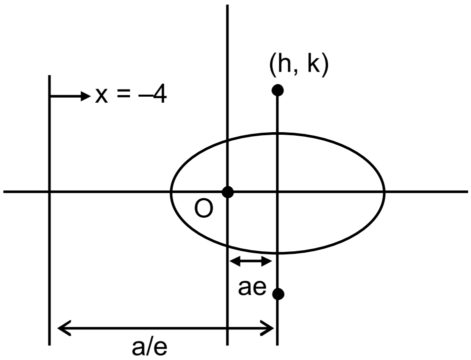
MATHEMATICS

1.



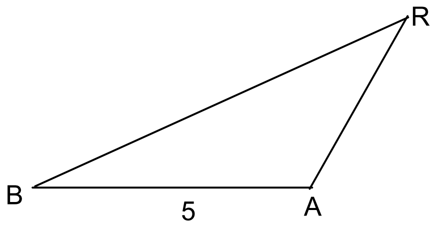
Now $k = b$ and $h = ae$

Also $\frac{a}{e} - ae = 4 \Rightarrow a(1 - e^2) = 4e \Rightarrow a^2(1 - e^2) = 4ae$

$\Rightarrow b^2 = 4h$

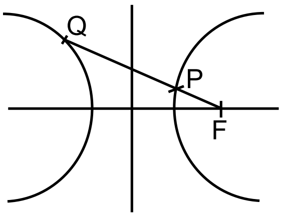
$\Rightarrow k^2 = 4h \Rightarrow \boxed{y^2 = 4x}$

2.



Since length of string is constant, RA + RB = 10, hence locus of R, i.e. conic C is an ellipse with eccentricity $\frac{5}{10} = \frac{1}{2}$.

3.



Let the parameteric equation of chord be

$x = a\sqrt{2} + r\cos\theta \Rightarrow y = r\sin\theta$

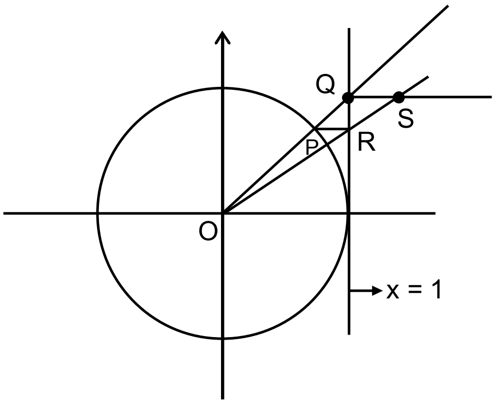
Solving it with $x^2 - y^2 = a^2$ We get $r^2(\cos 2\theta) + (2\sqrt{2}\ a\cos\theta)r + a^2 = 0$

$(PQ)^2 = (r_1 + r_2)^2 - 4r_1r_2 = \frac{8a^2\cos^2\theta - 4a^2\cos 2\theta}{\cos^2 2\theta} = 4a^2\sec^2 2\theta$

Hence $(RS)^2 = 4a^2\sec^2\left\{2\left(\frac{\pi}{2} + \theta\right)\right\} = 4a^2\sec^2 2\theta$

4. Line is $xe_2 + ye_1 - e_1e_2 = 0$ ...(i) also $4\left(e_1^2 + e_2^2\right) = e_1^2e_2^2$ ...(ii)

It is tangent to circle if $r = \frac{e_1e_2}{\sqrt{e_1^2 + e_2^2}} = 2$

5. 

Let point P be $(\cos\theta, \sin\theta)$, so equation of OP is $y = (\tan\theta)x$, hence point Q is $(1, \tan\theta)$. Equation of L_1 is $y = \tan\theta$. Now equation of line PR is $y = \sin\theta$, hence point R is $(1, \sin\theta)$. Therefore equation of OR is $y = (\sin\theta)x$. Point of intersection of OR and L_1 is $S(\sec\theta, \tan\theta)$. Hence locus of S is $x^2 - y^2 = 1$, a hyperbola

6. If the tangent at P does not meet the curve at any other point then the equation
$x^3 - 3x^2 + 2x + 1 = mx + c$ (where $y = mx + c$ in the equation tangent at P) has 3 coincident roots α.
Hence $x^3 - 3x^2 + (2 - m)x + (1 - c) \equiv (x - \alpha)^3$
$\Rightarrow \quad 3\alpha = 3 \;\Rightarrow\; \alpha = 1 \;\Rightarrow\; 2 - m = 3 \;\&\; 1 - c = -1$
$\Rightarrow \quad m = -1 \;\&\; c = 2$ hence point P is (1, 1)
and corresponding tangent is $x + y = 2$ & normal is $y = x$
So (A) (C) (D)

Aliter: Since $y = x^3 - 3x^2 + 2x + 1 = f(x)$ & $y = mx + c$ touch each other at $P(\alpha, \beta)$
$\Rightarrow$ $f'(\alpha) = m$, since the equation is a cubic equation these two curves when equated will give 3 roots and as 2 are real, third root too has to be real & as the given condition states 3rd root can be α only.
Hence $f''(\alpha) = 0 \quad \Rightarrow \quad \alpha = 1$ $m = -1$ and $c = 2 \quad \Rightarrow \quad$ (A)(C)(D)

7. Tangent at P is $xx_1^{n-1} + yy_1^{n-1} = a^n$
$\Rightarrow$ A is $(a^n x_1^{1-n}, 0)$ & B is $(0, a^n y_1^{1-n})$

$OA + OB = a^n(x_1^{1-n} + y_1^{1-n})$ = constant $\Rightarrow 1 - n = n \quad \Rightarrow \quad n = \frac{1}{2}$.

