METHOD OF DIFFERENTIATION

1.0 INTRODUCTION :

The essence of calculus is the derivative. The derivative is the instantaneous rate of change of a function with respect to one of its variables. This is equivalent to finding the slope of the tangent line to the function at a point. Let us use the view of derivatives as tangents to motivate a geometric definition of the derivative.

1.1 GEOMETRICAL MEANING OF A DERIVATIVE :

Let P $(x_0, f(x))$ and $Q(x_0 + h, f(x_0 + h))$ be two points very close to each other on the curve $y = f(x)$. Draw PM and QN perpendiculars from P and Q on x-axis, and draw PL as perpendicular from P on QN. Let the chord PQ produced meet the x-axis at R and $\angle QRN = \angle QPL = \phi$.
Now in right-angled triangle QPL

$$\tan\phi = \frac{QL}{PL} = \frac{NQ-NL}{MN} = \frac{NQ-MP}{ON-OM} = \frac{f(x_0+h)-f(x_0)}{(x_0+h)-x_0}$$

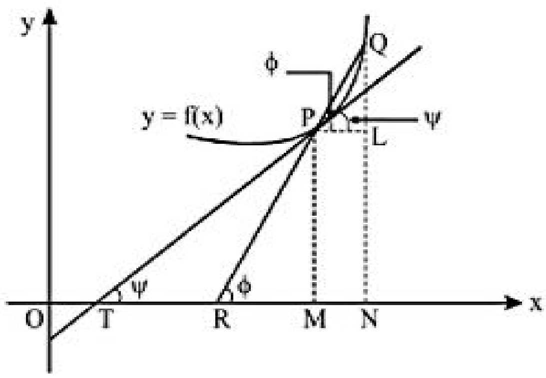
$$= \frac{f(x_0+h)-f(x_0)}{h} \qquad \text{.........(1)}$$

when $h \to 0$, the point Q moving along the curve tends to P, i.e., $Q \to P$. The chord PQ approaches the tangent line PT at the point P and then $\phi \to \psi$. Now applying $\lim\limits_{h\to 0}$ in equation (1), we get

$$\lim_{h\to 0}\tan\phi = \frac{f(x_0+h)-f(x_0)}{h}$$

$$\tan\psi = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

$$\Rightarrow \qquad f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$



This definition of derivative is also called the first principle of derivative. Clearly, the domain of definition of f'(x) is wherever the above limit exits.
Note that if $y = f(x)$ then the symbols

$$\frac{dy}{dx} = Dy = f'(x) = y_1 \text{ or } y' \text{ have the same meaning.}$$

However a dot, denotes the time derivative.

e.g. $\dot{S} = \frac{dS}{dt}$; $\dot{\theta} = \frac{d\theta}{dt}$ etc.

Illustration :

Find the derivative of $e^{\sqrt{x}}$ *w.r.t. x using first principle.*

Sol. *Let* $f(x) = e^{\sqrt{x}}$, *then* $f(x+h) = e^{\sqrt{x+h}}$

$$\therefore \quad \frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{e^{\sqrt{x+h}}-e^{\sqrt{x}}}{h} = e^{\sqrt{x}}\lim_{h\to 0}\left(\frac{e^{\sqrt{x+h}-\sqrt{x}}-1}{h}\right)$$

$$= e^{\sqrt{x}}\lim_{h\to 0}\left(\frac{e^{\sqrt{x+h}-\sqrt{x}}-1}{\sqrt{x+h}-\sqrt{x}}\right)\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)$$

$$= e^{\sqrt{x}}\lim_{h\to 0}\left(\frac{e^{\sqrt{x+h}-\sqrt{x}}-1}{\sqrt{x+h}-\sqrt{x}}\right)\times\frac{\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}+\sqrt{x}\right)}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= e^{\sqrt{x}}\lim_{h\to 0}\left(\frac{e^{y}-1}{y}\right)\lim_{h\to 0}\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)},$$

where $y = \sqrt{x+h}-\sqrt{x}$ $\quad (\because \text{ when } h\to 0,\ y\to 0)$

$$= e^{\sqrt{x}}\times 1\times\left(\frac{1}{\sqrt{x}+\sqrt{x}}\right) = \frac{e^{\sqrt{x}}}{2\sqrt{x}}.$$

Illustration :

Find the derivative of the following functions with respect to x using first principle.

(i) $y = \frac{x}{x^2+1}$; *(ii)* $y = \cos^2 x$; *(iii)* $y = \sin 3x$; *(iv)* $y = x^3 - 3^x$; *(v)* $y = \ln^2 x$.

Sol.

(i) $y = \frac{x}{x^2+1} = f(x)$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{x+h}{(x+h)^2+1}-\frac{x}{x^2+1}}{h}$$

$$= \lim_{h\to 0}\frac{(x+h)(x^2+1)-x\left[(x^2+1)+2hx+h^2\right]}{h(x^2+1)\left[(x+h)^2+1\right]} = \lim_{h\to 0}\frac{(x^2+1)-x(2x+h)}{(x^2+1)\left[(x+h)^2+1\right]}$$

$$= \lim_{h\to 0}\frac{(x^2+1)-2x^2}{(x^2+1)^2} = \lim_{h\to 0}\frac{1-x^2}{(x^2+1)^2}.$$

(ii) $y = \cos^2 x = f(x)$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\cos^2(x+h)-\cos^2 x}{h}$$

$$= \lim_{h\to 0}\frac{2\sin\left(x+\frac{h}{2}\right)\sin\left(\frac{-h}{2}\right)}{h}\ (\cos(x+h)+\cos x))$$

$$= 2\sin x\left(\frac{-1}{2}\right)(2\cos x) = -2\sin x\cos x = -\sin 2x.$$

(iii) $y = \sin 3x = f(x)$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(3x+3h)-\sin 3x}{h}$$

$$= \lim_{h\to 0}\frac{2\sin\left(\frac{3h}{2}\right)\cos\left(3x+\frac{3h}{2}\right)}{h} = 2\cdot\left(\frac{3}{2}\right)\cdot\cos(3x+0) = 3\cos 3x.$$

(iv) $y = x^3 - 3^x = f(x)$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^3-3^{x+h}-(x^3-3^x)}{h}$$

$$= \lim_{h\to 0}\frac{(x+h)^3-x^3}{h} - \lim_{h\to 0}\frac{(3^{x+h}-3^x)}{h} = \lim_{h\to 0}\frac{3xh(x+h)+h^3}{h} - 3^x\lim_{h\to 0}\frac{(3^h-1)}{h}$$

$$= 3x(x+0) - 3^x\ln 3 = 3x^2 - 3^x\ln 3.$$

(v) $y = f(x) = \ln^2 x$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\ln^2(x+h)-\ln^2 x}{h}$$

$$= \lim_{h\to 0}\left(\frac{\ln(x+h)-\ln x}{h}\right)(\ln(x+h)+\ln x)$$

$$= \lim_{h\to 0}\frac{\ln\left(1+\frac{h}{x}\right)}{h}(\ln(x+h)+\ln x) = \frac{2\ln x}{x}.$$

1.2 STANDARD DERIVATIVES :

(i) $\frac{d}{dx}x^n = nx^{n-1}, x \in R, n \in R, x > 0$

(ii) $\frac{d}{dx}(e^x) = e^x$

(iii) $\frac{d}{dx}(a^x) = a^x \ln a$

(iv) $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$

(v) $\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e$

(vi) $\frac{d}{dx}(\sin x) = \cos x$

(vii) $\frac{d}{dx}(\cos x) = -\sin x$

(viii) $\frac{d}{dx}(\tan x) = \sec^2 x$

(ix) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(x) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(xi) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

Illustration :

If $y = \left(1+x^{\frac{1}{4}}\right)\left(1+x^{\frac{1}{2}}\right)\left(1-x^{\frac{1}{4}}\right)$, *then find* $\frac{dy}{dx}$.

Sol. $y = \left(1+x^{\frac{1}{4}}\right)\left(1+x^{\frac{1}{2}}\right)\left(1-x^{\frac{1}{4}}\right) = \left(1+x^{\frac{1}{4}}\right)\left(1-x^{\frac{1}{4}}\right)\left(1+x^{\frac{1}{2}}\right) = \left(1-x^{\frac{1}{2}}\right)\left(1+x^{\frac{1}{2}}\right)$

$= 1 - x \Rightarrow \frac{dy}{dx} = -1.$

Illustration :

If $f(x) = x\,|x|$, *then prove that* $f'(x) = 2\,|x|$.

Sol. $f(x) = \begin{cases} -x^2, & x < 0 \\ x^2, & x > 0 \end{cases} \Rightarrow f'(x) = \begin{cases} -2x, & x < 0 \\ 2x, & x > 0 \end{cases}$

$\therefore \quad f'(x) = 2\,|x|.$

Illustration :

If $y = \sqrt{\frac{1-\cos 2x}{1+\cos 2x}}, x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, 0\right)$, *then find* $\frac{dy}{dx}$.

Sol. *We have*

$$y = \sqrt{\frac{1-\cos 2x}{1+\cos 2x}} = \sqrt{\frac{2\sin^2 x}{2\cos^2 x}} = \sqrt{\tan^2 x} \quad \Rightarrow \quad y = |\tan x|, \text{ where } x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$$

$$\Rightarrow \quad y = \begin{cases} \tan x, & x \in \left(0, \frac{\pi}{2}\right) \\ -\tan x, & x \in \left(\frac{\pi}{2}, \pi\right) \end{cases} \quad \Rightarrow \quad \frac{dy}{dx} = \begin{cases} \sec^2 x, \text{ if } x \in \left(0, \frac{\pi}{2}\right) \\ -\sec^2 x, \text{ if } x \in \left(\frac{\pi}{2}, \pi\right) \end{cases}$$

Illustration :

(i) $D\left(\frac{1}{\sin x}\right)$; (ii) $D(\tan(\tan^{-1}x))$; (iii) $D\left(\cos^4\frac{x}{2}-\sin^4\frac{x}{2}\right) =$;

(iv) $D(\cos^{-1}x + \sin^{-1}x)^n =$; (v) $D(\cos x \operatorname{cosec} x) =$ ____;

(vi) $D\left(\frac{1}{\log_x e}\right)$ when $x = \frac{1}{e}$ = ____ ; (vii) $D\left(\frac{1-\cos 2x}{\sin 2x}\right) =$ ____

Sol.

(i) $D\left(\frac{1}{\sin x}\right) = -\operatorname{cosec} x \cot x$

(ii) $D(\tan(\tan^{-1}x)) = 1$

(iii) $D\left(\cos^4\frac{x}{2}-\sin^4\frac{x}{2}\right) = \frac{d}{dx}\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right) = \frac{d}{dx}(\cos x) = -\sin x$

(iv) $D(\cos^{-1}x + \sin^{-1}x)^n = \frac{d}{dx}\left(\frac{\pi}{2}\right)^n = 0$

(v) $D(\cos x \operatorname{cosec} x) = \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(vi) $D\left(\frac{1}{\log_x e}\right) = \frac{1}{x}$ $\therefore$ $D\left(\frac{1}{\log_x e}\right)_{at\ x=\frac{1}{e}} = e$

(vii) $D\left(\frac{1-\cos 2x}{\sin 2x}\right) = D\left(\frac{2\sin^2 x}{2\sin x\cos x}\right) = D(\tan x) = \sec^2 x$

2.0 THEOREM ON DERIVATIVES :

T-1: $\frac{d}{dx}(f_1(x) \pm f_2(x)) = \frac{d}{dx}f_1(x) \pm \frac{d}{dx}f_2(x)$.

T-2: $\frac{d}{dx}(kf(x)) = k\frac{d}{dx}f(x)$, where k is any constant.

T-3 PRODUCT RULE :

$$\frac{d}{dx}\{f_1(x)\,f_2(x)\} = f_1(x)\frac{d}{dx}f_2(x) + f_2(x)\frac{d}{dx}f_1(x).$$

Illustration :

Find the derivative of the following functions

(i) $x \sin x$; *(ii)* $e^x \cdot \tan x$

Sol.

(i) $y = x \sin x$

$$y = x \sin x \text{ so } \frac{dy}{dx} = \sin x \frac{d}{dx}(x) + x\frac{d}{dx}(\sin x)$$

$$\frac{dy}{dx} = \sin x + x \cos x$$

(ii) $y = e^x \tan x$

$$\text{so } \frac{dy}{dx} = \tan x \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(\tan x) = e^x (\tan x + \sec^2 x)$$

Note : If 3 functions are involved then remember

$$D\left(f(x)\cdot g(x)\cdot h(x)\right) = f(x)\cdot g(x)\cdot h'(x) + g(x)\cdot h(x)\cdot f'(x) + h(x)\cdot f(x)\cdot g'(x)$$

$$= \frac{(fg)'(h) + (gh)'(f) + (hf)'(g)}{2}$$

This result can be generalised to the product of n functions

Illustration :

Let $F(x) = f(x)\cdot g(x)\cdot h(x)$. *If for some* $x = x_0$, $F'(x_0) = 21\,F(x_0)$; $f'(x_0) = 4f(x_0)$; $g'(x_0) = -7g(x_0)$ *and* $h'(x_0) = k\,h(x_0)$ *then find k.*

Sol. $F(x) = f(x)\, g(x)\, h(x)$

Given that $F'(x_0) = 21f(x_0)$, $f'(x_0) = 4f(x_0)$; $g'(x_0) = -7\, g(x_0)$ *and* $h'(x_0) = kh(x_0)$

$$F'(x_0) = f'(x_0)\, g(x_0)\, h(x_0) + f(x_0)\, g'(x_0)\, h(x_0) + f(x_0)\, g(x_0)\, h'(x_0)$$

$$21F(x_0) = (4 - 7 + k)\, f(x_0)\, g(x_0)\, h(x_0) \quad \Rightarrow 21 = -3 + k \Rightarrow k = 24.$$

Illustration :

Let f, g and h are differentiable functions. If $f(0) = 1$; $g(0) = 2$; $h(0) = 3$ *and the derivatives of their pair wise products at* $x = 0$ *are*

$(f\,g)'(0) = 6$; $(g\,h)'(0) = 4$ *and* $(h\,f)'(0) = 5$

then compute the value of $(fgh)'(0)$.

Sol. *Let* $wx = f(x)\, g(x)\, h(x)$

$$\frac{d}{dx}(w(x)) = f'(x)\, g(x)\, h(x) + f(x)\, g'(x)\, h(x) + f(x)\, g(x)\, h'(x)$$

$$2\frac{d}{dx}(w(x)) = 2f'(x)\, g(x)\, h(x) + 2f(x)\, g'(x)\, h(x) + 2f(x)\, g(x)\, h'(x)$$

$$= h(x)\frac{d}{dx}(f(x)\,g(x)) + g(x)\frac{d}{dx}(f(x)h(x)) + f(x)\frac{d}{dx}(g(x)h(x))$$

$$2\,\frac{d}{dx}(w(0)) = (3)\,(6) + (2)\,(5) + (1)\,(4) = 32$$

$$\frac{d}{dx}(w(0)) = 16.$$

Illustration :

If $f(x) = 1 + x + x^2 + ... + x^{100}$ *then* $f'(1) =$ ____.

Sol. $f(x) = 1 + x + x^2 + ... + x^{100}$
$f'(x) = 0 + 1 + 2x + 3x^2 + ... + 100\,x^{99}$

$$\therefore\ f'(1) = 1 + 2 + 3 + ... + 100 = \frac{100(100+1)}{2} = 5050$$

T-4 QUOTIENT RULE :

$$y = \frac{f(x)}{g(x)}$$

$$D\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{g^2(x)}$$, to be remembered as

$$D\left(\frac{N^r}{D^r}\right) = \frac{D^r\frac{d}{dx}(N^r) - N^r\frac{d}{dx}(D^r)}{(D^r)^2};$$

Note: If $y = \frac{1}{f(x)}$ then $D(y) = -\frac{f'(x)}{f^2(x)}$

Illustration :

Find the derivative of the following functions

(i) $y = \frac{1-\ln x}{1+\ln x}$; *(ii)* $y = \frac{x^3+2^x}{e^x}$; *(iii)* $y = \frac{x\sin x}{1+\tan x}$

Sol.

(i) $y = \frac{1-\ln x}{1+\ln x}$

$$\frac{dy}{dx} = \frac{(1+\ln x)\left(\frac{-1}{x}\right) - \frac{(1-\ln x)}{x}}{(1+\ln x)^2} = \frac{-(1+\ln x+1-\ln x)}{x(1+\ln x)^2} = \frac{-2}{x(1+\ln x)^2}$$

(ii) $y = \frac{x^3+2^x}{e^x} = x^3 e^{-x} + 2^x e^{-x}$

$$\frac{dy}{dx} = 3x^2e^{-x} - x^3e^{-x} - 2^xe^{-x} + 2^xe^{-x}(\ln 2) = (3x^2 - x^3 - 2^x + 2^x \ln 2)\,e^{-x}$$

(iii) $y = \dfrac{x \sin x}{(1 + \tan x)}$

$$\frac{dy}{dx} = \frac{(1+\tan x)(\sin x + x\cos x) - (x\sin x)\sec^2 x}{(1+\tan x)^2}$$

$$= \frac{\sin x + x\cos x + \sin x \tan x + x \sin x - x \tan x \sec x}{(1+\tan x)^2}$$

$$= \frac{(1+x)\sin x + x\cos x + (\sin x - x\sec x)\tan x}{(1+\tan x)^2}$$

Illustration :

(i) If $y = \dfrac{x^4 + x^2 + 1}{x^2 + x + 1}$ then $\dfrac{dy}{dx} = ax + b$ find a and b.

(ii) If $y = \dfrac{\sec x + \tan x - 1}{\tan x - \sec x + 1}$, find $\left.\dfrac{dy}{dx}\right]_{x=\pi/4}$

(iii) If $y = \dfrac{x^3 + x^2 + x}{1 + x^2}$, find $\dfrac{dy}{dx}$.

Sol.

(i) $y = \dfrac{x^4 + x^2 + 1}{x^2 + x + 1} = \dfrac{(x^4 + 2x^2 + 1) - x^2}{x^2 + x + 1} = x^2 - x + 1$

$$\frac{dy}{dx} = 2x - 1 = ax + b \Rightarrow a = 2, b = -1$$

(ii) $y = \dfrac{\sec x + \tan x - 1}{\tan x - \sec x + 1} = \dfrac{\sec x + \tan x - (\sec^2 x - \tan^2 x)}{\tan x - \sec x + 1}$

$$y = \frac{(\sec x + \tan x)(1 - \sec x + \tan x)}{(\tan x - \sec x + 1)}$$

$$y = \sec x + \tan x$$

$$\therefore \frac{dy}{dx} = \sec x \tan x + \sec^2 x = \sec x (\tan x + \sec x)$$

(iii) $y = \dfrac{x^3 + x^2 + x}{1 + x^2} = (x + 1) - \dfrac{1}{x^2 + 1}$

$$\frac{dy}{dx} = 1 - \frac{(x^2+1)\frac{d}{dx}(1) - 1 \cdot \frac{dy}{dx}(x^2+1)}{(x^2+1)^2} = 1 - \frac{(0 - 2x)}{(x^2+1)^2} = 1 + \frac{2x}{(x^2+1)^2}.$$

T-5 DIFFERENTIATION OF COMPOSITE FUNCTION (CHAIN RULE) :

If f(x) and g(x) are differentiable functions, then fog is also differentiable and $(fog)'(x) = f'(g(x)).g'(x)$.

or, $$\frac{d}{dx}\{fog(x)\} = \frac{d}{d\,g(x)}\{(fog)(x)\}\ \frac{d}{dx}(g(x))$$

or

If y is a function of t and t is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

Thus, if $y = f(t)$ and $t = \phi(x)$

$$\frac{dt}{dx} = \phi'(x)$$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx} = f'(t)\,\phi'(x)$. This rule is called Chain Rule.

