Fluid Mechanics

This lesson is devoted to fluids (liquids and gases). Find is a Latin word meanins 'to flow'. Liquids and gases can flow to take shape of the vessel that holds them. Another important property of fluids is that a tangential force causes continuous deformation of fluids whereas if we apply a tangential force on a solid there will be a particular deformation in the solid. First section of the chapter is study of fluid at rest called as fluid statics and Second section is study of fluid in motion called as fluid dynamics. In these two sections we shall see the concepts and their application for liquids only but these concepts and their application for liquids only but these concepts are equally good for gases also but for low-pressure variation.

What is a Fluid ?

A **fluid,** in contrast to a solid, is a substance that can flow. Fluids conform to the boundaries of any container in which we put them. They do so because a fluid cannot sustain a force that is tangential to its surface. a fluid is a substance that flows because it cannot withstand a shearing stress. It can, however, exert a force in the direction perpendicular to its surface.

FLUID MECHANICS

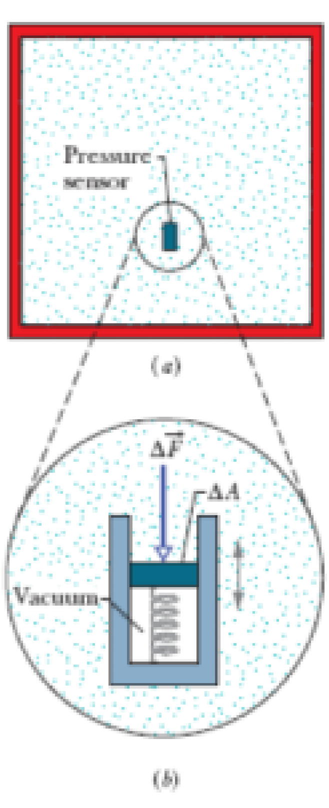
FLUID STATICS (Fluid remains stationary wrt container) | FLUID DYNAMICS | VISCOUS FLUID

Fluid includes property → (A) Density (B) Viscosity (C) Bulk modulus of elasticity (D) Pressure (E) Specific gravity.

Assumptions used in fluid mechanics

1. Fluid is incompressible means density remains constant and volume also remains constant.
2. Fluid is non viscous. There is no tangential force between two layers.

Pressure



Let a small pressure-sensing device be suspended inside a fluid-filled vessel, as in Fig. 14-1a. The sensor consists of a piston of surface area ΔA riding in a close-fitting cylinder and resting against a spring.A readout arrangement allows us to record the amount by which the (calibrated) spring is compressed by the surrounding fluid, thus indicating the magnitude ΔF of the force that acts normal to the piston.We define the **pressure** on the piston from the fluid as

$$p = \frac{\Delta F}{\Delta A}$$

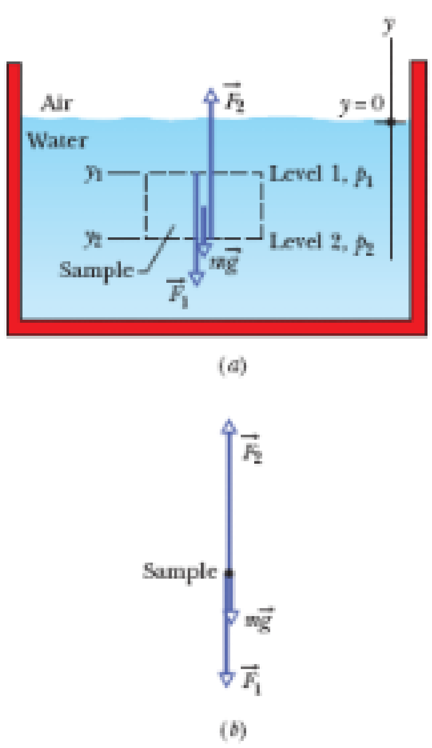
In theory, the pressure at any point in the fluid is the limit of this ratio as the surface area ΔA of the piston, centered on that point, is made smaller and smaller. However, if the force is uniform over a flat area A, we can write Eq. as

$$p = \frac{F}{A} \text{ (pressure of uniform force on flat area)}$$

where F is the magnitude of the normal force on area A. (When we say a force is uniform over an area, we mean that the force is evenly distributed over every point of the area.) We find by experiment that at a given point in a fluid at rest, the pressure p defined by Eq. has the same value no matter how the pressure sensor is oriented. Pressure is a scalar, having no directional properties. It is true that the force acting on the piston of our pressure sensor is a vector quantity, but Eq. involves only the magnitude of that force, a scalar quantity.

Let us look first at the increase in pressure with depth below the water's surface. We set up a vertical y axis in the tank, with its origin at the air–water interface and the positive direction upward. We next consider a water sample contained in an imaginary right circular cylinder of horizontal base (or face) area A, such that y_1 and y_2 (both of which are negative numbers) are the depths below the surface of the upper and lower cylinder faces, respectively.

Figure shows a free-body diagram for the water in the cylinder. The water is in static equilibrium; that is, it is stationary and the forces on it balance. Three forces act on it vertically: Force $\vec{F}_1$ acts at the top surface of the cylinder and is due to the water above the cylinder. Similarly, force $\vec{F}_2$ acts at the bottom surface of the cylinder and is due to the water below the cylinder. The gravitational force on the water in the cylinder is represented by $m\vec{g}$, where m is the mass of the water in the cylinder. The balance of these forces is written as



$$F_2 = F_1 + mg$$

We want to transform Eq. into an equation involving pressures. From Eq., we know that

$$F_1 = p_1A \text{ and } F_2 = p_2A$$

The mass m of the water in the cylinder is, from Eq., $m = \rho V$, where the cylinder's volume V is the product of its face area A and its height $y_1 - y_2$. Thus, m is equal to $\rho A(y_1 - y_2)$. Substituting this and Eq. into Eq., we find

$$p_2A = p_1A + \rho Ag(y_1 - y_2)$$

or

$$p_2 = p_1 + \rho g(y_1 - y_2)$$

This equation can be used to find pressure both in a liquid (as a function of depth) and in the atmosphere (as a function of altitude or height). For the former, suppose we seek the pressure p at a depth h below the liquid surface. Then we choose level 1 to be the surface, level 2 to be a distance h below it and p_0 to represent the atmospheric pressure on the surface. We then substitute

$$y_1 = 0,\ p_1 = p_0 \text{ and } y_2 = -h,\ p_2 = p$$

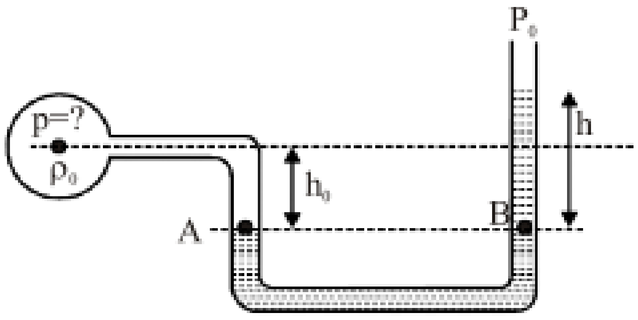
$$p = p_0 + \rho gh \text{ (Pressure at depth h)}$$

Note that the pressure at a given depth in the liquid depends on that depth but not on any horizontal dimension.

The pressure at a point in a fluid in static equilibrium depends on the depth of that point but not on any horizontal dimension of the fluid or its container.

Pressure Measuring Devices

A manmenter is a tube open at both the ends and bent into the shape of a "U" and partially filled with mercury. When one end of the tube is subjected to an unkonwon pressure p, the mercury level drops on that side of the tube and rises on the other so that the difference in murcury level is h as shown in the figure. When we move down in a fluid pressure increases with depth and when we move up the pressure decreases with depth. When we move horizontally in a fluid pressure remains constant.



Therefore,

$$p + \rho_0 g h_0 - \rho_m g h = p_0$$

Where p_0 the atmosheric pressure and

r_0 is the density of the fluid inside the vessel

The mercury barometer

It is a straight glass tube (closed at one end) completely filled with mercury and inserted into a dish which is also filled with mercury as shown in the figure. Atmosheric pressure supports the column of mercury in the tube to a height h. The pressure between the closed end of the tube and the column of mercury is zero. $p = 0$.

Therefore, pressure at points A and B are equal and thus $p_0 = 0 + \rho_m g h$

At the sea level, p_0 can support a column of mercury about 76 cm in height

Hence $p_0 = (13.6 \times 10^3)(9.81)(0.76) = 1.01 \times^5 \text{ Nm}^2$ (or Pa)

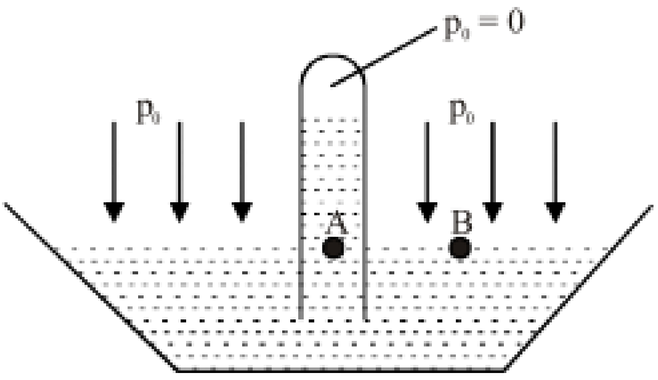
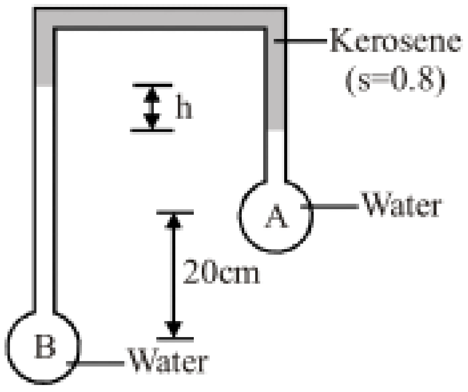


Illustration :

For the arrangement shown in the figure, determine h if the pressure difference between the vessels A and B is 3 kN/m²



Sol. *Let pressure in the horizontal tube is P So in left vertical tube*

$$P + \rho_k g h_1 + \rho_w g h_o = P_B$$

$$P + \rho_k g h_2 + \rho_w g (h_0 - h - 0.2) = P_A$$

here, $P_B - P_A = 3 \times 10^3\ N/m^2$,

$$\rho_W = 10^3\ kg/m^3$$

$$\rho_k = 800\ kg/m^3$$

Thus, $h = 0.5\ m = 50\ cm$

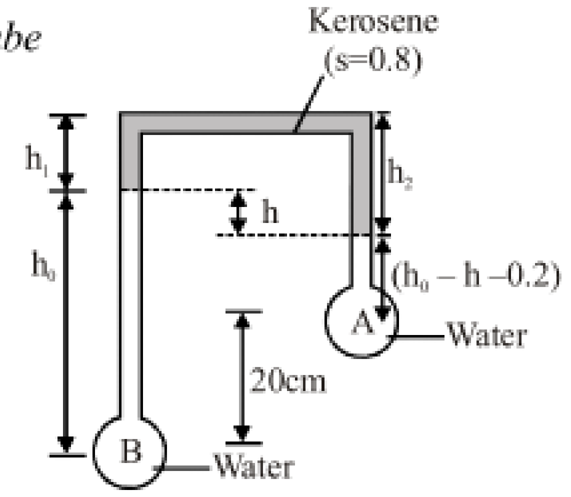


Illustration :

What must be the length of a barometer tube used to measure atmospheric pressure if we are to use water instead of mercury.

Sol. *We know that*

$$p_0 = \rho_m g h_m = \rho_w g h_w$$

where ρ_w *and* h_w *are the density and height of the water column supporing the atmospheric pressure* p_0.

$$\therefore \quad h_w = \frac{\rho_m}{\rho_w} h_m$$

Since $\frac{\rho_m}{\rho_w} = 13.6$ *and* $h_m = 0.76\ m$

$$\therefore \quad h_w = (13.6)(0.76) = 10.33\ m$$

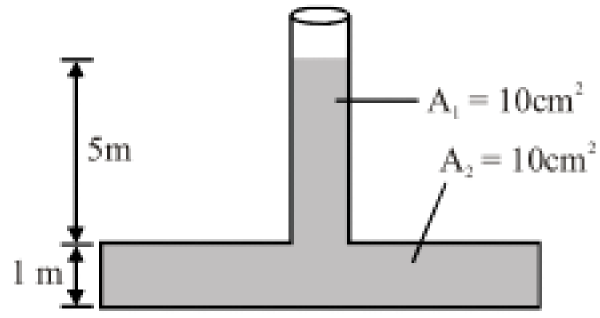
Illustration :

In the figure shown, find

(a) the total force on the bottom of the tank due to the water pressure.

(b) the total weight of water.

Why is there a difference between the two ?



Sol.

(a) Pressure at the base due to water is

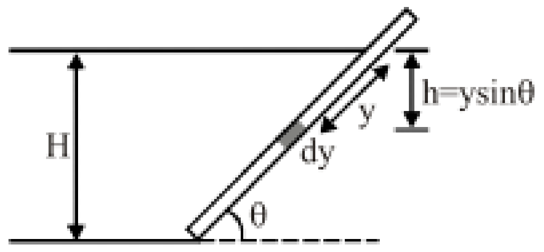
$$p = \rho_w g\,[5 + 1] = (10^3)\,(10)\,(5 + 1) = 6 \times 10^4\ N/m^2$$

$\therefore$ $\quad$ $Force = pA_2 = (6 \times 10^4)\,(100 \times 10^{-4}) = 600\ N$

(b) Weight of water $= \rho_w\, g\,[5A_1 + A_2] = 10^4\,[5 \times 10 \times 10^{-4} + 100 \times 10^{-4}] = 150\ N$

Illustration :

Find the force acting per unit width on a plane wall inclined at an angle θ with the horizontal as shown in the figure.



Sol. *Consider a small element of thickness dy at a distance y measured along the wall from the free surface. The pressure at the position of the element is*

$$p = \rho g h = \rho g y \sin\theta$$

The force is given by

$$dF = p\,(b\,dy) = \rho g b\,(y\,dy)\sin\theta$$

The total force per unit width b is given by

$$\frac{F}{b} = \rho g \sin\theta \int_0^{H/\sin\theta} y\,dy = \rho g \sin\theta \left[\frac{y^2}{2}\right]_0^{H/\sin\theta}$$

or $\quad \dfrac{F}{b} = \dfrac{1}{2}\rho g \dfrac{H^2}{\sin\theta}$

Note that the above formula reduces to $\dfrac{1}{2}\rho g H^2$ *for a vertical wall ($\theta = 90^o$)*

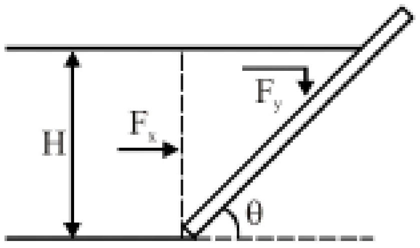
Alternatively, the force on the inclined wall may be obtained in two parts viz. Horizontal and vertical.

The horizontal force F_x acts on the vertical projection of the incline wall,

i.e. $$F_x = \frac{1}{2}\rho g b H^2$$

The vertical force F_y acts to weight of the liquid supported by the wall, i.e.

$$F_y = \frac{1}{2}\ \rho g b(H)\ (H \cot \theta) = \frac{1}{2}\rho g b H^2 \cot \theta$$



The magnitude of the resultant force is given by

$$F = \sqrt{F_x^2 + F_y^2}\ \frac{1}{2}\rho g b H^2 \operatorname{cosec}\theta$$

or $$F = \frac{1}{2}\rho g \frac{bH^2}{\sin\theta}$$

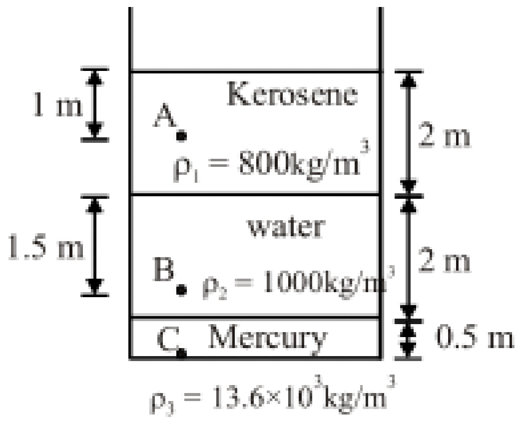
Illustration :

Atmosphericpressure is about 1.01×10^5 Pa. How large a force does the atmosphere exert on a $2cm^2$ area on the top of your head?

Sol. *Because p = F / A. where F is perpendicular to A, we have F = pA. Assuming that 2 cm^2 of your head is flat (nearly correct) and the force due to the atmosphere is perpendicular to the surface (as it is), we have*

$$F = pA = (1.01\times10^5\ N/m^2)\ (2\times10^{-4}m^2) \approx 20\ N$$

(1)6. Fin the absolute pressure and gauge pressure at pint A, B and C as hown in the figure (1 atm = 10^5 Pa)

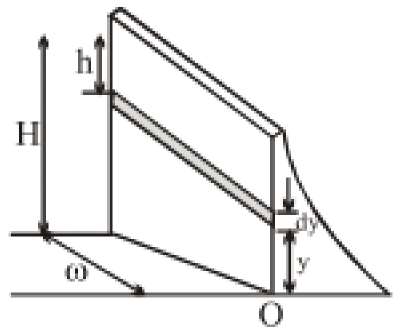


Sol. $p_{atm} = 10^5$ *Pa*

Points	*Gauge Pressure*	*Abolute Pressure*
A	$pA = \rho_1 gh_A = (800)\ (10)\ 1 = 8\ k\ Pa$	$p'A = pA + p_{atm} = 108\ kPa$
B	$pB = \rho_1 g(2) + \rho_2 g(1.5) = (800)(10)(2) + (10^3)(10)(1.5) = 31\ kPa$	$p'B = pB + p_{atm} = 131\ kPa$
C	$pC = \rho_1 g(2) + \rho_2 g(2) + \rho_3 g\ (0.5)$ $= (800)(10)(2) + (10^3)(10)(2) + (13.6\times10^3)(10)(0.5)$ $=104\ kPa$	$p'C = PC + p_{atm} = 204\ kPa$

Illustration :

Water is filled to a height H behind a dam of width w (fig.). Determine the resultant force exerted by the water on the dam.



Water exerts a force on a dam

Sol. *Let's consider a vertical y axis, starting from the bottom of the dam. Lets consider a thin horizontal strip at a height y above the bottom, such as shown in Fig. We need to consider force due to the pressure of the water only as atmospheric pressure acts on both sides of the dam.*

The pressure due to the water at the depth h : $P = \rho g h = \rho g(H - y)$

The force exerted on the shaded strip of area $dA = wdy$:

$$dF = P\,dA = \rho g(H - y)\,\omega\,dy$$

Integrate to find the total force on the dam :

$$F = \int P\,dA = \int_0^H \rho g(H - y)\omega\,dy = 1/2\ \rho g \omega H^2$$

Illustration :

In the previuous example find the total torque exerted by the water on dam about a horizontal axis through O. Also find the effective line of action of the total force exerted by the water is at a distance 1/3 H above O.

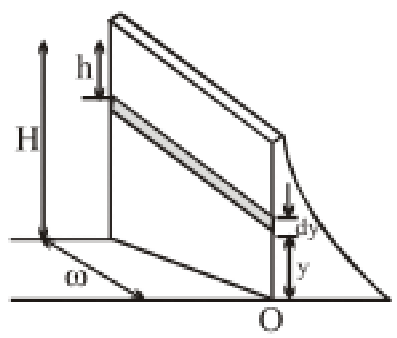
Sol. *The torque is* $\tau = \int d\tau = \int r dF$

From the figure $$\tau = \int_0^H y[\rho g(H - y)wdy] = \frac{1}{6}\ \rho g w H^3$$

The total force is given as $\frac{1}{2}\rho g w H^2$

If this were applied at a height y_{eff} *such that the torque remains unchanged, we have*

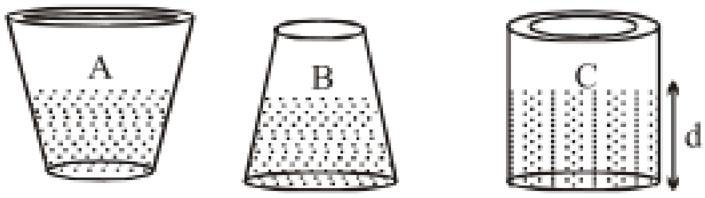
$$\frac{1}{6}\ \rho g w H^3 = y_{eff}\left[\frac{1}{2}\rho g w H^2\right] \quad \text{and} \quad y_{eff} = \frac{1}{3}\ H$$



Water exerts a force on a dam

Illustration :

Three vessels having different shapes are as shown in the figure below, they have same base area and the same weight when empty (Fig.). The vessels are filled with mercury to the same level. Neglect the effect of the atmosphere. (a) Which have the largest and which have the smallest pressures at the bottom of the vessel or are they same? (b)Which show the highest weight when weighed on a weighing scale or are they same?



Three differently shaped vessels filled with water to same level.

Sol. *(a) The mercury at the bottom of each vessel is at the same depth d below the surface. Neglecting the pressure at the surface, the pressures at the bottom must be equal hence:*

$$P = \rho g d$$

(b) The weight of each filled vessel is equal to the weight of the vessel itself plus the weight of the mercury inside. The vessels themselves are of equal weights, but vessel A holds more mercury than C, while vessel B holds less mercury than C. Vessel A weighs the most and vessel B weighs the least.

Illustration :

As the mercury exerts the same downward force on the bottom of each vessel, then why does the vessels weigh differently ?

Sol. *In vessel C ,forces due to fluid pressure on the sides of the container are horizontal. Forces on any two diametrically opposite points on the walls of the container are equal and opposite; thus, the net force on the container walls is zero. The force on the bottom is*

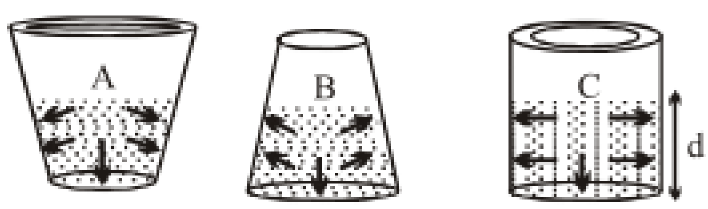
$$F = PA = (\rho g d)\,(\pi r^2)$$

The volume of water in the cylinder is $V = \pi r^2 d$, so

$$F = \rho g V = (\rho V)\,g = mg$$

The force on the bottom of vessel C is equal to the weight of the water, as expected. The forces due to fluid pressure on the sides of the containers A and B have vertical components also. Hence the force between the fluid and the base of container will not be equal to the weight of the fluid. These containers support the fluid by exerting an upward force equal in magnitude to the weight of the fluid but some force is being applied by the sidewalls and the remaining by the bottom. Fig. ( ) shows the forces acting on each container due to the water.

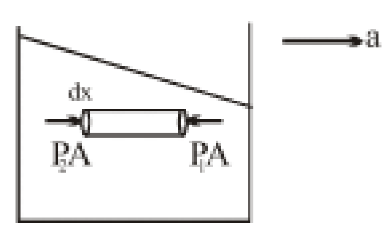
The force on the bottom of vessel A is less than the weight of the mercury in the container, while *the force on the bottom of vessel B is greater than the weight of the mercury. In vessel A, the forces on the container walls have downward components as well as horizontal components. The sum of the downward components of the forces on the walls and the downward force on the bottom of the container is equal to the weight of the water. Similarly, the forces on the walls of vessel B have upward components. In each case, the total force on the bottom and sides of the container due to the water is equal to the weight of the water.*



Forces exerted on the containers by the water.

Linear Accelerated Motion :

We consider an open container of a liquid that is moving along a straight line with a constant acceleration a as shown in Fig.



Lets consider a small horizontal cylinder of length dx and crossectional area A located y below the free surface of the fluid. This cylinder is accelerating in ground frame with accleration a hence the net horizontal force acting on it should be equal to the product of mass(dm) and acceleration.

$dm = Adx\rho$

$$P_2A - P_1A = (Adx\rho)a \qquad P_2 - P_1 = Padx$$

If we say that the right face of the cylinder is y below the free surface of the fluid then the left surface is y + dy below the surface of liquid. Thus

$$P_2 - P_1 = \rho g dy$$

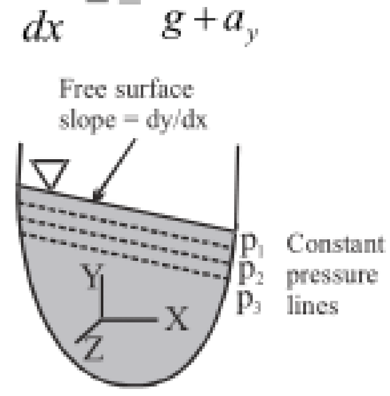
$$\therefore \qquad \frac{dy}{dx} = \frac{a}{g}$$

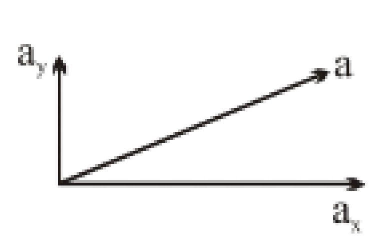
Since the slope of the free surface is coming out to be constant we can say that it must be straight line.

$$\tan\theta = \frac{a}{g}$$

If the container have acceleration along y also than the slope of this line is given by the relationship.

$$\frac{dy}{dx} = -\frac{a_x}{g + a_y}$$





Along a free surface the pressure is constant, so that for the accelerating mass shown in Fig. ( ) the free surface will be inclined if $a_x \neq 0$. In addition, all lines parallel to the free surface will have same presure. For the special circumstance in which $a_x = 0, a_y \neq 0$. which corresponds to the mass of fluid accelerating in the vertical direction, Eq. ( ) indicates that the fluid surface will be horizontal. However, from Eq. ( ) we see that the pressure variation is not ρgdy, but is given by the equation.

$$dP = \rho(g + a_y)dy$$

Thus, the pressure on the bottom of a liquid-filled tank which is resting on the floor of an elevator that is accelerating upward will be more than, if the, tank would have been at rest (or moving with a constant velocity). It is to be noted that for a freely falling fluid mass ($a_y = -g$), the pressure variation in all three coordinate directions are zero, which means that the pressure throughout will be same. The pressure throughout a "blob" of a liquid floating in an orbiting space shuttle (a form of free fall) is zero. The only force holding the liquid together is surface tension.

