

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A Distribution-Free Multivariate Sign Test Based on Interdirections

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A Distribution-Free Multivariate Sign Test Based on Interdirections

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Distribution-free tests are investigated for the one-sample multivariate location problem. Counts, called interdirections, which measure the angular distance between two observation vectors relative to the positions of the other observations, are introduced. These counts are invariant under nonsingular linear transformations and have a small-sample distribution-free property over a broad class of population models, called distributions with elliptical directions, which includes all elliptically symmetric populations and many skewed populations. A sign test based on interdirections is described, including, as special cases, the two-sided univariate sign test and Blumen's bivariate sign test. The statistic is shown to have a limiting χ_p^2 null distribution and, because it is based on interdirections, it is also seen to be invariant and to have a small-sample distribution-free property. Pitman asymptotic relative efficiencies and a Monte Carlo study show the test to perform well compared with Hotelling's T^2, particularly when the underlying population is heavy-tailed or skewed. In addition, it consistently outperforms the component sign test, which is often recommended in the nonparametric literature.

KEY WORDS: Location test; One-sample multivariate location.

1. INTRODUCTION

Let $X_j = (X_{1j}, \ldots, X_{pj})'$ $(j = 1, \ldots, n)$ denote a sample from a p-dimensional, absolutely continuous population with a p-vector location parameter θ. The population is assumed to be angularly symmetric around θ; that is, for any axis (a line infinite in both directions) through θ, the point θ is the median of the conditional distribution on that axis, so $(X - \theta)/\|X - \theta\| \stackrel{d}{=} -(X - \theta)/\|X - \theta\|$, where $\|\cdot\|$ is the Euclidean norm. [See Liu (in press).] We wish to test $H_0 : \theta = 0$ versus $H_a : \theta \neq 0$. Here 0 is used without loss of generality, since $H_0 : \theta = \theta_0$ can be tested by subtracting θ_0 from each observation vector and testing whether these differences are located at 0.

The classical procedure for this problem is Hotelling's T^2, which assumes that the underlying population is p-variate normal, that is, an $N_p(\theta, \Sigma)$ distribution with mean vector θ and variance–covariance matrix Σ, $(p \times p)$. It rejects H_0 in favor of H_a if $T^2 = n\overline{X}'S^{-1}\overline{X} \geq (n - 1)pF_\alpha(p, n - p)/(n - p)$, where $\overline{X}$ is the sample mean vector, $S = \sum_{i=1}^n (X_i - \overline{X})(X_i - \overline{X})'/(n - 1)$ is the unbiased estimator of the variance–covariance matrix, and $F_\alpha(\eta_1, \eta_2)$ is the upper αth quantile of an F distribution with η_1 and η_2 df. This test is quite effective and has many nice properties, including the intuitive property that it is invariant under all nonsingular linear transformations of the data. That is, if $Y_j = DX_j$ for $j = 1, \ldots, n$, where D is any nonsingular $p \times p$ matrix, then $T^2(Y_1, \ldots, Y_n) = T^2(X_1, \ldots, X_n)$. Thus, if the data points are rotated or if they are reflected around a $p - 1$ dimensional hyperplane or if the scales of measurement are altered, the value of T^2 stays the same. This property is intuitively appealing, and it also ensures that the performance of T^2 is the same for any variance–covariance matrix Σ.

Many nonparametric competitors to Hotelling's T^2 have been proposed. In this article we will concentrate on sign statistics, that is, ones that use the direction of the observations from 0 rather than the distances from 0. The most popular such statistic is the component sign test, which uses a sign statistic for each component of the vectors and combines them in a quadratic form. Let $S = (S_1, \ldots, S_p)'$, where $S_i = \sum_{j=1}^n \operatorname{sgn}(X_{ij})$ and $\operatorname{sgn}(t) = 1(0, -1)$ as $t > (=, <)0$. The test rejects H_0 for large values of $S_n^* = S'(n\hat{W})^{-1}S$, where $\hat{W}_{ii'} = n^{-1} \sum_{j=1}^n \operatorname{sgn}(X_{ij})\operatorname{sgn}(X_{i'j})$, for $1 \leq i$ and $i' \leq p$. The limiting distribution of S_n^* under H_0 is χ_p^2. Special cases of this statistic were discussed by Bennett (1962) and Chatterjee (1966). The general case was developed and studied extensively by Bickel (1965). The statistic S_n^* is not invariant under nonsingular linear transformations, and it does not have a small-sample distribution-free property unless a null distribution is generated by conditioning on the observed vectors, giving equal probability to each data vector being the observed one or a point on the opposite side of 0. The performance properties of S_n^* vary depending on Σ and the direction of shift from 0. Bickel demonstrated that it may not perform well when there are substantial correlations among variates in the vectors. Nevertheless, it is emphasized in the texts by Puri and Sen (1971) and Hettmansperger (1984). In an effort to stabilize the performance, Dietz (1982, 1984) proposed performing this test on transformed data. This created invariant (or asymptotically invariant) procedures that, for any Σ, had significance levels and power comparable with those of S_n^* when $\Sigma = I$. Bickel (1965) and Sen and Puri (1967) considered other score function versions of this statistic as well.