This chain rule can be extended as follows

Let $\quad y = f(t),\ t = \phi(z),\ z = \psi(x)$

then $\quad \dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dz} \times \dfrac{dt}{dx} = f'(t)\,\phi'(z)\,\psi'(x)$

Let $\quad y = \log\sin x^3 = \log t$

Putting $\ t = \sin x^3 = \sin z,\ z = x^3$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dz} \times \frac{dt}{dx} = (1/t)\cos z\ 3x^2 = \left(\frac{1}{\sin x^3}\right)(\cos x^3) \times 3x^2 = 3x^2 \cot x^3$$

Note : If $y = f(u)$ and $u = g(x)$ then $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx}$.

It can be extended for any number of chain. In general if $y = (E)^C$ where E = function of x and C = constant then

$$\frac{dy}{dx} = C(E)^{C-1}\cdot\frac{d}{dx}(E)$$

Illustration :

Find derivative of following functions ?

(i) $y = \sin^3\sqrt{x}$ *(ii)* $y = \ln(\sec x)$ *(iii)* $y = \cos(\ln x)$

(iv) $y = e^{ax}\sin bx.$ *(v)* $y = e^{ax}\cos bx$

Sol.

(i) $y = \sin^3\sqrt{x}$

Let $v = \sqrt{x}$; $u = \sin\sqrt{x} = \sin v$

$y = u^3$

$$\frac{dy}{dx} = \frac{d}{dx}(u^3) = \frac{d}{dx}(u^3)\cdot\left(\frac{du}{dx}\right) = 3u^2\cdot\frac{d}{dv}(\sin v)\left(\frac{dv}{dx}\right) = 3u^2\cos v\,\frac{d}{dx}\left(\sqrt{x}\right)$$

$$= 3\sin^2 v\cos v\left(\frac{1}{2\sqrt{x}}\right) = \frac{3\sin^2\sqrt{x}\cdot\cos\sqrt{x}}{2\sqrt{x}}$$

(ii) $y = \ln(\sec x)$

Let $u = \sec x$

$\therefore y = \ln u$

$$\frac{dy}{dx} = \frac{d}{dx}(\ln u) = \frac{d}{du}(\ln u)\frac{du}{dx} = \frac{1}{u}.\frac{d}{dx}(\sec x) = \frac{1}{\sec x}(\sec x \tan x) = \tan x$$

(iii) $y = \cos(\ln x)$

Let $u = \ln x$

$\therefore y = \cos u$

$$\frac{dy}{dx} = \frac{d}{dx}(\cos u) = \frac{d}{du}(\cos u)\frac{du}{dx} = -\sin u\,.\,\frac{d}{dx}(\ln x) = \frac{-\sin(\ln x)}{x}$$

(iv) $y = e^{ax}\sin bx$

$$\frac{dy}{dx} = e^{ax}\frac{d}{dx}(\sin bx) + \sin bx\,\frac{d}{dx}e^{ax}$$

$$= e^{ax}\cos bx\,\frac{d}{dx}(bx) + \sin bx\,.\,e^{ax}\frac{d}{dx}(ax)$$

$$= be^{ax}\cos bx + a\sin bx\,.\,e^{ax}$$

$$= (b\cos bx + a\sin bx)\,e^{ax}$$

(v) $y = e^{ax}\cos bx$

$$\frac{dy}{dx} = e^{ax}\frac{d}{dx}(\cos bx) + \cos bx\,\frac{d}{dx}(e^{ax}) = e^{ax}(-\sin bx)\frac{d}{dx}(bx) + \cos bx\,e^{ax}\,.\,a$$

$$= (a\cos bx - b\sin bx)\,e^{ax}$$

Illustration :

Find derivative of following functions ?

(i) $y = (fog)(x)$ (ii) $y = (gof)(x)$ (iii) $y = \dfrac{1}{(f(x))^n}$

(iv) $y = \sec^2\left(f^3(x)\right)$ (v) $y = \sqrt{f(x)}$ (vi) $y = f(1/x)$

Sol.

(i) $y = fog(x)\ \ f(g(x))$

$$\frac{dy}{dx} = \frac{d}{dx}f(g(x)) = f'(g(x))\,.\,\frac{d}{dx}g(x) \qquad = f'(g(x)).\,g'(x)$$

(ii) $y = gof(x) = g(f(x))$

$$\frac{dy}{dx} = \frac{d}{dx}g(f(x)) = g'(f(x))\,\frac{d}{dx}f(x) = g'(f(x))\,.\,f'(x)$$

(iii) $y = \dfrac{1}{(f(x))^n} = (f(x))^{-n}$

$$\frac{dy}{dx} = -n\,(f(x))^{-n-1}\,\frac{d}{dx}f(x) = \frac{-n\,f'(x)}{(f(x))^{n+1}}$$

(iv) $y = \sec^2 (f^3(x))$

$$\frac{dy}{dx} = 2\sec\left(f^3(x)\right) \frac{d}{dx}\sec\left(f^3(x)\right)$$

$$= 2 \sec (f^3(x)) \cdot \sec (f^3(x)) \tan (f^3 (x)) \frac{d}{dx}\left(f^3(x)\right)$$

$$= 2 \sec^2 (f^3(x)) \tan (f^3(x))\ 3f^2(x) \frac{d}{dx} f(x).$$

$$= 6 \sec^2 (f^3(x)) \tan (f^3(x)) f^2(x) \cdot f'(x)$$

(v) $y = \sqrt{f(x)}$

$$\therefore \frac{dy}{dx} = \frac{1}{2}(f(x))^{\frac{-1}{2}} \frac{d}{dx} f(x) = \frac{f'(x)}{2\sqrt{f(x)}}.$$

(vi) $y = f\left(\frac{1}{x}\right)$

$$\therefore \frac{dy}{dx} = f'\left(\frac{1}{x}\right)\frac{d}{dx}\left(\frac{1}{x}\right) = f'\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right) = \frac{-f'\left(\frac{1}{x}\right)}{x^2}.$$

Illustration :

Find derivative of following functions ?

(i) $y = \ln^3\tan^2(x^4)$ (ii) $y = \cos\left(\frac{ax}{b}\right)$ (iii) $y = e^{\sqrt{\sin(\ln(x^2+7)^5)}}$

(iv) $y = \sec x\sqrt{\tan x}$ (v) $\exp (\cos^3 (\tan^{-1}x^3)^2)$

Sol.

(i) $y = \ln^3 (\tan^2(x^4))$

$$\frac{dy}{dx} = 3 \ln^2 (\tan^2(x^4)) \frac{d}{dx} \ln (\tan^2(x^4))$$

$$= 3 \ln^2 (\tan^2(x^4)) \frac{1}{\tan^2(x^4)} \frac{d}{dx} \tan^2(x^4)$$

$$= \frac{3 \ln^2\left(\tan^2(x^4)\right)}{\tan^2(x^4)} \cdot 2 \tan (x^4) \cdot \frac{d}{dx} \tan\left(x^4\right)$$

$$= \frac{6 \ln^2\left(\tan^2(x^4)\right)}{\tan(x^4)} \sec^2(x^4) \frac{d}{dx}\left(x^4\right)$$

$$= \frac{24x^3 \ln^2\left(\tan^2(x^4)\right) \sec^2(x^4)}{\tan (x^4)}$$

(ii) $y = \cos\left(\frac{ax}{b}\right)$

$$\frac{dy}{dx} = -\sin\left(\frac{ax}{b}\right)\frac{d}{dx}\left(\frac{ax}{b}\right) = \frac{-a\sin\left(\frac{ax}{b}\right)}{b}$$

(iii) $y = e^{\sqrt{\sin\left(\ln(x^2+7)^5\right)}}$

$$\therefore\ y = e^{\sqrt{\sin\left(5\ln(x^2+7)\right)}}$$

$$\therefore\ \frac{dy}{dx} = e^{\sqrt{\sin\left(5\ln\left(x^2+7\right)\right)}} \cdot \frac{d}{dx}\sqrt{\sin\left(5\ln\left(x^2+7\right)\right)}$$

$$= e^{\sqrt{\sin\left(5\ln(x^2+7)\right)}} \cdot \frac{1}{2\sqrt{\sin\left(5\ln\left(x^2+7\right)\right)}} \cdot \frac{d}{dx}\sin\left(5\ln(x^2+7)\right)$$

$$= \frac{e^{\sqrt{\sin\left(5\ln(x^2+7)\right)}}}{2\sqrt{\sin\left(5\ln(x^2+7)\right)}}\ \cos\left(5\ln(x^2+7)\right) \cdot \frac{5}{(x^2+7)} \cdot \frac{d}{dx}(x^2+7)$$

$$= \frac{5x\cos\left(5\ln(x^2+7)\right)\ e^{\sqrt{\sin\left(5\ln(x^2+7)\right)}}}{(x^2+7)\sqrt{\sin\left(5\ln(x^2+7)\right)}}.$$

(iv) $y = \sec x\sqrt{\tan x}$

$$\frac{dy}{dx} = \sec x\frac{d}{dx}\sqrt{\tan x} + \sqrt{\tan x}\frac{d}{dx}(\sec x)$$

$$= \sec x\frac{1}{2\sqrt{\tan x}}\sec^2 x + \sqrt{\tan x}\cdot\sec x\cdot\tan x$$

$$= \left(\frac{\sec^3 x + 2\sec x\tan^2 x}{2\sqrt{\tan x}}\right)$$

(v) $y = \exp\left(\cos^3\left(\tan x^3\right)^2\right)$

$$\frac{dy}{dx} = \exp\left(\cos^3\left(\tan x^3\right)^2\right)\cdot\frac{d}{dx}\left(\cos^3\left(\tan x^3\right)^2\right)$$

$$= y\cdot 3\cos^2(\tan x^3)^2\cdot\ \frac{d}{dx}\cos(\tan x^3)^2$$

$$= 3y\cos^2(\tan x^3)^2\left(-\sin(\tan x^3)^2\right)\cdot\frac{d}{dx}\left(\tan x^3\right)^2$$

$$= -3y\cos^2(\tan x^3)^2\sin(\tan x^3)^2\left(2\tan(x^3)\cdot\sec^2 x^3\right)\cdot 3x^2$$

$$= -18y\,x^2\ \cos^2(\tan x^3)^2\cdot\sin(\tan x^3)^2\tan(x^3)\sec^2 x^3.$$

Illustration :

If $f(x) = (1+x)(3+x^2)^{1/2}(9+x^3)^{1/3}$ then $f'(-1)$ is equal to

(A) 0 *(B) $2\sqrt{2}$* *(C) 4* *(D) 6*

Sol. $f(x) = (1+x)(3+x^2)^{1/2}(9+x^3)^{1/3}$

$$f'(x) = (3+x^2)^{1/2}(9+x^3)^{1/3} + (1+x)\frac{1}{2}(3+x^2)^{-1/2}\cdot 2x\,(9+x^3)^{1/3}$$

$$+ (1+x)(3+x^2)^{1/2}\left(\frac{1}{3}(9+x^3)^{-2/3}\cdot 3x^2\right)$$

$f'(-1) = (3+1)^{1/2}(9-1)^{1/3} + 0 + 0$
$f'(-1) = 2\cdot 2 = 4$

Practice Problem

Q.1 Differentiate the following functions with respect to x using first principle

(i) $\sqrt{\sin x}$ (ii) $\cos^3 x$ (iii) $\tan^{-1} x$ (iv) $\log_e x$

Q.2 Find the derivative of the following functions with respect to x

(i) $\sin(x^2)$ (ii) $\sin(\cos(x^2))$ (iii) $\sin(x^2+5)$ (iv) $\cos(\sin x)$

(v) $\sin(ax+b)$ (vi) $\sec(\tan(\sqrt{x}))$ (vii) $\dfrac{\sin(ax+b)}{\cos(cx+d)}$ (viii) $\cos x^3\cdot\sin^2(x^5)$

(ix) $2\sqrt{\cos(x^2)}$ (x) $\cos(\sqrt{x})$

Q.3 Find the derivative of the following functions with respect to x

(i) e^{-x} (ii) $\sin(\log x), x>0$ (iii) $e^{\cos x}$

(iv) $\dfrac{e^x}{\sin x}$ (v) $\sin(\tan^{-1} e^{-x})$ (vi) $\log(\cos e^x)$

(vii) $e^x + e^{x^2} + ... + e^{x^5}$ (viii) $\sqrt{e^{\sqrt{x}}}, x>0$ (ix) $\log(\log x), x>1$

(x) $\dfrac{\cos x}{\log x}, x>0$ (xi) $\cos(\log x + e^x), x>0$

Q.4 Find the derivative of the following functions with respect to x

(i) x^2+3x+2 (ii) x^{20} (iii) $x\cdot\cos x$ (iv) $\log x$

(v) $x^3\log x$ (vi) $e^x\sin 5x$ (vii) $e^{6x}\cos 3x$ (viii) $\tan^{-1}x$

(ix) $\log(\log x)$ (x) $\sin(\log x)$ (xi) $\sqrt{3x+2} + \dfrac{1}{\sqrt{2x^2+4}}$ (xii) $\log_7(\log x)$

Answer key

Q.1 (i) $\frac{\cos x}{2\sqrt{\sin x}}$ (ii) $-3\cos^2 x \sin x$ (iii) $\frac{1}{1+x^2}$ (iv) $\frac{1}{x}$

Q.2 (i) $2x \cos x^2$ (ii) $-\cos(\cos x^2)\sin(x^2)\cdot 2x$ (iii) $2x\cos(x^2+5)$

(iv) $-\sin(\sin x)\cos x$ (v) $a\cos(ax+b)$ (vi) $\frac{\sec(\tan\sqrt{x})\tan(\tan\sqrt{x})\sec^2\sqrt{x}}{2\sqrt{x}}$

(vii) $a\cos(ax+b)\sec(cx+d)+c\sin(ax+b)\tan(cx+d)\sec(cx+d)$

(viii) $-3x^2\sin x^3\sin^2 x^5$ (ix) $\frac{-2\sqrt{2}x}{\sin x^2\sqrt{\sin 2x^2}}$ (x) $\frac{-\sin\sqrt{x}}{2\sqrt{x}}$

Q.3 (i) $-e^{-x}$ (ii) $\frac{\cos(\log x)}{x}$ (iii) $-e^{\cos x}\sin x$ (iv) $\frac{e^x(\sin x-\cos x)}{\sin^2 x}$

(v) $\frac{-e^{-x}\cos(\tan^{-1}e^{-x})}{1+e^{-2x}}$ (vi) $-e^x\tan e^x$, $e^x \neq (2n+1)\frac{\pi}{2}$, $n \in N$

(vii) $e^x+2xe^{x^2}+\ldots\ldots+5x^4e^{x^5}$ (viii) $\frac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}}, x>0$ (ix) $\frac{1}{x\log x}$

(x) $\frac{(x\sin x\log x+\cot x)}{x(\log x)^2}$ (xi) $-\left(\frac{1}{x}+e^x\right)\sin(\log x+e^x), x>0$

Q.4 (i) $2x+3$ (ii) $20x^{19}$ (iii) $\cos x - x\sin$ (iv) $\frac{1}{x}$ (v) $x^2+3x^2\log x$

(vi) $e^x(\sin 5x+5\cos 5x)$ (vii) $e^{6x}(6\cos 3x-3\sin 3x)$ (viii) $\frac{1}{x^2+1}$

(ix) $\frac{1}{x\log x}$ (x) $\frac{\cos(\log x)}{x}$ (ix) $\frac{3}{2\sqrt{(3x+2)}}-\frac{2x}{(2x^2+4)^{3/2}}$ (xii) $\frac{\log_7 e}{x\log x}$

T-6 LOGARITHMIC DIFFERENTIATION :

To find the derivative of :

(i) a function which is the product or quotient of a number of functions

$y = f_1(x)\,f_2(x)\,f_3(x)\ldots\ldots$ or $y=\frac{f_1(x)\,f_2(x)\,f_3(x)\ldots..}{g_1(x)\,g_2(x)\,g_3(x)\ldots..}$.

(ii) a function of the form $[f(x)]^{g(x)}$ where f & g are both derivable, it will be found convenient to take the logarithm of the function first & then differentiate

OR

$y=(f(x))^{g(x)}=e^{g(x).\ln(f(x))}$ and then differentiate.

Illustration :

If $y = \sin x \cdot \sin 2x \cdot \sin 3x .... \sin nx$, *find* y'.

Sol. $y = \sin x \sin 2x \sin 3x ..... \sin n x$

$\ln y = \ln (\sin x) + \ln (\sin 2x) + \ln (\sin 3x) + ... + \ln (\sin x)$

$$\frac{1}{y}\frac{dy}{dx} = \frac{\cos x}{\sin x} + \frac{2\cos 2x}{\sin 2x} + \frac{3\cos 3x}{\sin 3x} + ... + \frac{n\cos n x}{\sin nx}$$

$y' = y (\cot x + 2 \cot 2x + 3 \cot 3x + ... + n \cot n x)$.

Illustration :

If $f(x) = (x + 1)(x + 2)(x + 3)........(x + n)$ *then* $f'(0)$ *is*

(A) $n!$ (B) $\frac{n(n+1)}{2}$ (C) $(n!)(\ln n!)$ (D) $n!\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+....+\frac{1}{n}\right)$

Sol. $f(x) = (x + 1) (x + 2) (x + 3) .... (x + n)$

$\ln f(x) = \ln (x + 1) + \ln (x + 2) + \ln (x + 3) + ... + \ln (x + n)$

$$\frac{f'(x)}{f(x)} = \frac{1}{(x+1)} + \frac{1}{(x+2)} + \frac{1}{(x+3)} + ... + \frac{1}{(x+n)}$$

$f(0) = n!$

$$f'(0) = n!\left(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}\right).$$

Illustration :

$$\text{If } f(x) = \prod_{n=1}^{100}(x-n)^{n(101-n)} \text{ then find } \frac{f(101)}{f'(101)}.$$

Sol.

$$f(x) = \prod_{n=1}^{100}(x-n)^{n(101-n)}$$

$f(x) = (x - 1)^{1\,(101-1)} (x - 2)^{2(101-2)} (x - 3)^{3(101-3)} ..... (x - 100)^{(100)\,(1)}$.

