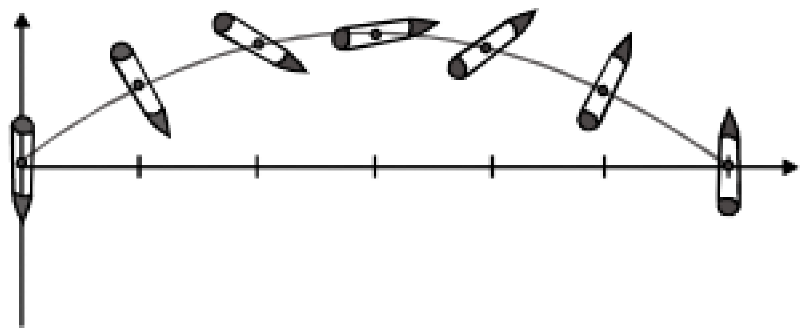
Centre of Mass, Momentum & Collision

Introduction

Until now we have been mainly concerned with the motion of single particles. When we have dealt with an extended body (that is a body that has size), we assumed that it can be approximated to be a point particle or that it underwent only translational motion. Real "extended" bodies, however, can undergo rotational and other types of motion as well. For example, if you flip a pen in air, you will find that its motion is indeed very complex as every part of the pen moves in a different way. Therefore, a pen can not be represented as a particle, but as a system of particles. However, if you closely look, you will find that one of the special points of the pen moves in a simple parabolic path, as if pen's entire mass is concentrated there. That point is called the 'center of mass' of the pen. Thus, precisely speaking centre of mass is the location where the entire mass of system of particles is assumed to be concentrated.

centre of mass is an imaginary point, which *may or may not* be located on the system.

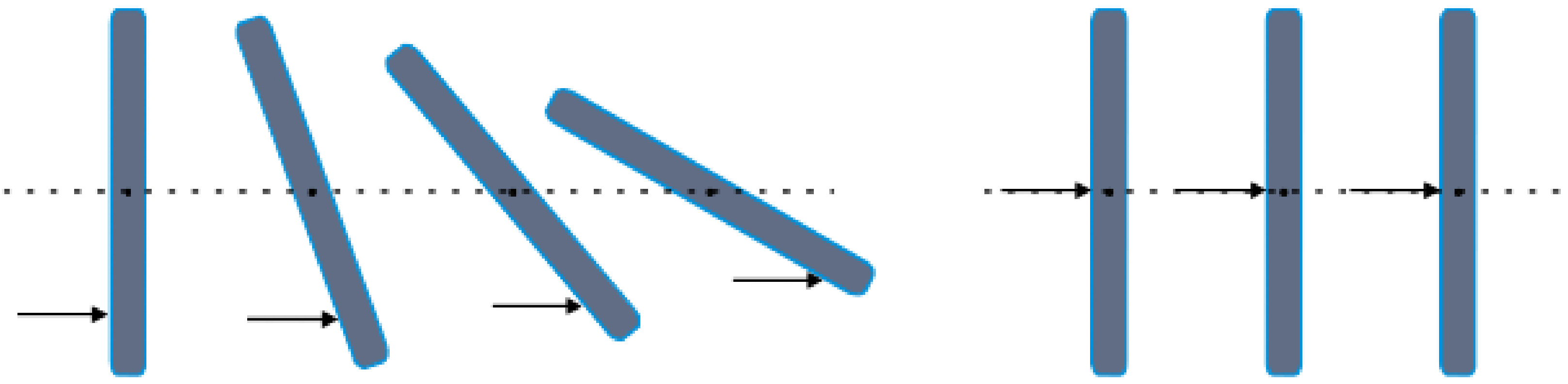


To locate, centre of mass, of a body, balance the body (let us say pen) on outstretched finger. The point on the axis, above your finger is centre of mass of the pen.

When a force is applied on a body apart from the magnitude & direction of $\vec{F}$ the motion of the body also depends upon the point of application.

Thus, $\vec{F} = m\vec{a}$ is not valid for all particles but for a special point i.e. centre of mass, $\vec{a}_{cm} = \frac{\vec{F}}{m}$.

Also, if a force is applied along a line passing through the centre of mass of the body, all the particles of the body move with same linear velocity and acceleration.



Centre of mass is useful concept and it can reduce the effort to solve many difficult problems. Let's see an example to understand this point.

Illustration:

Find the potential energy of a uniform rod of mass 'm' & length l kept vertically standing on the ground. Take potential energy at ground level to be zero.

Sol. *We can not write the potential energy of the rod directly. All the points of the rod are situated at different height from the ground. Therefore we divide it into many point masses with potential energy of a small mass being dU,*

$$dU = (dm)\, gy$$

Total P.E. would be the summation of P.E. of all the elements from y = 0 to y = l.

$$\Rightarrow \quad U = \int_0^l dm\; g\; y$$

Mass of small element, of length dy is dm :

m is mass of length l,

$$\therefore \quad \text{mass of unit length} = \frac{m}{l}$$

$$\text{mass of length } dy = \frac{m}{l}\, dy$$

$$dm = \frac{m}{l}\, dy$$

$$U = g\int_0^l \frac{m}{l}\, dy \cdot y$$

$$= \frac{mg}{l}\int_0^l y\, dy \quad = \frac{mg}{l} \times \left(\frac{l^2}{2}\right)$$

$$= mg\left(\frac{l}{2}\right)$$

In this case, rather than solving this problem, we could have said that the mass of the rod is concentrated at some height Y from the ground. And we shall replace the rod with a point mass at that height.

To determine that height y, compare U with Mg Y

$$U = Mg\, Y = mg\,\frac{l}{2}$$

$$\text{we get } Y = \frac{l}{2}$$

If we had known this position earlier, we could have solved the problem in no time without integration.

This position Y is the center of mass of the rod. Such problem would have been really cumbersome for more complicated bodies like ring, disc, sphere, cone etc. but if their centre of mass is known, we could have solved them easily without calculations.

Potential energy was an example, there are many more physical quantities of a system of particles which can be calculated by this concept of centre of mass. Now we shall explore the concept of centre of mass in detail calculation of its position and application on physical quantities.

Position of centre of mass

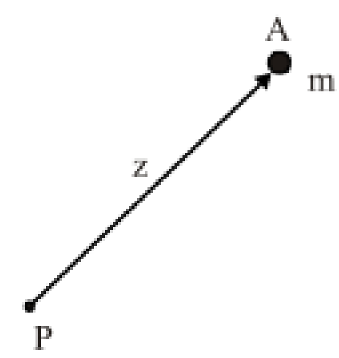
First of all we find the position of Centre of mass of a system of particles. Just to make the subject easy we classify a system of particles in three groups :

1. System of two particles
2. System of a large number of particles and
3. Continuous bodies.

Now, lest us take them separately.

Mass Moment

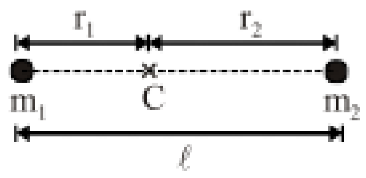
It is defined as the product of mass of the particle and distance of the particle from the point about which mass moment is taken. It is a vector quantity and its direction is directed from the point about which it is taken to the particle, as shown in figure, the mass moment of particle A (mass = m) about the point P is given by z.



There is a important property of centre of mass associated with the mass moments of the components of the system which forms the basis of analytical determination of centre of mass of a system. The property is *"The summation of mass moments of all the components of a system about its centre of mass is always equal to zero"*. This statement is an experimentally verified property which does not require any analytical proof. It can be used as a universal property in all type of system.

1. Position of Centre of mass of two particles

Consider the situation shown in figure. Two masses m_1 to m_2 are separated by a distance l, let C be the centre of mass of the system at a distance r_1 from m_1 and $(l-r_1)$ from m_2. According to the property of mass moments about centre of mass of system of two particles *The summation of mass moments of all the components of a system about its centre of mass is always equal to zero* we have



$$m_1\vec{r}_1 + m_2\vec{r}_2 = 0$$

in scalar form, $-m_1r_1 + m_2r_2 = 0$ (as r_1 is towards left we consider it –ve)

$$m_1r_1 = m_2r_2 \quad \text{.........(i)}$$

$$\frac{r_1}{r_2} = \frac{m_2}{m_1}$$

$$\frac{r_1}{r_1 + r_2} = \frac{m_2}{m_1 + m_2} \Rightarrow \frac{r_1}{\ell} = \frac{m_2}{m_1 + m_2}$$

$$r_1 = \frac{m_2\ell}{m_1 + m_2}$$

From equation (i). The distance of centre of mass from any of the particle (r) is inversely proportional to the mass of the particle (m).

$$r \propto \frac{1}{m}$$

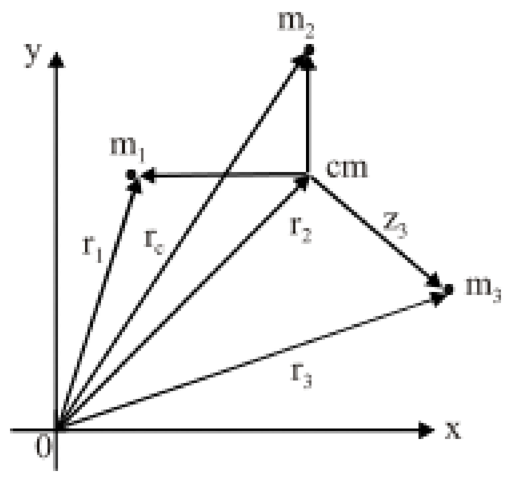
$r_1 = r_2 = \frac{d}{2}$ if $m_1 = m_2$, i.e. Centre of mass lies midway between the two particle of equal masses.

Similarly, $r_1 > r_2$ if $m_1 < m_2$ and $r_1 < r_2$ if $m_1 > m_2$ i.e. Centre of mass is nearer to the particle having larger mass.

2. Definition of Centre of mass for point particles :

Consider the situation shown in figure. There are three masses in a coordinate system with respective coordinates (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3). The position vectors of these masses with respect of origin can be given as

$$r_1 = x_1 i + y_1 j + z_1 k \qquad r_2 = x_2 i + y_2 j + z_2 k \qquad r_3 = x_3 i + y_3 j + z_3 k$$



In this system, we will now locate the position of centre of mass. Let the coordinates of centre of mass be (x_c, y_c, z_c) and so the position vector will be

$$r_c = x_c i + y_c j + z_c k$$

The mass moments of the masses m_1, m_2 and m_3 about the centre of mass can be given as

$$\vec{z}_1 = m_1 \vec{r}_{1/c} = m_1 \cdot (\vec{r}_1 - \vec{r}_c) \qquad \vec{z}_2 = m_2 \vec{r}_{2/c} = m_2 \cdot (\vec{r}_2 - \vec{r}_c) \qquad \vec{z}_3 = m_3 \vec{r}_{3/c} = m_3 \cdot (\vec{r}_3 - \vec{r}_c)$$

According to the property of mass moments The summation of mass moments of all the components of a system about its centre of mass is always equal to zero we have

$$\vec{z}_1 + \vec{z}_2 + \vec{z}_3 = 0$$

$$m_1 \cdot (\vec{r}_1 - \vec{r}_c) + m_2 \cdot (\vec{r}_2 - \vec{r}_c) + m_3 \cdot (\vec{r}_3 - \vec{r}_c) = 0$$

On solving we get

$$\vec{r}_c = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_1 + m_2 + m_3} \qquad \text{....(i)}$$

This relation can also be generalized for n mass system. Now by substituting the vector in terms of unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ and comparing the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$ we get

$$x_c\hat{i} + y_c\hat{j} + z_c\hat{k} = \frac{m_1(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + m_2(x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) + m_3(x_3\hat{i} + y_3\hat{j} + z_3\hat{k})}{m_1 + m_2 + m_3}$$

$$x_c = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} \qquad \text{....(ii)}$$

$$y_c = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3} \quad ....(iii)$$

$$z_c = \frac{m_1z_1 + m_2z_2 + m_3z_3}{m_1 + m_2 + m_3} \quad ....(iv)$$

Equation (ii), (iii), (iv) can also be extended to n - objects system.

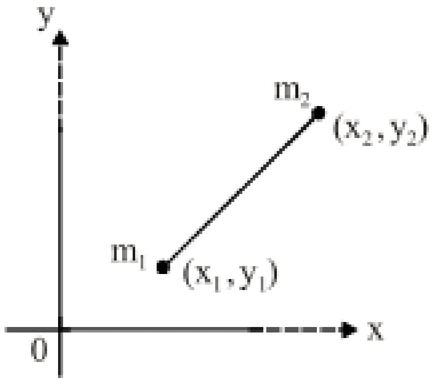
$$\vec{r}_{C.M.} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + ...m_n\vec{r}_n}{m_1 + m_2 + m_3 + ...m_n} = \frac{\Sigma m_i\vec{r}_i}{\Sigma m_i}$$

Thus $x_{cm} = \dfrac{m_1x_1 + m_2x_2 + m_3x_3 + ...}{m_1 + m_2 + m_3 + ...}$

for the body system this equation reduces to

$$x_{cm} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

$$y_{cm} = \frac{m_1y_1 + m_2y_2}{m_1 + m_2}$$



Note: *centre of mass divides two point masses in inverse ratio of their masses*

Illustration :

Two particles of mass 1 kg and 2 kg are located at x = 0 and x=3 m. Find the position of their centre of mass.

Sol. *Since both the particles lie on x-axis, the COM will also lie on x-axis. Let the COM is located at x = x, then*

r_1 = *distance of COM from the particle of mass 1 kg = x*

and r_2 = *distance of COM from the particle of mass 2 kg*

$= (3 - x)$

we know

$$x_c = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

Given ; $x_1 = 0$; $x_2 = 3$

$m_1 = 1$ kg ; $m_2 = 2$kg

$$x_c = \frac{1\times 0 + 2\times 3}{1 + 2}$$

$= 2$ cm

As expected, the centre of mass is nearer to the heavier mass.

using $m_1r_{1c} + m_2r_{2c} = 0$

$m_1(0 - x) + m(3 - x) = 0$

$-x + 6 - 2x = 0$

or $x = 2$ m

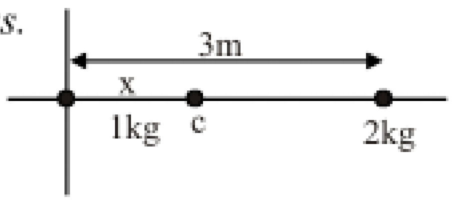
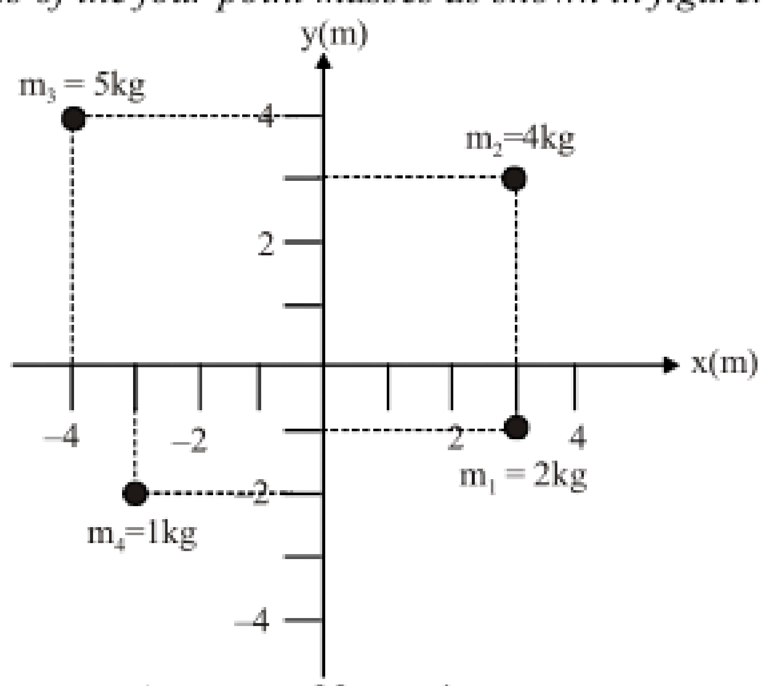


Illustration:

Find the center of mass of the four point masses as shown in figure.



A system of four point masses

Sol. *The total mass M = 12 kg,*

From equation we have, $x_{cm} = \dfrac{\Sigma m_i x_i}{\Sigma m_i}$, $y_{cm}\ \dfrac{\Sigma m_i y_i}{\Sigma m_i}$

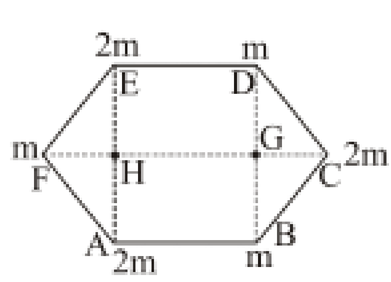
$$x_{cm} = \frac{(2kg)(3m) + (4kg)(3m) + (5kg)(-4m) + (1kg)(-3m)}{12kg} = \frac{-5}{12}m$$

$$y_{cm} = \frac{(2kg)(-1m) + (4kg)(3m) + (5kg)(4m) + (1kg)(-2m)}{12kg} = \frac{28}{12}m$$

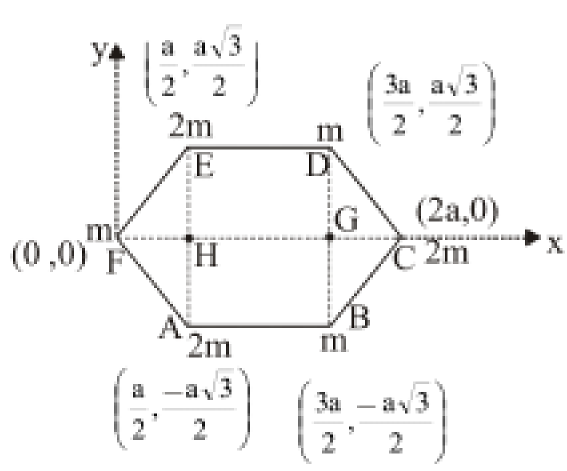
The position of the cm is $\vec{r}_{cm} = -0.42\hat{i} + 2.3\hat{j}m$

Illustration :

Find the position of centre of mass for a system of particles places at the vertices of a regular hexagon as shown in figure



Sol.



$$x_{cm} = \frac{\Sigma m_i x_i}{\Sigma m_i}$$

$$= \frac{m \times 0 + 2m \times \frac{a}{2} + m \times \frac{3a}{2} + 2m \times 2a + m \times \frac{3a}{2} + 2m \times \frac{a}{2}}{m + 2m + m + 2m + m + 2m}$$

$$x_{cm} = a$$

$$y_{cm} \frac{\Sigma m_i y_i}{\Sigma m_i}$$

$$= \frac{m \times 0 + 2m \times \frac{a\sqrt{3}}{2} + m \times \frac{a\sqrt{3}}{2} + 2m \times 0 + m \times \left(\frac{-a\sqrt{3}}{2}\right) + 2m \times \left(\frac{-a\sqrt{3}}{2}\right)}{9m}$$

$$y_{cm} = 0$$

c.m. : (a ,0)

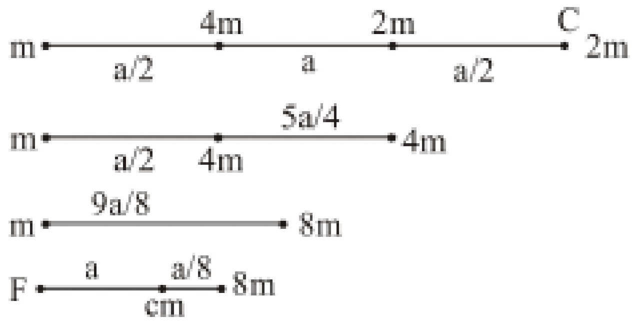
ALITER : *Masses at A & E can be placed at centre of AE, similarly masses at B & D can be placed at centre of BD.*

Objective *: While calculating centre of mass we can replace bodies by their centre of mass.*

Thus our problem reduces to :

Considering origin at F.

$y_{CM} = 0$ *(Because, centre of mass. of individual system lie on the x-axis as seen in the figure above.)*



$\therefore$ $x_{CM} = a$

$\therefore$ *centre of mass. : (a, 0)*

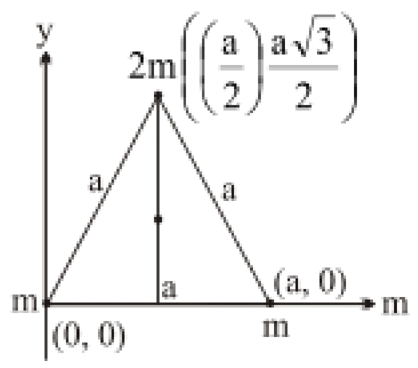
i.e. at the center of the hexagon.

Practice Exercise

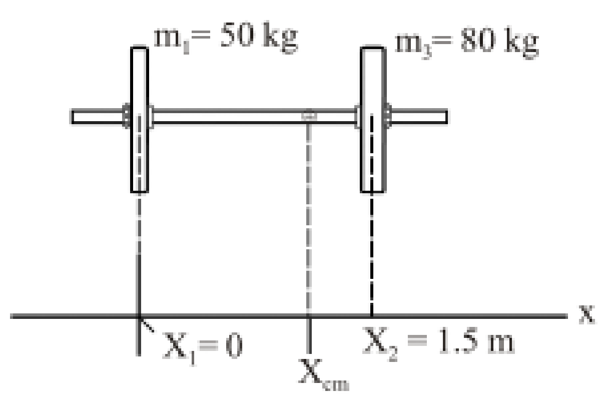
Q.1 If all the particles of a system lie in X - Y plane, is it necessary that the centre of mass be in X-Y plane ?