$AB = a^n\sqrt{x_1^{2-2n} + y_1^{2-2n}}$ = constant $\quad \Rightarrow \quad 2 - 2n = n \quad \Rightarrow \quad n = 2/3$

Mid-point of AB is $\left(\frac{a^n x_1^{1-n}}{2}, \frac{a^n y_1^{1-n}}{2}\right)$ remain same $\quad \Rightarrow \quad 1 - n = 0 \quad \Rightarrow \quad n = 1$

Slope of AB $= -\left(\frac{y_1}{x_1}\right)^{1-n} = -\frac{x_1}{y_1} \quad \Rightarrow \quad 1 - n = -1 \quad \Rightarrow \quad n = 2$

8. For $y^2 = x^3 + 1$, $\frac{dy}{dx} = \frac{3x^2}{2y}$ (if $y \neq 0$) $\quad \Rightarrow \quad \left.\frac{dy}{dx}\right|_{\text{for circle}} = \frac{4 - x}{y}$ (if $y \neq 0$)

If they touch each other for $y \neq 0$

$\frac{3x^2}{2y} = \frac{4 - x}{y} \quad \Rightarrow \quad 3x^2 + 2x - 8 = 0 \begin{cases} 4/3 \\ -2 \text{ (rejected)} \end{cases}$

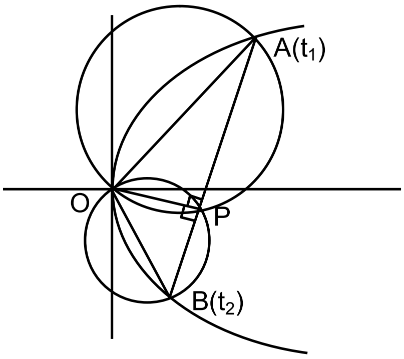
(Since $x = -2 \quad \Rightarrow \quad y^2 = -7$ for first curve)
If $y = 0$, $x = -1$ & tangents to both the curves are parallel to y-axis

9. Tangent at (p, q) to the hyperbola is $\frac{xp}{a^2} - \frac{yq}{b^2} = 1$

If they pass through $(\alpha, 0) \quad \Rightarrow \quad p = \frac{a^2}{\alpha} \quad \Rightarrow \quad x_1 = x_2 \quad \Rightarrow \quad y_1 + y_2 = 0$

If they pass through $(0, \beta)$

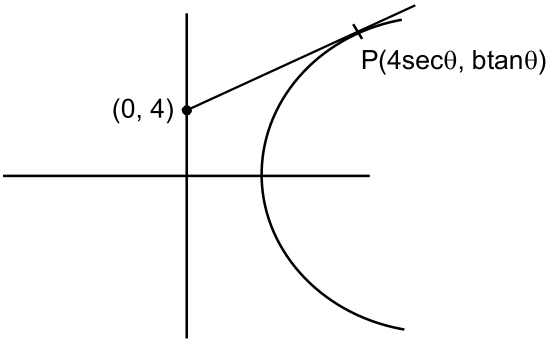
$q = -\frac{b^2}{\beta} \quad \Rightarrow \quad y_3 = y_4 \quad \Rightarrow \quad x_3 + x_4 = 0$

10. 

Since OA & OB are diameters of circles ∠OPA = ∠OPB = 90°
Hence A, P, B are collinear

Now $m = \dfrac{2}{t_1+t_2} = \dfrac{2}{\dfrac{1}{m_1}+\dfrac{1}{m_2}} = \dfrac{2m_1m_2}{m_1+m_2}$ $\left(m_1 = \dfrac{1}{t_1} \quad \& \quad m_2 = \dfrac{1}{t_2}\right)$

Hence (A), (B), (D)

11. 

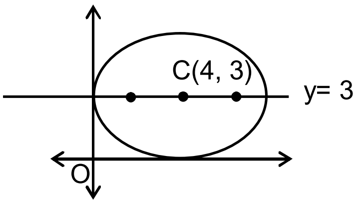
Tangent at P is $\dfrac{x\sec\theta}{4} - \dfrac{y\tan\theta}{b} = 1$.

