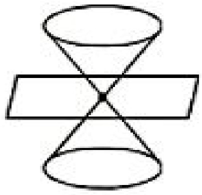
CONIC SECTION

Point, pair of straight lines, circle, parabola, ellipse and hyperbola are called conic section because they can be obtained when a cone (or double cone) is cut by a plane.

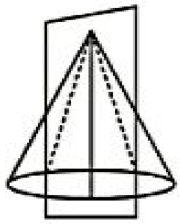
The mathematicians associated with the study of conics were Euclid, Aristarchus and Apollonius. Most of the objects around us and in space have shape of conic-sections. Hence study of these becomes a very important tool for present knowledge and further exploration.

Section of right circular cone by different planes :

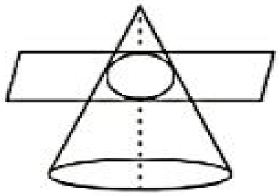
(1) When a double right circular cone is cut by a plane parallel to base at the common vertex, the cutting profile is a point.



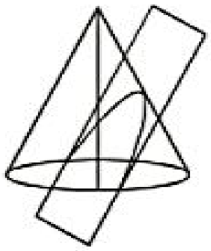
(2) When a right circular cone is cut by any plane through its vertex, the cutting profile is a pair of straight lines through its vertex



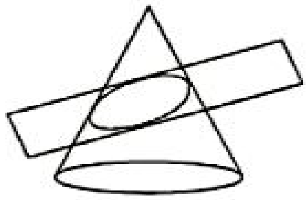
(3) When a right circular cone is cut by a plane parallel to its base the cutting profile is a circle.



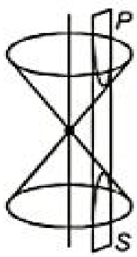
(4) When a right circular cone is cut by a plane parallel to a generator of cone, the cutting profile is a parabola.



(5) When a right circular cone is cut by a plane which is neither parallel to any generator / axis nor parallel to base, the cutting profile is an ellipse.



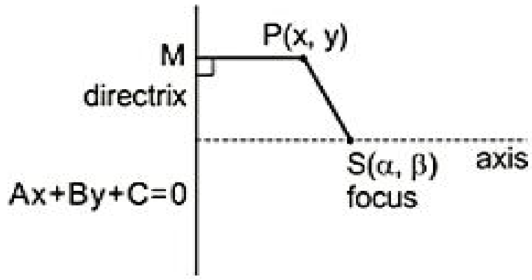
(6) When a double right circular cone is cut by plane, parallel to its common axis, the cut profile is hyperbola



Hence a point, a pair of intersecting straight lines, circle, parabola, ellipse and hyperbola, all are conic-

The conic section is the locus of a point which moves such that the ratio of its distance from a fixed point (focus) to perpendicular distance from a fixed straight line (directrix) is always constant (e). Here e is called eccentricity of conic i.e.,

$$\frac{PS}{PM} = e$$



A line through focus and perpendicular to directrix is called - axis. The vertex of conic is that point where the curve intersects its axis.

$$\frac{PS}{PM} = e \Rightarrow PS^2 = e^2 PM^2$$

$$\Rightarrow \ (x-\alpha)^2 + (y-\beta)^2 = e^2\left(\frac{Ax+By+C}{\sqrt{A^2+B^2}}\right)^2$$

Simplification shall lead to the equation of the form $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$

Distinguishing various conics :

The nature of the conic section depends upon the position of the focus S w.r.t. the directrix & also upon the value of the eccentricity e. Two different cases arise.

Case-I : When The Focus Lies On The Directrix (De-generated conic) :

In this case $\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ & the general equation of a conic represents a pair of straight lines if

$e > 1$ i.e. $h^2 > ab$ the lines will be real & distinct, intersecting at S.
$e = 1$ i.e. $h^2 = ab$ the lines will coincident.
$e < 1$ i.e. $h^2 < ab$ the lines will be imaginary.

Case-II : When The Focus Does Not Lie on the Directrix (Non de-generated conic) :

In this case $\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$ and conic represent

a parabola	an ellipse	a hyperbola	rectangular hyperbola	Circle
$e = 1$ $h^2 = ab$	$0 < e < 1$ $h^2 < ab$	$e > 1$ $h^2 > ab$	$e = \sqrt{2}$ $h^2 > ab;\ a + b = 0$	$e = 0$ $h = 0,\ a = b$

Note :

(i) For pair of straight lines $e \to \infty$

(ii) All second degree terms in parabola form a perfect square

Definition of various terms related to a conic :

(1) **Focus :** The fixed point is called a focus of the conic.

(2) **Directrix :** The fixed line is called a directrix of the conic.

(3) **Axis :** The line passing through the focus and perpendicular to the directrix is called the axis of the conic.

(4) **Vertex :** The points of intersection of the conic and the axis are called vertices of the conic.

(5) **Centre :** The point which bisects every chord of the conic passing through it, is called the centre of the conic.

(6) **Latus-rectum :** The latus-rectum of a conic is the chord passing through the focus and perpendicular to the axis.

(7) **Double ordinate :** A chord which is perpendicular to the axis of parabola or parallel to its directrix.

Illustration :

What conic does $13x^2 - 18xy + 37y^2 + 2x + 14y - 2 = 0$ represent ?

Sol. *Compare the given equation with*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$\therefore$ $a = 13, h = -9, b = 37, g = 1, f = 7, c = -2$

then $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= (13)(37)(-2) + 2(7)(1)(-9) - 13(7)^2 - 37(1)^2 + 2(-9)^2$

$= -962 - 126 - 637 - 37 + 162$

$= -1600 \neq 0$

and also $h^2 = (-9)^2 = 81$ *and* $ab = 13 \times 37 = 481$

Here $h^2 - ab < 0$

So we have $h^2 - ab < 0$ *and* $\Delta \neq 0$. *Hence the given equation represents an ellipse.*

Illustration :

For what value of λ the equation of conic $2xy + 4x - 6y + \lambda = 0$ represents two real intersecting straight lines? if $\lambda = 17$ then this equation represents ?

Sol. *Comparing the given equation of conic with*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$\therefore$ $a = 0, b = 0, h = 1, g = 2, f = -3, c = \lambda$

For two intersecting real lines

$$h^2 - ab \geq 0 \text{ and } \Delta = 0$$

here $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 0 + 2 \times (-3) \times 2 \times 1 - 0 - 0 - \lambda (1)^2$

$= -12 - \lambda = 0$

$\therefore$ $\lambda = -12$

and $h^2 - ab = 1$

hence for $\lambda = -12$ above equation always represent real intersecting lines.

if $\lambda = 17$ *then* $\Delta \neq 0$ *and* $h^2 - ab > 0$

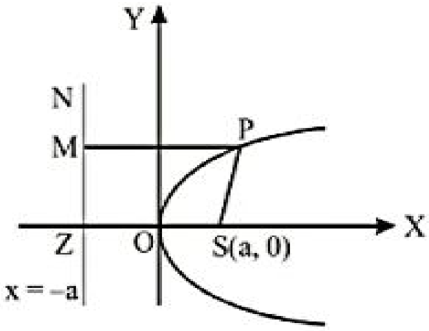
so we have $\Delta \neq 0$ and $h^2 - ab > 0$. Hence the given equation represents a Hyperbola.

PARABOLA

A parabola is the locus of a point which moves in a plane such that its distance from a fixed point (focus) is equal to its perpendicular distance from a fixed straight line (directrix). Eccentricity of parabola is 1.

Standard Equation of a parabola :

Let S be the focus and ZN is the directrix of the parabola.
From S, draw SZ perpendicular to the directrix.
Let O be the middle point of ZS. Take O as the origin and OS as x-axis and OY perpendicular to OS as the y-axis.
Let ZS = 2a, then ZO = OS = a
Now, $S \equiv (a, 0)$ and the equation of ZN is $x = -a$ or $x + a = 0$.
Let P(x, y) be any point on the parabola.
$\therefore$ PS = PM (by definition of parabola).



$$\Rightarrow \quad \sqrt{(x-a)^2+(y-0)^2} = \frac{|x+a|}{\sqrt{1^2+0}}$$

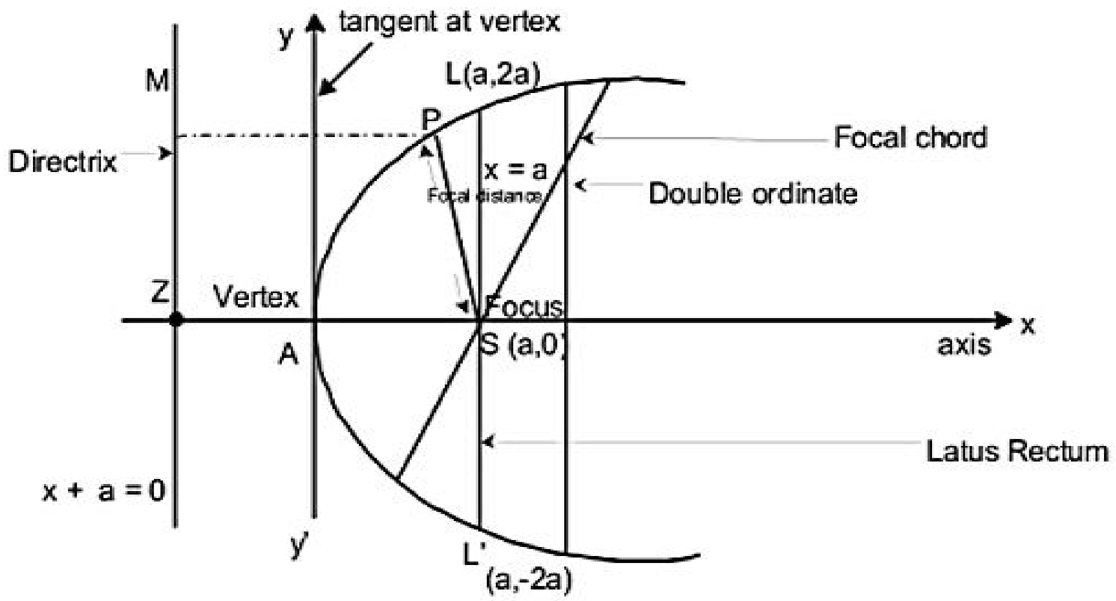
$$\Rightarrow \quad \sqrt{(x-a)^2+y^2} = |x+a|$$

or $\quad (x-a)^2 + y^2 = (x+a)^2$

or $\quad x^2 - 2ax + a^2 + y^2 = x^2 + 2xa + a^2$

or $\quad y^2 = 4ax \quad$ which is the required equation.

Terms related to Parabola :



(1) **Axis :** A straight line passes through the focus and perpendicular to the directrix is called the axis of parabola. For the parabola $y^2 = 4ax$, x-axis is the axis.
Since equation has even power of y therefore the parabola is symmetric about x-axis i.e. about its axis.

(2) **Vertex :** The point of intersection of a parabola and its axis is called the vertex of the Parabola. For the parabola $y^2 = 4ax$, O(0, 0) is the vertex.
The vertex is the middle point of the focus and the point of intersection of axis and directrix.

(3) **Focal Distance :** The distance of any point P (x, y) on the parabola from the focus is called the focal length (distance) of point P.
The focal distance of P = the perpendicular distance of the point P from the directrix.

(4) **Double Ordinate :** The chord which is perpendicular to the axis of Parabola or parallel to Directrix is called double ordinate of the Parabola.

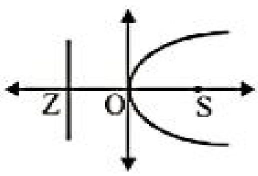
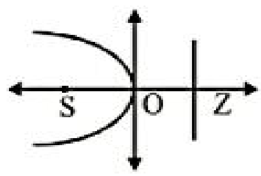
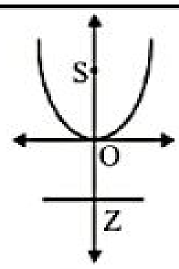
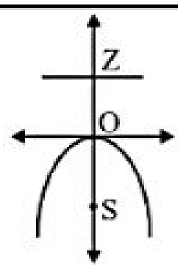
(5) **Focal Chord :** Any chord of the parabola passing through the focus is called Focal chord.

(6) **Latus Rectum :** If a double ordinate passes through the focus of parabola then it is called as latus rectum. The extremities of the latus rectum are L (a, 2a) and L'(a, –2a). Since LS = L'S = 2a, therefore length of the latus rectum LL' = 4a.

(7) **Parametric Equation of Parabola :** The parametric equation of Parabola $y^2 = 4ax$ are $x = at^2$, $y = 2at$. Hence any point on this parabola is $(at^2, 2\,at)$ which is also called as 't' point.

Note:
(i) The length of the latus rectum = 2 × perpendicular distance of focus from the directrix.
(ii) If $y^2 = lx$ then length of the latus rectum = l.
(iii) Two parabolas are said to be equal if they have same latus rectum.
(iv) The ends of a double ordinate of a parabola can be taken as $(at^2, 2\,at)$ and $(at^2, -2at)$.
(v) Parabola has no centre, but circle, ellipse, hyperbola have centre.

Other Standard parabola :

Equation of parabola	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
(a) Graphs				
(b) Eccentricity	e = 1	e = 1	e = 1	e = 1
(c) Focus	S(a, 0)	S(–a, 0)	S(0, a)	S(0, –a)
(d) Equation of directrix	$x + a = 0$	$x - a = 0$	$y + a = 0$	$y - a = 0$
(e) Equation of axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
(f) Vertex	O(0, 0)	O(0, 0)	O(0, 0)	O(0, 0)
(g) Extremities of latusrectum	(a, ±2a)	(–a, ±2a)	(±2a, a)	(±2a, –a)
(h) Length of latusrectum	4a	4a	4a	4a
(i) Equation of tangent at vertex	$x = 0$	$x = 0$	$y = 0$	$y = 0$
(j) Parametric coordinates of any point on parabola	$P(at^2, 2at)$	$P(-at^2, 2at)$	$P(2at, at^2)$	$P(2at, -at^2)$

REDUCTION TO GENERALIZED EQUATION OF PARABOLA :

If the equation of a parabola is either in the form $x = \ell y^2 + my + n$ or $y = \ell x^2 + mx + n$ then it can be reduced into generalised form. For this we change the given equation into the following forms-
$(y - k)^2 = 4a(x - h)$ or $(x - h)^2 = 4a(y - k)$

And then we compare from the standard equation of parabola to find all its parameters.

(A) When the equation of parabola is :

$$(y - k)^2 = 4a(x - h) \qquad \ldots(i)$$

Equation (i) is of the form $Y^2 = 4aX$
where $Y = y - k$ and $X = x - h$

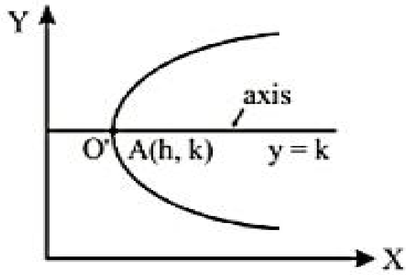
(1) Axis of parabola is $Y = 0$, i.e., $y - k = 0 \Rightarrow y = k$

(2) Coordinates of vertex of parabola are given by

$X = 0$ and $Y = 0$

i.e. $x - h = 0$ and $y - k = 0$

$\therefore$ Vertex is (h, k)



(3) Tangent at the vertex to parabola (i) is given by
$X = 0$, i.e., $x - h = 0$
Therefore, tangent at the vertex is $x = h$.

(4) Coordinates of focus of parabola are given by

$X = a$ and $Y = 0$

i.e. by $x - h = a$ and $y - k = 0$

$\therefore$ Focus of parabola is $(a + h, k)$.

(5) Equation of directrix of parabola is

$X = -a$

i.e., $x - h = -a$

Therefore, directrix of parabola is $x = h - a$

(6) Length of latus rectum of parabola is $|4a|$.

(7) Coordinates of ends of latus rectum of parabola are given by

$X = a$ & $Y = \pm 2a$

i.e., by $x - h = a$, $y - k = \pm 2a$ i.e. coordinate of latus rectum is $(a + h, k \pm 2a)$.

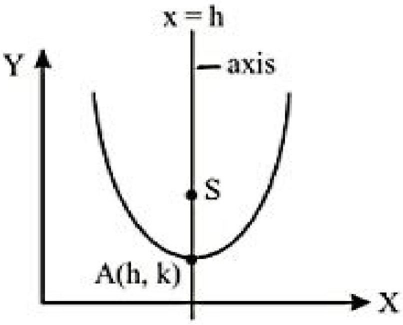
(8) Parametric equation is $x = h + at^2$ and $y = k + 2at$.

(B) When the equation of parabola is :

$(x-h)^2 = 4a(y-k)$ ...(i)

Equation (i) is of the form $X^2 = 4aY$
where $X = x - h$ and $Y = y - k$

(1) Axis of parabola is $X = 0$, i.e., $x - h = 0$



(2) Coordinates of vertex of parabola is given by
$X = 0$ and $Y = 0$
i.e., by $x - h = 0$ and $y - k = 0$
$\therefore$ $x = h$ and $y = k$
Hence vertex of parabola is (h, k)

(3) Equation of tangent at the vertex to parabola is
$Y = 0$ i.e., $y - k = 0$
or $y = k$

(4) Coordinates of focus of parabola are given by
$X = 0$ and $Y = a$
i.e., by $x - h = 0$ and $y - k = a$
$\therefore$ Focus of parabola is $(h, k + a)$.

(5) Equation of directrix of parabola (i) is given by
$Y = -a$ or $y - k = -a$ or $y = k - a$

(6) Length of latus rectum of parabola is $|4a|$.

(7) Coordinates of ends of latus rectum of parabola are given by
$Y = a$, $X = \pm 2a$
i.e., $y - k = a$, $X - h = \pm 2a$
$\therefore$ Ends of latus rectum are $(h \pm 2a, k + a)$

(8) Parametric equation is $x = h + 2at$ and $y = k + at^2$.

Equation of Parabola	Vertex	Axis	Focus	Directrix	Parametric equation
$(y-k)^2 = 4a(x-h)$	(h,k)	$y = k$	$(h + a, k)$	$x + a - h = 0$	$x = h + at^2, y = k + 2at$
$(x-h)^2 = 4a(y-k)$	(h, k)	$x = h$	$(h, a + k)$	$y + a - k = 0$	$x = h + 2at, y = k + at^2$

Note :

(i) For the parabola $y = Ax^2 + Bx + C$, the length of latus rectum is $\frac{1}{|A|}$ and axis is parallel to y-axis. If A is positive then it is concave up parbaola, if A is negative then it is concave down parabola.

(ii) For the parabola $x = Ay^2 + By + C$, the length of latus rectum is $\frac{1}{|A|}$ and axis is parallel to x-axis. If A is positive then it is opening right and if A is negative then it is opening left parabola.

Illustration :

Find the equation of the parabola whose focus is at (– 1, – 2) and the directrix is the line $x - 2y + 3 = 0$.

Sol. *Let P(x, y) be any point on the parabola whose focus is S(– 1, – 2) and the directrix $x - 2y + 3 = 0$. Draw PM perpendicular to directrix $x - 2y + 3 = 0$. Then by definition,*

$$SP = PM$$

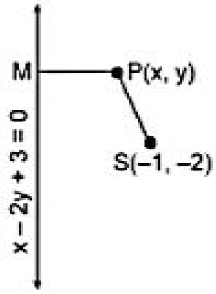
$$\Rightarrow \quad SP^2 = PM^2$$

$$\Rightarrow \quad (x+1)^2 + (y+2)^2 = \left(\frac{x-2y+3}{\sqrt{1+4}}\right)^2$$

$$\Rightarrow \quad 5\left[(x+1)^2 + (y+2)^2\right] = (x-2y+3)^2$$

$$\Rightarrow \quad 5(x^2 + y^2 + 2x + 4y + 5) = (x^2 + 4y^2 + 9 - 4xy + 6x - 12y)$$

$$\Rightarrow \quad 4x^2 + y^2 + 4xy + 4x + 32y + 16 = 0$$



This is the equation of the required parabola.

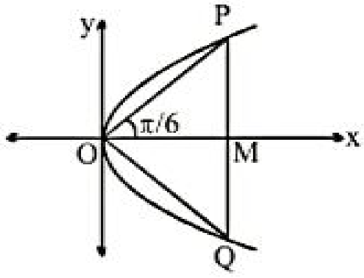
Illustration :

An equilateral triangle is inscribed in the parabola $y^2 = 4ax$, such that one vertex of this triangle coincides with the vertex of the parabola. Then find the side length of this triangle.

Sol. *If ΔOPQ is an equilateral triangle then $OP = OQ = PQ = k$*

$$\Rightarrow \quad \angle POM = \frac{\pi}{6}$$

$$\therefore \quad OM = k\cos\frac{\pi}{6} \text{ and } PM = k\sin\frac{\pi}{6}$$



$$\therefore \quad P\left(\frac{\sqrt{3}}{2}k, \frac{k}{2}\right) \text{ lie on the parabola } y^2 = 4ax$$

$$\Rightarrow \quad \left(\frac{k}{2}\right)^2 = 4a \cdot \frac{\sqrt{3}}{2}k \Rightarrow k = 8\sqrt{3}\,a.$$ ***Ans.***

Illustration :

The parametric equation of a parabola is $x = t^2 + 1$, $y = 2t + 1$. Then find the coordinate of vertex and length of latus -rectum.

Sol. *Eliminate t from parametric equation, we get equation of parabola. Hence*

$$x = \left(\frac{y-1}{2}\right)^2 + 1 \text{ or } (y-1)^2 = 4(x-1)$$

$\therefore$ *vertex is (1, 1) and length of latus rectum = 4.*

Illustration :

Find the parametric equation of the parabola $(x-1)^2 = -12\,(y-2)$

Sol. $\because$ $4a = -12 \Rightarrow a = -3,$

parametric equation is $y - 2 = -3t^2$

$x - 1 = -6t \Rightarrow x = 1 - 6t, \quad y = 2 - 3t^2$

Illustration :

The parametric equation of the curve $(y-2)^2 = 12\,(x-4)$ *are-*

(A) $6t, 3t^2$ (B) $2 + 3t, 4 + t^2$ (C) $4 + 3t^2, 2 + 6t$ (D) None of these

Sol. *Here* $a = 3$

$x - 4 = at^2 \Rightarrow x = 4 + 3.t^2 = 4 + 3t^2$ **Ans. [C]**

$y - 2 = 2at \Rightarrow y = 2 + 2.3t = 2 + 6t$

Illustration :

Find the equation of parabola, whose axis is parallel to y-axis and which passes through points (0, 2), (– 1, 0) and (1, 6).

Sol. *General equation of such parabola is* $y = Ax^2 + Bx + C$ .....(i)

Since it passes throught (0, 2) (– 1, 0) and (1, 6), then we have

$C = 2$ ....(ii)

$A - B + C = 0$ ......(iii)

$A + B + C = 6$ ......(iv)

from (ii), (iii) and (iv),

$A = 1, B = 3, C = 2$

$\therefore$ *equation of parabola is* $y = x^2 + 3x + 2$. **Ans.**

Illustration :

Find the value of λ *if the equation* $\sqrt{(x-1)^2 + (y-2)^2} = \sqrt{\lambda}\,(x + y + 3)$ *represent parabola. Find its focus, directrix, axis and equation of vertex.*

Sol. $(x-1)^2 + (y-2)^2 = \lambda\,(x + y + 3)^2$ *are given equation*

$$(x-1)^2 + (y-2)^2 = 2\lambda\left(\frac{x+y+3}{\sqrt{2}}\right)^2$$

$\Rightarrow$ *above equation represents parabola if* $2\lambda = 1 \Rightarrow \lambda = \dfrac{1}{2}$

In this case focus is (1, 2) and directrix is $x + y + 3 = 0$

Line perpendicular to directrix is $x - y + k = 0$ *and passes through (1, 2) gives axis of parabola*

$\Rightarrow$ $1 - 2 + k = 0 \Rightarrow k = 1$

$\therefore$ *axis is* $x - y + 1 = 0$

Now axis and directrix meet at (– 2, – 1) (called foot of directrix)

Thus vertex is the mid point of foot of directix and the focus.

i.e. vertex $\left(\dfrac{1-2}{2}, \dfrac{2-1}{2}\right)$ *i.e.,* $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$. **Ans.**

Illustration :

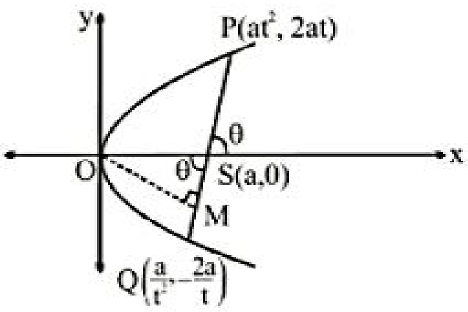
If the length of a focal chord of the parabola $y^2 = 4ax$ at a distance b from the vertex is c, then show that $b^2c = 4a^3$.

Sol. $OM = b$

focal length $PQ = 4a\ \text{cosec}^2\theta$ ....(i)

From right angled triangle OMS,

$$\sin\theta = \frac{OM}{OS} = \frac{b}{a}$$



$\therefore \quad \text{cosec}\ \theta = \dfrac{a}{b}$ ....(ii)

from (i) and (ii)

$$PQ = c = 4a \cdot \left(\frac{a}{b}\right)^2$$

$\Rightarrow \quad b^2c = 4a^3.$ ***Ans.***

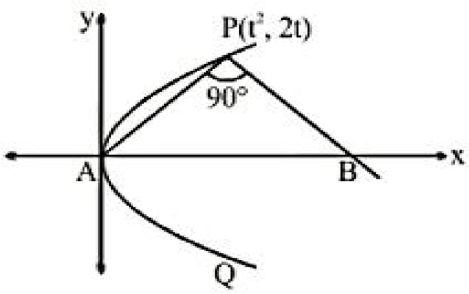
Illustration :

AP is perpendicular to PB, where A is the vertex of parabola $y^2 = 4x$ and P is on the parabola. If B lie on the axis of parabola, then find the locus of centroid of ΔPAB

Sol. Slope of $AP = \dfrac{2}{t}$

$\therefore \quad$ Slope of $BP = -\dfrac{t}{2}$

$\therefore \quad$ equation of line PB is $y - 2t = -\dfrac{t}{2}(x - t^2)$



$\therefore \quad$ point B is $(t^2 + 4, 0)$

Let centroid of ΔPAB is (h, k)

$$\therefore \quad h = \frac{t^2 + t^2 + 4}{3} \text{ and } k = \frac{2t}{3}$$

Now eliminate t from above two equation, we get

$$3h - 4 = 2\left(\frac{3k}{2}\right)^2$$

$\therefore \quad$ locus of centroid is $3x - 4 = \dfrac{9y^2}{2}$. ***Ans.***

Illustration :

Find the equation of the parabola whose vertex is (–3, 0) and directrix is $x + 5 = 0$.

Sol. *A line passing through the vertex (–3, 0) and perpendicular to directrix $x + 5 = 0$ is x-axis which is the axis of the parabola by definition. Let focus of the parabola is (a, 0). Since vertex, is the middle point of Z(–5, 0) and focus S, therefore*

$$-3 = \frac{(a-5)}{2} \Rightarrow a = -1$$

$\therefore$ Focus = (–1,0)

Thus the equation to the parabola is

$$(x + 1)^2 + y^2 = (x + 5)^2$$

$$\Rightarrow \quad y^2 = 8(x + 3)$$

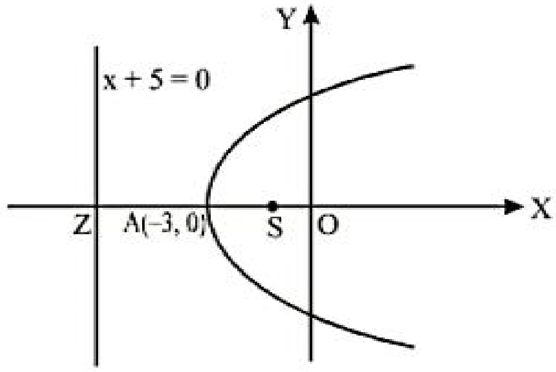


Illustration :

Find the equation of directrix and axis of the parabola $4y^2 - 6x - 4y = 5$.

Sol. Here $4y^2 - 4y = 6x + 5$

$$\Rightarrow \quad 4\left(y - \frac{1}{2}\right)^2 = 6(x + 1) \quad \Rightarrow \quad \left(y - \frac{1}{2}\right)^2 = \frac{3}{2}(x + 1)$$

Put $y - \frac{1}{2} = Y$ & $x + 1 = X$

The equation in standard form $Y^2 = \frac{3}{2}X$

$$4a = \frac{3}{2} \Rightarrow a = \frac{3}{8} \text{ \& length of latus rectum} = \frac{3}{2}$$

Directrix, $X + a = 0$

$$\Rightarrow \quad x + 1 + \frac{3}{8} = 0 \quad \Rightarrow \quad 8x + 11 = 0$$

Axis is $Y = 0 \Rightarrow y - \frac{1}{2} = 0 \quad \Rightarrow \quad 2y - 1 = 0.$

Parametric equation is $x + 1 = \frac{3}{8}t^2$ & $y - \frac{1}{2} = 2 \cdot \frac{3}{8} \cdot t$

i.e. $x = -1 + \frac{3}{8}t^2, y = \frac{1}{2} + \frac{3}{4}t.$

Illustration :

Find the vertex, axis, focus, directrix, latusrectum of the parabola, also draw their rough sketches.

$$4y^2 + 12x - 20y + 67 = 0$$

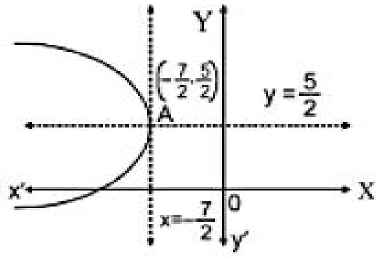
Sol. *The given equation is*

$$4y^2 + 12x - 20y + 67 = 0 \quad \Rightarrow \quad y^2 + 3x - 5y + \frac{67}{4} = 0$$

$$\Rightarrow \quad y^2 - 5y = -3x - \frac{67}{4} \quad \Rightarrow \quad y^2 - 5y + \left(\frac{5}{2}\right)^2 = -3x - \frac{67}{4} + \left(\frac{5}{2}\right)^2$$

$$\Rightarrow \quad \left(y - \frac{5}{2}\right)^2 = -3x - \frac{42}{4} \quad \Rightarrow \quad \left(y - \frac{5}{2}\right)^2 = -3\left(x + \frac{7}{2}\right) \qquad \text{....(i)}$$

Let $X = x + \frac{7}{2}$ & $Y = y - \frac{5}{2}$ ....(ii)



Using these relations, equation (i) reduces to

$$Y^2 = -3X \qquad \text{....(iii)}$$

This is of the form $Y^2 = -4aX$. *On comparing, we get* $4a = 3 \Rightarrow a = 3/4$.

The coordinates of the vertex are $(X = 0, Y = 0)$

i.e. the coordinates of the vertex are $\left(-\frac{7}{2}, \frac{5}{2}\right)$

The equation of the axis of the parabola is $Y = 0$.

i.e. the equation of the axis is $y = \frac{5}{2}$ *[Putting* $Y = 0$ *in (ii)]*

The coordinates of the focus are $(X = -a, Y = 0)$

$$\Rightarrow \quad x + \frac{7}{2} = -\frac{3}{4} \quad \& \quad y - \frac{5}{2} = 0$$

i.e. the coordinates of the focus are $(-17/4, 5/2)$

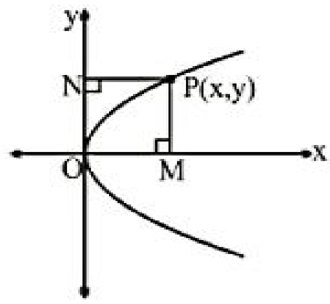
The equation of the directrix is $X = a$ *i.e.* $X = \frac{3}{4}$.

i.e. the equation of the directrix is $x = -\frac{11}{4}$

The length of the latusrectum of the given parabola is $4a = 3$.

Equation of parabola with respect to two perpendicular lines :

Let P(x, y) is any point on the parabola then equation of parabola $y^2 = 4ax$ is consider as

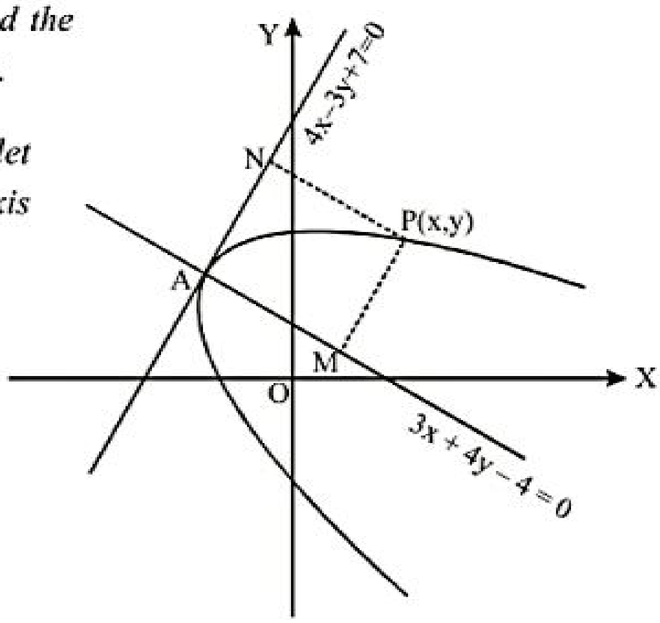


$$(PM)^2 = 4a(PN)$$

i.e. (The distance of P from its axis)^2 = (latus-rectum) × (The distance of P from the tangen at its vertex) where P is any point on the parabola.