Q.2 If all the particle of a system lie in a cube, is it necessary that the centre of mass be in the cube ?

Q.3 Find centre of mass of a system of three particle kept at the corner equilateral triangle as shown in figure



Q.4 Find the center of mass of a barbell consisting of 50 kg and 80 kg weights at the opposite ends of a 1.5- -long bar of negligible mass.



Our sketch of the barbell

[Hint] Choosing the origin at one the masses here conveniently makes one of the terms in the sum $\Sigma m_i x_i$ zero. But, as always, the choice of origin is purely for convenience and doesn't influence the actual physical location of the center of mass.

Q.5 Consider the previous problem, at what point must the rod be picked over a knife edge, so that the barbell remains horizontal.

Answers

Q.1 Yes Q.2 Yes Q.3 Coordinates of centre of mass. : $\left(\frac{a}{2}, \frac{a\sqrt{3}}{4}\right)$ Q.4 0.92 m

Q.5 0.92 m

Centre of mass of continuous bodies

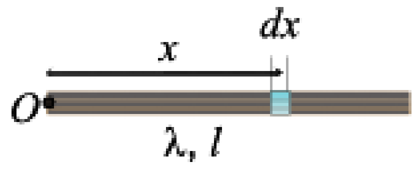
Mass Distribution in rigid bodies

Mass distribution in rigid bodies is often termed as density.

We have often heard about density as being mass per unit volume. But there are other densities as well.
Linear mass density (λ) mass/length

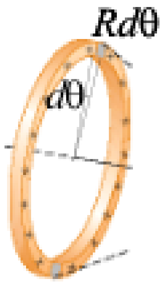
Superficial mass density (σ) mass/area
Volume mass density (r) mass/volume
Examples of linear mass density :
(a) Let λ be the linear mass density of a uniform rod of mass m and length l.

Then by definition $\lambda = \frac{m}{l}$



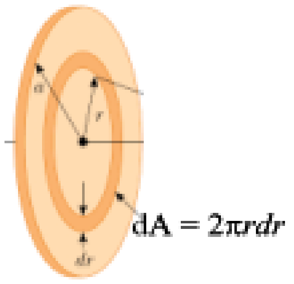
Thus, the mass of element chosen = λ dx
(b) Let λ be the linear mass density of a uniform ring of mass m & radius R

Then by definition $\lambda = \frac{m}{2\pi R}$



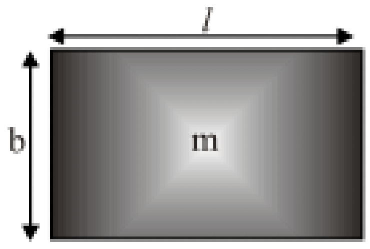
Thus, the mass of chosen arc = $\lambda R d\theta$
Examples of areal mass density :
(a) Let σ be the mass per unit area of the disc of mass m and radius R

Then by definition $\sigma = \frac{m}{\pi R^2}$



Area of ring chosen = $\pi (r + dr)^2 - \pi r^2 \approx 2\pi r\, dr$
Thus, mass of the chosen ring is = $\sigma\, 2\pi r\, dr$

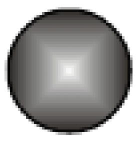
(b) Let σ be the superficial mass density of rectangular plate



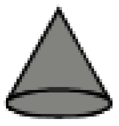
by definition , $\sigma = \frac{m}{lb}$

Examples of volume mass density (ρ)

(a) Sphere of mass m radius R



$\rho = \frac{m}{\frac{4}{3}\pi R^3}$

(b) Cone of mass m, radius R, height H,  $\rho = \dfrac{m}{\dfrac{\pi R^2 H}{3}}$

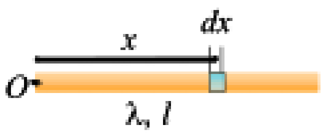
3 Centre of mass of rigid bodies $\vec{r}_{COM} = \dfrac{1}{m}\int \vec{r}\,dm$

While calculating the COM of rigid bodies, we consider small elements in the body and integrate, by replacing the element with equal mass placed at its centre of mass.
To summarize, meaning of each term

$$X_{COM} = \frac{1}{m}\int x\,dm$$

$m \to$ system's mass, $x \to$ position (co-ordinate) of COM of the element chosen
$dm \to$ mass of the chosen element

(a) Centre of Mass of Uniform Straight Rod



Let M and L be the mass and length of the rod respectively. Take the left end of the rod as the origin and the X-axis along the rod. Consider an element of the rod between the positions x and x + dx. If x = 0, the element is at its right end. x varies from 0 through L, the element covers the entire rod. As the rod is uniform, the mass per unit length is M/L. The coordinates of the element are (x, 0, 0). (The coordinates of different points of the element differ, but the difference is less than dx and that much is harmless as integration will automatically correct it.)

The x-coordinate of the centre of mass of the rod is

$$X = \frac{1}{M}\int x\,dm = \frac{1}{M}\int_0^L x\left(\frac{M}{L}dx\right)$$

$$= \frac{1}{L}\left[\frac{x^2}{2}\right]_0^L = \frac{L}{2}$$

The y-coordinate is

$$Y = \frac{1}{M}\int y\,dm = 0$$

and similarly Z = 0.

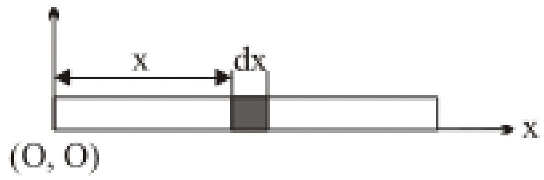
The centre of mass is at $\left(\dfrac{L}{2}, 0, 0\right)$, i.e., at the middle point of the rod.

Illustration :

Calculate Centre of mass of a non-uniform rod with linear mass density λ

$\lambda(\text{mass / length}) = kx^2$

Sol. *For a small element dx, the linear mass density can be considered constant :*



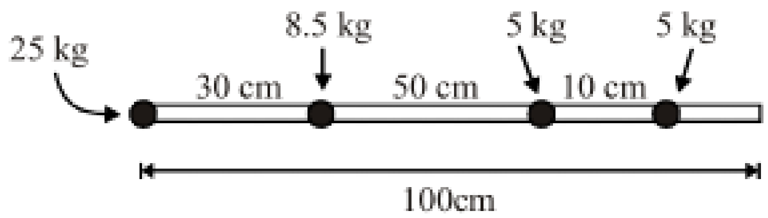
$$\frac{\text{Mass}}{\text{Length}} = \lambda = kx^2 \qquad \therefore \text{ mass of this small element } dm = \lambda\, dx = kx^2\, dx$$

$$M = k\int_0^L x^2\, dx = \frac{KL^3}{3}$$

$$x_{cm} = \frac{\int x\, dm}{\int dm} = \frac{K\int_0^L x^3\, dx}{K\dfrac{L^3}{3}} = \frac{3L}{4}$$

Illustration :

Figure shows a rod of mass 10 kg of length 100cm with some masses tied to it at different positions. Find the point on the rod at which if the rod is picked over a knife edge, it will be in equilibrium about that knife edge.



Sol. *Centre of mass of the system shown in figure, will be the point, at which if we place a knife edge, system will remain in equilibrium.*

To locate the centre of mass of the system, we consider origin at the left end of the rod. With respect to this origin the position of centre of mass of the system is

$$x_C = \frac{m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4 + m_{rod}x_{rod}}{m_1 + m_2 + m_3 + m_4 + m_{rod}}$$

$$x_C = \frac{25\times 0 + 8.5\times 30 + 10\times 50 + 5\times 80 + 5\times 90}{53.5} = 30cm$$

(b) Center of Mass of a Uniform Semicircular Wire

Let M be the mass and R the radius of a uniform semicircular wire. Take its centre as the origin, the line joining the ends as the X-axis, and the Y-axis in the plane of the wire. The centre of mass must be in the plane of the wire i.e. in the X-Y plane.

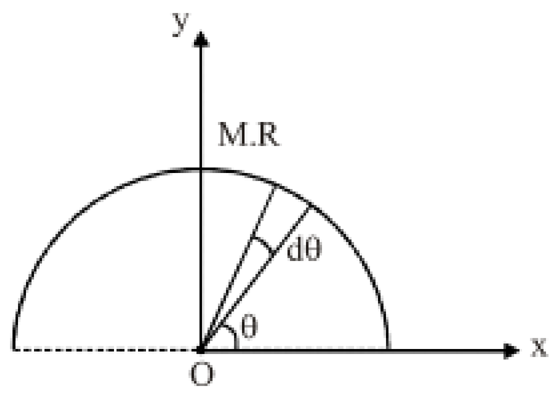
How do we choose a small element of the wire ?

First, the element should be so defined that we can vary the element to cover the whole wire. Secondly, if we are interested in $\int x\, dm$, the x-coordinates of different parts of the element should only infinitesimally differ in range. We select the element as follows. Take a radius making an angle θ with the X-axis and rotate it further by an angle $d\theta$. Note the points of intersection of the radius with the wire during this rotation. This gives an element of length $R\,d\theta$. When we take $\theta = 0$, the element is situated near the right edge of the wire. As θ is gradually increased to π, the element takes all positions on the wire i.e., the whole wire is covered. The coordinates of the element are $(R\cos\theta, R\sin\theta)$. Note that the coordinates of different parts of the element differ only by an infinitesimal amount.

As the wire is uniform, the mass per unit length of the wire $\lambda = \frac{M}{\pi R}$.

The mass of the element is, therefore,

$$dm = \left(\frac{M}{\pi R}\right)(R\,d\theta) = \frac{M}{\pi}\,d\theta$$



The coordinates of the centre of mass are

$$X = \frac{1}{M}\int x\,dm = \frac{1}{M}\int_0^{\pi}(R\cos\theta)\left(\frac{M}{\pi}\right)d\theta = 0$$

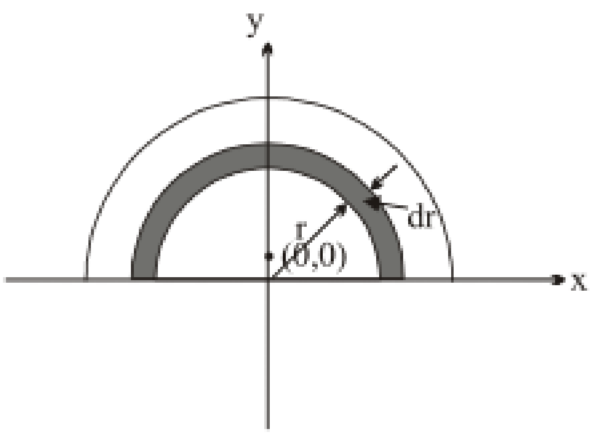
and

$$Y = \frac{1}{M}\int y\,dm = \frac{1}{M}\int_0^{\pi}(R\sin\theta)\left(\frac{M}{\pi}\right)d\theta = \frac{2R}{\pi}$$

The centre of mass is at $\left(0, \frac{2R}{\pi}\right)$.

(c) Centre of Mass of a Uniform Semicircular Plate

This problem can be worked out using the result obtained for the semicircular wire and that any part of the system (semicircular plate) may be replaced by a point particle of the same mass placed at the centre of mass of that part.



We take the origin at the centre of the semicircular plate, the X-axis along the straight edge and the Y-axis in the plane of the plate. Let M be the mass and R be its radius. Let us draw a semicircle of radius r on the plate with the centre at the origin. We increase radius to $r + dr$ and draw another semicircle with the same centre. Consider the part of the plate between the two semicircles of figure it may be considered as a semicircular wire.

If we take $r = 0$, the part will be formed near the centre and if $r = R$, it will be formed near the edge of the plate. Thus if r is varied from '0' to R the elemental parts will cover the entire semicircular plate.

We can replace the semicircular shaded part by a point particle of the same mass at its centre of mass for the calculation of the centre of mass of the plate.

The area of he shaded part = π rdr. The area of the plate is $\pi R^2/2$. As the plate is uniform, the mass per unit area $\sigma = \frac{M}{\pi R^2/2}$. Hence the mass of the semicircular element

$$dm = \frac{M}{\pi R^2/2}(\pi r\,dr) = \frac{2M\,rdr}{R^2}$$

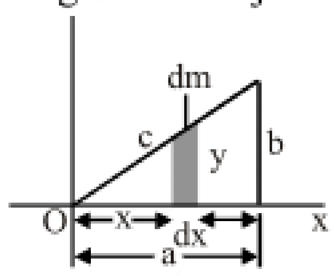
The y-coordinate of the centre of mass of this wire is $2r/\pi$. The y-coordinate of the centre of mass of the plate is, therefore,

$$Y = \frac{1}{M}\int_0^R \left(\frac{2r}{\pi}\right)\left(\frac{2Mr}{R^2}dr\right) = \frac{1}{M}\cdot\frac{4M}{\pi R^2}\frac{R^2}{3} = \frac{4R}{3\pi}$$

The x-coordinate of the centre of mass is zero by symmetry.

(d) Centre of mass of a right triangular sheet of

Dimensions are shown in figure. The object has a uniform mass per unit area.



The first step to find COM of a continuous body, is to determine the element.

We take small elements whose COM is known & then replace the element with equal mass placed at its COM, then we integrate to find COM of the body.

We divide the triangular lamina into narrow strips of width dx and height y as shown in the figure. The mass dm of each strip is

$$dm = \frac{\text{Total mass of the object}}{\text{Total area of the object}} \times \text{area of strip}$$

$$= \frac{M}{(1/2)ab}(y\,dx) = \frac{2M}{ab}y\,dx$$

Now x - coordinate of the centre of mass is

$$x_{CM} = \frac{1}{M}\int x\,dm = \frac{1}{M}\int_0^a x\left(\frac{2M}{ab}\right)y\,dx$$

$$= \frac{2}{ab}\int_0^a xy\,dx$$

To evaluate this integral we must express y in terms of x. From similar triangles in the figure we see that

$$\frac{y}{x} = \frac{b}{a} \quad \text{or} \quad y = \frac{b}{a}x$$

Hence, $$x_{CM} = \frac{2}{ab}\int_0^a x\left(\frac{b}{a}\right)x\,dx$$

$$\frac{2}{a^2}\int_0^a x^2dx$$

$$= \frac{2}{a^2}\left|\left(\frac{x^3}{3}\right)\right|_0^a = \frac{2}{3}a$$

On similar lines we can calculate the y-coordinate to be

$$y_{CM} = \frac{2}{3}b$$

(E) Centre of mass of a thin hemispherical shell of mass M and radius R.

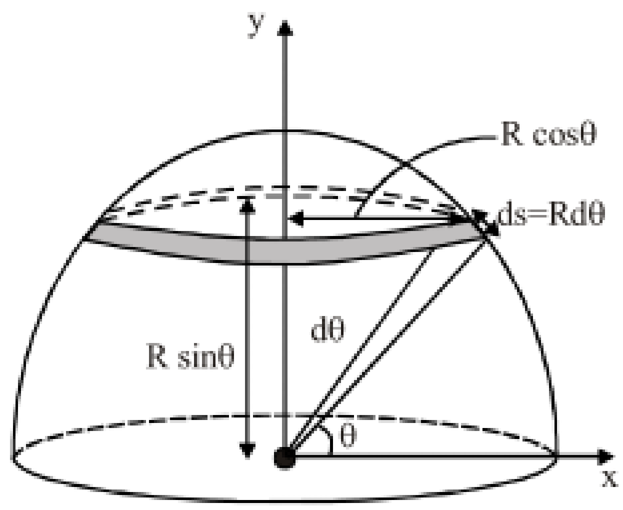
Assuming uniform mass distribution.

Sol. Here, the important part is determination of the element.

We can not take a ring element of thickness dy at a distance y from the center and integrate it from O to R.

Because the mass is distributed on the surface, by taking the above element, we do not cover the complete surface.

In this case element is a circular strip of thickness ds. The thickness of ring subtends angle $d\theta$ at the centre of the hemisphere as shown in figure. Radius of ring element is $R\cos\theta$. Mass of the element is



$$dm = \text{mass per unit area} \times \text{area of circular strip}$$

$$= \frac{M}{2\pi R^2} \times (2\pi R\cos\theta)R\ d\theta$$

Then $\quad x_{CM} = 0 \quad$ from symmetry

and

$$y_{CM} = \frac{1}{M}\int_0^{\pi/2} dm R\sin\theta$$

$$= \frac{1}{M}\int_0^{\pi/2} \frac{M}{2\pi R^2}(2\pi R\cos\theta R\, d\theta)R\sin\theta$$

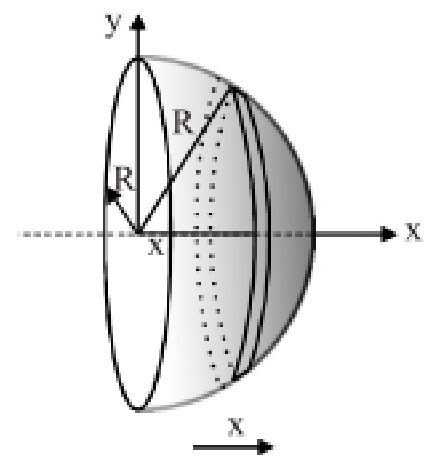
$$= R\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta \quad = \frac{R}{2}$$

(F) Center of mass of a hemispherical object of uniform density and radius R.

As shown in figure. We choose a coordinate system with the origin at the center of the flat face and we let the yz plane be the plane of the face. We now imagine slicing the hemisphere into discs parallel to the yz plane. A disk of thickness dx, located at a distance x from the plane face has a radius $\sqrt{R^2 - x^2}$. Therefore the mass of the disk $dm = \pi\rho(R^2 - x^2)dx$, where ρ is the mass density of the hemisphere. The center of mass of the hemisphere has an x coordinate x_{com} given by

$$x_{com} = \frac{\int_0^R x\,dm}{\int_0^R dm}$$

$$= \frac{\int_0^R \pi\rho x(R^2 - x^2)dx}{\int_0^R \pi\rho(R^2 - x^2)dx}$$

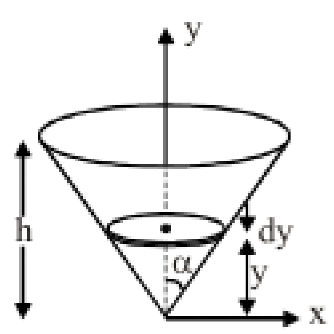


Hemispherical object of uniform density

$$x_{com} = \frac{\left(\frac{x^2R^2}{2} - \frac{x^4}{4}\right)\Big|_0^R}{\left(xR^2 - \frac{x^3}{3}\right)\Big|_0^R} = \frac{R^4/4}{2R^3/3} = \frac{3R}{8}$$

$$y_{com} = Z_{com} = 0$$

(G) Centre of mass of a uniform solid cone of height h and semi vertex angle α.



Sol. *We place the apex of the cone at the origin and axis of cone to be y axis. It is clear that the CM will lie along the y-axis. We divide the cone into disc of radius x and thickness dy. The volume of such a disc is $dV = \pi x^2\,dy = \pi\,(y\tan\alpha)^2\,dy$. The mass of the disc is $dm = \rho dV$. First we will determine the total mass of the cone.*

$$M = \int dm = \pi\rho\tan^2\alpha\int_0^h y^2dy$$

$$= \pi\rho\tan^2\alpha\,\frac{h^3}{3} \qquad \text{......(i)}$$

The position of the CM is given by

$$y_{CM} = \frac{1}{M}\int y\,dm$$

$$= \frac{1}{M}\pi\rho\tan^2\alpha\int_0^h y^3dy$$

$$= \frac{1}{M}\pi\rho\tan^2\alpha\,\frac{h^4}{4} \qquad \text{.....(ii)}$$

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From equation (i) and(ii), we have

$$y_{CM} = \frac{3h}{4}$$

1. Centre of mass of some commonly used systems :

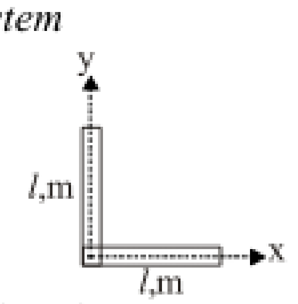
Body	Answer
a. Uniform rod of length L.	L/2
b. rod having linear mass density $\lambda = \alpha x$	2L/3
c. Quadrant of a uniform circular ring, radius R.	$2R/\pi$
d. Uniform semi circular ring of radius R.	$2R/\pi$
e. Uniform semi circular disc of radius R.	$4R/3\pi$
f. Uniform hemispherical shell of radius R.	R/2
g. Uniform solid hemisphere of radius R.	3R/8
h. Hollow cone of base radius R & height h.	h/3 from base of the cone.
i. Solid cone of base radius R and height h.	h/4 from base of the cone.

Practice Exercise

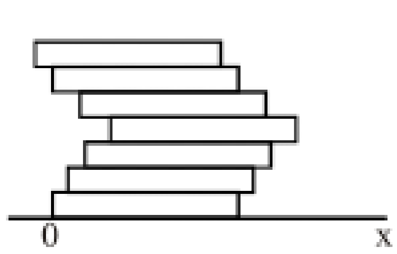
Q.1 A baseball bat of uniform density is cut at the location of its center of mass as shown in figure. The piece with the smaller mass is

(a) the piece on the right
(b) the piece on the left
(c) Both pieces have the same mass
(d) impossible to determine.