$\ln f(x) = 100 \ln (x - 1) + 2(101 - 2) \ln (x - 2) + 3(101 - 3) \ln (x - 3) + ... + 100 \ln (x - 100)$

$$\frac{f'(x)}{f(x)} = \frac{100}{(x-1)} + \frac{2(101-2)}{(x-2)} + \frac{3(101-3)}{(x-3)} + ... + \frac{100}{(x-100)}$$

$$\left.\frac{f'(x)}{f(x)}\right|_{x=101} = 1+2+3+...+100 = (100)\frac{(100+1)}{2} = (50)\,(101) = 5050$$

$$\text{So} \quad \frac{f(101}{f'(101)} = \frac{1}{5050}.$$

Illustration :

Find the derivative of the following functions

(i) $y = (\sin x)\left(e^{\sqrt{\sin x}}\right)(\ln x)(x^x)$; (ii) $y = (x^{\ln x})(\sec x)^{3x}$; (iii) $y = \dfrac{(\ln x)^x}{2^{3x+1}}$

Sol.

(i) $y = (\sin x)\left(e^{\sqrt{\sin x}}\right)(\ln x)\left(x^x\right)$ *take logarithm on both sides*

$\ln y = \ln(\sin x) + \sqrt{\sin x} + \ln(\ln x) + x \ln x$

$$\frac{1}{y}\frac{dy}{dx} = \frac{\cos x}{\sin x} + \frac{(\cos x)}{2\sqrt{\sin x}} + \frac{1}{(\ln x)} \cdot \frac{1}{x} + \ln x + 1$$

$$\frac{dy}{dx} = y\left(\cot x + \frac{1}{2}\frac{\cos x}{\sqrt{\sin x}} + \frac{1}{x \ln x} + \ln x + 1\right)$$

(ii) $y = x^{\ln x}(\sec x)^{3x}$

$\ln y = (\ln x)^2 + 3x \ln(\sec x)$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2\ln x}{x} + 3\ln(\sec x) + \frac{3x}{(\sec x)}(\sec x \tan x)$$

$$\frac{dy}{dx} = y\left(\frac{2\ln x}{x} + 3\ln(\sec x) + 3x\tan x\right).$$

(iii) $y = \dfrac{(\ln x)^x}{2^{3x+1}}$

$\ln y = x\ln(\ln x) - (3x + 1)\ln 2$

$$\frac{1}{y}\frac{dy}{dx} = \ln(\ln x) + \frac{x}{(\ln x)} \cdot \frac{1}{x} - 3\ln 2$$

$$\frac{dy}{dx} = y\left(\ln(\ln x) + \frac{1}{(\ln x)} - 3\ln 2\right).$$

Illustration :

If $y = (\sin x)^{\ln x} \operatorname{cosec}(e^x(a + bx))$ *and* $a + b = \dfrac{\pi}{2e}$ *then the value of* $\dfrac{dy}{dx}$ *at* $x = 1$ *is*

(A) $(\sin 1)\ln\sin(1)$ (B) 0 (C) $\ln\sin(1)$ (D) $1 + \ln(\sin 1)$

Sol. $y = (\sin x)^{\ln x} \operatorname{cosec}(e^x(a + bx))$

$\ln y = (\ln x)\ln(\sin x) - \ln\left(\sin\left(e^x(a+bx)\right)\right)$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\ln(\sin x) + (\ln x)\cot x - \frac{\cos\left(e^x(a+bx)\right)\left(ae^x + bxe^x + be^x\right)}{\sin\left(e^x(a+bx)\right)}$$

$$\frac{1}{y}\frac{dy}{dx}\bigg|_{at\ x=1} = ln\,(sin\,1) + 0\,\frac{cos\left(\frac{\pi}{2}\right)(a+2b)e}{sin\left(\frac{\pi}{2}\right)} = ln\,(sin\,1) + 0 - 0$$

$$\because\ a + b = \frac{\pi}{2e} \qquad \therefore\ y = 1\ \ at\ x = 1$$

so $\quad \dfrac{dy}{dx}\bigg|_{at\ x=1} = ln\,(sin\,1)\,y\,\big|_{at\ x=1} = ln\,(sin\,1)\cdot 1 = ln\,(sin\,1)$

Illustration :

If $y = 2^{log_2 x^{2x}} + \left(tan\dfrac{\pi x}{4}\right)^{\frac{4}{\pi x}}$ then $\dfrac{dy}{dx}\bigg]_{x=1}$ is :

(A) 4 (B) 5/2 (C) 3 (D) not defined

Sol. $y = 2^{log_2 x^{2x}} + \left(tan\dfrac{\pi x}{4}\right)^{\frac{4}{\pi x}}$

$$y = x^{2x} + \left(tan\frac{\pi x}{4}\right)^{\frac{4}{\pi x}}$$

Let $\quad u = x^{2x}$ so $ln\ u = 2x\ ln\ x$

$$\frac{1}{u}\frac{du}{dx} = 2\,(ln\ x + 1) \Rightarrow \frac{du}{dx}\bigg|_{x=1} = (1)(2) = 2$$

Let $\quad v = \left(tan\dfrac{\pi x}{4}\right)^{\frac{4}{\pi x}}$

$$ln\ v = \frac{4}{\pi x}\,ln\left(tan\left(\frac{\pi x}{4}\right)\right)$$

$$\frac{1}{v}\frac{dv}{dx} = -\frac{4}{\pi x^2}\,ln\left(tan\left(\frac{\pi x}{4}\right)\right) + \frac{1}{x}\,\frac{sec^2\left(\frac{\pi}{4}x\right)}{tan\left(\frac{\pi}{4}x\right)}$$

$$\frac{dv}{dx}\bigg|_{at\ x=1} = 0 + 2$$

So $\quad \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx} = 2 + 2 = 4.$

Illustration :

Find $\frac{dy}{dx}$ for (i) $y = (\sin x)^{\log x}$ (ii) $y = x^{\tan x} + (\sin x)^{\cos x}$

Sol.

(i) Given $y = (\sin x)^{\log x}$.

Then, $y = e^{\log x \log \sin x}$

Diff. both sides w.r.t. x, we get

$$\frac{dy}{dx} = e^{\log x \log \sin x} \frac{d}{dx} \{\log x \log \sin x\}$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\log x} \left\{\log \sin x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(\log \sin x)\right\}$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\log x}\left\{\frac{\log \sin x}{x} + \log x \frac{1}{\sin x} \cos x\right\}$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\log x}\left\{\frac{\log \sin x}{x} + \cot x \log x\right\}.$$

(ii) $y = x^{\tan x} + (\sin x)^{\cos x}$

Let $u = x^{\tan x}$, $v = (\sin x)^{\cos x}$

$\therefore$ $\ln u = \tan x \ln x$

$$\therefore \quad \frac{d}{dx}(\ln u) = \frac{d}{dx}(\tan x \cdot \ln x)$$

$$\frac{1}{u} \cdot \frac{du}{dx} = \tan x\left(\frac{1}{x}\right) + \sec^2 x \cdot \ln x$$

$$\frac{du}{dx} = x^{\tan x}\left(\frac{\tan x}{x} + \sec^2 x \cdot \ln x\right)$$

$v = (\sin x)^{\cos x}$

$\ln v = \cos x \cdot \ln \sin x$

$$\Rightarrow \quad \frac{d}{dx} \ln v = \frac{d}{dx}(\cos x \ln \sin x)$$

$$\frac{1}{v}\frac{dv}{dx} = \left(\cos x \frac{\cos x}{\sin x}\right) + \ln \sin x \cdot (-\sin x)$$

$$\frac{dv}{dx} = (\sin x)^{\cos x}\left(\frac{\cos^2 x}{\sin x} - \sin x \ln(\sin x)\right) \qquad \text{.......(ii)}$$

$y = u + v$

$$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

From (i) and (ii)

$$\Rightarrow \quad \frac{dy}{dx} = x^{\tan x}\left(\frac{\tan x}{x} + \sec^2 x \cdot \ln x\right) + (\sin x)^{\cos x}\left(\frac{\cos^2 x}{\sin x} - \sin x \cdot \ln(\sin x)\right).$$

Illustration :

Find the derivative of $\dfrac{\sqrt{x}\,(x+4)^{3/2}}{(4x-3)^{4/3}}$ *w.r.t. x,*

Sol. *Let* $y = \dfrac{\sqrt{x}\,(x+4)^{3/2}}{(4x-3)^{4/3}}$

Taking log of both sides, we get

$$\log y = \frac{1}{2}\log x + \frac{3}{2}\log(x+4) - \frac{4}{3}\log(4x-3)$$

Diff. both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2x} + \frac{3}{2x+4}\times 1 - \frac{4}{3}\times\frac{1}{4x-3}\times 4$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left\{\frac{1}{2x} + \frac{3}{2(x+4)} - \frac{16}{3(4x-3)}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sqrt{x}\,(x+4)^{3/2}}{(4x-3)^{4/3}}\left\{\frac{1}{2x} + \frac{3}{2(x+4)} - \frac{16}{3(4x-3)}\right\}$$

Illustration :

Find $\dfrac{dy}{dx}$ *for* $y = x^x$.

Sol. $y = x^x$

$$\ln y = x\ln x \qquad \Rightarrow \quad \frac{d}{dx}(\ln y) = \frac{d}{dx}(x\ln x)$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = x\cdot\left(\frac{1}{x}\right) + \ln x = 1 + \ln x \qquad \Rightarrow \quad \frac{dy}{dx} = x^x(1+\ln x).$$

Illustration :

If $y^x = x^y$, *then find* $\dfrac{dy}{dx}$.

Sol. $y^x = x^y$

$x\ln y = y\ln x$

Differentiating w.r.t. x.

$$\frac{d}{dx}(x\ln y) = \frac{d}{dx}(y\ln x)$$

$$x\cdot\left(\frac{1}{y}\cdot\frac{dy}{dx}\right) + \ln y = y\cdot\frac{1}{x} + \ln x\cdot\frac{dy}{dx} \qquad \Rightarrow \qquad \frac{dy}{dx}\left(\frac{x}{y} - \ln x\right) = \frac{y}{x} - \ln y$$

$$\frac{dy}{dx} = \frac{\frac{x}{y} - \ln y}{\frac{x}{y} - \ln x} = \frac{y(y - x\ln y)}{x(x - y\ln x)}$$

T-7 IMPLICIT FUNCTIONS :

If the variable x and y are connected by a relation of the form $f(x, y) = 0$ and it is not possible to express y as a function of x in the form $y = \phi(x)$, then such functions are said to be implicit functions.
For example,
(i) $x^2 + xy + y^3 = 1$ (ii) $x + y + \sin(xy) = 1$ (iii) $x^y + y^x = 1$

(iv) $16^{x^2+y} + 16^{x+y^2} = 1$

DERIVATIVE OF IMPLICIT FUNCTION :

(i) In order to find dy/dx, in the case of implicit functions, we differentiate each term w.r.t. x regarding y as a functions of x & then collect terms in dy/dx together on one side to finally find dy/dx.

(ii) Corresponding to every curve represented by an implicit equation, there exist one or more explicit functions representing that equation. It can be shown that dy/dx at any point on the curve remains the same whether the process of differentiation is done explictly or implicitly.

$$\frac{d}{dx}f(y) = \frac{d}{dy}f(y)\cdot\frac{dy}{dx} = f'(y)\cdot\frac{dy}{dx}$$

For example :

(i) $\frac{d}{dx}(\sin y) = \frac{d}{dy}(\sin y)\cdot\frac{dy}{dx} = \cos y\left(\frac{dy}{dx}\right)$ (ii) $\frac{d}{dx}(y^3) = \frac{d}{dy}(y^3)\cdot\frac{dy}{dx} = 3y^2\frac{dy}{dx}$

A DIRECT FORMULA FOR IMPLICIT FUNCTIONS :

Let $f(x, y) = 0$. Take all the terms of left side and put left side equal to $f(x, y)$.

Then $\frac{dy}{dx} = -\frac{\text{diff. of } f \text{ w.r.t. } x \text{ keeping } y \text{ as constant}}{\text{diff. of } f \text{ w.r.t. } y \text{ keeping } x \text{ as constant}}$

Illustration :

If $x^2 + 2xy + y^3 = 4$, find $\frac{dy}{dx}$.

Sol. *We have $x^2 + 2xy + y^3 = 4$*
Diff. both sides w.r.t. x,

$$\Rightarrow \frac{d}{dx}(x^2) + 2\frac{d}{dx}(xy) + \frac{d}{dx}(y^3) = \frac{d}{dx}(4)$$

$$\Rightarrow 2x + 2\left(x\frac{dy}{dx} + y.1\right) + 3y^2\frac{dy}{dx} = 0 \qquad \Rightarrow \frac{dy}{dx} = -\frac{2(x+y)}{(2x+3y^2)}$$

Alternative Method :

$$\frac{dy}{dx} = -\frac{\text{diff. of } f \text{ w.r.t. } x \text{ keeping } y \text{ as constant}}{\text{diff. of } f \text{ w.r.t. } y \text{ keeping } x \text{ as constant}} = -\frac{2x+2y}{2x+3y^2}$$

Illustration :

$$\text{If } y = x + \cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{x+\dots}}}, \text{ prove that } \frac{dy}{dx} = \frac{y}{2y-x}.$$

Sol. *We have,* $y = x + \cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{x+\dots}}}$,

$$\Rightarrow \quad y = x + \frac{1}{y} \qquad \Rightarrow \quad y^2 = xy + 1$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = y + x\frac{dy}{dx} + 0 \qquad \text{[Diff. both sides with respect to } x\text{]}$$

$$\Rightarrow \quad \frac{dy}{dx}(2y-x) = y \qquad \Rightarrow \quad \frac{dy}{dx} = \frac{y}{2y-x}.$$

Illustration :

If $\sqrt{x}+\sqrt{y} = 4$, *then find* $\frac{dx}{dy}$ *at* $y = 1$.

Sol. *Diff. both sides of the given equation w.r.t. y, we get*

$$\frac{1}{2\sqrt{x}}\frac{dx}{dy}+\frac{1}{2\sqrt{y}}=0 \quad \Rightarrow \quad \frac{dx}{dy}=-\frac{\sqrt{x}}{\sqrt{y}}=\frac{\sqrt{y}-4}{\sqrt{y}} \quad \Rightarrow \left[\frac{dx}{dy}\right]_{y=1} = \frac{1-4}{1} = -3.$$

Illustration :

If $x^y = e^{x-y}$ *then prove that* $\frac{dy}{dx}=\frac{\ln x}{(1+\ln x)^2}$.

Sol. $x^y = e^{x-y}$

$y \ln x = (x-y)$ ....(i)

now differentiating with respect to x.

$$\frac{y}{x}+\ln x\cdot\frac{dy}{dx}=1-\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1-\frac{y}{x}}{1+\ln x}=\frac{x-y}{x(1+\sin x)}=\frac{y\ln x}{x\left(1+\ln x\right)} \qquad \left(\text{From (i)}, \; y=\frac{x}{1+\ln x}\right)$$

$$\therefore \quad \frac{dy}{dx}=\frac{\ln x}{(1+\ln x)^2}.$$

Illustration :

If $\sin y = x\sin(a+y)$ *then prove that* $\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$.

Sol. $\sin y = x\sin(a+y)$ *differentiating both sides w.r.t.* x

$$\cos y\cdot\frac{dy}{dx} = \sin(a+y) + x\cos(a+y)\cdot\left(0+\frac{dy}{dx}\right).$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\cos y - x\cos(a+y)\right) = \sin(a+y)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sin(a+y)}{\cos y - x\cos(a+y)} = \frac{\sin(a+y)}{\cos y - \dfrac{\sin y}{\sin(a+y)}\cos(a+y)}$$

$$= \frac{\sin^2(a+y)}{\sin(a+y)\cos y - \sin y\cos(a+y)} = \frac{\sin^2(a+y)}{\sin a}$$

Illustration :

If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, *then prove that* $\dfrac{dy}{dx} = -\dfrac{1}{(1+x)^2}$

Sol. $x\sqrt{1+y} + y\sqrt{1+x} = 0$

$\Rightarrow \quad x^2(1+y) = y^2(1+x) \qquad \Rightarrow \quad (x^2-y^2) + x^2y - y^2x = 0$

$\Rightarrow \quad (x-y)(x+y+xy) = 0 \qquad \Rightarrow \quad x-y=0$ or $x+y+xy=0$

$\therefore \quad y = x$ or $\dfrac{-x}{x+1}$. $\qquad \Rightarrow \quad \dfrac{dy}{dx} = 1$ or $\dfrac{-1}{(x+1)^2}$

Illustration :

If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots\ldots\infty}}}$ *find* $\dfrac{dy}{dx}$ $(\sin x > 0)$.

Sol. $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots\ldots\infty}}} \quad \Rightarrow \quad y = \sqrt{\sin x + y}$

$$\Rightarrow \quad y^2 - y = \sin x \Rightarrow 2y\frac{dy}{dx} - \frac{dy}{dx} = \cos x \Rightarrow \quad \frac{dy}{dx} = \left(\frac{\cos x}{2y-1}\right).$$

Illustration :

If $ax^2 + 2hxy + by^2 = 0$ *then prove that* $\dfrac{dy}{dx} = -\dfrac{ax+hy}{hx+by} = \dfrac{y}{x}$

Sol. $ax^2 + 2hxy + by^2 = 0 \qquad \ldots\ldots\ldots(1)$

differentiating w.r.t. x.

$$2ax + 2h\left(x\frac{dy}{dx} + y\right) + 2by\frac{dy}{dx} = 0 \qquad \Rightarrow \quad (hx+by)\frac{dy}{dx} = -(ax+hy)$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{(ax+hy)}{(hx+by)} \quad ....(2)$$

From (1) $ax^2 + 2hxy + by^2 = 0$

$$\Rightarrow \quad x(ax + hy) + y\,(hx + by) = 0$$

$$\Rightarrow \quad \frac{ax+hy}{hx+by} = \frac{-y}{x}$$

From (ii) $\dfrac{dy}{dx} = \dfrac{-(ax+hy)}{(hx+by)} = \dfrac{y}{x}$.

Illustration :

A curve is discribed by the relation $ln(x + y) = xe^y$. Find the tangent to the curve at (0, 1).

Sol. $ln\,(x + y) = xe^y$

differentiating w.r.t. to x

$$\frac{1}{(x+y)}\left(1+\frac{dy}{dx}\right) = xe^y\,\frac{dy}{dx} + e^y \text{ at point } (0, 1),$$

$$\frac{1}{(0+1)}\left(1+\frac{dy}{dx}\right) = 0 + e^1$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{x=0,\,y=1} = e - 1$$

Equation of tangent $y - 1 = (e - 1)(x - 0) \Rightarrow (1 - e)\,x + y = 1$.

Illustration :

(i) $y = x^{x^{x^{\cdot^{\cdot^{\infty}}}}}$ (ii) $y = (ln\,x)^{(ln\,x)^{(ln\,x)^{\cdot^{\cdot^{\infty}}}}}$

Sol.