Hodges (1955) and Blumen (1958) proposed bivariate sign tests that are invariant and also have a small-sample (unconditional) distribution-free property. Hodges's statistic is the maximum number of points on one side of a line through the origin, where one maximizes over all such lines. Its local efficiency (Bahadur efficiency as $\theta \to 0$) is .636 when the underlying population is bivariate normal.

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See Joffe and Klotz (1962) and Killeen and Hettmansperger (1972). Blumen's procedure takes each data point and draws an axis through that point and 0. These axes are then rotated so that the angles they make with one another become evenly spaced around the origin and each axis retains its two adjacent axes. Each observed point is projected on its new axis to the closest point on the unit circle centered at 0. The square of the distance from 0 to the center of gravity (the mean) of these projected points is multiplied by $2n$ to form the test statistic. Its limiting null distribution is χ_2, and its Pitman asymptotic relative efficiency (ARE) relative to T^2 is .785 when the underlying population is bivariate normal. Klotz (1964) tabled portions of its null distribution for $n = 8$ through 12. Oja and Nyblom (1989) proposed a class of invariant bivariate sign tests that includes Hodges's and Blumen's tests as special cases. Their comparisons show, among other results, that Blumen's test is optimal against location shift for all elliptically symmetric distributions. Brown and Hettmansperger (in press) also proposed an invariant bivariate sign test related to the generalized median of Oja (1983). Liu (in press) suggested another invariant test statistic that uses simplical depth.

The purpose of this article is to describe a different method of producing a distribution-free property in this problem, one that can be used with data in any dimension. Counts, called interdirections, are introduced in Section 2. It is shown that they are invariant under nonsingular linear transformations and that they possess a small-sample distribution-free property under H_0 in the one-sample problem. Section 3 describes a multivariate sign test based on interdirections, which includes both the two-sided univariate sign test and Blumen's bivariate sign test as special cases. Properties of this test are also developed. Section 4 contains comparisons of this test to Hotelling's T^2 and the component sign test based on Pitman ARE's and a Monte Carlo study. This is followed by an example, conclusions, and remarks. Proofs of results are sketched in the Appendix.

2. INTERDIRECTIONS

Consider a pair of observations X_j and X_k. Let C_{jk} denote the number of hyperplanes formed by the origin 0 and $p - 1$ other points (not X_j or X_k) such that X_j and X_k are on opposite sides of the hyperplane formed. Thus each C_{jk} is an integer between 0 and $\binom{n-2}{p-1}$, inclusively. The counts $\{C_{jk} \mid 1 \le j < k \le n\}$ are called *interdirections*. Each count $C_{jk} \equiv C_{kj}$ measures the angular distance between X_j and X_k in relation to the origin. But it measures that angular distance relative to the positions of the other points. This is analogous to the way the difference between the ranks of two univariate observations measures the distance between those observations relative to the positions of the other values observed. In what follows we show some interesting and useful properties of interdirections.

Result 2.1. The interdirections are invariant under nonsingular linear transformations. (This is formally proved in the Appendix.)

Result 2.2. The interdirections use only the direction of each data point X_j from the origin and do not use its distance from the origin.

This follows from the fact that when a point is used together with the origin to form a hyperplane, we are really using its whole axis (the infinite line through that point and 0) to form the hyperplane. Moreover, when a point is on one side of a hyperplane through the origin, the whole half of its axis starting at 0 and going through the point and beyond also is on that same side of the hyperplane.

Result 2.3. Under H_0, the interdirections have a distribution-free property over a broad class of population models (defined subsequently), called distributions with elliptical directions.