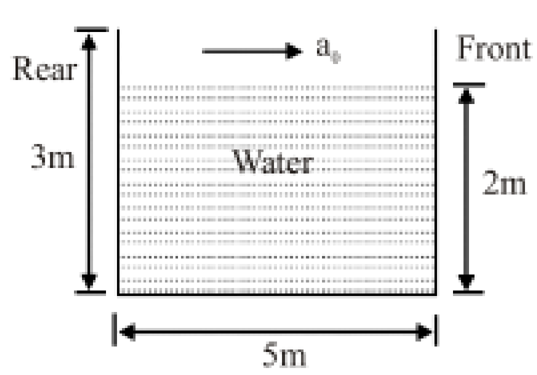
Illustration :

An open rectangular tank 5 m × 4 m × 3m high containing water upto a height of 2 m is accelerated horizontally along the langer side.

(a) Determine the maximum acceleration that can be given without spilling the water.

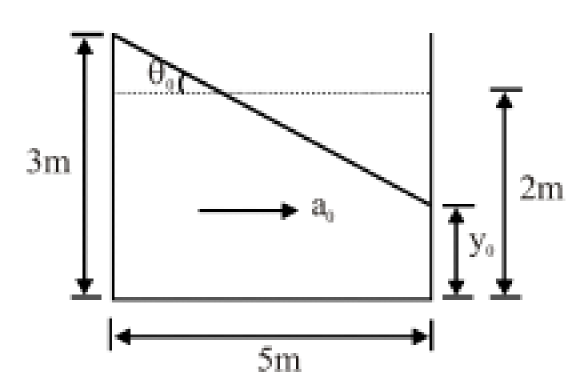
(b) Calculate the percentage of water spilt over, if this acceleration is increased by 20%

(c) If initially, the tank is closed at the top and is acceleration horizontally by 9 m/s², find the gauge pressure at the bottom of the front and rear walls of the tank. (Take g = 10 m/s²)



Sol. *(a) Volume of water inside that tank remains constant*

$$\left(\frac{3+y_0}{2}\right) 5 \times 4 = 5 \times 2 \times 4$$



or $y_0 = 1m$

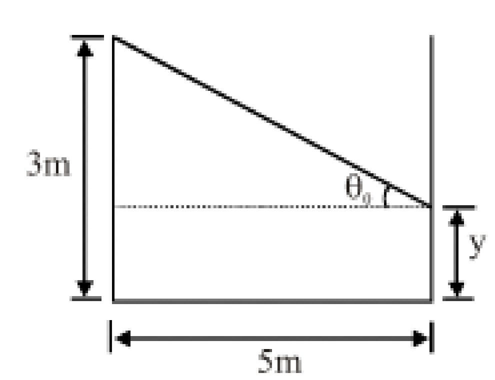
$\therefore \quad \tan\theta_0 = \dfrac{3-1}{5} = 0.4$

Since, $\tan\theta_0 = \dfrac{a_0}{g}$, *therefore* $a_0 = 0.4g = 4\ m/s^2$

(b) When acceleration is increased by 20%

$a = 1.2\ a_0 = 0.48\ g$

$\therefore \quad \tan\theta = \dfrac{a}{g} = 0.48$



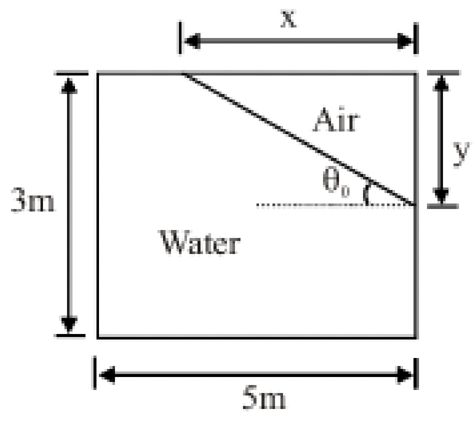
Now, $y = 3–5\tan\theta = 3 – 5\ (0.48) = 0.6\ m$

Fraction of water spilt over

$$= \frac{4 \times 2 \times 5 \frac{(3+0.6)}{2} \times 5 \times 4}{2 \times 5 \times 4} = 0.1$$

Percentage of water spilt over = 10%

(c) $a' = 0.9\,g$



$$tan\theta' = \frac{a'}{g} = 0.9$$

volume of air remins constant

$$4 \times \frac{1}{2} yx = (5)(1) \times 4$$

Since $y = x\ tna\ \theta' = 5$

or $x = 3.33\ m\ ;\ y = 3.0\ m$

Gauge presure at the bottom of the

(i) Front wall $p_f = zero$

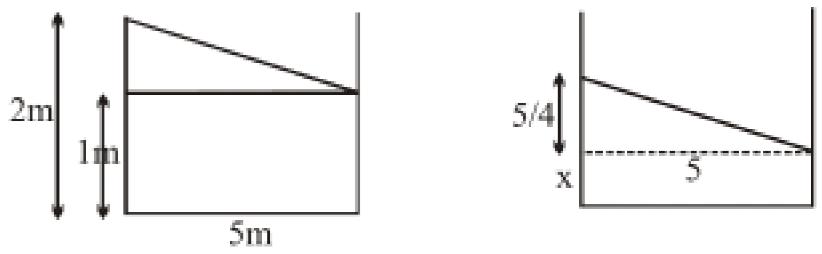
(ii) Rear wall $p_r = (5tan\theta')\,\rho_w g = 5\,(0.9)(10^3)(10) = 4.5 \times 10^4\ Pa$

Illustration :

The cross section of a tank kept on a vehicle is shown in Fig.. The rectangular tank is open to the atmosphere. During motion of the vehicle, the tank is subjected to a constant linear acceleration, $a = 2.5\ m/s^2$. How much fluid will be left inside the tank if initially the tank is half filled. The vessel is 5m wide and 2m high.

Sol. *If the height of the liquid on the left wall is greater than 2m the fluid will be spilled out. we can find the angle that the fluid will make with the horizontal.*

$$tan\ \theta\ = \frac{2.5}{10} = \frac{1}{4}$$



Lets assume that the dimension of tank in the plane perpendicular to the page is d.
From the geometry its easy to see that free surface on RHS will go down and will rise on LHS. Thus if we assume that fluid on RHS has not touched the floor, we will have fluid taking ashape as described in the diagram. The cuboid part will have volume $x \times 5 \times d$, where x is the height above the bottom.

The wedge part will have the volume $\frac{1}{2} \times h \times 5 \times d$ *where h can be found as following*

$(h/5) = tan\ \theta = (1/4)$

Thus total volume will be $\frac{1}{2} \times (5/4) \times 5 \times d + x \times 5 \times d$ *and if we assume there is no spilling than it must beequal to the final volume.*

$$\frac{1}{2} \times (5/4) \times 5 \times d + x \times 5 \times d = 1 \times 5 \times d$$

solving we get $x = \frac{3}{8}$

$\therefore$ *Total length* $\frac{5}{4} + \frac{3}{8} = \frac{10+3}{8} = \frac{13}{8} < 2$

Thus, height is less than 2.

Hence water will not spill.

Illustration :

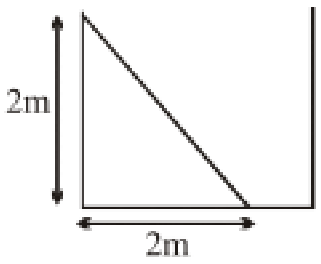
In previous question how much fluid will be left inside the tank if the vehicle accelerates at accleration, $a = 10\ m/s^2$ *?*

Sol. *(a) If the height of the liquid on the left wall is greater than 2m the fluid will be spilled out.*

If dimenion of tank in the plane perpendicular to page is d

$$\tan\theta = \frac{10}{10} = 1, \text{ thus } \quad \theta = \frac{\pi}{4}$$

In this case fluid can not remain inside. Fluid having an inclined free surface at 45^0 *angle, and covering the bottom of length 5m, will also be 5 m high. This will require the wall to be of 5 m height, which is just 2m for the given vessel. Instead if we think it other way round to keep in contact with the LHS wall, bottom will have to be covered only 2m with the fluid as as shown in the diagram,*



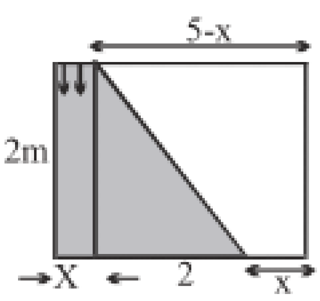
Fluid Inside $= (1/2) \times 2 \times 2 \times d\ m^3$

Remain inside $= 2d\ m^3$

$\therefore$ *Thus volume of fluid gone Outside* $= 3d\ m^3$

(b) *If the vessel is closed from the top and now accelerated at* $10\ m/s^2$*, then what length of the of the floor will be uncovered?*

Sol. *Looking at the figure it is clear*



$X + 2\ m + X = 5\ m$

$X = 1.5\ m$ *Ans*

Rotating Vessel

Consider a cylinderical vessel, rotating at constant angular velocity about its axis. If it contains fluid then after an initial irregular shape, it will rotate with the tank as a rigid body. The acceleration of fluid particles located at a distance r from the axis of rotation will be equal to $\omega^2 r$, and the direction of the acceleration is toward the axis of rotation as shown in the figure. The fluid particles will be undergoing circular motion.

Lets consider a small horizontal cylinder of length dr and crossectional area A located y below the free surface of the fluid and r from the axis. This cylinder is accelerating in ground frame with accleration $\omega^2 r$ towards the axis hence the net horizontal force acting on it should be equal to the product of mass (dm) and acceleration.

$dm = Adr\rho$

$$P_2A - P_1A = (Adr\rho)\omega^2 r$$

If we say that the left face of the cylinder is y below the free surface of the fluid then the right surface is y + dy below the surface of liquid. Thus

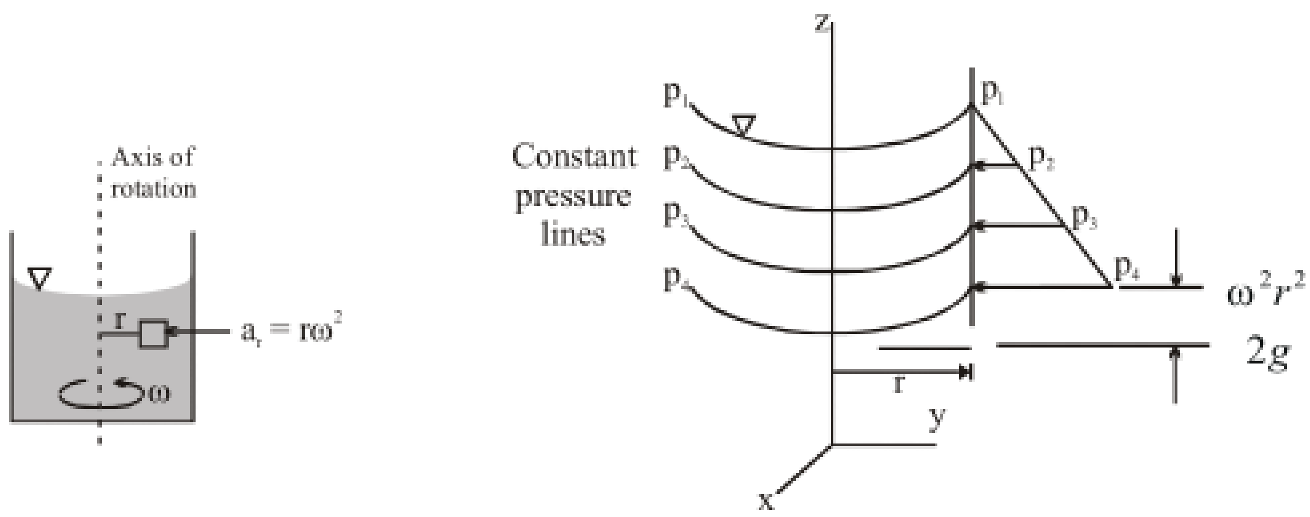
$$P_2 - P_1 = \rho g dy$$

Thus solving we get, $$\frac{dy}{dr} = \frac{r\omega^2}{g}$$

and, therefore, the equation for surfaces of constant pressure is

$$y = \frac{\omega^2 r^2}{2g} + \text{constant}$$

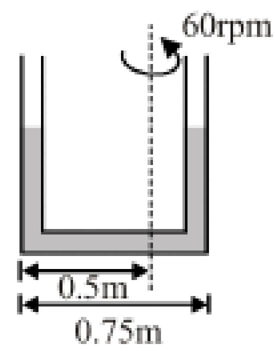
This equation means that these surfaces of constant pressure are parabolic as shown in Fig



The pressure varies with the distance from the axis of rotation, but at a fixed radius, the pressure varies hydrostatically in the vertical direction as shown in fig.

Illustration :

A vertical U - tube with the two limbs 0.75m apart is filled with water and rotated about a vertical axis 0.5m from the left limb, as shown in the figure. Determine the difference in elevation of the water levels in the two limbs. When speed of rotation is 60 rpm.

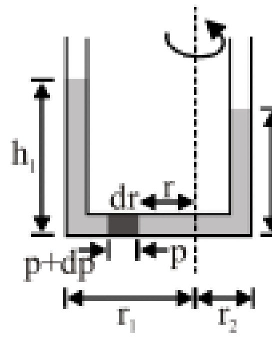


Sol. *consider a small element of length dr at a distance r from the axis of rotration considering the equilibrium of this element.*

$$(p + dp) - p = \rho\omega^2 r dr$$

or $$dp = \rho\omega^2 r\, dr$$

On integration between 1 and 2,



$$p_1 - p_2$$

$$= \rho\omega^2 \int_{-r_2}^{r_1} r dr$$

$$p_1 - p_2 = \frac{\rho\omega^2}{2}(r_1^2 - r_2^2)$$

or $$h_1 - h_2 = \frac{\omega^2}{2g}\left[r_1^2 - r_2^2\right] = \frac{(2\pi)^2}{2(10)}[(0.5)^2 - (0.25)^2] = 0.37\text{m}$$

Pascal's Principle

When you squeeze one end of a tube to get toothpaste out the other end, you are watching **Pascal's principle** in action. This principle is also the basis for the Heimlich maneuver, in which a sharp pressure increase properly applied to the abdomen is transmitted to the throat, forcefully ejecting food lodged there. The principle was first stated clearly in 1652 by Blaise Pascal (for whom the unit of pressure is named):

A change in the pressure applied to an enclosed incompressible fluid is transmitted undiminished to every portion of the fluid and to the walls of its container.

Demonstrating Pascal's Principle

Consider the case in which the incompressible fluid is a liquid contained in a tall cylinder, as in Fig. The cylinder is fitted with a piston on which a container of lead shot rests. The atmosphere, container, and shot exert pressure p_{ext} on the piston and thus on the liquid. The pressure p at any point P in the liquid is then

$$P = P_{ext} + \rho_{gh}$$

Let us add a little more lead shot to the container to increase P_{ext} by an amount P_{ext}. The quantities P_{ext}, g and h in Eq. are unchanged, so the pressure change at P is

$$\Delta p = \Delta p_{ext}$$

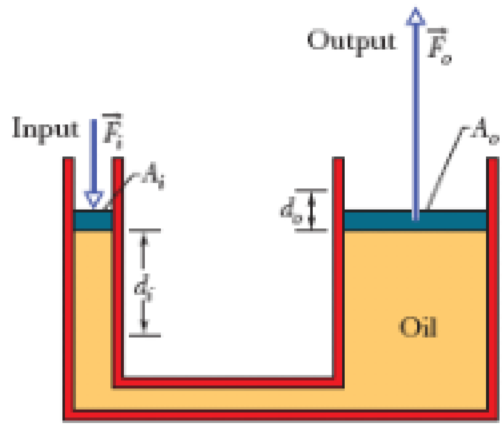
This pressure change is independent of h, so it must hold for all points within the liquid, as Pascal's principle states.

Pascal's Principle and the Hydraulic Lever

Figure shows how Pascal's principle can be made the basis of a hydraulic lever. In operation, let an external force of magnitude F_i be directed downwardon the left-hand (or input) piston, whose surface area is A_i. An incompressible liquid in the device then produces an upward force of magnitude F_0 on the right-hand (or output) piston, whose surface area is A_0. To keep the system in equilibrium, there must be a downward force of magnitude *Fo* on the output piston from an external load (not shown). The force $\vec{F}_i$ applied on the left and the downward force $\vec{F}_0$ from the load on the right produce a change Δp in the pressure of the liquid that is given by

$$\Delta p = \frac{F_i}{A_i} = \frac{F_0}{A_0}$$

$$F_0 = F_i \frac{A_0}{A_i}$$



Equation shows that the output force F_0 on the load must be greater than the input force F_i if $A_0 > A_i$ as is the case in figure.

If we move the input piston downward a distance d_i, the output piston moves upward a distance do, such that the same volume V of the incompressible liquid is displaced at both pistons.Then

$$V = A_i d_i = A_0 d_0$$

which we can write as

$$d_0 = d_i \frac{A_i}{A_0}$$

This shows that, if Ao > Ai (as in Figure), the output piston moves a smaller distance than the input piston moves.

From Eqs. we can write the output work as

$$W = F_0 d_0 = \left(F_i \frac{A_0}{A_i}\right)\left(d_i \frac{A_i}{A_0}\right) = F_i d_i$$

which shows that the work W done on the input piston by the applied force is equal to the work W done by the output piston in lifting the load placed on it.

The advantage of a hydraulic lever is this:

With a hydraulic lever, a given force applied over a given distance can be transformed to a greater force applied over a smaller distance.

The product of force and distance remains unchanged so that the same work is done. However, there is often tremendous advantage in being able to exert the larger force. Most of us, for example, cannot lift an automobile directly but can with a hydraulic jack, even though we have to pump the handle farther than the automobile rises in a series of small strokes.

Illustration :

Find the pressure in the air column at which the piston remains in equilibrium. Assume the piston to be massless and frictionless.

Sol. *Let p_a be the air pressure above the piston.*

Applying pascal's law at points A and B.

$$p_{atm} + \rho_w g(5) = p_a + \rho_k g(1.73)\sin 60^o$$

$$p_a = (10^3)(10)(5) + 10^5 - (800)(10)\frac{\sqrt{3}}{2}$$

$$= 138 \text{ kPa}$$

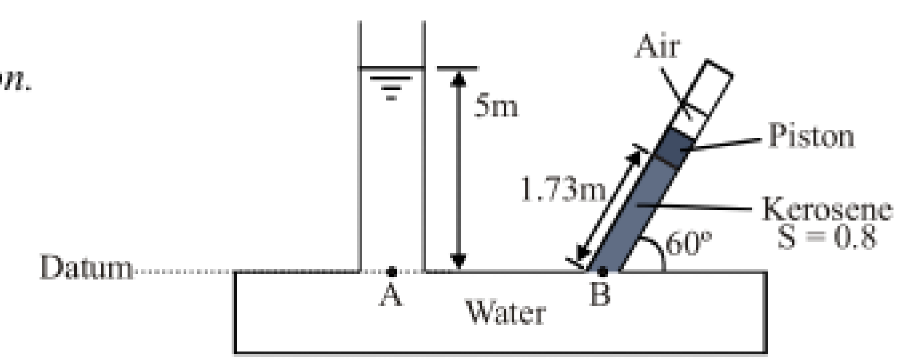
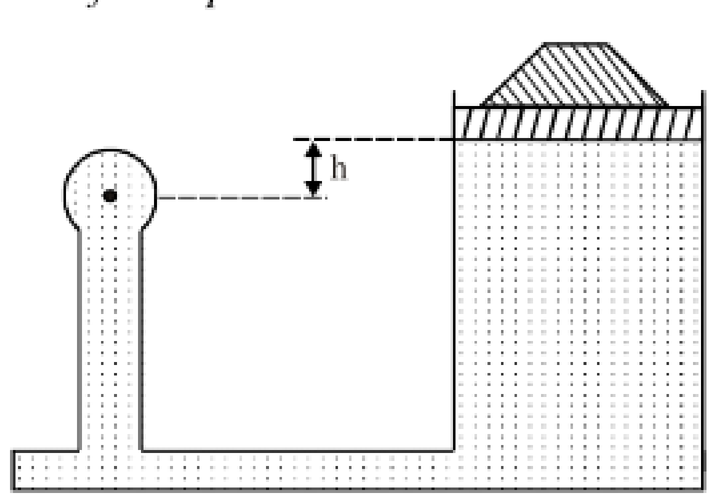


Illustration :

A weighted piston confines a fluid of density ρ in a closed container, as shown in the figure. The combined weight of piston and weight is $W = 200$ N, and the cross-sectional area of the piston is $A = 8$ cm^2. Find the total pressure at point B if the fluid is mercury and $h = 25$ cm ($\rho_m = 13600$ kg/m^3). What would an ordinary pressure gauge read at B ?

Sol. *Notice what Pascal's principle tells us about the pressure applied to the fluid by the piston and atmosphere. This added pressure is applied at all points within the fluid. Therefore, the total pressure at B is composed of three parts :*



Pressure of atmosphere $= 1.0 \times 10^5$ *Pa*

Pressure due to pistion and weight $= \frac{W}{A} = \frac{200N}{8\times10^{-4}m^2} = 2.5\times10^5 Pa$

Pressure due to height h of fluid $= h\rho g = 0.33 \times 10^5$ *Pa*

In this case, the pressure of the fluid itself is relatively small. We have

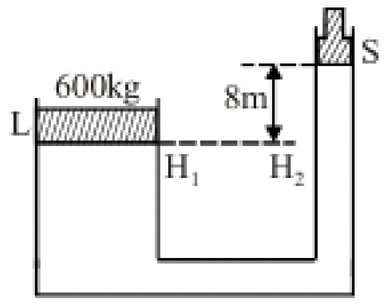
Total pressure at B $= 3.8 \times 10^5$ *Pa = 380 kPa*

The gauge pressure does not include atmospheric pressure. Therefore,

Gauge pressure at B = 280 kPa

Illustration :

For the system shown in figure, the cylinder on the left, at L, has mass of 600 kg and a cross-sectional area 25 cm^2 *and negligible weight. If the apparatus is filled with oil (*$\rho = 0.78$ *g/cm*^{3}*), find the force F required to hold the system in equilibrium as shown in figure.*



Sol. *The pressure at point* H_1 *and* H_2 *are equal because they are at the same level in the single connected fluid. Therefore,*

Pressure at H_1 *= Pressure at* H_2

(Pressure due to left piston) = (Pressure due to F and right piston) + (pressure due to 8 m of oil)

$$\frac{(600)(9.8)N}{0.08m^2} = \frac{F}{25\times10^{-4}m^2} + (8m)(780kg/m^3)(9.8m/s^2)$$

After solving, we get , F = 31 N.

Practice Exercise

Q.1 The passenger are advised to remove the ink from their pens whils going up in an aeroplane. Explain why?

Q.2 A hydraulic press has a ram (weight arm) 12.5cm in diameter and plunger (Force arm) of 1.25 cm diameter what force would be required by plunger to raise a weight of 1 tonn on the ram.

Q.3 Pressure 3 m below free surface of a liquid is 15 KN/m^2 in excess of atmosphere pressure. Determine its density and specific gravity. [g = 10 m/sec^2]

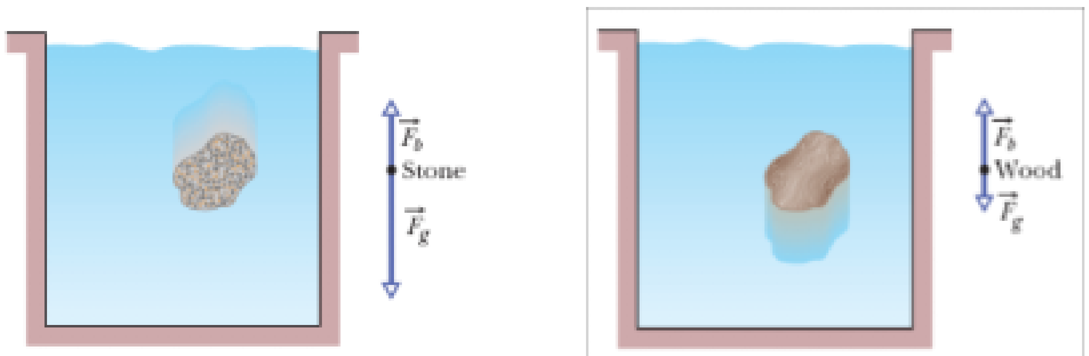
Answers

Q.1 Pressure at heights gets reduced, resulting rising of ink and leakage.