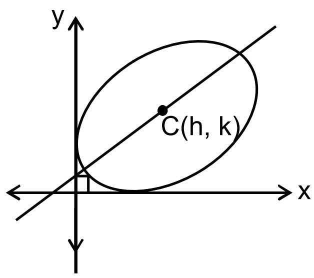
It passes through (0, 4) Hence $b = -4\tan\theta$ ...(1)

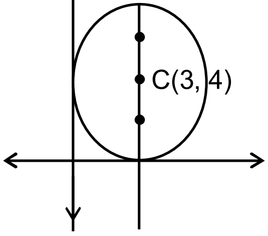
Now $h = 4\sec\theta$ and $k = b\tan\theta = -4\tan^2\theta$ (from (1))

$\Rightarrow \quad K = -4(\sec^2\theta - 1) \quad \Rightarrow \quad k = -4\left(\dfrac{h^2}{16} - 1\right)$.

$\Rightarrow \quad 4K - 16 = -h^2 \quad \Rightarrow \quad x^2 = -4(y-4) \Rightarrow$ (A) & (B)

12. 

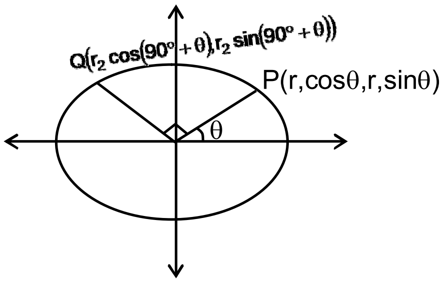




Consider the ellipse in some intermediate position with C being (h, k).

Now $h^2 + k^2 = a^2 + b^2 = 16 + 9 = 25$.
Hence C moves in a circle of radius 5 units whose centre is O. Initially major axis is along y = 3 hence C is (4, 3) and finally major axis is along x = 3 hence C is (3, 4)
Distance curved by C in this motion in $5\left(\tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}\left(\frac{3}{4}\right)\right) = 5\tan^{-1}\left(\frac{7}{24}\right)$

13. Let $OP = r_1$ & $OQ = r_2$



Now P & Q lie on the ellipse hence

$r_1^2\left(\frac{\cos^2\theta}{16} + \frac{\sin^2\theta}{9}\right) = 1 \quad \Rightarrow \quad \frac{\cos^2\theta}{16} + \frac{\sin^2\theta}{9} = \frac{1}{r_1^2}$ ....(1)

$r_2^2\left(\frac{\sin^2\theta}{16} + \frac{\cos^2\theta}{9}\right) = 1 \quad \Rightarrow \quad \frac{\sin^2\theta}{16} + \frac{\cos^2\theta}{9} = \frac{1}{r_2^2}$ .....(2)

Now (1) + (2) $\Rightarrow \frac{1}{16} + \frac{1}{9} = \frac{1}{r_1^2} + \frac{1}{r_2^2} = \frac{25}{144}$

Let equation of chord PQ be $x\cos\alpha + y\sin\alpha = p$, homogenizing the equation of ellipse with this chord gives

$$\frac{x^2}{16} + \frac{y^2}{9} - \left(\frac{x\cos\alpha + y\sin\alpha}{p}\right)^2 = 0$$

As OP & OQ are perpendicular
coefficient of x^2 + coefficient of $y^2 = 0$

$\Rightarrow \left(\frac{1}{16} - \frac{\cos^2\alpha}{p^2}\right) + \left(\frac{1}{9} - \frac{\sin^2\alpha}{p^2}\right) = 0 \quad \Rightarrow \quad \frac{1}{16} + \frac{1}{9} = \frac{1}{p^2} \Rightarrow p^2 = \frac{144}{25} \Rightarrow p = 12/5$

14. Let equation of tangent be $y = mx + \frac{1}{m}$, as this tangent passes through (h, k), we get $k = mh + \frac{1}{m}$

$\Rightarrow \quad hm^2 - km + 1 = 0 < \begin{matrix} m_1 \\ m_2 \end{matrix} \qquad m_1 + m_2 = \frac{k}{h}$

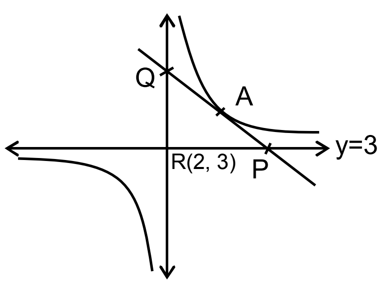
$m_1m_2 = \frac{1}{h}$

If $m_1, m_2 > 0 \Rightarrow \frac{k}{h} > 0$ & $\frac{1}{h} > 0 \Rightarrow k > 0$ & $h > 0 \Rightarrow hk > 0$

If $h < 0 \Rightarrow m_1m_2 < 0$

If $m_1m_2 < 0 \Rightarrow h < 0$ & if $hk > 0 \Rightarrow k < 0 \Rightarrow m_1 + m_2 = \frac{k}{h} > 0$

15.



$y = \dfrac{3(x-2)+7}{x-2} \quad \Rightarrow \quad (x-2)(y-3) = 7$

The given curve a rectangular hyperbola, now shifting origin at (2, 3) the curve transforms to xy = 7. We know that in a rectangular hyperbola portion between axes is bisected by point of tangency and area of triangle PQR is $2c^2$ (for $xy = c^2$). Hence, here area is 14 & circumcentre of PQR is mid-point of PQ which lies on the given curve.