(i) $y = x^{x^{x\ldots\infty}}$

$$\Rightarrow \quad y = x^y \quad \Rightarrow \quad ln\,y = y\,ln\,x \quad \Rightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx} = \frac{y}{x} + ln\,x\,\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \left(\frac{1}{y} - ln\,x\right) = \frac{y}{x} \qquad \Rightarrow \quad \frac{dy}{dx} = \frac{\frac{y}{x}}{\frac{(1-y\,ln\,x)}{y}} = \frac{y^2}{x(1-y\,ln\,x)}$$

(ii) $y = (ln\,x)^y$

$$\Rightarrow \quad ln\,y = y\,ln(ln\,x) \qquad \Rightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx} = \frac{y}{x\,ln\,x} + ln\,ln\,x\,\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{1}{y} - ln\,ln\,x\right) = \frac{y}{x\,ln\,x} \qquad \Rightarrow \quad \frac{dy}{dx}\,\frac{\frac{y}{x\,ln\,x}}{\frac{(1-y\,ln\,ln\,x)}{y}} = \frac{y^2}{x\,ln\,x\,(1-y\,ln\,ln\,x)}.$$

Illustration :

If $y^5 + xy^2 + x^3 = 4x + 3$, *then find* $\frac{dy}{dx}$ *at (2, 1)*

Sol. $y^5 + xy^2 + x^3 = 4x + 3$

differentiating w.r.t. x

$$5y^4\frac{dy}{dx} + \left(x\,2y\frac{dy}{dx} + y^2\right) + 3x^2 = 4 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{4 - 3x^2 - y^2}{(5y^4 + 2xy)}$$

$$\left.\frac{dy}{dx}\right|_{x=2,\,y=1} = \frac{4 - 3(2)^2 - (1)^2}{5(1)^4 + 2(2)(1)} = \frac{4 - 12 - 1}{5 + 4} = \frac{-9}{9} = -1.$$

Illustration :

If $y = \sqrt{x \log_e x}$, *then find* $\frac{dy}{dx}$ *at* $x = e$.

Sol. $$\frac{dy}{dx} = \frac{1}{2\sqrt{x \log_e x}}\,\frac{d}{dx}[x \log_e x] = \frac{1}{2\sqrt{x \log_e x}}\left[x \cdot \frac{1}{x} + 1 \cdot \log_e x\right]$$

$$\Rightarrow \quad \left[\frac{dy}{dx}\right]_{x=e} = \frac{1}{2\sqrt{e \cdot 1}}(1 + 1) = \frac{1}{\sqrt{e}} \qquad (\because \log_e e = 1)$$

Practice Problem

Q.1 Find the derivative of the following functions with respect to x :

(i) $\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}$ (ii) $x^{\sin x}, x > 0$ (iii) $y^x + x^y + x^x = a^b$

(iv) $\cos x \cdot \cos 2x \cdot \cos 3x$ (v) $\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$ (vi) $(\log x)^{\cos x}$

(vii) $x^x - 2^{\sin x}$ (viii) $(x+3)^2 \cdot (x+4)^3 \cdot (x+5)^4$ (ix) $\left(x + \frac{1}{x}\right)^x + x^{\left(1 + \frac{1}{x}\right)}$

(x) $(\log x)^x + x^{\log x}$ (xi) $x^{x \cos x} + \frac{x^2+1}{x^2-1}$ (xii) $x^{\sin x} + (\sin x)^{\cos x}$

Q.2 Find $\frac{dy}{dx}$ of the following functions :

(i) $(x \cos x)^x + (x \sin x)^{\frac{1}{x}}$ (ii) $x^y + y^x = 1$

(iii) $y^x = x^y$ (iv) $(\cos x)^y = (\cos y)^x$

(v) $xy = e^{(x-y)}$ (vi) $y = (x^2 - 5x + 8)(x^3 + 7x + 9)$

(vii) $y = (1+x)(1+x^2)(1+x^4)(1+x^8)$

Q.3 Find $\frac{dy}{dx}$ of the following functions :

(i) $2x + 3y = \sin x$ (ii) $2x + 3y = \sin y$

(iii) $ax + by^2 = \cos y$ (iv) $xy + y^2 = \tan x + y$

(v) $x^2 + xy + y^2 = 100$ (vi) $x^3 + x^2y + xy^2 + y^3 = 81$

(vii) $\sin^2 y + \cos xy = k$ (viii) $x^2 + xy + \cos^2 y = 1$

(ix) $y + \sin y = \cos x$

Q.4 Find the derivative of the following functions with respect to x :

(i) $(\sin x)^{\sin x}$, $0 < x < \pi$. (ii) $(\sin x - \cos x)^{(\sin x - \cos x)}$, $\frac{\pi}{4} < x < \frac{3\pi}{4}$

(iii) $x^x + x^a + a^x + a^a$, for some fixed $a > 0$ and $x > 0$

(iv) $x^{x^2-3} + (x-3)^{x^2}$, for $x > 3$ (v) $\log\left\{e^x\left(\frac{x-2}{x+2}\right)^{\frac{3}{4}}\right\}$

Answer key

Q.1 (i) $\frac{1}{2}\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}\left[\frac{1}{x-3}+\frac{2x}{x^2+4}-\frac{6x+4}{3x^2+4x+5}\right]$

(ii) $x^{\sin x - 1}\sin x + x^{\sin x}\cdot\cos x \log x$

(iii) $\dfrac{-[y^x \log y + y\cdot x^{y-1} + x^x(1+\log x)]}{x\cdot y^{x-1} + x^y \log x}$

(iv) $-\cos x \cos 2x \cos 3x\,[\tan x + 2\tan 2x + 3\tan 3x]$

(v) $\frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right]$

(vi) $(\log x)^{\cos x}\left[\frac{\cos x}{x\log x} - \sin x \log\log x\right]$

(vii) $x^x(1+\log x) - 2^{\sin x}\cos x \log 2$

(viii) $(x+3)(x+4)^2(x+5)^3(9x^2+70x+133)$

(ix) $\left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right] + x^{1+\frac{1}{x}}\left(\frac{x+1-\log x}{x^2}\right)$

(x) $(\log x)^{x-1}[1 + \log x\cdot\log\log x] + 2x^{\log x - 1}\log x$

(xi) $x^{x\cos x}[\cos x(1+\log x) - x\sin x\log x] - \frac{4x}{(x^2-1)^2}$

(xii) $x^{\sin x}\left[\frac{\sin x}{x} + \cos x \log x\right] + (\sin x)^{\cos x}[\cos x \cot x - \sin x \log \sin x]$

Q.2 (i) $(x\cos x)^x[1 - x\tan x + \log(x\cos x)] + (x\sin x)^{1/x}\left[\dfrac{x\cot x + 1 - \log(x\sin x)}{x^2}\right]$

(ii) $-\left(\dfrac{yx^{y-1} + y^x\log y}{x^y\log x + xy^{x-1}}\right)$ (iii) $\dfrac{y}{x}\left(\dfrac{y - x\log y}{x - y\log x}\right)$ (iv) $\dfrac{y\tan x + \log\cos y}{x\tan y + \log\cos x}$

(v) $\dfrac{y(x-1)}{x(y+1)}$ (vi) $5x^4 - 20x^3 + 45x^2 - 52x + 11$

(vii) $(1+x)(1+x^2)(1+x^4)(1+x^8)\left[\dfrac{1}{1+x} + \dfrac{2x}{1+x^2} + \dfrac{4x^3}{1+x^4} + \dfrac{8x^7}{1+x^8}\right]$

Q.3 (i) $\dfrac{\cos x - 2}{3}$ (ii) $\dfrac{2}{\cos y - 3}$ (iii) $\dfrac{-9}{2by + \sin y}$ (iv) $\dfrac{\sec^2 x - y}{x + 2y - 1}$

(v) $-\dfrac{(2x+y)}{(x+2y)}$ (vi) $\dfrac{-(3x^2 + 2xy + y^2)}{(x^2 + 2xy + 3y^2)}$ (vii) $\dfrac{y\sin(xy)}{\sin 2y - x\sin xy}$ (viii) $\dfrac{2x + y}{\sin 2y + x}$

(ix) $\dfrac{-\sin x}{1 + \cos y}$

Q.4 (i) $(1 + \log\sin x)(\sin x)^{\sin x}\cos x$

(ii) $(\sin x - \cos x)^{\sin - \cos x}(\cos x + \sin x)[1 + \log(\sin x - \cos x)], \sin x > \cos x$

(iii) $x^x(1 + \log x) + ax^{a-1} + a^x\log a$

(iv) $x^{x^2-3}\left[\dfrac{x^2-3}{x} + 2x\log x\right] + (x^2-3)^{x^2}\left[\dfrac{x^2}{x-3} + 2x\log(x-3)\right]$ (v) $\dfrac{x^2-1}{x^2-4}$

T-8 DERIVATIVE OF INVERSE FUNCTION :

Theorem :

Let the functions $f(x)$ and $g(x)$ be inverse of each other then

$$f(g(x)) = g(f(x)) = x$$

$$\therefore \quad f'(g(x))\,g'(x) = g'(f(x))\,f'(x) = 1$$

If $\dfrac{dy}{dx}$ exists and $\dfrac{dy}{dx} \neq 0$, then $\dfrac{dx}{dy} = \dfrac{1}{\frac{dy}{dx}}$ or $\dfrac{dy}{dx}.\dfrac{dx}{dy} = 1$ or $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$ $\left[\dfrac{dx}{dy} \neq 0\right]$

Illustration :

(a) *If* $y = f(x) = x^3 + x^5$ *and g is the inverse of f find* $g'(2)$

(b) *Let* $f(x) = \exp(x^3 + x^2 + x)$ *for any real number x, and let g be the inverse function for f. The value of* $g'(e^3)$ *is*

(A) $\frac{1}{6e^3}$ (B) $\frac{1}{6}$ (C) $\frac{1}{34e^{39}}$ (D) 6

(c) *If g is the inverse of f and* $f'(x) = \frac{1}{1+x^n}$, *prove that* $g'(x) = 1 + (g(x))^n$

Sol.

(a) $y = f(x) = x^3 + 5$

Let g(x) be the inverse of f(x) i.e., $g(x) = f^{-1}(x)$

$f(g(x)) = g(f(x)) = x$

differentiating w.r.t. x

$f'(g(x)) \cdot g'(x) = 1$

$$\Rightarrow \quad g'(x) = \frac{1}{f'(g(2))} \qquad \Rightarrow \quad g(2) = f^{-1}(2) = y \text{ (say)}$$

$$\Rightarrow \quad f(y) = 2 \quad \Rightarrow \quad y^3 + y^5 = 2 \Rightarrow \quad y = 1$$

$$\Rightarrow \quad g'(2) = \frac{1}{f'(1)} = \frac{1}{(3x^2 + 5x^4)_{x=1}} = \frac{1}{8}.$$

(b) $f(g(x)) = x$

$$\Rightarrow \quad f'(g(x))g'(x) = 1 \quad \Rightarrow \quad g'(x) = \frac{1}{f'(g(x))} \quad \Rightarrow \quad g'(e^3) = \frac{1}{f'(g(e^3))}$$

$$\Rightarrow \quad e^{x^3+x^2+x} = e^3 \quad \Rightarrow \quad x = 1.$$

$$\Rightarrow \quad g'(e^3) = \frac{1}{f'(1)} = \frac{1}{\left[e^{x^3+x^2+x}(3x^2+2x+1)\right]_{x=1}} = \frac{1}{e^3(3+2+1)} = \frac{1}{6e^3}.$$

(c) $f(g(x)) = x$

$$\Rightarrow \quad f'(g(x))g'(x) = 1 \quad \Rightarrow \quad g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\frac{1}{1+(g(x))^n}}$$

$$\Rightarrow \quad g'(x) = 1 + (g(x))^n.$$

DIFFERENTIATION OF INVERSE TRIGONOMETRIC FUNCTIONS :

(i) $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{(1-x^2)}}$ (ii) $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{(1-x^2)}}$

(iii) $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$ (iv) $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$

(v) $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$ (vi) $\frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}$

Proof :

(i) Proof of derivative of $f(x) = \sin^{-1}x$:
Let $y = \sin^{-1}x$. Then, $x = \sin y$.
Differentiating both sides w.r.t. x, we get

$$1 = \cos y \frac{dy}{dx}$$

which implies that $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos(\sin^{-1}x)}$

this is defined only for $\cos y \neq 0$, i.e., $\sin^{-1}x \neq -\frac{\pi}{2}, \frac{\pi}{2}$, i.e., $x \neq -1, 1$, i.e., $x \in (-1, 1)$

Recall that for $x \in (-1, 1)$, $\sin(\sin^{-1}x) = x$ and hence

$$\cos^2 y = 1 - (\sin y)^2 = 1 - (\sin(\sin^{-1}x))^2 = 1 - x^2$$

Also, since $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, cos y is positive and hence $\cos y = \sqrt{1-x^2}$

Thus, for $x \in (-1, 1)$,

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

(ii) Proof of derivative of $f(x) = \cos^{-1}x$

$$\frac{d}{dx}(\cos^{-1}x) = \frac{d}{dx}\left(\frac{\pi}{2} - \sin^{-1}x\right) \qquad \left[\because \cos^{-1}x + \sin x = \frac{\pi}{2}\right]$$

$$= 0 - \frac{d}{dx}(\sin^{-1}x) \quad = -\frac{1}{\sqrt{(1-x^2)}}$$

(iii) Proof of derivative of $f(x) = \tan^{-1}x$;
Let $y = \tan^{-1}x$. Then, $x = \tan y$,
Differentiating both sides w.r.t. x, we get

$$1 - \sec^2 y \frac{dy}{dx}$$

which implies that

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan y} = \frac{1}{1+\left(\tan\left(\tan^{-1}x\right)\right)^2} = \frac{1}{1+x^2}$$

(iv) Proof of derivative of $f(x) = \cot^{-1} x$.

$$\frac{d}{dx}(\cot^{-1} x) = \frac{d}{dx}\left(\frac{\pi}{2} - \tan^{-1} x\right) = 0 - \frac{d}{dx}(\tan^{-1} x)$$

$$= -\frac{1}{(1+x^2)}$$

Note :- $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

(v) Proof of derivative of $f(x) = \sec^{-1} x$;

Let $y = \sec^{-1} x$; $x = \sec y$

Differentiating both sides w.r.t. x, we get

$$1 = \sec y \tan y \frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{|x|\sqrt{(x^2-1)}}$$

(vi) Proof of derivative of $f(x) = \operatorname{cosec}^{-1} x$.

Let $y = \operatorname{cosec}^{-1} x$; $x = \operatorname{cosec} y$.

Differentiating both sides w.r.t. x, we get

$$1 = -\operatorname{cosec} y \cot y \frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{1}{\operatorname{cosec} y \cot y} = -\frac{1}{x\sqrt{(x^2-1)}}$$

Finding of the derivativesz of other inverse trigonometric function is left exercise. The following table gives the derivatives of the remaining inverse trigonometric functions.

f(x)	$\sin^{-1} x$	$\cos^{-1} x$	$\tan^{-1} x$	$\cot^{-1} x$	$\sec^{-1} x$	$\operatorname{cosec}^{-1} x$
f'(x)	$\frac{1}{\sqrt{(1-x^2)}}$	$\frac{-1}{\sqrt{1-x^2}}$	$\frac{1}{(1+x^2)}$	$\frac{1}{1+x^2}$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$
Domain of f'	$(-1, 1)$	$(-1, 1)$	R	R	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, -1) \cup (1, \infty)$

Note : Some Standard Substitutions

Expressions	**Substitution**
$\sqrt{a^2 - x^2}$	$x = a \sin\theta$ or $a \cos\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta$ or $a \cot\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta$ or $a \operatorname{cosec}\theta$
$\sqrt{\frac{a+x}{a-x}}$ or $\sqrt{\frac{a-x}{a+x}}$	$x = a \cos\theta$ or $a \cos 2\theta$

Illustration :

Prove that derivative of $sec^{-1}x$ *is* $\dfrac{1}{|x|\sqrt{x^2-1}}$ *if* $|x| > 1$, *using first principle method.*

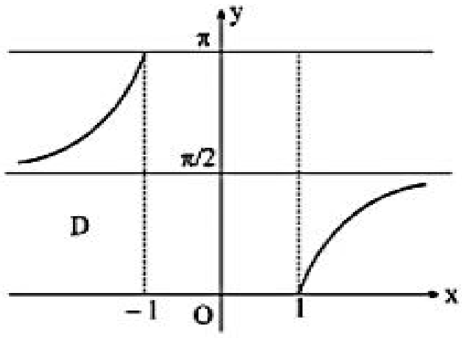
Sol. *Let,* $y = sec^{-1}x;\ \ |x| \ge 1\ \ ;\ y \in \left[0,\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

$sec\ y = x\ \ ;\ \ sec\ (y + \Delta y) = x + \Delta x$

$\therefore \quad \Delta x = sec(y + \Delta y) - sec\ y$

$$\frac{\Delta x}{\Delta y} = \frac{sec(y+\Delta y) - sec\ y}{\Delta y}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta y \to 0} \frac{1}{\dfrac{sec(y+\Delta y) - sec\ y}{\Delta y}} = \frac{1}{sec\ y\ tan\ y}$$



if $x > 1$ *then* $y \in \left(0, \frac{\pi}{2}\right)$ *(at* $x = 1$, $y = 0$ *and* $tan\ y = 0$ *therefore* $\frac{dy}{dx}$ *does not exist)*

$\Rightarrow$ *sign of tan y is + ve*

$$\therefore \quad \frac{dy}{dx} = \frac{1}{sec\ y\sqrt{sec^2 y - 1}} = \frac{1}{x\sqrt{x^2-1}},\ if\ x > 1 \qquad ....(1)$$

and if $x < -1$, *then* $y \in \left(\frac{\pi}{2}, \pi\right)$ *(at* $x = -1$, $y = \pi$ *and* $tan\ y = 0$ *and* $\frac{dy}{dx}$ *does not exist)*

hence sign of tan y is – ve

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{sec\ y\sqrt{sec^2 y - 1}} = -\frac{1}{x\sqrt{x^2-1}},\ if\ x < -1 \qquad ....(2)$$

from (1) and (2)

$$D(sec^{-1}x) = \begin{cases} \dfrac{1}{|x|\sqrt{x^2-1}}, & if\ |x| > 1 \\ does\ not\ exist, & if\ |x| = 1 \end{cases}$$

Illustration :

Find $\frac{dy}{dx}$ *for* $y = tan^{-1}\left\{\frac{1-cos\ x}{sin\ x}\right\}$, *where* $-\pi < x < \pi$.