Q.2 10 kg (98.1 N) Q.3 500 kg/m^3, 0.5

Archimedes' Principle

Figure shows a student in a swimming pool, manipulating a very thin plastic sack (of negligible mass) that is filled with water. She finds that the sack and its contained water are in static equilibrium, tending neither to rise nor to sink. The downward gravitational force $\vec{F}_g$ on the contained water must be balanced by a net upward force from the water surrounding the sack.



(b) (c)

This net upward force is a **buoyant force** $\vec{F}_b$. It exists because the pressure in the surrounding water increases with depth below the surface. Thus, the pressure near the bottom of the sack is greater than the pressure near the top, which means the forces on the sack due to this pressure are greater in magnitude near the bottom of the sack than near the top. Some of the forces are represented in Fig. (a), where the space occupied by the sack has been left empty. Note that the force vectors drawn near the bottom of that space (with upward components) have longer lengths than those drawn near the top of the sack (with downward components). If we vectorially add all the forces on the sack from the water, the horizontal components cancel and the vertical components add to yield the upward buoyant force $\vec{F}_b$ on the sack. (Force $\vec{F}_b$ is shown to the right of the pool in Fig. (a.) Because the sack of water is in static equilibrium, the magnitude $\vec{F}_b$ of is equal to the magnitude $m_f g$ of the gravitational force $\vec{F}_g$ on the sack of water: $F_b = m_f g$. (Subscript *f* refers to fluid, here the water.) In words, the magnitude of the buoyant force is equal to the weight of the water in the sack.

In Fig. (b), we have replaced the sack of water with a stone that exactly fills the hole in Fig. (a). The stone is said to displace the water, meaning that the stone occupies space that would otherwise be occupied by water. We have changed nothing about the shape of the hole, so the forces at the hole's surface must be the same as when the water-filled sack was in place. Thus, the same upward buoyant force that acted on the water-filled sack now acts on the stone; that is, the magnitude $\vec{F}_b$ of the buoyant force is equal to $m_f g$, the weight of the water displaced by the stone.

Unlike the water-filled sack, the stone is not in static equilibrium. The downward gravitational force $\vec{F}_g$ on the stone is greater in magnitude than the upward buoyant force, as is shown in the free-body diagram in Fig. (*b*). The stone thus accelerates downward, sinking to the bottom of the pool. Let us next exactly fill the hole in Fig. (a) with a block of lightweight wood, as in Fig. (c*)*. Again, nothing has changed about the forces at the hole's surface, so the magnitude $\vec{F}_b$ of the buoyant force is still equal to $m_f g$, the weight of the displaced water. Like the stone, the block is not in static equilibrium. However, this time the gravitational force $\vec{F}_g$ is lesser in magnitude than the buoyant force (as shown to the right of the pool), and so the block accelerates upward, rising to the top surface of the water. Our results with the sack, stone, and block apply to all fluids and are summarized in **Archimedes' principle:**

When a body is fully or partially submerged in a fluid, a buoyant force $\vec{F}_b$ from the surrounding fluid acts on the body.The force is directed upward and has a magnitude equal to the weight $m_f g$ of the fluid that has been displaced by the body. The buoyant force on a body in a fluid has the magnitude

$$F_b = m_f g \text{ (buoyant force)}$$

where m_f is the mass of the fluid that is displaced by the body.

Illustration :

Fin the density and specific gravity of gasoline if 51 g occupies 75 cm³ ?

Sol. $Density = \frac{\text{mass}}{\text{volume}} = \frac{0.051\text{kg}}{75\times10^{-6}\text{m}^3} = 680\,\text{kg}/\text{m}^3$

$Sp.gr = \frac{\text{density of gasoline}}{\text{density of water}} = \frac{680\text{kg}/\text{m}^3}{1000\,\text{kg}/\text{m}^3} = 0.68$

$Sp.\ gravity = \frac{\text{mass of } 75\,\text{cm}^3 \text{ gasoline}}{\text{mass of } 75\text{cm}^3 \text{ water}} = \frac{51\text{kg}}{75\,\text{kg}} = 0.68$

Illustration :

The mass of a liter of milk is 1.032 kg. The butterfat that it contains has a density of 865 kg/m³ when pure, and it constitutes 4 present of the milk by volume. What is the density of the fat-free skimmend milk?

Sol. *Volume of fat in* $1000cm^3$ *of milk* $= 4\% \times 1000\ cm^3 = 40cm^3$

Mass of $40\ cm^3$ *fat* $= v\rho = (40\times10^{-6}m^3)\ (865\ kg/m^3) = 0.0346\ kg$

Denstiy of skimmed milk $= \frac{\text{mass}}{\text{volume}} = \frac{(1.032-0.0346)\text{kg}}{(1000-40)\times10^{-6}\text{m}^3} = 1039\text{kg}/\text{m}^3$

Illustration :

An iceberg with a diensity of 920 kg m⁻³. What fraction of the iceberg is visible.

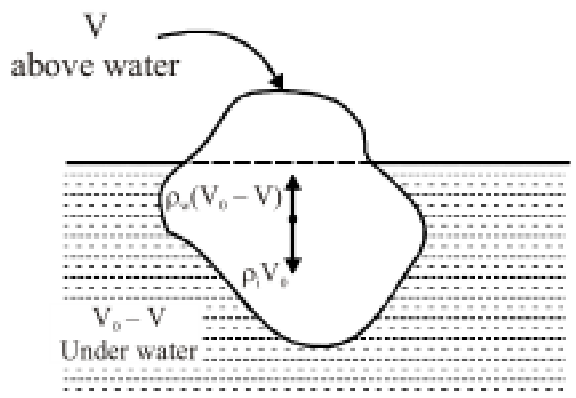
Sol. *Let V be the volume of the iceberg above the water surface, then the volume under water will be* $V_0 - V$.

Under floating conditions, the weight $(\rho_i V_0 g)$ *of the iceberg is balanced by the buoyant force* $\rho_w (V_0 - V)g$. *Thus,*

$$\rho_i V_0 g = \rho_w (V_0 - V)g$$

or $\rho_w V = (\rho_w - \rho_i)\ V_0$

or $\rho_w V = (\rho_w - \rho_i) V_0$



or $$\frac{V}{V_0} = \left(\frac{\rho_w - \rho_i}{\rho_w}\right)$$

Since $\rho_w = 1025\ kg\ m^{-3}$ and $\rho_i = 920\ kg\ m^{-3}$, therefore,

$$\frac{V}{V_0} = \frac{1025 - 920}{1025} = 0.10$$

Hence 10% of the total volume is visible.

Illustration :

When a 2.5 kg crown is immersed in water, it has an apparent weight of 22 N. What is the density of the crown ?

Sol. *Let W = actual weight of the crown*

W' = apparent weight of the crown

ρ = *density of crown*

ρ_0 = *density of water*

The buoyant force is given by

$$F_B = W - W'$$

or $$\rho_0 V g = W - W'$$

Since $W = \rho V g$ therefore, $V = \dfrac{W}{\rho g}$

Eliminating V from the above two equation, we get

$$\rho = \frac{\rho_0 W}{W - W'}$$

Here $W = 25N$; $W' = 22\ N$; $\rho_0 = 10^3\ kg\ m^{-3}$

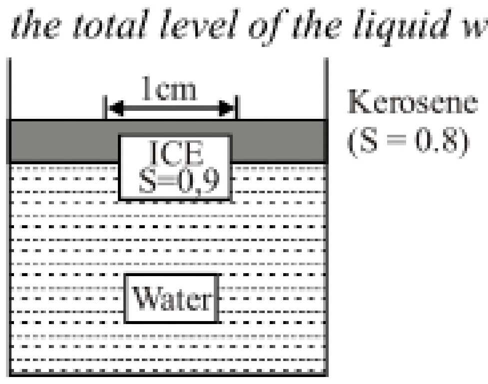
$$\therefore \quad \rho = \frac{(10^3)(25)}{25 - 22} = 8.3 \times 10^3\ kg\ m^{-3}$$

Illustration :

An ice cube of side 1 cm is floating at the interface of kerosene and water in a beaker of base area 10 cm². The level of kerosene is just covering the top surface of the ice cube.

(a) Find the depth of submergence in the kerosene and that in the water.

(b) Find the change in the total level of the liquid when the whole ice melts into water.



Sol. *(a) Condition of floating*

$$0.8\ \rho_w g h_k + \rho_w g h_w = 0.9\ \rho_w g h$$

or $$0.8 h_k + h_w = (0.9)\ h \qquad ...(i)$$

Where h_k *and* h_w *be the submerged depth of the ice in the kerosene and water, respectively.*

Also $$h_k + h_w = h \qquad ...(ii)$$

Solving equation (i) and (ii) we get $h_k = 0.5\ cm\ h_w = 0.5\ cm$

(b) $\underset{\text{(Ice)}}{1\text{cm}^3} \xrightarrow{\text{melts}} \underset{\text{(water)}}{0.9\text{cm}^3}$

Fall in the level of kerosene $\Delta h_k = \dfrac{0.5}{\text{A}}$

Rise in the level of water $\Delta h_w = \dfrac{0.9 - 0.5}{\text{A}} = \dfrac{0.4}{\text{A}}$

Net fall in the overall level.

$$\Delta h = \frac{0.1}{\text{A}} = \frac{0.1}{10} = 0.01\text{cm} = 0.1\text{mm}$$

Illustration :

Find the density and specific gravity of gasoline if 51 g occupies 75 cm³ ?

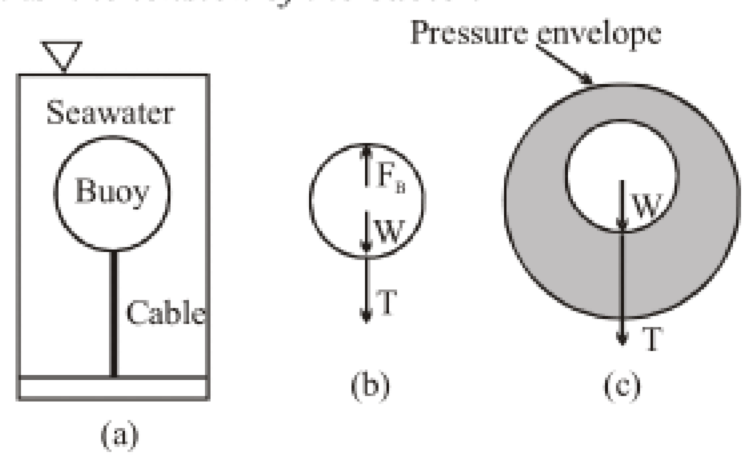
Sol. $$\textit{Density} = \frac{\text{mass}}{\text{volume}} = \frac{0.51\text{kg}}{75 \times 10^{-6}\text{m}^3} = 680\text{kg}/\text{m}^3$$

$$\textit{Sp. gr} = \frac{\text{denstiy of gasoline}}{\text{diensity of water}} = \frac{680\text{kg}/\text{m}^3}{1000\text{kg}/\text{m}^3} = 0.68$$

or $$\textit{Sp. graviy} = \frac{\text{mass of } 75\text{cm}^3 \text{ gasoline}}{\text{mass of } 75\text{cm}^3 \text{ water}} = \frac{51\text{g}}{75\text{g}} = 0.68$$

Illustration :

A spherical buoy has a diameter of 2 m, and mass 100 kg, and is anchored to the seafloor with a cable as is shown in Fig.(a). The buoy is completely immersed in water as illustrated. For this condition what is the tension of the cable ?



Sol. *We first draw a free-body diagram of the buoy as is shown in Fig. (b), where* F_B *is the buoyant force acting on the buoy, W is the weight of the buoy, and T is the tension in the cable.*

For equilibrium it follows that

$$T = F_B - W$$

From Eq. $F_B = \rho V g$

and for water with $\rho = 1000\ Kg/m^3$ *and* $V = \pi d^3/6$ *then*

$F_B = (1000\ Kg/m^3)\ [(\pi/6)(2m)^3] = 4188.8\ N$

The tension in the cable can now be calculated as

$T = 4188.8\ N - 1000\ N = 3188.8\ N$ *(Ans.)*

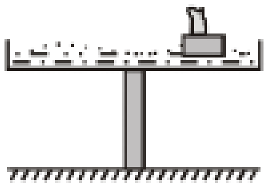
Illustration :

A wooden block floats vertically in a glass filled with water. How will the level of the water in the glass change if the block is kept in a horizontal position ?

Sol. *The level of the water will not change because the quantity of water displaced will remain the same.*

Illustration :

A vessel filled with water is placed exactly in middle of a thin wall(fig.). Will the system topple if a small wooden boat carrying some weight is floated in the vessel?



Sol. *The system will not topple, since according to Pascal's law the pressure on the bottom of the vessel will be the same everywhere thus the body will still remain in rotational equilibirium*

Illustration :

A homogeneous piece of ice floats in a glass filled with water. How will the level of the water in the glass change when the ice melts ?

Sol. *Since the piece of ice floats, the weight of the water displaced by it is equal to the weight of the ice itself or the weight of the water it produces upon melting. For this reason the water formed by the piece of ice will occupy a volume equal to that of the submerged portion, and the level of the water will not change.*

Illustration :

A piece of ice is floating in a tub filled with water. How will the level of the water in the tub change when the ice melts ? Consider the following cases :

(1) a stone is frozen in the ice

(2) the ice contains an air bubble

Sol. *(1) The volume of the submerged portion of the piece with the stone is greater than the sum of the volumes of the stone and the water produced by the melting ice. Therefore, the level of the water in the glass will drop.*

(2) The weight of the displaced water is equal to that of the ice (the weight of the air in the bubble may be neglected). For this reason, as in conceptual eg., the level of the water will not change.

Illustration :

A vessel with a body floating in it is kept in elevator accelerating downwards with acceleration a such that $a < g$. Will the body rise or sink further in the vessel?

Sol. *The force of bouyancy on the body can be written as $F = \rho V_2(g - a)$, where V_2 is the volume of the submerged portion of the body in the lift. As pressure at a point h below the surface will become $\rho(g - a)h$ instead of ρgh. Applying the Newton's second Law, remembering that the body was accelerating upwards at a.*

$$Mg - \rho V_2 (g - a) = Ma$$

Hence, $V_2 = \dfrac{M}{\rho}$ *thus* $V_2 = V$*, as in a stationary vessel,* $V = \dfrac{M}{\rho}$*. Thus the body does not rise to the surface.*

Illustration :

Mercury is poured into two communicating identical cylindrical vessels then equal amount of water is poured in both the vessels above the mercury. The level of the water in both vessels becomes same. Will the level of the water and the mercury be the same if a piece of wood is dropped into one vessel and some water equal in weight to this piece is added to the other ?

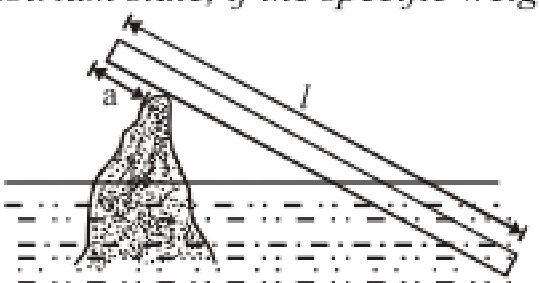
Sol. *Before the piece of wood is droped the level of the mercury was same, as the level of the water in both the vessel is same.*

The piece of wood applies the same force as the water added, on the mercury. So the level of the mercury in both vessels will be same.

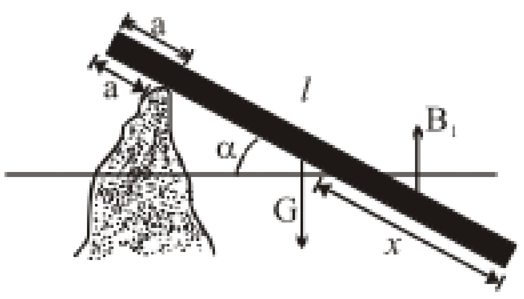
The submerged portion of the piece of wood occupies the same volume as the equal weight of the water. As water that will be displaced by this piece is equal to, its weight which is equal to the amount of water added. Therefore, if the crossections of the vessels are the same, the level of the water in both vessels will coincide.

Illustration :

One end of a board of length l is hinged on top of a stone protruding from water. Length a of the board is above the point of support (Fig.). What part of the board is below the surface of the water in equillibirium state, if the specific weight of wood is γ ?



Sol. *We can not solve this problem by equating net force to zero,as the force applied by the stone is unknown. Thus we will balance the torques of the forces acting on the board with respect to point C (Fig.). This will exclude the force applied by the stone, as the moment arm for this force will be zero.*



Here B_l is the bouyant force applied by the water on the submerged portion and isequal to the volume of submerged portion multiplied to density of water.

$B_l = Ax\gamma_0$ *The length of moment arm is* $\left(l-a-\frac{x}{2}\right)\cos\alpha$

Weight of the board is $W = Al\gamma$. Here A is the cross-sectional area of the board and γ_0 the specific weight of the water. Equating torque we get

$$B_l\left(l-a-\frac{x}{2}\right)\cos\alpha = W\left(\frac{1}{2}-a\right)\cos\alpha$$

Hence , $$x = (l-a) \pm \sqrt{(l-a)^2 - \frac{\gamma}{\gamma_0}l(l-2a)}$$

Since $x < l - a$, only one solution is valid :

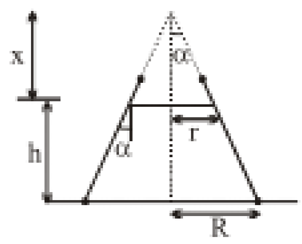
$$x = (l-a) - \sqrt{(l-a)^2 - \frac{\gamma}{\gamma_0}l(l-2a)}$$

Illustration :

A conical vessel without a bottom tightly stands on a table. A liquid is poured into the vessel and as soon as its level reaches the height h, the pressure of the liquid raises the vessel. The radius of the bottom greater base of the vessel is R, the semi vertex angle of the cone is α, and the weight of the vessel is W. What is the density of the liquid ?.

Sol. *We need to calculate the force mentioned above. The direct solution may appear to be taking elemental rings and integrating the force. But there is a faster approach. If we consider the liquid as a system in equilibibrium, then only unknown force acting on it will be the one applied by the sidewalls. According to Newton's third law, an identical force acts on the vessel.*

Lets assume when the height of the liquid is h in the vessel the vessel rises.



$$\tan\alpha = \frac{r}{x} = \frac{R}{x+h}$$

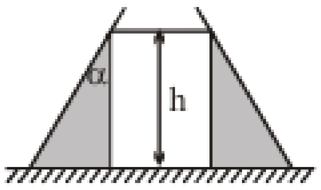
on solving $r = R - h\tan\alpha$ ....(1)

The pressure on the "bottom" of the vessel is ρgh and the force with which the hatched portion (the truncated cone minus the cylinder volume) of the liquid presses on the table is $\rho gh\pi(R^2 - r^2) = \rho gh\pi(2Rh\tan\alpha - h^2\tan^2\alpha)$

According to Newton's third law, an identical force acts on the liquid. For equilibrium of the liquid at the moment when the vessel starts rising

$$W + W_l = \rho gh\pi(2Rh\tan\alpha - h^2\tan^2\alpha)$$

where W_l is the weight of the hatched portion of the liquid



$$W_l = \frac{\rho g h}{3}\{\pi R^2 + \pi(R - h\tan\alpha)^2 + \pi R(R - h\tan\alpha)\} - \rho g h \pi (R - h\tan\alpha)^2$$

Therefore, $$\rho = \frac{W}{\pi g h^2 \tan\alpha\left(R - \frac{h\tan\alpha}{3}\right)}$$

Practice Exercise

Q.1 A boy is carrying a fish in one hand and a bucket full of water in the other hand. He then place the fish in the bucket and thinks that in accordance with Archimedes's prinicple he is now carrying less weight as weight of fish will reduce due to upthrust. Is he thinking right ?

Q.2 Ice flows in water nine tenth of its volume submerged. What is the frctional volume submerged for an iceberg floating on a fresh water lake of a (hypothetical) planet whose gravity is ten times of earth ?

Q.3 If the body is non-homogeneous, then the body rotates in the fluid why ?

Q.4 A cube of wood supporting a 200 gm mass just floats in water. When the mass is removed the cube rise by 2 cm. Find the size of cube

Q.5 A solid ball of density half that of water falls freely under gravity from a height of 19.6 m and then enter water. Upto what depht will the ball go ? How much time will it take to come again to the water surface ? Neglect air resistance and viscosity effects in water.

Q.6 A balloon filled with hydrogen has a volume of 1000 liters and its mass of 1 kg. What would be volume of the block of a very light material which it can just lift ? One litre of the material has a mass of 91.3 gm. (Density of air = 1.3 gm/ litre)

Q.7 An iceberg of density 915 kg/m^3 extends above the surface of sea water of density 1030 kg/m^3. What percent-age of the total volume of iceberg is visibel to an obserber.

Answers

Q.1 it's density is high because of salt Q.2 Same

Q.3 Centre of Buoyancy and centre of gravity are diffrent resulting torque.

Q.4 10 cm Q.5 19.6 m, 4 sec Q.6 3.33 litre. Q.7 11.15%

Floating

When we release a block of lightweight wood just above the water in a pool, the block moves into the water because the gravitational force on it pulls it downward.

As the block displaces more and more water, the magnitude F_b of the upward buoyant force acting on it increases. Eventually, F_b is large enough to equal the magnitude F_g of the downward gravitational force on the block, and the block comes to rest. The block is then in static equilibrium and is said to be floating in the water. In general, When a body floats in a fluid, the magnitude F_b of the buoyant force on the body is equal to the magnitude F_g of the gravitational force on the body.

We can write this statement as

$$F_b = F_g \text{ (Floating)}$$

From Eq. we know that $F_b = m_f g$. Thus,

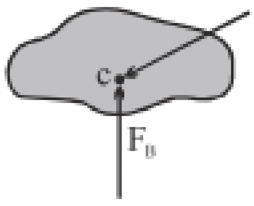
When a body floats in a fluid, the magnitude F_g of the gravitational force on the body is equal to the weight $m_f g$ of the fluid that has been displaced by the body.

We can write this statement as

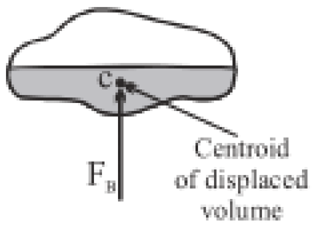
$$F_g = m_f g$$

In other words, a floating body displaces its own weight of fluid.

The location of the line of action of the buoyant force can be determined by adding torques of the forces due to pressure forces, with respect to some convenient axis. The buoyant force must pass through the center of mass of the desplaced volume, as shown in Fig. (c), as it was in translational and rotational equilibirium. The point through which the buoyant force acts is called the center of buoyancy.



(c)



(D)

These same results apply to floating bodies which are only partially submerged, as shown in Fig.(d), if the density of the fluid above the liquid surface is very small compared with the liquid in which the body floats. Since the fluid above the surface is usually air, for practial purposes this condition is satisfied.

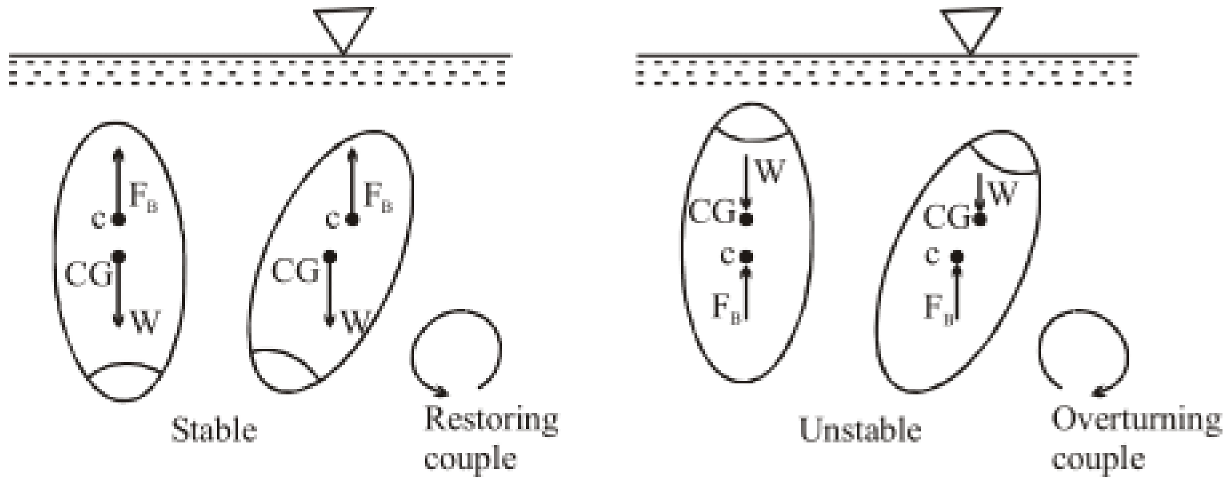
In the above discussion, the fluid is assumed to have a constant density. If a body is immersed in a fluid in which density varies with depth, such as having multiple layers of fluid, the magnitude of the buoyant force remains equal to the weight of the displaced fluid and the buoyant force passes through the center of mass of the displaced volume.

Stability

The center of buoyancy and center of gravity do not necessarily coincide so the floating or submerged body may not be in stable equilibirium. A small rotation can cause the buoyant force to produce either a restoring or overturning torque. For example, for the completely submerged body shown in Fig., which has a center of gravity below the center of buoyancy, a rotation from its equilibrium position will create a restoring torque by the buoyant force, F_B, which causes the body to rotate back to its original position. Thus, if the center of gravity falls below the center of buoyancy, the body is stable.

However, as shown in Fig., if the center of gravity of the completely submerged body is above the center of buoyancy, the resulting torque formed by the weight and the buoyant force will cause the body to overturn and move to a new equilibrium position. Thus, a completely submerged body with its center of gravity above its center of buoyancy is in an unstable equilibrium position.