16. Product of perpendicular from two foci on any tangent = $b^2 = \dfrac{3}{\sqrt{2}}.\dfrac{1}{\sqrt{2}} = \dfrac{3}{2} \quad \Rightarrow \quad b = \sqrt{\dfrac{3}{2}}$

Now ae = 1 $\Rightarrow$ $a^2 = b^2 + a^2e^2$ $\Rightarrow$ $a = \sqrt{\dfrac{5}{2}}$

We know that tangent and normal bisect the angle between focal distances of a point.

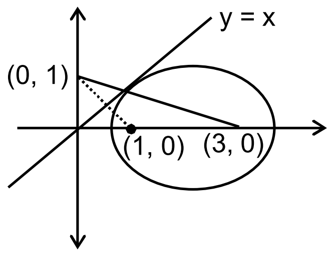


Image of (1, 0) in y = x is (0, 1), line joining (0, 1) & (3, 0) is x + 3y = 3. Point of contact of y = x & ellipse is the point of intersection of y = x and x + 3y = 3, i.e. $\left(\dfrac{3}{4}, \dfrac{3}{4}\right)$

17. As these tangents are perpendicular they meet on director circle of the ellipse, hence locus of point of intersection of these tangents is $x^2 + y^2 = 16 + 9 = 25$.

Let equation of a tangent be $y = mx + \sqrt{16m^2+9}$

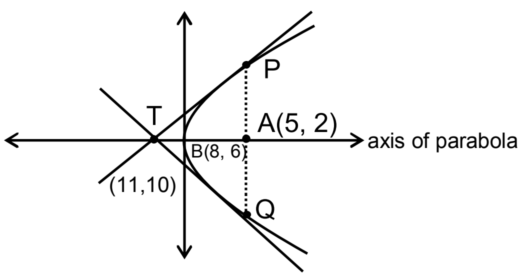
$\ell_1 = 2\sqrt{16-p^2}$, where p is perpendicular distance of the tangent from origin, here $p^2 = \dfrac{16m^2+9}{1+m^2}$

So $\ell_1^2 = 4\left(16 - \dfrac{16m^2+9}{1+m^2}\right) = \dfrac{28}{1+m^2}$

Similarly $\ell_2^2 = \dfrac{28m^2}{1+m^2}$ (replacing m by $-\dfrac{1}{m}$)

Hence $\ell_1^2 + \ell_2^2 = \dfrac{28(1+m^2)}{1+m^2} = 28$

18.



Note that B is mid-point of AT, hence tangents at the extremities of latus rectum meet at T.

Area of quadrilateral formed by tangents an normal at the extremities of latus rectum = $\dfrac{1}{2}$ (latus rectum)^2

$= \dfrac{1}{2} \times (20)^2 = 200$

19. Equation of chord AB is $\frac{x}{a}\cos\left(\frac{\theta_1+\theta_2}{2}\right) + \frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right) = \cos\left(\frac{\theta_1-\theta_2}{2}\right)$

It passes through (ae, 0) or (–ae, 0)

Hence e = $\left|\frac{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}\right|$

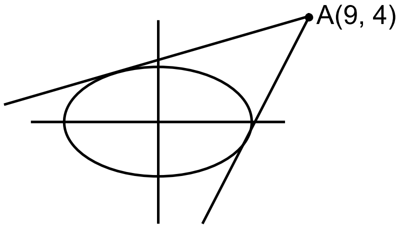
WLOG let it passes through (ae, 0) then BC passes through (–ae, 0)

Hence e = $\frac{1+\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}}{1-\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}}$

$\Rightarrow \quad \tan\frac{\theta_1}{2}\ \tan\frac{\theta_2}{2} = \frac{e-1}{e+1}$ & similarly $\tan\frac{\theta_2}{2}\ \tan\frac{\theta_3}{2} = \frac{-e-1}{-e+1} = \frac{e+1}{e-1}$

So, $\left(\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\right)\left(\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2}\right) = 1$

20.



$\frac{y-4}{x-9}$ is the slope of line joining A(9, 4) & (x, y)

For maximum & minimum value of this expression we have to determine the slope of tangents to the ellipse $\frac{x^2}{25}+\frac{y^2}{16} = 1$ from (9, 4)

Hence $y = Kx \pm \sqrt{16k^2+9}$ It passes through (9, 4)

Hence $(4-9k)^2 = 16K^2 + 9 \quad \Rightarrow \quad 65K^2 - 72K + 7 = 0$

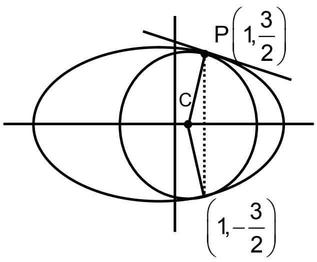
Hence K = 1 or $\frac{7}{65} \quad \Rightarrow \quad M = 1$ & $m = \frac{7}{65}$

21. If $\Delta \neq 0$, $h^2 = ab \Rightarrow$ curve is a parabola, hence S is a straight line

If $\Delta \neq 0$, h = 0, a = b $\neq$ 0 $\Rightarrow$ curve is a circle & S is a circle of radius $\sqrt{2\left(g^2+f^2-c\right)}$ (provided a = b = 1)

If Δ = 0, a + b = 0 $\Rightarrow$ curve is a pair of perpendicular straight lines for which S is a point which is the point of intersection of the two lines.