Illustration :

Find the equation of the parabola whose latus rectum is 4 units, axis is the line $3x + 4y - 4 = 0$ and the tangent at the vertex is the line $4x - 3y + 7 = 0$.



Sol. *Let P (x, y) be any point on the parabola and let PM and PN are perpendiculars from P on the axis and tangent at the vertex respectively then*

$$(PM)^2 = (\text{latus re-ctum})\ (PN)$$

$$\Rightarrow \left(\frac{3x+4y-4}{\sqrt{3^2+4^2}}\right)^2 = 4\left(\frac{4x-3y+7}{\sqrt{4^2+(-3)^2}}\right)$$

$$\Rightarrow (3x + 4y - 4)^2 = 20\,(4x - 3y + 7)$$

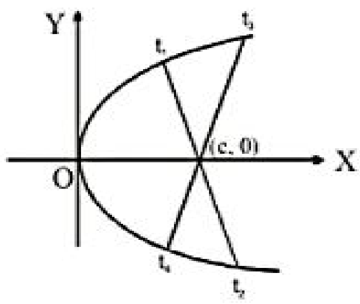
which is required parabola.

CHORD :

Line joining any two points on the parabola is called its chords.

Let the points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$, lie on the parabola then equation of chord is

$$(y - 2at_1) = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2}\,(x - at_1^2)$$



$$\Rightarrow \quad y - 2at_1 = \frac{2}{t_1 + t_2}\,(x - at_1^2)$$

$$\Rightarrow \quad (t_1 + t_2)\ y = 2x + 2at_1 t_2$$

If this chord meet the x-axis at point (c, 0) then from above equation

$$c + at_1t_2 = 0 \ \text{ i.e. } t_1t_2 = -c/a.$$

Note:

(i) If the chord joining t_1, t_2 & t_3, t_4 pass through a point (c, 0) on the axis, then $t_1t_2 = t_3t_4 = -c/a$.

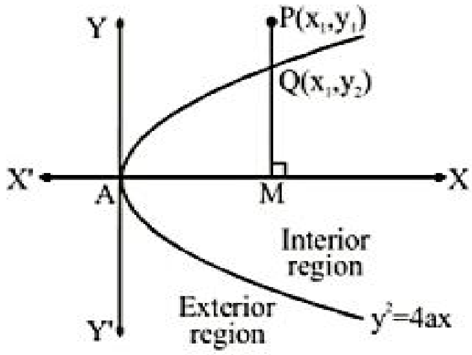
(ii) If PQ is a focal chord then $t_1t_2 = -1$ or $t_2 = -\frac{1}{t_1}$. which is required relation.

Hence if one extremity of a focal chord is $(at^2, 2at)$ then the other extremity will be $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$.

Position of a point with respect to a parabola $y^2 = 4ax$:

Let $P(x_1, y_1)$ be a point. From P draw $PM \perp AX$ (on the axis of parabola) meeting the parabola $y^2 = 4ax$ at $Q(x_1, y_2)$ where $Q(x_1, y_2)$ lie on the parabola therefore

$$y_2^2 = 4ax_1 \qquad \ldots(1)$$



Now, P will be outside, on or inside the parabola $y^2 = 4ax$ acoording as

$PM >, =, \text{or} < QM$

$\Rightarrow \quad (PM)^2 >, =, \text{or} < (QM)^2$

$\Rightarrow \quad y_1^2 >, =, \text{or} < y_2^2$

$\Rightarrow \quad y_1^2 >, =, \text{or} < 4ax_1 \qquad$ (from(1))

Hence $y_1^2 - 4ax_1 >, =, \text{or} < 0$

Hence in short, equation of parabola $S(x, y) = y^2 - 4ax$.

(i) If $S(x_1, y_1) > 0$ then $P(x_1, y_1)$ lie outside the parabola.

(ii) If $S(x_1, y_1) < 0$ then $P(x_1, y_1)$ lie inside the parabola.

(iii) If $S(x_1, y_1) = 0$ then $P(x_1, y_1)$ lie on the parabola.

This result holds true for circle, parabola and ellipse.

Illustration :

Show that the point (2, 3) lies outside the parabola $y^2 = 2x$.

Sol. *Let* $S(x, y) \equiv y^2 - 2x$

$\therefore \quad S(2, 3) = 9 - 2.2 = 5 =$ *positive*

$\Rightarrow$ *(2, 3) lie outside the parabola*

Illustration :

Find the position of the point (– 2, 2) with respect to the parabola $y^2 - 4y + 9x + 11 = 0$.

Sol. Let $S(x, y) = y^2 - 4y + 9x + 11$

$\therefore$ $S(-2, 2) = -11$ = negative

$\Rightarrow$ $(-2, 2)$ lie inside the parabola

Illustration :

If the point $(at^2, 2at)$ *be the extremity of a focal chord of parabola* $y^2 = 4ax$ *then show that the length of the focal chord is* $a\left(t + \frac{1}{t}\right)^2$.

Sol. Since one extremity of focal chord is $P(at^2, 2at)$ then the other extremity will be

$$Q\left(\frac{a}{t^2}, -\frac{2a}{t}\right) \qquad \text{(Replacing } t \text{ by } -1/t)$$

$\therefore$ Length of focal chord = PQ

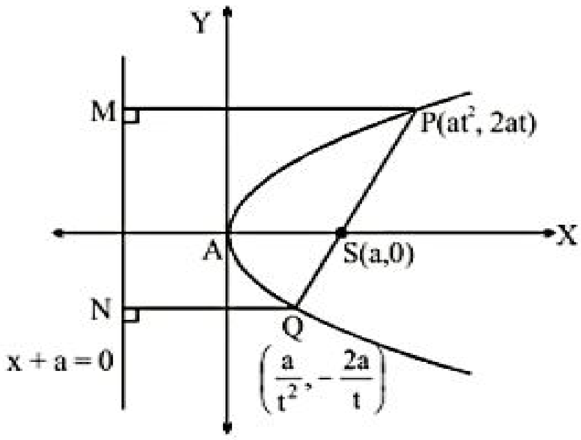
$= SP + SQ \qquad (\because SP = PM \text{ and } SQ = QN)$

$= PM + QN$

$= at^2 + a + \dfrac{a}{t^2} + a$

$= \left(t^2 + \dfrac{1}{t^2} + 2\right)a$

$= a\left(t + \dfrac{1}{t}\right)^2.$



Note :

(i) The length of focal chord having parameters t_1 and t_2 for its end points is $a(t_2 - t_1)^2$.

(ii) $\because$ $\left|t + \dfrac{1}{t}\right| \geq 2$ for all $t \neq 0$ $\qquad (\because AM \geq GM)$

$\therefore$ $a\left(t + \dfrac{1}{t}\right)^2 \geq 4a$

$\Rightarrow$ Length of focal chord $\geq$ latus rectum

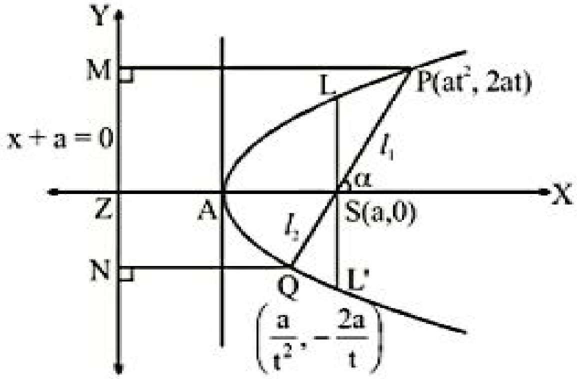
i.e., The length of smallest focal chord of the parabola is 4a, which is the latus rectum of a parabola.

Illustration :

Prove that the semi latus rectum of the parabola $y^2 = 4ax$ is the harmonic mean between the segments of any focal chord of the parabola.

Sol. Let parabola be $y^2 = 4ax$

If PQ be the focal chord then



$$P = (at^2, 2at) \text{ and } Q = \left(\frac{a}{t^2}, \frac{-2a}{t}\right)$$

If segment of focal chord are l_1 and l_2

then $l_1 = SP = PM = a + at^2 = a(1 + t^2)$

and $l_2 = SQ = QN = a + \dfrac{a}{t^2} = \dfrac{a(1+t^2)}{t^2}$

$\therefore$ Harmonic mean of l_1 and l_2

$$= \frac{2l_1 l_2}{l_1 + l_2} = \frac{1}{\dfrac{1}{l_2} + \dfrac{1}{l_1}} = \frac{2}{\dfrac{t^2}{a(1+t^2)} + \dfrac{1}{a(1+t^2)}} = \frac{2}{1/a} = 2a = \text{semi latus rectum.}$$

Note : *If l_1 and l_2 are the length of segments of a focal chord of a parabola, then its latus rectum is $\dfrac{4l_1 l_2}{l_1 + l_2}$.*

Illustration :

Show that the focal chord of parabola $y^2 = 4ax$ makes an angle α with the x-axis, then its length is equal to $4a\ \text{cosec}^2\alpha$.

Sol. Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be the end points of a focal chord PQ which makes an angle α with the axis of the parabola. Then

$PQ = a(t_2 - t_1)^2$

$= a((t_2 + t_1)^2 - 4t_1t_2)$

$= a((t_2 + t_1)^2 + 4)$ $\quad (\because t_1t_2 = -1)$ ....(1)

$\therefore$ $\tan\alpha = \text{slope of } PQ = \dfrac{2at_2 - 2at_1}{at_2^2 - at_1^2} = \dfrac{2}{t_2 + t_1}$

$t_2 + t_1 = 2\cot\alpha$ ....(2)

Substituting the value of $t_2 + t_1$ from (2) in (1) then

$PQ = a(4\cot^2\alpha + 4) = 4a\ \text{cosec}^2\ \alpha.$

Practice Problem

Single correct question

Q.1 The vertex of parabola $y^2 + 6x - 2y + 13 = 0$ is

(A) $(-2, 1)$ (B) $(2, -1)$ (C) $(1, 1)$ (D) $(1, -1)$

Q.2 The value of p such that the vertex of $y = x^2 + 2px + 13$ is 4 units above the x-axis is

(A) ± 2 (B) 4 (C) ± 3 (D) 5

Q.3 The length of the latus rectum of the parabola whose focus is $(3, 3)$ and directrix is $3x - 4y - 2 = 0$, is

(A) 1 (B) 2 (C) 4 (D) 8

Q.4 If the vertex and focus of a parabola are $(3, 3)$ and $(-3, 3)$ respectively, then its equation is

(A) $x^2 - 6x + 24y - 63 = 0$ (B) $x^2 - 6x + 24y + 81 = 0$

(C) $y^2 - 6x + 24x - 63 = 0$ (D) $y^2 - 6y - 24x + 81 = 0$

Q.5 The parabola having its focus at $(3, 2)$ and directrix along the y-axis has its vertex at

(A) $\left(\frac{3}{2}, 1\right)$ (B) $\left(\frac{3}{2}, 2\right)$ (C) $\left(\frac{3}{2}, \frac{1}{2}\right)$ (D) $\left(\frac{3}{2}, \frac{-1}{2}\right)$

Q.6 The directrix of the parabola $x^2 - 4x - 8y + 12 = 0$ is

(A) $y = 0$ (B) $x = 1$ (C) $y = -1$ (D) $x = -1$

Q.7 Which one of the following equations represented parametrically, represents equation to a parabolic profile?

(A) $x = 3\cos t$; $y = 4\sin t$ (B) $x^2 - 2 = -2\cos t$; $y = 4\cos^2\frac{t}{2}$

(C) $\sqrt{x} = \tan t$; $\sqrt{y} = \sec t$ (D) $x = \sqrt{1 - \sin t}$; $y = \sin\frac{t}{2} + \cos\frac{t}{2}$

Q.8 The locus of the point of trisection of all the double ordinates of the parabola $y^2 = lx$ is a parabola whose latus rectum is

(A) $\frac{l}{9}$ (B) $\frac{2l}{9}$ (C) $\frac{4l}{9}$ (D) $\frac{l}{36}$

Q.9 The straight line $y = m(x - a)$ will meet the parabola $y^2 = 4ax$ in two distinct real points if

(A) $m \in R$ (B) $m \in [-1, 1]$

(C) $m \in (-\infty, 1] \cup [1, \infty)R$ (D) $m \in R - \{0\}$

Q.10 The length of the intercept on y – axis cut off by the parabola, $y^2 - 5y = 3x - 6$ is
(A) 1 (B) 2 (C) 3 (D) 5

Q.11 If the line $x - 1 = 0$ is the directrix of the parabola $y^2 - kx + 8 = 0$ then one of the values of 'k' is
(A) 1/8 (B) 8 (C) 4 (D) 1/4

Q.12 Length of the latus rectum of the parabola $25[(x-2)^2 + (y-3)^2] = (3x - 4y + 7)^2$ is
(A) 4 (B) 2 (C) 1/5 (D) 2/5

Q.13 Maximum number of common chords of a parabola and a circle can be equal to
(A) 2 (B) 4 (C) 6 (D) 8

Multiple correct type question

Q.14 The locus of the mid point of the focal radii of a variable point moving on the parabola, $y^2 = 8x$ is a parabola whose
(A) Latus rectum is half the latus rectum of the original parabola
(B) Vertex is (1, 0)
(C) Directrix is y-axis
(D) Focus has the co-ordinates (2, 0)

Q.15 If from the vertex of a parabola $y^2 = 4x$ a pair of chords be drawn at right angles to one another and with these chords as adjacent sides a rectangle be made, then the locus of the further end of the rectangle is
(A) an equal parabola
(B) a parabola with focus at (9, 0)
(C) a parabola with directrix as $x - 7 = 0$
(D) a parabola having tangent at its vertex $x = 8$

Integer type question

Q.16 Find the vertex, focus, latus rectum, axis and the directrix of the parabola $x^2 + 8x + 12y + 4 = 0$.

Q.17 A parabola $y = ax^2 + bx + c$ crosses the x – axis at $(\alpha, 0)$ $(\beta, 0)$ both to the right of the origin. A circle also passes through these two points. Then find length of a tangent from the origin to the circle.

Answer key

Q.1	A	Q.2	C	Q.3	B	Q.4	C	Q.5	B
Q.6	C	Q.7	B	Q.8	A	Q.9	D	Q.10	A
Q.11	C	Q.12	D	Q.13	C				
Q.14	A, B, C, D	Q.15	A, B, C, D						

Q.16 V(–4, 1) S(–4, –2), axis : $x + 4 = 0$, directrix : $y - 4 = 0$, LR = 12 Q.17 $\sqrt{\frac{c}{a}}$

INTERACTION BETWEEN THE LINE AND PARABOLA :

Let the parabola be $y^2 = 4ax$ ... (i)

and the given line be $y = mx + c$ ... (ii)

then line may cut, touch or does not meet parabola.

The points of intersection of the line (1) and the parabola (2) will be obtained by solving the two equations simultaneously. By solving equation (i) and (ii), we get

$$my^2 - 4ay + 4ac = 0$$

this equation has two roots and its nature will be decided by the discriminent $D = 16a(a - cm)$

Now, if $D > 0$ i.e., $c < \frac{a}{m}$, then line intersect the parabola at two distinct points.

If $D = 0$ i.e., $c = \frac{a}{m}$, then line touches the parabola. **(It is condition of tangency)**

If $D < 0$ i.e., $c > \frac{a}{m}$, then line neither touch nor intersect the parabola.

EQUATION OF TANGENT :

1. Point Form :

Equation of parabola is $y^2 = 4ax$ ....(1)

Let $P \equiv (x_1, y_1)$ and $Q = (x_2, y_2)$ be any two points on parabola (1), then

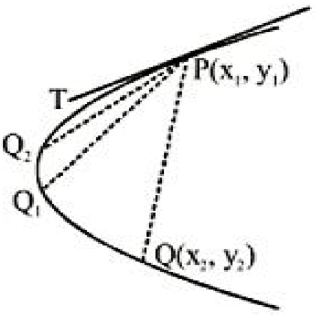
$$y_1^2 = 4ax_1 \qquad ....(2)$$

and $$y_2^2 = 4ax_2 \qquad ....(3)$$

Subtracting (2) from (3) then

$$y_2^2 - y_1^2 = 4a(x_2 - x_1)$$

or $$\frac{y_2 - y_1}{x_2 - x_1} = \frac{4a}{y_2 + y_1} \qquad ....(4)$$



Equation of PQ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \qquad ....(5)$$

From (4) and (5), then

$$y - y_1 = \frac{4a}{y_2 + y_1}(x - x_1) \qquad ......(6)$$

Now for tangent at P, Q → P, i.e., $x_2 \to x_1$ and $y_2 \to y_1$ then equation (6) becomes

$$y - y_1 = \frac{4a}{2y_1}(x - x_1)$$

or $\quad yy_1 - y_1^2 = 2ax - 2ax_1$

or $\quad yy_1 = 2ax + y_1^2 - 2ax_1$

or $\quad yy_1 = 2ax + 4ax_1 - 2ax_1$ [From (2)]

or $\quad yy_1 = 2ax + 2ax_1$

which is the required equation of tangent at (x_1, y_1).

The equation of tangent at (x_1, y_1) can also be obtained by replacing x^2 by xx_1, y^2 by yy_1, x by $\frac{x + x_1}{2}$, y by $\frac{y + y_1}{2}$ and xy by $\frac{xy_1 + yx_1}{2}$ and without changing the constant (if any) in the equation of curve.

This method (standard substitution) is apply to all conic when point lie on the conic.

Equation of tangent of standard parabola :

Equation of Parabolas	Tangent at (x_1, y_1)
$y^2 = 4ax$	$yy_1 = 2a(x + x_1)$
$y^2 = -4ax$	$yy_1 = -2a(x + x_1)$
$x^2 = 4ay$	$xx_1 = 2a(y + y_1)$
$x^2 = -4ay$	$xx_1 = -2a(y + y_1)$

2. Slope Form :

The equation of tangent to the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$yy_1 = 2a(x + x_1) \qquad \text{........(1)}$$

Since m is the slope of the tangent then

$$m = \frac{2a}{y_1} \quad \text{or} \quad y_1 = \frac{2a}{m}$$

Since (x_1, y_1) lies on $y^2 = 4ax$ therefore

$$y_1^2 = 4ax_1 \quad \text{or} \quad \frac{4a^2}{m^2} = 4ax_1 \quad \Rightarrow \quad x_1 = \frac{a}{m^2}.$$

Substituting the values of x_1 and y_1 in (1), we get

$$y = mx + \frac{a}{m} \qquad \text{........(2)}$$

Thus, $y = mx + \frac{a}{m}$ is a tangent to the parabola $y^2 = 4ax$ for all values of m, where m is the slope of the tangent and the co-ordinates of the point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

Thus $y = mx + c$ is the tangent of $y^2 = 4ax$ for all values of m if only if $c = \frac{a}{m}$

and $(y-k) = m(x-h) + \frac{a}{m}$ is tangent to the parabola $(y-k)^2 = 4a(x-h)$.

The equation of tangent, condition of tangency and point of contact in terms of slope (m) of standard parabolas are shown below in the table.

Equation of parabolas	Point of contact in terms of slope (m)	Equation of tangent in terms of slope (m)	Condition of tangnecy
$y^2 = 4ax$	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$	$y = mx + \frac{a}{m}$	$c = \frac{a}{m}$
$x^2 = 4ay$	$(2am, am^2)$	$y = mx - am^2$	$c = -am^2$

Alternative method :

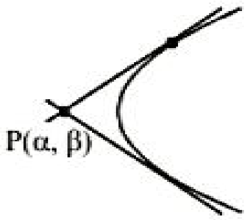
Let the parabola be $y^2 = 4ax$ ....(1)
and the given line by $y = mx + c$ ....(2)
Putting the value of y from (2) in (1), we get

$$m^2x^2 + 2x(mc - 2a) + c^2 = 0 \quad ....(3)$$

The line $y = mx + c$ is a tangent to parabola $y^2 = 4ax$ if the roots of equation (3) are equal. The condition for this is $4(mc-2a)^2 - 4m^2c^2 = 0$ (Discriminant of the quadratic equation = 0)

or $-4mca + 4a^2 = 0$ or $c = \frac{a}{m}$, which is the required condition of tangency.

Two tangents can be drawn from a point P (α, β) to a parabola if P lies outside the parabola :



Let the parabola be $y^2 = 4ax$ ....(1)
Let P (α, β) be the given point

The equation of a tangent to parabola (1) is $y = mx + \frac{a}{m}$ ....(2)

If line (2) passes through P (α, β), then $\beta = m\alpha + \frac{a}{m}$ or $m^2\alpha - \beta m + a = 0$ ....(3)

There will be two tangents to parabola (1) from P (α, β) if roots of equation (3) are real and distinct i.e., D > 0 i.e. if $\beta^2 - 4\alpha a > 0 \Rightarrow$ P(α, β) lies outside parabola (1).
We can also find the angle between two tangents from point P(α, β) using the formula

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

3. Parametric Form :

We have to find the equation of tangent to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ or 't'

Since the equation of tangent of the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$yy_1 = 2a(x + x_1) \qquad ....(1)$$

replacing x_1 by at^2 and y_1 by 2at, then (1) becomes

$$y(2at) = 2a(x + at^2)$$

$$ty = x + at^2$$

Point of intersection of tangents at any two points on the parabola :

Let the given parabola be $y^2 = 4ax$ and

two points on the parbaola are

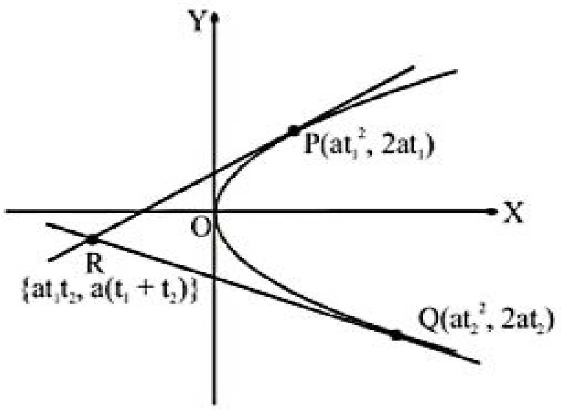
$$P = (at_1^2, 2at_1) \text{ and } Q = (at_2^2, 2\,at_2)$$

Equation of tangents at

$$P\,(at_1^2, 2at_1) \text{ and } Q(at_2^2, 2at_2)$$

are $\quad t_1y = x + at_1^2 \qquad ....(1)$

and $\quad t_2y = x + at_2^2 \qquad ....(2)$



Solving these equations we get

$$x = at_1t_2,\ y = a(t_1 + t_2)$$

Thus, the co-ordinates of the point of intersection of tangents at

$$P(at_1^2, 2at_1) \text{ and } Q(at_2^2, 2at_2) \text{ are } R(at_1t_2, a(t_1 + t_2)).$$

Note :

(i) The Arithmetic mean of the y-co-ordinates of P and Q $\left(\text{i.e., } \dfrac{2at_1 + 2at_2}{2} = a(t_1 + t_2)\right)$ is the y - co-ordinate of the point of intersection of tangents at P and Q on the parabola.

(ii) The Geometric mean of the x-co-ordinates of P and Q $\left(\text{i.e., } \sqrt{at_1^2 \times at_2^2} = at_1t_2\right)$ is the x co-ordinate of the point of intersection of tangents at P and Q on the parabola.

Illustration :

Show that the locus of the points of intersection of the tangents to a parabola at the extremity of focal chord are perpendicular and always meet at the directrix of the parabola.

Sol. Let the points $(P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$

on the parabola $y^2 = 4ax$ and tangents at P and Q are

$t_1y = x + at_1^2$ ....(1)

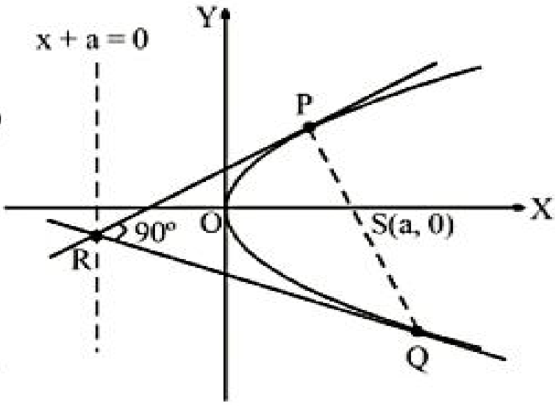
and $t_2y = x + at_2^2$ ....(2)

$\because$ point of intersection of these tangents is $(at_1t_2, a(t_1 + t_2))$

Let this point is (h, k)

then $h = at_1t_2$ ....(3)

and $k = a(t_1 + t_2)$ ....(4)



Slope of tangents (1) and (2) are $\frac{1}{t_1}$ and $\frac{1}{t_2}$ respectively.

$\therefore$ Product of slope $= \frac{1}{t_1} \times \frac{1}{t_2} = -1$ (Since PQ is focal chord)

$\Rightarrow h = -a$ from equation (3)

$\therefore$ Locus is $x + a = 0$ i.e. Directrix of parabola

Directrix is also called the Directer Circle of the parabola.

Illustration :

If the line $2x - 3y = k$ touches the parabola $y^2 = 6x$, then find the value of k.

Sol. Given $x = \frac{3y+k}{2}$ ....(1)

and $y^2 = 6x$ ....(2)

Solve (1) and (2), we get

$$\Rightarrow y^2 = 6\left(\frac{3y+k}{2}\right) \Rightarrow y^2 = 3(3y + k)$$

$\Rightarrow y^2 - 9y - 3k = 0$ ...(3)

If line (1) touches parabola (2) then roots of quadratic equation (3) is equal i.e. $D = 0$

$$\therefore (-9)^2 = 4 \times 1 \times (-3k) \Rightarrow k = -\frac{27}{4}$$

Alternative method :

Line $y = \frac{2}{3}x - \frac{k}{3}$ touch is $y^2 = 6x$ then use the condition of tangency

$$c = \frac{a}{m} \Rightarrow -\frac{k}{3} = \frac{9}{4} \Rightarrow k = -\frac{27}{4}$$

Illustration :

Find the equation to the tangents to the parabola $y^2 = 9x$ which goes through the point (4, 10).

Sol. *Equation of tangent to parabola $y^2 = 9x$ is*

$$y = mx + \frac{9}{4m}$$

Since it passes through (4, 10)

$$\therefore \quad 10 = 4m + \frac{9}{4m} \Rightarrow 16\,m^2 - 40\,m + 9 = 0$$

$$m = \frac{1}{4}, \frac{9}{4}$$

$\therefore$ *equation of tangent's are* $y = \frac{x}{4} + 9$ & $y = \frac{9}{4}x + 1.$ ***Ans.***

Illustration :

If the tangents at P and Q on a parabola (whose focus is S) meet in the point T, then prove that SP, ST and SQ are in geometric progression.

Sol. *Let P $(at_1^2, 2at_1)$ and Q $(at_2^2, 2at_2)$ be any two points on the parabola $y^2 = 4ax$, then point of intersection of tangents at P and Q will be*

$$T = [at_1t_2,\ a(t_1 + t_2)]$$

Now $SP = a(t_1^2 + 1)$

$$SQ = a(t_2^2 + 1)$$

$ST = a\sqrt{(t_1^2+1)(t_2^2+1)}$ *(use distance between two points formula)*

$\therefore \quad ST^2 = SP.SQ$

$\therefore$ SP, ST and SQ are in G.P. ***Ans.***

Illustration :

If two tangents are drawn from the pont (h, k) to the parabola $y^2 = 4x$ such that the slope of one tangent is double of the other, then prove that $9h = 2k^2$.

Sol. *Tangent to parabola $y^2 = 4x$ is*

$y = mx + \frac{1}{m}$ *and it passes through (h, k), so* $k = mh + \frac{1}{m}$

i.e. $hm^2 - km + 1 = 0$

Its roots are m_1 and $2m_1$

$$\therefore \quad m_1 + 2m_1 = \frac{k}{h} \quad \Rightarrow \quad 3m_1 = \frac{k}{h} \qquad ....(i)$$

$$m_1 \cdot 2m_1 = \frac{1}{h} \quad \Rightarrow \quad 2m_1^2 = \frac{1}{h} \qquad ....(ii)$$

from (i) and (ii) eliminate m, we get

$9h = 2k^2$ Ans.

Illustration :

A tangent to the parabola $y^2 = 8x$ makes an angle of 45° with the straight line $y = 3x + 5$. Then find the equation of tangent and point of contact also.

Sol. Here $a = 2$, equation of tangent at $(2t^2, 4t)$ is $yt = x + 2t^2$

slope of tangent $\quad m = \dfrac{1}{t}$

$$\therefore \quad \tan 45° = \pm \frac{\frac{1}{t} - 3}{1 + \frac{1}{t}\cdot 3} \Rightarrow \quad t = -\frac{1}{2} \text{ or } 2.$$

when $t = -\dfrac{1}{2}$, then equation of tangent is $\left(-\dfrac{1}{2}\right) y = x + 2\cdot\dfrac{1}{4} \Rightarrow 2x + y + 1 = 0$ and point of contact is $\left(\dfrac{1}{2}, -2\right)$. When $t = 2$, the tangent is $2y = x + 8$ and point of contact is $\left(2\cdot(2)^2, 4\cdot 2\right)$ i.e. $(8, 8)$.

Illustration :

Find the common tangent of the parabola $y^2 = 8ax$ and the circle $x^2 + y^2 = 2a^2$

Sol. Any tangent to parabola is $y = mx + \dfrac{2a}{m}$

Solving with the circle $x^2 + (mx + \dfrac{2a}{m})^2 = 2a^2$

$$x^2 (1 + m^2) + 4ax + \left(\frac{4a^2}{m^2} - 2a^2\right) = 0$$

The condition of tangency is $B^2 - 4AC = 0$ gives $m^4 + m^2 - 2 = 0 \Rightarrow m = \pm 1$
Hence equation of tangent is $y = x + 2a$ and $y = -x - 2a$.

Illustration :

Find the equations to the common tangents of the parabolas $y^2 = 4ax$ and $x^2 = 4by$.

Sol. Equation of tangent to $y^2 = 4ax$ is $y = mx + \dfrac{a}{m}$ ....(i)
If this line also touches parabola $x^2 = 4by$ ......(ii)
then solve (i) and (ii), we get

$$x^2 - 4bmx - \frac{4ab}{m} = 0$$

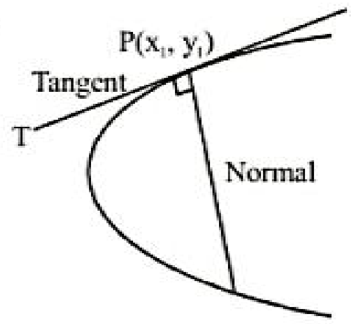
Now condition of tangency is $D = 0$.

this gives $\quad m^3 = -\dfrac{a}{b} \quad$ or $\quad m = -\left(\dfrac{a}{b}\right)^{1/3}$

$$\therefore \quad \text{equation of common tangent is } y = \left(-\frac{a}{b}\right)^{1/3} x + a\left(-\frac{b}{a}\right)^{1/3}. \qquad \textbf{Ans.}$$

EQUATION OF NORMALS :

1. Point form :



Since the equation of the tangent to the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$yy_1 = 2a(x + x_1) \quad \text{..........(1)}$$

The slope of the tangent at $(x_1, y_1) = 2a/y_1$.

$\therefore$ Slope of the normal at $(x_1, y_1) = -y_1/2a$

Hence the equation of normal at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

2. Slope form :

The equation of normal to the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1) \quad \text{.......(1)}$$

Since m is the slope of the normal

then $\quad m = -\dfrac{y_1}{2a} \quad$ or $\quad y_1 = -2am$

Since (x_1, y_1) lies on $y^2 = 4ax$ therefore

$y_1^2 = 4ax_1 \quad$ or $\quad 4a^2m^2 = 4ax_1$

$\therefore \quad x_1 = am^2$

Substituting the values of x_1 and y_1 in (1) we get

$$y + 2am = m(x - am^2) \quad \text{........(2)}$$

Thus, $y = mx - 2am - am^3$ is a normal to the parabola $y^2 = 4ax$ where m is the slope of the normal. The co-ordinates of the point of contact are $(am^2, -2am)$.