We define $X_1, \ldots, X_n$ to be observations from distributions with elliptical directions if they can be constructed in the following way. Let $U_1, \ldots, U_n$ be iid uniformly distributed over the p-dimensional unit sphere. Let $R_1, \ldots, R_n$ be any positive scalar random variables. Let D be any nonsingular $p \times p$ matrix and form: $X_1 = R_1 D U_1, \ldots, X_n = R_n D U_n$. Note that when $R_1, \ldots, R_n$ are iid the X_i's are a sample from some elliptically symmetric population. In fact, this generation process characterizes *all* elliptically symmetric populations when the R_i's are iid. But the R_i's need not have the same distributions and they need not be independent. In fact, R_i need not be independent of U_i. That is, the radius length can be a function of the direction chosen. So radii can be long in some directions and short in others, thus creating skewed population models. This class includes many possibilities.

To see that the interdirections are distribution free over this class is really quite simple. Result 2.2 shows that the interdirections use only the directions from 0 of the points and not their distances, that is, the R_i's are ignored. Result 2.1 shows that the interdirections are invariant under nonsingular linear transformations, so they will have the same null distribution as if D were I, the $p \times p$ identity matrix. Thus the C_{jk}'s have the same null distribution as though they were computed on $U_1, \ldots, U_n$.

3. A SIGN TEST BASED ON INTERDIRECTIONS

Consider the test for general p that rejects H_0 for large values of the statistic

$$V_n = \frac{p}{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \cos(\pi \hat{p}_{jk}), \tag{3.1}$$

where

$$\hat{p}_{jk} = (C_{jk} + d_n)/\binom{n}{p-1} \quad \text{if } j \ne k$$
$$= 0 \quad \text{if } j = k. \tag{3.2}$$

and $d_n = [\binom{n}{p-1} - \binom{n-2}{p-1}]/2$. The proportion $\hat{p}_{jk}$ is the observed fraction of times that X_j and X_k fall on opposite sides of data-based hyperplanes. In using $\binom{n}{p-1}$ and d_n, we examine *all* possible hyperplanes formed with 0 and $p - 1$ points. Via d_n, we count $\frac{1}{2}$ whenever one or both of X_j and X_k fall directly on a hyperplane. Using these corrections [i.e., $\binom{n}{p-1}$ and d_n] instead of the more natural estimator $\tilde{p}_{jk} = C_{jk}/\binom{n-2}{p-1}$ helps to make the test more conservative for smaller sample sizes and also creates Property 3.3, shown later. We now examine some characteristics of the test based on V_n. The first one is proved in the Appendix.

Property 3.1. If the observations are from distributions with elliptical directions and H_0 is true, then $V_n \xrightarrow{d} X_p^2$, as $n \to \infty$.

Thus large-sample critical values are easily obtained. The next two properties show some important special cases.

Property 3.2. When $p = 1$ the test based on V_n is the two-sided univariate sign test.

When $p = 1$, $d_n = 0$, $\binom{n}{p-1} = 1$, and $C_{jk} = 0(1)$ as X_j and X_k are on the same (opposite) side(s) of 0. Thus $V_n = 4n^{-1}(B - n/2)^2$, where B is the number of positive X_j's.

Property 3.3. When $p = 2$, the test based on V_n is Blumen's bivariate sign test.

When $p = 2$, $d_n = 1$ and $\hat{p}_{jk} = (C_{jk} + 1)/n$. Thus

$$V_n = \frac{2}{n}\sum_{j=1}^{n}\sum_{k=1}^{n} a_j a_k \cos\left(\frac{\pi(R_j - 1 - (R_k - 1))}{n}\right)$$

$$= 2n\left\{\left[\frac{1}{n}\sum_{j=1}^{n} a_j \cos\left(\frac{\pi(R_j - 1)}{n}\right)\right]^2 + \left[\frac{1}{n}\sum_{k=1}^{n} a_k \sin\left(\frac{\pi(R_k - 1)}{n}\right)\right]^2\right\},$$

which is Blumen's statistic. Here the axis of X_j is the line through X_j and 0 extending infinitely in both directions and R_j denotes the rank of the axis X_j, where the axes are ranked in a counterclockwise direction from the positive side of the horizontal axis to the negative side. In addition, $a_j = 1(-1)$ as X_j is above (below) the horizontal axis.

The next two properties are direct consequences of the fact that this sign test is based on interdirections.

Property 3.4. The sign test based on interdirections is invariant under nonsingular linear transformations.

Property 3.5. Under H_0, V_n has a small-sample distribution-free property over the class of distributions with elliptical directions.