Sol. $y = tan^{-1}\left\{\dfrac{1-cos\ x}{sin\ x}\right\} = tan^{-1}\left\{\dfrac{2\sin\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right\}$

$$= tan^{-1}\left(tan\frac{x}{2}\right) = \frac{x}{2} \qquad \left(\because -\pi < x < \pi \Rightarrow -\frac{\pi}{2} < \frac{x}{2} < \frac{\pi}{2}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}.$$

Illustration :

$$D\left(tan^{-1}x+tan^{-1}\left(\frac{1}{x}\right)\right)=\_\_\_\_$$

Sol. $tan^{-1}x + tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \frac{\pi}{2}, & x>0 \\ \frac{-\pi}{2}, & x<0 \end{cases}$

$$D\left(tan^{-1}+tan^{-1}\left(\frac{1}{x}\right)\right)=0\ \forall x\in R-\{0\}.$$

Illustration :

If $y=\frac{tan^{-1}x-cot^{-1}x}{tan^{-1}x+cot^{-1}x}$, find $\left.\frac{dy}{dx}\right]_{x=-1}$;

(A) 0 (B) 1 (C) $2/\pi$ (D) – 1

Sol. $y=\frac{tan^{-1}x-cot^{-1}x}{tan^{-1}x+cot^{-1}x}=\frac{2}{\pi}\left(tan^{-1}x-cot^{-1}x\right)$

$$\therefore \qquad \frac{dy}{dx}=\frac{2}{\pi}\left(\frac{1}{1+x}+\frac{1}{1+x^2}\right)=\frac{4}{\pi(1+x^2)}$$

$$\left.\frac{dy}{dx}\right|_{x=-1}\frac{4}{\pi(1+(-1)^2)}=\frac{2}{\pi}$$

Illustration :

If $y = tan^{-1}\left(\frac{x}{1\cdot 2+x^2}\right)+tan^{-1}\left(\frac{x}{2\cdot 3+x^2}\right)+tan^{-1}\left(\frac{x}{3\cdot 4+x^2}\right)+$ .... up to n times. Find $\frac{dy}{dx}$ expressing your answer in two terms and also find $\frac{dy}{dx}$ when $n\to\infty$.

Sol. $y=tan^{-1}\left(\frac{x}{1\cdot 2+x^2}\right)+tan^{-1}\left(\frac{x}{2\cdot 3+x^2}\right)+tan^{-1}\left(\frac{x}{3\cdot 4+x^2}\right)+......+tan^{-1}\left(\frac{x}{n(n+1)+x^2}\right)$

$$T_r=tan^{-1}\left(\frac{x}{r(r+1)+x^2}\right) \quad\Rightarrow\quad T_r=tan^{-1}\left(\frac{\frac{x}{r(r+1)}}{1+\frac{x^2}{r(r+1)}}\right)$$

$$T_r=tan^{-1}\left(\frac{\frac{x}{r}-\frac{x}{r+1}}{1+\frac{x}{r}\cdot\frac{x}{r+1}}\right)=tan^{-1}\left(\frac{x}{r}\right)-tan^{-1}\left(\frac{x}{r+1}\right).$$

$$y = S_n = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} tan^{-1}\left(\frac{x}{r}\right) - tan^{-1}\left(\frac{x}{r+1}\right)$$

$$= \left(tan^{-1}(x) - tan^{-1}\left(\frac{x}{2}\right) + \left(tan^{-1}\left(\frac{x}{2}\right) - tan^{-1}\left(\frac{x}{3}\right)\right)\right)$$

$$\left(tan^{-1}(x) - tan^{-1}\left(\frac{x}{2}\right) + \left(tan^{-1}\left(\frac{x}{2}\right) - tan^{-1}\left(\frac{x}{3}\right)\right)\right) + \left(tan^{-1}\left(\frac{x}{3}\right) - tan^{-1}\left(\frac{x}{4}\right)\right) +$$

$$\ldots\ldots + \left(tan^{-1}\left(\frac{x}{n}\right) - tan^{-1}\left(\frac{x}{n+1}\right)\right)$$

$$= tan^{-1}(x) - tan^{-1}\left(\frac{x}{n+1}\right).$$

$$\frac{dy}{dx} = \frac{1}{1+x^2} - \frac{1}{1+\left(\frac{x}{n+1}\right)^2} \cdot \left(\frac{1}{n+1}\right)$$

when $n \to \infty$

$$\frac{dy}{dx} = \frac{1}{1+x^2} - 0 = \frac{1}{1+x^2}.$$

Illustration :

(i) $y = tan^{-1}\left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right)$ *(ii)* $y = tan^{-1}\left(\frac{\sqrt{1+x^2} - 1}{x}\right)$

(iii) *If* $f(x) = sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$, *find* $f'(0)$.

Sol.

(i) $$y = tan^{-1}\left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right)$$

Let $x = cos\, 2\theta \Rightarrow \theta = \frac{1}{2} cos^{-1}(x)$

$$y = tan^{-1}\left(\frac{\sqrt{1+cos\,2\theta} - \sqrt{1-cos\,2\theta}}{\sqrt{1+cos\,2\theta} + \sqrt{1-cos\,2\theta}}\right) = tan^{-1}\left(\frac{\sqrt{2}\cos\theta - \sqrt{2}\sin\theta}{\sqrt{2}\cos\theta + \sqrt{2}\sin\theta}\right) = tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$y = tan^{-1}\left(tan\left(\frac{\pi}{4} - \theta\right)\right) = \frac{\pi}{4} - \theta$$

$$y = \frac{\pi}{4} - \frac{1}{2}cos^{-1}x$$

$$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{1-x^2}}.$$

(ii) $y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$

Put $x = \tan\theta \therefore \theta = \tan^{-1}x$

$$y = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$\Rightarrow y = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x \qquad \Rightarrow \frac{dy}{dx} = \frac{1}{2(1+x^2)}.$$

(iii) $f(x) = \sin^{-1}\left(\dfrac{2^{x+1}}{1+2^{2x}}\right)$

Let $2^x = \tan\theta$

$$f(x) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta)$$

$$f(x) = 2\theta = 2\tan^{-1}(2^x) \quad \Rightarrow \quad f'(x) = \frac{1}{1+2^{2x}}(2^x \ln 2)$$

$$f'(0) = \frac{2}{1+1}(2^0 \ln 2) = \ln 2.$$

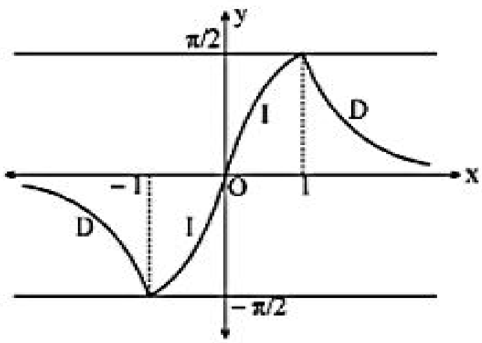
Derivative of $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$; $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$; $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$; $\sin^{-1}(3x-4x^3)$; $\cos^{-1}(4x^3-3x)$:

(i) $$y = f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \begin{bmatrix} 2\tan^{-1}x & |x| \le 1 \\ \pi - 2\tan^{-1}x & x > 1 \\ -(\pi + 2\tan^{-1}x) & x < -1 \end{bmatrix}$$

Highlights :

(a) Domain is $x \in R$ & range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Note: f is odd, aperiodic bound



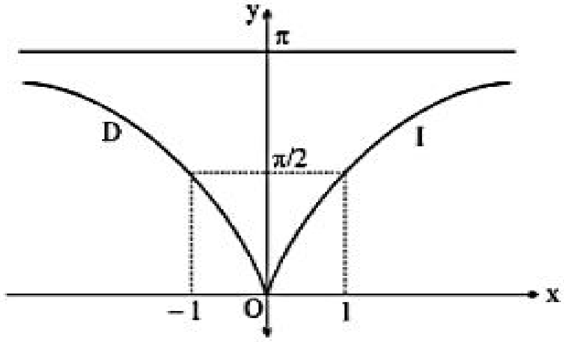
(b) f is continuous for all x but not diff. at $x = 1, -1$

(c) $$\frac{dy}{dx} = \begin{bmatrix} \frac{2}{1+x^2} & \text{for } |x| < 1 \\ \text{nonexistent} & \text{for } |x| = 1 \\ -\frac{2}{1+x^2} & \text{for } |x| > 1 \end{bmatrix}$$

(d) Increasing in $(-1, 1)$ & decreasing in $(-\infty, -1) \cup (1, \infty)$

(ii) Consider $y = f(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x & \text{if } x \ge 0 \\ -2\tan^{-1}x & \text{if } x < 0 \end{cases}$

Highlights :



(a) Domain is $x \in R$ & range is $(0, \pi)$

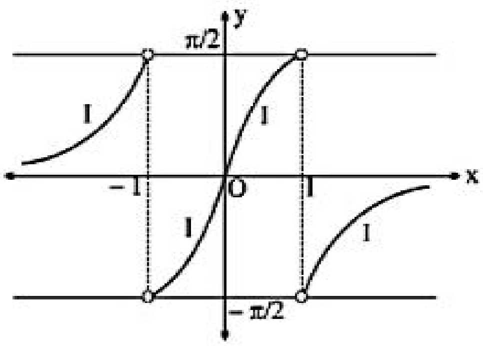
(b) Continuous for all x
but not differentiable at $x = 0$

(c) $$\frac{dy}{dx} = \begin{cases} \frac{2}{1+x^2} & \text{for } x > 0 \\ \text{nonexistent} & \text{for } x = 0 \\ -\frac{2}{1+x^2} & \text{for } x < 0 \end{cases}$$

(d) Increasing in $(0, \infty)$ & decreasing in $(-\infty, 0)$ Note: f is even, a periodic bound.

(iii) $y = f(x) = \tan^{-1}\frac{2x}{1-x^2} = \begin{cases} 2\tan^{-1}x & |x| < 1 \\ \pi + 2\tan^{-1}x & x < -1 \\ -(\pi - 2\tan^{-1}x) & x > 1 \end{cases}$

Highlights :



(a) Domain is $R - \{1, -1\}$ & range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(b) f is neither continuous
nor differentiable at $x = 1, -1$

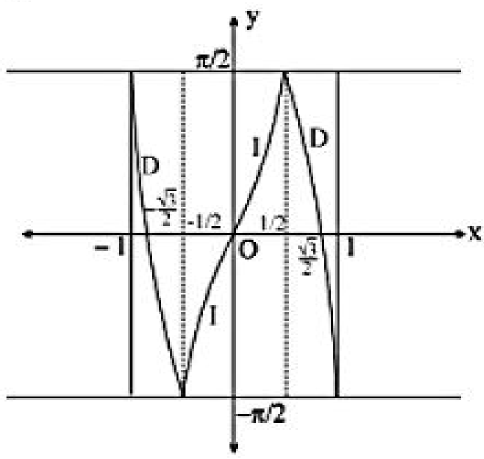
(c) $$\frac{dy}{dx} = \begin{cases} \frac{2}{1+x^2} & |x| \ne 1 \\ \text{nonexistent} & |x| = 1 \end{cases}$$

(d) Increasing $\forall$ x in its domain

(e) It is bound for all x

(iv) $y = f(x) = \sin^{-1}(3x - 4x^3) = \begin{cases} -(\pi + 3\sin^{-1}x) & \text{if } -1 \le x \le -\frac{1}{2} \\ 3\sin^{-1}x & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ \pi - 3\sin^{-1}x & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$

Highlights :



(a) Domain is $x \in [-1, 1]$ & range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(b) Not derivable at $|x| = \frac{1}{2}$

(c) $$\frac{dy}{dx} = \begin{cases} \frac{3}{\sqrt{1-x^2}} & \text{if } x \in \left(-\frac{1}{2}, \frac{1}{2}\right) \\ -\frac{3}{\sqrt{1-x^2}} & \text{if } x \in \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \end{cases}$$

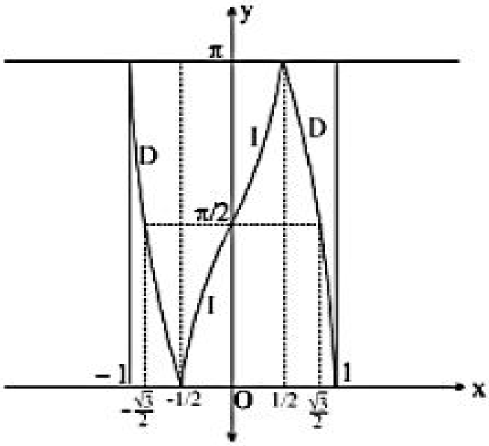
(d) Continuous everywhere in its domain

(v) $y = f(x) = \cos^{-1}(4x^3 - 3x) = \begin{cases} 3\cos^{-1}x - 2\pi & \text{if } -1 \le x \le -\frac{1}{2} \\ 2\pi - 3\cos^{-1}x & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ 3\cos^{-1}x & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$

Highlights :

(a) Domain is $x \in [-1, 1]$ & range is $[0, \pi]$

(b) Continuous everywhere in its domain but not derivable at $x = \frac{1}{2}, -\frac{1}{2}$



(c) Increasing in $\left(-\frac{1}{2}, \frac{1}{2}\right)$ &

Decreasing in $\left(\frac{1}{2}, 1\right] \cup \left[-1, -\frac{1}{2}\right)$

(d) $\frac{dy}{dx} = \begin{cases} \frac{3}{\sqrt{1-x^2}} & \text{if } x \in \left(-\frac{1}{2}, \frac{1}{2}\right) \\ -\frac{3}{\sqrt{1-x^2}} & \text{if } x \in \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \end{cases}$

Illustration :

If $y = \sin^{-1}\left[x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\right]$, *and* $0 < x < 1$, *then find* $\frac{dy}{dx}$

Sol. $y = \sin^{-1}\left[x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\right]$, *where* $0 < x < 1$

$= \sin^{-1}\left[x\sqrt{1-(\sqrt{x})^2} - \sqrt{x}\sqrt{1-x^2}\right] = \sin^{-1}x - \sin^{-1}\sqrt{x}$

Differentiating w.r.t. x, we get

$$\frac{d}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-(\sqrt{x})^2}}\frac{d}{dx}(\sqrt{x}) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x-x^2}}$$

Illustration :

Find $\frac{dy}{dx}$ *for* $y = \tan^{-1}\sqrt{\frac{a-x}{a+x}}$, *where* $-a < x < a$.

Sol. $y = \tan^{-1}\left\{\sqrt{\frac{a-x}{a+x}}\right\}$, *where* $-a < x < a$

Substituting $x = a\cos\theta$, *we have*

$$y = \tan^{-1}\left\{\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right\} = \tan^{-1}\left\{\sqrt{\tan^2\frac{\theta}{2}}\right\} = \tan^{-1}\left|\tan\frac{\theta}{2}\right|.$$

Also for $-a < x < a,\ -1 < \cos\theta < 1$

$$\Rightarrow \quad \theta \in (0, \pi) \Rightarrow \frac{\theta}{2} \in \left(0, \frac{\pi}{2}\right)$$

$$\therefore \quad y = \tan^{-1}\left|\tan\frac{\theta}{2}\right| = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\cos^{-1}\left(\frac{x}{a}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{2}\cdot\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\frac{d}{dx}\left(\frac{x}{a}\right) = -\frac{1}{2\sqrt{a^2-x^2}}$$

Practice Problem

Q.1 The function $f(x) = e^x + x$, being differentiable and one to one, has a differentiable inverse $f^{-1}(x)$. The value of $\frac{d}{dx}(f^{-1})$ at the point $f(\log 2)$ is

(A) $\frac{1}{\ln 2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) None of these

Q.2 Let $g(x)$ be the inverse of an invertible function $f(x)$ which is differentiable at $x = c$, then $g'(f(c))$ equals

(A) $f'(c)$ (B) $\frac{1}{f'(c)}$ (C) $f(C)$ (D) None of these

Q.3 If $f(x) = x + \tan x$ and f is inverse of g, then $g'(x)$ equals

(A) $\frac{1}{1+[g(x)-x]^2}$ (B) $\frac{1}{2-[g(x)-x]^2}$ (C) $\frac{1}{2+[g(x)-x]^2}$ (D) None of these

Q.4 Find $\frac{dy}{dx}$ of the following functions :

(i) $y = \sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right) + \sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)$

(ii) $y = \tan^{-1}\frac{4x}{1+5x^2} + \tan^{-1}\frac{2+3x}{3-2x}$.

(iii) $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$, where $x \neq 0$

(iv) $y = \tan^{-1}\frac{3a^2x - x^3}{a(a^2-3x^2)}$.

(v) $y = \sin^{-1}\left(\frac{5x+12\sqrt{(1-x)^2}}{13}\right)$.

(vi) $y = \tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right)$.

(vii) $y = \tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right)$, where $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and $\frac{a}{b}\tan x > -1$.

(viii) $y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$, where $-1 < x < 1, x \neq 0.$

(ix) $y = \sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right) + \cos^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right), 0 < x < \infty.$

(x) $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

Q.5 Find the derivative of the following functions with respect to x :

(i) $y = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

(ii) $y = \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), \dfrac{-1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}}$

(iii) $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), 0 < x < 1$

(iv) $y = \sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), 0 < x < 1$

(v) $y = \cos^{-1}\left(\dfrac{2x}{1+x^2}\right), -1 < x < 1$

(vi) $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right), -\dfrac{1}{\sqrt{2}} < x < \dfrac{1}{\sqrt{2}}$

(vii) $y = \sec^{-1}\left(\dfrac{1}{2x^2-1}\right), 0 < x < \dfrac{1}{\sqrt{2}}$

Q.6 Find the derivative of the following functions with respect to x :

(i) $\cos^{-1}(\sin x)$

(ii) $\tan^{-1}\left(\dfrac{\sin x}{1+\cos x}\right)$

(iii) $\sin^{-1}\left(\dfrac{2^{x+1}}{1+4^x}\right)$

Answer key

Q.1 B

Q.2 B

Q.3 C

Q.4 (i) 0

(ii) $\dfrac{1}{1+25x^2}$

(iii) $\dfrac{1}{2(1+x^2)}$

(iv) $\dfrac{3a}{(a^2+x^2)}$

(v) $\dfrac{1}{\sqrt{(1-x^2)}}$

(vi) $\dfrac{1}{2\sqrt{(1-x^2)}}$

(vii) -1

(viii) $-\dfrac{x}{\sqrt{(1-x^4)}}$

(ix) $\dfrac{2}{(1+x^2)}$

(x) $(\sin x)^x (x\cot x + \ln \sin x) + \dfrac{1}{2\sqrt{x(1-x)}}$

T-9 PARAMETRIC DIFFERENTIATION :

In some situation curves are represented by the equations e.g. $x = \sin t$ & $y = \cos t$
If $x = f(t)$ and $y = g(t)$ then

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{g'(t)}{f'(t)}$$

Illustration :

Find $\frac{dy}{dx}$ *if* $x = a(\theta - \sin\theta)$ *and* $y = a(1 - \cos\theta)$.

Sol. *We have,*
$x = a(\theta - \sin\theta)$ *and* $y = a(1 - \cos\theta)$

$$\Rightarrow \quad \frac{dy}{dx} = a(1 - \cos\theta) \text{ and } \frac{dy}{d\theta} = a\sin\theta$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\sin^2\left(\frac{\theta}{2}\right)} = \cot\frac{\theta}{2}.$$

Illustration :

If $x = a\sec^3\theta$ *and* $y = a\tan^3\theta$, *find* $\frac{dy}{dx}$ *at* $\theta = \frac{\pi}{3}$.