Hence $\quad y = mx + c$ will be normal to parabola. If and only if $c = -2am - am^3$

Note :

Equation of parabolas	Point of contact in terms of slope (m)	Equation of normals in terms of slope (m)	Condition of normality
$y^2 = 4ax$	$(am^2, -2am)$	$y = mx - 2am - am^3$	$c = -2am - am^3$
$x^2 = 4ay$	$\left(-\dfrac{2a}{m}, \dfrac{a}{m^2}\right)$	$y = mx + 2a + \dfrac{a}{m^2}$	$c = 2a + \dfrac{a}{m^2}$

3. Parametric form :

Equation of normal of the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1) \quad \text{.........(1)}$$

Replacing x_1 by at^2 and y_1 by $2at$ then (1) becomes

$y - 2at = -t(x - at)^2$

or $\quad y = -tx + 2at + at^3$

Three supplementry Results:

(a) Point of intersection of normals at any two points on the parabola :

Let the points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ lie on the parabola $y^2 = 4ax$

The equations of the normals at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ are

$$y = -t_1x + 2at_1 + at_1^3 \qquad ....(1)$$

and $$y = -t_2x + 2at_2 + at_2^3 \qquad ....(2)$$

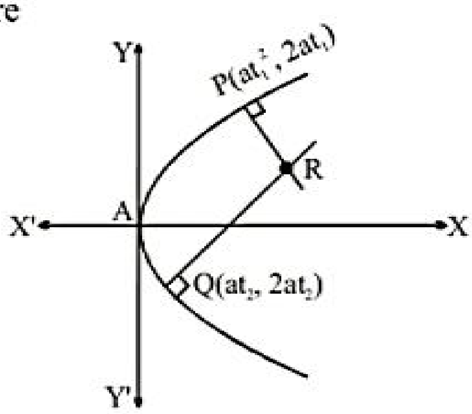
Hence point of intersection of above normals will be obtained by solving (1) and (2), we get

$$x = 2a + a(t_1^2 + t_2^2 + t_1t_2)$$

$$y = -at_1t_2(t_1 + t_2)$$

If R is the point of intersection then it is

$$R = [2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2)]$$



(b) Relation between t_1 and t_2 if normal at t_1 meets the parabola again at t_2 :

Let the parabola be $y^2 = 4ax$, equation of normal at $P(at_1^2, 2at_1)$ is

$$y = -t_1x + 2at_1 + at_1^3 \qquad ....(1)$$

Since normal meet the parabola again at $Q(at_2^2, 2at_2)$

$\therefore \quad 2at_2 = -at_1t_2^2 + 2at_1 + at_1^3$

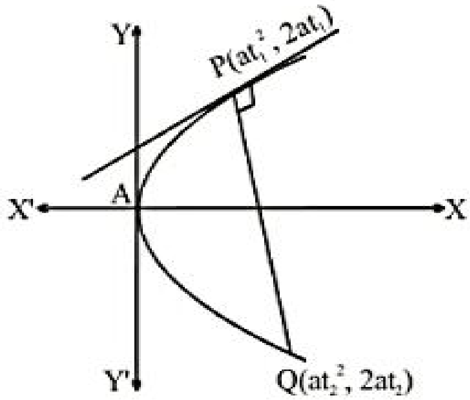
$\Rightarrow \quad 2a(t_2 - t_1) + at_1(t_2^2 - t_1^2) = 0$

$\Rightarrow \quad a(t_2 - t_1)[2 + t_1(t_2 + t_1)] = 0$

$\because \quad a(t_2 - t_1) \neq 0 \qquad (\because t_1$ and t_2 are different)

$\therefore \quad 2 + t_1(t_2 + t_1) = 0$

$\therefore \quad t_2 = -t_1 - \dfrac{2}{t_1}$



(c) If normal to the parabola $y^2 = 4ax$ drawn at any point $(at^2, 2at)$ meet the parabola at t_3 then

$$t_3 = -t - \frac{2}{t}$$

$\Rightarrow \quad t^2 + t\,t_3 + 2 = 0 \qquad ...(i)$

It has two roots t_1 & t_2. Hence there are two such point $P(t_1)$ & $Q(t_2)$ on the parabola from where normals are drawn and which meet parabola at $R(t_3)$

$\Rightarrow \quad t_1 + t_2 = -t_3 \quad \& \quad t_1t_2 = 2$

Thus the line joining $P(t_1)$ & $Q(t_2)$ meet x-axis at $(-2a, 0)$

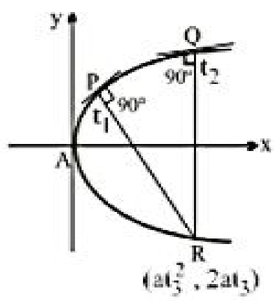
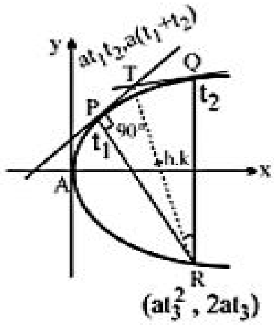


Illustration :

TP and TQ are tangents to the parabola $y^2 = 4ax$ and the normals at P and Q meet at a point R on the curve, prove that the centre of the circle circumscribing the triangle TPQ lies on the parabola $2y^2 = a(x - a)$.



Sol. *Normal at two points $P(t_1)$ & $P(t_2)$ meet parabola at $R(t_3)$.*

$\Rightarrow \quad t_1 + t_2 = -t_3 \quad \& \quad t_1t_2 = 2$

$\therefore \quad T(at_1t_2, a(t_1 + t_2)) \equiv T(2a, -at_3)$

Let centre of circle is (h, k) then it is mid point of T and R.

$\therefore \quad 2h = 2a + at_3^2 \qquad ...(i)$

$\quad 2k = -at_3 + 2at_3 \qquad ...(ii)$

i.e. $\quad 2k = at_3 \quad \Rightarrow \quad t_3 = \dfrac{2k}{a}$

Put value of t_3 in equation (i) we get the required locus. Now replace h by x and k by y

We get $2y^2 = a(x - a)$. **Ans.**

Illustration :

Find the equation of a normal at the parabola $y^2 = 4x$ which passes through (3,0).

Sol. *Equation of Normal $y = mx - 2am - am^3$*

Here a = 1 and it passes through (3,0)

$0 = 3m - 2m - m^3$

$\Rightarrow \quad m^3 - m = 0$

$\Rightarrow \quad m = 0, \pm 1$

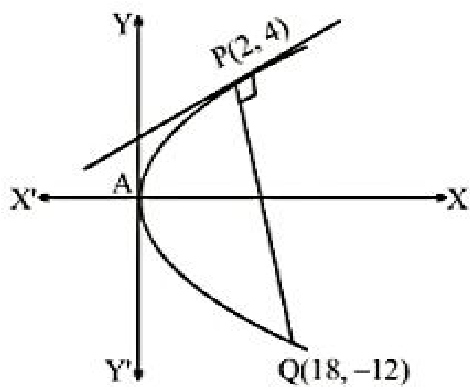
for $\quad m = 0 \quad \Rightarrow \quad y = 0$

$\quad m = 1 \quad \Rightarrow \quad y = x - 3$

$\quad m = -1 \quad \Rightarrow \quad y = -x + 3$ **Ans.**

Illustration :

Show that normal to the parabola $y^2 = 8x$ at the point (2, 4) meets it again at (18, – 12). Find also the length of the normal chord.



Sol. *Comparing the given parabola with $y^2 = 4ax$*

$\therefore \quad 4a = 8$

$\therefore \quad a = 2$

$P(at_1^2, 2at_1) \equiv (2, 4)$

$\Rightarrow \quad t_1 = 1$

$\Rightarrow \quad$ parameter at $Q(t_2) = -t_1 - \dfrac{2}{t_1} = -3$

$\therefore \quad Q(2(-3)^2, 2 \times 2(-3))$

i.e. $\quad Q(18, -12)$

$\therefore \quad$ *Length of normal chord PQ = Distance between points P and Q.*

$= PQ = \sqrt{(18-2)^2 + (-12-4)^2} = 16\sqrt{2}$.

Illustration :

If the normal to a parabola $y^2 = 4ax$, makes an angle ϕ with the axis show that it will cut the curve again at an angle $\tan^{-1}\left(\frac{1}{2}\tan\phi\right)$.

Sol. Let the normal at $P(at_1^2, 2at_1)$ be $y = -t_1x + 2at_1 + at_1^3$

$\therefore \quad \tan\phi = -t_1$ = slope of the normal ....(1)

It meet the curve again Q say $(at_2^2, 2at_2)$

$$\therefore \quad t_2 = -t_1 - \frac{2}{t_1} \qquad ....(2)$$

Now angle between the normal and parabola = Angle between the normal and tangent at Q (i.e., $t_2y = x + at_2^2$)

If θ be the angle, then

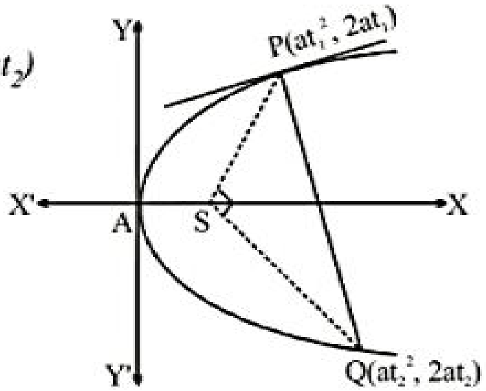
$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2} = \frac{-t_1 - \frac{1}{t_2}}{1 + (-t_1)\left(\frac{1}{t_2}\right)} = -\frac{t_1t_2 + 1}{t_2 - t_1} = \frac{t_1\left(-t_1 - \frac{2}{t_2}\right) + 1}{-t_1 - \frac{2}{t_2} - t_1} \quad \{\text{from equation (2)}\}$$

$$= \frac{-t_1^2 - 1}{-2\left(\frac{1 + t_1^2}{t_1}\right)} = \frac{t_2}{2} = \frac{\tan\phi}{2} \qquad \{\text{from equation (1)}\}$$

$$\theta = \tan^{-1}\left(\frac{1}{2}\tan\phi\right).$$

Illustration :

Prove that the normal chord to a parabola $y^2 = 4ax$ at the point whose ordinate is equal to abscissa subtends a right angle at the focus.



Sol. Let the normal at $P(at_1^2, 2at_1)$ meet the curve at $Q(at_2^2, 2at_2)$

$\therefore \quad$ PQ is a normal chord.

and $\quad t_2 = -t_1 - \dfrac{2}{t_1}$ ....(1)

By given condition $\quad 2at_1 = at_1^2$

$\therefore \quad t_1 = 2$ from equation (1), $t_2 = -3$

then $\quad P(4a, 4a)$ and $Q(9a, -6a)$

but focus $S(a, 0)$

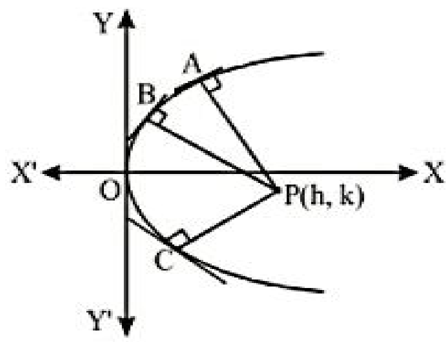
$$\therefore \quad \text{Slope of } SP = \frac{4a - 0}{4a - a} = \frac{4a}{3a} = \frac{4}{3} \text{ and slope of } SQ = \frac{-6a - 0}{9a - a} = \frac{6a}{8a} = -\frac{3}{4}$$

$$\because \quad \text{slope of } SP \times \text{slope of } SQ = \frac{4}{3} \times -\frac{3}{4} = -1$$

$$\therefore \quad \angle PSQ = \frac{\pi}{2} \quad \text{i.e., PQ subtends a right angle at the focus S.}$$

CO-NORMAL POINTS :

Maximum three normals can be drawn from a point to a parabola and their feet (points where the normal meet the parabola) are called co-normal points.



Let P(h, k) be any given point and $y^2 = 4ax$ be a parabola.

The equation of any normal to $y^2 = 4ax$ is

$$y = mx - 2am - am^3$$

If it passes through (h, k) then

$$k = mh - 2am - am^3$$

$\Rightarrow \quad am^3 + m(2a - h) + k = 0 \qquad$ ......(i)

This is a cubic equation in m, so it has three roots, say m_1, m_2 and m_3.

$\therefore \quad m_1 + m_2 + m_3 = 0, \qquad$ ...(ii)

$$m_1m_2 + m_2m_3 + m_3m_1 = \frac{(2a-h)}{a}, \qquad \text{...(iii)}$$

$$m_1m_2m_3 = -\frac{k}{a} \qquad \text{...(iv)}$$

Hence for any given point P(h, k), (i) has three real or imaginary roots. Correspoinding to each of these three roots, we have each normal passing through P(h, k). Hence we have three normals PA, PB and PC drawn through P to the parabola.

Points A, B, C in which the three normals from P(h, k) meet the parabola are called co-normal points.

Properties of co-normal points :

(1) The algebraic sum of the slopes of three concurrent normals is zero. This follows from equation (ii).

(2) The algebraic sum of ordinates of the feets of three normals drawn to a parabola from a given point is zero.

Let the ordiantes of A, B, C be y_1, y_2, y_3 respectively then

$$y_1 = -2\,am_1,\ y_2 = -2am_2 \text{ and } y_3 = -2am_3$$

$\therefore$ Algebraic sum of these ordinates is

$$\begin{aligned} y_1 + y_2 + y_3 &= -2am_1 - 2am_2 - 2am_3 \\ &= -2a(m_1 + m_2 + m_3) \\ &= -2a \times 0 \qquad \{\text{from equation (ii)}\} \\ &= 0 \end{aligned}$$

(3) If three normals drawn to any parabola $y^2 = 4ax$ from a given point (h, k) is real then $h > 2a$.

When normals are real, then all the three roots of equation (i) are real and in that case

$$m_1^2 + m_2^2 + m_3^2 > 0 \qquad (\text{for any values of } m_1, m_2, m_3)$$

$\Rightarrow \quad (m_1 + m_2 + m_3)^2 - 2 \quad (m_1m_2 + m_2m_3 + m_3m_1) > 0$

$\Rightarrow \quad (0)^2 - \dfrac{2(2a-h)}{a} > 0$

$\Rightarrow \quad h - 2a > 0$

or $\quad h > 2a$

(4) The centroid of the triangle formed by the feet of the three normals lies on the axis of the parabola. If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be vertices of ΔABC, then its centroid is

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right) = \left(\frac{x_1+x_2+x_3}{3}, 0\right)$$

Since $y_1 + y_2 + y_3 = 0$ (from result-2). Hence the centroid lies on the x-axis, which is the axis of the parabola also.

Now $\dfrac{x_1+x_2+x_3}{3} = \dfrac{1}{3}(am_1^2 + am_2^2 + am_3^2) = \dfrac{a}{3}\left(m_1^2 + m_2^2 + m_3^2\right)$

$$= \frac{a}{3}\{(m_1 + m_2 + m_3)^2 - 2(m_1m_2 + m_2m_3 + m_3m_1)\}$$

$$= \frac{a}{3}\left\{(0)^2 - 2\left\{\frac{2a-h}{a}\right\}\right\} = \frac{2h-4a}{3}$$

$\therefore$ Centroid of ΔABC is $\left(\dfrac{2h-4a}{3}, 0\right)$

Illustration :

If from point P(h, k) three normals are drawn to the parabola $(y-1)^2 = 8(x-2)$ then find the condition

Sol. *Here $a = 2$ and abscissa of point from where three normals are drawn must be greater than 2a.*
i.e. $x - 2 > 2a$ *i.e.* $x > 6$ *Hence* $h > 6$ *Ans.*

Illustration :

The ordinate of points P and Q on the parabola $y^2 = 12x$ are in the ratio of 1 : 2. Find the locus of the point of intersection of the normal to the parabola at P and Q.

Sol. *Here $a = 3$,*
Let $P(3t_1^2, 6t_1)$ and $Q(3t_2^2, 6t_2)$ lie on the parabola

According to the question, $\dfrac{6t_1}{6t_2} = \dfrac{1}{2} \Rightarrow t_2 = 2t_1$ ......(i)

Let $P(\alpha, \beta)$ be the point of intersection of normal to parabola at P and Q then

$\alpha = 2a + a(t_1^2 + t_2^2 + t_1t_2) = 6 + 21t_1^2$ ....(ii)

and $\beta = -at_1t_2(t_1 + t_2) = -18t_1^3$ ....(iii)

from (ii) $t_1^6 = \left(\dfrac{\alpha-6}{21}\right)^3$ *and from (iii)*

$$t_1^6 = \frac{\beta^2}{324}$$

equate t_1^6, we get

$343\beta^2 = 12(\alpha-6)^3$

$\therefore$ *locus is* $343y^2 = 12(x-6)^3$. ***Ans.***

Illustration :

Find the locus of points through which three normals to parabola $y^2 = 4ax$ passes and two of them are perperdicular to each other.

Sol. *Let P(h, k) be the point of intersection of three normals to be parabola $y^2 = 4ax$ then it will be*

$y = mx - 2am - am^3$ *and it passes through P(h, k)*

$k = mh - 2am - am^3$

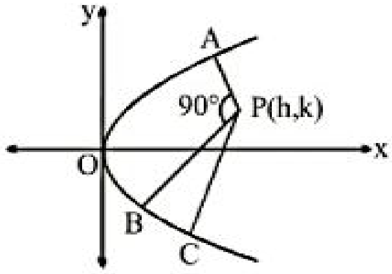
$\therefore \quad am^3 + m(2a - h) + k = 0 \qquad$ ....(i)

It has three roots m_1, m_2 and m_3

$m_1 + m_2 + m_3 = 0 \qquad$ ....(ii)

$$m_1m_2m_3 = \frac{-k}{a} \qquad ....(iii)$$

Given condition, $m_1m_2 = -1 \qquad$ ......(iv) *as given in figure.*



from (iii) and (iv)

$$m_3 = \frac{k}{a}$$

$\because \quad m_3$ *is also a root of equation (1) therefore it will satisfy equation (i)*

$$\therefore \quad a\left(\frac{k}{a}\right)^3 + (2a - h)\left(\frac{k}{a}\right) + k = 0$$

$\Rightarrow \quad k^3 + (2a - h)\, ka + ka^2 = 0$

$\Rightarrow \quad k^2 + 3a^2 - ah = 0$

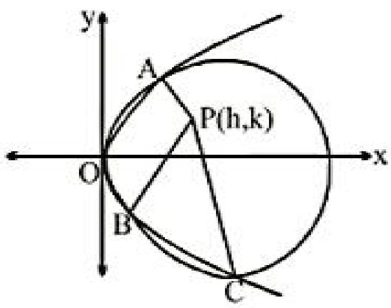
$\therefore \quad$ *locus of P(h, k) is*

$y^2 + 3a^2 - ax = 0$. ***Ans.***

Illustration :

Find the equation of circle passes through co-normal points.

Sol. *Let A$(am_1^2, -2am_1)$, B$(am_2^2, -2am_2)$ and C$(am_3^2, -2am_3)$ be the three points on the parabola and normal at these points intersect at P (h, k) then*



$am^3 + (2a - h)\, m + k = 0 \qquad$ ....(i)

$m_1 + m_2 + m_3 = 0 \qquad$ ....(ii)

$$m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a - h}{a} \qquad ......(iii)$$

$$m_1m_2m_3 = \frac{-k}{a} \qquad ......(iv)$$

Let the equation of circle through A, B and C is $x^2 + y^2 + 2gx + 2fy + c = 0$. *If the point* $(am^2, -2am)$ *lie on it then it will satisfy circle.*

$\Rightarrow \quad a^2m^4 + (4a^2 + 2ag)m^2 - 4afm + c = 0 \qquad$ (v)

It has four root m_1, m_2, m_3 and m_4

$\therefore \quad m_1 + m_2 + m_3 + m_4 = 0 \qquad ....(vi)$

$\Rightarrow \quad m_4 = 0$ *using equation (ii)*

$\Rightarrow \quad (am_4^2, -2am_4) \equiv (0, 0)$. *This is conform that above circle passes through vertex of parabola*

$\Rightarrow$ *equation (v) becomes*

$\therefore \quad a^2m^4 + (4a^2 + 2ag)m^2 - 4afm = 0$ *(No constant term)*

$\Rightarrow \quad am^3 + (4a + 2g)\, m - 4f = 0 \qquad .....(vi)$

equation (i) and (vi) are identical

$$\Rightarrow \quad \frac{1}{1} = \frac{4a+2g}{2a-h} = -\frac{4f}{k} \Rightarrow 2g = -(2a+h) \Rightarrow 2f = -\frac{k}{2}$$

$\therefore$ *equation of circle is*

$$x^2 + y^2 - (2a + h)x - \frac{k}{2}y = 0.$$

Thus circle through co-normal points.

CHORD OF THE PARABOLA $y^2=4ax$ WHOSE MIDDLE POINT IS GIVEN:

Equation of the parabola is $y^2 = 4ax \qquad ....(1)$

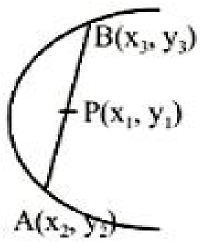
Let AB be a chord of the parabola whose middle point is $P(x_1, y_1)$.

Equation of chord AB is $y - y_1 = m(x - x_1) \qquad ....(2)$

where m = slope of AB

Let $A = (x_2, y_2)$ and $B = (x_3, y_3)$.

Since A and B lie on parabola (1)



$\therefore \quad y_2^2 = 4ax_2$ and $y_3^2 = 4ax_3$

$$\therefore \quad y_2^2 - y_3^2 = 4a(x_2 - x_3) \text{ or } \frac{y_2 - y_3}{x_2 - x_3} = \frac{4a}{y_2 + y_3} \qquad ....(3)$$

But $P(x_1, y_1)$ is the middle point of AB $y_2 + y_3 = 2y_1$

$$\therefore \text{ From (3), } \frac{y_2 - y_3}{x_2 - x_3} = \frac{4a}{2y_1} = \frac{2a}{y_1}$$

$$\therefore \text{ Slope of AB i.e., } m = \frac{2a}{y_1} \qquad ....(4)$$

$$\text{From (2), equation of chord AB is } y - y_1 = \frac{2a}{y_1}(x - x_1)$$

or $yy_1 - y_1^2 = 2ax - 2ax_1$ or $yy_1 - 2ax = y_1^2 - 2ax_1$

or $yy_1 - 2a(x - x_1) = y_1^2 - 4ax_1$ [Subtracting $2ax_1$ from both sides] $.......(5)$

(5) is the required equation. In usual notations, equation (5) can be written as $T = S_1$.

The same result holds true for cicle, ellipse and hyperbola also.

Illustration :

Find the locus of the mid points of the chords of the parabola $y^2 = 4ax$ which subtend a right angle at the vertex of the parabola.

Sol. *Let $P(h, k)$ be the mid point of a chord QR of the parabola $y^2 = 4ax$ then equation of the chord QR is $T = S_1$*

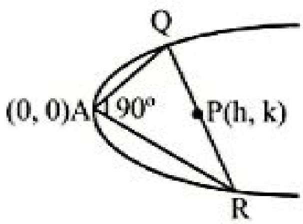
or $$yk - 2a(x + h) = k^2 - 4ah$$

$\Rightarrow$ $$yk - 2ax = k^2 - 2ah \quad ... (1)$$

If A is the vertex of the parabola. For combined equation of AQ and AR making homogeneous of $y^2 = 4ax$ with the help of (1)

$$y^2 = 4ax(1)$$

$$y^2 = 4ax\left(\frac{4k - 2ax}{k^2 - 2ah}\right)$$



$y^2(k^2 - 2ah) - 4akxy + 8a^2x^2 = 0$

Since $\angle QAF = 90^o$

$\therefore$ *Coefficient of x^2 + Co-efficient of $y^2 = 0$*

$$k^2 - 2ah + 8a^2 = 0$$

Hence the locus of P(h, k) is

$$y^2 - 2ax + 8a^2 = 0$$

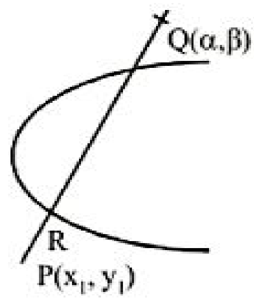
PAIR OF TANGENTS :

Let the parabola be $y^2 = 4ax$ ....(1)

Let $P(x_1, y_1)$ be a point outside the parabola.

Let a chord of the parabola through the point $P(x_1, y_1)$ cut the parabola at R and let $Q(\alpha, \beta)$ be an arbitrary point on line PR. Let R divide PQ in the ratio $\lambda : 1$,

then $R = \left(\frac{\lambda\alpha + x_1}{\lambda + 1}, \frac{\lambda\beta + y_1}{\lambda + 1}\right)$.



Since R lies on parabola (1), therefore,

$$\left(\frac{\lambda\beta + y_1}{\lambda + 1}\right)^2 - 4a\left(\frac{\lambda\alpha + x_1}{\lambda + 1}\right) = 0$$

or $(\lambda\beta + y_1)^2 - 4a(\lambda\alpha + x_1)(\lambda + 1) = 0$

or $(\beta^2 - 4a\alpha)\lambda^2 + 2[\beta y_1 - 2a(\alpha + x_1)]\lambda + (y_1^2 - 4ax_1) = 0$ ....(2)

Line PQ will become tangent to parabola (1) if roots of equation (2) are equal or if

$$4[\beta y_1 - 2a(\alpha + x_1)]^2 = 4(\beta^2 - 4a\alpha)(y_1^2 - 4ax_1)$$

Hence, locus of $Q(\alpha, \beta)$ i.e. equation of pair of tangents from $P(x_1, y_1)$ is

$$[yy_1 - 2a(x + x_1)]^2 = (y^2 - 4ax)(y_1^2 - 4ax_1)$$

$\Rightarrow$ $$SS_1 = T^2$$

where S, S_1 and T have usual meanings.

The same result holds true for cicle, ellipse and hyperbola also.

Illustration :

Tangents are drawn from the point (–1, 2) on the parabola $y^2 = 4x$. The length, these tangents will intercept on the line $x = 2$ is :

(A) 6 *(B) $6\sqrt{2}$* *(C) $2\sqrt{6}$* *(D) none of these*

Sol. $SS_1 = T^2$

$(y^2 - 4x)(y_1^2 - 4x_1) = (y y_1 - 2(x + x_1))^2$

$(y^2 - 4x)(4 + 4) = [2y - 2(x - 1)]^2 = 4(y - x + 1)^2$

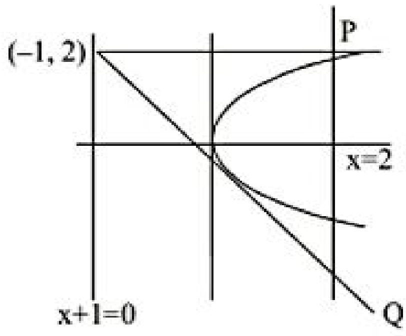
$2(y^2 - 4x) = (y - x + 1)^2$;

solving with the line $x = 2$ we get,

$2(y^2 - 8) = (y - 1)^2$ or $2(y^2 - 8) = y^2 - 2y + 1$

or $y^2 + 2y - 17 = 0$

where $y_1 + y_2 = -2$ and $y_1 y_2 = -17$



Now $|y_1 - y_2|^2 = (y_1 + y_2)^2 - 4y_1y_2$

or $|y_1 - y_2|^2 = 4 - 4(-17) = 72$

$\therefore\ (y_1 - y_2) = \sqrt{72} = 6\sqrt{2}$

CHORD OF CONTACT OF POINT WITH RESPECT TO A PARABOLA :

Two tangents PA and PB are drawn to parabola, then line joining AB is called the chord of contact to the parabola with respect to point P.

Let the parabola be $y^2 = 4ax$ ....(1)

Let $P(\alpha, \beta)$ be a point outside the parabola.

Let PA and PB be the two tangents from $P(\alpha, \beta)$ to parabola (1).

Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$

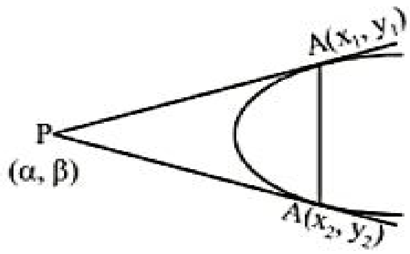
Equation of the tangent PA is $yy_1 = 2a(x + x_1)$ ....(2)

Equation of the tangent PB is $yy_2 = 2a(x + x_2)$ ....(3)

Since lines (2) and (3) pass through $P(\alpha, \beta)$, therefore

$\beta y_1 = 2a(a + x_1)$ ....(4)

and $\beta y_2 = 2a(a + x_2)$ ....(5)



Now we consider the equation $y\beta = 2a(x + \alpha)$ ......(6)

From (4) and (5), it follows that line (6) passes through $A(x_1, y_1)$ and $B(x_2, y_2)$.

Hence (6) is the equation of line AB which is the chord of contact of point $P(\alpha, \beta)$ with respect to parabola (1) i.e, chord of contact is $\mathbf{y\beta = 2a(x + \alpha)}$

The same result holds true for cicle, ellipse and hyperbola also.

Illustration :

Tangent are drawn to parabola $y^2 = 4ax$ at point where the line $lx + my + n = 0$ meets the parabola. Find the point of intersection of these tangents.

Sol. *Let the tangent intersect at $P(h, k)$, then $lx + my + n = 0$ will be the chord of contact of P. That means $lx + my + n = 0$ and $yk - 2ax - 2ah = 0$ will represent the same line. Thus,*

$$\frac{k}{m} = \frac{-2a}{l} = \frac{-2ah}{n} \quad \Rightarrow \quad h = \frac{n}{l}, k = -\frac{2am}{l}$$

POLAR & POLE :

Let P be any point inside or outside a parabola. Suppose in a straight line drawn through P intersect the parabola at Q and R. Then the locus of point of intersection of the tangents to the parabola at Q and R is called the polar of given point P with respect to the parabola and point P is called the pole of the polar.

Equation of the Polar of the point $P(x_1, y_1)$ w.r.t. the parabola $y^2 = 4ax$ is,

$$y y_1 = 2a(x + x_1)$$

The pole of the line $lx + my + n = 0$ w.r.t. the parabola $y^2 = 4ax$ is $\left(\frac{n}{l}, -\frac{2am}{l}\right)$.

Properties of polar :

(i) The polar of the focus of the parabola is the directrix.

(ii) When the point (x_1, y_1) lies without the parabola the equation to its polar is the same as the equation to the chord of contact of tangents drawn from (x_1, y_1) when (x_1, y_1) is on the parabola the polar is the same as the tangent at the point.

(iii) Two straight lines are said to be conjugated to each other w.r.t. a parabola when the pole of one lies on the other. Similarly two points P and Q are said to be conjugate points if polar of P passes through Q and vice versa.

(iv) Polar of a given point P w.r.t. any Conic is the locus of the harmonic conjugate of P w.r.t. the two points is which any line through P cuts the conic.