The form used for the statistic V_n can be motivated by recalling Raleigh's test, which is a very effective directional data procedure for testing the null hypothesis of a uniform distribution on the unit sphere versus an alternative distribution that is concentrated in an unknown direction. If $U_1, \ldots, U_n$ denote observations on the unit sphere, Raleigh's statistic is

$$V_n^* \equiv np\bar{U}'\bar{U} = \frac{p}{n}\sum_{j=1}^{n}\sum_{k=1}^{n} U_j' U_k$$

$$= \frac{p}{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \cos(\text{angle between } U_j \text{ and } U_k). \quad (3.3)$$

Thus V_n is formed by replacing the angle between U_j and U_k with an invariant estimator of that angle, namely $\pi\hat{p}_{jk}$.

4. COMPARISONS OF TESTS

Pitman ARE's will provide the first mode of comparing the test based on V_n with Hotelling's T^2 and the component sign test. First consider a power family of elliptically symmetric distributions with density functions

$$f(x) = k_p|\Sigma|^{-1/2}\exp\{[-(x - \theta)'\Sigma^{-1}(x - \theta)/c_0]^\nu\}, \quad x \in R^p, \quad (4.1)$$

where

$$c_0 = \frac{p\Gamma(p/2\nu)}{\Gamma[(p + 2)/2\nu]}, \qquad k_p = \frac{\nu\Gamma(p/2)}{\Gamma(p/2\nu)[\pi c_0]^{p/2}}, \quad (4.2)$$

Σ denotes the variance–covariance matrix, and R^p denotes Euclidean p space. This family includes the multivariate normal distribution ($\nu = 1$), heavier-tailed distributions ($0 < \nu < 1$), and lighter-tailed distributions ($\nu > 1$). The following result is proved in the Appendix.

Theorem 4.1. For the family of distributions given in Expression (4.1),

$$\text{ARE}(V_n, T^2) = \frac{4\nu^2\Gamma^2[(2\nu + p - 1)/2\nu]\Gamma[(p + 2)/2\nu]}{p^2\Gamma^3[p/2\nu]}. \quad (4.3)$$

This ARE is evaluated in Table 1 for selected values of ν and p. When $p = 1$, these efficiencies are those of the univariate sign test. For $p = 2$, they are efficiencies of Blumen's bivariate sign test and were derived by Peters (1988). When the underlying population is multivariate normal (i.e., $\nu = 1$) the efficiency of V_n is surprisingly good and increases relative to T^2 as p increases. For light-tailed distributions ($\nu = 2$ and 5) the sign test is not as effective as is T^2, yet the ARE's increase with the dimension. For heavy-tailed populations ($\nu = .5$, .25, and .1) V_n is more effective than T^2. Here the efficiencies decrease with p. In fact, the limit of each column as $p \to \infty$ is 1. But this is not a relevant limit, since dimension reduction methods should be used with very high-dimensional data. If we fix p, it is relevant to note that as $\nu \to 0$, the ARE $\to +\infty$ and as $\nu \to \infty$, the ARE $\to p/(p + 2)$, the latter being the efficiency comparison of these two procedures when the underlying population is uniform inside a unit sphere.

Table 1. ARE(V_n, T^2) for the Elliptically Symmetric Power Family

	ν					
Dimension	5.0	2.0	1.0	.5	.25	.10
1	.347	.411	.637	2.00	30.0	252,252
2	.519	.590	.785	1.50	5.83	367
3	.620	.688	.849	1.33	3.36	54.8
4	.687	.749	.884	1.25	2.53	21.0
5	.734	.790	.905	1.20	2.12	11.7
10	.849	.885	.951	1.10	1.47	3.53

A comparison of V_n with S_n^* via Pitman ARE yields the following result, the proof of which is sketched in the Appendix.

Theorem 4.2. If the underlying population is elliptically symmetric with density function

$$f(x) = k_p|\Sigma|^{-1/2}g((x - \theta)'\Sigma^{-1}(x - \theta)) \quad (4.4)$$

satisfying Assumptions A.1 and A.2 (see the Appendix), and if Σ is any diagonal matrix with positive elements on the diagonal,

$$\mathrm{ARE}_p(V_n, S_n^*) = \frac{\pi\Gamma^2[(p + 1)/2]}{p\Gamma^2(p/2)}. \quad (4.5)$$

Thus for the power family in Equation (4.1) and even a much broader class of population models, when Σ is diagonal, the ARE_p depends only on p. Some values are $\mathrm{ARE}_1 = 1.00$, $\mathrm{ARE}_2 = 1.23$, $\mathrm{ARE}_3 = 1.33$, $\mathrm{ARE}_4 = 1.39$, $\mathrm{ARE}_5 = 1.42$, and $\lim_{p\to\infty} \mathrm{ARE}_p = 1.57$. We see that in this situation V_n is uniformly better than S_n^* for $p \geq 2$ and the improvement increases with the dimension. The restriction of Σ to be diagonal is not severe. The efficiency expression for S_n^* is quite complicated. Making Σ diagonal is favorable to the efficiency of S_n^*, and it makes its efficiency the same regardless of the direction of shift. For other choices of Σ, it may be possible to slightly improve S_n^*'s efficiencies for some directions of shift and drastically depreciate its efficiencies for other directions. See Bickel (1965).