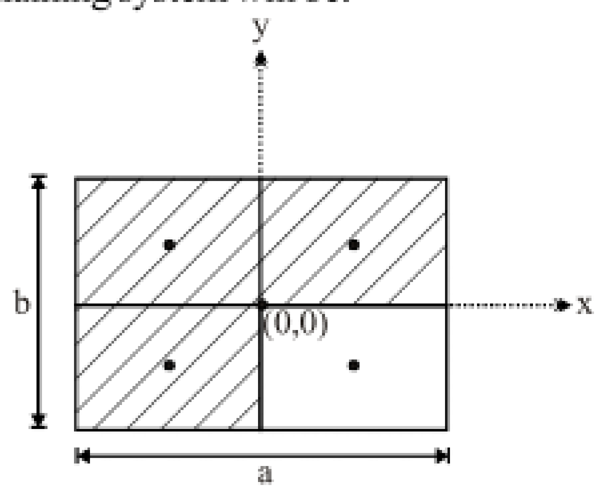
Q.2 *Calculate centre of mass of the system*



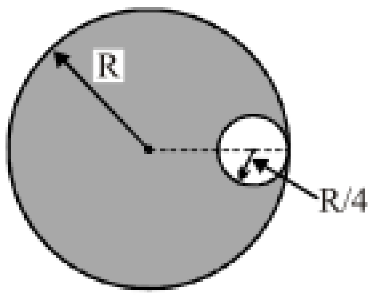
Q.3 Seven homogeneous bricks, each of length L, are arranged as shown in figure. Each brick is displaced with respect to the on in contact by L/10. Find the x-coordinate of the centre of mass relative ot the origin shown.



Q.4 Consider a rectangular plate of dimensions a ×b. If this plate is considered to be made up of four rectangles of dimensions $\frac{a}{2} \times \frac{b}{2}$ and we now remove one out of four rectangles. Find the position where the centre of mass of the remaining system will be.

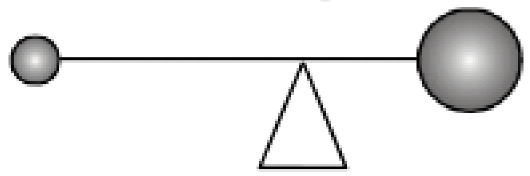


Q.5 Find the center of mass of the shaded portion of a disc



Answers

Q.1 (b) The piece with the handle will have less mass than the piece made up of the end of bat. To see why this is so, take the origin of coordinates as the center of mass before the bat was cut. Replace each cut piece by a small sphere located at the center of mass for each piece. The sphere representing the handle piece is farther from the origin, but the product of less mass and greater distance balance the product of greater mass and less distance from the end piece.



Q.2 $x_{cm} = \frac{l}{4}, y_{cm} = \frac{l}{4}$ Q.3 $\frac{22\ell}{35}$ Q.4 $x_{cm} = -\frac{a}{12}, y_{cm} = \frac{b}{12}$ Q.5 $\frac{R}{20}$ the left of disc

Motion of center of mass

Position vector of centre of mass

$$\vec{r}_{c.m.} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + ... + m_n\vec{r}_n}{m_1 + m_2 + ...m_n} \qquad ....(1)$$

Differentiating the above equation, we get the velocity of centre of mass in terms of velocity of individual particles.

Velocity vector of centre of mass

$$\vec{v}_{c.m.} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + ... + m_n\vec{v}_n}{m_1 + m_2 + ...m_n} = \frac{\sum m_i\vec{v}_i}{\sum m_i} \qquad ....(2)$$

On further differentiation, acceleration of centre of mass can be obtained.

$$\frac{d\vec{v}_{c.m.}}{dt} = \frac{m_1 \frac{d\vec{v}_1}{dt} + m_2 \frac{d\vec{v}_2}{dt} + ... + m_n \frac{d\vec{v}_n}{dt}}{m_1 + m_2 + ... + m_n}.$$

Acceleration vector of centre of mass

$$\vec{a}_{c.m.} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + ... + m_n\vec{a}_n}{m_1 + m_2 + ... + m_n} = \frac{\sum m_i\vec{a}_i}{\sum m_i} \qquad ....(3)$$

Let net force on particle of mass m_1 be $\vec{F}_1$, m_2 be $\vec{F}_2$

$\vec{F}_1 = m_1\vec{a}_1$, $\vec{F}_2 = m_2\vec{a}_2$ ....

substituting these values in equation (3)

$$\vec{a}_{cm} = \frac{\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + ...}{m_1 + m_2 + m_3 + ...}$$

We know that summation of internal forces is zero, thus

$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 .... \vec{F}_n$ is the net external force.

$$\vec{a}_{cm} = \frac{\left(\vec{F}_{external}\right)_{net}}{M_{total}}.$$

Displacement vector of centre of mass

$$\vec{s}_{c.m.} = \frac{m_1\vec{s}_1 + m_2\vec{s}_2 + ... + m_n\vec{s}_n}{m_1 + m_2 + ... + m_n} = \frac{\sum m_i\vec{s}_i}{\sum m_i}$$

Equation (ii) and (iii) are vector equations and thus can be solved separately for the three mutually perpendicular components $\left(\hat{i}, \ \hat{j} \text{ and } \hat{k}\right)$ as we did earlier in determining the position of centre of mass.

Components of velocity

$$v_{c.m.(x)} = \frac{\sum m_i v_{ix}}{\sum m_i}$$

$$v_{c.m.(y)} = \frac{\sum m_i v_{iy}}{\sum m_i}$$

Component of acceleration

$$a_{c.m.(x)} = \frac{\sum m_i a_{ix}}{\sum m_i}$$

$$a_{c.m.(y)} = \frac{\sum m_i a_{iy}}{\sum m_i}$$

Components of displacement

$$s_{c.m.(x)} = \frac{\sum m_i s_{ix}}{\sum m_i}$$

$$s_{c.m.(y)} = \frac{\sum m_i s_{iy}}{\sum m_i}$$

$$\frac{d\vec{v}_{cm}}{dt} = \vec{a}_{cm} = \frac{(\vec{F}_{ext})_{net}}{M_{total}}$$

If $\quad (\vec{F}_{ext})_{net} = 0 \Rightarrow a_{cm} = 0 \Rightarrow \vec{v}_{cm}$ is constant.

if $\quad (\vec{v}_{cm})_{initial} = 0$

During the course of motion it will remain zero, and thus displacement of the centre of mass of the system will also be zero.

Two bodies system

Initially m_1 and m_2 are at rest and they are free to move under the influence of internal forces only, then m_1 and m_2 may move with variable velocity and variable acceleration but centre of mass of system will remain at rest.

Initially

$$m_1 r_1 = m_2 r_2$$

After some time

$$m_1(r_1 - s_1) = m_2(r_2 - s_2)$$

$$m_1 r_1 - m_1 s_1 = m_2 r_2 - m_2 s_2$$

$$m_1 s_1 = m_2 s_2$$

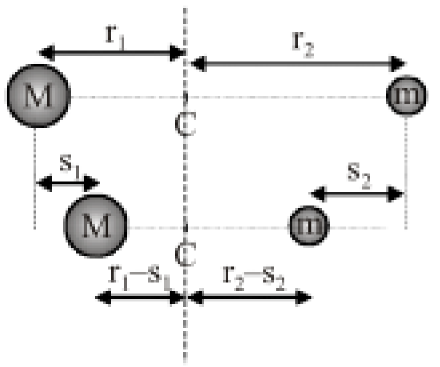
$$m_1 \frac{ds_1}{dt} = m_2 \frac{ds_2}{dt}$$

$$m_1 v_1 = m_2 v_2$$

$$m_1 \frac{dv_1}{dt} = m_2 \frac{dv_2}{dt}$$

$$m_1 a = m_2 a_2$$

$$\frac{s_1}{s_2} = \frac{v_1}{v_2} = \frac{a_1}{a_2} = \frac{m_2}{m_1} \text{ [numerically]}$$

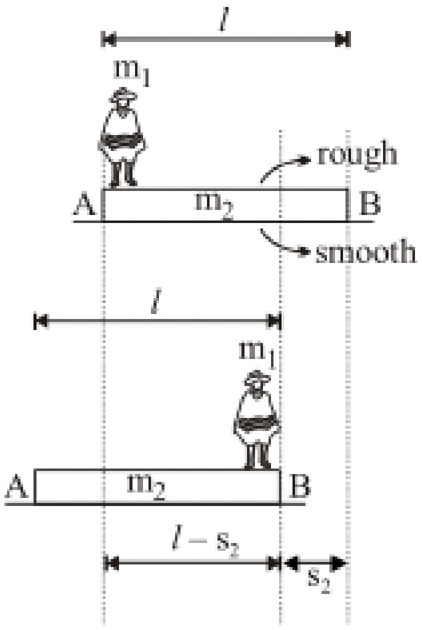


C is centre of mass of M & m

s_1 and s_2, v_1 and v_2 and a_1 and a_2 are oppositely directed.

Illustration :

A man of mass m_1 stands at an edge A of a plank of mass m_2 & length l which is kept on a smooth floor If man walks from A to the other edge B find displacement of plank.



Sol. *Let the displacement of the plank be s_2*

Initial momentum of system (man and plank) is zero.

Net external force on this system is zero.

Thus during the course of motion $\vec{P}_{sys} = 0 \Rightarrow \vec{v}_{cm} = 0 \Rightarrow \vec{s}_{cm} = 0$

$$\Rightarrow \quad m_1\vec{s}_1 + m_2\vec{s}_2 = 0$$

$$m_1\vec{s}_1 = -m_2\vec{s}_2$$

in scalar form

$$m_1 |\vec{s}_1| = m_2 |\vec{s}_2|$$

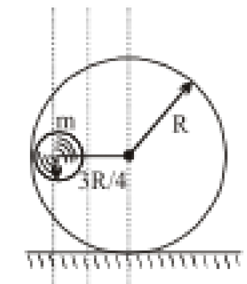
$$m_1 (l - s_2) = m_2 s_2$$

$$s_2 = \frac{m_1 \ell}{m_1 + m_2}$$

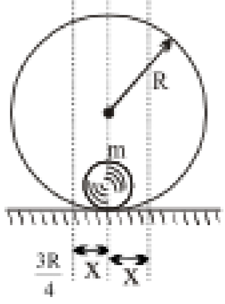
The plank moves backward because as the man moves forward, he pushes on the plank backwards.

Illustration :

Inside a smooth spherical shell of the radius R a ball of the same mass is released from the shown position (Fig.) Find the distance travelled by the shell on the horizontal floor when the ball comes to the lowest point of the shell.



Sol. *As initial momentum of the system in x-direction is zero, and there is no net external force in x-direction the momentum of system remains zero in x-direction and thus the center of mass of the system undergoes zero displacement in x-direction*

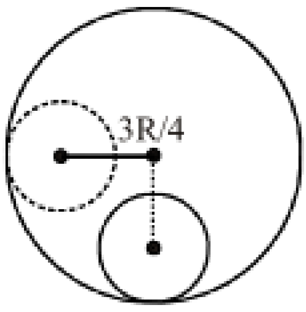


$$m_1 \vec{s}_1 + m_2 \vec{s}_2 = 0$$

When the ball comes to the lowest position ; shell moves backwards say by a distance x.

Displacement of ball in x-direction = Displacement of ball w.r.t. shell + displacement of shell.

Displacement of shell = (–x)



$\therefore$ *displacement of ball is x-direction is* $\left(\frac{3R}{4} + (-x)\right)$

$$m \left(\frac{3R}{4} - x\right) - mx = 0$$

$$\therefore \quad x = \frac{3R}{8}$$

If we do not consider that the shell moves back ward, we can take its forward displacement to be x,

$\therefore$ *displacement of ball in x-direction* $= \frac{3R}{4} + x$

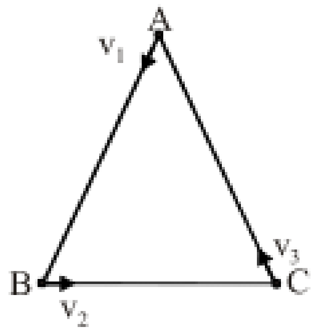
$$m \left(\frac{3R}{4} + x\right) + mx = 0$$

$x = -\frac{3R}{8}$ *(–ve sign indicates its backwards motion)*

Velocity of center of mass:

Illustration :

Let there are three equal masses situated at the vertices of an equilateral triangle, as shown in figure. Now particle-A starts with a velocity v_1 *towards particle B, particle-B starts with a velocity v_2 towards C and particle-C starts with velocity v_3 towards A. Find the displacement of the centre of mass of the three particles A, B, and C after time t. What it would be if $v_1 = v_2 = v_3$*



Sol *First we write the three velocities in vectorial form, taking right direction as positive x-axis and upwards as positive y-axis.*

$$v_1 = -\frac{1}{2} v_1 \hat{i} - \frac{\sqrt{3}}{2} v_1 \hat{j}$$

$$v_2 = v_2 \hat{i}$$

$$v_3 = -\frac{1}{2} v_3 \hat{i} + \frac{\sqrt{3}}{2} v_3 \hat{j}$$

Thus the velocity of centre of mass of the system is, $\vec{v}_{cm} = \dfrac{m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3}{m_1 + m_2 + m_3}$

$$\vec{v}_{cm} = \frac{\vec{v}_1 + \vec{v}_2 + \vec{v}_3}{3}$$

$$\vec{v}_{cm} = \frac{(v_2 - \frac{1}{2}v_1 - \frac{1}{2}v_3)\hat{i} + \frac{\sqrt{3}}{2}(v_3 - v_1)j}{3}$$

Which can be written as $\vec{v}_{cm} = v_x \hat{i} + v_y \hat{j}$

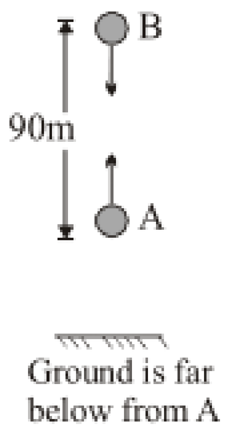
Thus displacement of the centre of mass in time t is $\Delta r = v_x t \hat{i} + v_y t \hat{j}$

If $v_1 = v_2 = v_3 = v$ *we have* $\vec{v}_{cm} = 0$

Therefore no displacement of centre of mass of the system.

Illustration :

Two particles A and B of mass 1 kg and 2kg respectively are projected in the direction shown in figure with speeds $u_A = 200$ m/s and $u_B = 50$ m/s. Initially they were 90m apart. Find the maximum height attained by the centre of mass of the particles from the initial position of A. Assume acceleration due to gravity to be constant ($g = 10$ m/s²)



Sol.

$$\vec{P}_{sys} = M_T \vec{V}_{CM}$$

$$\frac{d\vec{P}_{sys}}{dt} = M_T \vec{a}_{CM}$$

$$\vec{F}_{ext} = M_T \vec{a}_{CM}$$

Net external force is the gravitational force

$$F_{ext} = M_T \times g$$

$$\therefore \quad \vec{a}_{cm} = g \downarrow \text{ (downwards)}$$

$$(\vec{V}_{cm}) = \frac{m_A \vec{V}_A + m_B \vec{V}_B}{m_A + m_B} = \frac{1\times 200 - 2\times 50}{3} = \frac{100}{3} \text{ m/s} \uparrow \text{ (upwards)}$$

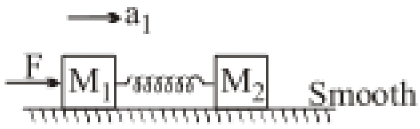
initial height of centre of mass from A, $h_0 = \dfrac{1\times 0 + 2\times 90}{1+2} = 60$ m

$$h_{max} = \text{initial height } (h_0) + \frac{v_{cm}^2}{2g} = 60 + \frac{\left(\frac{100}{3}\right)^2}{2\times 10} = 115.55 \text{ m}$$

Acceleration of centre of mass:

Illustration :

Two blocks of masses $M_1 = 1$kg and $M_2 = 2$kg kept on smooth surface, are connected to each other through a light spring ($k = 100$ N/m) as shown in the figure. When we push mass M_1 with a force $F = 10$N find the acceleration of centre of mass of system.

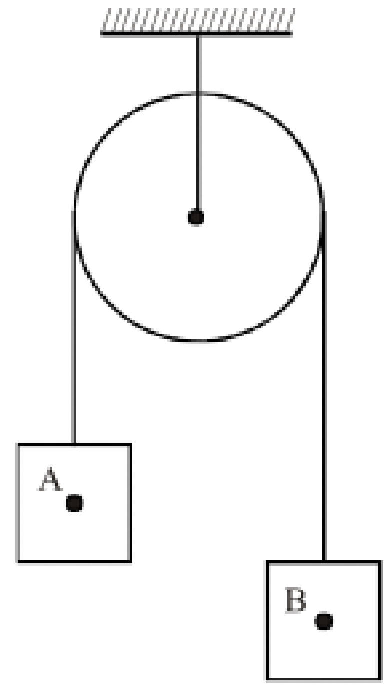


Sol. $a_{CM} = \dfrac{F_{ext}}{M_{Total}}$

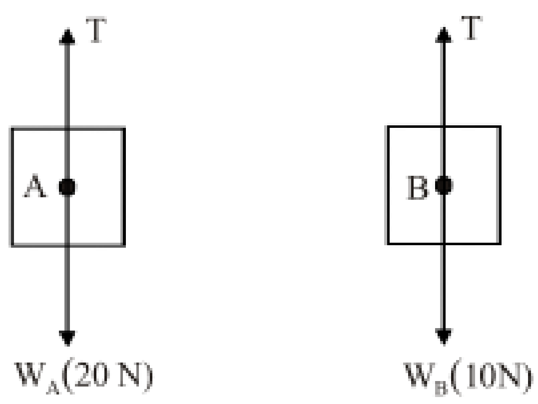
$$= \frac{10}{3} \text{ m/s}^2$$

Illustration :

In the arrangement shown in figure, $m_A = 2$ kg and $m_B = 1$ kg. The string is light and inextensible. Find the acceleration of COM of the blocks. Neglect friction everywhere.



Sol.



From Newton's II Law,

For block A, $20 - T = 2a$ .... (i)

For block B, $T - 10 = a$ .... (ii)

solving (i) and (ii), we obtain T

$$\Rightarrow T = \frac{40}{3}\text{ N}$$

$$F_{ext} = M_{total} \times a_{cm}$$

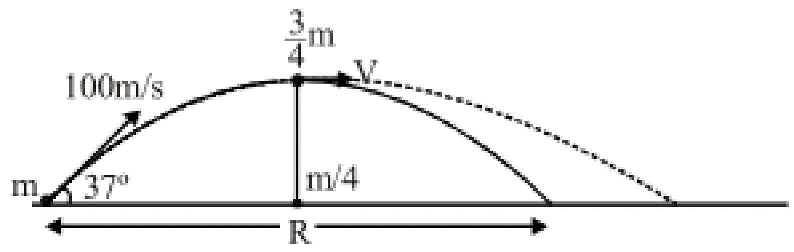
$$= m_1 g + m_2 g - 2T = 3 \times a_{CM}$$

$$\Rightarrow \quad 30 - \frac{80}{3} = 3 \times a_{CM}, \quad a_{CM} = \frac{10}{9}\ m/s^2$$

Illustration :

A projectile is fired at a speed of 100 m/s at an angle of 37° above the horizontal. At the highest point, the projectile breaks into two parts of mass ratio 1:3, the smaller coming to rest. Find the distance from the launching point to the point where the heavier piece lands.

Sol.



As there is no external force in horizontal direction, $(P_{sys})_x$ is conserved
let the horizontal velocity of heavier mass at the highest point be v
Applying conservation of momentum in x - direction, at highest point,

$$(m \times u \cos 37^\circ) = \left(\frac{m}{4} \times 0 + \frac{3m}{4} \times v\right)$$

$$\Rightarrow \qquad v = \frac{320}{3}\ m/s$$

for projectile time to reach highest point and time to reach ground are equal to $\frac{T}{2}$

for time of flight T, using concept of projectile motion,

$$T = \frac{2u\sin\theta}{g}$$

$$T = 12\ sec\ ;$$

$$\therefore \qquad T/2 = 6\ sec$$

$\therefore$ *horizontal distance travelled in first 6 sec. & next 6 sec is $(u \cos 37^o \times 6)$ & $(v \times 6)$ respectively*

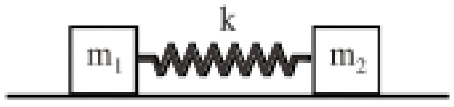
$$x = 480 + 640$$
$$= 1120\ m$$

Practice Exercise

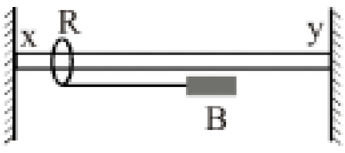
Q.1 A projectile is fired from a gun at an angle of 45° with the horizontal and with a speed of 20 m/s relative to ground. At the highest point in its flight the projectile explodes into two fragments of equal mass. One fragment, whose horizontal speed is zero falls vertically. How far from the gun does the other fragment land , assuming a level terrain ? Take $g = 10\ m/s^2$?

Q.2 Two particles of mass 2 kg and 4kg are approaching towards each other with acceleration $1\ m/sec^2$ and $2\ m/sec^2$ respectively on a smooth horizontal surface. Find the acceleration of centre of mass of the system.

Q.3 Figure shows two blocks of masses $m_1 = 5$ kg and $m_2 = 2$ kg placed on a frictionless surface and connected with a spring. An external kick gives a velocity 14 m/s to the heavier block in the direction of lighter one. Find (a) the velocity gained by the center of mass and (b) the velocities of the two blocks in the center of mass co-ordinate system just after the kick.