22.



By symmetry centre of circle lies on x-axis

Normal at P is $\frac{4x}{1}-\frac{3y}{3/2} = 1 \quad \Rightarrow \quad$ point C is $\left(\frac{1}{4},0\right)$

Radius $= \sqrt{\left(1-\frac{1}{4}\right)^2+\left(\frac{3}{2}\right)^2} = \sqrt{\frac{9}{16}+\frac{9}{4}} = \frac{3\sqrt{5}}{4}$

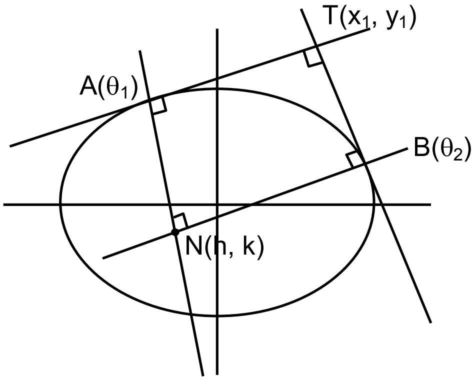
23. Equation of normal at P is $\frac{x}{x_1}+\frac{y}{y_1} = 2 \Rightarrow$ A$(2x_1, 0)$, B$(0, 2y_1)$

Hence P is mid-point of AB, i.e. circumcentre of ΔOAB

$m_{AB} = -\frac{y_1}{x_1}$, $m_{OP} = \frac{y_1}{x_1}$

Let (h, k) be centroid of the triangle OAB $\therefore$ $3h = 2a\sec\theta_1$ $3k = 2a\tan\theta$ $\Rightarrow$ $x^2 - y^2 = \frac{4a^2}{9}$

24.



$x_1 = \dfrac{a\cos\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_1}{2}\right)}$ ….. (1), $y_1 = \dfrac{b\sin\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_1}{2}\right)}$ ....(2)

As tangents are perpendicular $x_1^2 + y_1^2 = (a^2 + b^2)$

Hence $(a^2 + b^2)\cos^2\left(\frac{\theta_1-\theta_1}{2}\right) = a^2\cos^2\left(\frac{\theta_1+\theta_2}{2}\right) + b^2\sin^2\left(\frac{\theta_1+\theta_2}{2}\right)$

Now it is clear that ATBN is a rectangle, hence diagonals bisect each other, therefore.

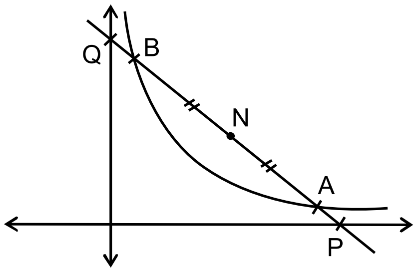
$h + x_1 = a(\cos\theta_1 + \cos\theta_2)$ & $k + y_1 = b(\sin\theta_1 + \sin\theta_2)$

$\Rightarrow \quad \frac{k+y_1}{h+x_1} = \frac{b}{a}\tan\frac{(\theta_1+\theta_2)}{2} = \frac{b}{a}\cdot\frac{ay_1}{bx_1}$ (from (1) & (2))

$\Rightarrow \quad \frac{k+y_1}{h+x_1} = \frac{y_1}{x_1} \quad \Rightarrow \quad kx_1 = hy_1 \quad \Rightarrow \quad \frac{y_1}{x_1} = \frac{k}{h}$

So origin (O), T, N are collinear

25.



Equation of AB is $x + t_1t_2y = a(t_1 + t_2)$

Hence point P is $(a(t_1 + t_2), 0)$ and Q is $\left(0, a\left(\frac{1}{t_1}+\frac{1}{t_2}\right)\right)$

N is $\left(\frac{a(t_1+t_2)}{2}, \frac{a}{2}\left(\frac{1}{t_1}+\frac{1}{t_2}\right)\right)$

Hence N bisects AB as well as PQ

$m_{ON} = \frac{a(t_1+t_2)}{2t_1t_2}\cdot\frac{2}{a(t_1+t_2)} = \frac{1}{t_1t_2}$

Now AN = BN and PN = QN $\Rightarrow$ AP + AN = BQ + BN $\Rightarrow$ AP = BQ

Further AP + AB = BQ + AB $\Rightarrow$ BP = AQ

26. Let the parabola be $y^2 = 4ax$ and $y^2 = -4b(x - a - b)$ (where $ab > 0$)
Let the point of intersection of parabola be (x_1, y_1)
then slope of tangents are say m_1 & m_2

$m_1 = \frac{2a}{y_1}$ & $m_2 = -\frac{2b}{y_1}$

Also, $4ax_1 = -4b(x_1 - a - b) \Rightarrow x_1 = b$ & $y_1 = \pm 2\sqrt{ab}$

Hence $y_1^2 = 4ab$ & PQ is perpendicular to axis, i.e. it is a double ordinate. further $m_1m_2 = -\frac{4ab}{y_1^2} = -1$.