For floating bodies the stability problem is more complicated, since as the body rotates the location of the center of buoyancy may change.



Fluid Dynamics

Ideal Fluids in Motion

The motion of real fluids is very complicated and not yet fully understood. Instead, we shall discuss the motion of an **ideal fluid,** which is simpler to handle mathematically and yet provides useful results. Here are four assumptions that we make about our ideal fluid; they all are concerned with flow:

1. **Steady flow** In steady (or laminar) flow, the velocity of the moving fluid at any fixed point does not change with time, either in magnitude or in direction. The gentle flow of water near the center of a quiet stream is steady; the flow in a chain of rapids is not. The speed of the smoke particles increases as they rise and, at a certain critical speed, the flow changes from steady to nonsteady.

2. **Incompressible flow** We assume, as for fluids at rest, that our ideal fluid is incompressible; that is, its density has a constant, uniform value.

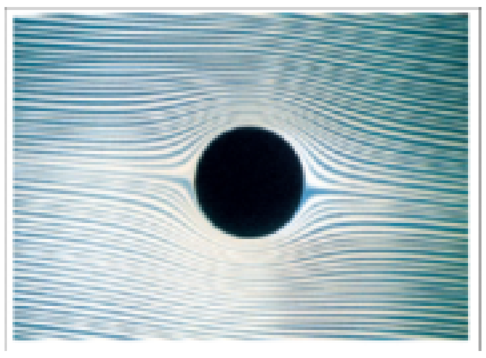
3. **Nonviscous flow** Roughly speaking, the viscosity of a fluid is a measure of how resistive the fluid is to flow. For example, thick honey is more resistive to flow than water, and so honey is said to be more viscous than water. Viscosity is the fluid analog of friction between solids; both are mechanisms by which

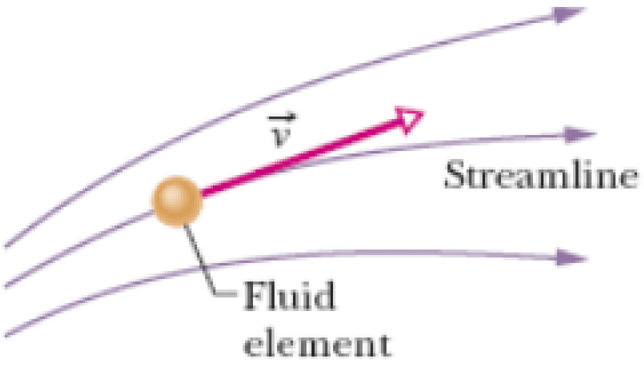
the kinetic energy of moving objects can be transferred to thermal energy. In the absence of friction, a block could glide at constant speed along a horizontal surface. In the same way, an object moving through a nonviscous fluid would experience no viscous drag force-that is, no resistive force due to viscosity; it could move at constant speed through the fluid.

4. **Irrotational flow** : Although it need not concern us further, we also assume that the flow is irrotational. To test for this property, let a tiny grain of dust move with the fluid. Although this test body may (or may not) move in a circular path, in irrotational flow the test body will not rotate about an axis through its own center of mass. For a loose analogy, the motion of a Ferris wheel is rotational; that of its passengers is irrotational. T8hat the velocity of a particle is always tangent to the path taken by the particle. Here the particle is the fluid element, and its velocity is $\vec{v}$ always tangent to a streamline (Figure). For this reason, two streamlines can never intersect; if they did, then an element arriving at their intersection would have two different velocities simultaneously an impossibility.

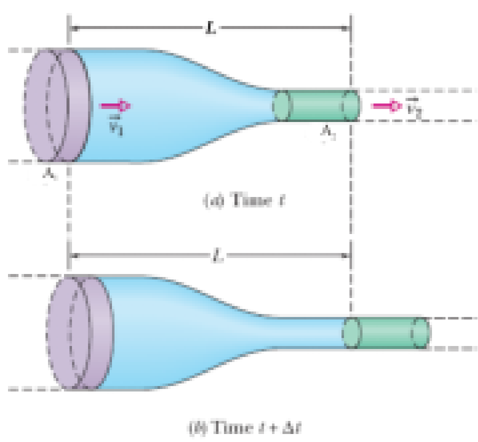
The Equation of Continuity

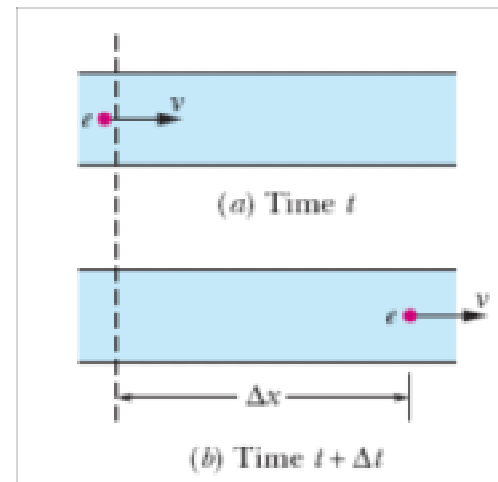
You may have noticed that you can increase the speed of the water emerging from a garden hose by partially closing the hose opening with your thumb. Apparently the speed v of the water depends on the cross-sectional area A through which the water flows.

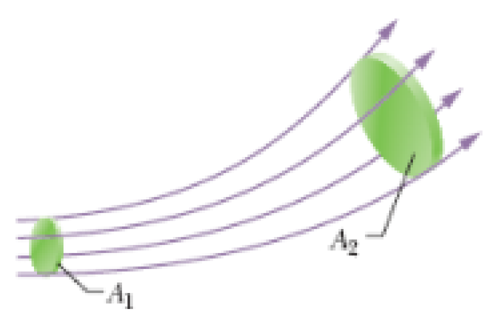




Here we wish to derive an expression that relates v and A for the steady flow of an ideal fluid through a tube with varying cross section, like that in Figure. The flow there is toward the right, and the tube segment shown (part of a longer tube) has length L. The fluid has speeds v_1 at the left end of the segment and v_2 at the right end. The tube has cross-sectional areas A_1 at the left end and A_2 at the right end. Suppose that in a time interval Δt a volume ΔV of fluid enters the tube segment at its left end (that volume is colored purple in Figure. Then, because the fluid is incompressible, an identical volume ΔV must emerge from the right end of the segment (it is colored green in Figure). We can use this common volume ΔV to relate the speeds and areas. To do so, we first consider Fig. , which shows a side view of a tube of uniform crosssectional area A. In Fig.(a), a fluid element e is about to pass through the dashed line drawn across the tube width.The element's speed is v, so during a time interval Δt, the element moves along the tube a distance $\Delta x = v\,\Delta t$.The volume ΔV of fluid that has passed through the dashed line in that time interval Δt is







$$\Delta V = A\,\Delta x = Av\,\Delta t.$$

Applying Eq. to both the left and right ends of the tube segment in Fig., we have

$$\Delta V = A_1 v_1\,\Delta t = A_2 v_2\,\Delta t$$

Or $$A_1 v_1 = A_2 v_2 \text{ (equation of continuity)}$$

This relation between speed and cross-sectional area is called the **equation of continuity** for the flow of an ideal fluid. It tells us that the flow speed increases when we decrease the cross-sectional area through which the fluid flows (as when we partially close off a garden hose with a thumb). Equation applies not only to an actual tube but also to any so-called tube of flow, or imaginary tube whose boundary consists of streamlines. Such a tube acts like a real tube because no fluid element can cross a streamline; thus, all the fluid within a tube of flow must remain within its boundary. Figure shows a tube of flow in which the cross-sectional area increases from area A_1 to area A_2 along the flow direction. From Eq. we know that, with the increase in area, the speed must decrease, as is indicated by the greater spacing between streamlines at the right in Fig. . Similarly, you can see that in Fig. the speed of the flow is greatest just above and just below the cylinder. We can rewrite Eq. as

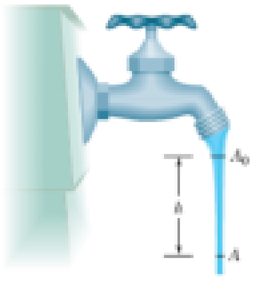
$R_V = Av$ = a constant (volume flow rate, equation of continuity), in which R_V is the **volume flow rate** of the fluid (volume past a given point per unit time). Its SI unit is the cubic meter per second (m^3/s). If the density ρ of the fluid is uniform, we can multiply Eq. by that density to get the **mass flow rate** R_m (mass per unit time):

$$R_m = \rho R_V = \rho A v = \text{a constant (mass flow rate).}$$

The SI unit of mass flow rate is the kilogram per second (kg/s). Equation says that the mass that flows into the tube segment of Fig. each second must be equal to the mass that flows out of that segment each second.

Illustration :

Figure shows how the stream of water emerging from a faucet "necks down" as it falls. The indicated cross-sectional areas are A_0 = 1.2 cm2 and A =0.35 cm^2. The two levels are separated by a vertical distance h = 45 mm. What is the volume flow rate from the tap?
The volume flow rate through the higher cross section must be the same as that through the lower cross section.



Sol. *where v_0 and v are the water speeds at the levels corresponding to A_0 and A. From Eq. we can also write, because the water is falling freely with acceleration g,*

$$v_2 = v^2{}_0 - 2gh.$$

Eliminating v between Eqs. and solving for v_0, we obtain

$$v_0 = \sqrt{\frac{2ghA^2}{A_0^2 - A^2}}$$

$$v_0 = \sqrt{\frac{(2)\times(9.8\text{m}/\text{s}^2)(0.045\text{m})(0.35\text{cm}^2)^2}{(1.2\text{cm}^2) - (0.35\text{cm}^2)^2}}$$

$$v_0 = 0.286 \text{ m/s} = 28.6 \text{ cm/s}.$$

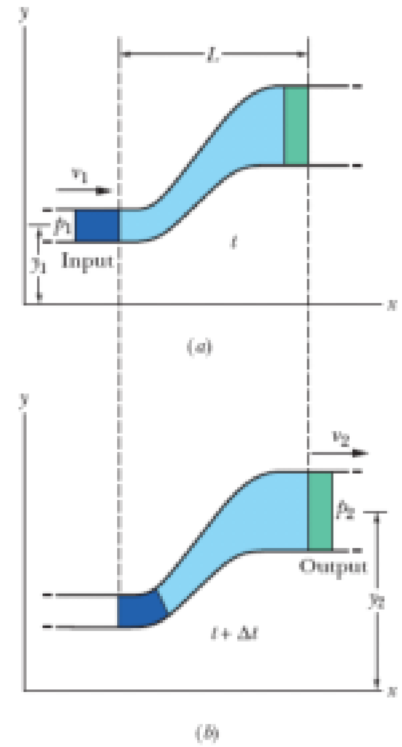
From Eq. , the volume flow rate R_V is then

$$R_V = A_0 v_0 = (1.2 \text{ cm}^2)(28.6 \text{ cm/s})$$
$$= 34 \text{ cm}^3/\text{s}.$$

Bernoulli's Equation

Figure represents a tube through which an ideal fluid is flowing at a steady rate. In a time interval Δt, suppose that a volume of fluid ΔV, colored purple in Fig. , enters the tube at the left (or input) end and an identical volume, colored green in Fig. , emerges at the right (or output) end. The emerging volume must be the same as the entering volume because the fluid is incompressible, with an assumed constant density ρ.

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$$



Let y_1, v_1, and p_1 be the elevation, speed, and pressure of the fluid entering at the left, and y_2, v_2, and p_2 be the corresponding quantities for the fluid emerging at the right. By applying the principle of conservation of energy to the fluid, we shall show that these quantities are related by

In general, the term $\frac{1}{2}\rho v^2$ is called the fluid's **kinetic energy density** (kinetic energy per unit volume).We can also write Eq. as

$$p + \frac{1}{2}\rho v^2 + \rho g y \text{ a constant (Bernoulli's equation).}$$

Equations are equivalent forms of **Bernoulli's equation,** after Daniel Bernoulli, who studied fluid flow in the 1700s.* Like the equation of continuity Eq., Bernoulli's equation is not a new principle but simply the reformulation of a familiar principle in a form more suitable to fluid mechanics. As a check, let us apply Bernoulli's equation to fluids at rest, by putting $v_1 = v_2 = 0$ in Eq. The result is

$$p_2 = p_1 + \rho g(y_1 - y_2)$$

Which is equation.

A major prediction of Bernoulli's equation emerges if we take y to be a constant ($y = 0$, say) so that the fluid does not change elevation as it flows. Equation then becomes

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$$

Which tells us that :

If the speed of a fluid element increases as the element travels along a horizontal streamline, the pressure of the fluid must decrease, and conversely.

Put another way, where the streamlines are relatively close together (where the velocity is relatively great), the pressure is relatively low, and conversely. The link between a change in speed and a change in pressure makes sense if you consider a fluid element. When the element nears a narrow region, the higher pressure behind it accelerates it so that it then has a greater speed in the narrow region. When it nears a wide region, the higher pressure ahead of it decelerates it so that it then has a lesser speed in the wide region. Bernoulli's equation is strictly valid only to the extent that the fluid is ideal. If viscous forces are present, thermal energy will be involved. We take no account of this in the derivation that follows.

Proof of Bernoulli's Equation

Let us take as our system the entire volume of the (ideal) fluid shown in Fig. .We shall apply the principle of conservation of energy to this system as it moves from its initial state to its final state. The fluid lying between the two vertical planes separated by a distance L in Fig. does not change its properties during this process; we need be concerned only with changes that take place at the input and output ends. First, we apply energy conservation in the form of the work–kinetic energy theorem,

$$W = \Delta K,$$

which tells us that the change in the kinetic energy of our system must equal the net work done on the system. The change in kinetic energy results from the change in speed between the ends of the tube and is

$$\Delta K = \frac{1}{2}\Delta m v_2^2 - \frac{1}{2}\Delta m v_1^2 = \frac{1}{2}\rho\Delta V(v_2^2 - v_1^2)$$

in which Δm $(= \rho\,\Delta V)$ is the mass of the fluid that enters at the input end and leaves at the output end during a small time interval Δt.

The work done on the system arises from two sources.The work W_g done by the gravitational force $(\Delta m\,\vec{g})$ on the fluid of mass Δm during the vertical lift of the mass from the input level to the output level is

$$W_g = \Delta m\, g(y_2 - y_1)$$
$$= -\rho\, g\Delta V(y_2 - y_1)$$

This work is negative because the upward displacement and the downward gravitational force have opposite directions.

Work must also be done on the system (at the input end) to push the entering fluid into the tube and by the system (at the output end) to push forward the fluid that is located ahead of the emerging fluid. In general, the work done by a force of magnitude F, acting on a fluid sample contained in a tube of area A to move the fluid through a distance Δx, is

$$F\Delta x = (pA)(\Delta x) = p(A\Delta x) = p\Delta v$$

The work done on the system is then $p_1\,\Delta V$, and the work done by the system is $-p_2\,\Delta V$. Their sum Wp is Wp

$$Wp = p_1\,\Delta V - p_2\,\Delta V$$
$$= -\,(p_2 - p_1)\,\Delta V.$$

The work–kinetic energy theorem of Eq. now becomes

$$W = Wg + Wp = \Delta K.$$

Substituting from Eqs. yields

$$-\rho g\Delta V(y_2 - y_1) - \Delta V(p_2 - p_1) = \frac{1}{2}\rho\Delta v(v_2^2 - v_1^2)$$

This, after a slight rearrangement, matches Eq. , which we set out to prove.

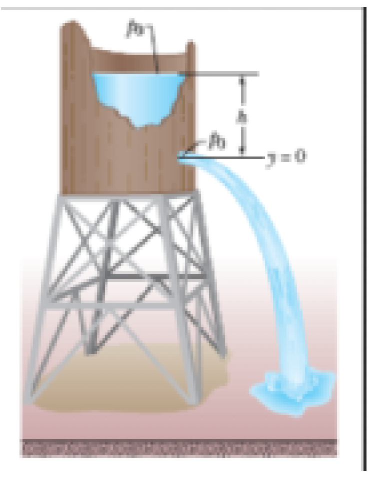
Illustration :

In the old West, a desperado fires a bullet into an open water tank (Fig.), creating a hole a distance h below the water surface. What is the speed v of the water exiting the tank?

Sol. *From Eq.* $R_V = av = Av_0$ *and thus*

$$v_0 = \frac{a}{A}v$$

Because $a << A$, *we see that* $v_0 << v$. *To apply Bernoulli's equation, we take the level of the hole as our reference level for measuring elevations (and thus gravitational potential energy). Noting that the pressure at the top of the tank and at the bullet hole is the atmospheric pressure* p_0 *(because both places are exposed to the atmosphere), we write Eq. as*



$$p_0 + \frac{1}{2}\rho v_0^2 + \rho g h = p_0 + \frac{1}{2}\rho v^2 + \rho g(0)$$

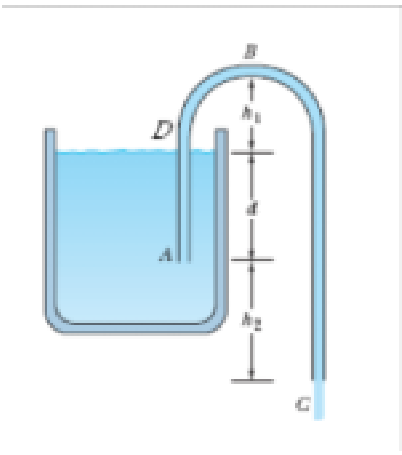
(Here the top of the tank is represented by the left side of the equation and the hole by the right side. The zero on the right indicates that the hole is at our reference level.) Before we solve Eq. for v, we can use our result that $v_0 << v$ *to simplify it: We assume that* v^2_0 *and thus the term* $\frac{1}{2}\rho v_0^2$ *in Eq. , is negligible relative to the other terms, and we drop it. Solving the remaining equation for v then yields*

$$v = \sqrt{2gh} \quad Ans.$$

This is the same speed that an object would have when falling a height h from rest. This is because the work done by atmospheric pressure is cancelling out at open surface and the hole.

Illustration :

Figure shows a siphon, which is a device for removing liquid from a container. Tube ABC must initially be filled, but once this has been done, liquid will flow through the tube until the liquid surface in the container is level with the tube opening at A. The liquid has density 1000 kg/m^3 *and negligible viscosity. The dis-tances shown are* $h_1 = 25$ *cm,* $d = 12$ *cm, and* $h_2 = 40$ *cm. (a) With what speed does the liquid emerge from the tube at C? (b) If the atmospheric pressure is* 1.0×10^5 *Pa, what is the pressure in the liquid at the topmost point B? (c) Theoretically, what is the greatest possible height* h_1 *that a siphon can lift water?*



Sol. *You may have used siphon and you may recollect that lower the exit point of the fluid is, faster the fluid flows out.*

We consider a point D on the surface of the liquid in the container, in the same tube of flow with points A, B and C. Applying Bernoulli's equation to points D and C, we obtain

$$p_D + \frac{1}{2}\rho v_D^2 + \rho g h_D = p_C + \frac{1}{2}\rho v_C^2 + \rho g h_C$$

$$v_C = \sqrt{\frac{2(p_D - p_C)}{\rho} + 2g(h_D - h_C) + v_D^2}$$

$$\approx \sqrt{2g(d + h_2)}$$

where in the last step we set $p_D = p_C = p_{air}$ *and* $v_D/v_C \approx 0$. *Plugging in the values, we obtain*

$$v_c = \sqrt{2(9.8\,\text{m/s}^2)(0.40\,\text{m} + 0.12\,\text{m})} = 3.2\ m/s$$

The result confirms our experience.

We now consider points B and C:

$$p_B + \frac{1}{2}\rho v_B^2 + \rho g h_B = p_C + \frac{1}{2}\rho v_C^2 + \rho g h_C$$

Since $v_B = v_C$ *by equation of continuity, and* $p_C = p_{air}$, *Bernoulli's equation becomes*

$$p_B = p_C + \rho g(h_C - h_g) = p_{air} - \rho g(h_1 + h_2 + d)$$

$$= 1.0 \times 10^5\ Pa\ (1.0 \times 10^3\ kg/m\ )(9.8\ m/s^2\ )(0.25\ m + 0.40\ m + 0.12\ m)$$

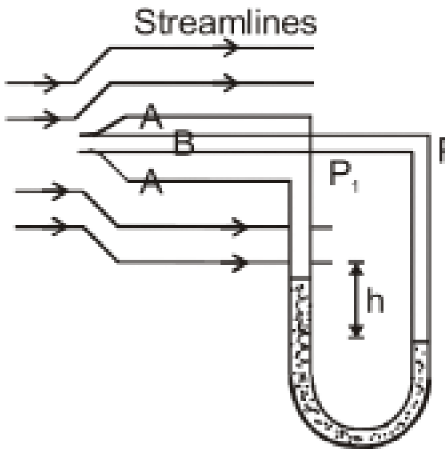
$$= 9.2 \times 10\ Pa.$$

Since $p_B \geq 0$, *we must let* $p_{air} - \rho g\,(h_1 + d + h_2) \geq 0$, *which yields*

$$h_1 \leq h_{1\max} = \frac{p_{air}}{\rho} - d - h_2 \leq \frac{p_{air}}{\rho} = 10.3\text{m}$$

Illustration :

Fig. shows a device called pitot's tube. It measures the velocity of moving fluids. Determine the velocity of the fluid in terms of the density ρ, the density of the fluid in manometer (U-tube) σ and the height 'h'.



figure

Sol. *The difference in the two tubes is that liquid will flow into the tube B with full Kinetic Energy while it will just pass over the tube A without directly entering into it.*

This problem is based on the use of Bernoulli's principle, on two different situations.

The fluid inside the right tube must be at rest as the fluid exactly at the end is in contact with the fluid in pitot tube, which is at rest.
The velocity υ_1 is the fluid velocity. The velocity υ_2 of the fluid at point B is zero and the pressure in the right arm is P_2 (called stagnation pressure).

Thus using Bernoulli's principle $P_1 + \frac{1}{2}\sigma\upsilon_1^2 = P_2 + \frac{1}{2}\sigma\upsilon_2^2$

We get $P_2 = P_1 + \frac{1}{2}\sigma\upsilon_1^2$

On the other hand the openings at point A is not along the flow lines, so we dont need to use Bernoulli's eqn. We can simply say that the pressure just outside the opening is same as that within the pitot tube.
Therefore the pressure at the left arm of the manometer is same as the fluid pressre P_f i.e., $P_1 = P_f$

Also $P_2 = P_1 + (\rho - \sigma)gh$ ....(3)

Generally $\sigma << \rho$, so it is ignored.

Thus $P_2 = P_1 + \rho gh$ ....(4)

From eqns. (4) and (3), $\frac{1}{2}\sigma\upsilon_f^2 = \rho gh$

or $\upsilon_f = \sqrt{\frac{2\rho gh}{\sigma}}$

Thus we can see that we have measured the fluid velocity as this was the only difference between the two tubes leading to the pressure difference between the tubes.

Illustration :

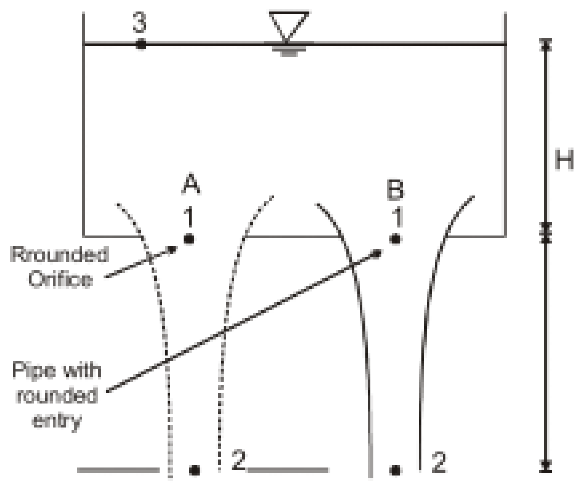
A tank has two oulets (i) a rounded orifice A of diameter D and (ii) a pipe B with well rounded entry and of length L, as shown in fig. For a height of water H in the tank determine the (i) discharge from the outlets A and B, (ii) velocities in the two outlets at levels 1 and 2 indicated in Fig.

Sol. *The difference in the two situations is that in part (i) pressure at the point 1 and all points below it will be atmospheric. On the other hand in part (ii) only at point 2 pressure will be atmospheric. This problem is based on the use of Bernoulli's principle, on two different situations.*
Part (i)
Rounded orifice A
Applying bernoulli equation to a point on the water surface 3 and point 1.

$$\frac{p_0}{\gamma} + 0 + H = \frac{p_1}{\gamma} + \frac{V_{1a}^2}{2g} + 0$$

and $V_1 = \sqrt{2gH}$. *The discharge* $Q_a = \frac{\pi}{4} D^2 \sqrt{2gH}$

At point 2, the pressure is atmospheric and hence by applying Bernoulli equation between points 3 and 2.



$$\frac{p_0}{\gamma} + 0 + (H + L) = \frac{p_0}{\gamma} + \frac{V_{2a}^2}{2g} + 0$$

or $$V_2 = \sqrt{2g(H+L)}$$

As the discharge is Q_a, the diameter at 2 will be smaller than D.