Sol. *We have* $\quad x = a\sec^3\theta$ *and* $y = a\tan^3\theta$

$$\frac{dx}{d\theta} = 3a\sec^2\theta \frac{d}{d\theta}(\tan\theta) = 3a\tan^2\theta\sec^2\theta$$

$$\Rightarrow \quad \frac{dt}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3q\tan^2\theta\sec^2\theta}{3a\sin^3\theta\tan\theta} = \frac{\tan\theta}{\sec\theta} = \sin\theta$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{\theta = \frac{\pi}{3}} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

Illustration :

Find $\frac{dy}{dx}$ *if*

(i) $x = a(\cos t + t\sin t)$ *and* $y = a(\sin t - t\cos t)$ *(ii)* $x = \frac{3at}{1+t^3}$; $y = \frac{3at^2}{1+t^3}$

(iii) $x = a\sec^2\theta$; $y = a\tan^3\theta$.

(iv) $x = a\sqrt{\cos 2t}\cos t$ *and* $y = a\sqrt{\cos 2t}\sin t$ *then, find* $\left.\frac{dy}{dx}\right|_{t=\pi/6}$.

(v) *If* $x = \sec\theta - \cos\theta$ & $y = \sec^n\theta - \cos^n\theta$, *then show that* $(x^2+4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2+4)$.

Sol.

(i) $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$

$$\frac{dx}{dt} = a(-\sin t + \sin t + t\cos t) = at\cos t$$

$$\frac{dy}{dt} = a(\cos t - \sin t + t\sin t) = at\sin t$$

$$\Rightarrow \frac{dy}{dx} = \tan t$$

(ii) $x = \dfrac{3at}{1+t^3}$; $y = \dfrac{3at^2}{1+t^3}$

$$\frac{dx}{dt} = \frac{3a\left((1+t^3)\cdot 1 - t(3t^2)\right)}{\left(1+t^3\right)^2} = \frac{3a(1-2t^3)}{\left(1+t^3\right)^2}$$

$$\frac{dy}{dt} = \frac{3a\left((1+t^3)2t - t^2(3t^2)\right)}{\left(1+t^3\right)^2} = \frac{3a(2t-t^4)}{\left(1+t^3\right)^2}$$

$$\frac{dy}{dx} = \frac{(2-t^3)t}{(1-2t^3)}$$

(iii) $x = a\sec^2\theta$; $y = a\tan^3\theta$

$$\frac{dx}{d\theta} = 2a\sec\theta(\sec\theta\tan\theta)$$

$$\frac{dy}{d\theta} = 3a\tan^2\theta\sec^2\theta$$

$$\frac{dy}{dx} = \frac{3\tan\theta}{2}.$$

(iv) $x = a\sqrt{\cos 2t}\,\cos t$ and $y = a\sqrt{\cos 2t}\,\sin t$

$$\frac{dx}{dt} = a\left(\sqrt{\cos 2t}\,(-\sin t) + \frac{\cos t(-2\sin 2t)}{2\sqrt{\cos 2t}}\right)$$

$$\frac{dx}{dt} = a\left(\frac{\sin t\cos 2t + \cos t\sin 2t}{\sqrt{\cos 2t}}\right)$$

$$\frac{dx}{dt} = -a\frac{\sin 3t}{\sqrt{\cos 2t}} \Rightarrow \left.\frac{dx}{dt}\right|_{\pi/6} = -\sqrt{2}\,a$$

$$\frac{dy}{dt} = a\left(\sqrt{\cos 2t}\,\cos t + \frac{\sin t}{2\sqrt{\cos 2t}}(-2\sin 2t)\right) = a\left(\frac{\cos 3t}{\sqrt{\cos 2t}}\right) \Rightarrow \left.\frac{dy}{dt}\right|_{\pi/6} = 0$$

$$\left.\frac{dy}{dx}\right|_{\pi/6} = 0.$$

(v) $x = \sec\theta - \cos\theta\,;\ y = \sec^n\theta - \cos^n\theta$

$$(x^2+4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2+4)$$

$$\frac{dx}{d\theta} = \sec\theta\tan\theta + \sin\theta = \sin\theta(\sec^2\theta + 1)$$

$$\frac{dy}{d\theta} = n\sec^{n-1}\theta\sec\theta\tan\theta + n\cos^{n-1}\theta\sin\theta \;= n\sec^n\theta\tan\theta + n\cos^n\theta\tan\theta$$

$$\frac{dy}{d\theta} = n\tan\theta(\sec^n\theta + \cos^n\theta)$$

$$\frac{dy}{dx} = \frac{n\tan\theta(\sec^n\theta+\cos^n\theta)}{\sin\theta(\sec^2\theta-1)} = \frac{n(\sec^n\theta+\cos^n\theta)}{\cos\theta(\sec^2\theta+1)} = \frac{n(\sec^n\theta+\cos^n\theta)}{(\sec\theta+\cos\theta)}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{n^2(\sec^{2n}\theta+\cos^{2n}\theta+2)}{(\sec^2\theta+\cos^2\theta+2)} = \frac{n^2(y^2+4)}{(x^2+4)}.$$

Illustration :

For the curve represented parametrically indicate the relation between the parameter t and the angle α between the tangent to the given curve and the x-axis.

(i) $\begin{cases} x = \cos t + t\sin t - \dfrac{t^2}{2}\cos t \\ y = \sin t - t\cos t - \dfrac{t^2}{2}\sin t \end{cases}$; *(ii)* $x = a\cos^3 t,\ y = a\sin^3 t$

Sol.

(i) $x = \cos t + t\sin t - \dfrac{t^2}{2}\cos t$

$$\frac{dx}{dt} = -\sin t + t\cos t + \sin t - \left(\frac{t^2}{2}\sin t + t\cos t\right) = \frac{t^2\sin t}{2}$$

$$y = \sin t - t\cos t - \frac{t^2}{2}\sin t$$

$$\frac{dy}{dt} = \cos t - (\cos t - t\sin t) - \frac{1}{2}(2t\sin t + t^2\cos t) = \frac{-t^2\cos t}{2}$$

$$\therefore\ \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = -\cot t = \tan\alpha \Rightarrow \tan\left(\frac{\pi}{2} - t\right) = \tan(-\alpha)$$

$$\Rightarrow \frac{\pi}{2} - t = -\alpha \Rightarrow t = \frac{\pi}{2} + \alpha.$$

(ii) $x = a\cos^3 t \Rightarrow \dfrac{dx}{dt} = -3a\cos^2 t\sin t$

$y = a\sin^3 t \Rightarrow \dfrac{dy}{dt} = 3a\sin^2 t\cos t$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\tan t = \tan\alpha$$

$\Rightarrow \tan(\pi - t) = \tan\alpha \Rightarrow \pi - t = \alpha \Rightarrow t = \pi - \alpha.$

T-10 DIFFERENTIATION OF ONE FUNCTION W.R.T. OTHER FUNCTION :

If $y = f(x)$ and $z = g(x)$ then derivative of $f(x)$ w.r.t. $g(x)$ is given by

$$\frac{dy}{dz} = \frac{dy}{dx}\cdot\frac{dx}{dz} = \frac{f'(x)}{g'(x)}$$

$$\therefore \quad \text{Differential coefficient of } f(x) \text{ w.r.t. } g(x) = \frac{\text{derivative of } f(x) \text{ w.r.t. } x}{\text{derivative of } g(x) \text{ w.r.t. } x} = \frac{f'(x)}{g'(x)}$$

Illustration :

Differentiate $\log\ \sin x$ *w.r.t.* $\sqrt{\cos x}$.

Sol. Let $u = \log\sin x$ and $v = \sqrt{\cos x}$

Then, $\dfrac{du}{dx} = \cot x$ and $\dfrac{dv}{dx} = -\dfrac{\sin x}{2\sqrt{\cos x}}$

$$\Rightarrow \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{\cot x}{-\dfrac{\sin x}{2\sqrt{\cos x}}} = -2\sqrt{\cos x}\cot x\,\text{cosec}\,x.$$

Illustration :

Differentiate $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ *w.r.t.* $\tan^{-1}x$,

Sol. Let $u = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ and $v = \tan^{-1}x$.

Putting $x = \tan\theta$,

we get $u = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$

$$= \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{1}{2}\theta = \frac{1}{2}\tan^{-1}x.$$

Thus, we have $u = \frac{1}{2}\tan^{-1} x$ *and* $v = \tan^{-1} x$.

$$\Rightarrow \quad \frac{du}{dx} = \frac{1}{2} \times \frac{1}{1+x^2} \text{ and } \frac{dv}{dx} = \frac{1}{1+x^2}$$

$$\Rightarrow \quad \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{1}{2(1+x^2)}(1+x^2) = \frac{1}{2}.$$

Illustration :

Find derivative of $(\ln x)^{\tan x}$ *w.r.t.* x^x.

Sol. $u = (\ln x)^{\tan x}$; $v = x^x$

$\ln (u) = \tan x \ln (\ln x)$

$$\Rightarrow \frac{1}{u}\left(\frac{du}{dx}\right) = (\sec^2 x) \ln (\ln x) + \tan x \left(\frac{1}{\ln x} \cdot \frac{1}{x}\right)$$

$$\frac{du}{dx} = \frac{u\left((x \ln x) \ln(\ln x) \sec^2 x + \tan x\right)}{(x \ln x)}$$

$\ln v = x \ln x$

$$\Rightarrow \frac{1}{v}\left(\frac{dv}{dx}\right) = (\ln x + 1)$$

$$\frac{du}{dx} = \frac{(\ln x)^{\tan x}}{x^x}\left(\frac{x \ln x \ln(\ln x) \sec^2 x + \tan x}{(x \ln x)(\ln x + 1)}\right).$$

Illustration :

Derivative of $\cos^{-1}(2x^2 - 1)$ *w.r.t.* $\sqrt{1-x^2}$ *when* $x = \frac{1}{2}$.

Sol. $u = \cos^{-1}(2x^2 - 1)$; $v = \sqrt{(1-x^2)}$

put $x = \cos\theta$ *as* $x = \frac{1}{2}$ *so* $\theta = \frac{\pi}{3}$

$u = \cos^{-1}(\cos 2\theta) = 2\theta$ *put* $v = \sin\theta$

$$\frac{du}{dv} = \frac{du/d\theta}{dv/d\theta} = \frac{2}{\cos\theta} = 4.$$

Illustration :

Define derivative of $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ *w.r.t.* $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ $\forall x \in R$.

Sol. *Let* $u = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ *and* $v = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Let $x = \tan\theta$

$u = \sin^{-1}(\sin 2\theta)$ *and* $v = \cos^{-1}(\cos 2\theta)$

$$u = \begin{cases} 2\theta, \dfrac{-\pi}{2} \le 2\theta \le \dfrac{\pi}{2} \Rightarrow \dfrac{-\pi}{4} \le \theta \le \dfrac{\pi}{4} \Rightarrow -1 \le x \le 1 \\ (-\pi - 2\theta), \dfrac{\pi}{2} \le 2\theta \le \dfrac{3\pi}{2} \Rightarrow \dfrac{\pi}{4} \le \theta \le \dfrac{3\pi}{4} \Rightarrow 1 \le x \le \propto \\ (\pi - 2\theta), \dfrac{3\pi}{2} \le 2\theta \le \dfrac{-\pi}{2} \Rightarrow \dfrac{-3\pi}{4} \le \theta \le \dfrac{-\pi}{4} \Rightarrow -\infty \le x \le -1 \end{cases}$$

$$v = \cos^{-1}(\cos 2\theta)$$

$$v = \begin{cases} 2\theta, 0 < 2\theta \le \pi \Rightarrow 0 < \theta \le \dfrac{\pi}{2} \Rightarrow 0 < x < \propto \\ -2\theta, -\pi < 2\theta \le 0 \Rightarrow \dfrac{-\pi}{2} \le \theta \le 0 \Rightarrow -\propto < x < 0 \end{cases}$$

$$\frac{du}{d\theta} = \begin{cases} 2, & -1 \le x \le 1 \\ -2, & 1 \le x < \infty \\ -2, & -\infty < x \le -1 \end{cases}$$

$$\frac{dv}{d\theta} = \begin{cases} 2, & 0 < x < \infty \\ -2, & -\infty < x < 0 \end{cases}$$

$$\frac{du}{dv} = \begin{cases} \left(\dfrac{-2}{-2}\right) = 1, & -\infty < x < -1 \\ \left(\dfrac{2}{-2}\right) = -1, & -1 < x < 0 \\ \left(\dfrac{2}{2}\right) = 1, & 0 < x < 1 \\ \left(\dfrac{-2}{2}\right) = -1, & 1 < x < \infty \end{cases}$$

$$\frac{du}{dv} = \begin{cases} 1 \ \forall x \in (-\infty, -1) \cup (0, 1) \\ -1 \ \forall x \in (-1, 0) \cup (1, \infty) \\ \text{not exists at } x = -1, 0, 1 \end{cases}$$

Illustration :

(i) *Differential coefficient of* $e^{\sin^{-1}x}$ *w.r.t.* $e^{-\cos^{-1}x}$ *is independent of x. (T or F)*

(ii) *Find the derivative of f(tan x) w.r.t. g (sec x) at* $x = \frac{\pi}{4}$, *where* $f'(1) = 2$ *and* $g'(\sqrt{2}) = 4$.

Sol.

(i) Let $u = e^{\sin^{-1}x}$ and $v = e^{-\cos^{-1}x}$

$$u = e^{\frac{\pi}{2} - \cos^{-1}x} \qquad [\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}]$$

$$u = e^{\frac{\pi}{2}} e^{-\cos^{-1}x} = e^{\frac{\pi}{2}} v$$

So, $\frac{du}{dv} = e^{\frac{\pi}{2}}$. *(True)*

(ii) Let $u = f(\tan x)$ and $v = g(\sec x)$

$$\Rightarrow \quad \frac{du}{dx} = f'(\tan x)\sec^2 x \quad \text{and} \quad \frac{dv}{dx} = g'(\sec x)\sec x \tan x$$

$$\Rightarrow \quad \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{f'(\tan x)\sec^2 x}{g'(\sec x)\sec x \tan x}$$

$$\Rightarrow \quad \left[\frac{du}{dv}\right]_{x=\frac{\pi}{4}} = \frac{f'\left(\tan\frac{\pi}{4}\right)}{g'\left(\sec\frac{\pi}{4}\right)\sin\frac{\pi}{4}} = \frac{f'(1)\sqrt{2}}{g'(\sqrt{2})} = \frac{2\sqrt{2}}{4} = \frac{1}{\sqrt{2}}.$$

General Note :

Concavity in each case is decided by the sign of 2nd derivative as :

$\frac{d^2y}{dx^2} > 0 \Rightarrow$ Concave upwards ; $\frac{d^2y}{dx^2} < 0 \Rightarrow$ Concave downwards

D = Decreasing; I = Increasing

T-11 SUCCESSIVE DIFFERENTIATION :

y = f(x) ; the popular symbols used to denote the derivatives are $\frac{dy}{dx} = Dy = f'(x) = y_1 = y'$. Higher order derivatives are denoted as $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = D^2y = f''(x) = y_2$ or y'' etc.

Note : A homogeneous equation of degree n represents '*n*' straight lines passing through the origin, hence

$$\frac{dy}{dx} = \frac{y}{x} \quad \text{and} \quad \frac{d^2y}{dx^2} = 0$$

e.g. If $x^3 + 3x^2y - 6xy^2 + 2y^3 = 0$, then $\left.\frac{d^2y}{dx^2}\right]_{(1,1)} = 0$

Illustration :

If $y = \sin(\sin x)$ *then prove that* $y_2 + (\tan x)y_1 + y\cos^2 x = 0$.

Sol. $y = \sin(\sin x)$

$y_1 = \cos(\sin x)\cdot\cos x$

$y_2 = -\cos(\sin x)\sin x - \sin(\sin x)\cos^2 x$

$\therefore\ y_2 + (\tan x)y_1 + y\cos^2 x$

$= -\cos(\sin x)\cdot\sin x - \sin(\sin x)\cos^2 x + \tan x(\cos(\sin x)\cdot\cos x) + \sin(\sin x)\cos^2 x$

$= 0.$

Illustration :

(i) *If* $y = a\cos(\ln x) + b\sin(\ln x)$ *then prove that* $x^2y_3 + 3xy_2 + 2y_1 = 0$

(ii) *If* $y = e^{a\cos^{-1}x}$, *then prove that* $(1-x^2)y_2 - xy_1 - a^2y = 0.$

(iii) *If* $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$ *then prove that* $(x^2-1)y_3 + 3xy_2 + (1-m^2)y_1 = 0$

Sol. (i) $y = a\cos(\ln x) + b\sin(\ln x)$

$$y_1 = \frac{-a\sin(\ln x)}{x} + \frac{b\cos(\ln x)}{x}$$

$\Rightarrow\ xy_1 = -a\sin(\ln x) + b\cos(\ln x) \Rightarrow xy_2 + y_1 = \dfrac{-a\cos(\ln x)}{x} - \dfrac{b\sin(\ln x)}{x}$

$\Rightarrow\ x^2y_2 + xy_1 = -a\cos(\ln x) - b\sin(\ln x) = -y \Rightarrow x^2y_2 + xy_1 + y = 0$

$\Rightarrow\ (x^2y_3 + 2xy_2) + (xy_2 + y_1) + y_1 = 0$

$\Rightarrow\ x^2y_3 + 3xy_2 + 2y_1 = 0.$

(ii) $y = e^{a\cos^{-1}x}$

$$\frac{dy}{dx} = e^{a\cos^{-1}x}\left(\frac{-a}{\sqrt{1-x^2}}\right) = y_1$$

$\Rightarrow y_1 = \dfrac{-ay}{\sqrt{1-x^2}} \Rightarrow (1-x^2)(y_1)^2 = a^2y^2$

now differentiating w.r.t. to x

$(1-x^2)\cdot 2y_1y_2 - 2x(y_1)^2 = a^2(2yy_1)$

$\Rightarrow (1-x^2)y_2 - xy_1 = a^2y.$

(iii) $y^{\frac{1}{m}} + y^{\frac{-1}{m}} = 2x$

$$\Rightarrow \; y^{2/m} - 2x\, y^{1/m} + 1 = 0 \quad \Rightarrow \; y^{1/m} = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = \left(x \pm \sqrt{x^2 - 1} \right)$$

$$\Rightarrow y = \left(x \pm \sqrt{x^2 - 1} \right)^m \;\Rightarrow y_1 = m \left(x \pm \sqrt{x^2 - 1} \right)^{m-1} \left(1 \pm \frac{x}{\sqrt{x^2 - 1}} \right)$$

$$\Rightarrow y_1 = \frac{m}{\sqrt{x^2 - 1}} \left(x \pm \sqrt{x^2 - 1} \right)^{m-1} \left(\sqrt{x^2 - 1} \pm x \right)$$

$$\Rightarrow y_1 = \pm \frac{my}{\sqrt{x^2 - 1}}$$

$$\Rightarrow (x^2 - 1)\,(y_1)^2 = m^2 y^2 \;\Rightarrow\; (x^2 - 1)\, 2y_1 y_2 + (y_1)^2\, 2x = m^2\, yy_1$$

$$\Rightarrow (x^2 - 1)\, y_2 + x\, y_1 = m^2 y$$

$$\Rightarrow (x^2 - 1)\, y_3 + 2x\, y_2 + x\, y_2 + y_1 = m^2\, y_1 \;\Rightarrow\; (x^2 - 1)\, y_3 + 3xy_2 + (1 - m^2)\, y_1 = 0.$$