Table 2 contains results from a Monte Carlo study of the power of T^2, V_n, and S_n^* when the dimension is p = 3, the sample size is $n = 20$, and the significance level is $\alpha = .05$. The statistic T^2 is performed as indicated in (1.1), and both V_n and S_n^* were compared with the upper αth quantile of a χ_3^2 distribution. Six different trivariate distributions were used. They were located at $\theta = (t\Delta, t\Delta, t\Delta)'$ for $t = 0, 1, 2, 3$. The value of Δ was adjusted for different distributions to examine somewhat similar points on the power curves. The six distributions were (1) uniform inside the three-dimensional unit sphere, generated via $X = ZU^{1/3}/(Z'Z)^{1/2}$, where Z is $N_3(0, I)$ and U is independent and uniform (0, 1); (2) a trivariate $N_3(0, I)$ distribution; (3) a t distribution with 3 df, generated via $X = Z/(S/\eta)^{1/2}$, where Z is $N_3(0, I)$ and S is independent with a χ^2 distribution with $\eta = 3$ df; (4) a Cauchy distribution that is the same as a t distribution with 1 df; and (5) a distribution that is skewed in the direction of the first component, generated via $X = 20Z(1 + Z_1(Z'Z)^{-1/2})$, where Z is $N_3(0, I)$ and Z_1 is the first component of Z. [If $Z = RU$, where U is uniform on the unit sphere, then $X = 20R(1 + U_1)U \equiv R^*U$, where U_1 is the first component of U and R^* is a nonnegative scalar, dependent on U.] In Case (6) the distribution is the same as in Case (5) except that the direction of shift was changed to $\theta = (-t\Delta, 0, 0)'$.

Table 2. Observed Relative Frequency of Rejecting H_0 With $\alpha = .05$

Test	Amount of shift 0	1Δ	2Δ	3Δ
	Uniform (Δ = .08)			
T^2	.048	.153	.484	.898
V_n	.038	.087	.302	.658
S_n^*	.052	.078	.236	.543
	Normal (Δ = .15)			
T^2	.056	.115	.376	.728
V_n	.045	.084	.283	.601
S_n^*	.039	.082	.214	.496
	t Distribution with 3 df (Δ = .19)			
T^2	.040	.109	.357	.653
V_n	.047	.121	.392	.719
S_n^*	.041	.098	.296	.620
	Cauchy (Δ = .21)			
T^2	.014	.035	.115	.252
V_n	.037	.111	.313	.544
S_n^*	.043	.106	.264	.518
	Skewed (Δ = .8)			
T^2	.157	.273	.480	.743
V_n	.045	.148	.463	.796
S_n^*	.039	.087	.281	.602
	Skewed (Δ = 4.0)			
T^2	.157	.073	.230	.589
V_n	.045	.152	.393	.688
S_n^*	.039	.113	.253	.512

Table 2 shows the fraction of times out of 1,000 that each procedure rejected H_0. Values near .05 and .95 have approximate standard errors of .007, values near .25 and .75 have .014, and values near .5 have .016 as an approximate standard error.

This study reinforces the conclusions found via the ARE's. For light-tailed elliptically symmetric populations, T^2 is the most effective test. It is much better than either V_n or S_n^*. When the underlying population is normal, T^2 is best, with V_n competing reasonably well and S_n^* a poor third place. With a moderately heavy-tailed distribution V_n is best with T^2 competing very well and S_n^* clearly in third place. With a very heavy-tailed population T^2 has difficulty maintaining the designated significance level. Both V_n and S_n^* do well here, with V_n the better of the two. In the skewed distribution cases, T^2 again has difficulty maintaining the desired significance level. The statistic V_n performs well here, far better than S_n^*.