Part (ii)

Pipe :

by applying Bernoulli equation between points 3 and 2.

$$\frac{p_0}{\gamma} + 0 + (H + L) = \frac{p_0}{\gamma} + \frac{V_{2b}^2}{2g} + 0$$

or $$V_{2b} = \sqrt{2g(H+L)}$$

As the pipe size is uniform from point 1 to 2, by continuity equation

$$V_{1b} = V_{2b} = \sqrt{2g(H+L)}$$

thus the results are :

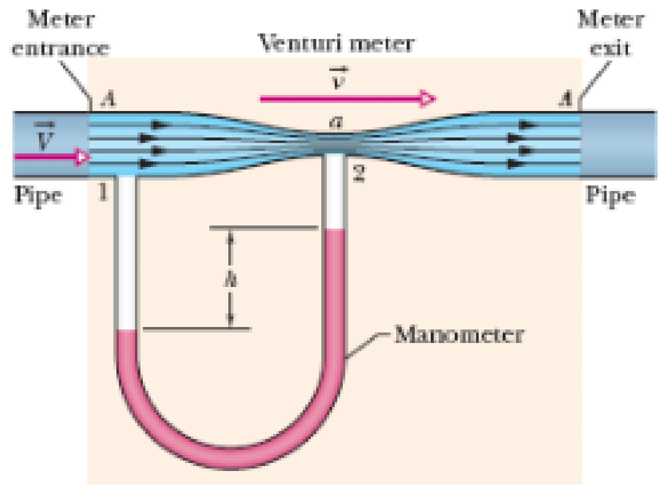
	Orifice	Pipe
Velocity at 1 =	$\sqrt{2gH}$	$\sqrt{2g(H+L)}$
Velocity at 2 =	$\sqrt{2g(H+L)}$	$\sqrt{2g(H+L)}$
Discharge Q =	$\frac{\pi}{4} D^2 \sqrt{2gH}$	$\frac{\pi}{4} D^2 \sqrt{2g(H+L)}$

We can see that velocity at point 2 is same in both the cases, this could have been directly concluded by applying Bernoulli equation between points 3 and 2 in both parts.

Conceptual example:A venturi meter is used to measure the flow speed of a fluid in a pipe. The meter is connected between two sections A venturi meter is used to measure the flow speed of a fluid in a pipe. The meter is connected between two sections of the pipe; the cross-sectional area A of the entrance and exit of the meter matches the pipe's crosssectional area. Between the entrance and exit, the fluid flows from the pipe with speed V and then through a narrow "throat" of cross-sectional area a with speed v A manometer connects the wider portion of the meter to the narrower portion. The change in the fluid's speed is accompanied by a change Dp in the fluid's pressure, which causes a height difference h of the liquid in the two arms of the manometer. (Here Δp means pressure in the throat minus pressure in the pipe.) (a) By applying Bernoulli's equation and the equation of continuity to points 1 and 2 in Fig. , show that

$$v = \sqrt{\frac{2a^2\Delta p}{\rho(a^2 - A^2)}}$$

where ρ is the density of the fluid. (b) Suppose that the fluid is fresh water, that the cross-sectional areas are 64 cm^2 in the pipe and 32 cm^2 in the throat, and that the pressure is 55 kPa in the pipe and 41 kPa in the throat. What is the rate of water flow in cubic meters per second?



(a) The continuity equation yields AV = aV, and Bernoulli's equation yields $\Delta p + \frac{1}{2}\rho v^2 = \frac{1}{2}\rho V^2$ where $\Delta p = p_1 - p_2$. The first equation gives AV = aV. We use this to substitute for V in the second equation, and obtain $\Delta p + \frac{1}{2}\rho v^2 = \frac{1}{2}\rho(A/a)^2 v^2$ We solve for v. The result is

$$v = \sqrt{\frac{2\Delta p}{\rho((A/a)^2 - 1)}} = \sqrt{\frac{2a^2\Delta p}{\rho(A^2 - a^2)}}$$

(b) We substitute values to obtain

$$v = \sqrt{\frac{2(32\times10^{-4}\,m^2)^2(55\times10^3\,Pa - 41\times10^3\,Pa)}{(1000kg/m^3)((64\times10^{-4}\,m^2)^2 - (32\times10^{-4}\,m^2)^2)}} = 30.06\ m/s$$

Consequently, the flow rate is

$$Av = (64 \times 10^{-4} m^2)(3.06\ m/s) = 2.0 \times 10^{-2} m^3/s.$$

Illustration :

A siphon tube discharging a liquid of specific gravity 0.9 from a rbeservoir as shown in the figure.

(a) Find the velocity of the lique through the siphon

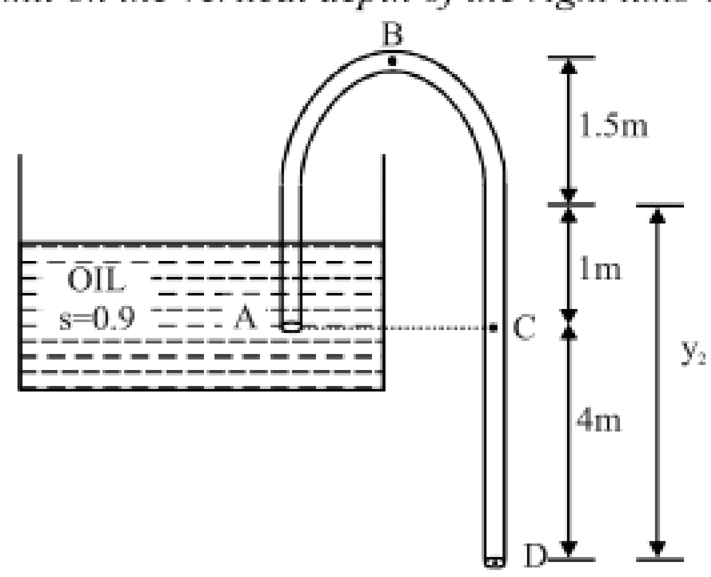
(b) Fin dthe presswure of highest point B.

(c) Find the pressure at the points A (out side the tube) and C.

(d) Would the rate of flow be more, less or the same if the liquid were water ?

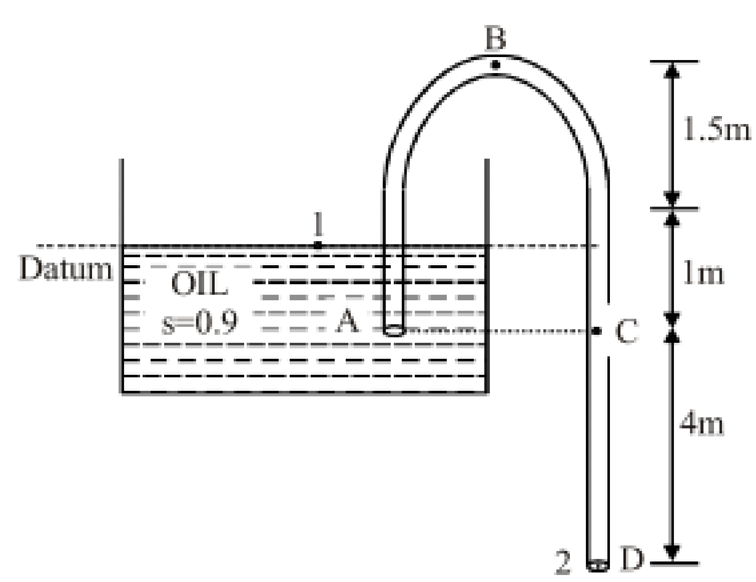
(e) Is there a limit on the maximum height of B above the liquid level in the reservoir.

(f) Is there a limit on the veritcal depth of the right limb of the siphon.



Sol. *Assume datum at the free surface of the liquid.*

(a) Applying Bernoulli's equation on point 1 and 2, as shown in the figure.



$$\frac{p_1}{\rho g}+\frac{v_1^2}{2g}+y_1=\frac{p_2}{\rho g}+y_2$$

Here $p_1 = p_2 = p_0 = 10^5\ N/m^2$; $y_1 = -5m$

Since area of the tube is very small as compared to the reservoir, therefore,

$$v_1 << v_2.\ thus\ \frac{v_1^2}{2g} \approx 0$$

$$\therefore v_2 = \sqrt{2g(y_1 - y_2)} = \sqrt{2(10)(5)} = 10\ m/s$$

(b) Applying Bernoulli's equation at 1 and B.

$$\frac{p_B}{\rho g}+\frac{v_B^2}{2g}+y_B=\frac{p_1}{\rho g}+\frac{v_1^2}{2g}+y_1$$

Here, $p_1 = 10^5\ N/m^2\ ;\ \dfrac{v_1^2}{2g} \approx 0\ ;$

$y_1 = 0,\ v_B = v_2 = 10\ m/s,\ y_B = 1.5\ m$

$\therefore \quad p_B = p_1 - \dfrac{1}{2}\rho v_2^2 - \rho g y_B$

or $\quad p_B = 10^5 - \dfrac{1}{2}\ (900)(10)^2 - (900)(10)(1.5) = 41.5\ kN/m^2$

(c) Applying bernoulli's equation at 1 and A

$p_A = p_1 + \rho g\ (y_1 - y_A)$

or $\quad p_A = 10^5 + (900)(10)(1) = 109\ kN/m^2$

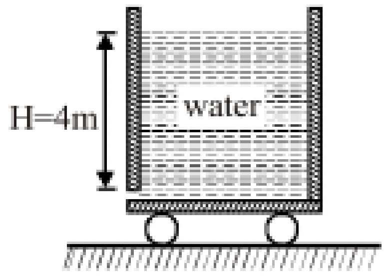
Applying Bernoulli's equation at 1 and C,

$$p_C = p_1 - \frac{1}{2}\rho v_2^2 - \rho g y_C$$

$$= 10^5 - \frac{1}{2}(900)(10)^2 - (900)(10)(-) = 10^5 - 45000 + 9000 = 64\ kN/m^2$$

Illustration :

A tank initaially at rest, is filled with water to a height H = A m. A small orifice is made at the bottom of the wall. Find the velocity attained by the tank when it becomes completely empty. Assume mss of the tank to be negligible. Friction is negligible.



Sol. *Let v be the intantaneous velocity of the tank and c be the instantaneous velocity of efflux with respect to the tank.*

Thrust exrted on the tank is

$F = \rho a c^2$

Where a is the crross-sectional area of the orifice.

$c = \sqrt{2gh}$

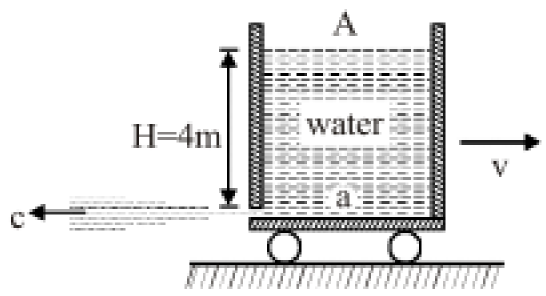
Where h is the instantaneous height of water in the tank.

Mass of the tank at any time t is

$m = \rho A h$

A = cross-sectional area of the tank.

Using Newton's second law



$$F = m\frac{dv}{dt} = \rho A h\frac{dv}{dt}$$

$\therefore \quad \rho A h\ \dfrac{dv}{dt} = \rho a c^2 = 2\rho g a h$

or $\quad \dfrac{dv}{dt} = 2g\left(\dfrac{a}{A}\right) \qquad$ *...(i)*

In a time dt if the water level falls by dh, then according to the conservation of mass.

$-\rho A dh = \rho a c dt \qquad$ *or* $\qquad \dfrac{dh}{dt} = -\dfrac{ac}{A}$

Equation (i) can be written as

$$\frac{dv}{dh}\frac{dh}{dt} = 2g\left(\frac{a}{A}\right) \qquad or \qquad \frac{dv}{dh}\left(-\frac{ac}{A}\right) = 2g\left(\frac{a}{A}\right)$$

or $$\frac{dv}{dh} = -\frac{2g}{c} = -\frac{2g}{\sqrt{2gh}} = -\sqrt{\frac{2g}{h}}$$

On integrating

$$\int_0^v dv = -\sqrt{2g}\int_H^0 \frac{dh}{\sqrt{h}}$$

$$v = 2\sqrt{2gH}$$

Since $H = 4$ *m, therefore* $v = 2\sqrt{2(10)(4)} = 17.9$ *m/s*

Practice Exercise

Q.1 During wind storm. light roofs are blown off. Why ?

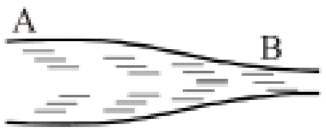
Q.2 A man standing on the platform just near the raliway line be sucked in by a fast moving train. Explain.

Q.3 Air is streaming past a horizontal airplane wing such that its speed is 120 ms^{-1} over the upper surface and 90 ms^{-1} at the lower surface. If the density of air is 1.3 kgm^{-3}, find the difference in pressure between the top and bottom of the wing. If the wing is 10 m long and has average width of 2 m. Calculate the gross lift of the wing.

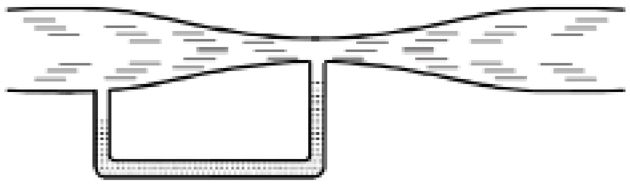
Q.4 A liquid is kept in cylindrical vessel which is rotated along its axis. The liquid rises at the sides. If the radius of the vessel is 0.05 m and the speed of rotation is 2 rev per sec. Find the difference in the height of the liquid at the cntre of the vessel and at its sides.

Q.5 The pressures of water in a water pipe when tap is open and closed respectively 3×10^5 N/m^2 and 3.5×10^5 N/m^2. If tap is opened, then find out-
(a) velocity of water flowing (b) rate of volume of water flowing if area of cross-section of tap is 2 cm^2.

Q.6 Water flows through a horizontal tube of variable cross-section (figure). The area of cross-section at A and B are 4 mm^2 and 2 mm^2 respectively. If 1 cc of water eneters persecond through A, find (a) the speed of water at A, (b) the speed of water at B and (c) the pressre differnece $P_A - P_B$.



Q.7 Water flows through the tube shown infigure. The areas of cross-section of the wide and the narrow portion of the tube are 5 cm^2 and 2 cm^2 respectively. The rate of flow of water through the tube is 500 cm^3/s. Find the difference of mercury levels in the u-tube.
(density of mercury = 13.6gm/cm^3)



Answers

Q.1 Due to high velocity of wind above roof, pressure decreases resulting upward force.

Q.2 Due to decreases in air pressure between preson and train.

Q.3 Due to decrease in pressure in between.

Q.4 $h = 2$ cm

Q.5 (a) 10 m/s (b) 2×10^{-3} m^3/s

Q.6 (a) 25 cm/s (b) 50 cm/s (c) 94 N/m^2 Q.7 2.13 cm

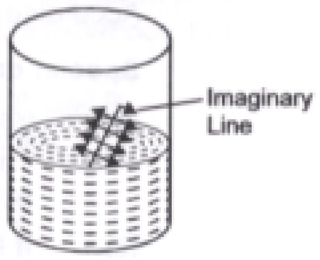
Surface Tension & Viscosity

Surface Tension

Surface Tension is a property of liquid at rest by virtue of which a liquid surface gets contracted to a minimum area and behaves like a stretched membrane.

Surface Tension of a liquid is measured by force per unit length on either side of any imaginary line drawn tangentially over the liquid surface, force being normal to the imaginary line as shown in fig. i.e. Surface tension.

$$(T) = \frac{\text{Total force on either of the imginary line (F)}}{\text{Length of the line } (l)}$$



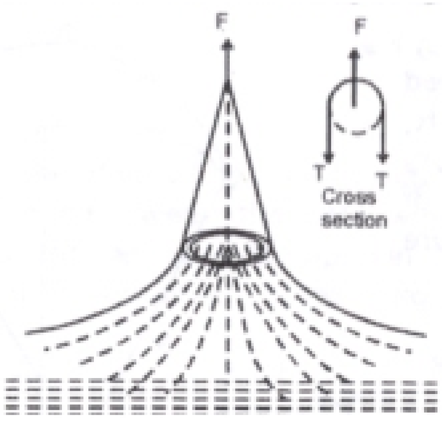
Unit of Surface Tension

In C. G. S. system the unit of surface tension is dyne/cm (dyne cm^{-1}) and SI system its units is Nm^{-1}

Illustration :

A ring is cut form a platinum tube of 8.5 cm internal and 8.7 cm external diameter. It is supported horizontally from a pan of a balance so that it comes in contact with the water in a glass vessel. What is the surface tension of water if an extra 3.97 g weight is required to pull it away from water? ($g = 980\ cm/s^2$).

Sol.



The ring is in contact with water along its inner and outer circumference ; so when pulled out the total force on it due to surface tension will be

$$F = T(2\pi r_1 + 2\pi r_2)$$

So, $$T = \frac{mg}{2\pi(r_1 + r_2)} \qquad [\because F = mg]$$

i.e., $$T = \frac{3.97 \times 980}{3.14 \times (8.5+8.7)} = 72.13\ dyne/cm$$

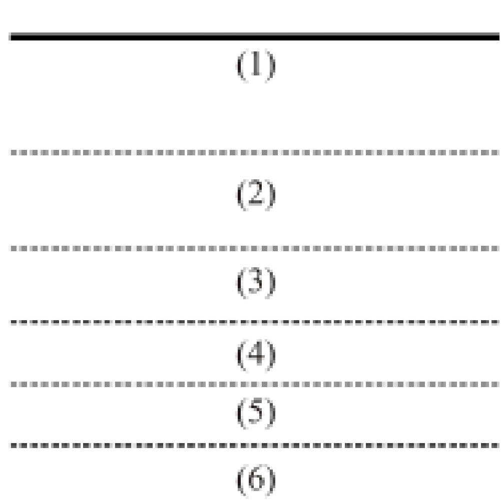
Explanation of some observed phenomena

1. Lead balls are spherical in shape.
2. Rain drops and a globule of mercury placed on glass plate are spherical.
3. Hair of a shaving brush/painting brush, when dipped in water spread out, but as soon as it is taken out. Its hair stick together.
4. A greased needle placed gently on the free surface of water in a beaker does not sink.
5. Similarly, insects can walk on the free surface of water without drowning.
6. Bits of Camphor gum move inregularly when placed on water surface.

Surface energy

The course of reasoning given below is usually followed to prove that the molecules of the surface layer of a liquid have surplus potential energy. A molecule inside the liquid is acted upon by the forces of attraction from the other molecules which compensate each other on the average. If a molecule is singled out on the surface, the resulting force of attraction from the other molecule is directed into the liquid. For this reason the molecule tends to move into the liquid, and definite work should be done to bring it to the surface. Therefore, each molecule of the surface layer has excess potential energy equal to this work. The average force that acts on any molecule from the side of all the others, however, is always equal to zero if the liquid is in equilibrium. This is why the work done to move the liquid from a depth to the surface should also be zero. What is the origin, in this case, of the surface energy ?
The forces of attraction acting on a molecule in the surface layer from all the other molecules produce a resultant directed downward. The closest neighbours, however, exert a force of repulsion on the molecule which is therefore in equilibrium.

Owing to the forces of attraction and repulsion, the density of the liquid is smaller in the surface layer than inside. Indeed, molecule 1 (figure) is acted upon by the force of repulsion from molecule 2 and the forces of attraction from all the other molecules (3, 4, ...). Molecule 2 is acted upon by the forces of repulsion from 3 and 1 and the forces of attraction from the molecules in the deep layers. As a result, distance 1-2 should be greater than 2-3, etc.



This course of reasoning is quite approximate (thermal motion, etc. is disregarded), but nevertheless it gives a qualitatively correct result.

An increase in the surface of the liquid causes new sections of the rarefied surface layer to appear. Here work should be performed against the forces of attraction between the molecules. It is this work that constitutes the surface energy.

We know that the molecules on the liquid surface experience net downward force. So to bring a molecule from the interior of the liquid to the free surface, some work is required to the done against the intermolecular force of attraction, which will be stored as potential energy of the molecule on the surface. The potential energy of surface molecules per unit area of the surface is called surface is called surface energy. Unit of surface energy is erg cm^{-2} in C.G..S. system and Jm^{-2} in SI system. Dimensional formula of surface energy is $[ML^{\circ}T^{-2}]$ surface energy depends on number of surfaces e.g. a liquid drop is having one liquid air surface while bubble is having two liquid air surface.

Relation between surface tension and surface energy

Consider a rectangular frame PQRS of wire, whose arm RS can slide on the arms PR and QS. If this frame is dipped in a soap solution. then a soap film is produced in the frame PQRS in fig. Due to surface tension (T), the film exerts a force on the frame (towards the interior of the film). Let l be the length of the arm RS, then the force acting on the arm RS towards the film is $F = T \times 2l$ [Since soap film has two surfaces, that is way the length is taken twice.]

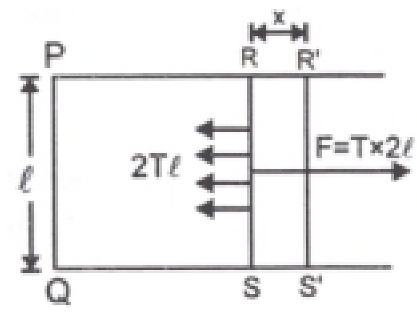
$\therefore$ work done, $W = Fx = 2Tlx$

Increase in potential energy of the soap film.

$= EA = 2Elx =$ work done in increasing the area (ΔW)

where E = surface energy of the soap film per unit area.

According the law of conservation of energy, the work done must be equal to the increase in the potential energy.



$$\therefore \quad 2Tlx = 2Elx \text{ or } T = E = \frac{\Delta W}{A}$$

Thus, surface tension is numerically equal to surface energy or work done per unit increase surface area.

Illustration :

A mercury drop of radius 1 cm is sprayed into 10^6 droplets of equal size. Calculate the energy expanded if surface tension of mercury is 35×10^{-3} N/m.

Sol. *If drop of radius R is sprayed into n droplets of equal radius r, then as a drop has only one surface, the initial surface area will be $4\pi R^2$ while final area is n $(4\pi r^2)$. So the increase in area*

$$\Delta S = n(4\pi r^2) - 4\pi R^2$$

So energy expended in the process.

$$W = T\Delta S = 4\pi T\,[nr^2 - R^2] \qquad \text{... (1)}$$

Now since the total volume of n droplets is the same as that of initial drop, i.e.

$$\frac{4}{3}\pi R^3 = n[(4/3)\,\pi r^3] \text{ or } \qquad r = R/n^{1/3} \qquad \text{... (2)}$$

Putting the value of r from equation (2) in (1)

$$W = 4\pi R^2 T\,[(n)^{1/3} - 1$$

Illustration :

If a number of little droplets of water, each of radius r, coalesce to form a single drop of radius R, show that the rise in temperature will be given by

$$\frac{3T}{J}\left(\frac{1}{r}-\frac{1}{R}\right)$$

where T is the surface tension of water and J is the mechanical equivalent of heat.

Sol. *Let n be the number of little droplets.*

Since volume will remain constant, hence volume of n little droplets = volume of single drop

$$\therefore \quad n \times \frac{4}{3}\pi r^3 = \frac{4}{3}\pi R^3 \quad or \quad nr^3 = R^3$$

Decrease in surface area $= n \times 4\pi^3 - 4\pi R^2$

$$or \quad \Delta A = 4\pi\,[nr^2 - R^2] = 4\pi\left[\frac{nr^3}{r} - R^2\right] = 4\pi\left[\frac{R^3}{r} - R^2\right] = 4\pi R^3\left[\frac{1}{r}-\frac{1}{R}\right]$$

Energy evolved $W = T \times$ *decrease in surface area* $= T \times 4\pi R^3\left[\frac{1}{r}-\frac{1}{R}\right]$

Heat produced, $Q = \frac{W}{J} = \frac{4\pi TR^3}{J}\left[\frac{1}{r}-\frac{1}{R}\right]$ *But* $Q = ms\,d\theta$

where m is the mass of big drop, s is the specific heat of water and $d\theta$ *is the rise in temperature.*

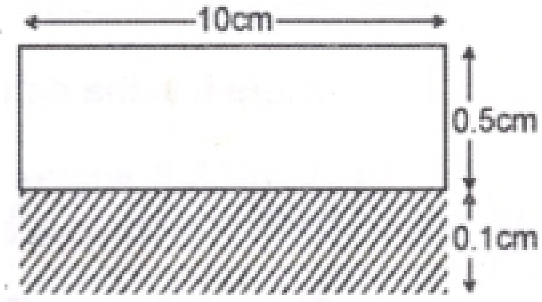
$$\therefore \quad \frac{4\pi TR^3}{J}\left[\frac{1}{r}-\frac{1}{R}\right] = \text{volume of big drop} \times \text{density of water} \times \text{sp. heat of water} \times d\theta$$

$$or, \quad \frac{4}{3}\pi R^3 \times 1 \times 1 \times d\theta = \frac{4\pi TR^3}{J}\left(\frac{1}{r}-\frac{1}{R}\right) \quad or, \quad d\theta = \frac{3T}{J}\left[\frac{1}{r}-\frac{1}{R}\right]$$

Illustration :

A film of water is formed between two straight parallel wires each 10cm long and at a separation 0.5 cm. Calculate the work required to increase 1mm distance between them. Surface tension of water 72×10^{-3} *N/m.*

Sol. *Here the increase in area is shown by shaded portion in the figure.*



Since this a water film, it has two surface, therefore increase in area, $\Delta S = 2 \times 10 \times 0.1 = 2cm^2$

$\therefore$ *Work required to be done*

$W = \Delta S \times T$

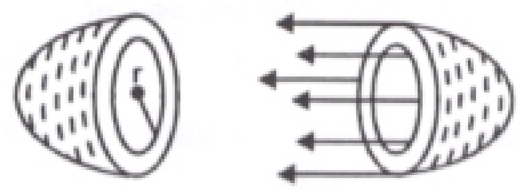
$= 2 \times 10^{-4} \times 72 \times 10^{-3}$

$= 144 \times 10^{-7}$ *joule*

$= 1.44 \times 10^{-5}$ *joule*

Excess pressure inside A liquid drop and a bubble

1. Inside a bubble : Consider a soap bubble of radius r. Let p be the pressure inside the bubble and p_a outside. The excess pressure $= p - p_a$. Imagine the bubble broken into two halves, and consider one half of it as shown in fig. Since there are two surface, inner and outer, so the force due to surface tension is



$F =$ surface tension $\times$ length $= T \times 2$ (circumference of the bubble) $= T \times 2\,(2\pi r)$ ... (1)

The excess pressure $(p - p_a)$ acts on a cross-sectional area πr^2, so the force due to excess pressure is

$\Rightarrow \quad F = (p - p_a)\,\pi r^2$ ... (2)

The surface tension force given by equation (1) must balance the force due to excess pressure given by equation (2) to maintain the equilibrium , i.e. $(p - p_a)\,\pi r^2 = T \times 2(2\pi\, r)$

or $$(p - p_a) = \frac{4T}{r} = p_{excess}$$

above expression can also be obtained by equation of excess pressure of curve surface by putting $R_1 = R_2$.