Q.4 The ring R of mass m in the arrangement shown can slide along a smooth fixed, horizontal rod XY. It is attached to the block B of mass m by a light string of length L. The block is released from rest, with the string horizontal. Find the displacement of ring when the string becomes vertical.



Answers

Q.1 60 m   Q.2 1 m/sec^2 (towards 2 kg block)

Q.3 (a) $V_{CM} = \dfrac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = \dfrac{5 \times 142 \times 0}{7} = 10$ m/s

(b) $v_{m_1/CM} = 14 - 10 = 4\,m/s$; $v_{m_2/CM} = 0 - 10 = -10\,m/s$

(c) 30 m

Q.4 $\dfrac{ML}{m+M}$

Linear momentum and its conservation principle

The (linear) momentum of a particle is defined as $\vec{p} = m\vec{v}$. The momentum of an N-particle-system is the (vector) sum of the momenta of the N particles i.e.,

$$\vec{P}_{sys} = \sum_i \vec{p}_i = \sum_i m_i \vec{v}_i \,.$$

But $$\sum_i m_i \vec{v}_i = \frac{d}{dt}\sum_i m_i \vec{r}_i = \frac{d}{dt} M\vec{R}_{CM} = M\vec{V}_{CM}$$

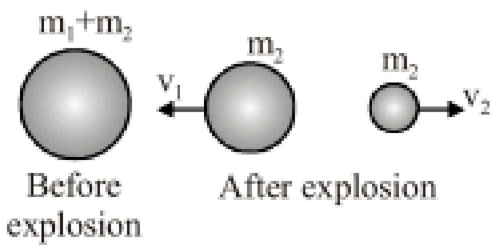
Thus, $\vec{P}_{sys} = M\,\vec{V}_{CM}$

As we have seen, if the external forces acting on the system add up to zero, the centre of mass moves with constant velocity, which means $\vec{p}$ = constant. Thus the linear momentum of a system remains constant (in magnitude and direction), if the external forces acting on the system add up to zero. This is known as the principle of conservation of linear momentum.

Let as see a simple example of a bomb explosion.

Consider a bomb placed on a horizontal surface which suddenly explodes into two parts of masses m_1 and m_2. The forces that are responsible for the explosion are internal. As there is no external force on the system, momentum of system remains conserved. The initial momentum of the system is zero, Thus the final momentum of the system must also be zero.

i.e. After explosion, if mass m_1 moves with velocity $\vec{v}_1$, and mass m_2 moves with velocity $\vec{v}_2$, then by conservation of linear momentum.



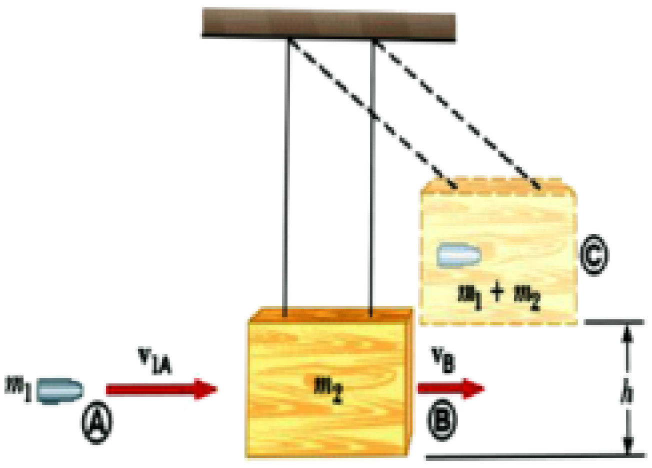
$$m_1\vec{v}_1 + m_2\vec{v}_2 = 0$$

in scalar form,

$$m_1v_1 + m_2(-v_2) = 0$$
$$m_1v_1 = m_2v_2.$$

Illustration :

The ballistic pendulum is an apparatus used to measure the speed of a fast moving projectile, such as a bullet. A bullet of mass m_1 is fired into a large block of wood of mass m_2 suspended from light wires. The bullet embeds in the block and the entire system swings through a height h. Determine the initial speed of the bullet in terms of h?



Sol. *As there is no external force in x-direction, applying conservation of momentum during collision*

$$m_1v_1 = (m_1 + m_2)\, V_B$$

$$\therefore \quad V_B = \frac{m_1v_1}{m_1 + m_2} \qquad .....(i)$$

Applying conservation of mechanical energy after collision

$$\frac{1}{2}(m_1 + m_2)\, V_B^2 = (m_1 + m_2)\, g\, h$$

here h is the displacement of center of mass :

$$V_B = \sqrt{2gh} \qquad ....(ii)$$

From (i) & (ii)

$$\therefore \quad V_1 = \left(1 + \frac{m_2}{m_1}\right)\sqrt{2gh}$$

Illustration :

A light spring of constant k is kept compressed between two blocks of masses m & M on a smooth horizontal surface. When released, the blocks acquire velocities in opposite directions. The spring looses contact with the blocks when it acquires natural length. If the spring was initially compressed through distance x, find the speed of two blocks.

Sol. *As net external force acting on the system is zero (net $F_{ext} = 0$)*

$\therefore$ *Applying conservation of momentum in horizontal direction*

$$P_f = P_i$$

$$MV_2 - mv_1 = 0 \qquad ....(i)$$

applying conservation of mechanical energy (as no work is done)

$$\frac{1}{2}kx^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}MV_2^2 \qquad .....(ii)$$

$$v_1 = \sqrt{\frac{MK}{m(m+M)}} \;;\; v_2 = \sqrt{\frac{mK}{M(m+M)}}$$

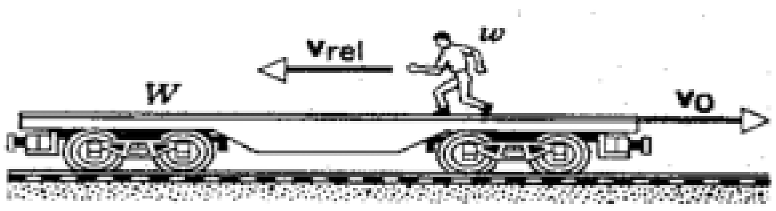
Illustration :

A railroad flatcar of mass M can roll without friction along a straight horizontal track. Initially, a man of mass 'm' is standing on the car, which is at rest. What is the velocity of the car if the man runs to the left so that his speed relative to the car is v_{rel}.

Sol. *Let velocity of man w.r.t. ground be V_m and the velocity attained by car be v_0 in backward direction.*

$V_{rel} = v_m - (-v_0)$ *(considering left to be positive)*

$v_m = v_{rel} - v_0$



Applying conservation of momentum in x-direction,

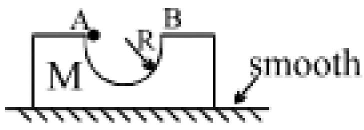
$$mv_m + M(-v_0) = 0$$

$$m(v_{rel} - v_0) - Mv_0 = 0$$

$$\therefore \qquad v_0 = \frac{mv_{rel}}{m+M}$$

Illustration :

In the figure shown the wedge of mass M has a semicircular groove. A particle of mass $m = \frac{M}{2}$ *is released from A. It slides on the smooth circular track and starts climbing on the right face.*



Find the maximum velocity of wedge during process of motion.

Sol. *Maximum velocity of wedge will be when the ball is at the lowest point in the wedge as till this point the horizontal component of normal on the wedge will be speeding the wedge but after this it will be opposite to the direction of motion of wedge, thereby slowing it down.*

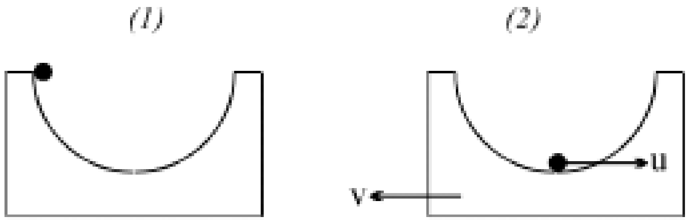
Applying conservation of linear momentum in horizontal direction at positions 1 & 2.

Initial momentum, $p_i = 0$

Fianl momentum, $p_f = -Mv + mu$

$$p_i = p_f$$

$$u = \frac{Mv}{m} = 2v$$



Applying conservation of mechanical energy

$$U_i + K_i = U_f + K_f$$

$$mgR + 0 = 0 + \frac{1}{2}mu^2 + \frac{1}{2}Mv^2$$

$$2mgR = m(2v)^2 + Mv^2$$

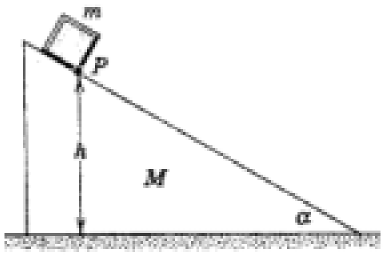
$$2 \times \frac{M}{2} \times gR = 4\frac{M}{2}v^2 + Mv^2$$

$$MgR = 3Mv^2$$

$$v = \sqrt{\frac{gR}{3}}$$

Illustration :

A block of mass m rests on a wedge of mass M which, in turn, rests on a horizontal table, as shown in figure. All surfaces are frictionless. If the system starts at rest with point P of the block a distance h above the table, find the speed of the wedge the instant point P touches the table.



Sol. As net external force in horizontal direction is zero, (F_{ext}) horizontal = 0

applying momentum conservation in horizontal direction,

$$mu_x = Mv \qquad \text{.........(i)}$$

from constraint we obtain(refere figure - (2)),

$$u_y = (u_x + v)\tan\alpha \qquad \text{.........(ii)}$$

applying conservation of mechanical energy,

$$mgh = \frac{1}{2}m(u_x^2 + u_y^2) + \frac{1}{2}Mv^2 \qquad \text{.........(iii)}$$

Solving equation (i), (ii) and (iii), we get

$$v = \sqrt{\frac{2mgh}{\frac{M^2}{m} + \frac{M}{2} + \frac{\tan^2\alpha(m+M)^2}{m}}}$$

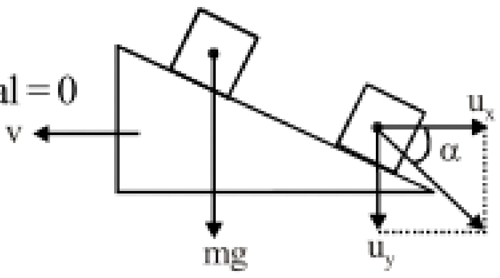
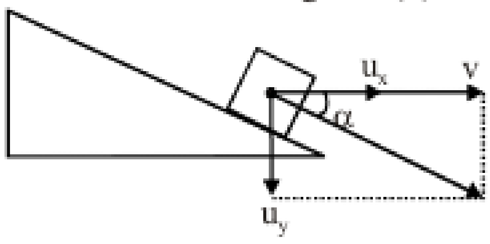


figure - (1)

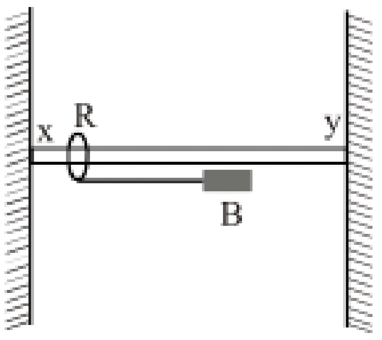


motion of block w.r.t. wedge

figure - (2)

Practice Exercise

Q.1 The ring R in the arrangement shown can slide along a smooth fixed, horizontal rod XY. It is attached to the block B by a light string. The block is released from rest, with the string horizontal. Then which of the following are true.



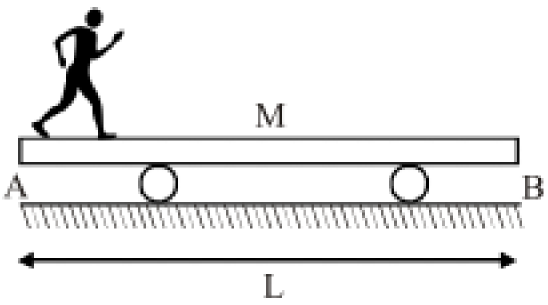
(a) One point in the string will have only vertical motion

(b) R and B will always have momentum of same magnitude

(c) When the string becomes vertical, the speeds of R and B will be inversely proportional to their masses

(d) R will lose contact with the rod at some point

Q.2 The figure shows a man of mass m standing at the end A to a trolley of mass M placed at rest on a smooth horizontal surface. The man starts moving towards the end B with a velocity u_{rel} with respect to the trolley. The length of the trolley is L.



(a) Find the time taken by the man to reach the other end.

(b) As the man walk on the trolley, find the velocity centre of mass of the system (man + trolley).

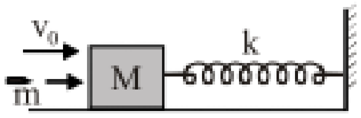
(c) When the man reaches the end B, find the distance moves by the trolley with respect to ground.

(d) Find the distance moved by the man with respect to ground.

Q.3 A man of mass 60 kg jumps from a trolley of mass 20 kg standing on smooth surface with absolute velocity 3 m/s. Find the velocity of trolley and total energy produced by man.

Q.4 Three particles of mass 20g, 30 g, and40 g are initially moving along the positive direction of the three coordinate axes respectively with the same velocity of 20 cm/s, when due to their mutual interactional the first particle comes to rest, the second acquires a velocity $10\hat{i} + 20\hat{k}$. What is then the velocity of the third particle ?

Q.5 A bullet of mass m strikes a block of mass M connected to a light spring of stiffness k, with a speed v_0 and gets into it. Find the loss of K.E. of the bullet.



Answers

Q.1 (a), (c)

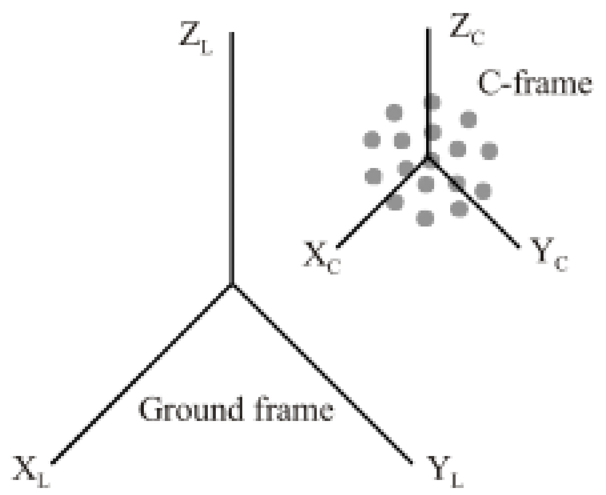
Q.2 (a) $\frac{L}{u_{rel}}$ (b) zero (c) $\frac{mL}{m+M}$ (d) $\frac{ML}{m+M}$

Q.3 9 m/s, 1.08 kJ Q.4 $2.5\hat{i}+15\hat{j}+5\hat{k}$ Q.5 $\frac{Mmv_0^2}{2(M+m)}$

C-Frame :

The total momentum of a system of particles in the C-frame of reference is always zero. We can attach a frame of reference to the center of mass of a system, this is called the center of mass or C-frame of reference (figure). Relative to this frame, the center of mass is at rest ($v_{com}=0$) and according to equation $P = M\vec{v}_{com}$, the total momentum of a system of particles in the C-frame of reference is always zero.

$$\vec{P} = \sum_i \vec{P}_i = 0 \text{ in the C-frame of reference}$$



C-frame of reference

The C-frame is important because many situations can be more simply analyzed in this frame. It is clear that the C-frame moves with a velocity v_{com} relative to the ground frame. When no external forces act on a system, the C-frame becomes an inertial frame.

Illustration :

The velocities of two particles of masses m_1 and m_2 relative to an observer in an inertial frame are v_1 and v_2. Determine the velocity of the center of mass relative to the observer and the velocity of each particle relative to the center of mass.

Sol. *From definition*

$$v_{com} = \frac{dr_{com}}{dt} = \frac{1}{M}\sum_i m_i \frac{dr_i}{dt} = \frac{\sum_i m_i v_i}{M}$$

The velocity of the center of mass relative to the observer is

$$v_{com} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

The velocity of each particle relative to the center of mass (figure) using the relative motion equations for velocities is

$$v_{1c}' = v_1 - v_{com} = v_1 - \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

Answers

Q.1 (a), (c)

Q.2 (a) $\frac{L}{u_{rel}}$ (b) zero (c) $\frac{mL}{m+M}$ (d) $\frac{ML}{m+M}$

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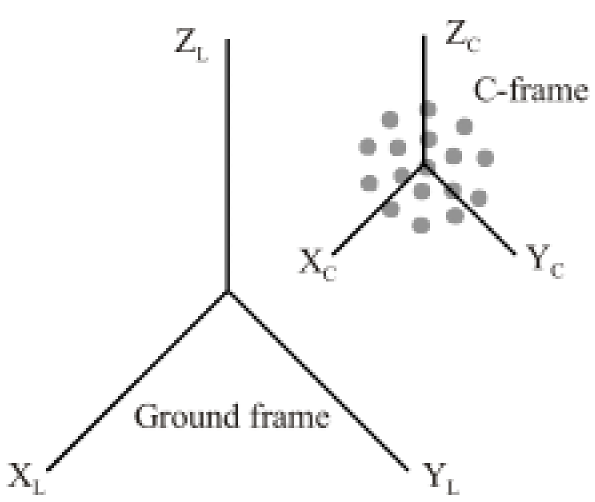
$$\vec{P}=\sum_i \vec{P}_i = 0 \text{ in the C-frame of reference}$$



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Sol. *From definition*

$$v_{com}=\frac{dr_{com}}{dt}=\frac{1}{M}\sum_i m_i\frac{dr_i}{dt}=\frac{\sum_i m_i v_i}{M}$$

The velocity of the center of mass relative to the observer is

$$v_{com}=\frac{m_1v_1+m_2v_2}{m_1+m_2}$$

The velocity of each particle relative to the center of mass (figure) using the relative motion equations for velocities is

$$v_{1c}'=v_1-v_{com}=v_1-\frac{m_1v_1+m_2v_2}{m_1+m_2}$$

$$= \frac{m_2(v_1 - v_2)}{m_1 + m_2} = \frac{m_2 v_{12}}{m_1 + m_2}$$

$$v_{2c}' = v_2 - v_{com} = \frac{m_1(v_2 + v_1)}{m_1 + m_2} = -\frac{m_1 v_{12}}{m_1 + m_2}$$

where $v_{12} = v_1 - v_2$ is the relative velocity of the two particles.
Thus, in the C-frame, the two particles appear to be moving in opposite directions with velocities inversely proportional to their masses.

Also relative to the center of mass, the two particles move with equal but opposite momentum since

$$p_1' = m_1 v_{1c}' = \frac{m_1 m_2 v_{12}}{(m_1 + m_2)} = p_2'$$

The expressions for two particle problems are much simpler when they are related to the C-frame of reference.

Kinetic energy of system of particles

Let us find relation between kinetic energy of a system from ground frame and C-frame. We have a system consisting of many particles, let's say speed of the ith particle is v_i. Then kinetic energy of system, K, in ground frame will be summation of individual kinetic energies.

$$K_{sys} = \Sigma\left(\frac{1}{2} m_i v_i^2\right)$$

now $\quad v_i = v_{i/c} + v_c$

where v_i is velocity of the ith particle in ground frame, $v_{i/c}$ is velocity of the ith particle in reference to frame attached to the center of mass and v_c is velocity of center of mass in ground frame.

$$K_{sys} = \frac{1}{2}\Sigma m_i \left(\vec{v}_{i/c} + \vec{v}_c\right)^2$$

$$K_{sys} = \frac{1}{2}\Sigma m_i v_{i/c}^{\ 2} + \frac{1}{2}\Sigma m_i \vec{v}_c^{\,2} + \frac{1}{2} \times 2\left(\Sigma m_i \cdot \vec{v}_{i/c} \cdot \vec{v}_c\right)$$

$$K_{sys} = \frac{1}{2}\Sigma m_i v_{i/c}^{\ 2} + \frac{1}{2}(\Sigma m_i)\vec{v}_c^{\,2} + \vec{v}_c^{\,2}\left(\Sigma m_i \cdot \vec{v}_{i/c}\right)\vec{v}_c$$

We can take v_c out of summation in second and third term as it is constant. Now third term becomes zero, as $\Sigma m_i \vec{v}_{i/c} = M\vec{v}_{c/c} = 0$ (total momentum of a system of particles in the C-frame of reference is always zero.)

$v_{c/c}$ is velocity of center of mass in frame of com. Which is zero. Also it represents momentum of system in C-frame which is zero.

$$\left(\frac{1}{2}\Sigma m_i \vec{v}_{i/c}^{\,2}\right) = K_{sys/c}$$

Thus, we get $\quad K_{sys} = K_{sys/c} + \frac{1}{2} m v_c^2$

Where $K_{sys/c}$ means kinetic energy of system in C-frame. This important conclusion will be useful again in rotational dynamics; we can do little manipulation to write the equation as

$$K_{sys} = K_{sys/c} + \frac{p_c^2}{2M}$$

A system of two Particles

Suppose the masses of the particles are equal to m_1 and m_2 and their velocities in the K reference frame be $\vec{v}_1$ and $\vec{v}_2$. respectively. Let us find the expressions defining their moment and the total kinetic energy in the C-frame.