5. AN EXAMPLE, COMMENTS, AND CONCLUSIONS

Example. Merchant et al. (1975) studied changes in pulmonary function of 12 workers after six hours of cotton dust exposure. Table 3 displays the changes in forced vital capacity (FVC), forced expiratory volume (FEV_3) and closing capacity (CC). Testing the mean vector of changes equal to 0, we find $T^2 = 3.823$ ($p = .051$, using its F distribution), $V_n = 8.359$ ($p = .039$, using a χ_3^2), and $S_n^* = 3.667$ ($p = .300$, using a χ_3^2). Thus the interdirection sign test and Hotelling's T^2 both indicate degrees of significance in these changes, but the component sign test does not.

We note that the conditional null distribution of V_n is very simple to compute and is appropriate under weak population assumptions. If the underlying population is not one with elliptical directions, we can compare the value

Table 3. Changes in Pulmonary Function After Six Hours of Exposure to Cotton Dust

Subject	FVC	FEV_3	CC
1	−.11	−.12	−4.3
2	.02	.08	4.4
3	−.02	.03	7.5
4	.07	.19	−0.3
5	−.16	−.36	−5.8
6	−.42	−.49	14.5
7	−.32	−.48	−1.9
8	−.35	−.30	17.3
9	−.10	−.04	2.5
10	.01	−.02	−5.6
11	−.01	−.17	2.2
12	−.26	−.30	5.5

NOTE: FVC denotes forced vital capacity, FEV_3 denotes forced expiratory volume, and CC denotes closing capacity.

of V_n with its conditional null distribution provided that under H_0 the origin is the median of the conditional distribution on any observable axis through the origin. To create this null distribution, condition on the collection of observed axes. Let $\delta = (\delta_1, \ldots, \delta_n)'$, where $\delta_j = 0$ if X_j is on the observed side of the origin on the jth axis, and $\delta_j = 1$ if X_j is on the unobserved side of the origin on the jth axis. Under H_0, the 2^n possible δ vectors are equally likely and the corresponding values of V_n can be generated via

$$V_n(\delta) = \frac{p}{n} \sum_{j=1}^{n} \sum_{k=1}^{n} (-1)^{\delta_j+\delta_k} \cos(\pi \hat{p}_{jk}).$$

The $\cos(\pi \hat{p}_{jk})$ terms do not depend on δ and need only be computed once. The foregoing form shows how simple it is to generate the 2^n values of V_n. This is in contrast to Hotelling's T^2, for which one must assume under H_0 that the conditional distribution on any observable axis through the origin is *symmetric* around the origin and which requires *recomputation* of S^{-1} for each of the 2^n equally likely values, unless S is replaced by $\Sigma x_i x_i'/n$.

The comparisons in Section 4 did not include the component signed rank test. It is best seen in competition with a signed rank test based on interdirections as developed by Peters (1988). These and multisample versions will be reported separately. The first purpose of this article was to introduce interdirections and to describe their invariance and distribution-free properties. The second purpose was to develop a sign test based on interdirections. This test performs well relative to T^2 when the underlying population is heavy-tailed or skewed. It consistently outperforms the component sign test, which is frequently recommended in the nonparametric literature. Not only does it have better power, but it has the desirable natural invariance and a small-sample distribution-free property.

APPENDIX: PROOFS OF RESULTS

Proof of Result 2.1. Let $X_1, \ldots, X_{p-1}$ denote $p - 1$ vectors used along with 0 to form a hyperplane in R^p. Let $\delta \neq 0$ denote a vector satisfying $\delta' X_t = 0$ for $t = 1, \ldots, p-1$; that is, δ is orthogonal to the hyperplane and is unique up to a nonzero multiplicative constant. Let $Y_t = DX_t$ for $t = 1, \ldots, n$, where D is any nonsingular $p \times p$ matrix. Note that $\delta^* \equiv (D^{-1})'\delta$ satisfies $\delta^{*\prime} Y_t = 0$, for $t = 1, \ldots, p-1$; that is, δ^* is orthogonal to the transformed hyperplane. In addition, $\delta^{*\prime} Y_j = \delta' X_j$ and $\delta^* Y_k = \delta' X_k$. Thus, if X_j and X_k were on the same (opposite) side(s) of the hyperplane before transformation, they will be on the same (opposite) side(s) after transformation.

Under H_0, $X_1, \ldots, X_n$ are iid with density $f(x)$. Under a sequence of alternatives let $X_1, \ldots, X_n$ be iid with density $f(x - cn^{-1/2})$, where $c \in R^p - \{0\}$ is arbitrary but fixed. Assume the following.