2. Inside the drop : In a drop, there is only one surface and hence excess pressure can be written as

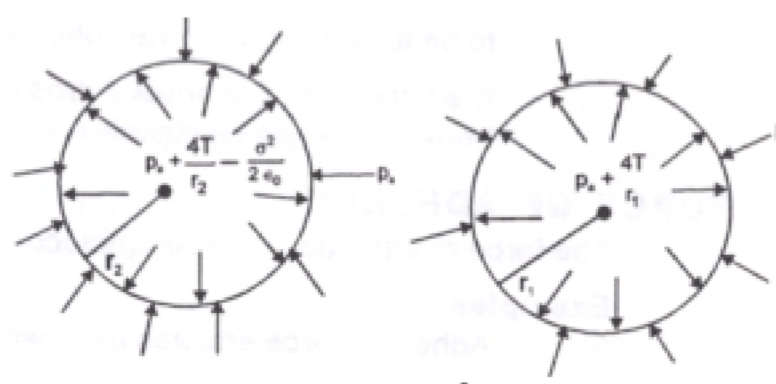
$$(p - p_a) = \frac{2T}{r} = p_{excess}$$

3. Inside air bubble in a liquid :

$$(p - p_a) = \frac{2T}{r} = p_{excess}$$

4. A charged bubble : If bubble is charged, it's radius increases. Bubble has pressure excess due to charge too. Initially pressure inside the bubble

$$= p_a + \frac{4T}{r_1}$$



for charge bubble, pressure inside $= p_a + \dfrac{4T}{r_2} - \dfrac{\sigma^2}{2\epsilon_0}$, where σ surface is surface charge density. Taking temperature remains constant, then from Boyle's law

$$\left(p_a + \frac{4T}{r_1}\right)\frac{4}{3}\pi r_1^{\,3} = \left[p_a + \frac{4T}{r_2} - \frac{\sigma^2}{2\epsilon_0}\right]\frac{4}{3}\pi r_2^{\,3}$$

From above expression the radius of charged drop may be calculated. It can conclude that radius of charged bubble increases, i.e. $r_2 > r_1$.

Illustration :

A minute spherical air bubble is rising slowly through a column of mercury contained in a deep jar. If the radius of the bubble at a depth of 100cm is 0.1 mm, calculate its depth where its radius is 0.126 mm, given that the surface tension of mercury is 567 dyne/cm. Assume that the atmospheric pressure is 76cm of mercury.

Sol. *The total pressure inside the bubble at depth h_1 is (P atmospheric pressure)*

$$= (P + h_1\rho g) + \frac{2T}{r_1} = P_1$$

and the total pressure inside the bubble at depth h_2 is $= (P + h_2\rho g) + \frac{2T}{r_2} = P_2$

Now, according to Boyl's Law

$$P_1V_1 = P_2V_2 \text{ where } V_1 = \frac{4}{3}\pi r_1^3, \text{ and } V_2 = \frac{4}{3}\pi r_2^3$$

Hence we get $$\left[(P+h_1\rho g)+\frac{2T}{r_1}\right]\frac{4}{3}\pi r_1^3 = \left[(P+h_2\rho g)+\frac{2T}{r_2}\right]\frac{4}{3}\pi r_2^3$$

or, $$\left[(P+h_1\rho g)+\frac{2T}{r_1}\right]r_1^3 = \left[(P+h_2\rho g)+\frac{2T}{r_2}\right]r_2^3$$

Given that : $h_1 = 100$ cm, $r_1 = 0.1$ mm $= 0.01$ cm, $r_2 = 0.126$ cm, $T = 567$ dyne / cm, $P = 76$ cm of mercury. Substituting all the values, we get

$$h_2 = 9.48 \text{ cm.}$$

The force of cohesion

The force of attraction between the molecules of the same substance is called cohesion.

In case of solids, the force of cohesion is very large and due to this solids have definite shape and size. On the other hand, the force of cohesion in case of liquids is weaker than that of solids. Hence liquids do not have definite shape but have definite volume. The force of cohesion is negligible in case of gases. Because of this fact, gases have neither fixed shape nor volume.

Example

(i) Two drops of a liquid Two drops of a liquid coalesce into one when brought in mutual contact because of the cohesive force.

(ii) It is difficult to separate two sticky plates of glass wetted with water because a large force has to be applied against the cohesive force between the molecules of water.

(iii) It is very difficult to break a drop of mercury into small droplets because of large cohesive force between mercury molecules.

Force of Adhesion

The force of attraction between molecules of different substance is called adhesion.

Example

(i) Adhesive force enables us to write on the black board with a chalk.

(ii) Adhesive force helps us to write on the paper with ink.

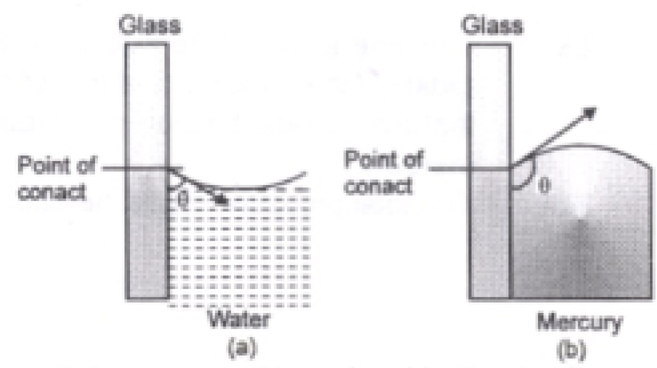
(iii) Large force of adhesion between cement and bricks helps us in construction work.

(iv) Due to force of adhesive, water wets the glass plate.

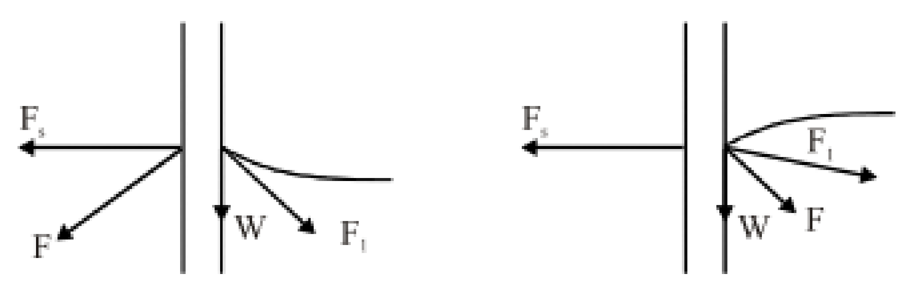
(v) Fevicol and gum are used in gluing two surfaces together because of adhesive force.

Angle of contact

The angle which the tangent to the liquid surface at the point of contact makes with the solid surface inside the liquid is called angle of contact. Those liquids which we the wall of the container (say in case of water and glass) have meniscus concave upwards and their value of angle of contact is less than 90° (also called acute angle). However, those liquids which don't wet the walls of the container (say in case of mercury and glass) have meniscus convex upwards and their value of angle of contact is greater than 90° (also called obtuse angle). The angle of contact of mercury with glass about 140°, whereas the angle of contact of water with glass is about 8°. But, for pure water, the angle of contact θ with glass is taken as 0°.



Let us now see why the liquid surface bands near the contact with a solid. A liqued in equilibrium cannot sustain tangential stress.The resultant force onany small part of the surface layer must be perpendicular to the surface there. Consider a small part of the liquid surface near its contact with the solid



(a) (b)

The ;force acting on this part are

(a) F_s, attracting due to the molecules of the solid surface near it,

(b) F_l, the force due to the liquid molecules near this part, and

(b) W, the weight of the part considered.

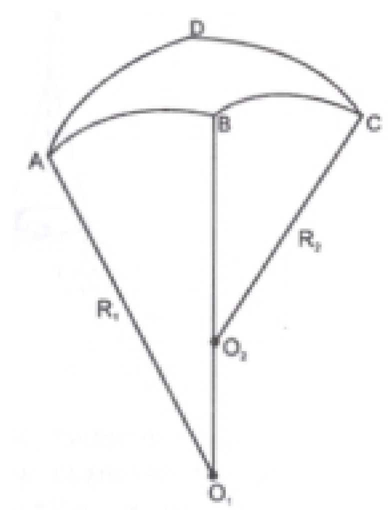
The force between the molecules of the same material is known as cohesive force and the force between the molecules of different kinds of material is called adhesive force. Here F_s is adhesive force and F_l is cohesive force.

As is clear from the figure, the adhesive force F_s is perpendicular to the solid surface and is into the solid. The cohesive force F_l is in the liquid, its direction and magnitude depends on the shape of the liquid surface as this determines the distribution of the molecules attracting the part considered. Of course, F_s and F_l depend on the nature of the substance especialy on their densitites.

The direction of the resultant of F_s, F_l and W decides the shape of the surface near the constact. The liquid rests in such a way that the surface is through the solid, the surface is concave upward and the liquid rises along the solid. If the resultant passes through the liquid, the surface is convex upwards and the liquid is demressed near the solid.

Relation between surface tension, radii of curvature and excess pressure on a curved surface

Let us consider a small element ABCD (fig.) of a curved liquid surface which is convex on the upper side. R_1 and R_2 are the maximum and minimum radii of curvature respectively. They are called the 'principal radii of curvature' of the surface. Let p be the excess pressure on the concave side.



then $p = T\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$. If instead of a liquid surface,

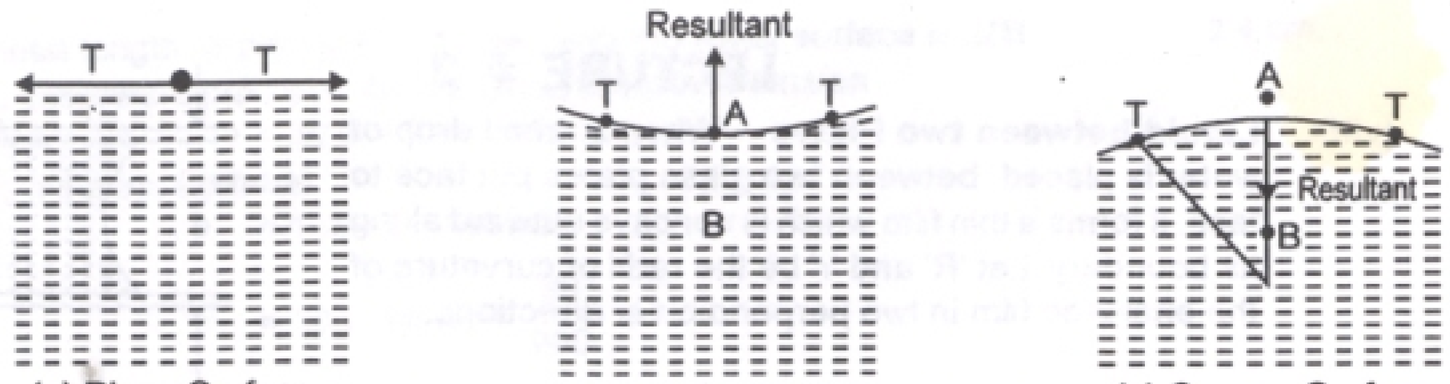
we have a liquid film, the above expression will be

$p = 2T\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$, because a film has two surface.

Excess of pressure inside a curved surface

1. Plane surface : If the surface of the liquid is plane [as shown in fig. (a)], the molecule on the liquid surface is attracted equally in all directions. The resultant force due to surface tension is zero. The pressure, therefore on the liquid surface is normal.

2. Concave surface : If the surface is concave upward [as shown in fig. (b)], there will be upward resultant force due to surface tension acting on the molecule. Since the molecule on the surface is in equilibrium, there must be an excess of pressure on the concave side in the downward direction to balance the resultant force of surface tension $p_A - p_B = \frac{2T}{r}$.



(a) Plane Surface (a) Concave Surface (c) Convex Surface

3. Convex surface : If the surface is convex [as shown fig.(c)], the resultant force due to surface tension acts in the downward direction. Since the molecule on the surface are in equilibrium, there must be an excess of pressure on the concave side of the surface acting in the upward direction to balance the downward resultant force of surface tension. Hence there is always in excess of pressure on concave side of a curved surface over that on the convex side.

$$p_B - p_A = \frac{2T}{r}$$

Illustration :

A barometer contains two uniform capillaries of radii 1.44×10^{-3} m and 7.2×10^{-4} m. if the height of the liquid in the tube is 0.2m more than that in the wide tube, calculate the true pressure difference. Density of liquid = 10^3 kg/m^3, surface tension = 72×10^{-3} N/m and g = 9.8 m/s^2.

Sol. *Let the pressure in the wide and narrow capillaries of radii r_1 and r_2 respectively be P_1 and P_2. Then pressure just below the mensiscus in the wide and narrow tubes respectively are :*

$$\left(P_1 - \frac{2T}{r_1}\right) \text{ and } \left(P_2 - \frac{2T}{r_2}\right) \qquad \left[\text{excess pressure} = \frac{2T}{r}\right]$$

Difference in these pressure $= \left(P_1 - \frac{2T}{r_1}\right) - \left(P_2 - \frac{2T}{r_2}\right) = h\rho g$

$\therefore$ *True pressure difference* $= P_1 - P_2$

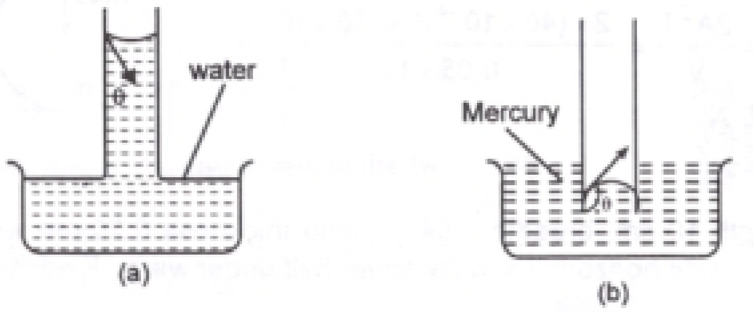
$$= h\rho g + 2T\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$= 0.2 \times 10^3 \times 9.8 + 2 \times 72 \times 10^{-3}\left[\frac{1}{1.44 \times 10^{-3}} - \frac{1}{7.2 \times 10^{-4}}\right]$$

$$= 1.86 \times 10^3 = 1860 \text{ N/m}^2$$

Capillarity

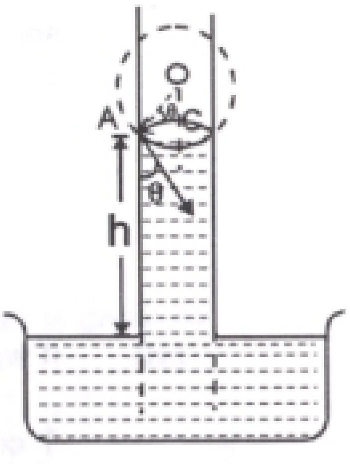
A glass tube of very fine bore throughout the length of the tube is called capillary tube. If the capillary tube is dipped in water, the water wets the inner side of the tube and rises in it [shown in figure (a)]. If the same capillary tube is dipped in the mercury, then the mercury is depressed [shown in figure (b)]. The phenomenon of rise or fall of liquids in a capillary tube is called capillarity.



Practical applications of capillarity

1. The oil in a lamp rises in the wick by capillary action.
2. The tip of nib of a pen is split up, to make a narrow capillary so that the ink rises upto the tin or nib continuously.
3. Sap and water rise upto the top of the leaves of the tree by capillary action.
4. If one end of the towel dips into a bucket of water and the Other end hangs over the bucket the towel soon becomes wet throughout due to capillary action.
5. Ink is absorbed by the blotter due to capillary action.
6. Sandy soil is more dry than clay. It is because the capillaries between sand particles are not so fine as to draw the water up by capillaries.
7. The moisture rises in the capillaries of soil to the surface, where it evaporates. To preserve the moisture m the soil, capillaries must be broken up. This is done by ploughing and leveling the fields.
8. Bricks are porous and behave like capillaries.

Capillary rise (height of a liquid in a capillary tube) ascent formula



consider the liquid which wets the wall of the tube, forms a concave meniscus shown in figure. Consider a capillary tube of radius r dipped in a liquid of surface tension T and density ρ. Let h be the height through which the liquid rises in the tube. Let p be the pressure on the concave side of the meniscus and p_a be the pressure on the convex side of the meniscus. The excess pressure

$(p - p_a)$ is given by $(p - p_a) = \dfrac{2T}{R}$

Where R is the radius of the meniscus. Due to this excess pressure, the liquid will rise in the capillary tube till it becomes equal to the hydrostatic pressure hρg. Thus in equilibrium state.

Excess pressure = Hydrostatic pressure or $\dfrac{2T}{R} = h\rho g$

Let θ be the angle of contact and r be the radius of the capillary tube shown in the fig.

$$\text{From } \Delta OAC,\ \frac{OC}{OA} = \cos\theta \text{ or } R = \frac{r}{\cos\theta} \Rightarrow h = \frac{2T\cos\theta}{r\rho g}$$

The expression is called Ascent formula.

Discussion.

(i) For liquids which wet the glass tube or capillary tube, angle of contact $\theta < 90^\circ$. Hence $\cos\theta$ = positive $\Rightarrow$ h = positive. It means that these liquids rise in the capillary tube. Hence, the liquids which wet capillary tube rise in the capillary tube. For example, water milk, kerosene oil, patrol etc.

Illustration :

A liquid of specific gravity 1.5 is observed to rise 3.0 cm in a capillary tube of diameter 0.50 mm and the liquid wets the surface of the tube. Calculate th excess pressure inside a spherical bubble of 1.0 cm diameter blown from the same liquid. Angle of contact = 0°.

Sol. *The surface tension of the liquid is*

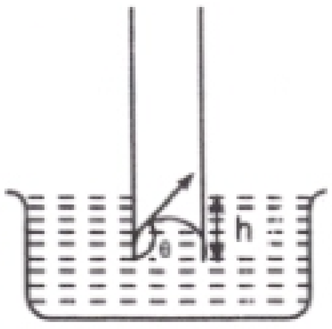
$$T = \frac{rh\rho g}{2} = \frac{(0.025\text{cm})(3.0\text{cm})(1.5\,\text{gm}/\text{cm}^3)(980\,\text{cm}/\text{sec}^2)}{2}$$

$$= 55 \text{ dyne/cm.}$$

Hence excess pressure inside a spherical bubble

$$p = \frac{4T}{R} = \frac{4\times 55\text{dyne}/\text{cm}}{(0.5\text{cm})} = 440 \text{ dyne/cm}^2.$$

(ii) For liquids which do not wet the glass tube or capillary tube, angle of contact $\theta > 90°$. Hence $\cos\theta$ negative $\Rightarrow$ h negative. Hence, the liquids which do not wet capillary tube are depressed in the capillary tube. For example, mercury.

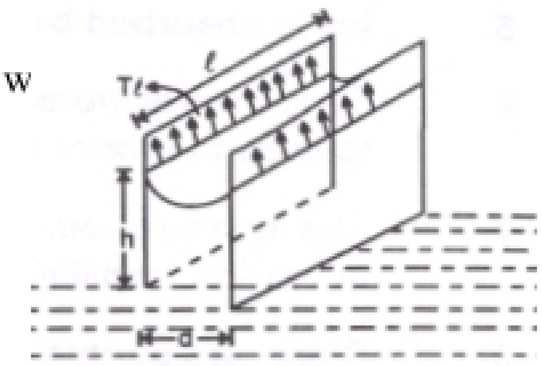


(iii) T, θ, ρ and g are constant and hence $h \propto \frac{1}{r}$. Thus, the liquid rises more in a narrow tube and less in a wider tube. This is called Jurin's Law.

(iv) If two parallel plates with the spacing 'd' are placed in w then height of rise

$$\Rightarrow \quad 2Tl = \rho l h d g$$

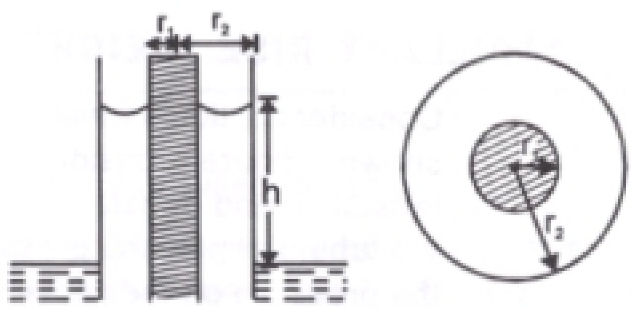
$$h = \frac{2T}{\rho d g}$$



(v) If two concentric tube of radius 'r_1' and 'r_2' (inner one is solid) are placed in water reservoir, then height of rise

$$\Rightarrow \quad T\,[2\pi r_1 + 2\pi r_2] = [\pi r_2^2 h - \pi r_1^2 h]\,\rho g$$

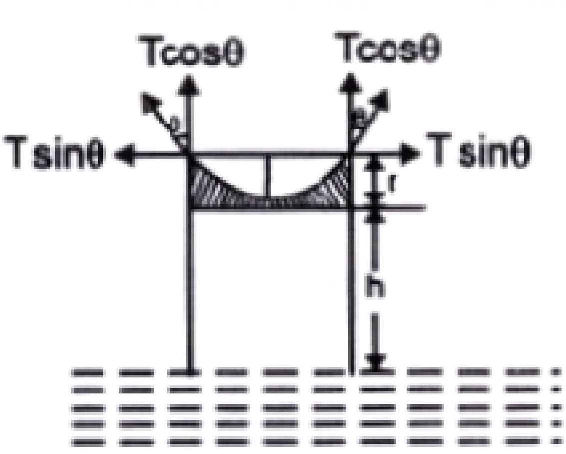
$$h = \frac{2T}{(r_2 - r_1)\rho g}$$



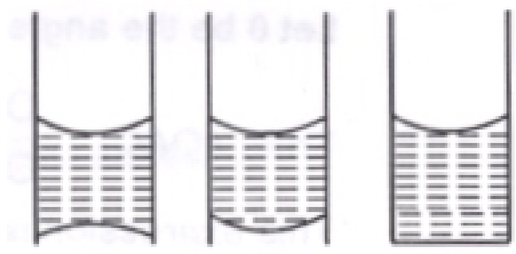
(vi) If weight of the liquid in the menscus is to be consider :

$$T\cos\theta \times 2\pi r = [\pi r^2 h + \frac{1}{3}\,\pi r^2 \times \pi r_1^2 h]\,\rho g$$

$$\left[h + \frac{r}{3}\right] = \frac{2T\cos\theta}{r\rho g}$$



(vii) When capillary tube (radius, 'r') is in vertical position, the upper meniscus is concave and pressure due to surface tension is directed vertically upward and is given by $p_1 = 2T / R_1$ where R_1 = radius of curvature of upper meniscus.



(a) (b) (c)

The hydrostatic pressure $p_2 = h\rho g$ is always directed downwards.

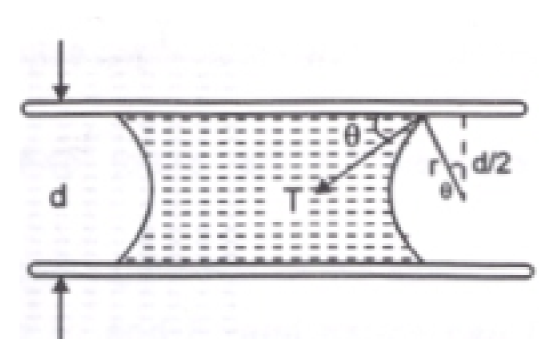
If $p_1 > p_2$ i.e. resulting pressure is directed upward. For equilibrium, the pressure due to lower meniscus should be downward. This makes lower meniscus concave downward (fig. (a)). The radius of lower meniscus R_2 can be given by $\frac{2T}{R_2} = (p_1 - p_2)$.

If $p_1 < p_2$ i.e. resulting pressure is directed downward for equilibrium, the pressure due to lower meniscus should be upward. This makes lower meniscus convex upward (fig. b).

The radius of lower meniscus can be given by $\frac{2T}{R_2} = p_2 - p_1$.

If $p_1 = p_2$, then is no resulting pressure, then $p_1 - p_2 = \frac{2T}{R_2} = 0$ or $R_2 = \infty$ i.e. lower surface will be FLAT. (fig.c).

(viii) **Liquid between two plates :** When a small drop of water is placed between two glass plates put face to face, it forms a thin film which is concave outward along its boundary. Let 'R' and 'r' be the radii of curvature of the enclosed film in two perpendicular directions.