The momentum of the first particle in the C-frame is

$$\vec{P}_{1/c} = m\vec{v}_{1/c} = m_1(\vec{v}_1 - \vec{v}_c)$$

Where v_c is the velocity of the center of mass of the system in the ground frame. Substituting in this formula expression.

$$\vec{v}_c = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$$

$$\vec{P}_{1/c} = \mu(\vec{v}_1 - \vec{v}_2)$$

Where μ is the reduced mass of the system, given by

$$m = \frac{m_1 m_2}{m_1 + m_2}$$

Similarly, the momentum of the second particle in the C-frame is

$$\vec{P}_{2/c} = \mu(\vec{v}_2 - \vec{v}_1)$$

Thus, the momenta of the two particles in the C-frame are equal in magnitude and opposite in direction; the modulus of the momentum of each particle is

$$\vec{P}_{1/c} = \mu v_{rel}$$

Where $v_{rel} = |\vec{v}_1 - \vec{v}_2|$ is the velocity of one particle relative to another.

Finally, let us consider kinetic energy. The total kinetic energy of the two particles in the C-frame is

$$K_{sys/c} = K_1 + K_2 = \frac{\vec{P}^2}{2m_2} + \frac{\vec{P}^2}{2m_2}$$

we know $\quad \mu = \frac{m_1 m_2}{m_1 + m_2} \quad$ or $\quad \frac{1}{m_2} + \frac{1}{m_2} = \frac{1}{\mu}$

Then $\quad K_{sys/c} = \frac{\vec{P}^2}{2\mu} = \frac{\mu v_{rel}^2}{2}$

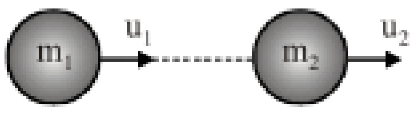
$K_{sys} = K_{sys/c} + K_c$, we get

$$K_{sys} = \frac{\mu v_{rel}^2}{2} + \frac{m v_c^2}{2} \qquad \text{(where } m = m_1 + m_2\text{)}$$

Read this Illustration after collision

Illustration :

Two particles of mass m_1, m_2 moving with initial velocity u_1 and u_2 collide head-on. Find minimum kinetic energy that system has during collision. Thus. Prove that maximum kinetic energy is lost in perfectly inelastic collision



Sol. *Particles moving with velocity u_1 and u_2 in the same direction.*
In C-frame initial kinetic energy of system is

$$K_{sys} = K_{sys/c} + K_c, \text{ we get}$$

$$K_{sys} = \frac{\mu v_{rel}^2}{2} + \frac{m v_c^2}{2}$$

$$\frac{1}{2}\mu(u_2 - u_1)^2 + \frac{m v_c^2}{2}$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

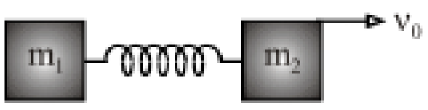
During collision, at the instant of maximum deformation. we get minimum kinetic energy in C-frame as particles attain same. velocity, thus relative velocity becomes zero.
When an isolated system has minimum kinetic energy in C-frame, it will also have minimum kinetic energy in ground frame, as velocity of center of mass is constant.
Thus, minimum kinetic energy during collision is

$$\frac{1}{2}(m_1 + m_2)v_c^2$$

Where $$v_c = \frac{(m_1 u_1 + m_2 u_2)}{m_1 + m_2}$$

In perfectly inelastic collision, since both the particles move together, the relative velocity becomes zero. Thus, final kinetic energy is $\frac{1}{2}(m_1 + m_2)v_c^2 (m_s = m_1 + m_2)$, as velocity of center of mass is constant. This is the minimum possible kinetic energy that a system will have because in all other case there will be one more term adding in the kinetic energy of system because of particles having relative velocity.

Two block of mass m_1 and m_2 connected by an ideal spring of spring constant k are kept on a smooth horizontal surface. Find maximum extension of the spring when the block m_2 is given an initial velocity of v_o toward right as shown in figure.



Blocks of masses m_1 and m_2
connected by an ideal spring

When a block of mass m_2 is given an initial velocity of v_0 toward right, the spring extends and pulls the block toward left and the same extended spring will pull the block m1 toward right. Initially the force acting on m_2 will reduce its speed and the force acting on m_1 will increase its speed. Thus, we can see that initially the extension be increasing.

If we consider the two blocks and spring as one system, then total mechanical energy must be conserved as there is no dissipative force present. Also, momentum will be conserved as there is no external force present.

Now there will be an instant when the block will have same velocity, that is, velocity of m_1 has increased sufficiently to become equal to the velocity of m_2 which has been decreasing continuously. At this moment. The spring will have the maximum extension x_{max}, as till this point distance between the blocks was continuously increasing because m_2 had larger velocity. Now it will start decreasing as m_1 will be moving faster than m_2 and it will reduce the distance between the two blocks. Thus when $v_1 = v_2$, extension is maximum. This can also be understood alternatively by looking at m_1 from reference frame attached to m_2. To an observer sitting on m_1 the block m_2 will be closest or farthest when it is relative at rest.

Since no external force is present, velocity of canter of mass is given as

$$v_{com} = \frac{(m_2 v_0)}{m_1 + m_2}$$

From the reference frame of center of mass, the initial kinetic energy is given by

$$K = \frac{1}{2}\mu(v_{2f} - v_{1f})^2 = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}(u)^2$$

From the reference frame of center of mass, the final kinetic energy is given by

$$K = \frac{1}{2}\mu(v_{2f} - v_{1f})^2 = 0$$

Thus, equating initial and final energies in C-frame, we get

$$\frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}(v_0)^2 = 0 + \frac{1}{2}kx_{max}^2$$

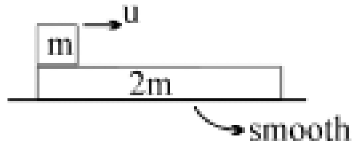
Thus, maximum extension

$$x_{max} = v_0\sqrt{\frac{m_1 m_2}{k(m_1 + m_2)}}$$

This problem can be thought exactly as the opposite of the previous Illustration as here the maximum extension is occurring the relative velocity is zero.

Illustration:

Find total work done by friction assuming plank is sufficiently long.



Solve this question in both ground frame and C-frame.

Sol. ***In ground frame*** *:*

let v be the final velocity of block & plank when relative motion ceases between block and plank; applying conservation of linear momentum, mu = (2m + m)v

$\therefore \quad v = u/3$

w_f = change in K.E. = $K.E._f - K.E._i$

$$w_f = \frac{1}{2}3m\left(\frac{u}{3}\right)^2 - \frac{1}{2}mu^2 = -\frac{mu^2}{3}$$

In C-frame :

Considering block and plank as a system

Work done by friction is change in kinetic energy

w_f = change in K.E. = $K.E._f - K.E._i$

$$K.E._i = \frac{1}{2}\mu(u-0)^2 + \frac{(2m+m)v_c^2}{2} \qquad \left(\mu = \frac{2m \times m}{2m+m}\right)$$

$$K.E._f = 0 + \frac{(2m+m)v_c^2}{2}$$ (v_c is constant as external force is zero)

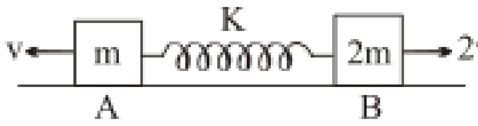
$$w_f = K.E._f - K.E._i = -\frac{mu^2}{3}$$

Note: *P_{sys} = conserved if $f_{ext.} = 0$ although internal friction are doing work.*

Illustration :

. *Two blocks A and B of masses m & 2m placed on smooth horizontal surface are connected with a light spring. The two blocks are given velocities as shown when spring is at natural length.*

(i) Find velocity of centre of mass (b) maximum extension in the spring



Sol. (a) $V_{CM} = \frac{2m \times 2v - m_x v}{3m} = v$

(b) There will be maximum extension in the spring when $v_{rel} = 0$

$\therefore$ applying conservation of mechanical energy,

$$\frac{1}{2}MV_{CM}^2 + 0 + \frac{1}{2}kx^2 = \frac{1}{2}MV_{CM}^2 + \frac{1}{2}\mu v_{rel}^2$$

$$x = \sqrt{\frac{\mu}{k}}v_{rel} = \sqrt{\frac{2m}{3k}} \times 3v = \sqrt{\frac{6m}{k}}v$$

Impulse

The impulse of the net force acting on a particle during a given time interval is equal to the change in momentum of the particle during that interval. The Impulse is a vector quantity.

For any arbitrary force. The impulse $\vec{J}$ is defined as

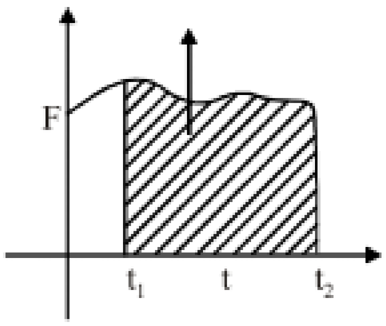
$$\vec{J} = \int_{t_1}^{t_2} \vec{F}_{ext}\,dt$$

$$\vec{F}_{ext}\,dt = d\vec{P}_{sys}$$

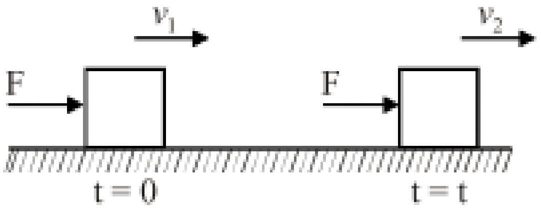
$$\vec{J} = \int_{P_i}^{P_f} \overrightarrow{dP}_{sys}$$

$$\vec{J} = \vec{P}_f - \vec{P}_i$$

ared under the curve is impulse



The concept of impulse can be better explained by an example shown in figure



A is a block of mass m moving with a velocity v_1, at time t = 0, a constant force F is applied on it in the direction of velocity for a time t. Due to this force the velocity of the body increases hence momentum increase. If after time t the velocity of the body becomes v_2, then according to momentum conservation we have

Initial momentum + momentum imparted = final momentum

$$mv_1 + F\,t = mv_2 \qquad \text{.......(i)}$$

If applied force is opposite to the direction of v_1 then we'll have

$$mv_1 - F\,t = mv_2 \qquad \text{.......(ii)}$$

Equation (i) and (ii) are similar to the equations written for work - energy theorem as work done by the system or on the system are subtracted or added to the initial kinetic energy, gives the final kinetic energy of the system. Similar to that in initial momentum impulse due the forces acting on the system are added or subtracted, gives the final momentum of the system. If force is in the direction of the initial velocity of the particle, impulse is added to the initial momentum and if it is against the velocity, impulse is subtracted from the initial momentum

Impulsive Force

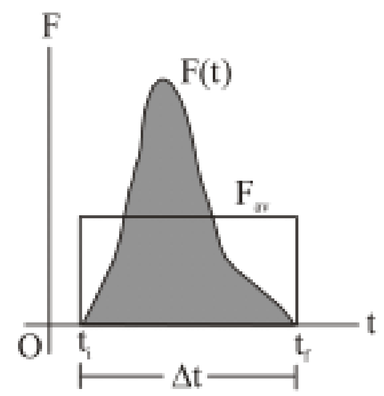
When a force of high magnitude acts for a time that is short compared with the time of observation of the system, it is referred as an impulsive force. An impulsive force can change the momentum of a body by a finite magnitude in a very short time interval.

An impulsive force can only be balanced by another impulsive force.

We defined the impulse in terms of a single force, but the impulse-momentum theorem deals with the change in momentum due to the impulse of the net force –the is, the combined effect of all the forces that act on the particle. In the case of a collision involving two particles, there is often no distinction because each particle is acted upon by only one force, which is due to the other particle. In this case, the change in momentum of one particle is equal to the impulse of the force exerted by the other particle.

Average Force

The impulsive force whose magnitude is represented in figure is assumed to have a constant direction. The magnitude of the impulse of this force is represented by the area under the F(t) curve. We can represent that same area by the rectangle in figure of width Δt and height F_{av} where F_{av} is the magnitude of the average force that acts during the interval Δt. Thus



$$J = F_{av}\, \Delta t.$$

Impulsive force is a relative term. There is no clear differentiation between an impulsive and non-impulsive force.

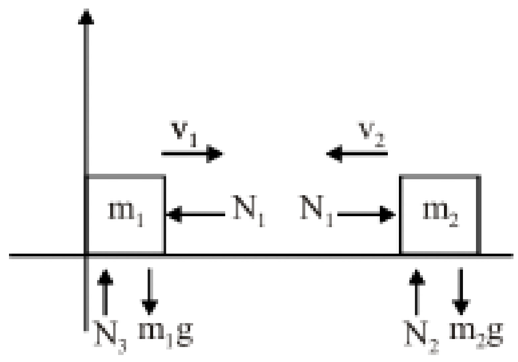
1. Gravitational force and spring force are always non-impulsive.
2. Normal, tension and friction are case dependent.

1. Impulsive Normal :

In case of collision, normal forces at the surface of collision are always impulsive.

eg. (i) N_1 is Impulsive ;

Normal reaction due to ground N_2 & N_3 are Non-impulsive

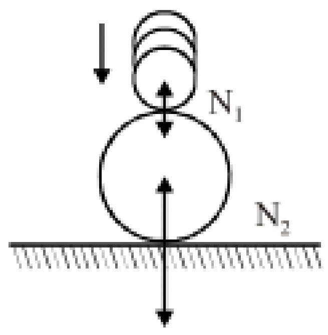


(a)

(ii) Consider a ball dropped on a large ball.

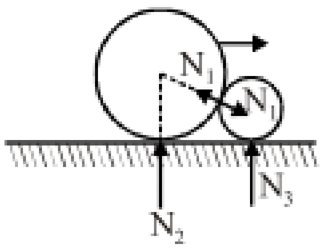
Both normal forces N_1 and N_2 are impulsive

N_2 is impulsive, as it balance N_1 for the large ball.



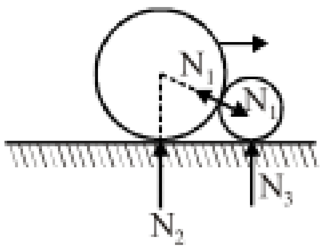
(a)

(iii) Consider a large ball colliding with small ball N_1, N_3 are impulsive ; N_2 is non-impulsive
N_1 can be easily seen to be impulsive, as it is the normal force during collision here N_3 balance a component of N_1, therefore it is also impulsive.

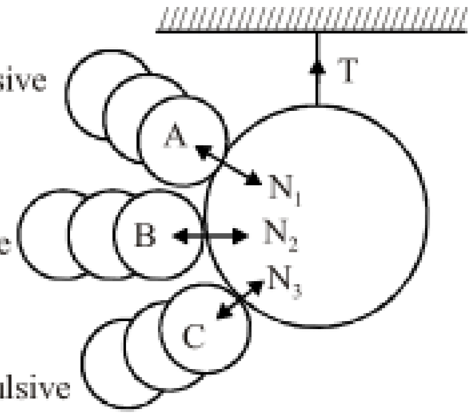


(b)

2. **Impulsive Friction :** If the normal between the two objects is impulsive, then the friction between the two will also be impulsive.



3. **Impulsive Tensions In a string :** When a string is jerked out equal and opposite tension acts suddenly at each end and impulses act on the bodies attached with the string in the direction of the string.



All normal are impulsive but tension
T is impulsive only for the ball A

One end of the string is fixed : The impulsive force which acts at the fixed end of the string cannot change the momentum of the fixed object attached at the other end. The object attached to the free end however will undergo a change in momentum in the direction of the string. The momentum remains unchanged in a direction perpendicular to the string. In this direction string cannot exert impulsive forces.
Both ends of the string attached to movable objects : In this case equal and opposite impulses act on the two object, producing equal and opposite changes in momentum. The total momentum of the system therefore remains constant, although the momentum of each individual object is changed in the direction of the string. Perpendicular to the string however, no impulse acts and the momentum of each particle in this direction is unchanged.

In case of rod : Tension is always impulsive.

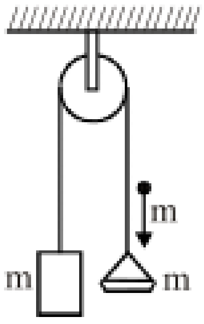
In case of spring : Tensions always non-impulsive.

Illustration :

A block of mass m and a pan of equal mass are connected by a string going over a smooth light pulley. Initially the system is at rest when a particle of mass m falls on the pan and sticks to it. If the particle strikes the pan with a speed u, find the speed with which the system moves just after the collision.

Sol. *Let the required speed be V.*

As there is sudden change in the speed of the block, the tension must change by a large amount during the collision.



Let N = magnitude of the contact force between the particle and the pan

T = tension in the string

Consider the impulse imparted to the particle. The force is N is in upward direction and the impulse is $\int N \, dt$. *This should be equal to the change in its momentum.*

Thus, $\int N \, dt = mu - mV.$ ...(i)

Similarly considering the impulse imparted to the pan,

$$\int (N - T)\,dt = mV \quad ....(ii)$$

and that to the block,

$$\int T\,dt = mV \quad ....(iii)$$

Adding (ii) and (iii),

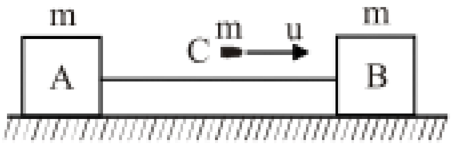
$$\int N\,dt = 2mV$$

comparing with (i)

$$mu - mV = 2\,mV \qquad or \qquad V = u/3.$$

Illustration :

Two identical block A and B connected by a massless string are placed on a frictionless horizontal plane. A bullet having same mass, moving with speed u strikes block B from behind as shown. If the bullet gets embedded into the block B then find :



(a) The velocity of A,B,C after collision.

(b) Impulse on A due to tension in the string

(c) Impulse on C due to normal force of collision.

(d) Impulse on B due to normal force of collision.

Sol. (a) After collision, the blocks & the bullet will move together with same velocity, say v
By conservation of linear momentum $mu = 3mv$

$$v = \frac{u}{3}$$

(b) Net impulse on A is due to tension force;
Impulse on $A = P_f - P_i$

$$\int T\,dt = \frac{mu}{3} - 0$$

(c) On the bullet C, net impulse is due to N

$$-\int N\,dt = P_f - P_i$$

$$= \frac{mu}{3} - mu$$

$$= \frac{-2mu}{3}$$

(d) On B two impulsive forces act i.e. Normal & Tension.

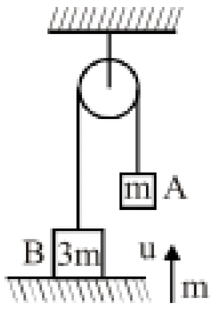
$$\vec{J} = \int (N - T)\,dt = \int N\,dt - \int T\,dt = \frac{mu}{3}$$

$$\Rightarrow \quad \int N\,dt = \frac{2mu}{3}$$

The impulse due to normal force on both the colliding bodies is equal. Thus we can directly say impulse on B due to normal is same as impulse on C due to normal.

Illustration :

Two blocks of masses m and 3m are connected by an inextensible string and the string passes over a fixed pulley which is massless and frictionless. A bullet of mass m moving with a velocity 'u' hits the hanging block of mass 'm' and gets embedded in it. Find the height through which block A rises after the collision.



Sol As soon as the collision occurs, the string becomes slack, tension becomes zero. Gravitational force is acting vertically downwards. But as gravitational force is a weaker force than impulse force, therefore a possible change of momentum due to gravitational force during collision can be neglected.

Hence, conserving momentum of the system during collision along vertical:

If v = velocity of the combined mass block A & bullet just after collision, then

$$mu = 2mv \quad \text{or} \quad v = \frac{u}{2}$$

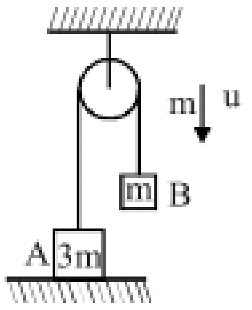
Block A & bullet start moving with velocity $\frac{u}{2}$

Now, maximum height through which the combined mass rises,

$$h_{max} = \frac{v^2}{2g} = \frac{u^2}{8g}$$

Illustration :

A system of two blocks A and B are connected by an inextensible massless strings as shown. The pulley in massless and frictionless. A bullet of mass 'm' moving with a velocity 'u' (as shown) hits the block 'B' and gets embedded into it. Find the impulse imparted by tension force to the system to block A



Sol. *Let velocities of B and A after collision have magnitude v.*
At the time of collision, tension is T and normal force between bullet and block B is F.
Impulse provided by tension = $\int T\ dt$
For bullet: considering downward direction to be positive

$$-\int F\ dt = mv - mu \quad \ldots (i)$$

For block B:

$$\int (F - T)\ dt = mv \quad \ldots (ii)$$

For block A

$$\int T\ dt = 3mv \quad \ldots (iii)$$

Adding (i), (ii) & (iii)

$$mu = 5\ mv$$

or $$v = \frac{u}{5}$$

Hence, impulse imparted by tension force, $\int T\ dt = 3mv$

$$= 3m\left(\frac{u}{5}\right) = \frac{3mu}{5}$$

Collisions

Introduction

Newton's laws are useful for solving a wide range of problems in dynamics. However, there is one class of problem in which, even though Newton's laws still apply as we have defined them, we may have insufficient knowledge of the forces to permit us to analyze the motion. These problems involve collisions between two or more objects

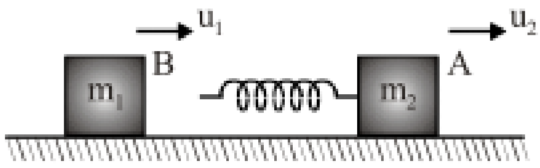
In this section we will learn how to analyze collisions between two objects. In doing so, we will find that we need a new dynamic variable apart from velocity, acceleration, force and energy, called linear momentum. We will see that the law of conservation of linear momentum, one of the fundamental conservation laws of physics, can be used to study the collisions of objects from the scale of subatomic particles to the scale of galaxies.