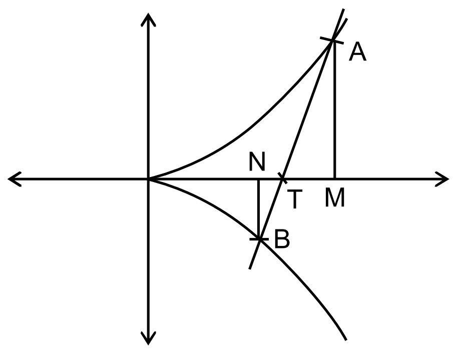
27. Let point A be (t_1^2, t_1^3), hence equation of tangent is $y - t_1^3 = \frac{3t_1}{2}(x - t_1^2)$

If point B is $(t_2^2\ t_2^3)$, then $t_2^3 - t_1^3 = \frac{3t_1}{2}(t_2^2 - t_1^2)$.

$\Rightarrow \quad t_1^2 + 2t_1t_2 + 2t_2^2 = 3t_1t_2 + 3t_1^2$

$\Rightarrow \quad 2t_2^2 - t_1t_2 - t_1^2 = 0 \quad \Rightarrow \quad (t_2 - t_1)(2t_2 + t_1) = 0 \quad \Rightarrow \quad t_2 = -\frac{t_1}{2}$

So point B is $\left(\frac{t_1^2}{4}, \frac{-t_1^3}{8}\right)$



M is $(t_1^2, 0)$, N is $\left(\frac{t_1^2}{4}, 0\right)$ and T is $\left(\frac{t_1^2}{3}, 0\right)$

Triangle AMT and BNT are similar triangle

Hence $\frac{\Delta(AMT)}{\Delta(BNT)} = \left(\frac{AM}{BN}\right)^2 = 8^2 = 64$

28. Since the above conic has a centre it must be a hyperbola or an ellipse
Let origin be shifted to M(p,q) and axis be so rotated that it coincides with the principle axis of conic S, hence its equation is $Ax^2 + By^2 = 1$, and new-co-ordinates of N be (α', β')
Equation of chord whose mid-point is (h, k) is $T = S_1$, i.e.
$Axh + Byk = Ah^2 + Bk^2$
it passes through (α', β')
Hence $A(x^2 - x\alpha') + B(y^2 - y\beta') = 0$

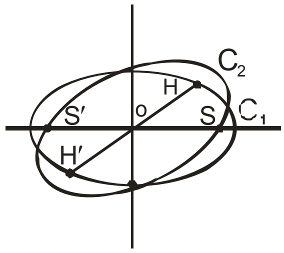
$\Rightarrow \quad A\left(x - \frac{\alpha'}{2}\right)^2 + B\left(y - \frac{\beta'}{2}\right)^2 = \frac{A(\alpha')^2}{4} + \frac{B(\beta')^2}{4}$

Hence locus is a similar conic whose centre is $\left(\frac{\alpha'}{2}, \frac{\beta'}{2}\right)$

i.e. mid-point of MN.

29. For major axis to be x-axis,
$f(k^2 + 2k + 5) > f(k + 11)$
$\Rightarrow \quad k^2 + 2k + 5 < k + 11 \quad \Rightarrow \quad k \in (-3, 2)$

30.



HH′ & SS′ have same mid-point $\Rightarrow$ HSH′S′ is a parallelogram

let one ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ ...(i) $\therefore$ H lies on it

also H $\equiv$ ($ae_2 \cos\theta$, $ae_2 \sin\theta$)

putting in equation (i)

$$\cos^2\theta = \frac{1}{e_1^2}+\frac{1}{e_2^2}-\frac{1}{e_1^2e_2^2}$$

$\therefore$ (A), (B) & (C) are correct. (C) follows from (B)

(31 to 32)

Considering a point (at^2, 2at) and substitute it in equation of circle, S, we get

$$a^2\,t^4 + 2a(2a + g)t^2 + 4aft + c = 0 \;\Big\langle\; t_1,\ t_2,\ t_3,\ t_4$$

$t_1 + t_2 + t_3 + t_4 = 0$

$\Sigma t_1t_2 = \frac{2a(2a+g)}{a^2}$

$\Sigma y_i = 2a\Sigma t_i = 0$

$\Sigma x_i = a\Sigma t_i^2 = a\{(\Sigma t_i)^2 - 2\Sigma t_1t_2\} = -4(2a + g)$

$\Pi t = \frac{c}{a^2} \Rightarrow \frac{\Pi y_i}{16a^4} = \frac{c}{a^2} \Rightarrow \Pi y_i = 16a^2c$

If A, B, C are co-normal point $t_1 + t_2 + t_3 = 0$ and as $X_L = 2$ (radius of S), centre lies on x-axis

$\Rightarrow$ f = 0 and $t_4 = 0$ (as $\Sigma t_i = 0$)