Illustration :

Prove that the area of the triangle formed by the tangets drawn from (x_1, y_1) to $y^2 = 4ax$ and their chord of contact is $(y_1^2 - 4ax_1)^{3/2} / 2a$

Sol. *Equation of QR (chord of contact) is*

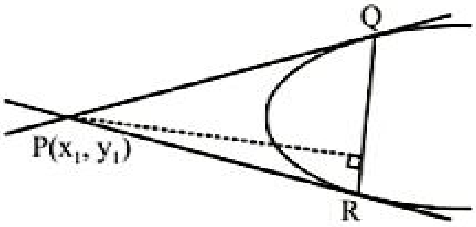
$$yy_1 = 2a(x + x_1)$$
$$yy_1 - 2a(x + x_1) = 0$$

$\because$ $\quad PM$ = *Length of perpendicular from* $P(x_1, y_1)$ *on* QR

$$= \frac{|y_1y_1 - 2a(x_1 + x_1)|}{\sqrt{(y_1^2 + 4a^2)}} = \frac{|(y_1^2 - 4ax_1)|}{\sqrt{(y_1^2 + 4a^2)}}$$

[Since $P(x_1, y_1)$ *lies outside the parabola* $\therefore y_1^2 - 4ax_1 > 0]$

Now area of $\Delta PQR = \frac{1}{2} QR \cdot PM$



$$= \frac{1}{2}\frac{1}{|a|}\sqrt{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}\,\frac{(y_1^2 - 4ax_1)}{\sqrt{y_1^2 - 4a^2}}$$

$$= (y_1^2 - 4ax)^{3/2} / 2a, \text{ if } a > 0.$$

Note : *Length of the chord of contact* $QR = \dfrac{\sqrt{y_1^2 + 4a^2}\,\sqrt{y_1^2 - 4ax_1}}{a}$

Practice Problem

Single correct question

Q.1 From an external point P, pair of tangent lines are drawn to the parabola, $y^2 = 4x$. If θ_1 & θ_2 are the inclinations of these tangents with the axis of x such that, $\theta_1 + \theta_2 = \frac{\pi}{4}$, then the locus of P is :

(A) $x - y + 1 = 0$ (B) $x + y - 1 = 0$ (C) $x - y - 1 = 0$ (D) $x + y + 1 = 0$

Q.2 If m_1, m_2 are slopes of the two tangents that are drawn from (2, 3) to the parabola $y^2 = 4x$ then the value of $\frac{1}{m_1} + \frac{1}{m_2}$ is

(A) –3 (B) 3 (C) $\frac{2}{3}$ (D) $\frac{3}{2}$

Q.3 The points of contact Q and R of tangent from the point P (2, 3) on the parabola $y^2 = 4x$ are

(A) (9, 6) and (1, 2) (B) (1, 2) and (4, 4) (C) (4, 4) and (9, 6) (D) (9, 6) and $\left(\frac{1}{4}, 1\right)$

Q.4 If the lines $(y - b) = m_1(x + a)$ and $(y - b) = m_2(x + a)$ are the tangents to the parabola $y^2 = 4ax$, then

(A) $m_1 + m_2 = 0$ (B) $m_1m_2 = 1$ (C) $m_1m_2 = -1$ (D) $m_1 + m_2 = 1$

Q.5 If $2x + y + \lambda = 0$ is a normal to the parabola $y^2 = -8x$ then the value of λ is

(A) –24 (B) –16 (C) –8 (D) 24

Q.6 The set of points on the axis of the parabola $y^2 - 4x - 2y + 5 = 0$ from which all three normals to the parabola are real is

(A) $(\lambda, 0); \lambda > 1$ (B) $(\lambda, 1); \lambda > 3$ (C) $(\lambda, 2); \lambda > 6$ (D) $(\lambda, 3); \lambda > 8$

Q.7 The slope of a chord of the parabola $y^2 = 4ax$ which is normal at one end and which subtends a right angle at the origin is

(A) $\frac{1}{\sqrt{2}}$ (B) $\sqrt{2}$ (C) $-\frac{1}{\sqrt{2}}$ (D) $-\sqrt{2}$

Q.8 Equation of the other normal to the parabola $y^2 = 4x$ which passes through the intersection of those at $(4, -4)$ and $(9, -6)$ is

(A) $5x - y + 115 = 0$ (B) $5x + y - 135 = 0$

(C) $5x - y - 115 = 0$ (D) $5x + y + 115 = 0$

Q.9 Which one of the following lines cannot be the normals to $x^2 = 4y$?

(A) $x - y + 3 = 0$ (B) $x + y - 3 = 0$ (C) $x - 2y + 12 = 0$ (D) $x + 2y + 12 = 0$

Q.10 Normal to the parabola $y^2 = 8x$ at the point P (2, 4) meets the parabola again at the point Q. If C is the centre of the circle described on PQ as diameter then the coordinates of the image of the point C in the line $y = x$ are

(A) $(-4, 10)$ (B) $(-3, 8)$ (C) $(4, -10)$ (D) $(-3, 10)$

Q.11 Normals are concurrent drawn at points A, B, and C on the parabola $y^2 = 4x$ at P(h, k). The locus of the point P if the slope of the line joining the feet of two of them is 2, is

(A) $x + y = 1$ (B) $x - y = 3$ (C) $y^2 = 2(x - 1)$ (D) $y^2 = 2\left(x - \frac{1}{2}\right)$

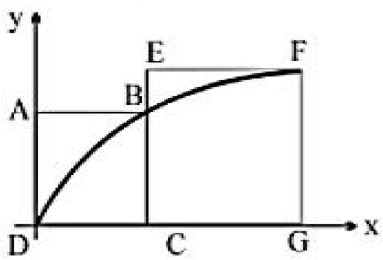
Q.12 If (a, b) is the mid-point of chord passing through the vertex of the parabola $y^2 = 4x$, then

(A) $a = 2b$ (B) $2a = b$ (C) $a^2 = 2b$ (D) $2a = b^2$

Q.13 Tangents are drawn from the points on the line $x - y + 3 = 0$ to parabola $y^2 = 8x$. Then the variable chords of contact pass through a fixed point whose coordinates are :

(A) (3, 2) (B) (2, 4) (C) (3, 4) (D) (4, 1)

Q.14 ABCD and EFGC are squares and the curve $y = k\sqrt{x}$ passes through the origin D and the points B and F. Then find the ratio of $\frac{FG}{BC}$.



(A) $\frac{\sqrt{5}+1}{2}$ (B) $\frac{\sqrt{3}+1}{2}$

(C) $\frac{\sqrt{5}+1}{4}$ (D) $\frac{\sqrt{3}+1}{4}$

Integer type question

Q.15 Find the y-intercept of the common tangent to the parabola $y^2 = 32x$ and $x^2 = 108y$.

Answer key

Q.1	C	Q.2	B	Q.3	B	Q.4	C	Q.5	A, D
Q.6	B	Q.7	D	Q.8	B	Q.9	D	Q.10	A
Q.11	B	Q.12	D	Q.13	C	Q.14	A	Q.15	– 12

DIAMETER OF A PARABOLA :

Diameter of a conic is the locus of middle points of a series of its parallel chords.

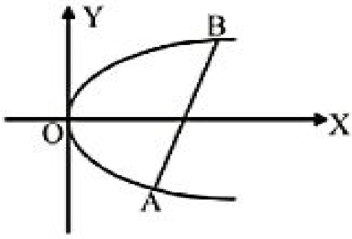
Equation of diameter of a parabola :

Let the parabola be $y^2 = 4ax$ ....(1)

Let AB be one of the chords of a series of parallel chords having slope m.

Let $P(\alpha, \beta)$ be the middle point of chord AB, then equation of AB will be $T = S_1$.

or $y\beta - 2a(x + \alpha) = \beta^2 - 4\alpha a$ ....(2)

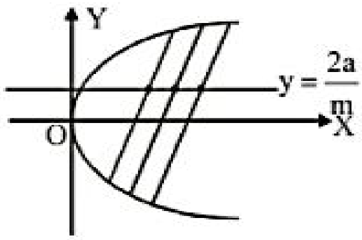


Slope of line (2) $= \dfrac{2a}{\beta}$

but slope of line (1) i.e. line AB is m.

$$\therefore \quad \frac{2a}{\beta} = m \quad \text{or} \quad \beta = \frac{2a}{m}$$

Hence locus of $P(\alpha, \beta)$ i.e. equation of diameter (which is the locus of a series of a parallel chords having slope m) is

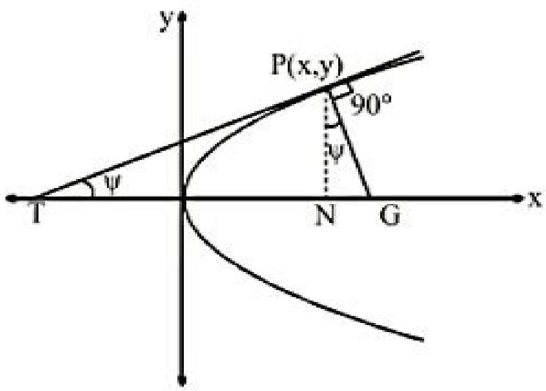


$$y = \frac{2a}{m} \qquad ....(3)$$

Clearly line (3) is parallel to the axis of the parabola. Thus a diameter of a parabola is parallel to its axis.

Length of tangent, subtangnet, normal and sub-normal :

Let the parabola is $y^2 = 4ax$. Let the tangent at any point P(x, y) meet the axis of parabola at T and G respectively and tangent makes an angle ψ with x-axis.



$$\therefore \qquad \tan\psi = \left(\frac{dy}{dx}\right)_{P(x,y)} \text{ and } PN = y$$

$\therefore$ PT = length of tangent = $PN \operatorname{cosec}\psi = y \operatorname{cosec}\psi$

PG = length of normal = $y \sec\psi$

TN = length of sub-tangent = $PN \cot\psi = y \cot\psi$

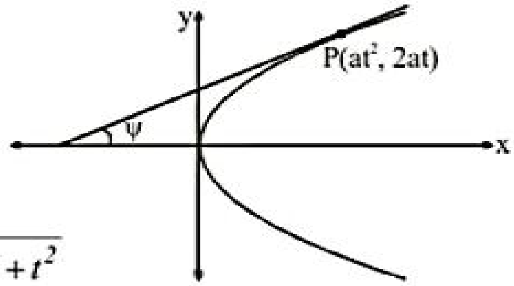
NG = length of sub-normal = $y \tan\psi$

Illustration :

Find the length of tangent, sub-tangent, normal and sub-normal to $y^2 = 4ax$ at $(at^2, 2at)$.

Sol. $y^2 = 4ax$

$$2y\frac{dy}{dx} = 4a$$



$$\therefore \quad \left(\frac{dy}{dx}\right)_{(at^2,2at)} = \frac{1}{t} = \tan\psi \Rightarrow \cot\psi = t$$

$$\therefore \quad l\,(\text{tangent}) = y\,\text{cosec}\,\psi = 2at\sqrt{1+\cot^2\psi} = 2at\sqrt{1+t^2}$$

$$l\,(\text{normal}) = y\sec\psi = 2at\sqrt{1+\tan^2\psi} = 2a\sqrt{1+t^2}$$

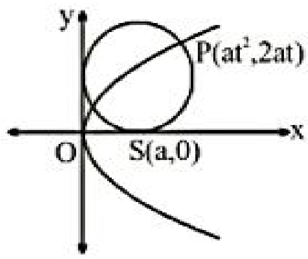
$$l\,(\text{sub-tangent}) = y\cot\psi = 2at\cdot t = 2at^2$$

$$l\,(\text{sub-normal}) = y\tan\psi = 2at\cdot\frac{1}{t} = 2a$$

Thus length of sub-normal of parabola is constant and equal to semi-latus rectum.

PROPERTIES OF PARABOLA :

(1) Circle described on the focal length (distance) as diameter touches the tangent at the vertex.



Equation of the circle described on SP as diameter is

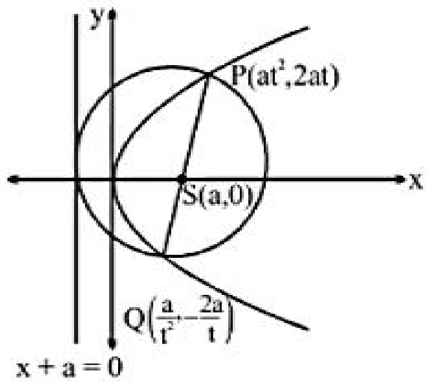
$$(x - at^2)(x - a) + (y - 2at)(y - 0) = 0$$

Solve it with y-axis i.e. $x = 0$, we get

$$y^2 - 2aty + a^2t^2 = 0 \Rightarrow (y - at)^2 = 0$$

circle touches y-axis at $(0, at)$.

(2) Circle described on the focal chord as diameter touches directrix



Equation of the circle described on PQ as diameter is

$$(x - at^2)\left(x - \frac{a}{t^2}\right) + (y - 2at)\left(y + \frac{2a}{t}\right) = 0$$

Solving it with $x = -a$

$$(-a - at^2)\left(-a - \frac{a}{t^2}\right) + (y - 2at)\left(y + \frac{2a}{t}\right) = 0$$

$$\Rightarrow \quad y^2 - 2a\left(t - \frac{1}{t}\right)y + a^2\left(t - \frac{1}{t}\right)^2 = 0$$

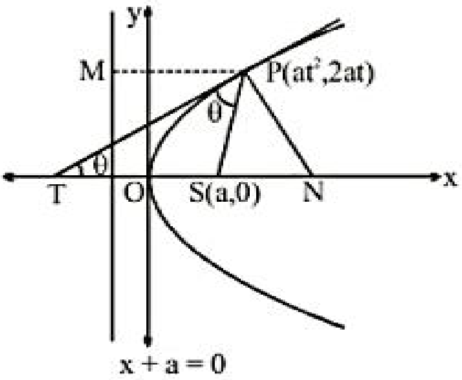
$$\Rightarrow \quad \left[y - a\left(t - \frac{1}{t}\right)\right]^2 = 0$$

$\Rightarrow$ circle tocuhes the directrix.

(3) Tangent at P is

$yt = x + at^2$, meet x-axis at T, then $T(-at^2, 0)$

Normal at P is $y + xt = 2at + at^3$, meet x-axis at N, then $N(2a + at^2, 0)$



$\Rightarrow$ $ST = SN = a + at^2 = PM = PS$

$\Rightarrow$ $\angle PTS = \angle TPS = \theta$

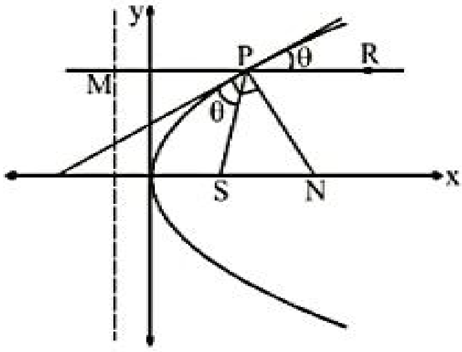
$\because$ $TS = PS = PM \Rightarrow \angle TPM = \theta$

Tangent and Normal at any point P bisect the angle between PS and PM internally and externally. This property leads to the *reflection property of parabola.*

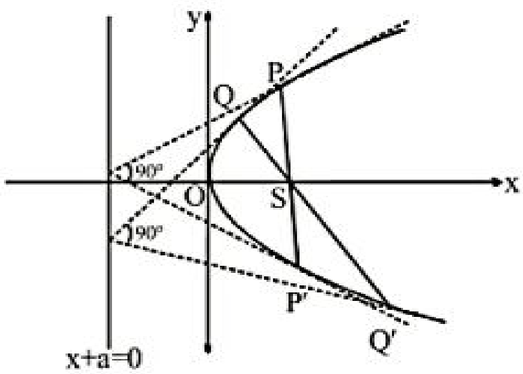
Circle circumscribing the triangle formed by any tangent normal and x-axis, has its centre at focus.

If we extend MP, then from figure $\angle RPN = \angle SPN = 90 - \theta$

Thus ray parallel axis meet parabola at P and after reflection from P it passes through the focus.



(4) The tangents at the extremities of a focal chord intersect at right angles on the directrix.



(5) The portion of tangent to the parabola intercepted between the directrix and the curve subtends a right angle at the focus.

tangent at $P(at^2, 2at)$ is $yt = x + at^2$ meet the directrix at $x = -a \Rightarrow Q\left(-a, \dfrac{at^2 - a}{t}\right)$ and $S(a, 0)$.

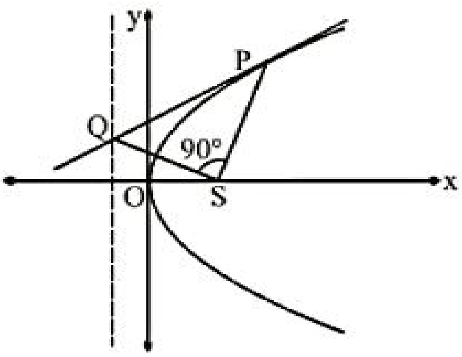
$$\text{Slope at SP} = \frac{2at - 0}{at^2 - a} = \frac{2t}{t^2 - 1} = m_1$$

$$\text{Slope at SQ} = \frac{\dfrac{at^2 - a}{t} - 0}{-a - 0} = \frac{t^2 - 1}{-2t} = m_2.$$

$\Rightarrow$ $m_1 m_2 = -1$

$\Rightarrow$ $SP \perp SQ$

$\Rightarrow$ $\angle PSQ = 90°$



(6) Tangent at P is $yt = x + at^2$ ......(i)

Line perpendicular to above line is $xt + y = \lambda$

and passes through (a, 0) gives $\lambda = at$

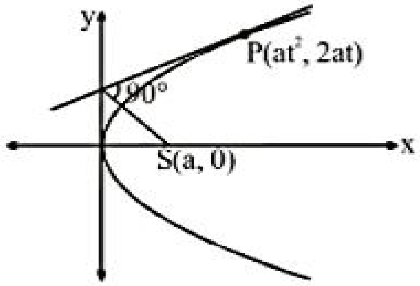
$\therefore$ perpendicular line will be $xt + y = at$ .....(ii)

Solve (i) and (ii), we get

$x = 0$

i.e., these two line intersect at y-axis i.e. tangnet at the vertex.

The foot of the perpendicular from the focus on any tangent to a parabola lies on the tangent at vertex.



(7) **Tangents and Normals** at the extremities of the latus rectum of a parabola $y^2 = 4ax$ **constitute a square**, their points of intersection being (– a, 0) & (3a, 0).

(8) The **circle circumscribing the triangle** formed by **any three tangents** to a parabola passes through the focus.

(9) The **orthocentre** of any triangle formed by three tangents to a parabola $y^2 = 4ax$ lies on the directrix & has the co–ordinates $(-a,\ a(t_1 + t_2 + t_3 + t_1t_2t_3))$.

(10) The area of the triangle formed by three points on a parabola is **twice the area** of the triangle formed by the tangents at these points.

Illustration :

If incident ray from point (– 3, 2) parallel to the axis of parabola $y^2 = 4x$ strike the parabola, then find the equation of reflected ray.

Sol. *Since incident ray strikes parabola at P(1, 2) i.e. extremity of latus rectum and it will pass through the focus of parabola therefore reflected ray will be parallel to y-axis and its equation will be x = 1.*

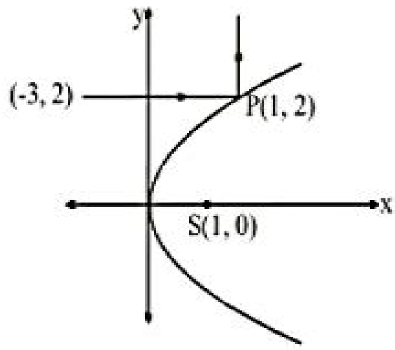


Illustration :

A ray of light moving parallel to the x-axis get reflected from a parabolic mirror $(y-2)^2 = 4(x+1)$. Find the point on the axis of parabola through which the ray must pass after reflection.

Sol. *Axis of parabola is y = 2 i.e., parallel to x-axis. As we know if incident ray is parallel to x-axis then after reflextion it will pass through the focus of parabola and focus is (0, 2).* ***Ans.***

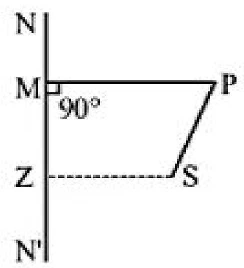
ELLIPSE

DEFINITION :

An ellipse is the locus of the point which moves in a plane such that the ratio of its distance from a fixed point (focus) to fixed straight line (directrix) is always constant (called eccentricity).

In the given figure, S is the focus and NN' is the directrix.
Let P be a point on the ellipse, then

$$\frac{PS}{PM} = e, \quad e < 1 \quad \text{(for ellipse)}$$



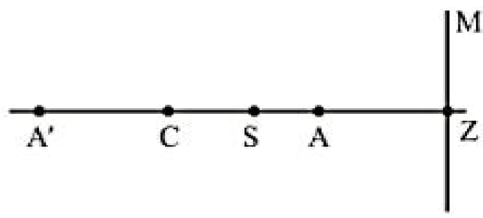
Thus, we can find the equation of an ellipse when the coordinates of its focus, equation of the directrix and eccentricity (e) are given.

STANDARD EQUATION OF AN ELLIPSE :

Let S be the focus & ZM is the directrix of an ellipse. Draw perpendicular from S to the directrix which meet it at Z. A moving point is on the ellipse such that

$$PS = ePM$$

then there is point lies on the line SZ and which divide SZ internally at A and externally at A' in the ratio of e : 1.



therefore $\quad SA = e\,AZ \quad$ ...(i)

$\quad SA' = eA'Z \quad$ ...(ii)

Let AA' = 2a & take C as mid point of AA'

$\therefore \quad CA = CA' = a$

Add (i) & (ii)

$$SA + SA' = e\,(AZ + A'Z)$$

$\Rightarrow \quad AA' = e[CZ - CA + CA' + CZ]$

$$2a = 2eCZ$$

$\Rightarrow \quad CZ = \frac{a}{e} \quad$ ...(iii)

Subtract (ii) & (i), we get

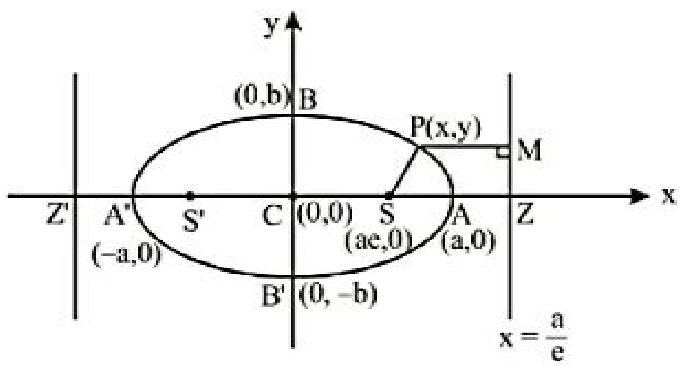
$$SA' - SA = e\,(A'Z - AZ)$$

$\Rightarrow \quad (CA' + CS) - (CA - CS)$

$\quad = e\,[(CA' + CZ) - (CZ - CA)]$

$\Rightarrow \quad 2CS = 2e\,CA$

$\therefore \quad CS = ae \quad$ ...(iv)



Result (iii) & (iv) are independent of axis.

Consider CZ line as x-axis, C as origin & perpendicular to this line & passes through C is considered as y-axis. Let P(x, y) is a moving point, then

By defintion of ellipse.

$PS = ePM \quad \Rightarrow \quad (PS)^2 = e^2\,(PM)^2$

$$\Rightarrow \quad (x-ae)^2 + (y-0)^2 = e^2\left(\frac{a}{e}-x\right)^2 \quad \Rightarrow \quad (x-ae)^2 + y^2 = (a-ex)^2$$

$$\Rightarrow \quad x^2 + a^2e^2 - 2xae + y^2 = a^2 + e^2x^2 - 2xae \quad \Rightarrow \quad x^2\,(1-e^2) + y^2 = a^2\,(1-e^2)$$

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1 \quad \text{or} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{where } b^2 = a^2\,(1-e^2)$$

Tracing of an ellipse :

Given ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \qquad \ldots(1)$

(i) If we put $y = 0$ then we see that ellipse cuts x-axis at $(\pm a, 0)$.

(ii) If we put $x = 0$ then we see that ellipse cuts y-axis at $(0, \pm b)$.

(iii) Equation of ellipse does not change when y is replaced by $-y$. Hence, ellipse is symmetrical about x-axis. (Since equation contain even power of y therefore curve is symmetric about x-axis).

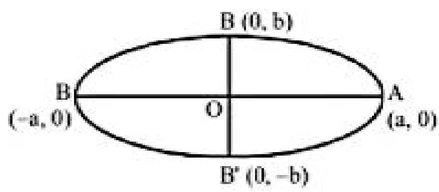
(iv) When x is replaced by $-x$, the equation of curve does not change therefore ellipse is symmetrical about y-axis. (Since equation contain even power of x therefore curve is symmetric about y-axis).

(v) From (1), $y = \pm\dfrac{b}{a}\sqrt{a^2-x^2}$, Since y is real. $\therefore\ a^2 - x^2 \geq 0$ or $-a \leq x \leq a$

Also from (1), $x = \pm\dfrac{a}{b}\sqrt{b^2-y^2}$, Since x is real. $\therefore\ b^2 - y^2 \geq 0$ or $-b \leq y \leq b$

Hence ellipse lies entirely between the lines $x = -a$ and $x = a$ and the lines $y = -b$ and $y = b$,

Thus an ellipse is a closed curve. Since curve is symmetrical about both axis, therefore first of all we draw its graph only in the first quadrant and then we will take its image in both axis.



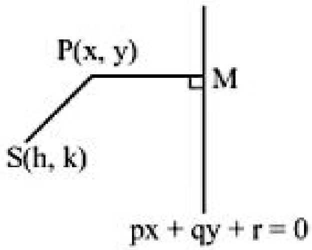
FACTS ABOUT AN ELLIPSE :

(1) By the symmetry of equation of ellipse, if we take second focus S'(–ae, 0) & second directrix $x = -\frac{a}{e}$ & perform same calculation, we get same equation of ellipse, therefore there are two focii & two directrix of an ellipse. The two focii of ellipse are (ae, 0) and (–ae, 0) and the two corresponding directrices are lines $x = \frac{a}{e}$ and $x = \frac{-a}{e}$. If focus of the ellipse is taken as (ae, 0), then corresponding directrix is $x = \frac{a}{e}$ and if focus is (–ae, 0), then corresponding directrix is $x = \frac{-a}{e}$.

(2) If equation of directrix is $px + qy + r = 0$ & focus is (h, k) then its equation will be

$$PS^2 = e^2\ PM^2$$

$$(x-h)^2 + (y-k)^2 = e^2 \cdot \left(\frac{px+qy+r}{\sqrt{p^2+q^2}}\right)^2$$



(3) Distance between foci $SS' = 2ae$ & distance between directrix $ZZ' = 2\frac{a}{e}$.

(4) Degree of flatness of an ellipse is also called on eccentricity & written as

$$e = \frac{CS}{CA} = \frac{\text{Distance from centre to focus}}{\text{Distance from centre to vertex}}$$

If $e \to 0 \Rightarrow b \to a \Rightarrow$ foci becomes closer & move towards centre and ellipse becomes circle.
If $e \to 1 \Rightarrow b \to 0 \Rightarrow$ ellipse get thinner & thinner

(5) Two ellipse are said to be similar if they have same eccentricity.

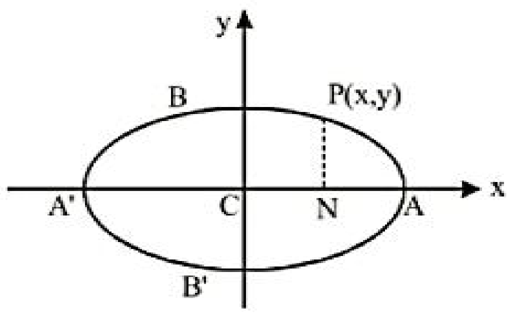
(6) Distance of focus from the extremity of minor axis is equal to 'a' because $a^2e^2 + b^2 = a^2$

(7) Let P(x, y) be any point on the ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \therefore \qquad \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\Rightarrow \quad \frac{y^2}{b^2} = \frac{(a-x)(a+x)}{a^2} \quad \Rightarrow \quad \frac{PN^2}{b^2} = \frac{AN \cdot A'N}{a^2}$$

$$\Rightarrow \quad \frac{PN^2}{AN \cdot A'N} = \frac{b^2}{a^2}$$



(8) By definition of ellipse, the distance of any point P on the ellipse from focus = e (the distance of point P from the corresponding directrix).

BASIC TERMS RELATED TO AN ELLIPSE :

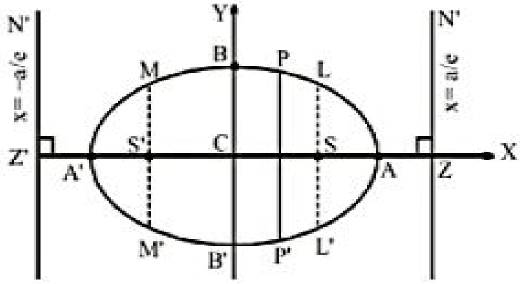
Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$ ....(1)

1. Centre :

In the figure, C is the centre of the ellipse. All chords passing through C are called diameter and bisected at C.

2. Foci :

S and S' are the two foci of the ellipse and their coordinates are (ae, 0) and (–ae, 0) respecitvely. The line containing two foci are called the focal axis and the distance between S & S' the focal length



3. Directrices :

ZN and Z'N' are the two directices of the ellipse and their equations are $x=\frac{a}{e}$ and $x=\frac{-a}{e}$ respectively.

Here Z and Z' are called foot of directrix.

4. Axes :

The line segments A'A and B'B are called the major and minor axes respectively of the ellipse.
The point of intersection of major and minor axis is called centre of the ellipse. Major and minor axis together are called principal axis of ellipse.

Here Semi-major axis are $CA = CA' = a$
and Semi-minor axis are $CB = CB' = b$

5. Vertex :

The points where major axis meet the ellipse is called its vertices. In the given figure, A' and A are the vertices of the ellipse.

6. Ordinate and double ordinates :

Let P be a point on the ellipse. From P we draw PM perpendicular to major axis of the ellipse. Produce PM to meet the ellipse at P', then PM is called an ordinate and PMP' is called the double ordinate of the point P.
It is also defined as any chord perpendicular to major axis is called its double ordinate.

7. Latus rectum :

When double ordinate passes through focus then it is called the Latus rectum.
Let $L'L = 2k$, then $LS = k$ so $L = (ae, k)$.
Here LL' and MM' are called latus rectum.

Since L (ae, k) lies on the ellipse (1), therefore $\frac{a^2e^2}{a^2}+\frac{k^2}{b^2}=1$ or $\frac{k^2}{b^2}=1-e^2$

or $\quad k^2 = b^2(1-e^2) = b^2 \cdot \dfrac{b^2}{a^2} = \dfrac{b^4}{a^2} \qquad [\because b^2 = a^2(1-e^2)]$

$\therefore \quad k = \dfrac{b^2}{a}$

Hence length of semi latus rectum LS $= \dfrac{b^2}{a} =$ MS'

i.e. length of the latus rectum LL' or MM' $= \dfrac{2b^2}{a} = \dfrac{(\text{minor axis})^2}{\text{major axis}}$

$= 2a(1-e^2)$

$= 2e$ (distance from focus to the corresponding directrix).

And the end points of latus rectum are $L\left(ae, \dfrac{b^2}{a}\right)$, $L'\left(ae, \dfrac{-b^2}{a}\right)$, $M\left(-ae, \dfrac{b^2}{a}\right)$ & $M'\left(-ae, -\dfrac{b^2}{a}\right)$

8. Focal chord :

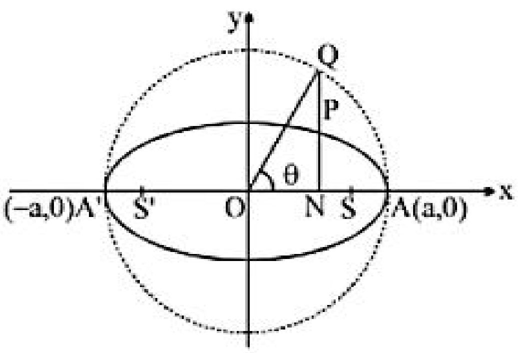
A chord of the ellipse passing through its focus is called a focal chord.

AUXILIARY CIRCLE AND ECCENTRIC ANGLE :

A circle described on major axis as diameter is called the **auxiliary circle** of the given ellipse & its equation is

$$x^2 + y^2 = a^2 \qquad \text{... (1)}$$

and given ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \qquad$ ... (2)



Let Q be a point on the auxiliary circle $x^2 + y^2 = a^2$ then line through Q and perpendicular to x-axis meet the ellipse at P then P and Q are called the CORRESPONDING POINTS on the ellipse & the auxiliary circle respectively. Here $\angle QOA = \theta$ is called the ECCENTRIC ANGLE of the point P on the ellipse $(0 \le \theta < 2\pi)$.

Since Q lie on the circle therefore $Q(a\cos\theta, a\sin\theta)$

So coordinate of $P(a\cos\theta, y)$, which satisfy the equation of ellipse.

$\therefore \quad \dfrac{a^2\cos^2\theta}{a^2} + \dfrac{y^2}{b^2} = 1 \Rightarrow y = b\sin\theta$

$\therefore$ coordinate of P will be $(a\cos\theta, b\sin\theta)$ and this is called parametric equation of ellipse.