Hence the pressure inside the film is less than the atmospheric pressure outside it by an amount p given by $p = T\left(\frac{1}{r} + \frac{1}{r = \infty}\right)$ and we have, $p = \frac{T}{r}$.

If d be the distance between the two plates and θ the angle of contact for water and glass, then, from the figure, $\cos\theta = \frac{\frac{1}{2}d}{r}$ or $\frac{1}{r} = \frac{2\cos\theta}{d}$.

Substituting for $\frac{1}{r}$ in, we get $p = \frac{2T}{d}\cos\theta$.

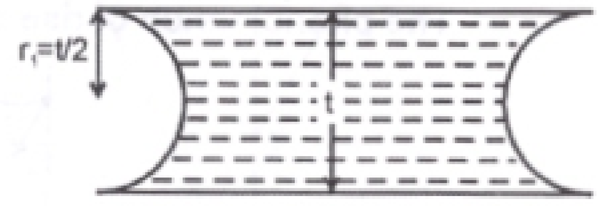
θ can be taken zero for water and glass, i.e. $\cos\theta = 1$. Thus the upper plate is pressed downward by the atmospheric pressure minus $\frac{2T}{d}$. Hence the resultant downward pressure acting on the upper plate is

$\frac{2T}{d}$. If A be the are of the plate wetted by the film, the resultant force F pressing the upper plate downward is given by F = resultant pressure × area = $\frac{2TA}{d}$. For very nearly plane surface, d will be very small and hence the pressing force F very large. Therefore it will be difficult to separate the two plates normally.

Illustration :

A drop of water volume 0.05 cm³ is pressed between two glass-plates, as a consequence of which, it spreads and occupies an area of 40 cm². If the surface tension of water is 70 dyne/cm, find the normal force required to separate out the two glass plates in newton.

Sol. *Pressure inside the film is less than outside by an amount, $P = T\left[\frac{1}{r_1}+\frac{1}{r_2}\right]$, where r_1 and r_2 are the radii of curvature of the meniscus. Here r_1 = 1/2 and $r_2 = \infty$, then the force required to separate the two glass plates, between which a liquid film is enclosed (figure) is, $F = P \times A = \frac{2AT}{t}$, where t is the thickness of the film, A = area of film.*



$$F = \frac{2A^2T}{At} = \frac{2A^2T}{V} = \frac{2\times(40\times10^{-4})^2\times(70\times10^{-3})}{0.05\times10^{-6}} = 45\text{ N}$$

Illustration :

A glass plate of length 10cm, breadth 1.54 cm and thickness 0.20 cm weigh 8.2 gm in air. It is held vertically with the long side horizontal and the lower half under water. Find the apparent weight of the plate. Surface tension of water 73 dyne per cm, g = 980 cm/sec².

Sol. *Volume of the portion of the plate immersed in water is*

$$10 \times \frac{1}{2}(1.54)\times 0.2 = 1.54\ cm^3.$$

Therefore, if the density of water is taken as 1, then upthrust

= wt. of the water displaced

= 1.54 × 1 × 980 = 1509.2 dynes.

Now, the total length of the plate in contact with the water surface is 2 (10 + 0.2) = 20.4 cm,

∴ *downward pull upon the plate due to surface tension*

= 20.4 × 73 = 1489.2 dynes

$\therefore$ resultant upthrust

$$= 1509.2 - 1489.2$$

$$= 20.0 \text{ dynes} = \frac{20}{980}$$

$$= 0.0204 \text{ gm. wt.}$$

$\therefore$ apparent weight of the plate in water

= weight of the plate in air – resultant upthrust

$$= 8.2 - 0.0204 = 8.1796 \text{ gm}$$

Illustration :

A glass tube of circular cross-section is closed at one end. This end is weighted and the tube floats vertically in water, heavy end down. How far below the water surface is the end of the tube? Give : Outer radius of the tube 0.14 cm, mass of weighted tube 0.2 gm, surface tension os water 73 dyne/cm and g = 980 cm/sec^2.

Sol. Let l be the length of the tube inside water. The forces acting on the tube are :

(i) Upthrust of water acting upward

$$= \pi r^2 l \times 1 \times 980 \qquad = \frac{22}{7} \times (0.14)^2\, l \times 980 = 60.368\, l \text{ dyne.}$$

(ii) Weight of the system acting downward

$$= mg = 0.2 \times 980 = 196 \text{ dyne.}$$

(iii) Force of surface tension acting downward

$$= 2\pi rT$$

$$= 2 \times \frac{22}{7} \times 0.14 \times 73 = 64.24 \text{ dyne.}$$

Since the tube is in equilibrium, the upward force is balanced by the downward forces. That is,

$$60.368\, l = 196 + 64.24 = 260.24.$$

$$\therefore \quad l = \frac{260.24}{60.368} = 4.31 \text{ cm}$$

Illustration:

A glass U-tube is such that the diametre of one limb is 3.0 mm and that of the other is 6.00 mm. The tube is inverted vertically with the open ends below the surface of water in a beaker. What is the difference between the heights to which water rises in the two limbs? Surface tension of water is 0.07 N/m. Assume that the angle of contact between water and glass is 0°.

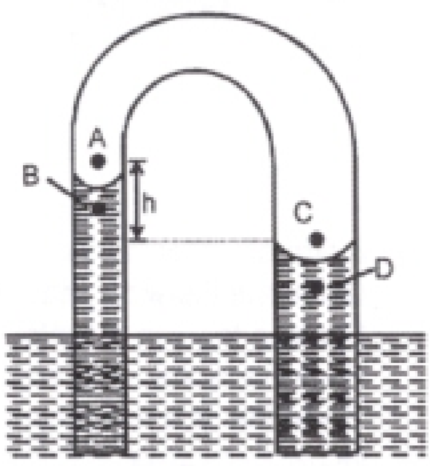
Sol. Suppose pressure at the points A, B, C and D be P_A, P_B, P_C and P_0 respectively.

The pressure on the concave side of the liquid surface is greater than that on the other side by 2T/R.

Ang angle of contact θ is given to be 0°, hence $\qquad R \cos 0° = r$ or $R = r$

$$\therefore \quad P_A = P_B + 2T/r_1 \quad \text{and} \quad P_C = P_0 + 2T/r_2$$

where r_1 and r_2 are the radii of the two limbs



But $\quad P_A = P_C$

$$\therefore \quad P_B + \frac{2T}{r_1} = P_D + \frac{2T}{r_2}$$

$$\text{or} \quad P_D - P_B = 2T\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

where h is the difference in water levels in the two limbs

$$\text{Now,} \quad h = \frac{2T}{\rho g}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Given that $\quad T = 0.07\ Nm^{-1},\ \rho = 1000\ kgm^{-3}$

$$r_1 = \frac{3}{2}\ mm = \frac{3}{20}\ cm = \frac{3}{20 \times 100}\ m = 1.5 \times 10^{-3}\ m,\ r_2 = 3 \times 10^{-3}\ m$$

$$\therefore \quad h = \frac{2 \times 0.07}{1000 \times 9.8}\left(\frac{1}{1.5 \times 10^{-3}} - \frac{1}{3 \times 10^{-3}}\right) m = 4.76 \times 10^{-3}\ m = 4.76\ mm$$

Illustration :

Two narrow bores of diameters 3.0 mm and 6.0 mm are joined together to form a U-shaped tube open at both ends. If the U-tube contains water, what is the difference in its levels in the two limbs of the tube? Surface tension of water at the temperature of the experiment is $7.3 \times 10^{-2}\ Nm^{-1}$. Take the angle of contact to be zero.

Sol. *Given that* $r_1 = \frac{3.0}{2} = 1.5\ mm = 1.5 \times 10^{-3}\ m,\ \ r_2 = \frac{6.0}{2} = 3.0 \times 10^{-3}\ m,$

$T = 7.3 \times 10^{-3}\ Nm^{-1},\ \ \theta = 0°\ \ \rho = 1.0 \times 10^3\ kg\ m^{-1},\ g = 9.8\ ms^{-2}$

When angle of contact is zero degree, the radius of the meniscus equals radius of bore.

Excess pressure in the first bore, $P_1 = \frac{2T}{r_1} = \frac{2 \times 7.3 \times 10^{-3}}{1.5 \times 10^{-3}} = 97.3$ *Pascal*

Excess pressure in the second bore, $P_2 = \frac{2T}{r_2} = \frac{2 \times 7.3 \times 10^{-2}}{3 \times 10^{-3}} = 48.7$ *Pascal*

Hence, pressure difference in the two limbs of the tube

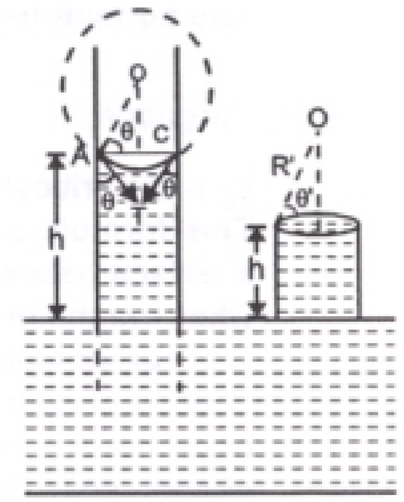
$$\Delta P = P_1 - P_2 = h\rho g$$

$$\text{or} \quad h = \frac{P_1 - P_2}{\rho g} = \frac{97.3 - 48.7}{1.0 \times 10^3 \times 9.8} = 5.0\ mm.$$

Capillary rise in a tube of insufficient length

We know, the height through which a liquid rises in the capillary tube of radius r is given by

$$\therefore \qquad h = \frac{2T}{R\rho g} \quad \text{or} \quad h\,R = \frac{2T}{\rho g} = \text{constant}$$



When the capillary tube is cut and its length is less then h (i.e. h′). then the liquid rises upto the top of the tube and spreads in such a way that the radius (R′) of the liquid meniscus increases and it becomes more flat so that hR = h′R′ = Constant. Hence the liquid does not overflow.

$$\text{If} \quad h' < h \text{ then } R' > R \quad \text{or} \quad \frac{r}{\cos\theta'} > \frac{r}{\cos\theta}$$

$$\Rightarrow \quad \cos\theta' < \cos\theta \quad \Rightarrow \quad \theta' > \theta$$

Illustration:

If a 5cm long capillary tube with 0.1 mm internal diameter open at both ends is slightly dipped in water having surface tension 75 dyne cm^{-1}, state whether (i) water will rise half way in the capillary. (ii) Water will rise up to the upper end of capillary (iii) What will overflow out of the upper end of capillary. Explain your answer.

Sol. *Given that surface tension of water, T = 75 dyne/cm*

$$\text{Radius } r = \frac{0.1}{2} \text{ mm} = 0.05 \text{ mm} = 0.005 \text{ cm}$$

density $\rho = 1$ *gm/cm*^{3}*, angle of contact,* $\theta = 0°$

Let h be the height to which water rise in the capillary tube. Then

$$h = \frac{2T\cos\theta}{r\rho g} = \frac{2 \times 75 \times \cos 0°}{0.005 \times 1 \times 981} \text{ cm} = 30.58 \text{ cm}$$

But length of capillary tube, $h' = 5$*cm*

(i) *Because* $h > \frac{h'}{2}$ *therefore the first possibility does not exist.*

(ii) *Because the tube is of insufficient length therefore the water will rise upto the upper end of the tube.*

(iii) *The water will not overflow out of the upper end of the capillary. It will rise only upto the upper end of the capillary.*

The liquid meniscus will adjust its radius of curvature R′ in such a way that

$$R'h' = Rh \qquad \left[\because hR = \frac{2T}{\rho g} = \text{constant}\right]$$

where R is the radius of curvature that the liquid meniscus would possess if the capillary tube were of sufficient length.

$$\therefore \quad R' = \frac{Rh}{h'} = \frac{rh}{h'} \qquad \left[\because R = \frac{r}{\cos\theta} = \frac{r}{\cos 0°} = r\right] = \frac{0.005 \times 30.58}{5} = 0.0306 \text{ cm}$$

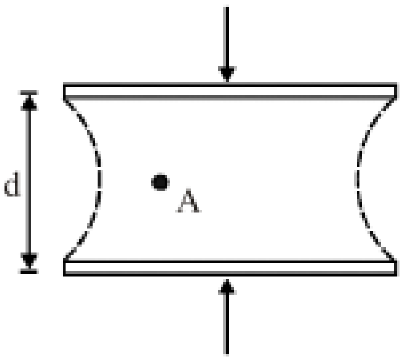
Illustration :

A drop of water of mass m = 0.2 g is placed between two clean glass plates, the distance between which is 0.01 cm. Find the force of attraction between the plates. Surface tension of water = 0.07 Nm^{-1}.

Sol. *Let R be the radius of the circular layer of water. Then* $\pi R^2 d \times \rho = m$

Pressure at A $= p_0 - \dfrac{2T}{d}$ *($\because$ meniscus is cylindrical in shape)*

Thus pressure between the plates is less than the atmospheric pressure and so the plates are pressed together as though attracted towards each other.



F, force of attaraction $= \Delta p \times$ *area* $\Rightarrow \quad F = \dfrac{2T}{d} \times \pi R^2$

$$\Rightarrow \quad F = \frac{2T}{d} \times \frac{m}{d.\rho} = \frac{2Tm}{d^2.\rho} = \frac{2 \times 0.2 \times 10^{-3} \times 0.07}{0.01^2 \times 10^{-4} \times 100} = 2.8N$$

Illustration :

A glass capillary sealed at the upper end is of length 0.11 m and internal diameter 2×10^{-5} m. The tube is immersed vertically into a liquid of surface tension 5.0×10^{-2} N/m. To what length has the capillary to be immersed so that liquid level inside and outside the capillary becomes same ? What will happen to the water level inside the capillary if the seal is now broken ?

Sol. *If A is the cross-sectional area of the tube and L its length, the inital volume of air inside it will be $V_1 = AL$ While pressure $p_1 = p_0$ = atmospheric pressure.*

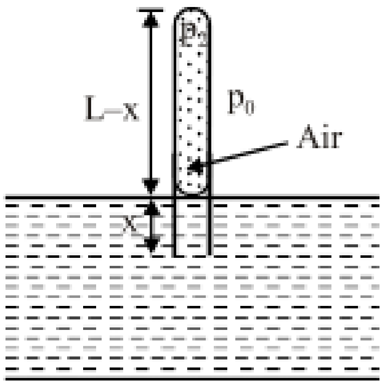
Now when the tube is immersed in water with its length x in water, the level of water inside and outside is same, so the volume of air in the tube will be $V_2 = A\,(L - x)$. Futher if p_2 is the pressure of gas in the tube,

$$p_2 - \frac{2T}{r} = p_0, \qquad \text{i.e. } p_2 = p_0 + \frac{2T}{r}$$

Now if temperature is constant

$$P_1V_1 = P_2\,V_2$$

$$p_0\,AL = \left[p_0 + \frac{2T}{r}\right] A\,(L - x) \quad or \quad x\left[1 + \frac{rp_0}{2T}\right] = L$$



$$\text{i.e.} \quad x\left[1 + \frac{1.012 \times 10^5 \times 1 \times 10^{-5}}{2 \times 5.06 \times 10^{-2}}\right] = 0.11 \quad or \quad x = \frac{0.11}{11} = 0.01m$$

If the seal is broken the pressure inside the capillary become atmospheric, i.e. p_0 while capillarity will take place and the rise will be

$$h = \frac{2T}{r\rho g} = \frac{2 \times 5.06 \times 10^{-2}}{10^{-5} \times 10^3 \times 9.8} = 1.03\,m$$

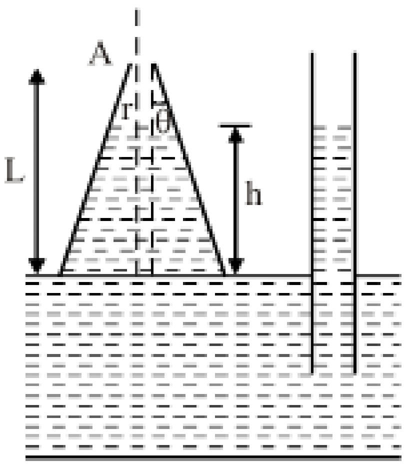
However, the length of the tube outside the water is 0.11 – 0.01 = 0.1 m; so the tube will be of insufficient length and so the liquid will rise to the top of the tube and will stay with radius of meniscus,

$$r = \frac{h}{L} = \frac{1.03 \times 10^{-3}}{0.1} = 1.03 \times 10^{-4}\,m$$

Illustration:

A conical glass capillary tube of length 0.1 m has diameters 10^{-3} and 5×10^{-4} m at the ends. When it is just immersed in a liquid at 0°C with larger diameter in contact with it, the liquid rises to 8×10^{-2} m in the tube. If another cylindrical glass capillary tube B is immersed inthe same liquid at 0°C, the liquid rises to 6×10^{-2} m height. The rise of liquid in the tube B is only 5.5×10^{-2} m when the linquid is at 50°C. Find the rate at which the surface tension changes with temperature considering the change to be linear. The density of the liquid is $(1/14) \times 10^4$ kg/m³ and angle of contact is zero Effect of temperature on density of liquid and glass is negligible.

Sol. *If r is the radius of the meniscus in the conical tube, then as shown in figure.*



$$\tan\theta = \frac{r - r_1}{L - h} = \frac{r_2 - r_1}{L}$$

i.e. $$\frac{r - 2.5\times10^{-4}}{0.1 - 0.08} = \frac{(5.2.5)\times10^{-4}}{0.1}$$

i.e., $r \times 10^4 - 2.5 = 0.2 \times 2.5$ *i.e.,* $r = 3 \times 10^4$ m

Now as capillarity is independent of the shape of tube so at same temperature $\theta = 0^o$ C.

$$h_A r_A = h_B r_B = (2T_0 / \rho g) = \text{constant}$$

so $r_B = (0.08 \times 3 \times 10^{-4})/(6 \times 10^{-2}) = 4 \times 10^{-4}$ m

Now as from $h = (2T/r\rho g)$ *for cylindrical tube,*

$$T_0 = \frac{h_0 \rho g r}{2} = \frac{1}{2}\left[6\times10^{-2} \times \frac{1}{14} \times 10^4 \times 9.8 \times 4 \times 10^{-4}\right]$$

$$= 8.4 \times 10^{-2} \text{ N/m}$$

Now as for a given tube and liquid $T \propto h$ *(as* $T = h\rho g r/2$*)*

$$\frac{T}{T_0} = \frac{h_{50}}{h_0} \qquad \text{so} \qquad T_{50} = \frac{5.5\times10^{-2}}{6\times10^{-2}} \times 8.4 \times 10^{-2} \times 10^{-2} = 7.7 \times 10^{-2}\,\text{N/m}$$

So rate of change of surface tension with temperature assuming lineartiy,

$$\frac{\Delta T}{\Delta\theta} = \frac{T_{50} - T_0}{50 - 0} = \frac{(7.7 - 8.4)\times10^{-2}}{50} = 1.4 \times 10^{-2}\,\text{N/m}^o\text{C}$$

Negative sign shows that with with rise in temperature surface tension decreases.

Applications of surface tension

(i) The wetting property is made use of in detergents and waterproofing. When the detergent materials are added to liquids, the angle of contact decreases and hence the wettability increases. On the other hand, when water proofing material is added to a fabric, it increases the angle of contact, making the fabric water-repellant.

(ii) The antiseptics have very low value of surface tension. The low value of surface tension prevents the formation of drops that may otherwise block the entrance to skin or a wound. Due to low surface tension the antiseptics spreads properly over the wound. The lubricating oils and paints also have low surface tension. So they can spread properly.

(iii) Surface tension of all lubricating oils and paints is kept low so that they spread over a large area.

(iv) Oil spreads over the surface of water because the surface tension of oil is less than the surface tension of cold water.

(v) A rough sea can be calmed by pouring oil on its surface.

Effect of temperature and impurities on surface tension

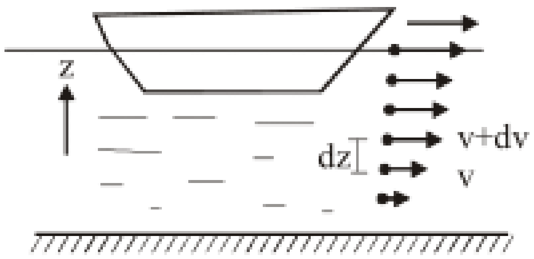
The surface tension of a liquid decreases with the rise in temperature and vice versa. According to Ferguson, $T = T_0\left(1-\frac{\theta}{\theta_C}\right)^n$ where T_0 is surface tension at 0°C, θ is absolute temperature of the liquid, θ_C is the critical temperature and n is a constant varies slightly from liquid and has means value 1.21. This formula shows that the surface tension becomes zero at the critical temperature, that is why machinery parts get jammed in winter.

The surface tension of a liquid change appreciably with addition of impurities. For example, surface tension of water increases with addition of highly soluble substances like NaCl, $ZnSO_4$ etc. On the other hand surface tension of water gets reduced with addition of sparingly soluble substances like phenol, soap etc.

Viscosity

When a layer of a fluid slips or tens to slip on another layer in contact, the two layers exert tangential forces on each other. The directions are such that the relative motion between the layers is opposed. this property of a fluid to oppose relative motion between its layers is called viscosity. The forces between the layers opposing relative motion between them are known as the forces of viscosity. Thus, viscosity may be thought of as the internal friction of a fluid in motion.

If a solid surface is kept in contact with a fluid and is moved, forces of viscosity appear between the solid surface and the fluid layer in contact. the fluid in contact is dragged with the solid. If the viscosity is sufficient, the layer moves with the solid and there is no relative slipping. When a boat moves slowly on the water of a calm river, the water in contact with the boat is dragged with it, whereas the water in contact with the bed of the river remains at rest. Velocities of different layers are different. Let v be the velocity of the layer at a distance z from the bed and v +dv be the velocity at a distance z +dz (figure).



Thus, the velocity differs by dv in going through a distance dz perpendicular to it. The quantity dv/dz is called the velocity gradient.

The force of viscosity between two layers of a fluid is proportional to the velocity gradient in the direction perpendicular to the layers. Also the force is proportional to the area of the layer.

Thus, if F is the force exerted by a layer of area A on a layer in contact,

$$F \propto A \text{ and } F \propto dv/dz$$

or, $$F = -\eta A\, dv/dz$$

The negative sign is included as the force is frictional in nature and opposes relative motion. The constant of proportionality η is called the coefficient of viscosity.

The SI unit of viscosity can be easily worked out from equation. It is N-s/m^2 . However, the corresponding CGS unit dyne-s/cm^2 is in common use and is called a poise in honour of the French scientist Poiseuille. We have

$$1 \text{ poise} = 0.1 \text{ N-s/m}^2$$

Terminal velocity

The viscous force on a solid moving through a fluid is proportional to its velocity. When a solid is dropped in a fluid, the forces acting on it are

(a) weight W acting vertically downward,

(b) the viscous force F acting vertically upward and

(c) the buoyancy force B acting vertically upward.

The weight W and the buoyancy B are constant but the force F is proportional to the velocity v, initially, the velocity and hence the viscous force F is zero and the solid is accelerated due to the force W – B. Because of the acceleration, the velocity increases. Accordingly, the viscous force also increases. At a certain instant the viscous force becomes equal to W-B. the net force then becomes zero and the solid falls with constant velocity. This constant velocity is known as the terminal velocity.

Consider a spherical body falling through a liquid. Suppose the density of the body = ρ, density of the liquid = σ, radius of the sphere = r and the terminal velocity = v_0 . The viscous force is

$$F = 6\pi\eta r v_0$$

the weight $$W = \frac{4}{3}\pi r^3 \rho g$$

and the buoyancy force $$B = \frac{4}{3}\pi r^3 \sigma g$$

We have

$$6\pi\eta r v_0 = W - B = \frac{4}{3}\pi r^3 \rho g - \frac{4}{3}\pi r^3 \sigma g$$

or

$$v_0 = \frac{2r^2(\rho - \sigma)g}{9\eta}$$

Illustration:

A large wooden plate of area 10 m^2 floating on the surface of a river is made to move horizontally with a speed of 2 m/s by applying a tangential force. If the river is 1 m deep and the water in contact with the bed is stationary, find the tangential force needed to keep the plate moving. Coefficient of viscosity of water at the temperature of the river = 10^{-2} poise.

Sol. *The velocity decreases from 2 m/s to zero in 1 m of perpendicular length. Hence, velocity gradient.*

$$= dv/dx = 2\ s^{-1}$$

Now,

$$\eta = \left|\frac{F/A}{dv/dx}\right|$$

or,

$$10^{-3}\ \frac{N-s}{m^2} = \frac{F}{(10m)^2(2s^{-1})}$$

or,

$$F = 0.02\ N.$$

Illustration :

The velocity of water in a river is 18 km/hr near the surface. If the river is 5 m deep, find the shearing stress between the horizontal layers of water. The coefficient of viscosity of water = 10^{-2} poise.

Sol. *The velocity gradient in vertical direction is*

$$\frac{dv}{dx} = \frac{18\,km/hr}{5m} = 1.0\,s^{-1}$$

The magnitude of the force of viscosity is

$$F = \eta A \frac{dv}{dx}.$$

The shearing stress is

$$F/A = \eta \frac{dv}{dx} = (10^{-2}\ \text{poise})\ (1.0\ s^{-1}) = 10^{-3}\ N/m^2$$

Solved Example

Q.1 A conical glass capillary tube A of length 0.1 m has diameters 10^{-3} m and 5×10^{-4} m at the ends. When it is just immersed in a liquid at 0ºC with larger in contact with it, te liquid rises to 8×10^{-2} m in the tube. In another cylindrical capillary tube B, when immersed in the ame liquid at 0ºC, the liquid rises to 6×10^{-2} m height. The rise of liquid in tube B is only 5.5×10^{-2} m when the liquid is at 50ºC. Find the rate at which the surface tension chages with temperature considering the chage to be linear. The density of liquid is $(1/14) \times 10^4$ kg/m^3 and the angle of contact is zero. Effect of temperature on the density of liquid and glass is negligible.