Assumption A.1. $f(x)$ is absolutely continuous with respect to Lebesgue measure.

Assumption A.2. $0 < I_c(f) \equiv \int [c'\partial f(x)/f(x)]^2 f(x)\, dx < \infty$ for all $c \neq 0$.

Under these assumptions, the rationalé of Hájek and Šidák (1967, pp. 212–213 and earlier), show that the alternatives are contiguous to the null hypothesis.

Because V_n does not depend on R_i and is invariant with respect to D, we will assume under H_0 that $X_1 = U_1, \ldots, X_n = U_n$ are iid uniformly distributed on the unit p sphere. Define $A_j = \operatorname{sgn}(U_{1j})U_j$ and $B_j = \operatorname{sgn}(U_{1j})$ so that $U_j = B_j A_j$ for $j = 1, \ldots, n$. Under H_0, B_j is independent of A_j and $B_1, \ldots, B_n$ are iid with $\Pr[B_j = 1] = \Pr[B_j = -1] = \frac{1}{2}$. Direct computation yields the following lemma.

Lemma A.1. Let $h(x_1, x_2)$ satisfy (i) $h(x_1, x_2) = -h(-x_1, x_2) = -h(x_1, -x_2) = h(-x_1, -x_2)$, (ii) $h(x_1, x_2) = h(x_2, x_1)$, and (iii) $h(x_1, x_1) = 0$, for every x_1 and x_2. Then

$$E_B\left[\left\{\sum_{j=1}^{n}\sum_{k=1}^{n} h(B_jA_j, B_kA_k)\right\}^2\right] = 2\sum_{j=1}^{n}\sum_{k=1}^{n} h^2(A_j, A_k)$$
$$= 2\sum_{j=1}^{n}\sum_{k=1}^{n} h^2(B_jA_j, B_kA_k).$$

Using V_n^* defined in (3.3), we now establish an asymptotic approximation to V_n and its limiting null distribution.

Theorem A.1. If the observations are from distributions with elliptical directions and H_0 is true, then (a) $V_n - V_n^* \xrightarrow{p} 0$ and (b) $V_n \xrightarrow{d} \chi_p^2$ as $n \to \infty$.

Proof. Because $\hat{p}_{jk}$ depends on the B's only through B_j and B_k, Lemma A.1 yields

$$E_{H_0}[(V_n - V_n^*)^2] = \frac{2p^2}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{n} E_{H_0}[\{\cos(\pi\hat{p}_{jk}) - U_j'U_k\}^2]$$
$$\leq 2p^2\, E_{H_0}[\{\cos(\pi\hat{p}_{jk}) - U_j'U_k\}^2]$$
$$\leq 2\pi^2p^2E[\{\hat{p}_{jk} - \arccos(U_j'U_k)/\pi\}^2], \quad j \neq k.$$

Note that the conditional expected value given U_j and U_k goes to 0 as $n \to \infty$. Hence the Lebesgue dominated convergence theorem yields Part (a). Part (b) follows from $V_n^* \xrightarrow{d} \chi_p^2$. See, for example, Watson (1983, p. 42).

Proof of Theorem 4.1. Define $L_n^* = -n^{-1/2}\sum_{j=1}^{n} c'\partial f(X_j)/f(X_j)$. With $f(x)$ given by (4.1) and a sequence of alternatives as in Assumptions A.1 and A.2, standard arguments show that, under H_0,

$$(L_n^*, \sqrt{n}\,\lambda'\overline{U}) \xrightarrow{d} N_2\left(0, \begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{pmatrix}\right),$$

where $\sigma_{11} = I_c(f)$, $\sigma_{22} = \lambda'\lambda/p$, and $\sigma_{12} = c'\lambda 2\nu E_{H_0}[R_1^{2\nu-1}]/(pc_0^\nu)$. Thus under the sequence of alternatives, $V_n^* = pn\overline{U}'\overline{U} \xrightarrow{d} \chi_p^2(4\nu^2E_{H_0}^2[R_1^{2\nu-1}]c'c/(pc_0^{2\nu}))$ and hence $\mathrm{ARE}(V_n, T^2) = 4\nu^2E_{H_0}^2[R_1^{2\nu-1}]/(pc_0^{2\nu})$. Evaluation of the expected value yields (4.3).