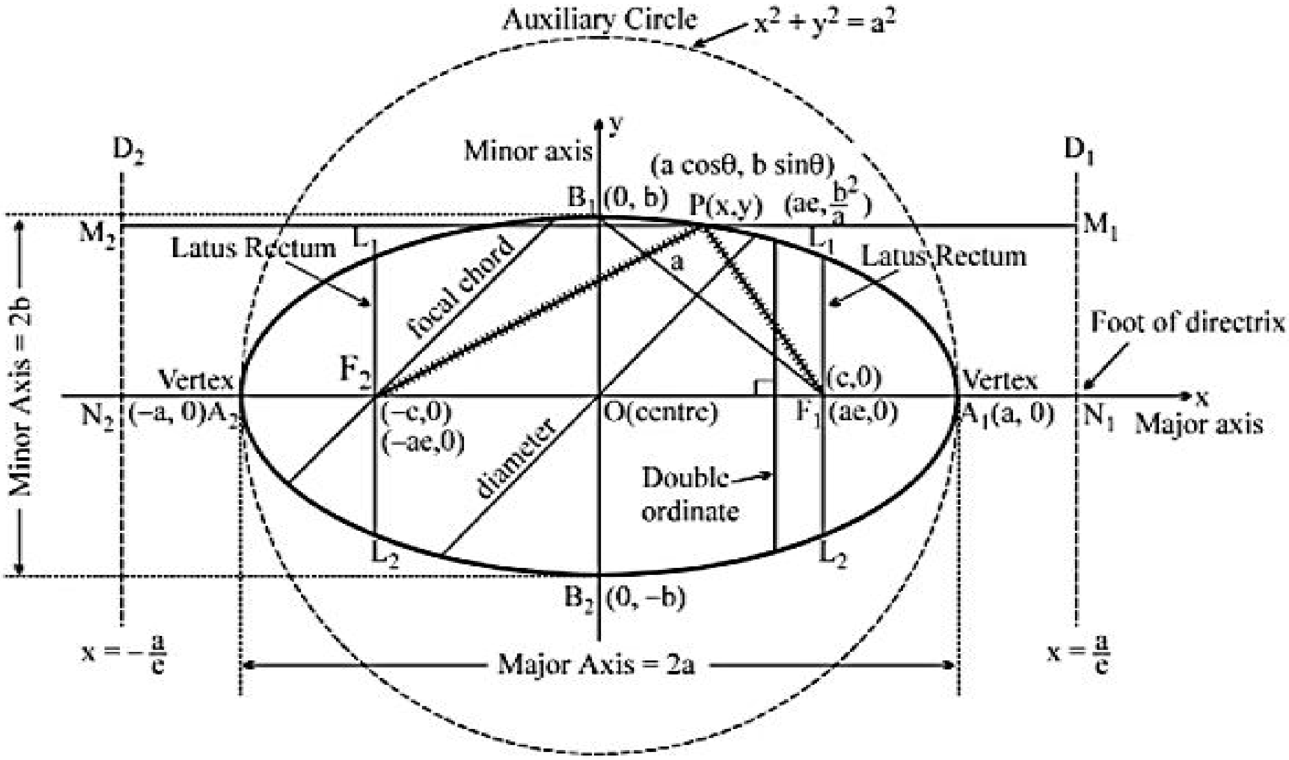
The equations $x = a\cos\theta$ & $y = b\sin\theta$ together represent the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Where θ is an eccentric angle of point P

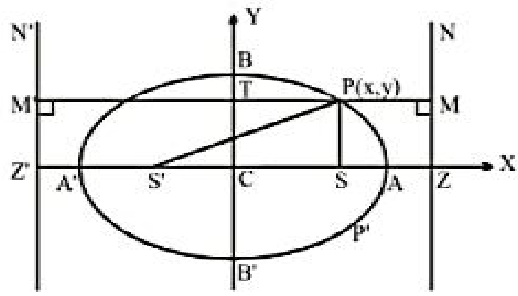
We observe that $\dfrac{\ell(PN)}{\ell(QN)} = \dfrac{b}{a} = \dfrac{\text{Semi minor axis}}{\text{Semi major axis}}$

Hence "If from each point of a circle perpendiculars are drawn upon a fixed diameter then the locus of the points dividing these perpendiculars in a given ratio is an ellipse of which the given circle is the auxiliary circle". This another defintion of ellipse.

ELLIPSE AT A GLANCE :



FOCAL DISTANCE OF A POINT :



Let P(x, y) be any point on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad ...(1)$$

Then by definition of ellipse,

$$SP = ePM = e(MT - PT) = e\left(\frac{a}{e} - x\right) = a - ex$$

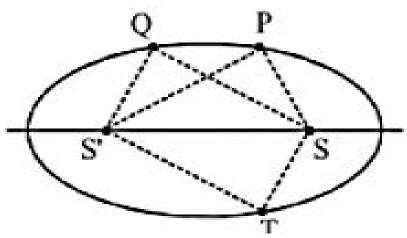
& $$S'P = ePM' = e(M'T + PT) = e\left(\frac{a}{e} + x\right) = a + ex$$

Hence $SP + S'P = 2a$

Because of the above property, **ellipse is also defined** as the locus of a point which moves in a plane such that the sum of its distance from two fixed points (called foci) is a contant (Length of major axis).

This defintion is called the physical definition of the ellipse.

Hence $PS + PS' = QS + QS' = TS + TS' =$ length of major axis



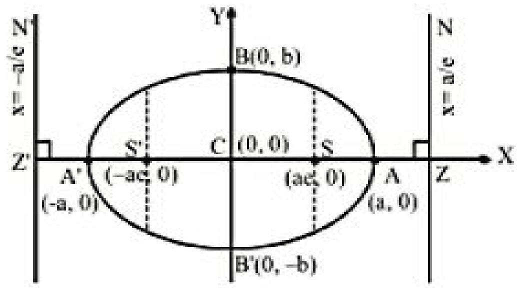
TWO STANDARD FORMS OF ELLIPSE :

There are two standard forms of ellipse with centre at the origin and axes along coordinate axes. The foci of the ellipse are either on the x-axis or on the y-axis.

1. Major axis along x-axis :

The equation of this type of ellipse is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a>b>0$ and $b=a\sqrt{1-e^2}$.

For this ellipse :

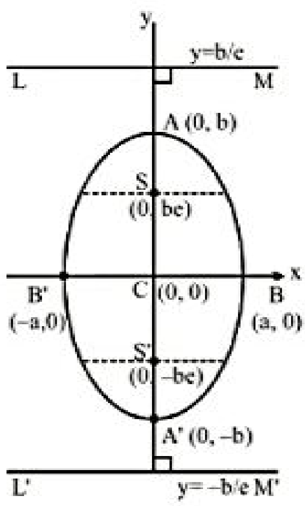


(i) Major axis is 2a
(ii) Minor axis is 2b.
(iii) Centre is (0, 0)
(iv) Vertices are $(\pm a, 0)$
(v) Foci are $(\pm ae, 0)$
(vi) Equation of directrices are $x=\pm\frac{a}{e}$
(vii) Equation of major axis is $y=0$
(viii) Equation of minor axis is $x=0$
(ix) Length of latus rectum $=\frac{2b^2}{a}$
(x) Extremity of latus rectum is $\left(\pm ae, \pm\frac{b^2}{a}\right)$

2. Major axis along y-axis :

The equation of this type of ellipse is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. where $0<a<b$ and $a=b\sqrt{1-e^2}$.

For this ellipse :



(i) Major axis is 2b
(ii) Minor axis is 2a.
(iii) Centre is (0, 0)
(iv) Vertices are $(0, \pm b)$
(v) Foci are $(0, \pm be)$
(vi) Equation of directrices are $y=\pm\frac{b}{e}$
(vii) Equation of major axis is $x=0$
(viii) Equation of minor axis is $y=0$
(ix) Length of latus rectum $=\frac{2a^2}{b}$
(x) Extremity of latus rectum is $\left(\pm\frac{a^2}{b}, \pm be\right)$

COMPARISON CHART BETWEEN STANDARD ELLIPSE :

Basic Elements	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	
	a > b	**a < b**
Centre	(0, 0)	(0, 0)
Vectrex	$(\pm a, 0)$	$(0, \pm b)$
Length of major axis	2a	2b
Length of minor axis	2b	2a
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrix	$x = \pm\frac{a}{e}$	$y = \pm\frac{b}{e}$
Relation among a, b, & e	$b^2 = a^2(1-e^2)$	$a^2 = b^2(1-e^2)$
Equation of major axis	$y = 0$	$x = 0$
Equation of minor axis	$x = 0$	$y = 0$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
End of latus rectum	$\left(\pm ae, \pm\frac{b^2}{a}\right)$	$\left(\pm\frac{a^2}{b}, \pm be\right)$
Focal distances of $P(x_1, y_1)$	$a \pm ex_1$	$b \pm ey_1$
SP + SP'	2a	2b
Distance between foci	2ae	2be
Distance between directrix	$\frac{2a}{e}$	$\frac{2b}{e}$
Parametric equation	$(a\cos\theta, b\sin\theta)$ $(0<\theta<2\pi)$	$(a\cos\theta, b\sin\theta)$

If the equation of the ellipse is given as $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ & nothing is mentioned then the rule is to assume that $a > b$.

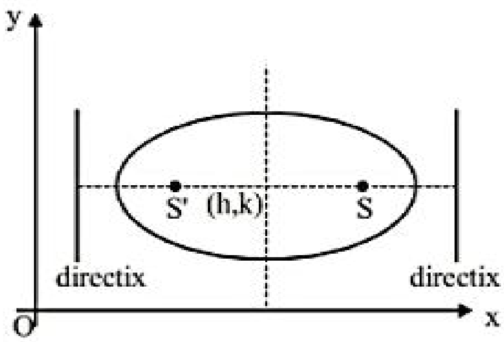
To find the Various Parameter of an ellipse :

Equation of an ellipse whose axis are parallel to coordiante axis & its centre is (h, k). The foci of the ellipse are either on x-axis or on the y-axis.

(I) Major axis parallel to x-axis :

Here the equation of ellipse is of the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, where $a > b$ & $b^2 = a^2(1-e^2)$

Here the equation of the ellipse is of the form $\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$, where $X = x - h$ and $Y = y - k$.



(1) Equation of major axis is $Y = 0$, i.e., $y - k = 0$
Equation of minor axis is $X = 0$, i.e., $x - h = 0$

(2) Coordinate of centre of the ellipse are given by
$X = 0$ and $Y = 0$ i.e, $x - h = 0$ and $y - k = 0$
$\therefore$ Centre of the ellipse is (h, k)

(3) Coordinate of foci of the ellipse are given by
$X = \pm ae, Y = 0$ i.e., $x - h = \pm ae$ and $y - k = 0$
$\therefore$ Hence foci of the ellipse are $(h \pm ae, k)$

(4) Equation of the directrices of the ellipse are $X = \pm\frac{a}{e}$, i.e., $x - h = \pm\frac{a}{e}$.

Thus directrices are $x = h \pm \frac{a}{e}$

(5) Coordinate of ends of latera recta are given by $X = \pm ae, Y = \pm\frac{b^2}{a}$ i.e. $x - h = \pm ae, y - k = \pm\frac{b^2}{a}$

Therefore ends of latera recta are given by $\left(h \pm ae,\ k \pm \frac{b^2}{a}\right)$

(6) Coordinate of vertices of the ellipse are given by $X = \pm a, Y = 0$ i.e., $x - h = \pm a, y - k = 0$.
Hence vertices are $(h \pm a, k)$

(II) Major axis parallel to y-axis :

Here the equation of ellipse is of the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, where $a < b$ & $a^2 = b^2(1 - e^2)$

Equation (1) is of the form $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$, where $X = x - h$ and $Y = y - k$

(1) Equation of major axis is $X = 0$, i.e., $x - h = 0$
Equation of minor axis is $Y = 0$, i.e., $y - k = 0$

(2) Coordinate of centre of the ellipse are given by $X = 0$ and $Y = 0$
$\Rightarrow$ $x - h = 0$ and $y - k = 0$
$\therefore$ Centre of the ellipse is (h, k)

(3) Coordinate of foci of the ellipse are given by $X = 0, Y = \pm be$
$x - h = 0$ & $y - k = \pm be$
$x = h$ & $y = k \pm be$
$\therefore$ Foci are $(h, k \pm be)$

(4) Equation of the directrices of the ellipse are $Y = \pm\frac{b}{e}$, i.e,. $y - k = \pm\frac{b}{e}$

Thus directrices are $y = k \pm \frac{b}{e}$

(5) Coordinate of ends of latera recta are given by

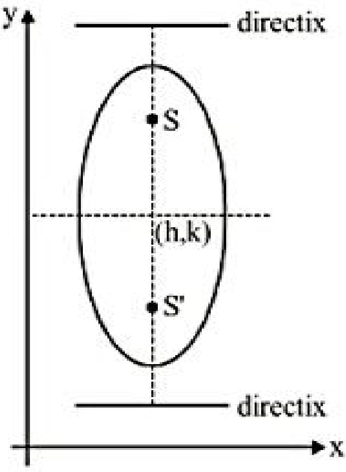
$$X = \pm\frac{a^2}{b} \quad \& \quad Y = \pm ae,$$

i.e. $$x - h = \pm\frac{a^2}{b} \quad \& \quad y - k = \pm be,$$

or $$x = h \pm \frac{a^2}{b} \quad \& \quad y = k \pm be$$

∴ Coordinates of ends of latera recta are given by

$$\left(h \pm \frac{a^2}{b},\ k \pm be\right)$$



(6) Coordinates of vertices of the ellipse is given by $X = 0$ & $Y = \pm b$
i.e., $x - h = 0$ and $y - k = \pm b$ therefore coordinates of vertex are $(h, k \pm b)$

Comparison chart between above two ellipse :

Basic Elements		$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	
		a > b	**a < b**
1.	Length of major axis Length of minor axis	$2a$ $2b$	$2b$ $2a$
2.	Equation of major axis Equation of minor axis	$y - k = 0$ $x - h = 0$	$x - h = 0$ $y - k = 0$
3.	Centre of ellipse	(h, k)	(h, k)
4.	Eccentricity	$e = \sqrt{1 - \frac{b^2}{a^2}}$	$e = \sqrt{1 - \frac{a^2}{b^2}}$
5.	Foci	$(h \pm ae, k)$	$(h, k \pm be)$
6.	Equation of directrix	$x = h \pm \frac{a}{e}$	$y = k \pm \frac{b}{e}$
7.	Extremities of latus rectum	$\left(h \pm ae, k \pm \frac{b^2}{a}\right)$	$\left(h \pm \frac{a^2}{b}, k \pm be\right)$
8.	Vertices of an ellipse	$(h \pm a, k)$	$(h, k \pm b)$
9.	Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$

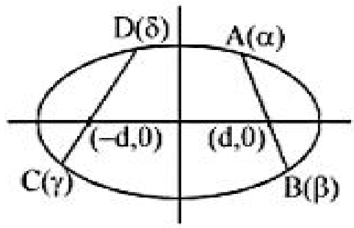
EQUATION OF CHORD OF AN ELLIPSE :

Equation of a chord of an ellipse joining two points P(α) and Q(β) on it is equal to

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$$

(use formula of line joining points P(a cosα, b sinα) and Q(a cosβ, b sinβ))
If this particular chord passes through (d, 0) then we have

$$\frac{d}{a}\cos\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)\ ;\ \frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}=\frac{a}{d}$$



Using componendo and dividendo rule

$$\frac{\cos\left(\frac{\alpha+\beta}{2}\right)-\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)+\cos\left(\frac{\alpha-\beta}{2}\right)}=\frac{a-d}{a+d}$$

or $\quad -\dfrac{2\sin\alpha/2\,\sin\beta/2}{2\cos\alpha/2\,\cos\beta/2}=\dfrac{a-d}{a+d}$ i.e. $\quad \tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}=\dfrac{d-a}{d+a}$

if $d = ae$ i.e. **PQ is a focal chord** then $\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}=\dfrac{e-1}{e+1}$

Illustration :

Find the equation of the ellipse (referred to its axis as the x-axis & y-axis) whose foci are (±2, 0) & eccentricity = $\dfrac{1}{2}$.

Sol. *Let* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ *represent ellipse where* $e=\dfrac{1}{2}$

given $(\pm ae, 0) = (\pm 2, 0)$

$\Rightarrow \quad ae = 2 \quad \Rightarrow \quad a = 4 \quad \Rightarrow \quad b^2 = a^2(1-e^2)$ *gives* $b^2 = 12$

$\therefore$ *Equation of ellipse is* $\dfrac{x^2}{16}+\dfrac{y^2}{12}=1$ ***Ans.***

Illustration :

Find the equation of an ellipse, referred to its axes as the axes of coordinates, with foci (± 2, 0) and latus rectum is 6 units.

Sol. $ae = 2$

Mid point of focus is centre (0, 0); $\frac{2b^2}{a} = 6 \Rightarrow b^2 = 3a$

$$e^2 = 1 - \frac{b^2}{a^2} \quad \Rightarrow \quad a^2e^2 = a^2 - b^2$$

$\Rightarrow \quad 4 = a^2 - 3a \quad \Rightarrow \quad a^2 - 3a - 4 = 0$

$\Rightarrow \quad (a-4)(a+1) = 0$

$a = 4, -1$

a cannot be negative, hence $a = 4$

$\therefore \quad b = 2\sqrt{3}$

So the equation of ellipse is $\frac{x^2}{16} + \frac{y^2}{12} = 1$

$\boxed{3x^2 + 4y^2 = 48}$ **Ans.**

Illustration :

Find the eccentricity of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose latus rectum is half of its major axis.

Sol. According to question $\frac{2b^2}{a} = a$

$\Rightarrow \; 2b^2 = a^2 \;\Rightarrow\; 2a^2(1 - e^2) = a^2 \;\Rightarrow\; e = \frac{1}{\sqrt{2}}$

Illustration :

If the focal distance of the end of minor axis of an ellipse is q & distance between its foci is 2p, then find its equation.

Sol. Let the equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\quad (a^2 + b^2 = a^2e^2,\ b^2 = p^2 - q^2)$

According to question $2ae = 2p$ ...(i)

& $a = q$

$\Rightarrow \quad b^2 = a^2e^2 - a^2 \Rightarrow \quad b^2 = q^2 - p^2$

$\Rightarrow \quad$ Equation is $\frac{x^2}{q^2} + \frac{y^2}{q^2 - p^2} = 1$ **Ans.**

Illustration :

Find the equation of the ellipse having axes along the coordinate axes and passing through the points (4, 3) and (–1, 4).

Sol. *Let the equation of the required ellipse be* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ... (1)

(4, –3) is on the ellipse $\therefore \frac{4^2}{a^2}+\frac{(-3)^2}{b^2}=1$ *or* $\frac{16}{a^2}+\frac{9}{b^2}=1$ ... (2)

(–1, 4) is also on the ellipse $\therefore \frac{(-1)^2}{a^2}+\frac{4^2}{b^2}=1$ *or* $\frac{1}{a^2}+\frac{16}{b^2}=1$ ... (3)

$(3)\times 16 \Rightarrow \frac{9}{a^2}+\frac{256}{b^2}=16$ ... (4)

$(4)-(2) \Rightarrow \frac{247}{b^2}=15 \Rightarrow b^2=\frac{247}{15}$

From (3), $\frac{1}{a^2}+16\left(\frac{15}{247}\right)=1 \Rightarrow \frac{1}{a^2}=1-\frac{240}{247}=\frac{7}{247} \Rightarrow a^2=\frac{247}{7}$

$\therefore$ *The equation of the required ellipse is* $\frac{x^2}{\frac{247}{7}}+\frac{y^2}{\frac{247}{15}}=1$ *or* $7x^2+15y^2=247$

Illustration :

Find the equation of curve whose parametric equation are $x = 1 + 4\cos\theta, y = 2 + 3\sin\theta$. *Also find its eccentricity.*

Sol. $\cos\theta=\frac{x-1}{4}$ & $\sin\theta=\frac{y-2}{3}$

$\because \sin^2\theta+\cos^2\theta=1 \Rightarrow \frac{(x-1)^2}{16}+\frac{(y-2)^2}{9}=1$

$\therefore e=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{9}{16}}=\frac{\sqrt{7}}{4}$ **Ans.**

Illustration :

Find the eccentric angle of the point on the ellipse $\frac{x^2}{6}+\frac{y^2}{2}=1$ *whose distance from the centre is 2.*

Sol. *Any point on the ellipse is* $\left(\sqrt{6}\cos\phi, \sqrt{2}\sin\phi\right)$, *where* ϕ *is an eccentric angle of the point.*
It's distance from the center (0, 0) is given and equal to 2, therefore

$6\cos^2\phi+2\sin^2\phi=4$ *or* $3\cos^2\phi+\sin^2\phi=2$

$2\cos^2\phi=1 \Rightarrow \cos\phi=\pm\frac{1}{\sqrt{2}}$; $\phi=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$. **Ans.**

Illustration :

Find the eccentric angles of the extremity of latus rectum lie in the first quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol. *The coordinate of any point on the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *whose eccentric angle* θ *are* $(a\cos\theta, b\sin\theta)$ *.The coordinate of the end point of latus rectum are* $\left(ae, \pm\frac{b^2}{a}\right)$

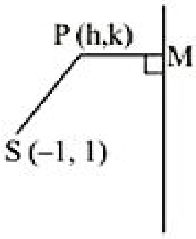
$\therefore$ *For* I^{st} *quadrant*

$a\cos\theta = ae$ and $b\sin\theta = \frac{b^2}{a}$; $\tan\theta = \frac{b}{ae} \Rightarrow \theta = \tan^{-1}\left(\frac{b}{ae}\right)$. ***Ans.***

Illustration :

Find the equation to the ellipse whose focus is the point (– 1, 1), whose directrix is the straight line $x - y + 3 = 0$ *and eccentricity is* $\frac{1}{2}$.

Sol. *Let* $P \equiv (h, k)$ *be moving point,*



$$e = \frac{PS}{PM} = \frac{1}{2}$$

$$\Rightarrow \quad (h+1)^2 + (k-1)^2 = \frac{1}{4}\left(\frac{h-k+3}{\sqrt{2}}\right)^2$$

$\Rightarrow$ *locus of* $P(h, k)$ *is*

$$8\{x^2 + y^2 + 2x - 2y + 2\} = (x^2 + y^2 - 2xy + 6x - 6y + 9)$$

$$7x^2 + 7y^2 + 2xy + 10x - 10y + 7 = 0.$$ ***Ans.***

Illustration :

Find the equation to the ellipse whose centre is origin, axes are the axes of co-ordinate and passes through the points (2, 2) and (3, 1).

Sol. *Let the equation to the ellipse is* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Since it passes through the points (2, 2) and (3, 1)

$$\therefore \quad \frac{4}{a^2} + \frac{4}{b^2} = 1 \qquad \text{..........(i)}$$

$$\text{and} \quad \frac{9}{a^2} + \frac{1}{b^2} = 1 \qquad \text{.........(ii)}$$

from (i) and (ii), we get

$$a^2 = \frac{32}{3} \quad \text{and} \quad b^2 = \frac{32}{5}$$

Ans.

Illustration :

Find the length and equation of major and minor axes, centre, eccentricity, foci, equation of directrices, vertices and length of the ellipse $16x^2 + y^2 = 16$.

Sol. *Given equaiton of the ellipse is* $16x^2 + y^2 = 16$ *or* $\frac{x^2}{1^2} + \frac{y^2}{4^2} = 1$ ... (1)

Here $a = 1$, $b = 4$ *and* $a < b$

Length of major axis $= 2b = 8$

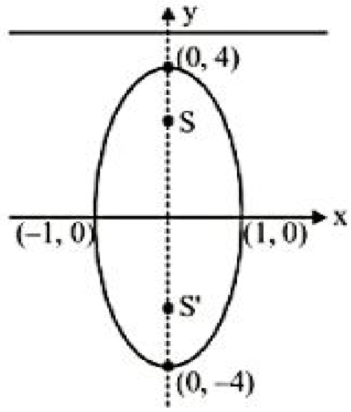
Length of minor axis $= 2a = 2$

Equation of major axis is $x = 0$

Equation of minor axis is $y = 0$

Coordiantes of centre are $(0, 0)$

Eccentricity of the ellipse,



$$e = \sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{1}{16}} = \frac{\sqrt{15}}{4}$$

Coordinate of foci are given by

$y = \pm be, x = 0$ *i.e.,* $y = \pm\sqrt{15}, x = 0$

Hence foci are $(0, \pm\sqrt{15})$

Equation of directrices are $y = \pm\frac{b}{e}$ *or* $y = \pm\frac{16}{\sqrt{15}}$

Coordinates of vertices are given by $y = \pm b$ *and* $x = 0$ *i.e.,* $y = \pm 4, x = 0$

Hence vertices are $(0, \pm 4)$.

Illustration :

Find the equation of the ellipse whose focii are (4, 0) and (– 4, 0) and eccentricity is $\frac{1}{3}$

Sol. *Since both focus lies on x-axis, therefore x-axis is major axis and mid point of focii is origin which is centre and a line perpendicular to major axis and passes throguh centre is minor axis which is y-axis, then equation of ellipse is*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\because$ $2ae = 8$ *and* $e = \frac{1}{3}$ *(Given)*

$\therefore$ $a = 12$ *and* $b^2 = a^2(1 - e^2)$

$\Rightarrow$ $b^2 = 144\left(1 - \frac{1}{9}\right)$ $\Rightarrow$ $b^2 = 16 \times 8$ $\Rightarrow$ $b = 8\sqrt{2}$

Equation of ellipse is $\frac{x^2}{144} + \frac{y^2}{128} = 1$ ***Ans.***

Illustration :

A rod of length 12cm moves with its ends always touching the coordiante axes. Determine the equation of the locus of a point on the rod, which is 3cm from the end in contact with the x-axis.

Sol. Let AB be the rod of length 12cm touching the coordiante axes at points A and B.

Let $A \equiv (a, 0)$, $B \equiv (0, b)$

Now $AB^2 = 12^2$

$\Rightarrow \quad a^2 + b^2 = 144 \qquad \dots (1)$

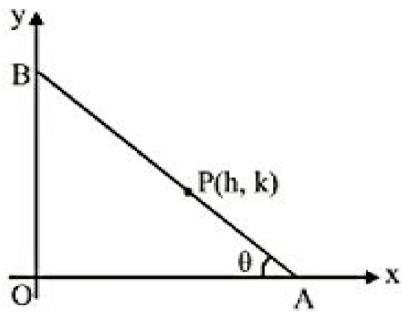
Let P be a point on AB such that

$AP = 3cm$

then $BP = 12cm - 3cm = 9cm$

$\therefore AP : PB = 1 : 3$

Hence P divides AB internally in the ratio 1 : 3.



$$\therefore \quad P \equiv \left(\frac{1.0 + 3.a}{1+3}, \frac{1.b + 3.0}{1+3}\right) \quad or \quad P \equiv \left(\frac{3a}{4}, \frac{b}{4}\right)$$

$$\therefore \quad h = \frac{3a}{4} \text{ and } k = \frac{b}{4} \quad \Rightarrow \quad a = \frac{4h}{3} \text{ and } b = \frac{k}{3}$$

Put value of a and b in equaiton (1) we get $\dfrac{16h^2}{9} + 16k^2 = 144$

$$\therefore \quad \text{Locus of } P(h, k) \text{ is } \frac{16}{9}x^2 + 16y^2 = 144,$$

i.e. $x^2 + 9y^2 = 81$, which is the equation of the required locus.

Illustration :

If (5, 12) & (24, 7) are the foci of an ellipse passing through origin, then find the eccentricity of the ellipse.

Sol. Let the S(5, 12) & S(24, 7) are two foci & ellipse passes through origin O.

$\therefore \quad OS + OS' = 2a$

$\Rightarrow \quad \sqrt{25 + 144} + \sqrt{576 + 49} = 2a$

$\Rightarrow \quad 2a = 13 + 25 \quad \Rightarrow \quad a = 19 \quad \& \quad 2ae = SS' = \sqrt{386}$

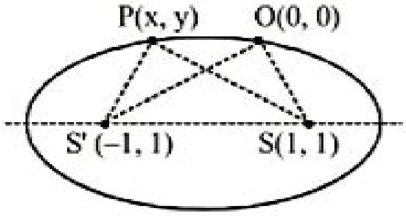
$$\therefore \quad e = \frac{\sqrt{386}}{38} \quad \textbf{Ans.}$$

Illustration :

Find the equaton of the ellipse that passes through the origin and has the foci at the points (–1, 1) and (1, 1).

Sol. *Let P(x, y) be any point on the ellipse and the foci be S(–1, 1) and S′(1, 1).*
and O lie on the ellipse
OS + OS′ = constant = 2a

$$=\sqrt{(-1-0)^2+(1-0)^2}+\sqrt{(1-0)^2+(1-0)^2}=2\sqrt{2}$$



Let P(x, y) be any point on the ellipse $\Rightarrow PS + PS' = 2\sqrt{2}$

$\Rightarrow \quad \sqrt{(x+1)^2+(y-1)^2}+\sqrt{(x-1)^2+(y-1)^2}=2\sqrt{2}$

$\Rightarrow \quad \sqrt{(x+1)^2+(y-1)^2}=2\sqrt{2}-\sqrt{(x-1)^2+(y-1)^2}$

$\Rightarrow \quad (x+1)^2+(y-1)^2=8+[(x-1)^2+(y-1)^2]-4\sqrt{2}\sqrt{(x-1)^2+(y-1)^2}$

$\Rightarrow \quad x^2+2x+1+y^2-2y+1=8+x^2-2x+1+y^2-2y+1-4\sqrt{2}\sqrt{(x-1)^2+(y-1)^2}$

$\Rightarrow \quad 4x-8=-4\sqrt{2}\sqrt{(x-1)^2+(y-1)^2}$

$\Rightarrow \quad x-2=-\sqrt{2}\sqrt{(x-1)^2+(y-1)^2}$

$\Rightarrow \quad x^2-4x+4=2[x^2-2x+1+y^2-2y+1]$

$\Rightarrow \quad 0=x^2+2y^2-4y$

$\Rightarrow \quad x^2+2y^2-4y=0$ *is the required equation of the ellipse.*

Illustration :

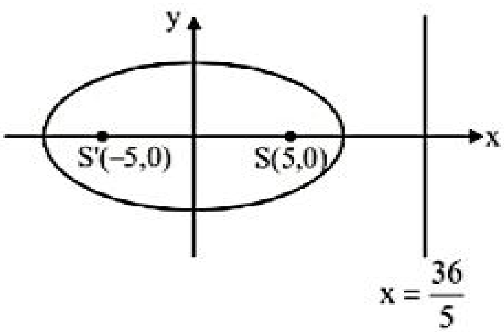
Find the equation of the ellipse with foci at (±5, 0) and $x=\frac{36}{5}$ *as one directrix.*

Sol. *Let S(5, 0) and S′ (–5, 0) be the two foci. Centre of the ellipse will be C(0, 0).*

Clearly, foci S and S′ lie on x-axis. Therefore, major axis of the ellipse will be the x-axis.
Let a and b be the length of semi-major and semi-minor axes respectively of the ellipse.
Then, $2ae = 10$ *or* $ae = 5$ *... (i)*

Also equation of one directrix is given to be $x=\frac{36}{5}$

$\therefore \quad \frac{a}{e}=\frac{36}{5}$ *...(ii)*



(i) × (ii) $\Rightarrow a^2 = 36 \;\therefore a = 6$ *From (i),* $e=\frac{5}{6}$

Now, $b^2=a^2(1-e^2)=36\left(1-\frac{25}{36}\right)=11$

The required equation of the ellipse will be

$$\frac{(x-0)^2}{a^2}+\frac{(y-0)^2}{b^2}=1 \text{ or } \frac{x^2}{36}+\frac{y^2}{11}=1$$

Illustration :

Find the equation of the ellipse having major and minor axes along x and y axes respectively, the distance between whose foci is 8 units and the distance between the directrices is 18 units.

Sol. Given, 8 = distance between foci $= 2ae$ ... (1)

and 18 = distance betwwen directrices $= \dfrac{2a}{e}$ ... (2)

$$(1) \times (2) \Rightarrow (8)\,(18) = (2ae)\left(\frac{2a}{e}\right) = 4a^2$$

$\Rightarrow \quad a^2 = 36 \Rightarrow a = 6$ ... (3)

Again, $8 = 2ae = 2(6)e = 12e$

$\Rightarrow \quad e = \dfrac{8}{12} = \dfrac{2}{3}$ ... (4)

Also, $b^2 = a^2(1 - e^2) = (6)^2\left[1 - \left(\dfrac{2}{3}\right)^2\right] = 36\left(\dfrac{5}{9}\right) = 20$ ... (5)

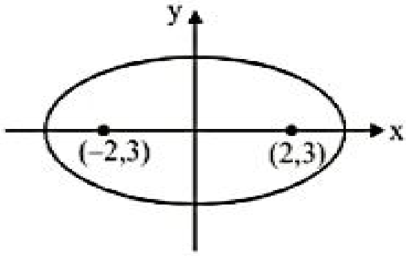
$\therefore$ Required equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ or } \frac{x^2}{36} + \frac{y^2}{20} = 1 \qquad \text{[From (3) and (5)]}$$

Illustration :

Find the equation of the ellipse whose foci are (2, 3), (–2, 3) and whose semi-minor axis is of length $\sqrt{5}$.

Sol. Let $S_1(2, 3)$ and $S_2(-2, 3)$ be the two foci and let $2a$ and $2b$ denote the lengths of major and minor axes respectively, then, $b = \sqrt{5}$ and $2ae = S_1S_2 = 4$, where e is the eccentricity of the ellipse.



$\therefore \quad ae = 2$

$$b^2 = a^2(1 - e^2) \;\Rightarrow\; b^2 = a^2 - a^2c^2$$

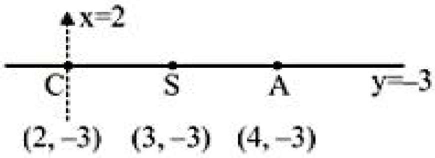
$\Rightarrow \quad S = a^2 - 4 \Rightarrow a = 3$

The major axes is $y = 3$ and centre is $(0, 3)$ – the mid point of the foci. Hence, equation of the ellipse is $\dfrac{x^2}{9} + \dfrac{(y-3)^2}{5} = 1$ **Ans.**

Illustration :

Find the equation of the ellipse having its centre at the point (2, –3), one focus at (3, –3) and one vertex at (4, –3).