Sol. The situation is shown if figure.

Let r_1 and r_2 be radii of upper and lower ends of the conical capillary tube. The radius r at the meniscus is given by

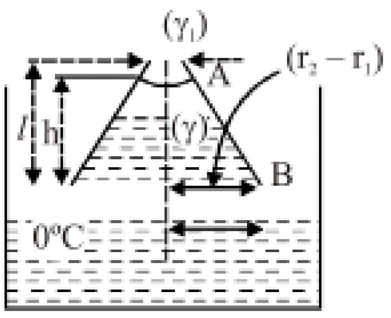
$$r = r_1 + (r_2 - r_1)\left(\frac{l-h}{l}\right)$$

$$= (2.5 \times 10^{-4}) + (2.5\times10^{-4})\left(\frac{0.1-0.08}{0.1}\right)$$

$$= 3.0 \times10^{-4} \text{ m}$$

The surface tension at 0ºC is given by

$$T_0 = \frac{rh\rho g}{2}$$



$$= \frac{(3.0\times10^{-4})(8\times10^{-2})(1/4\times10^{4})(9.8)}{2}$$

$$= 0.084 \text{ N/m}$$

For tube B

$$\frac{T_0}{T_{50}} = \frac{h_0}{h_{50}} = \frac{6\times10^{-3}}{5.5\times10^{-3}} = \frac{12}{11}$$

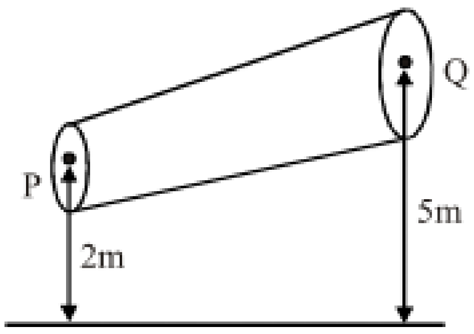
or $$T_{50} = \frac{11}{12}\times T_0 = \frac{11}{12}\times 0.084 = 0.077 \text{ N/m}$$

Considering the change in surface tension as linear, the change ; in surface tension with temperature is given by

$$\alpha = \frac{T_{50} - T_0}{T_0 T_{50}} = \frac{0.077-(0.084)}{0.084\times0.077}$$

$$= \frac{1}{60} \text{ per K.}$$

Q.2 A non-viscous liquid of constant density 1000 kg/m^3 flows in a streanline motion along a tube of variable cross-section. The tube is kept inclined inthe vertical plane as shown in the figure.the area of cross-sectionof the tube at two points P and Q at heights of 2 mater and 5 meter are respectively 4×10^{-3} m^2 and 8×10^{-3} m^2. The velocity of the liquid at point P is 1 m/s. Find the work done per unit volume by the pressure and the gravity forces as the fluid flows from point P and Q.



Sol. As gravitational field is conservative i.e., W = – U

So, $\left(\frac{dW}{dV}\right)_g = -\frac{dU}{dV} = -\frac{mg(h_2 - h_1)}{V} = -\rho g(h_2 - h_1)$

So work done by the force of gravity per unit volume

$$\left(\frac{dW}{dV}\right)_g = \rho g\,(h_2 - h_1) = -10^3 \times 9.8\,(5-2) = -2.94 \times 10^4\ \frac{J}{m^3} \qquad ...(i)$$

Now in case of ideal fluid motion by conservation of mass, i.e.

$$\left(\frac{dm}{dt}\right)_1 = \left(\frac{dm}{dt}\right)_2 \qquad \text{and} \qquad (\rho A \upsilon)_1 = (\rho A \upsilon)_2$$

or $(\rho A\upsilon)_1 = (\rho A\upsilon)_2$ [as ρ = constant (given)]

so $$\upsilon_2 = \frac{A_1\upsilon_1}{A_2} = \frac{4\times 10^{-3} \times 1}{8\times 10^{-3}} = \frac{1}{2}\,m/s \qquad ...(ii)$$

Now as work done per unit volume by pressure,

$$\left(\frac{dW}{dV}\right)_P = \frac{PdV}{dV} = P = (p_1 - p_2) \qquad [\text{as } dW = PdV]$$

But by Bernoulli's theorem,

$$p_1 + \rho g h_1 + \frac{1}{2}\rho\upsilon_1^2 = p_2 + rgh_2 + \frac{1}{2}\rho\upsilon_2^2$$

so $$\left(\frac{dW}{dV}\right)_P = (p_1 - p_2) = \rho g(h_2 - h_1) + \frac{1}{2}\rho(\upsilon_2^2 - \upsilon_1^2) \qquad ...(iii)$$

Using (i), (ii) and (iii) we get

$$\left(\frac{dW}{dV}\right)_P = 2.94\times 10^4 + \frac{1}{2} - 10^3[(0.5)^2 - 1^2] = 29025\,J$$

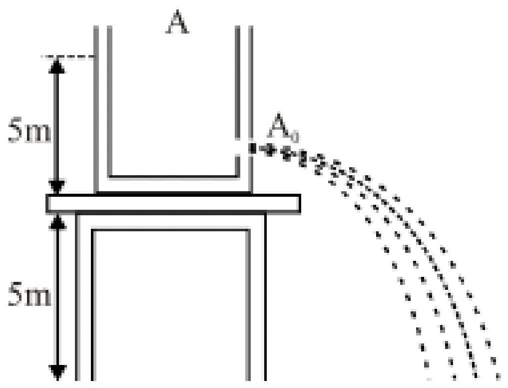
Q.3 A cylindrical tank I m in radius rests on a platform 5 m high. Initially the tank is filled with water up to a height of 5 m. A plug whose area is 10^{-4} m^2 is removed from an orifice on the side of the tank at the bottom. Calculate (a) initial speed with which the water flows from the orifice
(b) initial speed with which the water strikes the ground (c) time taken to empty the tank to half its original value (d) Does the time to empty the tank depend upon the height of stand.

Sol. (a) As speed of efflux is given by

$$\upsilon_H = \sqrt{(2gh)} \text{ so here } u = \sqrt{2 \times 10 \times 5} = 10 \text{ m/s}$$

(b) As vertical speed with which water strikes the gournd,

$$\upsilon_v = \sqrt{\upsilon_H^2 + \upsilon_v^2} = 10\sqrt{2} = 14.1 \text{m/s}$$



(c) When the height of water level above the hole is y,

velocity of flow will be $\upsilon = \sqrt{2gy}$ and so rate flow

$$\frac{dV}{dt} = A_0\upsilon = A_0\sqrt{2gy}$$

or $-Ady = (\sqrt{2gu})\, A_0\, dt$ [as $dV = -Ady$]

Which on integration gives

$$t = \frac{A}{A_0}\sqrt{\frac{2}{g}}[\sqrt{H} - \sqrt{H'}]$$

So

$$t = \frac{\pi \times 1^2}{10^{-4}}\sqrt{\frac{2}{10}}[\sqrt{5} - \sqrt{(5/2)}] = 9.2 \times 10^3 \text{s} = 2.5\text{h}$$

(d) No, as expression of t is independent of height of stand.

Q.4 Under isothermal condition two soap bubbles of radii a and b coalesce to from a single bubble of radius c. If the external pressure is p_0 show that surface tension,

$$T = \frac{p_0(c^3 - a^3 - b^3)}{4(a^2 + b^2 - c^2)}$$

Sol. As excess pressure for a soap bubble is (4T/r) and exteranal pressure p_0,

$$p_i = p_0 + (4T/r)$$

so $p_a = \left[p_0 + \frac{4T}{a}\right]$, $p_b = \left[p_0 + \frac{4T}{b}\right]$ and $p_c = \left[p_0 + \frac{4T}{c}\right]$ ...(i)

and $V_a = \frac{4}{3}\pi a^3$, $V_b = \frac{4}{3}\pi b^3$ and $V_c = \frac{4}{3}\pi c^3$ ...(ii)

Now as mass is conserved ,

i.e., $$\frac{p_a V_a}{Rt_a} + \frac{p_b V_b}{RT_b} = \frac{p_c V_c}{RT_c} \qquad \left[\text{as } PV = \mu RT, \text{i.e.,} \mu = \frac{pV}{RT}\right]$$

As temp, is constant, i.e., $T_a = T_b = T_c$, the above expression reduces to

$$p_a V_a + p_b V_b = p_c V_c$$

Which in the light of Eqn. (i) and (ii) becomes

$$\left[p_0 + \frac{4T}{a}\right]\left[\frac{4}{3}\pi a^3\right] + \left[p_0 + \frac{4T}{b}\right]\left[\frac{4}{3}\pi b^3\right] = \left[p_0 + \frac{4T}{c}\right]\left[\frac{4}{3}\pi c^3\right]$$

i.e., $4T\,(a^2+b^2-c^2) = p_0(c^3-a^3-b^3)$

i.e., $$T = \frac{p_0(c^3 - a^3 - b^3)}{4(a^2 + b^2 - c^2)}.$$

Q.5 The fresh water behind a reservoir dam is 15 m deep. A horizontal pipe 4.0 cm in diameter passed through the dam 6.0 m below the water surface as shown in figure. A plug secures the pipe opening. (a) Find the friction force between theplug and pipe wall. (b) The plug is remove. What volume of water flows out of the pipe in 3.0 hour ?

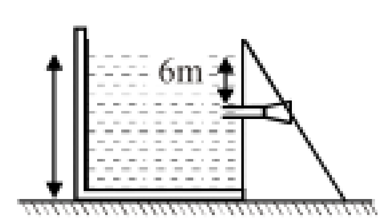
Sol. (a) As the plug secures the pipe opening, the force of friction between plug and pipe wall.

$$F = A\,(p_2 - p_1)$$

But $p_1 = p_0$ and $p_2 = p_0 + h\rho g$

so $F = Ah\rho g$

i.e., $F = \pi \times (2\times10^{-2})^2 \times 6\ 5\ 10^3 \times 9.8 = 74\ N$



(b) As the velocity of efflux.

$$\upsilon = \sqrt{2gh} = \sqrt{2\times10\times6} = 11 m/s$$

so assuming the level of water in the tank to be constant (i.e., area = ∞) as it is not given the volume coming out per second will be

$$R = \frac{dV}{dt} = A_0\upsilon = \pi(2\times10^{-2})^2 \times 11\ m^3/s$$

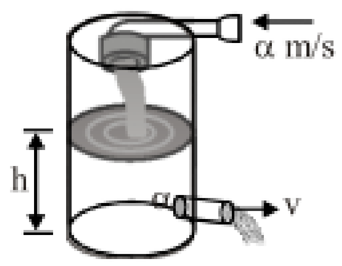
so the volume of the water flowing through the pipe in 3 hours

$$V = R \times t = 44 \times 3.14 \times 10^{-4} \times 3 \times (60 \times 60) = 150\ m^3.$$

Q.6 A cylindrical vessel of base are A has a small hole of cross-section 'a' punched near its base. At time t = 0, water is supplied into the vessel at a constant rate 'α' m³/s. Find

(a) The maximum water level h_{max} in the vessel

(b) The time 't' when water level becomes h ($< h_{max}$).



Sol. (a) Water level will have maximum height when inflow rate = outflow rate and there will be no futher change in level.

$\therefore \quad \alpha = av$

or, $\quad \alpha = a\sqrt{2gh_{max}}$ $\qquad [\because v = \sqrt{2gh}\,]$

or, $\quad h_{max} = a^2/2ga^2$

(b) Let the water level by y at time t.

$$\therefore \quad A\left(\frac{dy}{dt}\right) = \alpha - av = \alpha - \sqrt{2gy}$$

Here $\frac{dy}{dt}$ is positive as y increase with time instantaneous effux velocity $v = \sqrt{2gy}$.

Rearranging the above equation

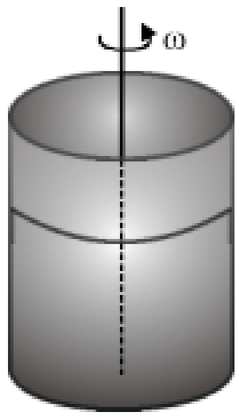
$$\int_0^h \frac{dy}{\alpha - \alpha\sqrt{2gy}} = \frac{1}{A}\int_0^t dt$$

Integration under the given limits, we get the required time

$$t = \frac{A}{ag}\left[\frac{\alpha}{a} In\left(\frac{\alpha - a\sqrt{agh}}{\alpha}\right) - \sqrt{2gh}\right]$$

This gives the time t as a function of h. For any valume $h(\leq h_{max})$, the corresponding time t and be evaluated.

Q.7 A cylindrical vessel of (radius r) containing a liquid spins continously with constant angular velocity ω as shown in the figure. Shown that the pressure at a radial distance r from the axis is



$$P = P_0 + \frac{1}{2}\rho\omega^2 r^2,$$

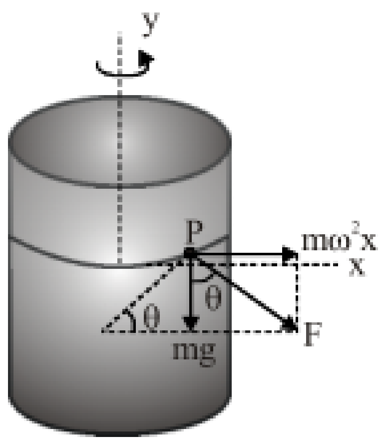
where P_0 = atmospheric pressure.

Sol. Consider a particel of the fluid at a point P (x, y) w.r.t. the corrdinate axes as shown in the figure. The force acting on This particle are $m\omega^2 x$ (the centrifugal force) and the weight mg.

The net force F acting at P should be perpendicular to the free surface, so that

$$\tan\theta = \frac{m\omega^2 x}{mg} = \frac{\omega^2 x}{g}$$

or, $$\frac{dy}{dx} = \frac{m\omega^2}{g} \qquad \left[\because \text{slope} = \tan\theta = \frac{dy}{dx}\right]$$



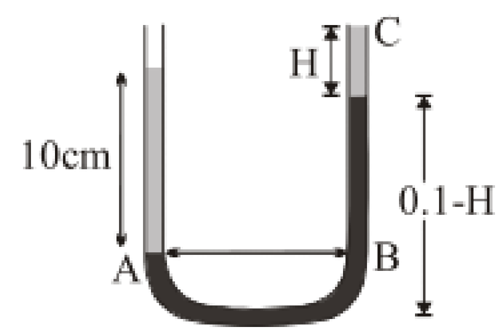
or, $$y = \frac{\omega^2}{2g}x^2.$$

This equation represents a parabola; for which the elevation from origin at $x = r$ will be $y = \frac{\omega^2}{2g}r^2$.

$\therefore$ $$\text{Pressure } P(r) = P_0 + \rho g y = P_0 + \frac{\rho\omega^2 r^2}{2}$$

Q.8 A vertical U-tube of uniform cross-section contains mercury in both arms. A glycerine (relative density = 1.3) column of length 10 cm is introduced into one of the arms. Oil of density 800 kg m^{-3} is poured into the other arm until the upper surface of the oil and glycerine are at the same horizontal level. Find the length of the oil column. Density of mercury is 13.6×10^3 kgm^{-3}.

Sol. Pressure at A and B must be same



Pressure at A = $p_0 + 0.1 \times (1.3 \times 1000) \times g$

where p_0 = atmospheric pressure

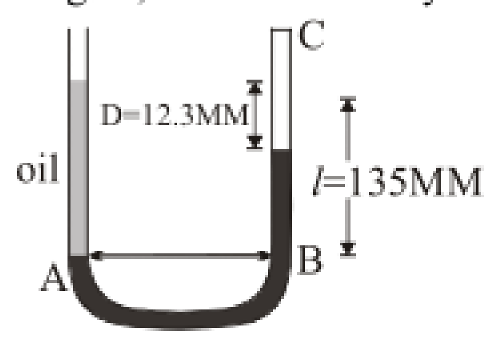
Pressure at B = $p_0 + h \times 800 \times g + (0.1 - h) \times 13.6 \times 1000\ g$

$\therefore$ $p_0 + 0.1 \times 1300 \times g$

$= p_0 + 800\ gh + 1360\ g - 13600 \times g \times h$

or $h = 9.6$ cm

Q.9 For the arrangement shown in the figure, what is the density of oil?



Sol. $p_{surface} = p_0 + \rho_w . g l$

$p_{surface} = p_0 + \rho_{oil}(l + d)\ g$

$$\Rightarrow \rho_{oil} = \frac{\rho_\omega \cdot l}{(l + d)} = \frac{1000(135)}{(135 + 12.3)} = 916\ kg/m^3$$

Q.10 A pipe of copper having an internal cavity weighs 264 gm in air and 221 gm in water. Find the volume of the cavity. [Density of copper is 8.8 gm/cc.]

Sol. The buoyant force on the copper piece, $F = V\rho g$

Hence, volume of the copper piece $V = \frac{F}{\sigma g} = \frac{(264 - 221)\,g}{1 \times g} = 43\,cc$

The volume of the material of the copper piece

$$V_0 = \frac{\text{mass of copper piece}}{\text{density of material}} = \frac{264}{8.8} = 30\text{ cc}$$

Hence, volume of the cavity $= V - V_0 = 43 - 30 = 13$ cc

Q.11 A piece of brass (alloy of copper and zinc) weighs 12.9 gm in air. When completely immersed in water, it weighs 11.3 gm. What is the mass of copper contained in the alloy?

[Specific gravities of copper and zinc are 8.9 and 7.1, respectively.]

Sol. Let the mass of copper in alloy = x gm.

$\therefore$ Amount of zinc = (12.9 – x) gm

Volume of copper, $V_{Cu} = \frac{x}{\rho_{Cu}} = \frac{x}{8.9}$ and

Volume of zinc $V_{Zn} = \frac{(12.9 - x)}{7.1}$

$\therefore$ Total volume of the alloy, $V = V_{Cu} + V_{Zn}$

or $$V = \frac{(12.9 - x)}{7.1} + \frac{x}{8.9} \qquad \text{... (i)}$$

Buoyant force $F = V_\rho g$ = loss of weight

$= (12.9 - 11.3)g = 1.6\,g$

Substituting the value of V in equation (i), we get

$x = 7.6$ gm

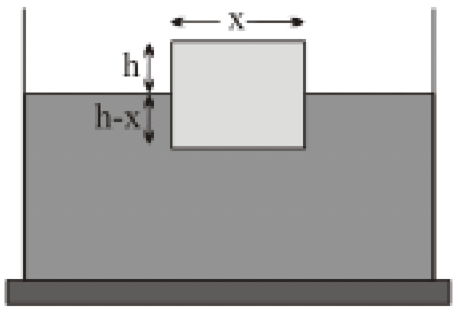
Q.12 A cubical block of iron of edge 5 cm is floating on mercury in a vessel.

(a) What is the height of the block above mercury level?

(b) Water is poured into the vessel so that it just covers the iron block. What is the height of the water column?

[Relative density of Hg = 13.6 and that of Fe = 7.2]

Sol. (a) Let h be the height of the iron block above mercury.



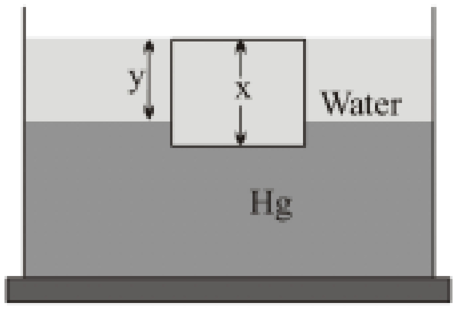
In case of flotation,

Weight of the block = buoyant force

i.e., $x^3 \rho g = [(x-h)\sigma g]\, x^2$

or $h = x\left(1-\dfrac{\rho}{\sigma}\right) = 5\left(1-\dfrac{7.2}{13.6}\right) = 2.35 \text{ cm}$

(b) Let y be the height of the water level.



For equilibrium of the block,

$x^3\rho g = [\sigma_w g y + \sigma_{Hg}\, g(x-y)]\, x^2$

$x\sigma = (x-y)\,\sigma_{Hg} + y\sigma_w$

or $\quad y = x\left(\dfrac{\sigma_{Hg}-\sigma}{\sigma_{Hg}-\sigma_w}\right) = 5\left(\dfrac{13.6-7.2}{13.6-1}\right) = 2.54 \text{ cm}$

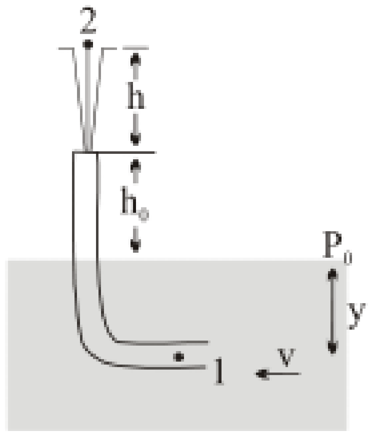
Q.13 The 'tip of the iceberg' in popular speech has come to mean a small visible fraction of something that is mostly hidden. For real icebergs, what is this fraction? ($\rho_{ice} = 917 \text{ kg/m}^3$, $\rho_{sea\ water} = 1024 \text{ kg/m}^3$)

Sol. $W_{ice} = \rho_i V_i g$, $W_{sea\ water} = \rho_w V_w g$

For floatation, $\rho_i V_i g = \rho_w V_w g$

Fraction of the volume submerged $= \dfrac{V_i - V_w}{V_i} = 1 - \dfrac{917}{1024} = \dfrac{107}{1024} = 10.45\%$

Q.14 A bent tube is lowered into the stream as shown. The velocity of the stream relative to the tube is equal to V. The closed upper end of the tube is located at height h_0. To what height h will the water jet spurt?



Sol. Let tube's entrance be a depth 'y' below the surface. Take point 1 at entry and point 2 at the maximum height of the fountain. This is a tube of flow. Now let's apply Bernoulli's theorem,

$$P_1 + \rho g h_1 + \frac{1}{2}\rho v_1^{\,2} = P_2 + \rho g h_2 + \frac{1}{2}\rho v_2^{\,2}$$

Taking, $h_1 = 0$, $h_2 = (y + h_0 + h)$, $V_1 = V$, $V_2 = 0$

$P_1 = P_0 + \rho gy$, $P_2 = P_0$,

Substituting, $P_0 + \rho gy + \rho g \times 0 + \frac{1}{2}\rho V^2 = P_0 + \rho g\,(y + h_0 + h) + \frac{1}{2}\rho \times 0^2$

$\Rightarrow \frac{1}{2}\rho V^2 = \rho g\,(h_0 + h)$ or $h = \left(\frac{V^2}{2g} - h_0\right)$

Q.15 Water enters a house through a pipe with inlet diameter of 2.0 cm at an absolute pressure of 4.0×10^5 Pa (about 4 atm). A 1.0 cm diameter pipe leads to the second floor bathroom 5.0 above. When flow speed at the inlet pipe is 1.6 m/s, find the flow speed, pressure and volume flow rate in the bathroom.

[Sol. Let point 1 and 2 be at the inlet pipe and the bathroom, then from continuity equation

$a_1 v_1 = a_2 v_2 \quad \Rightarrow v_2 = 6.0$ m/s

Now, applying Bernoulli's equation at the inlet (y = 0) and at the bathroom ($y_2 = 5.0$ m).

As $p + \frac{1}{2}\sigma v^2 + \sigma gy = \text{constant}$

Hence, $p_2 - p_1 = \frac{1}{2}p\left(v_2^2 - v_1^2\right) - \rho g\,(y_2 - y_1)$

Which gives $p_2 = 3.3 \times 10^5$ Pa

The volume flow rate $= A_2 v_2 = A_1 v_1 = \frac{\pi}{4}(0.1)^2\, 6 = 4.7 \times 10^{-4}$ m^3/s.]

Q.16 Water coming out the jet having across sectional area a, with a speed v strikes a stationary plate and stops after striking. Find the force exerted by the water jet on the plate.

Sol. The change of momentum of water in time dt $= 0 - \rho av^2 dt\,\hat{i} = -\rho av^2 dt\,\hat{i}$ where $\hat{i}$ is a unit vector in the direction of the velocity of the jet. The rate of change of momentum of water jet $= -\rho av^2\,\hat{i}$

Thus the force exerted on the water jet by the plate $= -\rho av^2\,\hat{i}$

The force exerted on the plate by the water jet $= \rho av^2\,\hat{i}$.

Q.17 What is the surface energy of an air bubble inside a soap solution?

Sol. $E = T \times A = 4\pi r^2 T$, as it has only one surface.

Q.18 A metal plate 0.04 m^2 in area is lying on liquid layer of thickness 10^{-3} m and co-efficient of viscosity 140 poise. Calculate the horizontal force needed to move the plate with a speed of 0.040 m/s.

Sol. Area of the plate, $A = 0.04\ m^2$

Thickness, $\Delta x = 10^{-3}$ m

Δx is the distance of the free surface with respect to the fixed surface.

Velocity gradient, $\dfrac{\Delta v}{\Delta x} = \dfrac{0.04}{10^{-3}} = 40\,s^{-1}$

Co-efficient of viscosity, $\eta = 14\ kg\ ms^{-1}\ s^{-1}$

Let F be the required force.

Then, $F = \eta A \dfrac{\Delta v}{\Delta x} = 22.4\,N$