In a collision, two object exert forces on each other for an identifiable time interval, so that we can separate the motion into three parts : before, during and after the collision. Before and after the collision, we assume that the objects are far enough apart that they do not exert any force on each other. During the collision, the objects exert forces on each other; these forces are equal in magnitude and opposite in direction, according to Newton' third law. We assume that these forces are much larger than any forces exerted on the two objects by other objects in their environment. The motion of the objects (or at least one of them) changes rather abruptly during the collision, so that we can make a relatively clear separation of the situation before the collision from the situation after the collision.

When a bat strikes a baseball, for example, the bat is in contact with the ball for an interval that is quite short compared with the time during for which we are watching the ball. During the collision the bat exerts a large force on the ball. This force varies with time in a complex way that we can measure only with difficulty.

When an alpha particle (^{4}He nucleus) collides with another nucleus, the force exerted on each by the other may be the repulsive electrostatic force associated with the charges on the particles. The particles may not actually come into direct contact with each other, but we still may speak of this interaction as collision because a relatively strong force, acting for a time that is short compared with the time that the alpha particle is under observation, has a substantial effect on the motion of the alpha particle.

Collision



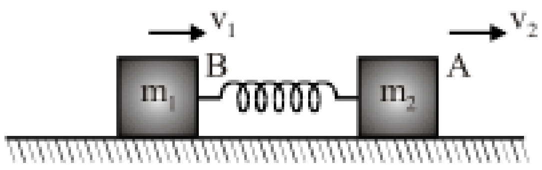
Consider the situation shown in figure . Two block of masses m_1 and m_2 are moving on the same straight line on a frictionless horizontal table. The block m_2, which is ahead of m_1 is going with a speed u_2 smaller than the speed u_1 of m_1 . A spring is attached to the rear end of m_2. Since $u_1 > u_2$, the block m_1 will touch the rear end the spring at some instant.

Since m_1 moves faster then m_2, the length of the spring will decrease. The spring will compress, it pushes back both the blocks with force kx. This force is in the direction of the velocity of m_2, hence m_2 will accelerate. However, this is opposite to the velocity of m_1 and so m_1 will decelerate. The velocity of the front block A (Which was slower initially will gradually increase, and the velocity of the rear block B (which was faster initially) will gradually decrease. The spring will continue to become more and more compressed as long as the rear block B is faster than the front block A. There will be an instant when the

two blocks will have equal velocities.
This corresponds to the maximum compression of the spring. Thus ***"the spring compression is maximum when the two blocks attain equal velocities"***
Now, the spring being already compressed, continues to push back the two blocks. Thus, the front block A will still be accelerated and the rear block B will still be decelerated. At the instant of maximum compression velocities were equal and hence after this the front block will move faster than the rear block. And so do the ends of the spring as they are in contact with the blocks. The spring will thus increase its length. This process will continue till the spring acquires its natural length. Once the spring regains its natural length, it stops exerting any force on the blocks. As the two blocks are moving with different velocities by this time, the rear one slower, the rear block will leave contact with the spring and the blocks will move with constant velocities. Their separation will go on increasing.

During the whole process, the momentum of the two - blocks system remains constant.
This is because there is no resultant external force acting on the system. Note that the spring being massless, exerts equal and opposite forces on the blocks.
Next consider the energy of the system. As there is no friction anywhere, the sum of the kinetic energy and the elastic potential energy remains constant. The elastic potential energy is $\frac{1}{2}kx^2$ when the spring is compressed by x. If v_1 and v_2 are the speeds at this time, we have,

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}kx^2 = E$$



Compression in the spring is x

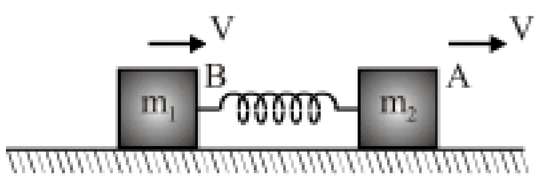
Where E is the total energy of the system.
Initially the spring is at its natural length so that,

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = E \qquad ....(i)$$

At the time when the compression of the spring is maximum.

$$v_1 = v_2 = v$$

$$\frac{1}{2}(m_1 + m_2)V^2 + \frac{1}{2}kx_{max}^2 = E$$



During maximum compression $v_1 = v_2 = V$

when the spring acquires its natural length, so that,

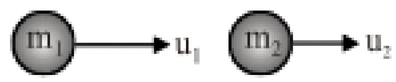
$$\frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2 = E \qquad ....(ii)$$

From (i) and (ii),

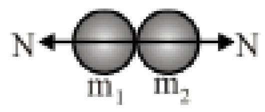
$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1u_1'^2 + \frac{1}{2}m_2u_2'^2$$

The kinetic energy before the collision is the same as the kinetic energy after the collision. However, we can not say that the kinetic energy remains constant because it changes as a function of time, during the process.

Now consider a very similar situation as shown figure. The two bodies of masses m_1 and m_2 are moving along a line with velocities u_1 and u_2 respectively, u_1 should be greater then u_2 for two bodies to collide.

Fig. I 

After some time the two bodies will come in contact as in the previous situation.

Fig. II 

As the velocity of m_1 (u_1) is greater than that of m_2 (u_2), there will be a deformation in the bodies (as we saw in the previous example of spring mass system). This deformation continues till they acquire same velocity. Now a question arises, that why will they acquire same velocity.

As shown figure II both m_1 and m_2 exert force on each other equal in magnitude (Newton's IIIrd Law) This force is in the direction of the velocity of m_2, hence m_2 will accelerate. However, this is opposite to the velocity of m_1 and so m_1 will decelerate. The velocity of the front body m_2 (which was slower initially) will gradually increase, and the velocity of the rear body m_1 (which was faster initially) will gradually decrease.

When they have attained a common velocity ; if bodies do not possess elastic properties, the two bodies will continue to move together and the process of collision ends here at maximum deformation. This is known as perfectly inelastic collision.

As net external force is zero therefore by conservation momentum, we can find the common velocity.

$$m_1u_1 + m_2u_2 = (m_1 + m_2)\,V$$

$$V = \frac{m_1u_1 + m_2u_2}{m_1 + m_2}$$

If bodies possess elastic property they will try to regain their shape that is the potential energy stored in deformation period will get converted into kinetic energy. If bodies are perfectly elastic then total potential energy will get converted into kinetic energy that is kinetic energy will be same before and after the collision, this type of collision in known as perfectly elastic collision.

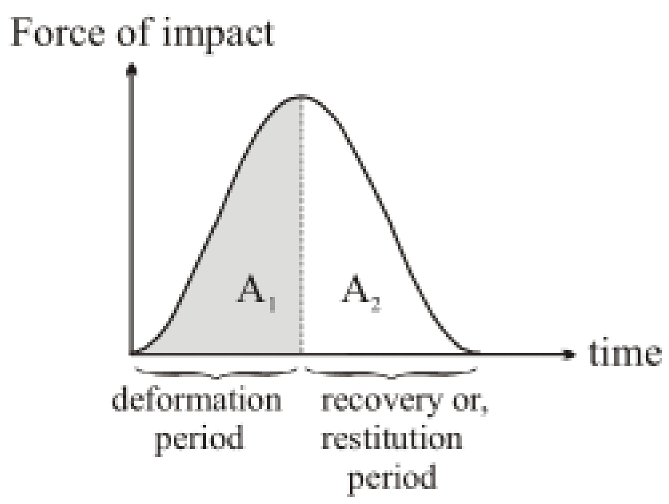
By conservation momentum.

$$m_1u_1 + m_2u_2 = m_1\,v_1 + m_2\,v_2$$

During the whole process, the momentum of the two -block system remains constant. Before and after collision kinetic energy may be same but not during the whole process.

Force of impact

The total time period of impact Δt is divided into period of deformation and period of restitution (recovery). Impulse of deformation and restitution (recovery) are finite and appreciable although the time interval of impact is extremely small.



Area A_1 represents impulse of deformation

Area A_2 represent impulse of reformation (restitution).

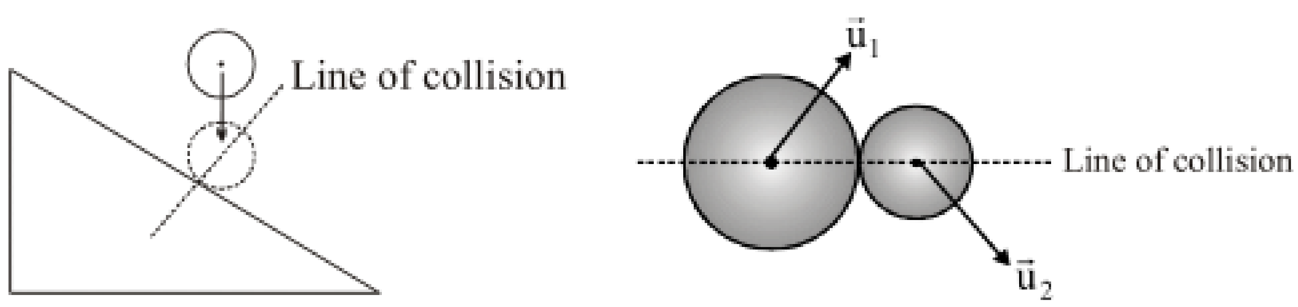
Deformation is maximum at the end of deformation period or at the start of recovery period.

In collision (impact) we consider the situation just before and just after collision. Just before collision is the moment when deformation period starts and just after collision is the moment when recovery period ends. At these moments one may say that force of impact becomes effectively zero.

Since time interval of impact is very small so impulse due to weight of body or weight dependent force is neglected during collision.

Line of collision (LOC)

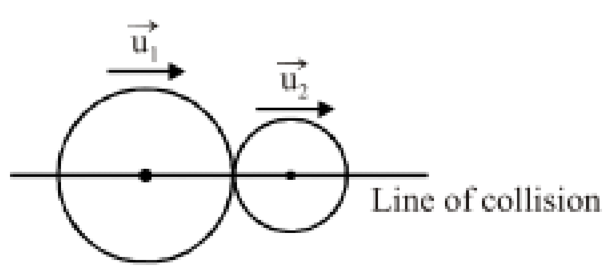
When two bodies collide, they exert force on each other through point of contact, perpendicular to the plane of contact. The direction of force of interaction is line of collision. Line of collision is independent of the direction of velocities of colliding bodies.



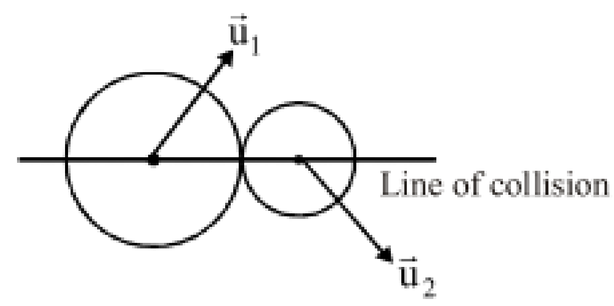
After collision, only the components of velocity along line of collision changes, the perpendicular components of velocity remain unaffected.

According to initial velocities and line of collision, collision is of two types :

(A) Head on	(B) Oblique
Line of collision is same as the line of motion (velocities) of the two colliding bodies.	When the lines of motion of two bodies are different, then the collision is oblique.



Head on collision



Oblique collision

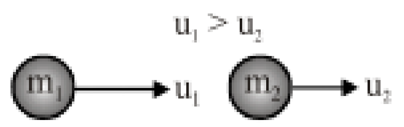
Laws of collision

(i) Conservation of momentum

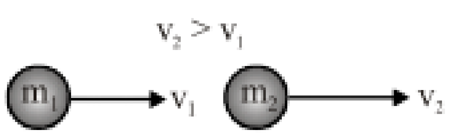
If no external impulsive forces act in a particular direction, the total momentum of system in that direction remains conserved.

Here we talk about external impulsive force because the force of impact is impulsive and changes the momentum of the individual bodies in a very short interval of time. The effect of non-impulsive forces such as gravity, spring force in such a small interval of time is negligible. Thus momentum remains conserved

$$m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{u}_1 + m_1\vec{u}_1$$



Just before collision

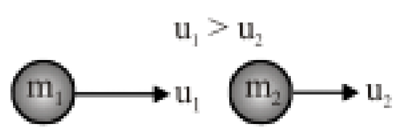


Just after collision

(ii) Newton's Experimental Law.

This is an experimental law which relates the velocity of approach of the bodies before collision and the velocity of separation after the impact.

When the collision is neither perfectly elastic nor perfectly inelastic, for such cases, Newton did certain experiments and gave another law which is now called as Newton's Experimental Law.



Just before collision

Just after collision

$$\frac{v_2 - v_1}{u_2 - u_1} = -e \text{ (coefficient of restitution) or } \frac{v_2 - v_1}{u_1 - u_2} = e$$

$v_2 - v_1$ is velocity of separation and $u_1 - u_2$ is velocity of approach.

Relative velocity of impact = e [Relative velocity of approach before impact]

v_1 and v_2 : Components of velocities of masses colliding, along the line of contact, after collision (with sign).

u_1 and u_2 : Components of velocities of colliding masses, along the line of contact, before collision (with sign)

Newton's Experimental Law can be stated **"velocity of separation is e times velocity of approach"**

Coefficient of Restitution is the property of colliding bodies which depends on their elastic behaviour.

Note : This Law is valid even when momentum is not conserved i.e. external impulsive forces act.

(iii) Type of collision depends on e

For e = 1, collision is perfectly elastic

for 0 < e < 1, collision is inelastic

for e = 0, collision is perfectly inelastic (Bodies will move together)

Illustration:

A ball of mass 4 kg moving with a velocity of 12 m/s impinges directly on another ball of mass 8 kg moving with a velocity of 4 m/s in the same direction. Find their velocities after impact if e = 0.5.

Sol.

Just before collision   Just after collision

$u_1 = 12\ m/s \quad m_1 = 4\ kg$

$u_2 = 4 m/s \quad m_2\ 8\ kg$

Let v_1 *and* v_2 *be the velocity after impact.*

By conservation of momentum :

$$m_1v_1 + m_2v_2 = m_1u_1 + m_1u_2$$

$$\Rightarrow \quad 4v_1 + 8v_2 = 80 \qquad \text{....(i)}$$

By Newton's experimental Law :

$$v_2 - v_1 = e\,(u_1 - u_2)$$

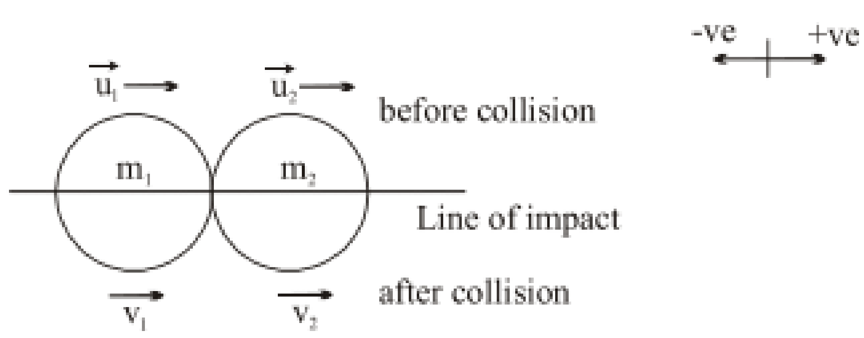
$$v_2 - v_1 = 0.5\,(12 - 4) = 4 \qquad \text{....(ii)}$$

Solving (i) and (ii), we get :

$$v_1 = 4\ m/s \text{ and } v_2 = 8\ m/s$$

Head-on collision

Two bodies m_1 & m_2 are moving with velocities $\vec{u}_1$ and $\vec{u}_2$ along the same line as shown.



1. By conservation of momentum

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \qquad \text{... (i)}$$

2. By Newton's experimental law

$$\frac{v_2 - v_1}{u_1 - u_2} = e \qquad \text{... (ii)}$$

From (i) & (ii)

$$v_1 = \frac{(m_1 - em_2)u_1}{m_1 + m_2} + \frac{(1+e)m_2 u_2}{m_1 + m_2}$$

$$v_2 = \frac{(1+e)m_1 u_1}{m_1 + m_2} + \frac{(m_2 - em_1)u_2}{m_1 + m_2}$$

for perfectly elastic collision velocities can be obtained by substituting e = 1.

$$v_1 = \frac{m_1 - m_2}{m_1 + m_2}u_1 + \frac{2m_2}{m_1 + m_2}u_2 \qquad \& \qquad v_2 = \frac{2m_1}{m_1 + m_2}u_1 + \frac{m_2 - m_1}{m_1 + m_2}u_2$$

Special cases:

Case 1: $m_1 = m_2$: In this case the velocities of particles are exchanged.

Just before collision Just after collision

Case 2: $m_1 >> m_2$: $v_1 \approx u_1$ & $v_2 = 2u_1 - u_2$
$e = 1$

Loss in kinetic energy

The loss in kinetic energy is equal to initial kinetic energy (k_i) minus final kinetic energy (k_f)

$$\Delta KE = k_i - k_f$$

$$k_i = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2$$

$$k_f = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

Substituting the values of v_1 & v_2 we obtain

$$\Delta KE = \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}(u_1 - u_2)^2(1 - e^2)$$

This result can also be obtained by using concept of C-frame.

For e = 1, i.e. perfectly elastic collision

$$\Delta KE = 0$$

As we discussed earlier there is no loss of kinetic energy

However, in elastic collision, KE is same before & after the collision, i.e. $\frac{1}{2}m_1u_1^2+\frac{1}{2}m_2u_2^2=\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2$

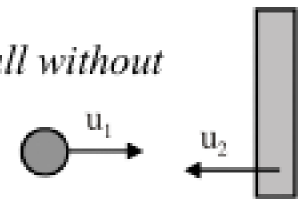
This equation is not independent, using Newton's experimental law & momentum conservation we can solve for final velocities more easily.
For $e=0$, i.e. perfectly in elastic collision

$$\Delta KE=\frac{1}{2}\frac{m_1m_2}{m_1+m_2}(u_1-u_2)^2$$

in this case, there is maximum loss in kinetic energy.

Illustration :

A particle of mass m moves with velocity $u_1 = 20$ m/s towards a wall that is moving with velocity $u_2 = 5$ m/s. If the particle collides with the wall without losing its energy, find the speed of the particle just after the collision.



Sol. *Velocity of approach*

$$u_{approch}=(u_1+u_2)$$

Let the velocity of the particle just after the collision be $\vec{v}_1$.

The velocity of separation,

$$v_{seperation}=(v_1-v_2)$$

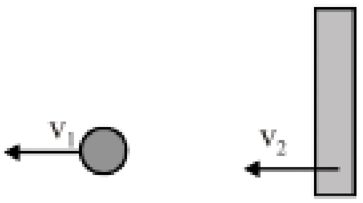
Since, according to Newton's experimental law

$$e=\frac{v_1-v_2}{u_1+u_2}$$

As there is no loss of energy, collision is elastic

Putting the value of e = 1, we obtain

$$\frac{v_1-v_2}{(u_1-u_2)}=1 \text{ or } u_1+u_2=v_1-v_2$$



or $v_1 = u_1 + u_2 + v_2$

Putting $u_1 = 20$ m/s, $u_2 = 5$ m/s and $v_2 = u_2$, since the wall being very heavy (infinite mass), moves with constant velocity, we obtain

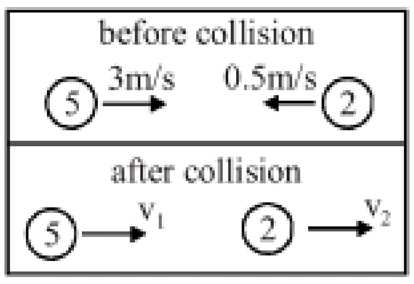
$$v_1 = v_0 + 2v = 20 + 2(5) = 30 \text{ m/sec.}$$

Solving (i) & (ii), we obtain $v_1 = \frac{12}{7}$ m/s, $v_2 = \frac{26}{7}$ m/s

Illustration :

A ball of mass 5 kg moving velocity 3 m/s impinges direction on another ball of mass 2 kg moving with velocity 0.5 m/s towards the first ball. Find the velocity after impact, if $e = \frac{4}{7}$

Sol. *By conservation of momentum*



$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

$$5\times 3 + 2\,(-0.5) = 5v_1 + 2v_2$$

$$14 = 5v_1 + 2v_2 \qquad \text{..........}(i)$$

By Newton's Law of collision $e = \dfrac{v_2 - v_1}{u_1 + u_2} = \dfrac{v_2 - v_1}{3-5}$

$$v_2 - v_1 = \frac{4}{7} \times 3.5 = 2 \text{ m/s} \qquad \text{..........}(ii)$$

Practice Exercise

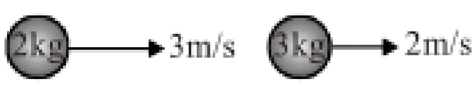
Q.1 Find the final velocities of the masses after in collision the given situations.