Sol. Let $C = (2, -3)$, $S = (3, -3)$ and $A = (4, -3)$



Now $CA = \sqrt{(4-2)^2+(-3+3)^2} = 2 \therefore a = 2$

Again, $CS = \sqrt{(3-2)^2+(-3+3)^2} = 1$

$\therefore \quad ae = 1; \ \therefore\ e = \frac{1}{a} = \frac{1}{2} \quad \Rightarrow b^2 = a^2(1-e^2) = 4\left(1-\frac{1}{4}\right) = 3$

Major axis is $y = -3$ and parallel to x-axis

$\therefore$ Equation of ellipse is $\frac{(x-2)^2}{4}+\frac{(y+3)^2}{3} = 1$

Illustration :

Find the equation of axes, directrix, co-ordinate of focii, centre, vertices, length of latus - rectum and eccentricity of an ellipse $\frac{(x-3)^2}{25} + \frac{(y-2)^2}{16} = 1$.

Sol. Let $x - 3 = X$, $y - 2 = Y$, so equation of ellipse becomes as $\frac{X^2}{5^2} + \frac{Y^2}{4^2} = 1$,

$a = 5$, $b = 4$ and $a > b$

equation of major axis is $Y = 0 \quad \Rightarrow \quad y = 2$.

equation of minor axis is $X = 0 \quad \Rightarrow \quad x = 3$.

centre $(X = 0, Y = 0) \quad \Rightarrow \quad x = 3, y = 2$

$C \equiv (3, 2)$

Length of major axis $2a = 10$

Length of minor axis $= 2b = 8$.

Let 'e' be eccentricity, then

$\therefore \quad b^2 = a^2(1-e^2)$

$$e = \sqrt{\frac{a^2-b^2}{a^2}} = \sqrt{\frac{25-16}{25}} = \frac{3}{5}.$$

Length of latus rectum $= LL' = \frac{2b^2}{a} = \frac{2\times 16}{5} = \frac{32}{5}$

Co-ordinates focii are $X = \pm ae$, $Y = 0$

$x - 3 = \pm 5 \cdot \frac{3}{5}$ and $y - 2 = 0 \quad \Rightarrow \quad x = 3 \pm 3$ and $y = 2$ i.e. $(6, 2)$ and $(0, 2)$

Illustration :

Find the centre, the length of the axes, and the eccentricity of the ellipse
$2x^2+3y^2-4x+12y+13=0$.

Sol. $2x^2+3y^2-4x+12y+13=0$

$2(x^2-2x)+3(y^2+4y)+13=0$

$2(x-1)^2+3(y+2)^2=1$

$$\frac{(x-1)^2}{\left(\frac{1}{2}\right)}+\frac{(y+2)^2}{\left(\frac{1}{3}\right)}=1$$

$\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$; where $X=x-1$, $Y=y+2$, $a=\frac{1}{\sqrt{2}}$, $b=\frac{1}{\sqrt{3}}$ & $a>b$

Centre is : $X=0$, $Y=0$; $x=1$, $y=-2$

Length of major axis $=2a=\sqrt{2}$ *and* *Length of minor axis* $=2b=\frac{2}{\sqrt{3}}$

If e denotes the eccentricity, then, $b^2=a^2(1-e^2)$

$\therefore$ $\frac{1}{3}=\frac{1}{2}(1-e^2)$; $e=\frac{1}{\sqrt{3}}$ ***Ans.***

Illustration :

Find the centre, the length of the axes, eccentricity and the foci of the ellipse.
$12x^2+4y^2+24x-16y+25=0$

Sol. *The given equation can be written in the form*
$12(x+1)^2+4(y-2)^2=3$

$$\Rightarrow \quad \frac{(x+1)^2}{1/4}+\frac{(y-2)^2}{3/4}=1 \qquad ...(1)$$

here $a^2=\frac{1}{4}$ *and* $b^2=\frac{3}{4} \Rightarrow a<b$

Co- ordinates of centre of the ellipse are given by $x+1=0$ *and* $y-2=0$
Hence centre of the ellipse is $(-1, 2)$

$\therefore$ *Length of major axis* $=2a=\sqrt{3}$ *and* *Length of minor axis* $=2b=1$

i.e. $a=\frac{\sqrt{3}}{2}$, $b=\frac{1}{2}$

Since $b^2=a^2(1-e^2)$ $\therefore 1/4=3/4(1-e^2) \Rightarrow e=\sqrt{2/3}$ $\therefore ae=\frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{\sqrt{3}}=\frac{1}{\sqrt{2}}$

Co-ordinates of foci are given by $x+1=0$, $y-2=\pm ae$

Thus foci are $\left(-1, 2\pm\frac{1}{\sqrt{2}}\right)$ ***Ans.***

Illustration :

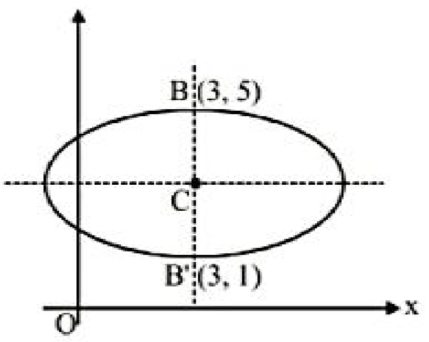
Find the equation of the ellipse the extremities of whose minor axis are (3, 1) and (3, 5) and whose eccentricity is 1/2.

Sol. *Let C be the centre of the ellipse.*

Let $B' \equiv (3, 1)$ *and* $B \equiv (3, 5)$*, then* $C \equiv (3, 3)$ *[Since C is the mid-point of BB']*

Also $BB' = 4$ $\therefore\ 2b = 4 \Rightarrow b = 2$

Also, slope of $BB' = \frac{5-1}{0}$ *(not defined)*



Hence minor axis is parallel to y-axis and therefore, major axis will be parallel to x-axis. Let a be the length of semi-major axis of the ellipse, then

$b^2 = a^2 (1 - e^2)$

$\therefore\quad 4 = a^2\left(1 - \frac{1}{4}\right)$ or $a^2 = \frac{16}{3}$

Since centre of the ellipse is (3, 3), therefore, its equation will be

$$\frac{(x-3)^2}{16/3} + \frac{(y-3)^2}{4} = 1$$

or $\quad 3x^2 + 4y^2 - 18x - 24y + 47 = 0$

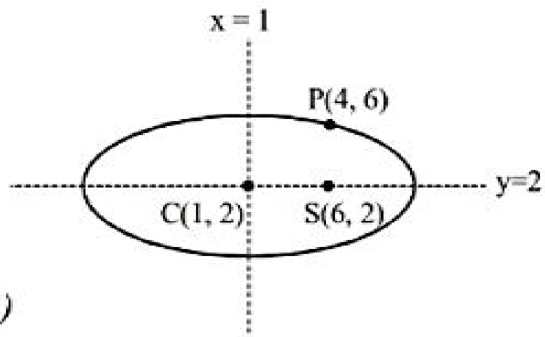
Illustration :

Find the equaiton of the ellipse with its centre at (1, 2), one focus at (6, 2) and passing through the point (4, 6).

Sol. *Let* $S \equiv (6, 2)$ *and* $C \equiv (1, 2)$*. Sloep of CS = 0, therefore major axis of the ellipse is parallel to x-axis and minor axis is parallel to y-axis.*

Since centre of the ellipse is (1, 2), therefore its equation will be of the form

$$\frac{(x-1)^2}{a^2} + \frac{(y-2)^2}{b^2} = 1 \qquad \dots (1)$$



Since (4, 6) lies on (1), therefore $\frac{(4-1)^2}{a^2} + \frac{(6-2)^2}{b^2} = 1$

or $\quad \frac{9}{a^2} + \frac{16}{b^2} = 1 \qquad \dots (2)$

Since ae = distnace between centre and focus = 5,

$\therefore\quad b^2 = a^2(1 - e^2) = a^2 - (ae)^2 = a^2 - 25 \qquad \dots (3)$

Substituting this value in (2), we have

$$\frac{9}{a^2} + \frac{16}{a^2 - 25} = 1 \text{ or } a^4 - 50a^2 + 225 = 0$$

or $\quad (a^2 - 45)(a^2 - 5) = 0$ or $a^2 = 45, 5.$

When $a^2 = 5$*, from (3),* $b^2 < 0$ *(not possible)*

$\therefore\quad a^2 = 45$*, and from (3),* $b^2 = 45 - 25 = 20$

Hence from (1), equation of required ellipse is $\frac{(x-1)^2}{45} + \frac{(y-2)^2}{20} = 1$

Practice Problem

Single correct question

Q.1 The eccentricity of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose latus rectum is half of its major axis, is

(A) $\frac{1}{\sqrt{3}}$ (B) $\frac{1}{\sqrt{2}}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\sqrt{\left(\frac{2}{3}\right)}$

Q.2 The length of the major axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is three times the length of minor axis, it eccentricity is

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{\sqrt{3}}$ (C) $\frac{2\sqrt{2}}{3}$ (D) $\frac{2\sqrt{2}}{5}$

Q.3 The centre of the ellipse $\frac{(x+y-2)^2}{9}+\frac{(x-y)^2}{16}=1$, the locus of its pole is

(A) (0, 0) (B) (1, 0) (C) (0, 1) (D) (1, 1)

Q.4 The equation $\frac{x^2}{10-a}+\frac{y^2}{4-a}=1$ represents an ellipse, if

(A) $a<4$ (B) $a>4$ (C) $4<a<10$ (D) $a>10$

Q.5 The equation, $2x^2+3y^2-8x-18y+35=K$ represents
(A) no locus if $K>0$ (B) an ellipse if $K<0$
(C) a point if $K=0$ (D) a hyperbola if $K>0$

Q.6 If $\tan\alpha\tan\beta=-\frac{a^2}{b^2}$, then the chord joining two points α and β on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ will subtend a right angle at
(A) focus (B) centre (C) end of major axis (D) end of minor axis

Q.7 The eccentric angle of one end of a diameter of $x^2+3y^2=3$ is $\frac{\pi}{6}$, then the eccentric angle of the other end will be

(A) $\frac{5\pi}{6}$ (B) $-\frac{5\pi}{6}$ (C) $-\frac{2\pi}{3}$ (D) $\frac{2\pi}{3}$

Q.8 The eccentricity of the ellipse $(x-3)^2+(y-4)^2=\frac{y^2}{9}$ is

(A) $\frac{\sqrt{3}}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{3\sqrt{2}}$ (D) $\frac{1}{\sqrt{3}}$

Q.9 F_1 and F_2 are the two foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$. Let P be a point on the ellipse such that $|PF_1| = 2|PF_2|$, where F_1 and F_2 are the two foci of the ellipses. The area of ΔPF_1F_2 is

(A) 3 (B) 4 (C) $\sqrt{5}$ (D) $\frac{\sqrt{13}}{2}$

Q.10 A circle has the same centre as an ellipse & passes through the foci F_1 & F_2 of the ellipse, such that the two curves intersect in 4 points. Let 'P' be any one of their point of intersection. If the major axis of the ellipse is 17 & the area of the triangle PF_1F_2 is 30, then the distance between the foci is :

(A) 11 (B) 12 (C) 13 (D) none

Q.11 Let S(5, 12) and S'(– 12, 5) are the foci of an ellipse passing through the origin. The eccentricity of ellipse equals

(A) $\frac{1}{2}$ (B) $\frac{1}{\sqrt{3}}$ (C) $\frac{1}{\sqrt{2}}$ (D) $\frac{2}{3}$

Q.12 An ellipse is inscribed in a circle and a point within the circle is chosen at random. If the probability that this point lies outside the ellipse is 2/3 then the eccentricity of the ellipse is :

(A) $\frac{2\sqrt{2}}{3}$ (B) $\frac{\sqrt{5}}{3}$ (C) $\frac{8}{9}$ (D) $\frac{2}{3}$

Multiple correct type question

Q.13 Consider the ellipse $\frac{x^2}{\tan^2\alpha}+\frac{y^2}{\sec^2\alpha}=1$ where $\alpha \in (0, \pi/2)$.

Which of the following quantities would vary as α varies?

(A) degree of flatness (B) ordinate of the vertex

(C) coordinates of the foci (D) length of the latus rectum

Integer type question

Q.14 Find the latus rectum, eccentricity, co-ordinates of the foci, co-ordinates of the vertices, the length of the axes and the centre of the ellipse

$$4x^2 + 9y^2 - 8x - 36y + 4 = 0$$

Q.15 The y-axis is the directrix of the ellipse with eccentricity $e = 1/2$ and the corresponding focus is at (3, 0), then find the equation to its auxilary circle.

Q.16 Find the eccentricity of the ellipse which meets the straight line $2x - 3y = 6$ on the X-axis and the straight line $4x + 5y = 20$ on the Y-axis and whose principal axes lie along the coordinate axes.

Q.17 A bar of length 20 units moves with its ends on two fixed straight lines at right angles. A point P marked on the bar at a distance of 8 units from one end describes a conic. Find its eccentricity.

Q.18 If one extremity of the minor axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the foci form an equilateral triangle, then find its eccentricity.

Q.19 There are exactly two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose distance from the centre of the ellipse are greatest and equal to $\sqrt{\frac{a^2+2b^2}{2}}$. Then find the eccentricity of this ellipse.

Answer key

Q.1	B	Q.2	C	Q.3	D	Q.4	A	Q.5	C
Q.6	B	Q.7	B	Q.8	B	Q.9	B	Q.10	C
Q.11	C	Q.12	A	Q.13	A, B, D				

Q.14 $\frac{8}{3}, \frac{\sqrt{5}}{3}$ $(1 \pm \sqrt{5}, 2)$; $(-2, 2)$ and $(4, 2)$; 6 and 4; $(1, 2)$ Q.15 $x^2 + y^2 - 8x + 12 = 0$

Q.16 $e=\frac{\sqrt{7}}{4}$ Q.17 $e=\frac{\sqrt{5}}{3}$ Q.18 $e=\frac{1}{2}$ Q.19 $e=\frac{1}{\sqrt{2}}$

POSITION OF A POINT w.r.t. AN ELLIPSE :

Let $S(x, y)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$ be the given ellipse

and $P(x_1, y_1)$ is the given point.

(i) If $S(x_1, y_1) > 0$ then $P(x_1, y_1)$ lie outside the ellipse.

(ii) If $S(x_1, y_1) < 0$ then $P(x_1, y_1)$ lie inside the ellipse.

(iii) If $S(x_1, y_1) = 0$ then $P(x_1, y_1)$ lie on the ellipse.

This result holds true for circle and parabola also.

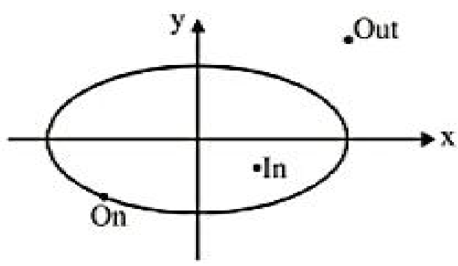


Illustration :

Check whether the point P(3, 2) lies inside, on or outside of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1.$

Sol. $S(3, 2) \equiv \frac{9}{25}+\frac{4}{16}-1=\frac{9}{25}+\frac{1}{4}-1<0$

$\therefore$ *Point P(3, 2) lies inside the ellipse.* ***Ans.***

Illustration :

Find the set of values of 'α' for which the point P(α, – α) lies inside the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

Sol. *If P(α, – α) lies inside the ellipse then*

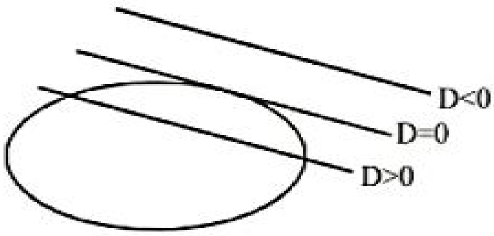
$$\therefore \quad S(\alpha, -\alpha) < 0 \Rightarrow \frac{\alpha^2}{16} + \frac{\alpha^2}{9} - 1 < 0 \Rightarrow \frac{25}{144} \cdot \alpha^2 < 1$$

$$\Rightarrow \quad \alpha^2 < \frac{144}{25}; \quad \therefore \alpha \in \left(-\frac{12}{5}, \frac{12}{5}\right). \quad \textit{Ans.}$$

INTERACTION OF A LINE AND AN ELLIPSE :

Let the equations of the line is $y = mx + c$ ......(1)

and equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ......(2)



The points of intersection of the line and the ellipse can be obtained by solving the two equations simultaneously. Hence by eliminating y from (1) & (2), we get

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1.$$

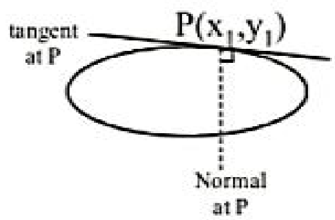
i.e. $(b^2 + a^2m^2)\, x^2 + 2a^2\, cmx + a^2\, (c^2 - b^2) = 0$ ......(3)

Let x_1, x_2 be the roots of the quadratic equation (3). The line meets the ellipse in real and distinct points if the roots x_1 and x_2 are real and different. The line is a tangent to the ellipse if $x_1 = x_2$ and the line does not meet the ellipse if the roots x_1 and x_2 are imaginary. All these will be decided by the discriminant of quadratic equation (3).

TANGENTS :

(i) Point form :

$\frac{x x_1}{a^2} + \frac{y y_1}{b^2} = 1$ is tangent to the ellipse at (x_1, y_1).



Since point (x_1, y_1) lie on the curve therefore we can use standard substitution to obtain the equation of tangent.

(ii) Slope form :

Let the given line is $y = mx + c$ and given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

If line touch is the ellipse then by solving the two equations simultaneously (by eliminating y from (1) & (2)), we get $\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$.

i.e. $\quad (b^2 + a^2m^2)\,x^2 + 2a^2\,cmx + a^2\,(c^2 - b^2) = 0 \qquad$ ......(3)

Since line is tangent to the ellipse therefore its $D = 0$

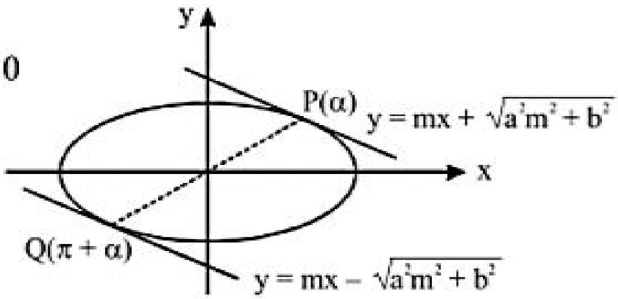
$$4a^4c^2m^2 - 4\,(b^2 + a^2m^2)\cdot a^2(c^2 - b^2) = 0$$

or $\quad 4a^2\,[a^2c^2m^2 - b^2c^2 - a^2c^2m^2 + b^4 + a^2b^2m^2] = 0$

or $\quad b^2\,(-c^2 + b^2 + a^2m^2) = 0$

or $\quad c^2 = b^2 + a^2m^2$ or $c = \pm\sqrt{a^2m^2 + b^2}$

which is the required **condition of tangency.**



Substituting this value of c in $y = mx + c$, we have

$y = mx + \sqrt{a^2m^2 + b^2}$ or $y = mx - \sqrt{a^2m^2 + b^2}$, which are tangents to the ellipse for all values of m.

Here ± sign represents two tangents to the ellipse having the same m, i.e. there are two tangents parallel to any given direction.

The equation of any tangent to the ellipse $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

$$(y - k) = m\,(x - h) \pm \sqrt{a^2m^2 + b^2}$$

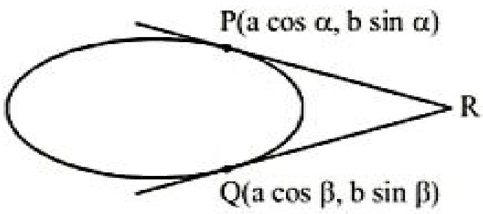
(iii) Parametric form :

$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ is tangent to the ellipse at the point $(a\cos\theta, b\sin\theta)$.

NOTE:

(i) Point of intersection of the tangents at the point α & β is $\left(a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}},\, b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$ can be deduced by comparing chord joining $P(\alpha)$ and $Q(\beta)$ with C.O.C. of the pair of tangents from $R(x_1, y_1)$ on the ellipse, where

$$x_1 = a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\ ;\ y_1 = b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$$



(ii) The eccentric angles of point of contact of two parallel tangents differ by π. Conversely if the difference between the eccentric angles of two points is π then the tangents at these points are parallel.

Illustration :

Find the equations of the tangents to the ellipse $3x^2 + 4y^2 = 12$ which are perpendicular to the line $y + 2x = 4$.

Sol. Slope of tangent to the given line $= -2$

Given ellipse is $\frac{x^2}{4} + \frac{y^2}{3} = 1$

Equation of tangent whose slope is 'm' is $y = mx \pm \sqrt{4m^2+3}$

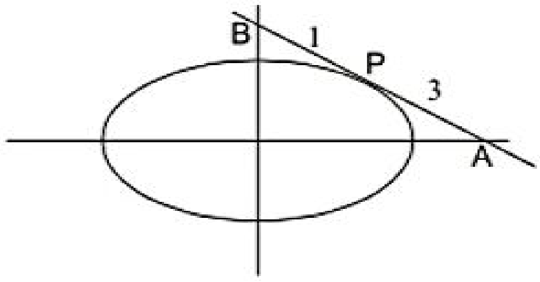
$\because \quad m = \frac{1}{2} \qquad \therefore \quad y = \frac{1}{2}x \pm \sqrt{1+3}$

$2y = x \pm 4$ **Ans.**

Illustration :

A tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ touches at the point P on it in the first quadrant and meets the co-ordinate axes in A and B respectively. If P divides AB in the ratio 3 : 1, find the equation of the tangent.

Sol. Let $P = (a\cos\theta, b\sin\theta)$



$\therefore$ equation of tangent is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

$A = (a\sec\theta, 0)$

$B = (0, b\,\text{cosec}\,\theta)$

$\because$ P divide AB internally in the ratio 3 : 1

$\therefore \quad a\cos\theta = \frac{a\sec\theta}{4} \quad \Rightarrow \quad \cos^2\theta = \frac{1}{4} \quad \Rightarrow \quad \cos\theta = \frac{1}{2}$

and $\quad b\sin\theta = \frac{3b\,\text{cosec}\,\theta}{4} \quad \Rightarrow \quad \sin\theta = \frac{\sqrt{3}}{2}$

$\therefore$ tangent is $\frac{x}{2a} + \frac{\sqrt{3}y}{2b} = 1 \quad \Rightarrow \quad bx + \sqrt{3}\,ay = 2ab$ **Ans.**

Illustration :

How many real tangents can be drawn from the point (4, 3) to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$. Find the equation these tangents & angle between them.

Sol. Given point $P = (4, 3)$

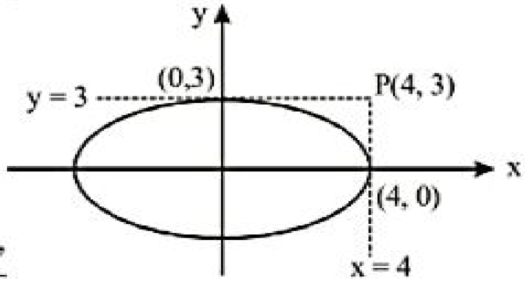
ellipse $\quad S = \frac{x^2}{16} + \frac{y^2}{9} - 1 = 0$

$\because \quad S_1 = \frac{16}{16} + \frac{9}{9} - 1 = 1 > 0$

$\Rightarrow$ *Point P = (4, 3) lies outside the ellipse.*

$\therefore$ *Two tangents can be drawn from the point P(4, 3).*

Equation of pair of tangents is



$$SS_1 = T^2 \Rightarrow \left(\frac{x^2}{16}+\frac{y^2}{9}-1\right) \cdot 1 = \left(\frac{4x}{16}+\frac{3y}{9}-1\right)^2$$

$$\Rightarrow \quad \frac{x^2}{16} + \frac{y^2}{9} - 1 = \frac{x^2}{16} + \frac{y^2}{9} + 1 + \frac{xy}{6} - \frac{x}{2} - \frac{2y}{3}$$

$\Rightarrow \quad -xy + 3x + 4y - 12 = 0$

$\Rightarrow \quad (4-x)(y-3) = 0 \quad \Rightarrow \quad x = 4 \;\&\; y = 3$

and angle between them $= \dfrac{\pi}{2}$ ***Ans.***

Illustration :

Find the equations of the tangents to the ellipse $x^2 + 16y^2 = 16$ *each one of which makes an angle of* 60^o *with the x- axis.*

Sol. *We have,* $x^2 + 16y^2 = 16 \quad \Rightarrow \quad \dfrac{x^2}{4^2} + \dfrac{y^2}{1^2} = 1$

This is of the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, *where*

$a^2 = 16$ *and* $b^2 = 1$

So, the equations of the tangents are

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

i.e. $\quad y = \sqrt{3}\,x \pm \sqrt{16\times3+1} \Rightarrow y = \sqrt{3}\,x \pm 7$ ***Ans.***

Illustration :

For what value of λ *does the line* $y = x + \lambda$ *touches the ellipse* $9x^2 + 16y^2 = 144$.

Sol. $\because$ *Equation of ellipse is*

$$9x^2 + 16y^2 = 144 \quad \text{or} \quad \frac{x^2}{16} + \frac{y^2}{9} = 1$$

Comparing this with $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ *then we get* $a^2 = 16$ *and* $b^2 = 9$

& comparing the line $y = x + \lambda$ *with* $y = mx + c$

$\therefore \quad m = 1$ *and* $c = \lambda$

therefore condition of tangency $c^2 = a^2m^2 + b^2 \Rightarrow \lambda^2 = 16 \times 1^2 + 9 \Rightarrow \lambda^2 = 25$

$\therefore \quad \lambda = \pm 5$ ***Ans.***

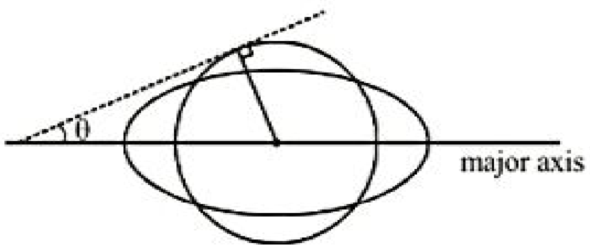
Illustration :

A circle of radius r is concentric with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. *Prove that the common tangent is inclined to the major axis at an angle* $\tan^{-1}\left(\sqrt{\frac{r^2-b^2}{a^2-r^2}}\right)$.

Sol. *Let the equation of circle is* $x^2+y^2=r^2$ ......(i)

& $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ......(ii)

Tangent to be ellipse is $y=mx+\sqrt{a^2m^2+b^2}$ *i.e.* $mx-y+\sqrt{a^2m^2+b^2}=0$. *Since it is tangent to the circle also, therefore perpendicular distance from centre (0, 0) will be equal to radius r,*



$\therefore \quad \left|\frac{\sqrt{a^2m^2+b^2}}{\sqrt{m^2+1}}\right|=r$

$\Rightarrow \quad a^2m^2+b^2=r^2(1+m^2)$

$\Rightarrow \quad m^2=\frac{r^2-b^2}{a^2-r^2}$

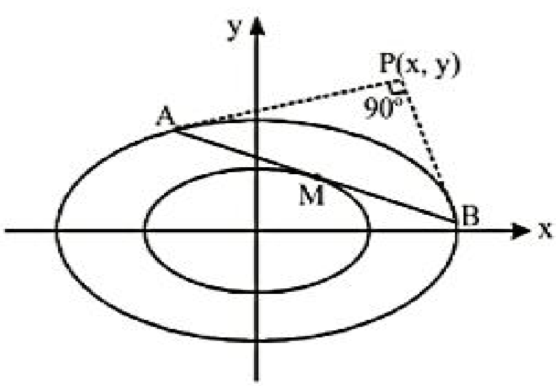
$\therefore \quad m=\tan\theta=\sqrt{\frac{r^2-b^2}{a^2-r^2}}$

$\therefore \quad \theta=\tan^{-1}\sqrt{\frac{r^2-b^2}{a^2-r^2}}$ ***Ans.***

Illustration :

Show that the tangents drawn at those points of the ellipse $\frac{x^2}{a}+\frac{y^2}{b}=(a+b)$, *where it is cut by any tangent to* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, *intersect at right angles.*

Sol. *Given ellipse are* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...... (i)



& $\frac{x^2}{a(a+b)}+\frac{y^2}{b(a+b)}=1$ ......(ii)

The chord of contact (x_1, y_1) *w.r.t. (ii) ellipse is*

$\frac{xx_1}{a(a+b)}+\frac{yy_1}{b(a+b)}=1$

i.e. $$y = \frac{-bx_1}{ay_1}x + \frac{b(a+b)}{y_1} \qquad \text{......(iii)}$$

(iii) is tangent to ellipse (i) which is also given as

$$y = mx + \sqrt{a^2m^2 + b^2} \qquad \text{...(iv)}$$

Hence, (iii) & (iv) is identical

$$\therefore \quad m = \frac{bx_1}{ay_1} \quad \text{...(v)} \qquad \& \qquad a^2m^2 + b^2 = \frac{b^2(a+b)^2}{y_1^2} \quad \text{...(vi)}$$

Now eliminate m from (v) & (vi), we get

$$x_1^2 + y_1^2 = (a+b)^2$$

$\therefore$ Locus of $P(x_1, y_1)$ is $x^2 + y^2 = (a + b)^2$ **Ans.**

Illustration :

Prove that in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ the locus of the middle points of the portions of tangents included between the axes is the curve $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 4$.

Sol. *Equation of any tangent to the given ellipse may be taken as*

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1 \qquad \text{... (1)}$$

Let this tangent meet the x-axis in P and y-axis in Q.

Putting $y = 0$ in (1), we get $x = a\sec\theta$.

$\therefore$ Co-ordinates of P are $(a\sec\theta, 0)$

Similarly co-ordinates of Q are $(0, b\,\mathrm{cosec}\,\theta)$

Let (h, k) be the mid. point of PQ.

$$\therefore \quad h = \frac{a\sec\theta + 0}{2} = \frac{a\sec\theta}{2} \quad \text{and} \quad k = \frac{0 + b\,\mathrm{cosec}\,\theta}{2} = \frac{b\,\mathrm{cosec}\,\theta}{2}$$

Hence $$2\cos\theta = \frac{a}{h} \qquad \text{... (2)}$$

and $$2\sin\theta = \frac{b}{k} \qquad \text{... (3)}$$

Squaring and adding (2) and (3), we have $4 = \frac{a^2}{h^2} + \frac{b^2}{k^2}$.

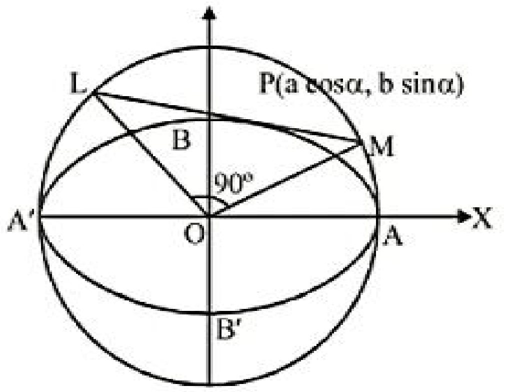
$\therefore$ Locus of (h, k) is $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 4$.

Illustration :

The tangent at any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the auxiliary circle in two points which subtend a right angle at the centre. Show that the eccentricity of the ellipse is $\frac{1}{\sqrt{1+\sin^2\alpha}}$.

Sol. Given ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{... (1)}$$



Its auxiliary circle is

$$x^2 + y^2 = a^2 \qquad \text{... (2)}$$

Let $P \equiv (a \cos \alpha, b \sin \alpha)$

Equation of tangent to the ellipse at $P(a \cos \alpha, b \sin \alpha)$ is

$$\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}=1 \qquad \text{...(3)}$$

Making equation (2) homogeneous with the help of (3), we get

$$x^2+y^2-a^2\left(\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}\right)^2=0$$

or $\quad (1-\cos^2\alpha)x^2+\left(1-\frac{a^2}{b^2}\sin^2\alpha\right)y^2-2\frac{a}{b}\cos a \sin a\, xy=0 \text{ ... (4)}$

(4) is the joint equaiton of OL and OM.