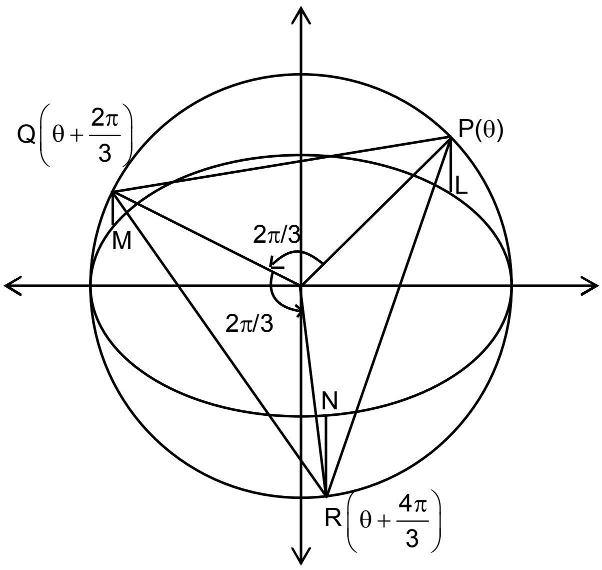
Hence t = 0 is a repeated root of circle and parabola

$\Rightarrow$ one of A, B, C is origin apart from D being origin, i.e. O coincides with one of the points amongst A,B,C

$\Rightarrow$ one of t_1, t_2, t_3 is zero $\Rightarrow$ $t_1 + t_2 = 0$ or $t_2 + t_3 = 0$ or $t_3 + t_1 = 0$

and circle has double contact with parabola at origin.

(33 to 34)



Note that PQR must be an equilateral triangle hence if P is ($5\cos\theta$, $5\sin\theta$), Q & R would be $\left(5\cos\left(\theta+\frac{2\pi}{3}\right), 5\sin\left(\theta+\frac{2\pi}{3}\right)\right)$ & $\left(5\cos\left(\theta+\frac{4\pi}{3}\right), 5\sin\left(\theta+\frac{4\pi}{3}\right)\right)$.

Also area of $\Delta PQR = \frac{\sqrt{3}}{4}(10\sin 60°)^2 = \frac{\sqrt{3}}{4}.100\times\frac{3}{4} = \frac{75\sqrt{3}}{4}$

Now $\frac{\text{area of }\Delta PQR}{\text{area of }\Delta LMN} = \frac{5}{4}$ $\Rightarrow$ Area of $\Delta LMN = 15\sqrt{3}$

Now normals at L, M, N are

$(5\sin\theta)x - (4\cos\theta)y = \frac{9}{2}\sin 2\theta$

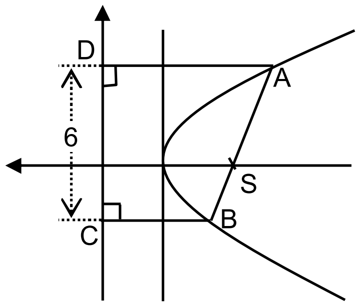
$5\sin\left(\theta+\frac{2\pi}{3}\right)x - 4\cos\left(\theta+\frac{2\pi}{3}\right)y = \frac{9}{2}\sin\left(2\theta+\frac{4\pi}{3}\right)$

$5\sin\left(\theta+\frac{4\pi}{3}\right)x - 4\cos\left(\theta+\frac{4\pi}{3}\right)y = \frac{9}{2}\sin\left(2\theta+\frac{8\pi}{3}\right)$

Now $\frac{1}{2}\begin{vmatrix} 5\sin\theta & 4\cos\theta & 9\sin 2\theta \\ 5\sin\left(\theta+\frac{2\pi}{3}\right) & 4\cos\left(\theta+\frac{2\pi}{3}\right) & 9\sin\left(2\theta+\frac{4\pi}{3}\right) \\ 5\sin\left(\theta+\frac{4\pi}{3}\right) & 4\cos\left(\theta+\frac{4\pi}{3}\right) & 9\sin\left(2\theta+\frac{8\pi}{3}\right) \end{vmatrix} = 0$ (by $R_1 \to R_1 + R_2 + R_3$)

Hence the normals are concurrent

35.



Let S be focus AS = AD & BS = BC

Area of trapezium

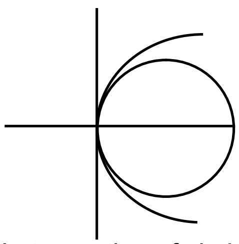
$= \frac{1}{2}\{AD + BC\}.6$

= 3 (AS + BS)

= 3AB

hence AB = 8 units

36.