(i) 

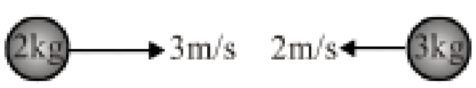
$e = 1$

(ii) 

$e = 5$

(iii) 

$e = 5$

(iv) 

$e = 3/5$

(v) 

$e = 1$

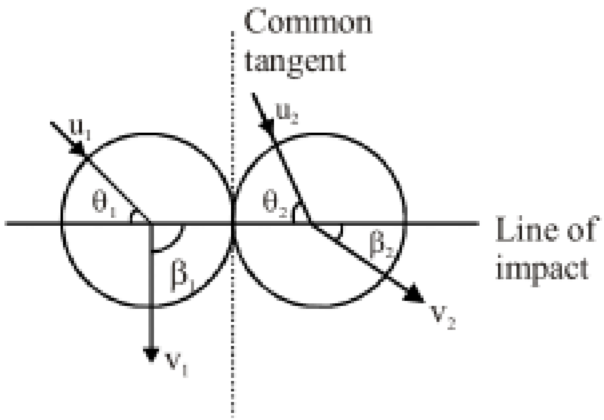
Answers

Q.1
(i) $v_1 = 0$, $v_2 = 3$ (Rightwards)
(ii) $v_1 = .3$ (Rightwards), v_2 1.8(Rightwards)
(iii) $v_1 = 2.1$ (Rightwards), $v_2 = 2.6$(Rightwards)
(iv) $v_1 = 1.8$ (Leftwards), $v_2 = 1.2$ (Rightwards)
(v) v_2 (Leftwards) , v_1 (Rightwards)

Oblique collision

In, oblique impact, the relative velocity of approach of the bodies doesn't coincide with the line of impact.

Conserving the momentum of the system along and perpendicular to the line of impact (due to absence of any other external impulsive force) we obtain



$$m_1 u_1 \cos\theta_1 + m_2 u_2 \cos\theta_2 = m_1 v_1 \cos\beta_1 + m_2 v_2 \cos\beta_2 \qquad \ldots(1)$$

and, $m_1 u_1 \sin\theta_1 + m2\ u2 \sin\theta_2 = m_1 v_1 \sin\beta_1 + m_2 v_2 \sin\beta_2$

Since no force is acting on m_1 and m_2 along the tangent, the individual momentum of m_1 and m_2 remains conserved.

$\Rightarrow \quad m_1 u_1 \sin\theta_1 = m_1 v_1 \sin\beta_1 \qquad \ldots(2)$

and $\quad m_2 u_2 \sin\theta_2 = m_2 v_2 \sin\beta_1 \qquad \ldots(3)$

Newton's experimental Law:
(We consider velocities along the line of collision)

$$e = \frac{v_2 \cos\beta_2 - v_1 \cos\beta_1}{u_1 \cos\theta_1 - u_2 \cos\theta_2} \qquad \ldots(4)$$

Now we have four equations and four unknown's v_1, v_2, β_1 and β_2. Solving four equations for four unknown we obtain.

$$v_1 \cos\beta_1 = \frac{(m_1 - em_2)u_1 \cos\theta_1 + m_2(1+e)u_2 \cos\theta_2}{m_1 + m_2} \qquad \ldots(5)$$

and $$v_2 \cos\beta_2 = \frac{m_1(1+e)\cos\theta_1 + (m_2 - em_2)u_2 \cos\theta_2}{m_1 + m_2} \qquad \ldots(6)$$

$\therefore \quad v_1 = \sqrt{(v_1 \sin\beta_1)^2 + (v_1 \cos\beta_1)^2}$

and $\quad \tan\beta_1 = \dfrac{v_1 \sin\beta_1}{v_1 \cos\beta_1}$

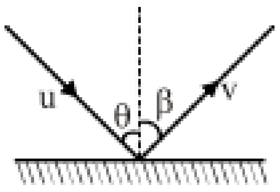
$\Rightarrow \quad \beta_1 = \tan^{-1}\left(\dfrac{v_1 \sin\beta_1}{v_1 \cos\beta_1}\right)$

[Put $v_1 \sin\beta_1$ from (2) and $v_1 \cos\beta_1$ from (5)]
Similarly, find v_2 and β_2

$$\text{Impulse} = \frac{m_1 m_2}{m_1 + m_2}(1+e)(u_1 \cos\theta_1 - u_2 \cos\theta_2)$$

$$\text{Energy loss} = \frac{m_1 m_2}{2(m_1 + m_2)}(1 - e^2)(u_1\cos\theta_1 - u_2\cos\theta_2)^2$$

Oblique collision on a Fixed Plane



Let a small ball collides with a smooth horizontal floor with a speed u at an angle θ to the vertical as shown in the figure. Just after the collision, let the ball leaves the floor with a speed v at an angle β to vertical.

It is quite clear that the line of action is perpendicular to the floor. Therefore, the impact takes place along the (normal) vertical. Now we can use Newton's experimental law as

$$e = \frac{\text{velocity of seperation}}{\text{velocity of approach}}$$

$\Rightarrow \quad e[\text{velocity of approach}] = \text{velocity of seperation}$

$\Rightarrow \quad e(u\cos\theta)(-\hat{j}) = -(v\cos\beta)(+\hat{j})$

or $\quad v\cos\beta = e\,u\cos\theta \qquad \ldots(i)$

Since impulsive force N acts on the body along the normal, we cannot conserve its momentum. Since along horizontal the component of N is zero, therefore we can conserve the horizontal momentum of the body.

Velocity

$\Rightarrow \quad (P_x)\text{ body} = \text{Constant}$

$\Rightarrow \quad (P_x)_{initial} = (P_x)_{final}$

$\Rightarrow \quad m\,u\sin\theta = m\,v\sin\beta$

$\Rightarrow \quad v\sin\beta = u\sin\theta \qquad \ldots(ii)$

Squaring equations (i) and (ii) and adding,

$$v^2\cos^2\beta + v^2\sin^2\beta = e^2u^2\cos^2\theta + u^2\sin^2\theta$$

$\Rightarrow \quad v^2 = u^2[e^2\cos^2\theta + \sin^2\theta]$

$\Rightarrow \quad v = u\sqrt{\sin^2\theta + e^2\cos^2\theta}$

Dividing equation (i) by (ii)

$$= \frac{v\cos\beta}{v\sin\beta} = \frac{eu\cos\theta}{u\sin\theta}$$

$\Rightarrow \quad \cot\beta = e\cot\theta$

$\Rightarrow \quad = \cot^{-1}(e\cot\theta).$

Impulse

Impulse of the blow = change of momentum of the body

$$= \{mv\sin\beta\,\hat{i} + (mv\cos\beta)\}\,\hat{j}\} - (mu\sin\theta\,\hat{i} - mu\cos\theta\,\hat{j})$$

$\Rightarrow \quad \text{Impulse} = m(v\sin\beta - u\sin\theta)\,\hat{i} + m(v\cos\beta + u\cos\theta)\,\hat{j}$

Since $v\sin\beta = u\sin\theta$

$\Rightarrow$ Impulse = m (v cosβ + u cos θ) $\hat{j}$

Putting v cosβ = eu cosθ from eq (i), we obtain

Impulse = m(1 + e)u cosθ $\hat{j}$

∴ Magnitude of the impulse = m(1 + e) u cosθ (in the direction of line of collision)

Change in kinetic energy

$$\Delta K.E. = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$

Putting the value of v we obtain

$$= \frac{1}{2}m\left[\left[u\sqrt{\sin^2\theta + e^2\sin^2\theta}\right]^2 - u^2\right]$$

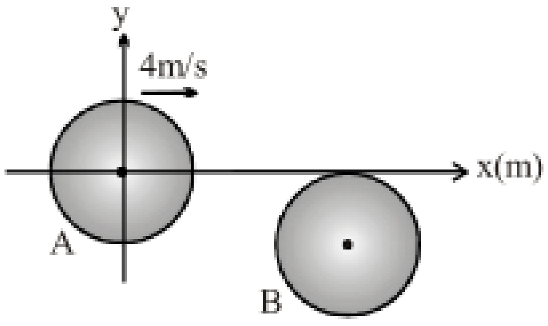
$$= \frac{1}{2}mu^2\,[\sin^2\theta + e^2\cos^2\theta - 1]$$

$$\Delta\, K.E. = \frac{1}{2}(1 - e^2)\,mu^2\cos^2\theta$$

Negative sign indicates the loss of K.E. i.e. final K.E. is less than initial K.E.

Illustration :

Two smooth balls A and B each of mass m and radius R, have their centre at (0, 0) and at (5R, – R) respectively, in a coordinate system as shown. Ball A, moving along positive x-axis, collides with ball B. Just before the collision, speed of ball A is 4m/s and ball B is stationary. The collision between the balls is elastic.

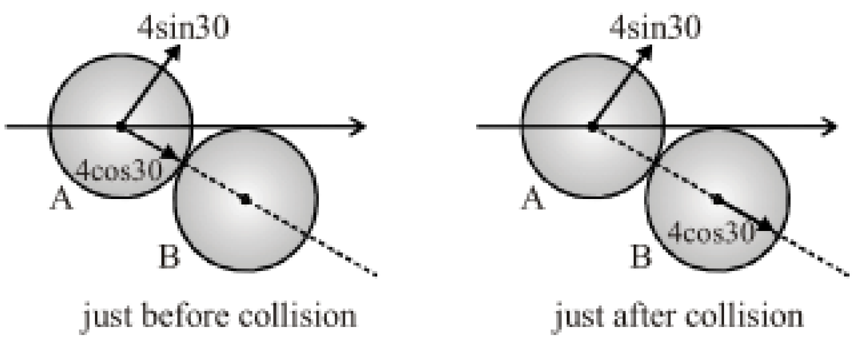


(a) *Find speed of the ball A just after the collision.*

(b) *Find impulse of the force exerted by A on B during the collision.*

Sol.

(a) As the collision is elastic & bodies are of equal mass, the velocity along the line of collision will interchange. As no force acts perpendicular to line of impact, therefore velocity of A perpendicular to the line of impact (4 sin 30) will remain same.

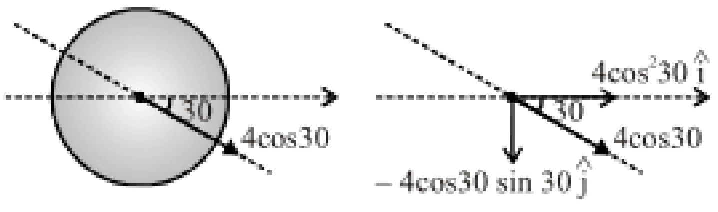


from the figure it is clear that velocity of A is 4 sin 30 and velocity of B is 4 cos 30 and both move at 90° to each other.

(b) $\vec{J}_{A\,on\,B} = m(\vec{v}_{Bf} - \vec{v}_{Bi})$

$m[4\cos 30^o(\cos 30^o\,\hat{i} - \sin 30^o\,\hat{j}] - 0]$

$(3m\hat{i} - \sqrt{3}m\,\hat{j})\ kg - m/s$



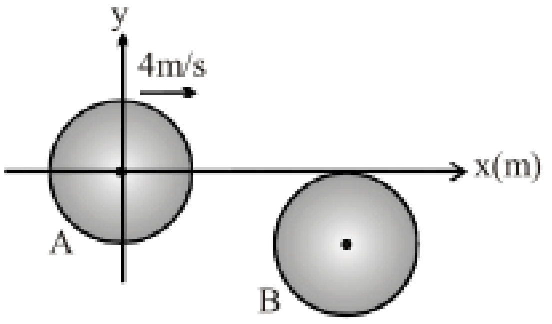
Final velocity of B (v_{bf})

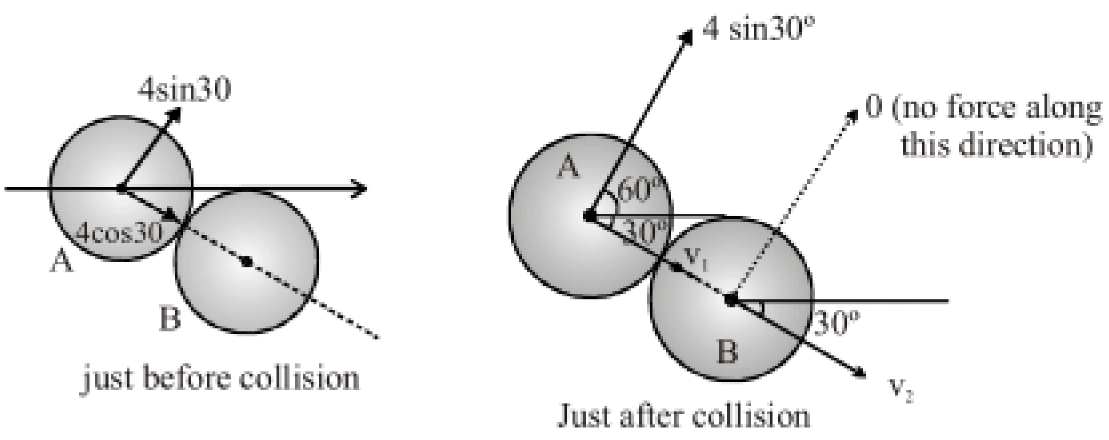
Impulse of collision is always along the line of collision.

Illustration :

In the previous illustration if coefficient of restitution during the collision is changed to 1/2, keeping all other parameters unchanged. What is the velocity of the ball B after the collision ?

Sol.





By Newton's experimental law

$$\frac{1}{2} = \frac{(v_2 - v_1)}{(4cos30^o)}$$

$$v_2 - v_1 = \sqrt{3} \qquad \text{........(i)}$$

By conservation of momentum along line of collision

$$m\frac{4\sqrt{3}}{2} = mv_1 + mv_2$$

$$v_1 + v_2 = 2\sqrt{3} \qquad \text{........(ii)}$$

From equation (i) & (ii),

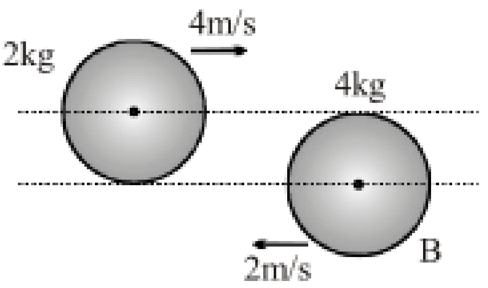
$$v_1 = \frac{\sqrt{3}}{2}\,m/s, \quad v_2 = \frac{3\sqrt{3}}{2}\,m/s$$

$$\vec{v}_2 = \frac{3\sqrt{3}}{2}[cos30^o\,\hat{i} + sin30^o\,(-\hat{j})]\ m/s$$

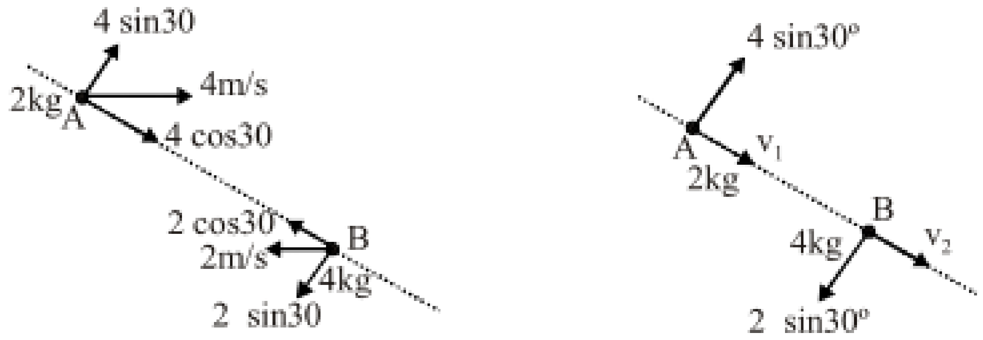
$$= \left(\frac{9}{4}\hat{i} - \frac{3\sqrt{3}}{4}\hat{j}\right)$$

Illustration :

Two spheres are moving towards each other. Both have same radius but their masses are 2kg and 4kg. If the velocities are 4 m/s and 2m/s respectively and coefficient of restitution is e = 1/3, Find final velocities along line of impact.



Sol. *Let v_1 and v_2 be the final velocities of A and B respectively then by conservation of momentum along line of impact.*



Before collision

Just After collision

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2 v_2$$

$$2\ (4 \cos 30^o) - 4\ (2 \cos 30^o) = 2(v_1) + 4\ (v_2)$$

or $$0 = v_1 + 2v_2 \qquad ....(1)$$

By Newton's Experimental Law,

$$e = \frac{\text{velocity of separation along LOC}}{\text{velocity of approach along LOC}}$$

or $$\frac{1}{3} = \frac{v_2 - v_1}{4\cos 30^o + 2\cos 30^o}$$

or $$v_2 - v_1 = \sqrt{3} \qquad ....(2)$$

From the above two equations,

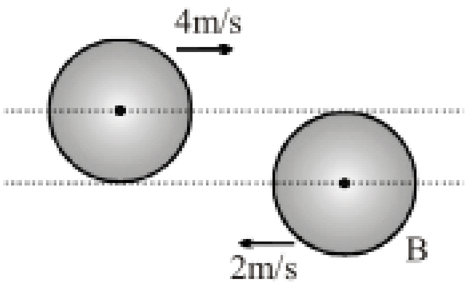
$$v_1 = -\frac{2}{\sqrt{3}}\ m/s$$

Negative answer denotes that we have chosen the wrong direction, actual direction of final velocity will be opposite to the direction we assumed in figure

$$v_2 = \frac{1}{\sqrt{3}}\ m/s$$

Practice Exercise

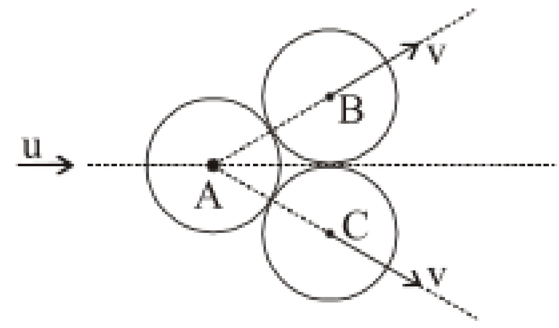
Q.1 Two spheres are moving towards each other. Both have same radius but their masses are 4kg and 2kg. If the velocities are 4 m/s and 2m/s respectively and coefficient of restitution is e = 1. Find, final velocity along line of impact.



Q.2 Two spheres are moving towards each other. Both have same radius but their masses are 4kg and 4kg. If the velocities are 4 m/s and 2m/s respectively and coefficient of restitution is e = 1, find. Final velocity along line of impact.

Q.3 A ball of mass m moving with a speed u_1 collides elasticity with another identical ball moving with velocity u_2.
(a) Find the velocities of the balls after collision if the impact is direct.
(b) Find the angle between velocities after collision if they collide obliquely and $u_2 = 0$.

Q.4 Two equal sphere of mass m are in contact on a smooth horizontal table. A third identical sphere impinges symmetrically on them and is reduced to rest. Find e and the loss of KE.



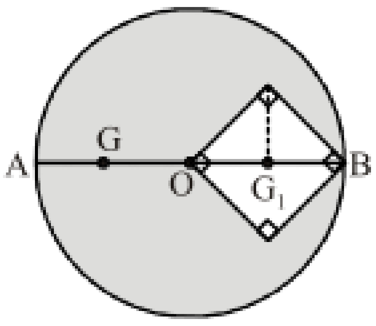
Answers

Q.1 $v_1 = 0,\ \ v_2 = 3\sqrt{3}\ m/s$	Q.2 $v_1 = -\sqrt{3}\ m/s,\ \ v_2 = 2\sqrt{3}\ m/s$
Q.3 (a) $v_1 = u_2$, $v_2 = u_1$; (b) $\frac{\pi}{2}$	Q.4 $\frac{2}{3}, \frac{mu^2 6}{6}$

Solved Examples

Q.1 A square hole is punched out of a circular lamina, the diagonal of the square being the radius of the circle. If 'a' be the diameter of the circle, find the distance of centre of mass of the remainder from the centre of the circle.

Sol. Consider the figure shown below. Let AB be the diameter passing through the diagonal OB of the square portion where O is the centre of the circle.



As mass is proportional to area of a uniform laninar body,
Mass of the portions can be replaced by their respective areas at their centre of mass

Area of circular potion $= \dfrac{\pi a^2}{4}$

Area of square portion $= \dfrac{a^2}{8}$

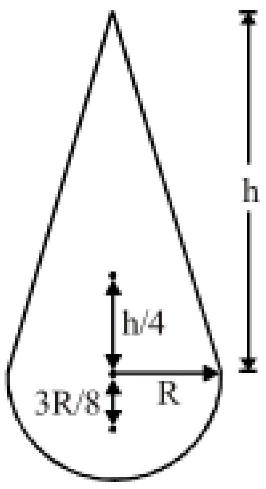
If G_1 and G the positions of centre of mass of the cut square portion and remaining portion.

Then
$$OG = \frac{-\frac{\pi a^2}{4}(0) - \frac{a^2}{8}\left(\frac{a}{4}\right)}{\frac{\pi a^2}{4} - \frac{a^2}{8}} = \frac{\frac{a}{32}}{\left(\frac{2\pi - 1}{8}\right)}$$

$$= \frac{a}{4(2\pi - 1)}$$

$\therefore$ The centre of mass of the remaining parting is at a distance of $\dfrac{a}{4(2\pi - 1)}$ from the centre.

Q.2 Find out the centre of mass of a composite object shown in figure. Object consists of a cone with its base joint with the base of a hemisphere. The dimensions of the object are shown in figure. Assume uniform density of the system.