Since $\angle LOM = 90^o$

$\therefore \quad$ Coefficient of x^2 + coeff. of $y^2 = 0$

$\therefore \quad 1-\cos^2\alpha+1-\frac{a^2}{b^2}\sin^2\alpha=0$

or $\quad \sin^2\alpha\left(\frac{a^2}{b^2}-1\right)$ or $\sin^2\alpha\left(\frac{1}{1-e^2}-1\right)=1 \qquad [\because b^2=a^2(1-e^2)]$

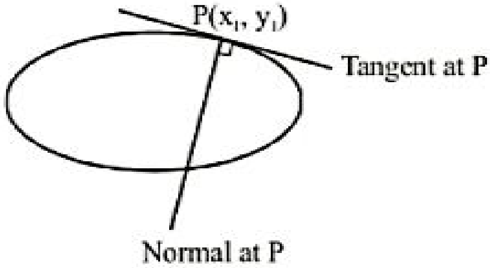
or $\quad e^2\sin^2\theta = 1-e^2$ or $e^2(1+\sin^2\alpha)=1$

or $\quad e=\frac{1}{\sqrt{1+\sin^2\alpha}}$

NORMALS :

(i) Point form :

Equation of the tangent to the ellipse at (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.



The slope of the tangent at $(x_1, y_1) = \frac{-x_1}{a^2} \times \frac{b^2}{y_1}$

$\therefore$ Slope of the normal at $(x_1, y_1) = \frac{a^2}{x_1} \times \frac{y_1}{b^2} = \frac{a^2 y_1}{b^2 x_1}$

Hence the equation of the normal at (x_1, y_1) is $y - y_1 = \frac{a^2 y_1}{b^2 x_1}(x - x_1)$

or $\quad \dfrac{x - x_1}{\frac{x_1}{a^2}} = \dfrac{y - y_1}{\frac{y_1}{b^2}}$

(ii) Parametric form :

In above equation if we put $x = a\cos\theta$ and $y = b\sin\theta$ then we will get normal equation in parametric form.

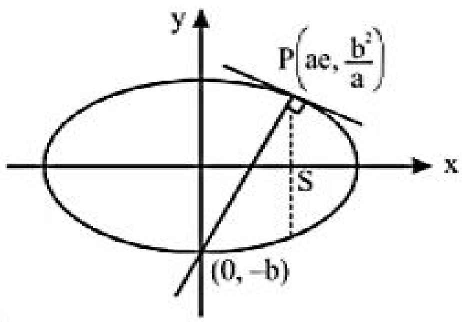
$\therefore \quad ax\sec\theta - by\,\mathrm{cosec}\,\theta = a^2 - b^2 = a^2e^2$

This is equation of normal in parametric form.

(iii) Equation of a normal in terms of its slope 'm' is $y = mx - \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2m^2}}$.

Illustration :

If the normal at one end of a latus-rectum of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through other extremity of the minor axis, then show that $e^2 = \frac{\sqrt{5} - 1}{2}$.



Sol. *Equation of normal at* $P\left(ae, \frac{b^2}{a}\right)$ *is* $\dfrac{x - ae}{\frac{ae}{a^2}} = \dfrac{y - \frac{b^2}{a}}{\frac{b^2/a}{b^2}}$

$\Rightarrow \quad \frac{ax}{e} - ya = a^2 - b^2$

Since it passes through (0, –b)

$\Rightarrow \quad 0 + ab = a^2 - b^2 \qquad \Rightarrow \quad a^2b^2 = (a^2 - b^2)^2$

$\Rightarrow \quad a^2 \cdot a^2(1 - e^2) = (a^2 \cdot e^2)^2 \qquad \Rightarrow \quad e^4 + e^2 - 1 = 0$

$\Rightarrow \quad e^2 = \frac{\sqrt{5} - 1}{2}$ ***Ans.***

Illustration :

Find the condition that the line lx + my = n may be a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Sol. *Equation of normal to the given ellipse at (a cosθ, b sinθ) is*

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2 \qquad ...(1)$$

If the line lx + my = n is also normal to the ellipse then there must be a value of θ for which line (1) and line lx + my = n are identical. For that value of θ we have

$$\frac{l}{\left(\frac{a}{\cos\theta}\right)}=\frac{m}{\left(\frac{-b}{\sin\theta}\right)}=\frac{n}{(a^2-b^2)} \quad \text{or} \quad \frac{l}{a}\cos\theta=-\frac{m\sin\theta}{b}=\frac{n}{(a^2-b^2)}$$

$$\therefore \quad \cos\theta=\frac{an}{l.(a^2-b^2)} \qquad ...(3)$$

$$\text{and} \quad \sin\theta=\frac{-bn}{(a^2-b^2)m} \qquad ...(4)$$

Squaring and adding (3) and (4), we get

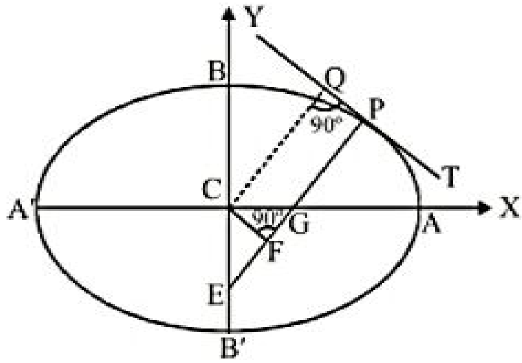
$$1=\frac{n^2}{(a^2-b^2)^2}\left(\frac{a^2}{l^2}+\frac{b^2}{m^2}\right)$$ *which is the required solution.*

Illustration :

If the nromal at any point P of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *meets the major and minor axes at G and E. respectively, and if CF is perpendicular upon this normal from the centre C of the ellipse, show that : PF.PG =* b^2 *and PF.PE =* a^2

Sol. *Given ellipse is*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad ...(1)$$



Let P(a cos θ, b sin θ) be any point on ellipse (1)
Equation of normal to ellipse (1) at P is

$$ax\sec\theta-by\,\mathrm{cosec}\,\theta=a^2-b^2 \qquad ...(2)$$

since line (2) meets the major axis (x – axis) and minor axis (y – axis) at G and E respectively, therefore

$$G\equiv\left(\frac{a^2-b^2}{a}\cos\theta,0\right) \quad \text{and } E\equiv\left(0,\frac{-(a^2-b^2)}{b}\sin\theta\right)$$

PF = CQ ≡ length of perp. form C(0, 0) on the tangent at P,

i.e. on the line $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}-1=0$

$$=\frac{|-1|}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}}=\frac{ab}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}} \quad \text{... (3)}$$

Also $PG=\sqrt{\left\{\frac{(a^2-b^2)\cos\theta}{a}-a\cos\theta\right\}^2+(0-b\sin\theta)^2}$

$$=\sqrt{\frac{b^4\cos^2\theta}{a^2}+b^2\sin^2\theta}=\frac{b}{a}\sqrt{b^2\cos^2\theta+a^2\sin^2\theta} \quad \text{... (4)}$$

$$PE=\sqrt{a^2\cos^2\theta+\left(b\sin\theta+\frac{a^2-b^2}{b}\sin\theta\right)^2}$$

$$=\sqrt{a^2\cos^2\theta+\frac{a^4\sin^2\theta}{b^2}}=\frac{a}{b}\sqrt{b^2\cos^2\theta+a^2\sin^2\theta} \quad \text{... (5)}$$

From (3) and (4), $PF.PG = b^2$
From (3) and (5), $PF.PE = a^2$

DIRECTOR CIRCLE :

Locus of the point of intersection of the tangents which meet at right angles is called the **Director Circle**.

Let equation of any tangent is $y=mx+\sqrt{a^2m^2+b^2}$
If it passes through (h, k) then

$$k=mh\pm\sqrt{a^2m^2+b^2}$$
$$(k-mh)^2=a^2m^2+b^2$$
$$(h^2-a^2)m^2-2khm+k^2-b^2=0 \quad \text{....(3)}$$

Equation (3) has two roots m_1 & m_2

$$m_1+m_2=\frac{2hk}{h^2-a^2} \quad \text{... (4)}$$

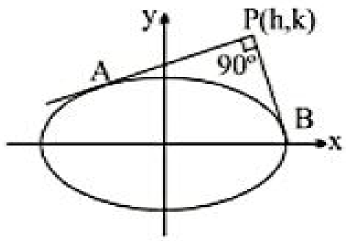
$$m_1m_2=\frac{k^2-b^2}{h^2-a^2} \quad \text{... (5)}$$

Hence passing through a given point there can be a maximum of two tangents.

If $PA \perp PB$ then $m_1 m_2 = -1$

i.e. $m_1 m_2 = \dfrac{k^2 - b^2}{h^2 - a^2} = -1$

i.e. $k^2 - b^2 = a^2 - h^2$; i.e. $x^2 + y^2 = a^2 + b^2$



which is the director circle of the ellipse. Hence **director circle of an ellipse is a circle** whose centre is the centre of ellipse and whose radius is the length of the line joining the ends of the major and minor axis.

Equation (3) can be used to determine the locus of the point of intersection of two tangents enclosing.

If from any point P (h, k) pair of tangents are drawn to the ellipse which include an angle α, then

$$\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \frac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2}$$

By putting value of $m_1 + m_2$ and $m_1 m_2$ in above equation we will get the angle between pair of tangents.

Note :

If a right triangle, right angled at A circumscribes an ellipse then locus of the point A is the director circle of the ellipse.

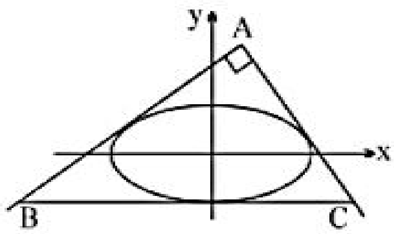


Illustration :

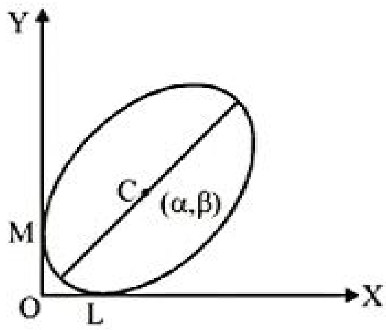
An ellipse slides between two lines at right angles to another. Show that the locus of its centre is a circle.

Sol. *Let the two given perpendicular lines be taken as the x and y axes respectivley.*

Let $C(\alpha, \beta)$ be the centre of the ellipse in any position. Here the position of centre C changes as the ellipse slides.

Let a and b be the semimajor and minor axes of the elipse.
Equation of the director circle of the ellipse is

$$(x - \alpha)^2 + (y - \beta)^2 = a^2 + b^2 \qquad \text{... (1)}$$



Since OX and OY are mutually perpendicular tangents to sliding ellipse for all its positions, therefore, O(0, 0) will lie on its director circle (1)

$\therefore \quad \alpha^2 + \beta^2 = a^2 + b^2$

Hence locus of C(a, b) is $x^2 + y^2 = a^2 + b^2$ .... (2)

CHORD OF CONTACT :

Pair of tangents drawn from outside point P (x_1, y_1) to the ellipse which meet it at A and B. Now line joining A and B is called the chord of contact of point P (x_1, y_1) w.r.t. the ellipse.
The equation of chord of contact is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$

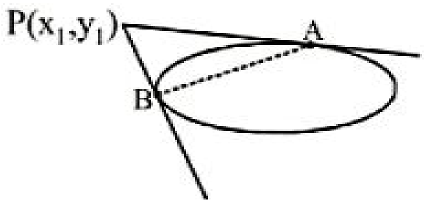


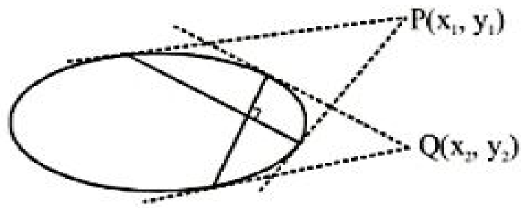
Illustration :

If the chords of contact of tangents from two points (x_1,y_1) and (x_2,y_2) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are at right angles, then Find the value of $\frac{x_1x_2}{y_1y_2}$.

Sol. *The equations of the chords of contact of tangents drawn from (x_1,y_1) and (x_2,y_2) to the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ are } \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \qquad \dots (i)$$

$$\frac{xx_2}{a^2}+\frac{yy_2}{b^2}=1 \qquad \dots (ii)$$



It is given that (i) and (ii) are at right angles.

$$\therefore \frac{-b^2}{a^2}\frac{x_1}{y_1}x\frac{-b^2}{a^2}\frac{x_2}{y_2}=-1 \quad \Rightarrow \frac{x_1x_2}{y_1y_2}=-\frac{a^4}{b^4} \qquad \textit{Ans.}$$

PAIR OF TANGENTS :

Pair of tangents PA and PB are drawn from outside point P (x_1, y_1), which is shown below. Hence joint equation of line PA and PB is given by $SS_1 = T^2$.

Here $\quad S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}-1$

$$S_1 \equiv \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$$

$$T \equiv \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$$

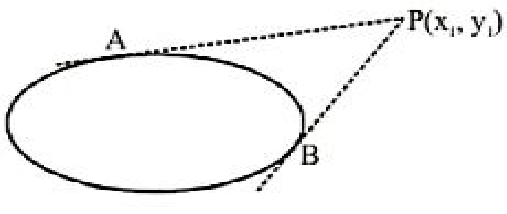


Illustration :

Find the locus of point of intersection of perpendicular tangents to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

Sol. *Let P(h, k) be the point of intersection of two perpendicular tangents*

$$\Rightarrow \quad \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)=\left(\frac{hx}{a^2}+\frac{ky}{b^2}-1\right)^2$$

$$\Rightarrow \quad \frac{x^2}{a^2}\left(\frac{k^2}{b^2}-1\right)+\frac{y^2}{a^2}\left(\frac{h^2}{a^2}-1\right)+\ldots\ldots=0 \qquad \ldots\ldots\ldots(i)$$

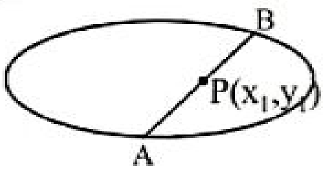
Since equation (i) represents two perpendicular lines

$$\therefore \quad \frac{1}{a^2}\left(\frac{k^2}{b^2}-1\right)+\frac{1}{b^2}\left(\frac{h^2}{a^2}-1\right)=0$$

$\Rightarrow \quad k^2-b^2+h^2-a^2=0 \qquad \Rightarrow \qquad$ *locus is* $x^2+y^2=a^2+b^2$ ***Ans.***

CHORD WITH A GIVEN MIDDLE POINT :

Here chord AB is shown in the figure whose mid point is $P(x_1, y_1)$.
Then equation of this chord AB is $T=S_1$



Here $\quad S_1\equiv\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}-1$

$$T\equiv\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$$

Illustration :

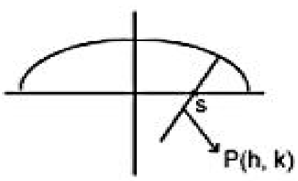
Find the locus of the mid - point of focal chords of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1.$

Sol. *Let P = (h, k) be the mid-point*

$\therefore \quad$ *Equation of chord whose mid-point is given* $\dfrac{xh}{a^2}+\dfrac{yk}{b^2}-1=\dfrac{h^2}{a^2}+\dfrac{k^2}{b^2}-1$

Since it is a focal chord, therefore it passes through focus, either (ae, 0) or (–ae, 0)
If it passes through (ae, 0)

$\therefore \quad$ *locus is* $\dfrac{ex}{a}=\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}$



If it passes through (–ae, 0)

$\therefore \quad$ *locus is* $-\dfrac{ex}{a}=\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}$ ***Ans.***

Practice Problem

Single correct question

Q.1 If $\frac{x}{a}+\frac{y}{b}=\sqrt{2}$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ then its, eccentricity angle θ is equal to

(A) 0 (B) 45° (C) 60° (D) 90°

Q.2 The equations of the tangents to the ellipse $3x^2+y^2=3$ making equal intercepts on the axes are

(A) $y=\pm x\pm 2$ (B) $y=\pm x\pm 4$ (C) $y=\pm x\pm \sqrt{30}$ (D) $y=\pm x\pm \sqrt{35}$

Q.3 The common tangent of $x^2+y^2=4$ and $2x^2+y^2=2$ is

(A) $x+y+4=0$ (B) $x-y+7=0$ (C) $2x+3y+8=0$ (D) None of these

Q.4 If the normal at any point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the axes in G and g respectively, then

PG : Pg =

(A) a : b (B) $a^2 : b^2$ (C) b : a (D) $b^2 : a^2$

Q.5 From the point $(\lambda, 3)$ tangents are drawn to $\frac{x^2}{9}+\frac{y^2}{4}=1$ and are perpendicular to each other than λ is

(A) ±1 (B) ±2 (C) ±3 (D) ±4

Q.6 The locus of the point of intersection of two perpendicular tangents of the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ is

(A) $x^2+y^2=4$ (B) $x^2+y^2=9$ (C) $x^2+y^2=13$ (D) $x^2+y^2=5$

Q.7 The angle between the pair of tangents drawn from the point (1, 2) to the ellipse $3x^2+2y^2=5$ is

(A) $\tan^{-1}\left(\frac{12}{5}\right)$ (B) $\tan^{-1}\left(\frac{6}{\sqrt{5}}\right)$ (C) $\tan^{-1}\left(\frac{12}{\sqrt{5}}\right)$ (D) $\tan^{-1}(12\sqrt{5})$

Q.8 If a quadrilateral is formed by four tangents to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ is a square,

then the area of the square is equal to

(A) 26 (B) 24 (C) 22 (D) 20

Q.9 The Locus of the middle point of chords of an ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$ passing through P(0, 5) is another ellipse E. The coordinates of the foci of the ellipse E, is

(A) $\left(0,\frac{3}{5}\right)$ and $\left(0,\frac{-3}{5}\right)$ (B) $(0,-4)$ and $(0,1)$

(C) $(0,4)$ and $(0,1)$ (D) $\left(0,\frac{11}{2}\right)$ and $\left(0,\frac{-1}{2}\right)$

Q.10 The area of the rectangle formed by the perpendiculars from the centre of the standard ellipse to the tangent and normal at its point whose eccentric angle is $\pi/4$ is :

(A) $\frac{\left(a^2-b^2\right)ab}{a^2+b^2}$ (B) $\frac{(a^2-b^2)}{(a^2+b^2)ab}$ (C) $\frac{\left(a^2-b^2\right)}{ab\left(a^2+b^2\right)}$ (D) $\frac{a^2+b^2}{\left(a^2-b^2\right)ab}$

Integer type question

Q.11 $x-2y+4=0$ is a common tangent to $y^2=4x$ & $\frac{x^2}{4}+\frac{y^2}{b^2}=1$. Then find the value of b and also find the other common tangent.

Q.12 If a tangent having slope 2 of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a>0, b>0)$ is normal to the circle $x^2+y^2+4x+1=0$, then find the maximum value of ab.

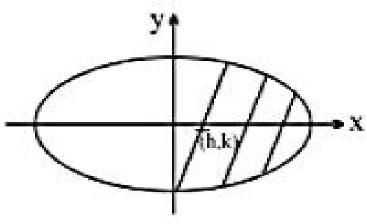
Answer key

Q.1	B	Q.2	A	Q.3	D	Q.4	D	Q.5	B
Q.6	B	Q.7	C	Q.8	A	Q.9	C	Q.10	A
Q.11	$b=\sqrt{3}$; $x+2y+4=0$			Q.12	4				

DIAMETER :

The locus of the middle points of a system of parallel chords with slope 'm' of an ellipse is a straight line passing through the centre of the ellipse, called its diameter and has the equation

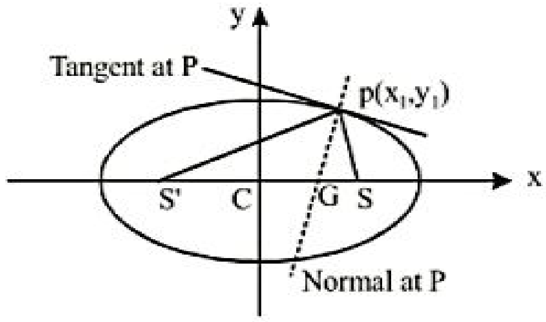
$$y=-\frac{b^2}{a^2m}x.$$



Chord of contact, pair of tangents, chord with a given middle point, pole & polar are to be interpreted as same as they are in parabola.

Properties of the ellipse :

(1) Let the equation of an ellipse is



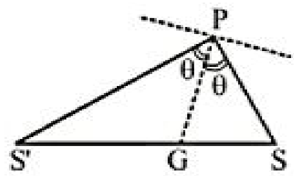
$$\frac{x^2}{a^2}+\frac{y^2}{b^2} \qquad \ldots(i)$$

& Normal at $P(x_1, y_1)$ is

$$\frac{x-x_1}{x_1/a^2}=\frac{y-y_1}{y_1/b^2} \qquad \ldots(ii)$$

The normal meet x-axis at G $\Rightarrow$ Put $y = 0$ in equation (ii) we get

$$CG = x = \left(\frac{a^2-b^2}{a^2}\right)x_1 = e^2x_1$$



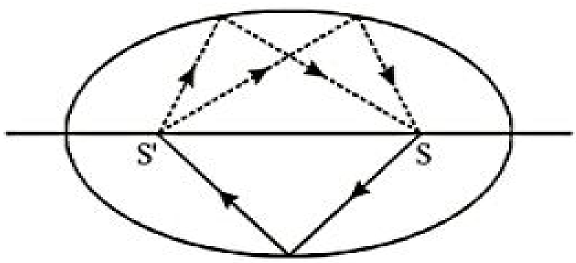
$\therefore$ $SG = CS - CG = ae - e^2x_1 = e(a - ex_1) = eSP$

Similarly $S'G = eS'P$

$\therefore$ $$\frac{SG}{S'G}=\frac{eSP}{eS'P}=\frac{SP}{S'P}$$

$\Rightarrow$ PG is bisector of angle $\angle P$ in $\Delta S'PS$.

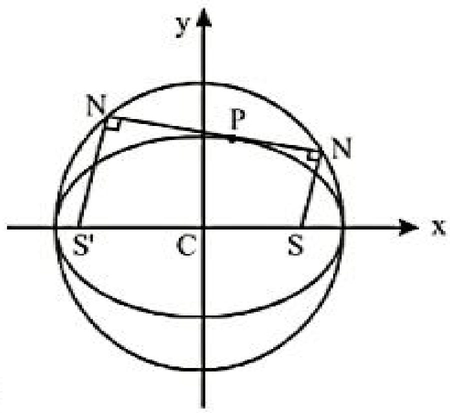
Tangent & normal at any point P bisect the external & internal angles between the focal distances of SP & S'P.



This lead to reflextion property of ellipse.

If incoming light ray passes through focus S'(or S), strike the concave side of ellipse the after reflextion it will pass through other focus.

(2) Let ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(i)



its tangent is $y-mx=\sqrt{a^2m^2+b^2}$ ...(ii)

Equation of perpendicular to above the passes through focus (ae, 0) is

$$my + x = ae \qquad \ldots(iii)$$

Eliminate m from (ii) & (iii) we will get focus of intersection point.

For that square & add (ii) & (iii) we will get an answer

$\therefore \quad y^2 + m^2x^2 - 2mxy + x^2 + my^2 + 2mxy = a^2m^2 + b^2 + a^2e^2$

$\Rightarrow \quad x^2(1 + m^2) + y^2(1 + m^2) = a^2m^2 + a^2$

$\Rightarrow \quad x^2 + y^2 = a^2$ which is the auxilairy circle

The locus of the point of interseciton of feet of perpendicular from foci on any tangent to an ellipse is the auxiliary circle.

(3) From previous equation of any tangent is $mx - y + \sqrt{a^2m^2 + b^2} = 0$

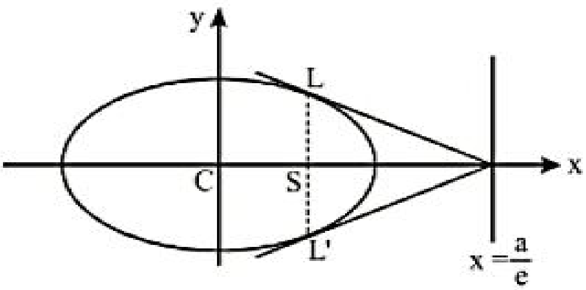
$$SN = \left|\frac{\sqrt{a^2m^2+b^2} + ame}{\sqrt{1+m^2}}\right| \quad \& \quad S'N' = \left|\frac{\sqrt{a^2m^2+b^2} - ame}{\sqrt{1+m^2}}\right|$$

$$SN \cdot S'N' = \frac{(a^2m^2+b^2) - a^2m^2e^2}{(1+m^2)} = \frac{a^2m^2+b^2-(a^2-b^2)m^2}{1+m^2} = b^2 \qquad \because a^2e^2 = a^2 - b^2$$

The product of perpendicular distance from the foci to any tangent of an ellipse is equal to square of the semi minor axis.

(4) Equation of tangent at $L\left(ae, \frac{b^2}{a}\right)$ is

$$\frac{x(ae)}{a^2} + \frac{y\left(\frac{b^2}{a}\right)}{b^2} = 1 \Rightarrow xe + y = a \quad \ldots(i)$$



& equation of tangent at $L'\left(ae, -\frac{b^2}{a}\right)$ is

$$ex - y = a \quad \ldots(ii)$$

Solve (i) & (ii) we get $x = \frac{a}{e}$ & $y = 0$ i.e. at the directrix of ellipse.

Tangents at the extremities of latus-rectum of an ellipse intersect on the corresponding directrix.

(5) If P be any point on the ellipse with S & S′ as its foci then $\ell(SP) + \ell(S'P) = 2a$.

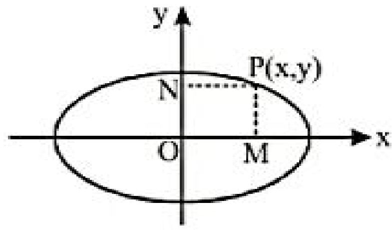
(6) If the normal at any point P on the ellipse with centre C meet the major & minor axes in G & g respectively & if CF be perpendicular upon this normal then

(i) $PF. PG = b^2$ (ii) $PF. Pg = a^2$

(7) The portion of the tangent to an ellipse between the point of contact & the directrix subtends a right angle at the corresponding focus.

Equation of an ellipse referred to two perpendicular lines :

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is given ellipse}$$



Let P(x, y) be any point on the ellipse, then PM = y & PN = x

$\therefore$ above equation can be written as $\dfrac{PN^2}{a^2}+\dfrac{PM^2}{b^2}=1$

From above we conclude that if perpendicular distances p_1 & p_2 of a moving point P(x, y) from two mutually perpendicular straight lines $L_1 \equiv lx+my+n_1=0$ & $L_2 \equiv mx-ly+n_2=0$ respectively then equation of ellipse in the plane of line will be

$$\frac{p_2^2}{a^2}+\frac{p_1^2}{b^2}=1$$

$$\Rightarrow \frac{\left(\frac{mx-ly+n_2}{\sqrt{l^2+m^2}}\right)^2}{a^2}+\frac{\left(\frac{lx+my+n_1}{\sqrt{l^2+m^2}}\right)^2}{b^2}=1$$

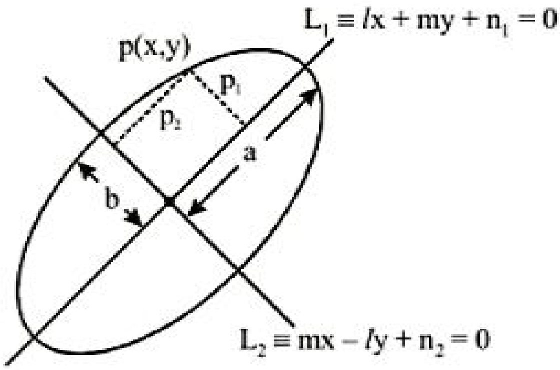


Illustration :

Find the equation the ellipse whose axis are of length 6 & $2\sqrt{6}$ & their equations are $x-3y+3=0$ & $3x+y-1=0$.

Sol. *Equation of ellipse will be*

$$\frac{\left(\frac{x-3x+3}{\sqrt{10}}\right)^2}{(\sqrt{6})^2}+\frac{\left(\frac{3x+y-1}{\sqrt{10}}\right)^2}{(3)^2}=1$$

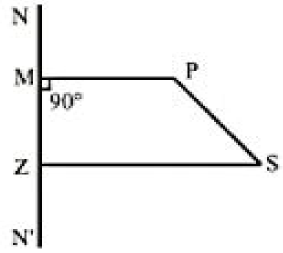
HYPERBOLA

DEFINITON OF HYPERBOLA :

A hyperbola is the locus of a point which moves in a plane such that the ratio of its distance from a fixed point and a given straight line is always constant.

The fixed point is called the focus, the fixed line is called the directrix and the constant ratio is called the eccentricity of the hyperbola and denoted by e.

In the given figure, S is the focus and N'N the directrix.



Let P be any point on the hyperbola, then $\frac{PS}{PM} = e, e > 1$.

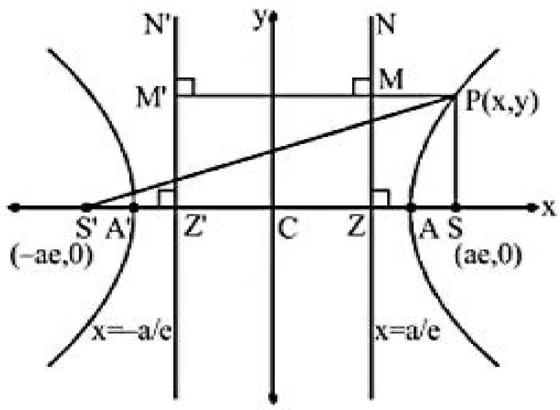
Equation of a hyperbola can be obtained if the coordinates of its focus, equation of its directrix and eccentricity are given.

STANDARD EQUATION OF A HYPERBOLA :

Let S be the focus & ZN is the directrix of an ellipse. Draw perpendicular from S to the directrix which meet it at Z. A moving point is on the hyperbola such that

$$PS = ePM$$

then there is point lies on the line SZ and which divide SZ internally at A and externally at A' in the ratio of e : 1.



therefore $\quad SA = e\,AZ \quad \ldots(i)$

$\quad SA' = eA'Z \quad \ldots(ii)$

Let AA' = 2a & take C as mid point of AA'

$\therefore \quad CA = CA' = a$

Add (i) & (ii)

$$SA + SA' = e\,(AZ + A'Z)$$

$$(CS - CA) + (CA' + CS) = e\,[CA - CZ + CA' + CZ]$$

$$2CS = 2e.\ CA$$

$$CS = ae$$

Subtract (ii) & (i), we get

$$SA' - SA = e\,(A'Z - AZ)$$

$$(CA' + CS) - (CS - CA) = e\,[(CA' + CZ) - (CA - CZ)]$$

$$2CA = 2e \cdot CZ \Rightarrow CZ = \frac{a}{e}.$$

Consider CZ line as x-axis, C as origin & perpendicular to this line & passes through C is considered as y-axis. Now represent important parameters on coordinates plane. Let P(x, y) is a moving point, then By defintion of ellipse.

PS = ePM $\Rightarrow$ $(PS)^2 = e^2 (PM)^2$

$$\Rightarrow \quad (x - ae)^2 + (y - 0)^2 = e^2\left(x - \frac{a}{e}\right)^2 \Rightarrow (x - ae)^2 + y^2 = (a - ex)^2$$

$$\Rightarrow \quad x^2 + a^2e^2 - 2xae + y^2 = a^2 + e^2x^2 - 2xae \Rightarrow x^2 (1 - e^2) + y^2 = a^2 (1 - e^2)$$

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1 \quad \text{or} \quad \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1 \quad \text{where } b^2 = a^2 (e^2 - 1)$$

Hence equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $b^2 = a^2 (e^2 - 1)$

TRACING OF HYPERBOLA :

Equation of given hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ....(1)

(i) If we put y = 0, then we see that hyperbola cuts x-axis at $(\pm a, 0)$

(ii) If we put x = 0, then $y^2 = -b^2$. Hence hyperbola does not cut y-axis.

(iii) When y is replaced by – y, the equation of hyperbola does not change, hence hyperbola is symmetric about x-axis. (Since equation contain even power of y therefore curve will be symmetric about x-axis.

(iv) When x is replaced by – x, then equation of hyprbola does not change, hence hyperbola is symmetric about y-axis. (Since equation contain even power of x therefore curve will be symmetric about y-axis.

(v) From (1), $y = \pm \frac{b}{a}\sqrt{x^2 - a^2}$, since y is real $\Rightarrow x^2 - a^2 \ge 0$ or $x \in (-\infty, -a] \cup [a, \infty)$

$\Rightarrow$ curve don't lie in $(-a, a)$

For each $x \ge a$ or $x \le -a$ there are two values of y symmetrically situated on both side of x-axis. Hence, the curve denoted by (1) consist of two symmetrical branches, each extending to infinite in two direction.

FACTS ABOUT THE HYPERBOLA :

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

(1) By symmetry of equation of hyperbola, if we take second focus (–ae, 0) and second directrix $x = -\frac{a}{e}$ and perform same calculation then we get same equation of hyperbola. This suggest that their are two foci are (ae, 0) and (– ae, 0) and the two corresponding directrices as $x = \frac{a}{e}$ and $x = -\frac{a}{e}$. If focus is (ae, 0), then corresponding directrix is $x = \frac{a}{e}$ and if focus is (– ae, 0), then corresponding directrix is $x = -\frac{a}{e}$.