Proof of Theorem 4.2. Let $F(x)$ be given by (4.4). Since both tests are invariant under changes in scale, we can without loss of generality take $\Sigma = I$. Standard arguments show that, under H_0,

$$(L_n^*, \sqrt{n}\,\lambda'\overline{U}) \xrightarrow{d} N_2\left(0, \begin{pmatrix} \sigma_{11}^* & \sigma_{12}^* \\ \sigma_{12}^* & \sigma_{22}^* \end{pmatrix}\right),$$

where $\sigma_{11}^* = I_c(f)$, $\sigma_{22}^* = \lambda'\lambda/p$, and $\sigma_{12}^* = 2c'\lambda E_{H_0}[R_1 g'(R_1^2)/g(R_1^2)]/p$. Thus, under the sequence of alternatives, $V_n \xrightarrow{d} \chi_p^2(4E_{H_0}^2[R_1 g'(R_1^2)/g(R_1^2)]c'c/p)$. From Bickel (1965), we see that, under the sequence of alternatives, $S_n^* \xrightarrow{d} \chi_p^2(4h^2(0)c'c)$, where $h(0)$ is the common marginal density evaluated at 0. In our case $h(0) = k_p/k_{p-1}$ and $\text{ARE}(V_n, S_n^*) = k_{p-1}^2 E_{H_0}^2[R_1 g'(R_1^2)/g(R_1^2)]/(pk_p^2)$. Evaluation of this expected value produces (4.5).

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REFERENCES

Bennett, B. M. (1962), "On Multivariate Sign Tests," *Journal of the Royal Statistical Society*, Ser. B, 24, 159–161.

Bickel, P. J. (1965), "On Some Asymptotic Competitors to Hotelling's T^2," *The Annals of Mathematical Statistics*, 36, 160–173.

Blumen, I. (1958), "A New Bivariate Sign Test," *Journal of the American Statistical Association*, 53, 448–456.

Brown, B. M., and Hettmansperger, T. P. (in press), "An Affine Invariant Bivariate Version of the Sign Test," *Journal of the Royal Statistical Society*, 51.

Chatterjee, S. K. (1966), "A Bivariate Sign Test for Location," *The Annals of Mathematical Statistics*, 37, 1771–1782.

Dietz, E. J. (1982), "Bivariate Nonparametric Tests for the One-Sample Location Problem," *Journal of the American Statistical Association*, 77, 163–169.

——— (1984), "Linear Signed Rank Tests for Multivariate Location," *Communications in Statistics*, 13, 1435–1451.

Hájek, J., and Šidák, Z. (1967), *Theory of Rank Tests*, New York: Academic Press.

Hettmansperger, T. P. (1984), *Statistical Inference Based on Ranks*, New York: John Wiley.

Hodges, J. L., Jr. (1955), "A Bivariate Sign Test," *The Annals of Mathematical Statistics*, 26, 523–527.

Joffe, A., and Klotz, J. (1962), "Null Distribution and Bahadur Efficiency of the Hodges Bivariate Sign Test," *The Annals of Mathematical Statistics*, 33, 803–807.

Killeen, T. J., and Hettmansperger, T. P. (1972), "Bivariate Tests for Location and Their Bahadur Efficiencies," *The Annals of Mathematical Statistics*, 43, 1507–1516.

Klotz, J. (1964), "Small Sample Power of the Bivariate Sign Tests of Blumen and Hodges," *The Annals of Mathematical Statistics*, 35, 1576–1582.

Liu, R. Y. (in press), "On a Notion of Data Depth Based on Random Simplices," *The Annals of Mathematical Statistics*, 17.

Merchant, J. A., Halprin, G. M., Hudson, A. R., Kilburn, K. H., McKenzie, W. N., Jr., Hurst, D. J., and Bermazohn, P. (1975), "Responses to Cotton Dust," *Archives of Environmental Health*, 30, 222–229.

Oja, H. (1983), "Descriptive Statistics for Multivariate Distributions," *Statistics and Probability Letters*, 1, 327–333.

Oja, H., and Nyblom, J. (1989), "Bivariate Sign Tests," *Journal of the American Statistical Association*, 84, 249–259.

Peters, D. (1988), "Rank Tests for the One- and Two-Sample Bivariate Location Problems," unpublished Ph.D. dissertation, University of Florida, Dept. of Statistics.

Puri, M. L., and Sen, P. K. (1971), *Nonparametric Methods in Multivariate Analysis*, New York: John Wiley.

Sen, P. K., and Puri, M. L. (1967), "On the Theory of Rank Order Tests for Location in the Multivariate One Sample Problem," *The Annals of Mathematical Statistics*, 38, 1216–1228.

Watson, G. S. (1983), *Statistics of Spheres*, New York: Wiley-Interscience.