Illustration :

If $(x - a)^2 + (y - b)^2 = c^2 \;\; (c > 0)$ then $\dfrac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}}$ equals

(A) c (B) c^2 (C) c^3 (D) c^4

Sol. $(x - a)^2 + (y - b)^2 = c^2$ $\quad (c > 0)$ $\quad$ ....(1)

Now differentiating w.r.t. to x

$2\,(x - a) + 2\,(y - b)\, y_1 = 0$

$\Rightarrow (x - a) + (y - b)\, y_1 = 0$ $\quad$ ....(2)

$\Rightarrow 1 + (y - b)\, y_2 + (y_1)^2 = 0$ $\quad$ ....(3)

$$1 + (y_1)^2 = 1 + \frac{(x-a)^2}{(y-b)^2} \qquad \text{from (2)}$$

$$1 + (y_1)^2 = \frac{c^2}{(y-b)^2}$$

$$\frac{\left(1 + (y_1)^2\right)^{3/2}}{y_2} = \frac{\left(\dfrac{c^2}{(y-b)^2}\right)^{3/2}}{\dfrac{-\left(1 + (y_1)^2\right)}{(y-b)}} \qquad \text{from (3)}$$

$$= \frac{\left| c^3/(y-b)^3 \right|}{-c^2/(y-b)^3} = \pm\, c.$$

Illustration :

Use the substitution $x = \tan\theta$ to show that the equation,

$$\frac{d^2y}{dx^2}+\frac{2x}{1+x^2}\frac{dy}{dx}+\frac{y}{(1+x^2)^2}=0 \text{ changes to } \frac{d^2y}{d\theta^2}+y=0.$$

Sol. $$\frac{d^2y}{dx^2}+\frac{2x}{1+x^2}\left(\frac{dy}{dx}\right)+\frac{y}{\left(1+x^2\right)^2}=0 \qquad \text{.......(1)}$$

$$x=\tan\theta \Rightarrow dx=\sec^2\theta\, d\theta \Rightarrow \frac{d\theta}{dx}=\frac{1}{\sec^2\theta}=\cos^2\theta$$

$$\frac{dy}{dx}=\frac{dy}{d\theta}\left(\frac{d\theta}{dx}\right)=\frac{1}{\sec^2\theta}\left(\frac{dy}{d\theta}\right)=\cos^2\theta\left(\frac{dy}{d\theta}\right)$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{\sec^2\theta}\cdot\frac{dy}{d\theta}\right)=\frac{d}{d\theta}\left(\cos^2\theta\frac{dy}{d\theta}\right)\cdot\frac{d\theta}{dx}=\left(\cos^2\theta\frac{d^2y}{d\theta^2}-2\cos\theta\sin\theta\frac{dy}{d\theta}\right)\cos^2\theta$$

From equation (1) putting $x = \tan\theta$

$$\cos^2\theta\left(\cos^2\theta\frac{d^2y}{d\theta^2}-2\cos\theta\sin\theta\frac{dy}{d\theta}\right)+\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\left(\cos^2\theta\frac{dy}{d\theta}\right)+\left(\frac{y}{\left(1+\tan^2\theta\right)^2}\right)=0$$

$$\Rightarrow\left(\frac{d^2y}{d\theta^2}-2\tan\theta\frac{dy}{d\theta}\right)+\left(2\tan\theta\frac{dy}{d\theta}\right)+y=0 \Rightarrow \frac{d^2y}{d\theta^2}+y=0.$$

Illustration :

Starting with $\dfrac{dx}{dy}=\dfrac{1}{dy/dx}$. *Prove that* $\dfrac{d^2x}{dy^2}=-\dfrac{d^2y/dx^2}{(dy/dx)^3}$ *and deduce that for the parabola*

$$y^2=4ax,\quad \frac{d^2y}{dx^2}\cdot\frac{d^2x}{dy^2}=-\frac{2a}{y^3}$$

Sol. $$\frac{dx}{dy}=\frac{1}{\left(\frac{dy}{dx}\right)}$$

$$\frac{d^2x}{dy^2}=\frac{d}{dy}\left(\frac{1}{\left(\frac{dy}{dx}\right)}\right)=\frac{1}{dx}\left(\left(\frac{dy}{dx}\right)^{-1}\right)\cdot\frac{dx}{dy}=-1\left(\frac{dy}{dx}\right)^{-2}\left(\frac{d^2y}{dx^2}\right)\cdot\frac{1}{\left(\frac{dy}{dx}\right)}=\frac{-\left(\frac{d^2y}{dx^2}\right)}{\left(\frac{dy}{dx}\right)^3}$$

Given $y^2 = 4ax$

$$\Rightarrow 2y\frac{dy}{dx} = 4a \Rightarrow y\frac{dy}{dx} = 2a \Rightarrow \frac{dy}{dx} = \frac{2a}{y} \Rightarrow \frac{d^2y}{dx^2} = \frac{-2a}{y^2}\left(\frac{dy}{dx}\right) = \frac{-4a^2}{y^3}$$

$$\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2x}{dy^2}\right) = \left(\frac{d^2y}{dx^2}\right)\left(\frac{-\left(\frac{d^2y}{dx^2}\right)}{\left(\frac{dy}{dx}\right)^3}\right) = \frac{-\left(\frac{4a^2}{y^3}\right)^2}{\left(\frac{2a}{y}\right)^3} = \frac{-2a}{y^3}.$$

Illustration :

If $y = \left(\frac{1}{x}\right)^x$ then prove that $y_2(1) = 0$ i.e. $\frac{d^2y}{dx^2} = 0$

Sol. $y = x^{-x} \Rightarrow \ln y = -x \ln x$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = -(1 + \ln x) \Rightarrow \frac{dy}{dx} = -y(1 + \ln x) \qquad \text{so } \left.\frac{dy}{dx}\right|_{x=1} = -y$$

$$\frac{d^2y}{dx^2} = -\left(\frac{dy}{dx}\right)(1 + \ln x) - \frac{y}{x}$$

$$\left.\frac{d^2y}{dx^2}\right|_{at\ x=1} = -\left.\frac{dy}{dx}\right|_{x=1} - y = y - y = 0.$$

Illustration :

If $e^{x+y} = y^2$ then prove that $y'' = \frac{2y}{(2-y)^3}$.

Sol. $e^{x+4} = y^2 \Rightarrow x + y = 2\ln y$

$$1 + \frac{dy}{dx} = \frac{2}{y}\left(\frac{dy}{dx}\right)$$

$$1 = \left(\frac{2-y}{y}\right)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{y}{2-y}$$

$$\frac{d^2y}{dx^2} = \frac{(2-y)\frac{dy}{dx} - y\left(\frac{-dy}{dx}\right)}{(2-y)^2} = \frac{2}{(2-y)^2}\left(\frac{dy}{dx}\right) = \frac{2y}{(2-y)^3}.$$

Illustration :

(i) Use induction to prove that $D(x^n) = n\,x^{n-1}\ \forall\ n \in N$

(ii) If $y = e^{\tan^{-1}x}$ then prove that $(1 + x^2)y_{n+1} + (2nx - 1)y_n + n(n-1)y_{n-1} = 0$.

(iii) If $y = \sin(m \sin^{-1}x)$ then show that $(1 - x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2 - m^2)y_n$.

Sol.

(i) $D(x^n) = n\,x^{n-1}$

$D(x^{n+1}) = D(x^n \cdot x) = D(x^n)\,x + x^n D(x)$

$= nx^{n-1}\,x + x^n = (n+1)\,x^n$

(ii) $y = e^{\tan^{-1}x}$ then $(1 + x^2)\,y_{n+1} + (2nx - 1)\,y_n + n\,(n-1)\,y_{n-1} = 0$

$$\frac{dy}{dx} = \frac{e^{\tan^{-1}x}}{(1+x^2)} \Rightarrow (1 + x^2)\,y_1 = e^{\tan^{-1}x} = y$$

Differentiating again

$(1 + x^2)\,y_2 + 2xy_1 = y_1$

so at $n = 1$; $(1 + x^2)\,y_2 + (2x - 1)\,y_1 + 0 = 0$.

Let this equation is true for n ; then

$(1 + x^2)\,y_{n+1} + (2nx - 1)\,y_n + n\,(n-1)\,y_{n-1} = 0$

Now again differentiating w.r.t. to x.

$(1 + x^2)\,y_{n+2} + 2x\,y_{n+1} + (2n\,x - 1)\,y_{n+1} + 2n\,y_n + n\,(n-1)\,y_n = 0$

$(1 + x^2)\,y_{n+2} + [2x\,(n+1) - 1]\,y_{n+1} + n\,(n-1)\,y_n = 0$

so the equation is valid for $(n + 1)$ also

So from M.I. $(1 + x^2)\,y_{n+1} + (2nx - 1)\,y_n + n\,(n+1)\,y_{n-1} = 0$ is valid for all n.

(iii) $y = \sin(m \sin^{-1}x)$ then $(1 - x^2)\,y_{n+2} = (2n+1)\,xy_{n+1} + (n^2 - m^2)\,y_n$

Now, differentiating w.r.t. to x

$$\frac{dy}{dx} = \cos(m \sin^{-1}x)\,\frac{m}{\sqrt{(1-x^2)}}$$

$$y_1 = \sqrt{(1-y^2)}\,\frac{m}{\sqrt{(1-x^2)}}$$ Now, squaring we get

$$y_1^2 = m^2\,\frac{1-y^2}{1-x^2} \Rightarrow (1-x^2)\,y_1^2 = m^2\,(1-y^2)$$

$(1 - x^2)\,2y\,y_2 - 2x\,y_1^2 = -m^2\,2yy_1 \Rightarrow (1 - x^2)\,y_2 - xy_1 + m^2y = 0$

So for $n = 0$; $(1 - x^2)\,y_2 - x\,y_1 + m^2y = 0$

Let the equation $(1 - x^2)\,y_{n+2} = (2n+1)\,xy_{n+1} + (n^2 - m^2)\,y_n$ is true for any n then again differentiating

$(1 - x^2)\,y_{n+3} - 2x\,y_{n+2} = (2n+1)\,xy_{n+2} + (2n+1)\,y_{n+1} + (n^2 - m^2)\,y_{n+1}$

$\Rightarrow (1 - x^2)\,y_{n+3} - 2x\,y_{n+2} - (2n+1)\,xy_{n+2} = [(n+1)^2 - m^2]\,y_{n+1}$

$\Rightarrow (1 - x^2)\,y_{n+3} = [2\,(n+1) + 1]\,xy_{n+2} + [(n+1)^2 - m^2]\,y_{n+1}$.

Practice Problem

Q.1 Find $\frac{dy}{dx}$, if

(i) $x = a\cos\theta, y = a\sin\theta$ (ii) $x = at^2, y = 2at$

(iii) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ (iv) $x = 2at^2, y = at^4$

(v) $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}, y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$ (vi) $x = a\cos\theta, y = b\cos\theta$

(vii) $x = \cos\theta - \cos 2\theta, y = \sin\theta - \sin 2\theta$ (viii) $x = a(\theta - \sin\theta), y = a(1 + \cos\theta)$

(ix) $x = a\left(\cos t + \log\tan\frac{t}{2}\right), \ y = a\sin t$ (x) $x = a\sec\theta, y = b\tan\theta$

(xi) $x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$

Q.2 If $x = \sqrt{a^{\sin^{-1}t}}, y = \sqrt{a^{\cos^{-1}t}}$, show that $\frac{dy}{dx} = -\frac{y}{x}$..

Q.3 Find $\frac{d^2y}{dx^2}$, if $y = x^3 + \tan x$.

Q.4 (i) If $y = 3e^{2x} + 2e^{3x}$, prove that $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$.

(ii) If $y = \sin^{-1}x$, show that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$.

(iii) If $y = 5\cos x - 3\sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$.

(iv) If $y = 3\cos(\log x) + 4\sin(\log x)$, show that $x^2y_2 + xy_1 + y = 0$

(v) If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$

(vi) If $e^y(x + 1) = 1$. Show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

(vii) If $y = (\tan^{-1}x)^2$, show that $(x^2 + 1)^2 y_2 + 2x(x^2 + 1) y_1 = 2$.

Q.6 For a positive constant a find $\frac{dy}{dx}$, where $y = a^{t+\frac{1}{t}}$ and $x = \left(t + \frac{1}{t}\right)^a$.

Q.7 Differentiate $\sin^2 x$ w.r.t. $e^{\cos x}$.

Q.8 Find $\frac{dy}{dx}$, if $y = 12(1 - \cos t), x = 10(t - \sin t), -\frac{\pi}{2} < t < \frac{\pi}{2}$.

Q.9 Find $\frac{dy}{dx}$, if $y = \sin^{-1}x + \sin^{-1}\sqrt{1-x^2}, -1 \le x \le 1$

Q10 If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, for, $-1 < x < 1$, prove that $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$.

Q.11 If $\cos y = x \cos(a + y)$, with $\cos a \ne \pm 1$, prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$.

Q.12 If $x = a(\cos t + t \sin t)$ and $y = a(\sin t - t \cos t)$, find $\frac{d^2y}{dx^2}$.

Q.13 If $y = e^{a\cos^{-1}x}, -1 \le x \le 1$, show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$.

Q.14 If $y = \begin{vmatrix} \sin x & \cos x & \sin x \\ \cos x & -\sin x & \cos x \\ x & 1 & 1 \end{vmatrix}$, find $\frac{dy}{dx}$.

Q.15 If $f(x) = \begin{vmatrix} x^n & n! & 2 \\ \cos x & \cos\frac{n\pi}{2} & 4 \\ \sin x & \sin\frac{n\pi}{2} & 8 \end{vmatrix}$, then find the value of $\frac{d^n}{dx^n}[f(x)]_{x=0}$

Answer key

Q.1 (i) $-\cot\theta$ (ii) $\frac{1}{t}$ (iii) $-\left(\frac{y}{x}\right)^{1/3}$ (iv) $t^2(v)$ $-\cot 3t$

(vi) $\frac{b}{a}$ (vii) $\frac{(\cos\theta - 2\cos 2\theta)}{(2\sin 2\theta - \sin\theta)}$ (viii) $-\cot\left(\frac{\theta}{2}\right)$ (ix) $\tan t$

(x) $\frac{b}{a}\operatorname{cosec}\theta$ (xi) $\tan\theta$

Q.3 $6x + 2\sec^2 x \tan x$ **Q.6** $\frac{a^{t+\frac{1}{t}}\log a}{a\left(t+\frac{1}{t}\right)^{a-1}}$ **Q.7** $-\frac{2\cos x}{e^{\cos x}}$ **Q.8** $\frac{6}{5}\cot\left(\frac{t}{2}\right)$

Q.9 0 **Q.12** $\frac{\sec^3 t}{at}, 0 < t < \frac{\pi}{2}$ **Q.14** 1 **Q.15** 0

DEDUCTION OF NEW IDENTITIES BY DIFFERENTIATING A GIVEN IDENTITY :

Illustration :

If $\cos\frac{x}{2}\cdot\cos\frac{x}{2^2}\cdot\cos\frac{x}{2^3}.......\cos\frac{x}{2^n}=\frac{\sin x}{2^n\sin\left(x/2^n\right)}$, then prove that

$$\sum_{r=1}^{n}\frac{1}{2^r}\tan\frac{x}{2^r}=\frac{1}{2^n}\cot\frac{x}{2^n}-\cot x$$

Sol. $\cos\frac{x}{2}\cdot\cos\frac{x}{2^2}\cdot\cos\frac{x}{2^3}.......\cos\frac{x}{2^n}=\frac{\sin x}{2^n\sin\left(x/2^n\right)}$ ......(1)

Taking logarithm of both sides

$$\log\left(\cos\frac{x}{2}\right)+\log\left(\cos\frac{x}{2^2}\right)+\log\left(\cos\frac{x}{2^3}\right)+.......+\log\left(\cos\frac{x}{2^n}\right)=\log(\sin x)-\log\left(2^n\sin\left(\frac{x}{2^n}\right)\right)$$

Differentiating w.r.t. to x

$$-\left(\frac{1}{2}\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}+\frac{1}{2^2}\frac{\sin\left(\frac{x}{2^2}\right)}{\cos\left(\frac{x}{2^2}\right)}+.......+\frac{1}{2^n}\frac{\sin\left(\frac{x}{2^n}\right)}{\cos\left(\frac{x}{2^n}\right)}\right)=\cot x-\frac{1}{2^n\sin\left(\frac{x}{2^n}\right)}\cdot\cos\left(\frac{x}{2^n}\right)$$

so $$\sum_{r=1}^{n}\frac{1}{2^r}\tan\left(\frac{x}{2^r}\right)=\frac{1}{2^n}\cot\left(\frac{x}{2^r}\right)-\cot x.$$

Illustration :

If $(1+x)^n=C_0+C_1x+C_2x^2+.......+C_nx^n$, then prove that

$C_1+2C_2+3C_3+.......+nC_n=n2^{n-1}$

and $C_0+2C_1+3C_2+.......+(n+1)C_n=(n+2)2^{n-1}$

Sol. Given $(1+x)^n=C_0+C_1x+C_2x^2+..........+C_nx^n$ ....(1)

(i) Differentiating equation (1) w.r. to x

$n(1+x)^{n-1}=C_1+2C_2x+3C_3x^2+........+nC_nx^{n-1}$.

put $x=1$

$n2^{n-1}=C_1+2C_2+3C_3+........+nC_n$

(ii) Now multiply x on both sides of equation (1)

$x(1+x)^n=C_0x+C_1x^2+C_2x^3+.........+C_nx^{n+1}$

Differentiating equation (1) w.r.t to x,

$(1+x)^n+nx(1+x)^{n-1}=C_0+2C_1x+3C_2x^2+..........+(n+1)C_nx^n$

put $x=1$

$2^n+n\,2^{n-1}=C_0+2C_1+3C_2+.........+(n+1)C_n$

$2^{n-1}(n+2)=C_0+2C_1+3C_2+.......+(n+1)C_n$

DERIVATIVE OF FUNCTIONS EXPRESSED IN THE DETERMINANT FORM :

Let $F(x) = \begin{vmatrix} f & g & h \\ u & v & w \\ l & m & n \end{vmatrix}$ where all functions are differentiable then

$$D'(x) = \begin{vmatrix} f' & g' & h' \\ u & v & w \\ l & m & n \end{vmatrix} + \begin{vmatrix} f & g & h \\ u' & v' & w' \\ l & m & n \end{vmatrix} + \begin{vmatrix} f & g & h \\ u & v & w \\ l' & m' & n' \end{vmatrix}$$

This result may be proved by first principle and the same operation can also be done column wise.