Sol. The shown object is made up of joining a solid cone and a hemisphere. We already know the location of the centre of mass of a cone and that of a hemisphere. The masses of the two are in proportion of their volume. The masses of cone and hemisphere are

Mass of cone is $m_1 = \rho \frac{1}{3} \pi R^2 h$

and that of hemisphere is $m_2 = \rho \frac{2}{3} \pi R^3$

Now we apply the result of two body system to find the centre of mass of the composite body. Let l be the distance between the independent centre of mass of the bodies cone and hemisphere, then

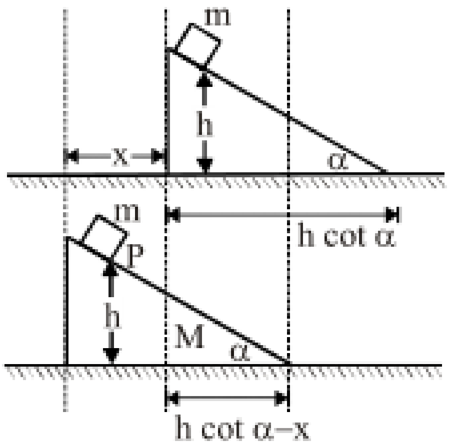
$$l = \frac{3R}{8} + \frac{h}{4}$$

The position of centre of mass from m_2 is

$$x = \frac{m_1 l}{m_1 + m_2} = \frac{\rho \frac{1}{3} \pi R^2 h \left(\frac{3R}{8} + \frac{h}{4} \right)}{\rho \frac{1}{3} \pi R^2 h + \rho \frac{2}{3} \pi R^3}$$

$$x = \frac{h(3R + 2h)}{8(h + 2R)}$$

Q.3 For the figure shown, block of mass m is released from the rest. Find the distance of the wedge from initial position, when block m arrives at the bottom of the wedge. All surfaces are frictionless.



Sol. As there is no net external force in the x direction, thus the momentum of system in x direction $(P_{sys})_x$ is conserved.

$Mv_1 = mv_2$, Initially $(P_{sys})_x = 0$

$\therefore$ Displacement of centre of mass in x-direction = 0

i.e. $Mx_1 = mx_2$ ....(i)

Let the displacement of wedge M be x backwards

$\therefore$ displacement of block = $h \cot \alpha - x$

Using equation (i)

$$m(h \cot\alpha - x) = Mx$$

$$x = \frac{m}{m + M} h \cot \alpha$$

Q.4 Two bodies A and B of masses m and 2m respectively are placed on a smooth floor. They are connected by a light spring of stiffness k. A third body C of mass m moves with velocity v_0 along the line joining A and B and collides elastically with A. If ℓ_0 be the natural length of the spring then find the minimum separation between the blocks.

Sol. Initially there will be collision between C and A which is elastic, therefore, by using conservation of momentum we obtain,

$$mv_0 = mv_A + mv_C \quad ; \quad v_0 = v_A + v_C$$

Since $e = 1$, $v_0 = v_A - v_C$

Solving the above two equation, $v_A = v_0$ and $v_C = 0$

Now A will move and compress the spring which in turn acceleration B and retard A and finally both A and B will move with same velocity v.

(a) Since net external force is zero, therefore momentum of the system (A and B) is conserved.

Hence $mv_0 = (m + 2m)v$

$\Rightarrow \quad v = v_0/3$

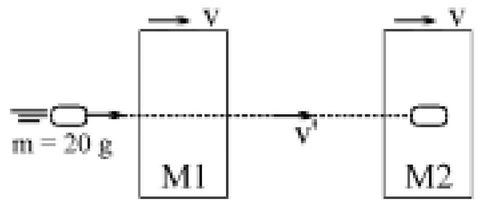
(b) If x_0 is the maximum compression, then using energy conservation

$$\frac{1}{2}mv_0^2 = \frac{1}{2}(m+2m)v^2 + \frac{1}{2}kx_0^2$$

$$\Rightarrow \quad \frac{1}{2}mv_0^2 = \frac{1}{2}(3m)\frac{v_0^2}{9} + \frac{1}{2}kx_0^2 \quad \Rightarrow \quad x_0 = v_0\sqrt{\frac{2m}{3k}}$$

Hence minimum distance $D = \ell_0 - x_0 = \ell_0 - v_0\sqrt{\frac{2m}{3k}}$

Q.5 A 20 g bullet pierces through a plate of mass $M_1 = 1$ kg and then comes to rest inside a second plate of mass $M_2 = 2.98$ kg (refer figure). It is found that the two plates, initially at rest, now move with equal velocity v. Find the velocity of the bullet (in m/s) when it is between M_1 and M_2. Given that it entered M_1 with 100 m/s.



Sol. From the principle of conservation of linear momentum we have

$$mu = M_1v + mv'$$

and $\quad mv' = (m + M_2)v$

or $\quad 20u = 1000v + 20v'$

and $\quad 20v' = (20 + 2980)v$

or $\quad u = 50v + v' \quad ...(i)$

and $\quad v' = 150\,v \quad ...(ii)$

From (i) and (ii), we get $\quad 3u = v' + 3v' = 4v' \quad$ or $\quad v' = 3u/4 = 75$

Q.6 A block of mass 4 kg is moving with a velocity of 7 m/s on a surface. It collides with another block of mass 3 kg elastically. The surface is smooth for 4 kg block but rough for 3 kg block ($\mu = 0.4$). Find the time (in sec) after which next collision will occur.

Sol. 

Let after collision their velocities are v_1 & v_2

applying conservation of momentum

$$4 \times 7 + 0 = 4v_1 + 3v_2$$

$$4v_1 + 3v_2 = 28 \qquad \text{... (1)}$$

also $\quad e = -\left(\dfrac{v_2 - v_1}{u_2 - u_1}\right)$

$$1 = -\left(\frac{v_2 - v_1}{0 - 7}\right)$$

$$v_2 - v_1 = 7 \qquad \text{....(2)}$$

solving (1) & (2) $\Rightarrow$ $v_2 = 8$ m/s

$v_1 = 1$m/s

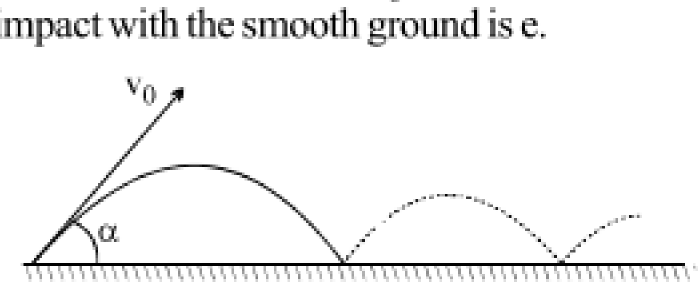
time after 2nd block stops $= \dfrac{u}{a} = \dfrac{8}{4} = 2$ sec

distance travelled by 2nd block till this moment $s = 8t + \dfrac{1}{2}at^2$

$$s = 8 \times 2 - \frac{1}{2} 4 \times 2^2 = 8\text{m}$$

so time elasped till 2nd collision $= \dfrac{8}{1} = 8$ sec.

Q.7 A particle of mass 'm' is projected with velocity v_0 at an angle 'α' with the horizontal. The coefficient of restitution for any of its impact with the smooth ground is e.



Find total time taken by the particle before it stops moving vertically ?

Sol. Total time taken by the particle to stop

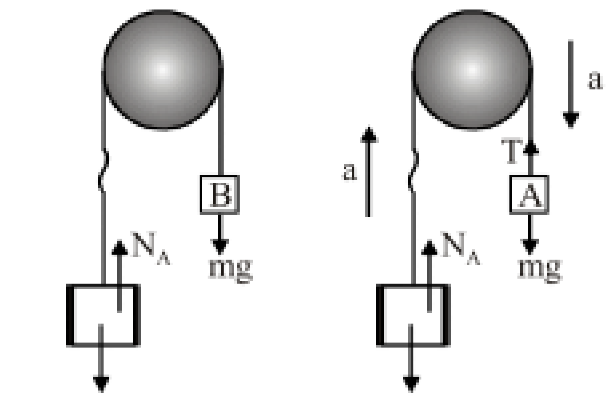
$$T = \frac{2v_0 \sin\alpha}{g} + \frac{2ev_0 \sin\alpha}{g} + \frac{2e^2v_0 \sin\alpha}{g} + \ldots$$

$$= \frac{2v_0 \sin\alpha}{g}(1 + e + e^2 + \ldots) \qquad = \frac{2v_0 \sin\alpha}{g(1-e)}$$

Q.8 After falling from rest through a height h a body of mass m begins to raise a body of mass M (M > m) connected to it through a pulley.

(a) Determine the time it will take for the body of mass M to return to its original position.

(b) Find the fraction of kinetic energy lost when the body of mass M is jerked into motion.



(a)
Before the string tautens

(b)
after the string tautens

Sol. (a) The speed of the body B just before the string becomes taut is $v = \sqrt{2gh}$. When the string is jerked, large impulsive reactions are generated in the string. At this moment effect of gravity is negligible. So momentum of the system is conserved at this instant. Let v' be the common speed of the two bodies after they are jerked into motion. From conservation of momentum, we have

$$mv = (M + m)\,v' \qquad \text{or} \qquad v' = \frac{m}{M+m}v$$

The acceleration of the system is

$$\Sigma F = Mg - mg = (M + m)\,a \qquad \text{or} \qquad a = -\frac{M-m}{M+m}g$$

The acceleration is negative, (opposite to v')

Let the system return to original position at time t.

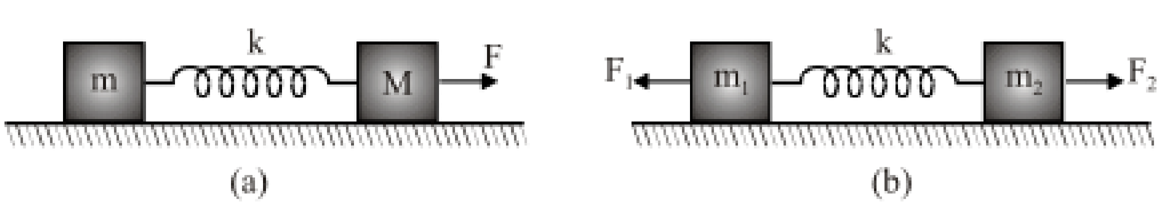
$$0 = v't + \frac{1}{2}at^2$$

$$\text{or} \qquad t = -\frac{2v'}{a} = \frac{2m}{M-m}\sqrt{\frac{2h}{g}}$$

(b) The fractional loss of kinetic energy is

$$\frac{\frac{1}{2}mv^2 - \frac{1}{2}(M+m)v'^2}{\frac{1}{2}mv^2} = \frac{M}{M+m}$$

Q.9 A block of mass m is connected to another block of mass M by massless spring constant k The blocks are kept on a smooth horizontal plane. Initially, the blocks are at rest and the spring is unstretched when a constant force F starts acting on the block of mass M to pull it. Find the maximum extension of the spring.



Sol. Let us take the two blocks plus the spring as the system. The centre of mass of the system moves with an acceleration $a = \frac{F}{m+M}$. Let us work from a reference frame with its origin at the centre of mass. As this frame is acceleration with respect to the ground we have to apply a pseudo force ma towards left on the block of mass m and Ma towards left on the block of mass M. The net external force on m is

$$F_1 = ma = \frac{mF}{m+M} \text{ towards left}$$

and the net external force on M is

$$F_2 = F - Ma = F = \frac{MF}{m+M} = \frac{mF}{m+F} \text{ towards right}$$

The situation from this frame is shown in figure. As the centre of mass is at rest in this frame, the block move in opposite direction and come to instantaneous rest at some instant. The extension of the spring will be maximum of this instant. Suppose the left block through a distance x_2 from the initial positions. The total work done by the external forces F_1 and F_2 in this period are

$$W = F_1x_1 + F_2x_2 = \frac{mF}{m+F}(x_1 + x_2).$$

This should be equal to the increase in the potential energy of the spring as there is no change in the kinetic energy. Thus,

$$\frac{mF}{m+F}(x_1 + x_2) = \frac{1}{2}k(x_1 + x_2)^2$$

or, $$x_1 + x_2 = \frac{2mF}{k(m+M)}$$

This is the maximum extension of the spring.

Q.10 A glass ball collides with a smooth horizontal surface with a velocity $\vec{v} = a\hat{i} - b\hat{j}$. If the coefficient of restitution of collision be e, find the velocity of the ball just after the collision. (Take y-axis along vertical)

Sol. Collision takes place along the normal. Therefore the magnitude normal component (v_y) of the velocity of the glass ball is changed to $v_{y'} = e\,v_y$ just after the collision whereas the horizontal component (v_x) of its velocity remains constant due to the absence of any horizontal force.

$\Rightarrow$ The velocity of the ball just after the impact

$= \vec{v} = \vec{v}_x + \vec{v}_y$

$\Rightarrow$ $\vec{v}' = v'_x\,\hat{i} + v'_y\,\hat{j}$

where, $v'_x = a$ & $v'_y = eb$

$\Rightarrow$ $\vec{v}' = a\,\hat{i} + eb\hat{j}$

Therefore the magnitude of the velocity $\vec{v}' = |\vec{v}'| = \sqrt{a^2 + e^2b^2}$ and the direction is given as

$$\theta = \tan^{-1}\left(\frac{v_{x'}}{v_{y'}}\right) = \tan^{-1}\left(\frac{a}{eb}\right) \text{ to the normal (vertical)}$$

Q.11 A stationary body explodes into four identical fragments such that three of them fly off mutually perpendicular to each other each with same kinetic energy (E_0). Find the energy of explosion.

Sol. Let three of the fragments move along X,Y & Z axes. Therefore their velocities can be given as

$\vec{v}_1 = v\hat{i}, \vec{v}_2 = v\hat{j}$ & $\vec{v}_3 = v\hat{k}$ where v = speed of each of the three fragments. Let the velocity of the fourth fragment be $\vec{v}$. Since, in explosion no net external force is involved, the net momentum of the system remains conserved just before and after the explosion.

$\Rightarrow \quad (\vec{p})_f = (\vec{p})_i$

$\Rightarrow \quad m\vec{v}_1 + m\vec{v}_2 + m\vec{v}_3 + m\vec{v}_4 = 0$ ($P_i = 0$ because the body was stationary), putting the values of $\vec{v}_1, \vec{v}_2$ & $\vec{v}_3$, we obtain,

$$\vec{v}_4 = -v(\hat{i} + \hat{j} + \hat{k})$$

Therefore, $v_4 = \sqrt{3}v$

The energy of explosion (ΔKE) system

$$\Rightarrow \quad E = KE_f - KE_i$$

$$= \left(\frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 + \frac{1}{2}mv_3^2 + \frac{1}{2}mv_4^2\right) - (0)$$

Putting $v_1 = v_2 = v_3 = v, v_4 = \sqrt{3}v$

$$E = \left(\frac{1}{2}mv^2 + \frac{1}{2}mv^2 + \frac{1}{2}mv^2 + \frac{1}{2}m(\sqrt{3}v)^2\right) - (0)$$

$$E = 3mv^2 \qquad \text{.........(i)}$$

As we know from question that kinetic energy of three are equal and equal to E_0

$\therefore \quad \frac{1}{2}mv^2 = E_0$, Putting this value in equation (i)

we obtain, $E = 6E_0$.

Q.12 A man of mass m climbs a rope of length L suspended below a balloon of mass M. The balloon is stationary with respect to ground. If the man begins to climb up the rope at a speed v (relative to rope) in what direction and with speed (relative to ground) will the balloon move?

Sol. Balloon is stationary

$\Rightarrow$ No net external force acts on it.

$\Rightarrow$ The conservation of linear momentum of the system (balloon + man) is valid

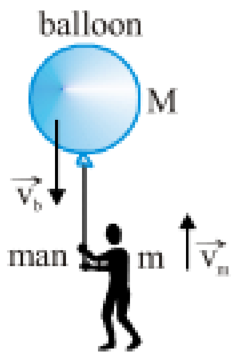
$\Rightarrow \quad M\vec{v}_b + m\vec{v}_{mb} = 0$

where $\vec{v}_m = \vec{v}_{mb} + \vec{v}_b$

$\Rightarrow \quad M\vec{v}_b + m[\vec{v}_{mb} + \vec{v}_b] = 0$

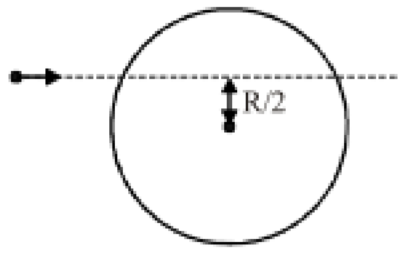
where v_{mb} = velocity of man relative to the balloon (rope)

$$\Rightarrow \quad \vec{v}_b = -\frac{m\vec{v}_{mb}}{M+m}$$

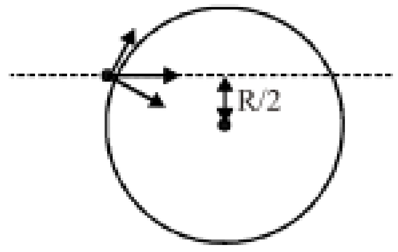


Where $v_{mb} = v \Rightarrow v_b = \frac{mv}{M+m}$ and directed opposite to that of the man.

Q.13 A particle of mass m strikes elastically with a disc of radius R, with a velocity $\vec{v}$ as shown in the figure. If the mass of the disc is equal to that of the particle and the surface of the contact is smooth, find the velocity of the disc just after the collision.



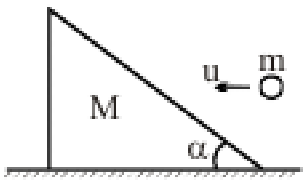
Sol. We see that impact takes place along the normal. Therefore, the particle and the disc change their momentum along that line. However, no external force acts on the system along the normal line. Hence we can conserve the linear momentum of the system (disc + particle) along the normal. Since the masses of the disc and particle are equal, so the exchange of momentum takes place along the normal. That means, the particle completely delivers the part (component) of its momentum (m v cos θ) along the normal



$\Rightarrow$ Velocity of the disc, $\vec{v}_1 = (v\cos\theta)\,\hat{j}$ where, $\cos\theta = \cos 30^\circ = \frac{\sqrt{3}}{2}$

$\Rightarrow$ $\vec{v}_1 = \frac{\sqrt{3}v}{2}\,\hat{j}$

Q.14 A wedge of mass m rest on horizontal surface. The inclination of the wedge is α. A ball of mass m moving horizontal with speed u hits the inclined face of the wedge inelastically and after hitting slides up the inclined face of the wedge. Find the velocity of the wedge just after collision. Neglect any friction.



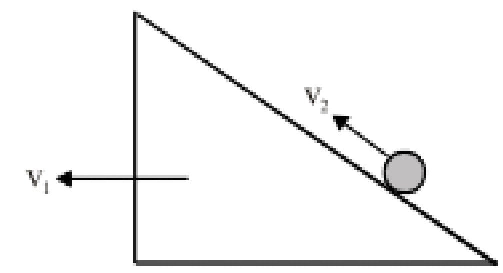
Sol. Let velocity of the ball after collision is $\vec{v}_2$ (w.r. to wedge) in directions as shown in the figure. Conserving momentum along horizontal, we get

$mu = m[v_2 \cos\alpha + v_1] + Mv_1$

$\Rightarrow mu = mv_2 \cos\alpha + (M + m)v_1$ ......(i)

Since common normal is along y', therefore momentum of ball remains constant along the incline (along x' $\because \vec{F}_{x'} = 0$)

$\Rightarrow \quad u\cos\alpha = v_2 + v_1 \cos\alpha$ .....(ii)



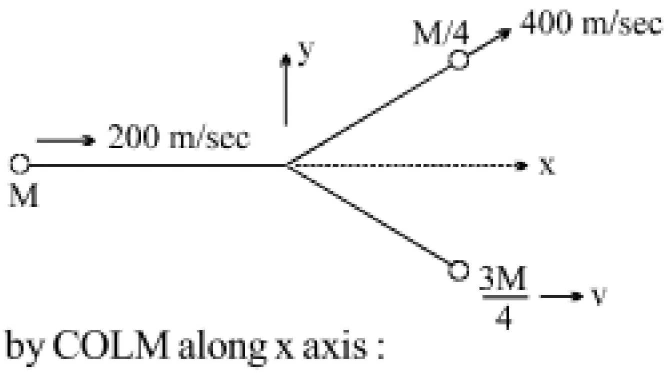
from equation (i) and (ii), we get

$mu = mu\cos^2 a - mv_1\cos^2 a + (M+m)v_1$

$$\Rightarrow \quad v_1 = \frac{mu\sin^2\alpha}{M + m\sin^2\alpha}$$

Q.15 A missile of mass M moving with velocity v = 200 m/s explodes in midair breaking into two parts of mass M/4 & 3M/4. If the smaller piece flies off at an angle of 60° with respect to the original direction of motion with an initial speed of 400 m/s, what is the magnitude and direction of the initial velocity of the other piece.

Sol.



by COLM along x axis :

$$200M = 400 \times \frac{M}{4}\cos 60° + \frac{3M}{4} V\cos\theta$$

$$\frac{3}{4} v\cos\theta = 150 \quad \Rightarrow v\cos\theta = 200 \quad \text{... (i)}$$

by COLM along y axis :

$$0 = 400\frac{M}{4}\sin 60° - \frac{3M}{4} V\sin\theta$$

$$3v\sin\theta = \frac{400\sqrt{3}}{2}$$

$$v\sin\theta = \frac{200}{\sqrt{3}} \quad \text{... (ii)}$$

(ii) / (i) gives

$$\tan\theta = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \theta = 30°$$

$$\Rightarrow \quad v = \frac{400}{\sqrt{3}} \text{ m/s}$$