Let equation of circle be $(x - r)^2 + y^2 = r^2$

Solving at with $y^2 = 8x_1$ we get

$x^2 - 2rx + r^2 + 8x = r^2$ $\Rightarrow$ x = 0 or x = 2r – 8

Now $2r - 8 \le 0$ $\Rightarrow$ $r \le 4$ Hence $r_{max} = 4$

Alter: Normal at $(2t^2, 4t)$ to $y^2 = 8x$, meets x-axis at $(4 + 2t^2, 0)$, So x-coordinate of centre should be such that $r \le 4 + 2t^2 \le 4$, hence $r_{max} = 4$

37. Tangent to a curve at (x_1, y_1) meets y-axis at $(0, y_1 - mx_1)$ where. $m = \left.\dfrac{dy}{dx}\right|_{(x_1,\, y_1)}$

Hence $PQ = |x_1|\sqrt{1+m^2}$

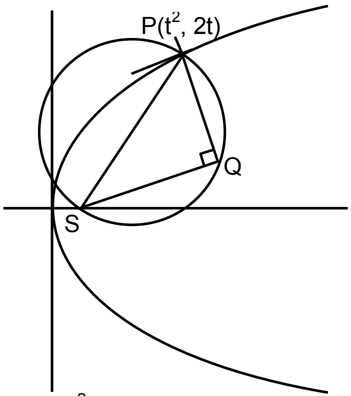
Now $\dfrac{dy}{dx} = \dfrac{1}{1+\sqrt{1-x^2}}.\left(\dfrac{-x}{\sqrt{1-x^2}}\right) - \dfrac{1}{x} + \dfrac{x}{\sqrt{1-x^2}}$

$$= \frac{-x^2 - \sqrt{1-x^2}(1+\sqrt{1-x^2}) + x^2(1+\sqrt{1-x^2})}{(1+\sqrt{1-x^2}).x.\sqrt{1-x^2}} = \frac{-\cancel{x^2} - \sqrt{1-x^2} - 1 + \cancel{x^2} + x^2 + x^2\sqrt{1-x^2}}{x(1+\sqrt{1-x^2})\sqrt{1-x^2}}$$

$$= \frac{-(1-x^2)(1+\sqrt{1-x^2})}{x(1+\sqrt{1-x^2})\sqrt{1-x^2}} = \frac{-\sqrt{1-x^2}}{x}$$

Hence $PQ^2 = x_1^2\left(1+\dfrac{1-x_1^2}{x_1^2}\right) = 1 \qquad \Rightarrow \qquad PQ = 1$

38.



Let P be $(t^2, 2t)$, then equation of normal is $y + tx = 2t + t^3$

Therefore $SQ = \left|\dfrac{t(t^2+1)}{\sqrt{1+t^2}}\right| = |t\sqrt{1+t^2}|$

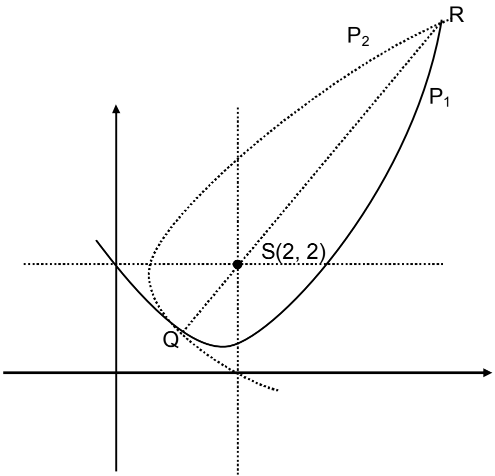
Now $SP = (1 + t^2)$

So $\quad PQ = \sqrt{SP^2 - SQ^2}$

$= \sqrt{(1+t^2)^2 - t^2(1+t^2)} = \sqrt{(1+t^2)(1+t^2-t^2)} = \sqrt{1+t^2}$

Now $PQ = 2 \qquad \Rightarrow \qquad t^2 = 3$ Hence abscissa of point P is 3.

39.



$$\left.\begin{aligned} P_1 \quad (y-2)^2 &= 4(x-1) \\ P_2 \quad (x-2)^2 &= 4(y-1) \end{aligned}\right\}$$ Subtracting them we get $(x - y)(x + y) = 0 \Rightarrow$ line QR is $y = x$

Hence x_1 & x_2 are roots of the equation

$(x-2)^2 = 4(x-1) \quad \Rightarrow \quad x^2 - 8x + 8 = 0 \left\langle \begin{array}{l} 4-2\sqrt{2} = x_1 \\ 4+2\sqrt{2} = x_2 \end{array} \right.$ (given $x_2 > x_1$)

So $\dfrac{RS}{QS} = \dfrac{2\sqrt{2}+2}{2\sqrt{2}-2} = \dfrac{\sqrt{2}+1}{\sqrt{2}-1} = 3+2\sqrt{2}$

40. Let PA = r_1 and PB = r_2 where r_1 and r_2 are roots of equation $(2 + r\sin\theta)^2 = 4\left(-\dfrac{5}{4} + r\cos\theta\right)$,

$\Rightarrow \quad r^2\sin^2\theta + 4(\sin\theta - \cos\theta)r + 9 = 0$

Let Q be (h, k) and PQ = r $\quad \Rightarrow \quad h = -\dfrac{5}{4} + r\cos\theta,\ k = 2 + r\sin\theta$

then $r_1r_2 = \dfrac{9}{\sin^2\theta} = r^2$

$\Rightarrow \quad 9 = (k-2)^2 \quad \Rightarrow \quad k^2 - 4k - 5 = 0$

Hence k = 5 or k = – 1(rejected)

Hence locus is y = 5 $\quad \Rightarrow \quad$ a = 0, b = 5 $\quad \Rightarrow \quad \boxed{a+b=5}$