Illustration :

If f, g & h are differentiable functions of x & D= $\begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2f)'' & (x^2g)'' & (x^2h)'' \end{vmatrix}$

prove that $D' = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'')' & (x^3h'')' \end{vmatrix}$

Sol. $D = \begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2f)'' & (x^2g)'' & (x^2h)'' \end{vmatrix}$

$$D = \begin{vmatrix} f & g & h \\ f+xf' & g+xg' & h+xh' \\ 2f+4xf'+x^2f'' & 2g+4xg'+x^2g'' & 2h+4xh'+x^2h'' \end{vmatrix}$$

After dowing row and column operation. This determinant simplifies to

$$D = \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ x^2f'' & x^2g'' & x^2h'' \end{vmatrix} = x^3 \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix}$$

$$D = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ x^3f'' & x^3g'' & x^3h'' \end{vmatrix}$$

$$D' = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'')' & (x^3h'')' \end{vmatrix}$$

Illustration :

If $f(x) = \begin{vmatrix} \cos(x+x^2) & \sin(x+x^2) & -\cos(x+x^2) \\ \sin(x-x^2) & \cos(x-x^2) & \sin(x-x^2) \\ \sin 2x & 0 & \sin 2x^2 \end{vmatrix}$ *then find* $f'(x)$.

Sol. $f(x) = \begin{vmatrix} \cos(x+x^2) & \sin(x+x^2) & -\cos(x+x^2) \\ \sin(x-x^2) & \cos(x-x^2) & \sin(x-x^2) \\ \sin 2x & 0 & \sin 2x^2 \end{vmatrix}$

Expending the determinant

$f(x) = \cos(x+x^2)\cos(x-x^2)\sin 2x^2 - \sin(x+x^2)$

$$\left(\sin(x-x^2)\right)\left(\sin 2x^2 - \sin 2x\right) + \cos(x+x^2)\cos(x-x^2)\sin 2x$$

$= \sin 2x^2 \cos 2x + \sin 2x \cos(2x^2) = \sin(2x^2 + 2x)$

$f'(x) = \cos(2x^2 + 2x)\, 2(2x+1)$

Illustration :

If α *be a repeated root of a quadratic equation* $f(x) = 0$ & $A(x)$, $B(x)$, $C(x)$ *be the polynomials of degree 3, 4 & 5 respectively, then show that* $\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ *is divisible by* $f(x)$, *where dash denotes the derivative.*

Sol. $g(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ *and* $g'(x) = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$

so $g(\alpha) = 0$ *and* $g'(\alpha) = 0$

so α *is a repeated root of* $g(x)$.

so $g(x) = (x-\alpha)^2 h(x) = f(x)\, h(x)$

so $g(x)$ *is divisible by* $f(x)$.

Note : If $(x - r)$ is a factor of the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_0$ repeated m times where $1 \le m \le n$ then r is a root of the equation $f'(x) = 0$ repeated $(m-1)$ times.

General things to remember :

Illustration :

Prove that the derivative of an even differentiable function is an odd functions and the derivative of an odd differentiable function is an even function.

Sol. *Let* $f(x)$ *be even function*

$f(-x) = f(x)$

$f'(-x)\,(-1) = f'(x)$

$\Rightarrow$ $f'(-x) = -f'(x) \Rightarrow f'(x)$ *is an odd function.*

Let $f(x)$ *be odd function*

$f(-x) = -f(x)$

$f'(-x)(-1) = -f'(x)$

$\Rightarrow$ $f'(-x) = f'(x) \Rightarrow f'(x)$ *is an even function.*

3.0 L'HOSPITAL'S RULE (0^0 / ∞^0):

e.g. $f(x) = x^x$ or $\left(-\frac{1}{x}\right)^{\sin x}$

If $f(x)$ and $g(x)$ are two function such that

(i) $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$

(ii) f and g are derivable / continuous at $x = a$

i.e. $\lim_{x\to a} f(x) = f(a) = 0$; $\lim_{x\to a} g(x) = g(a) = 0$

(iii) $f'(x)$ and $g'(x)$ are continuous at $x = a$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)}$$

Illustration :

Evaluate $\lim_{x\to 0} \frac{x\cos x - \ln(1+x)}{x^2}$.

Sol. $\lim_{x\to 0} \frac{x\cos x - \ln(1+x)}{x^2}$ $\left(\frac{0}{0}\right)$ *form*

$$= \lim_{x\to 0} \frac{(\cos x - x\sin x) - \frac{1}{1+x}}{2x} \quad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{x\to 0} \frac{-\sin x - (\sin x + x\cos x) + \frac{1}{(1+x)^2}}{2} = \frac{-0-(0+0)+1}{2} = \frac{1}{2}.$$

Illustration :

Evaluate find a and b if $\lim_{x\to 0} \frac{x(1+a\cos x) - b\sin x}{x^3} = 1$

Sol. *Let* $L = \lim_{x\to 0} \frac{x(1+a\cos x) - b\sin x}{x^3}$ $\left(\frac{0}{0}\right)$ *form*

$$L = \lim_{x\to 0} \frac{(1+a\cos x) - ax\sin x - b\cos x}{3x^2} \quad \left(\frac{0}{0}\right) \text{ form}$$

$1 + a - 0 - b = 0 \Rightarrow b - a = 1$ ....(1)

$$L = \lim_{x\to 0} \frac{-a\sin x - a(x\cos x + \sin x) + b\sin x}{6x}$$

$$= \lim_{x\to 0} \frac{1}{6}\left[-a\left(\frac{\sin x}{x}\right) - a\cos x - \frac{a\sin x}{x} + \frac{b\sin x}{x}\right] = \frac{1}{6}(-a-a-a+b) = 1$$

$\Rightarrow$ $-3a + b = 6$ ....(2)

From equation (1) and (2)

$$a = \frac{-5}{2},\ b = \frac{-3}{2}.$$

Illustration :

$$\lim_{x\to 0}\frac{e^x \sin x - x - x^2}{x^2 + x - \ln(1-x)}$$

Sol. $L = \lim_{x\to 0}\frac{e^x \sin x - x - x^2}{x^2 + x - \ln(1-x)}$ $\qquad \left(\frac{0}{0}\right)$ *form*

$$= \lim_{x\to 0}\frac{e^x(\cos x + \sin x) - 1 - 2x}{2x + 1 + \frac{1}{(1-x)}} \qquad \left(\frac{0}{0}\right) \text{ form}$$

$$= \frac{1-1-0}{0+1+1} = 0.$$

Illustration :

Evaluate $\lim_{x\to 0}\frac{1+\sin x - \cos x + \ln(1-x)}{x\tan^2 x}$

(A) $-\frac{1}{2}$ (B) $-\frac{1}{3}$ (C) $\frac{1}{6}$ (D) DNE

Sol. $\lim_{x\to 0}\frac{1+\sin x - \cos x + \ln(1-x)}{x\tan^2 x}$

$$= \lim_{x\to 0}\frac{1+\sin x - \cos x + \ln(1-x)}{x^3\left(\frac{\tan^2 x}{x^2}\right)}$$

$$= \lim_{x\to 0}\frac{1+\sin x - \cos x + \ln(1-x)}{x^3} \qquad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{x\to 0}\frac{\cos x + \sin x - \frac{1}{1-x}}{3x^2} \qquad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{x\to 0}\frac{-\sin x + \cos x - \frac{1}{(1-x)^2}}{6x} \qquad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{x\to 0}\frac{-\cos x + \sin x - \frac{2}{(1-x)^3}}{6} = \frac{-1+0-2}{6} = \frac{-1}{2}.$$

Illustration :

Evaluate $\lim\limits_{x\to 0} \dfrac{\log_{\sec\frac{x}{2}}(\cos x)}{\log_{\sec x}(\cos(x/2))}$

(A) 1 (B) 16 (C) 4 (D) 2

Sol. $\lim\limits_{x\to 0} \dfrac{\log_{\sec\left(\frac{x}{2}\right)}\cos x}{\log_{\sec x}\cos\left(\frac{x}{2}\right)} = \lim\limits_{x\to 0} \dfrac{\left(\dfrac{\ln\cos x}{\ln\sec\frac{x}{2}}\right)}{\left(\dfrac{\ln\cos\frac{x}{2}}{\ln\sec x}\right)} = \lim\limits_{x\to 0}\left(\dfrac{\ln\cos x}{\ln\cos\frac{x}{2}}\right)^2$

$$= \left(\lim_{x\to 0}\frac{\ln\cos x}{\ln\cos\left(\frac{x}{2}\right)}\right)^2 \left(\frac{0}{0}\right) form = \left(\lim_{x\to 0}\frac{-\tan x}{\frac{-1}{2}\tan\frac{x}{2}}\right)^2 = \left(\lim_{x\to 0} 4\,\frac{\frac{\tan x}{x}}{\frac{\tan\frac{x}{2}}{\left(\frac{x}{2}\right)}}\right)^2 = 4^2 = 16.$$

Illustration :

Evaluate $\lim\limits_{x\to 1} \dfrac{x^x - x}{x-1-\ln x}$

Sol. $\lim\limits_{x\to 1} \dfrac{x^x - 1}{x-1-\ln x} = \lim\limits_{x\to 1} \dfrac{e^{x\ln x} - x}{x-1-\ln x} \left(\dfrac{0}{0}\right) form$

$\lim\limits_{x\to 1} \dfrac{e^{x\ln x}(1+\ln x)-1}{\left(1-\dfrac{1}{x}\right)} \left(\dfrac{0}{0}\right) form$

$$= \lim_{x\to 1} \frac{e^{x\ln x}(1+\ln x)^2 + e^{x\ln x}\cdot\frac{1}{x} - 0}{\left(\frac{1}{x^2}\right)} \left(\frac{0}{0}\right) form$$

$= 1 + 1 - 0 = 2.$

Illustration :

Evaluate $\lim\limits_{x\to 0} \dfrac{1-\cos x\cdot\cos 2x\cdot\cos 3x}{x^2}$

Sol. $\lim\limits_{x\to 0} \dfrac{1-\cos x\cos 2x\cos 3x}{x^2} = \lim\limits_{x\to 0} \dfrac{1-e^{(\ln\cos x+\ln\cos 2x+\ln\cos 3x)}}{x^2}$

$$= \lim_{x\to 0} \frac{-e^{\ln(\cos x)+\ln\cos 2x+\ln\cos 3x}\cdot(-\tan x - 2\tan 2x - 3\tan 3x)}{2x} = \frac{e^0}{2}(1+4+9) = 7.\ \textbf{Ans.}$$

Illustration :

Evaluate $\displaystyle \lim_{x\to 0}\frac{(\cos ax)^{1/m}-(\cos bx)^{1/n}}{x^2}$

Sol. $\displaystyle \lim_{x\to 0}\frac{(\cos ax)^{1/m}-(\cos bx)^{1/n}}{x^2}\quad\left(\frac{0}{0}\right)$ *form*

$$= \lim_{x\to 0}\frac{\frac{1}{m}(\cos ax)^{\frac{1}{m}-1}\;(-a\sin ax)-\frac{1}{n}(\cos bx)^{\frac{1}{n}-1}(-b\sin bx)}{2x}$$

$$= \frac{1}{2}\left[\frac{-a}{m}\lim_{x\to 0}(\cos ax)^{\frac{1}{m}-1}\left(\frac{\sin ax}{x}\right)+\frac{b}{n}\lim_{x\to 0}(\cos bx)^{\frac{1}{n}-1}\frac{\sin bx}{x}\right]$$

$$= \frac{1}{2}\left[\frac{-a}{m}\cdot a+\frac{b}{n}\cdot b\right]=\frac{1}{2}\left(\frac{b^2}{n}-\frac{a^2}{m}\right)=\left(\frac{mb^2-na^2}{2mn}\right).$$

Note : If $\lim_{x\to a} f(x)\to\infty$ and $\lim_{x\to a} g(x)\to\infty$ then also $\displaystyle \lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$

Illustration :

Evaluate the following limits

(i) $\displaystyle \lim_{x\to 0^+}(\operatorname{cosec} x)^{\frac{1}{\ln x}}$ (ii) $\displaystyle \lim_{x\to \pi/2}(\sec x)^{\cot x}$ (iii) $\displaystyle \lim_{x\to 0^+}(\cot x)^{\frac{1}{\ln x}}$

(iv) $\displaystyle \lim_{x\to 0}\left(\frac{1}{x}\right)^{\tan x}$ (v) $\displaystyle \lim_{x\to 0} x^x$ (vi) $\displaystyle \lim_{x\to 1}(1-x^2)^{\frac{1}{\ln(1-x)}}$

(vii) $\displaystyle \lim_{x\to 4}\frac{\sqrt{2x+1}+\sqrt{x-3}-4}{\sqrt{(3x+4)}+\sqrt{5x+5}-9}$

Sol.

(i) $\displaystyle L=\lim_{x\to 0^+}(\operatorname{cosec} x)^{\frac{1}{\ln x}}$

$$\ln L=\lim_{x\to 0^+}\frac{\ln\operatorname{cosec} x}{\ln x}\quad\left(\frac{\infty}{\infty}\right)\text{ form}$$

$$=\lim_{x\to 0^+}\frac{\frac{1}{\operatorname{cosec} x}(-\operatorname{cosec} x\cdot\cot x)}{\left(\frac{1}{x}\right)}=\lim_{x\to 0^+}-\left(\frac{x}{\tan x}\right)=-1.$$

$= L = e^{-1}$

(ii) $$L = \lim_{x\to\frac{\pi}{2}} (\sec x)^{\cot x}$$

$$\ln L = \lim_{x\to\frac{\pi}{2}} \frac{\ln \sec x}{\tan x} = \lim_{x\to\frac{\pi}{2}} \frac{\frac{1}{\sec x}\cdot(\sec x\cdot\tan x)}{\sec^2 x} = \lim_{x\to\frac{\pi}{2}} \sin x\cdot\cos^2 x = 0$$

$$L = e^0 = 1.$$

(iii) $$L = \lim_{x\to 0^+} (\cot x)^{\frac{1}{\ln x}} \quad\Rightarrow\quad \ln L = \lim_{x\to 0^+} \frac{\ln(\cot x)}{\ln x}$$

$$\ln L = \lim_{x\to 0^+} \frac{\frac{1}{\cot x}\left(-\mathrm{cosec}^2 x\right)}{\left(\frac{1}{x}\right)} = \lim_{x\to 0^+} \frac{-x}{\left(\frac{\cos x}{\sin x}\cdot\sin^2 x\right)} = \lim_{x\to 0^+} \frac{-2x}{\sin 2x} = -1$$

$$\Rightarrow \quad L = \frac{1}{e}.$$

(iv) $$L = \lim_{x\to 0} \left(\frac{1}{x}\right)^{\tan x}$$

$$\ln L = \lim_{x\to 0} \tan x\,(-\ln x) = \lim_{x\to 0} \frac{-\ln x}{\cot x}\left(\frac{\infty}{\infty}\right) \text{ form}$$

$$= \lim_{x\to 0} \frac{-\frac{1}{x}}{-\mathrm{cosec}^2 x} = \lim_{x\to 0}\left(\frac{\sin x}{x}\right)\cdot\sin x = 0$$

$$L = 1.$$

(v) $$L = \lim_{x\to 0} x^x$$

$$\ln L = \lim_{x\to 0} x\ln x = \lim_{x\to 0} \frac{\ln x}{(1/x)}\left(\frac{\infty}{\infty}\right) \text{ form}$$

$$= \lim_{x\to 0} \frac{\frac{1}{x}}{\left(\frac{-1}{x^2}\right)} = \lim_{x\to 0}(-x) = 0$$

$$L = 1.$$

(vi) $$L = \lim_{x\to 1} \left(1-x^2\right)^{\frac{1}{\ln(1-x)}}$$

$$\ln L = \lim_{x\to 1} \frac{\ln\left(1-x^2\right)}{\ln(1-x)} \qquad \left(\frac{\infty}{\infty}\right) \text{ form}$$

$$= \lim_{x\to 1} \frac{\left(\frac{-2x}{1-x^2}\right)}{\left(\frac{-1}{1-x}\right)} = \lim_{x\to 1} \frac{2x}{1+x} = 1.$$

$$L = e$$

(vii) $\lim\limits_{x \to 4} \dfrac{\sqrt{2x+1}+\sqrt{x-3}-4}{\sqrt{3x+4}+\sqrt{5x+5}-9}$ $\left(\dfrac{0}{0}\right)$ form

$$= \lim_{x \to 4} \frac{\dfrac{2}{2\sqrt{2x+1}}+\dfrac{1}{2\sqrt{x-3}}-0}{\dfrac{3}{2\sqrt{3x+4}}+\dfrac{5}{2\sqrt{5x+5}}-0}\left(\frac{0}{0}\right) form$$

$$= \frac{\dfrac{2}{3}+\dfrac{1}{1}}{\dfrac{3}{4}+1} = \frac{\dfrac{5}{3}}{\dfrac{7}{4}} = \left(\frac{20}{21}\right).$$

Practice Problem

Q.1 Evaluate the following limits using L'hospital's Rule

(i) $\lim\limits_{x \to \frac{\pi}{2}} \tan x \log \sin x$

(ii) $\lim\limits_{x \to 0} \dfrac{\log \cos x}{x}$

(iii) $\lim\limits_{x \to 0} \dfrac{2^x - 1}{(1+x)^{1/2}-1}$

(iv) $\lim\limits_{x \to \frac{\pi}{4}} (2-\tan x)^{\frac{1}{ln(\tan x)}}$

Q.2 If $\lim\limits_{x \to a} \dfrac{a^x - x^a}{x^x - a^a} = -1$ and $a > 0$, then find the value of a.

Q.3 If $f(x) = \dfrac{ln(x^2+e^x)}{ln(x^4+e^{2x})}$. Then $\lim\limits_{n \to \infty} f(x)$ is equal to

(A) 1 (B) 1/2 (C) 2 (D) None of these

Q.4 Let $f(x+y) = f(x)\cdot f(y)$ for all x and y. Suppose $f(5) = 2$ and $f'(0) = 3$, find $f'(5)$.

Q.5 Let $f(xy) = f(x)\, f(y)\ \forall\, x, y \in R$ and f is differentiable at $x = 1$ such that $f'(1) = 1$ also $f(1) \neq 0$, $f(2) = 3$, then find $f'(2)$.

Q.6 If $f\left(\dfrac{x+2y}{3}\right) = \dfrac{f(x)+2f(y)}{3}\ \forall\, x, y \in R$ and $f'(0) = 1$, $f(0) = 2$, then find $f(x)$.

Answer key

Q.1	(i)	0	(ii)	0	(iii)	$ln\,4$	(iv)	e^{-1}		
Q.2	$a = 1$				**Q.3**	B			**Q.4**	6
Q.5	6				**Q.6**	$x + 2$				