(2) By definition of hyperbola, the distance of any point P on the hyperbola from focus
= e(the distance of P from the corresponding directrix)

(3) Distance between foci $SS' = 2ae$ & distance between directrix $ZZ' = 2\frac{a}{e}$.

(4) Two hyperbola are said to be similar if they have same eccentricity.

(5) The hyperbola has two branches neither of them cut the y-axis (conjugate axis).

(6) Since the fundamental equation to the hyperbola only differs from that to the ellipse is having $-b^2$ instead of b^2. It is observed that many propositions for the hyperbola are derived from those for the ellipse by simply changing the sign of b^2.

(7) Eccentricity $e = \sqrt{1 + \frac{b^2}{a^2}}$. Also, $b^2 = a^2(e^2 - 1)$.

The smaller the e, the smaller will be the value of b for a given a. Therefore, as e decreases for a given a, the branches of the hyperbola would be bending towards x-axis. As e increases, the branches open up.

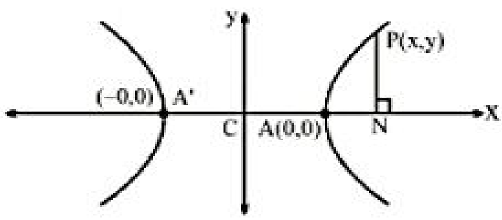
(8) Equation of hyperbola when its transverse and conjugate axes are x and y-axes respectively is

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1 \quad \text{or} \quad x^2 - \frac{y^2}{(e^2-1)} = a^2$$

If e kept constant and $a \to 0$, then hyperbola will tend to pair of straight lines $x^2 - \frac{y^2}{(e^2-1)} = 0$, both passing through the origin.

Thus in the situated limiting case of a hyperbola is a pair of straight lines.

(9) Since $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



$$\therefore \quad \frac{y^2}{b^2} = \frac{x^2}{a^2} - 1 = \frac{(x-a)(x+a)}{a^2}$$

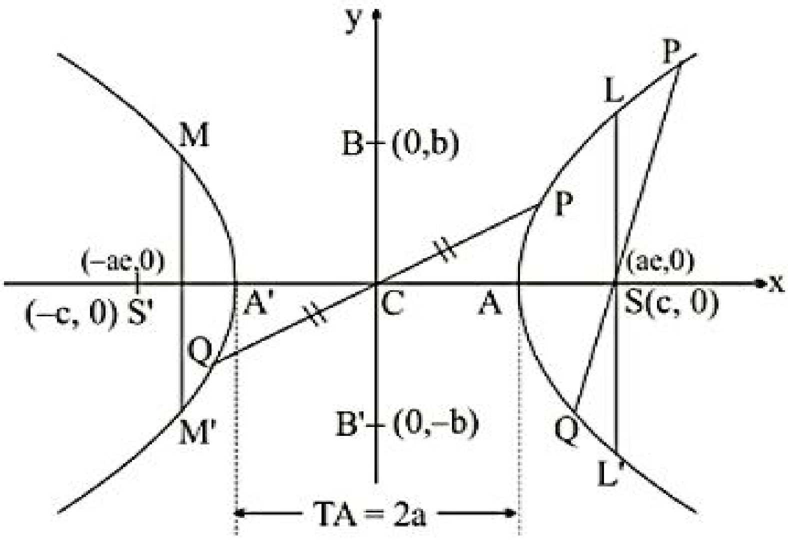
From figure, AN = CN – CA = x – a
A'N = CN + CA = x + a and PN = y

$$\therefore \quad \frac{(PN)^2}{} = \frac{b^2}{}$$

TERMS RELATED TO HYPERBOLA :

(1) Centre :

In the figure, C is the centre of the ellipse. All chords passing through C are called diameter and bisected at C.



(2) Foci :

S(ae, 0) and S'(–ae, 0) are two foci of hyperbola. Line containing the fixed points S and S' (called Foci) is called Transverse Axis (TA) or Focal Axis and the distance between S and S' is called Focal Length.

(3) Axes :

The line AA' is called transverse axis and the line BB' is perpendicular to it and passes through the centre (0, 0) of the hyperbola is called conjugate axis.

The length of transverse and conjugate axes are taken as 2a and 2b respectively.

The transverse and conjugate axes together are called principal axes of hyperbola and their intersection point is called the centre of hyperbola.

The points of intersection of the directrix with the transverse axis are known as Foot of the directrix (Z and Z').

(4) Vertex :

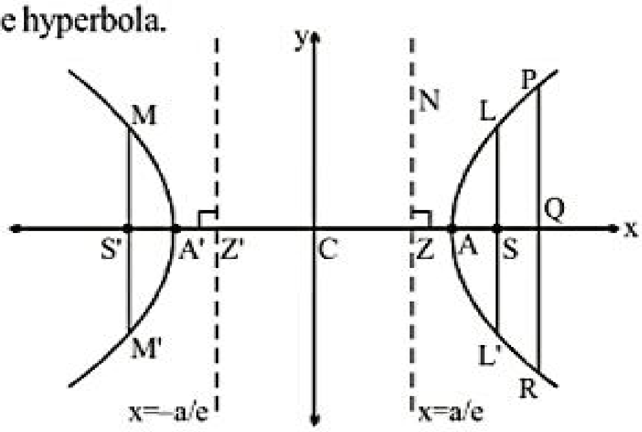
The points of intersection (A, A') of the curve with the transverse axis are called Vertices of the hyperbola.

(5) Double ordiante :

Any chord perpendicular to the Transverse axis is called a Double Ordinate.

(6) Latus-rectum :

When double ordinates passes through the focus of parabola then it is called the latus rectum. In the given figure LL' and MM' are the latus-rectums of the hyperbola.



let LL' = 2k then LS = L'S = k

Let L(ae, k) lie on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\therefore \quad \frac{a^2e^2}{a^2} - \frac{k^2}{b^2} = 1$$

$$\text{or} \quad k^2 = b^2(e^2 - 1) = b^2\left(\frac{b^2}{a^2}\right) \qquad [\because b^2 = a^2(e^2 - 1)]$$

$$\therefore \quad k = \frac{b^2}{a} \qquad (\because k > 0)$$

$$\therefore \quad 2k = \frac{2b^2}{a} = LL'$$

$$\text{Length of latus rectum} = LL' = \frac{2b^2}{a} = \frac{(CA)^2}{TA} = 2a(e^2 - 1) = 2e\left(ae - \frac{a}{e}\right)$$

= (2e) (distance between the focus and the foot of the corresponding directrix)

End points of latus -rectums are

$$L = \left(ae, \frac{b^2}{a}\right), L' = \left(ae, -\frac{b^2}{a}\right); M = \left(-ae, \frac{b^2}{a}\right); M' = \left(-ae, -\frac{b^2}{a}\right) \text{ respectively.}$$

(7) Focal chord :

A chord of hyperbola passing through its focus is called a focal chord.

ECCENTRICITY :

For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ we have

$$b^2 = a^2(e^2 - 1)$$

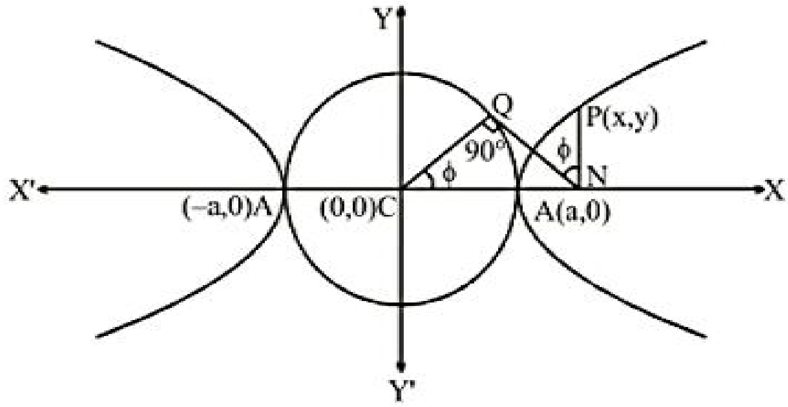
$$\Rightarrow \quad e = \sqrt{1 + \left(\frac{b^2}{a^2}\right)} \Rightarrow e = \sqrt{\left\{1 + \frac{(\text{conjugate axis})^2}{(\text{transverse axis})^2}\right\}}$$

Eccentricity defines the curvature of the hyperbola and is mathematically spelled as :

$$e = \frac{\text{distance from centre to focus}}{\text{distance from centre to vertex}}$$

PARAMETRIC EQUATIONS OF THE HYPERBOLA :

Let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be the hyperbola with centre C and transverse axis A'A. Therefore circle drawn with centre C and segment A'A as a diameter is called auxiliary circle of the hyperbola.



$\therefore$ Equation of the auxiliary circle is

$$x^2 + y^2 = a^2$$

Let P(x, y) be any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Draw PN perpendicular to x-axis.

Let NQ be a tangent to the auxiliary circle $x^2 + y^2 = a^2$. Join CQ and let $\angle QCN = \phi$ then P and Q are the corresponding points of the hyperbola and the auxiliary circle. Here ϕ is the eccentric angle of P. $(0 \le \phi < 2\pi)$.

Since $\quad Q = (a \cos \phi, a \sin \phi)$

Now $\quad x = CN = CQ \sec \phi = \sec \phi \cdot a$

$\therefore \quad P(x, y) = (a \sec \phi, y)$

$\because \quad$ P lies on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\therefore \quad \frac{a^2 \sec^2 \phi}{a^2} - \frac{y^2}{b^2} = 1 \text{ or } \frac{y^2}{b^2} = \sec^2 \phi - 1 = \tan^2 \phi$$

$\therefore \quad y = \pm b \tan \phi$

$\therefore \quad y = b \tan \phi \qquad$ (P lies in I quadrant)

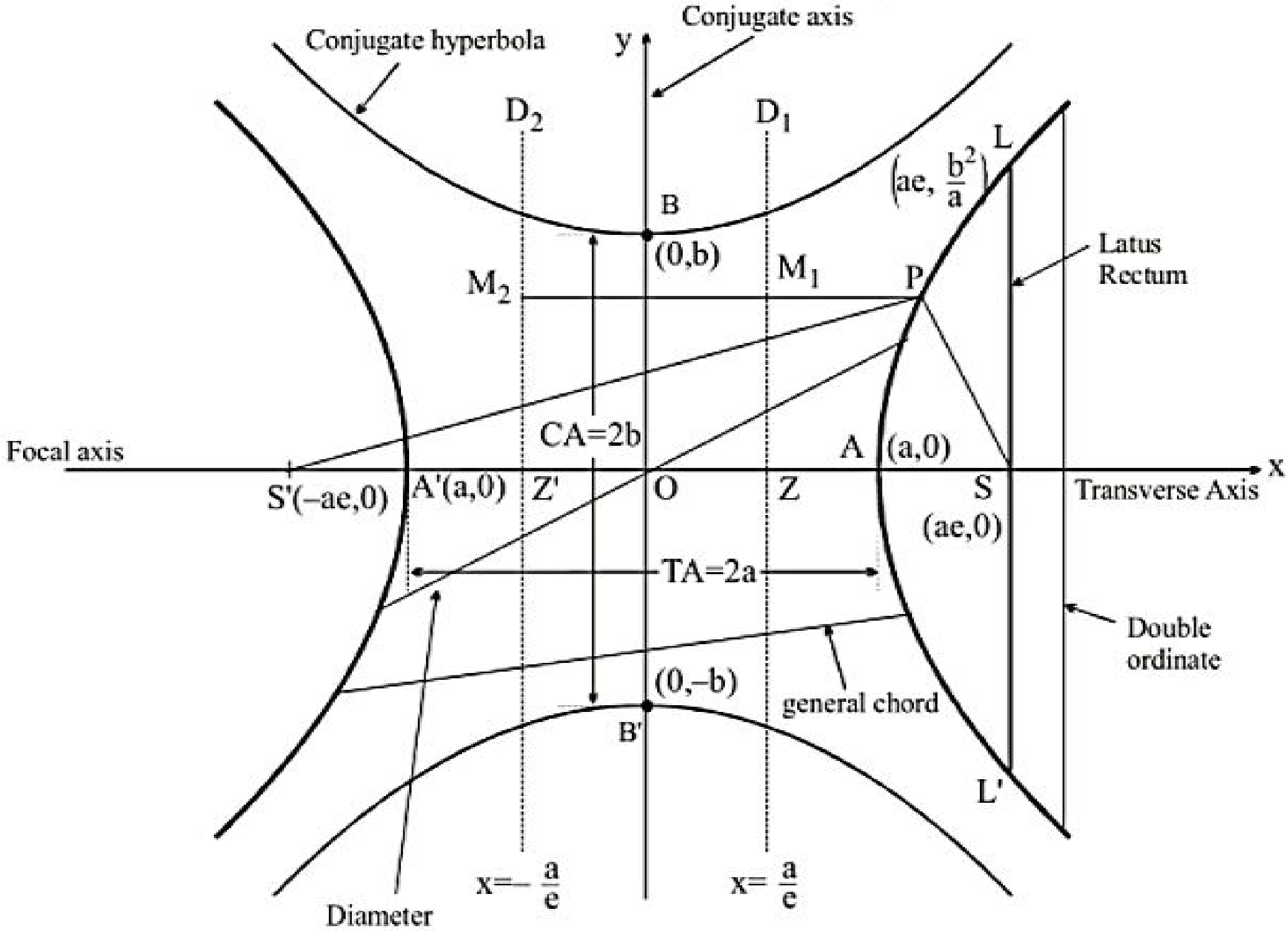
The equations of $x = a \sec \phi$ and $y = \tan \phi$ are known as the parametric equations of the hyperbola.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

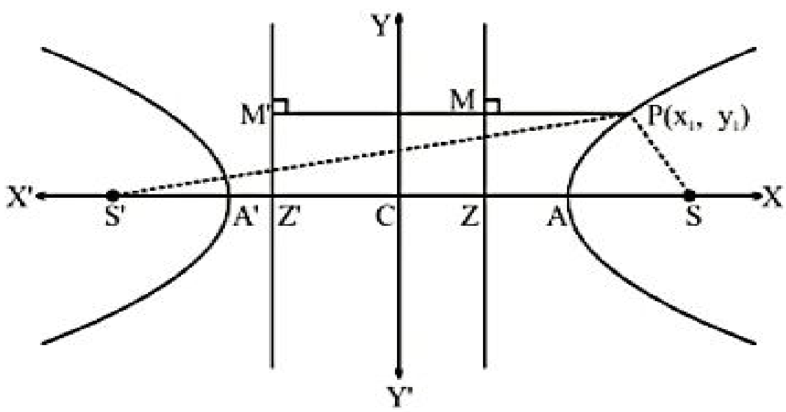
Position of points Q an auxiliary circle and corresponding point P which describes the hyperbola or shown below in the table. Here $0 \le \phi < 2\pi$.

ϕ varies from	$Q(a \cos \phi, a \sin \phi)$	$P(a \sec \phi, b \tan \phi)$
0 to $\frac{\pi}{2}$	I	I
$\frac{\pi}{2}$ to π	II	III
π to $\frac{3\pi}{2}$	III	II
$\frac{3\pi}{2}$ ro 2π	IV	IV

HYPERBOLA AT A GLANCE :



FOCAL DISTANCE OF A POINT ON HYPERBOLA :



The hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ....(1)

The foci S and S' are (ae, 0) and (– ae, 0) & corresponded directrix are $x = \frac{a}{e}$ and $x = -\frac{a}{e}$ respectively.

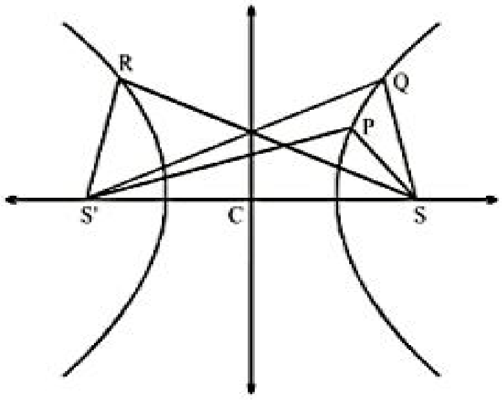
Let $P(x_1, y_1)$ be any point on (1).

Now SP = ePM $= e\left(x_1 - \frac{a}{e}\right) = ex_1 - a$

and $\quad S'P = ePM' = e\left(x_1 + \frac{a}{e}\right) = ex_1 + a$

$\therefore \quad S'P - SP = (ex_1 + a) - (ex_1 - a) = 2a$

$= AA' = $ Transverse axis



Thus hyperbola is the locus of a point which moves in a plane such that the difference of its distances from two fixed point (foci) is constant and always equal to transverse axis.

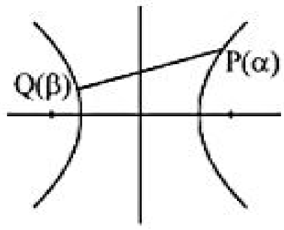
Hence, in the given figure

PS' – PS = QS' – QS = RS – RS' = length of transverse axis.

CHORD OF HYPERBOLA :

Chord joining two points with eccentric angles α and β is given by

$$\frac{x}{a}\cos\frac{\alpha-\beta}{2} - \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha+\beta}{2} \qquad ....(1)$$



if (1) passes through (d, 0) then

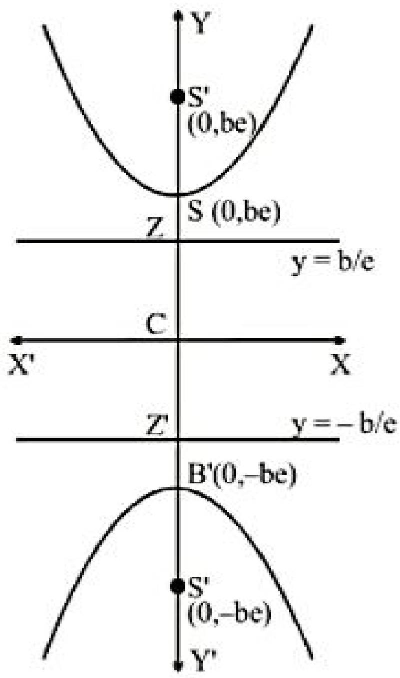
$$\frac{d}{a}\cos\frac{\alpha-\beta}{2} = \cos\frac{\alpha+\beta}{2}$$

$$\frac{d}{a} = \frac{\cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2}}$$

$$\frac{d+a}{d-a} = -\frac{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}}{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}} \quad \Rightarrow \quad \tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{a-d}{a+d}$$

CONJUGATE HYPERBOLA :

Corresponding to every hyperbola there exist a hyperbola such that, the conjugate axis and transverse axis of one is equal to the transverse axis and conjugate axis of other, such hyperbolas are known as conjugate to each other.



Hence for the hyperbola, $\quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad .....(1)$

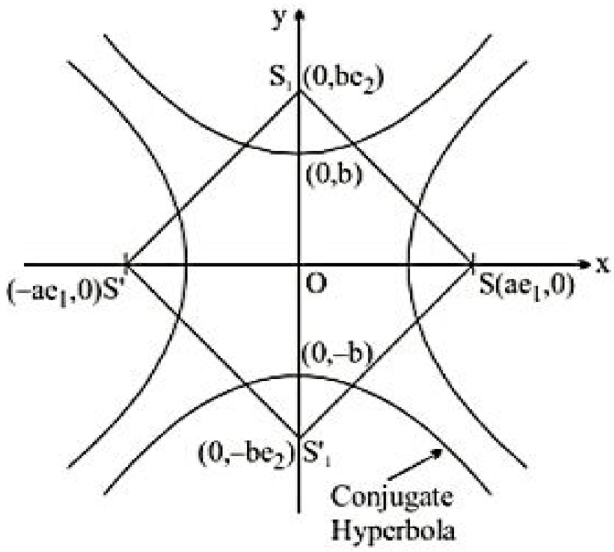
the conjugate hyperbola is, $\quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \qquad ....(2)$

Comparison between hyperbola and its conjugate hyperbola		
Basic Elements	**Hyperbola**	**Conjugate Hyperbola**
	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$
Centre	(0, 0)	(0, 0)
Length of transeverse axis	2a	2b
Length of conjugate axis	2b	2a
Eccentricity	$b^2 = a^2(e^2-1)$	$a^2 = b^2(e^2-1)$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrix	$x = \pm \frac{a}{e}$	$y = \pm \frac{b}{e}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Parametric coordinate	$(a \sec \phi, b \tan \phi)$ $0 < \phi < 2\pi$	$(a \tan \phi, b \sec \phi)$ $0 \le \phi < 2\pi$
Focal distances	$ex_1 \pm a$	$ey_1 \pm b$
Diffference of focal distances	2a	2b
Equation of transverse axis	$y = 0$	$x = 0$
Equation of conjugates	$x = 0$	$y = 0$
Tangent at vertices	$x = \pm a$	$y = \pm b$

Note :

(1) The foci of a hyperbola and its conjugate are concylic and form the vertices of a square.

(2) If e_1 & e_2 are the eccentrcities of the hyperbola & its conjugate then $e_1^{-2} + e_2^{-2} = 1$.



RECTANGULAR HYPERBOLA :

If the lengths of transverse and conjugate axes of any hyperbola be equal then it is called rectangular or equilateral hyperbola.
Since length of transverse axis and conjugate axis are same i.e. $a = b$

then, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ becomes $x^2 - y^2 = a^2$.

Eccentricity, $e = \sqrt{\left(1 + \frac{b^2}{a^2}\right)} = \sqrt{2}$.

All the results of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are applicable to the hyperbola $x^2 - y^2 = a^2$ after changing b by a.

FIND ALL THE PARAMETERS OF A HYPERBOLA $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$:

When equation of the hyperbola is $\frac{(x-\alpha)^2}{a^2} - \frac{(y-\beta)^2}{b^2} = 1$

This equation is the form of $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$, where $X = x - h$ and $Y = y - k$

(1) Length of semi-transverse axis = a, length of semi-conjugate axis = b

(2) Equation of transverse axis is $Y = 0$, i.e, $y - k = 0$
Equation of conjugate axis is $X = 0$, i.e, $x - h = 0$

(3) Coordinates of centre is given by $X = 0$ and $Y = 0$, i.e., $x - h = 0$ and $y - k = 0$
Therefore, centre is (h, k)

(4) Eccentricity of the hyperbola $e = \sqrt{1 + \frac{b^2}{a^2}}$

(5) Coordinates of vertices of the hyperbola are given by $X = \pm a, Y = 0$ i.e., $x - h = \pm a, y - k = 0$. Hence vertices are $(h \pm a, k)$.

(6) Coordinate of foci are given by $X = \pm ae, Y = 0$
i.e., $x - h = \pm ae, y - k = 0$. Hence foci are $(h \pm ae, k)$

(7) Equation of directrices of the hyperbola are $X = \pm \dfrac{a}{e}$, i.e., $x - h = \pm \dfrac{a}{e}$.

Hence directrices are $x = h \pm \dfrac{a}{e}$

(8) Length of latus rectum $= \dfrac{2b^2}{a}$

(9) Coordinate of ends of latera recta are given by $X = ae, Y = \pm \dfrac{b^2}{a}$ i.e. $x - h = \pm ae, y - h = \pm \dfrac{b^2}{a}$

$\therefore$ end if LR is $\left(h \pm ae, k \pm \dfrac{b^2}{a}\right)$

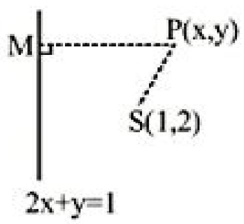
Illustration :

Find the equation of the hyperbola whose directrix is $2x + y = 1$, focus $(1, 2)$ and eccentricity $\sqrt{3}$.

Sol. *Let $P(x, y)$ be any point on the hyperbola.*
Draw PM perpendicular from P on the directrix.
Then by definition $SP = e\,PM$

$\Rightarrow \quad (SP)^2 = e^2 (PM)^2$

$$\Rightarrow \quad (x-1)^2 + (y-2)^2 = 3\left\{\frac{2x + y - 1}{\sqrt{4+1}}\right\}^2$$



$\Rightarrow \quad 5(x^2 + y^2 - 2x - 4y + 5) = 3(4x^2 + y^2 + 1 + 4xy - 2y - 4x)$

$\Rightarrow \quad 7x^2 - 2y^2 + 12xy - 2x + 14y - 22 = 0$

Which is the required hyperbola.

Illustration :

Find the eccentricity of the hyperbola whose latus-rectum is half of its transverse axis.

Sol. *Let the equation of hyperbola be* $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

Then transverse axis = 2a and latus-rectum $= \dfrac{2b^2}{a}$

According to question $\frac{2b^2}{a} = \frac{1}{2}(2a)$

$\Rightarrow 2b^2 = a^2 \qquad (\because b^2 = a^2 (e^2 - 1))$

$\Rightarrow 2a^2 (e^2 - 1) = a^2 \quad \Rightarrow \quad 2e^2 - 2 = 1 \quad \Rightarrow \quad e^2 = \frac{3}{2}$

$\therefore e = \sqrt{\frac{3}{2}}$.

Hence the required eccentricity is $\sqrt{\frac{3}{2}}$.

Illustration :

Find the equaiton of the hyperbola, the length of whose latus rectum is 8, eccentricity is $\frac{3}{\sqrt{5}}$ and whose transverse and conjugate axes are along the x and y axes respectively.

Sol. Given, $8 = \frac{2b^2}{a} = \frac{2a^2(e^2-1)}{a}$

$$\Rightarrow \quad 8 = 2a(e^2 - 1) = 2a\left[\left(\frac{3}{\sqrt{5}}\right)^2 - 1\right] = 2a\left[\frac{9}{5} - 1\right] = \frac{8a}{5}$$

$\Rightarrow \quad a = 5$

Again, $8 = \frac{2b^2}{a} = 8 \times 5 \quad \therefore b^2 = 20$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad$ or $\quad \frac{x^2}{25} - \frac{y^2}{20} = 1$

Illustration :

Find the equaiton of the hyperbola whose eccentricity is $\sqrt{2}$ and the distance between the foci is 16, taking transverse and conjugate axes of the hyperbola as x and y-axes respectively.

Sol. Let the equation of the hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad ...(1)$$

The coordinates of the foci are (ae, 0) and (–ae, 0)

Given $2ae = 16$ or $2.a\sqrt{2} = 16 \quad \therefore a = 4\sqrt{2}$

Also $\quad b^2 = a^2 (e^2 - 1) = 32 (2 - 1) = 32 \quad [\because e = \sqrt{2}]$

$\therefore$ The required equation of the hyperbola is $\frac{x^2}{32} - \frac{y^2}{32} = 1$ or $x^2 - y^2 = 32$

Illustration :

Prove that the point $\left\{\frac{a}{2}\left(t+\frac{1}{t}\right), \frac{b}{2}\left(t-\frac{1}{t}\right)\right\}$ *lies on the hyperbola for all values of t* $(t \neq 0)$.

Sol. Let $x = \frac{a}{2}\left(t+\frac{1}{t}\right)$ or $\frac{2x}{a} = t + \frac{1}{t}$ or $\left(\frac{2x}{a}\right)^2 = t^2 + \frac{1}{t^2} + 2$ ....(1)

and let $y = \frac{b}{2}\left(t-\frac{1}{t}\right)$ or $\frac{2y}{b} = t - \frac{1}{t}$ or $\left(\frac{2y}{b}\right)^2 = t^2 + \frac{1}{t^2} - 2$ ....(2)

Subtracting (2) from (1), $\frac{4x^2}{a^2} - \frac{4y^2}{b^2} = 4$ or $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

Which is hyperbola.

Illustration :

Find the lengths of transverse axis, conjugate axis, eccentricity, the co-ordinates of foci, vertices, lengths of the latus-rectum and equations of the directrices of the following hyperbolas.

(i) $9x^2 - y^2 = 1$ (ii) $16x^2 - 9y^2 = -144.$

Sol.

(i) The equation $9x^2 - y^2 = 1$ can be written as $\frac{x^2}{(1/9)} - \frac{y^2}{1} = 1 \Rightarrow a = \frac{1}{3}, b = 1$

The length of transverse, axis $= 2a = \frac{2}{3}$

The length of conjugate axis $= 2b = 2.$

Eccentricity $e = \sqrt{\left(1+\frac{b^2}{a^2}\right)}\sqrt{1+\frac{1}{(1/9)}} = \sqrt{10}$

The co-ordinates of the foci are $(\pm ae, 0)$ i.e., $\left(\pm\frac{\sqrt{10}}{3}, 0\right)$.

The co-ordinates of the vertices are $(\pm a, 0)$ i.e., $\left(\pm\frac{1}{3}, 0\right)$

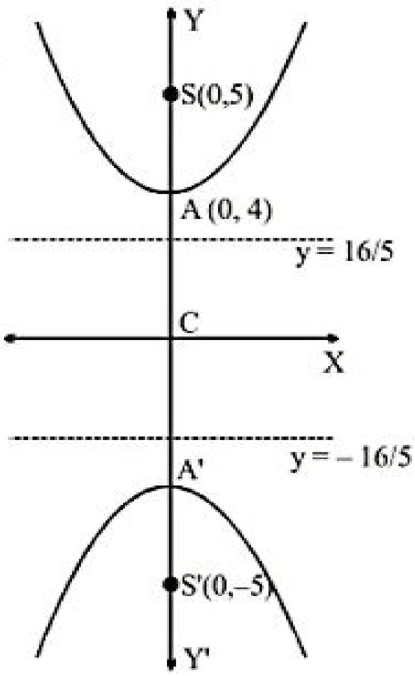
The length of latus - rectum $= \frac{2b^2}{a} = \frac{2(1)^2}{1/3} = 6.$

The equations of the directrices are

$$x = \pm\frac{a}{e} \text{ i.e., } x = \pm\frac{1/3}{\sqrt{10}} \text{ or } x = \pm\frac{1}{3\sqrt{10}}$$

(ii) The equation $16x^2 - 9y^2 = -144$ can be written as $\frac{x^2}{9} - \frac{y^2}{16} = -1$.

$a = 3$, $b = 4$, This conjugate hyperbola
The length of transverse axis $= 2b = 8$.
The length of conjugate axis $= 2a = 6$.



Eccentricity $e = \sqrt{\left(1+\frac{a^2}{b^2}\right)} = \sqrt{\left(1+\frac{9}{16}\right)} = \frac{5}{4}$

The co-ordinates of the foci are $(0, \pm be)$ i.e., $(0, \pm 5)$
The co-ordinates of the vertices are $(0, \pm b)$ i.e., $(0, \pm 4)$.

The length of latus-rectum $= \frac{2a^2}{b} = \frac{2(3)^2}{4} = \frac{9}{2}$

The equation of directrices are $y = \pm\frac{b}{e} = \pm\frac{4}{(5/4)} \Rightarrow y = \pm\frac{16}{5}$.

Illustration :

If e and e' are the eccentricities of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and its conjugate hyperbola, prove that $\frac{1}{e^2} + \frac{1}{e'^2} = 1$.

Sol. Given hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ....(1)

The eccentricity e of hyperbola (1) is given by
$b^2 = a^2(e^2 - 1)$

$\therefore \quad \frac{b^2}{a^2} = e^2 - 1$ ....(2)

The equation of the conjugate hyperbola is $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$.

$\therefore$ Its eccentricity e' is given by $a^2 = b^2(e'^2 - 1)$

$\therefore \quad (e'^2 - 1) = \frac{a^2}{b^2}$ ....(3)

Multiple (2) and (3), we get

$$(e^2 - 1) \times (e'^2 - 1) = \frac{a^2}{b^2} \times \frac{b^2}{a^2}$$

$\therefore \quad e^2e'^2 - e^2 - e'^2 + 1 = 1$

$\therefore \quad e^2e'^2 = e^2 + e'^2$

$\therefore \quad 1 = \frac{1}{e^2} + \frac{1}{e'^2}$.

Illustration :

If the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{b^2}=1$ *and hyperbola* $\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25}$ *coincide, then the value of* b^2 *is-*

(A) 1 (B) 5 (C) 7 (D) 9

Sol. *For hyperbola* $\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25} \Rightarrow \frac{x^2}{144/25}-\frac{y^2}{81/25}=1$

$e^2 = 1+\frac{b^2}{a^2} = 1+\frac{81}{144} = \frac{225}{144}$; $e = \frac{15}{12} = \frac{5}{4}$

Hence the foci are

$(\pm ae, 0) = \left(\pm\frac{12}{5}.\frac{5}{4},0\right) = (\pm 3, 0)$

Now the foci coincide therefore for ellipse

$ae = 3$ or $a^2e^2 = 9$ or $a^2\left(1-\frac{b^2}{a^2}\right) = 9$

$a^2 - b^2 = 9$ or $16 - b^2 = 9 \Rightarrow b^2 = 7.$ **Ans. [C]**

Illustration :

The foci of a hyperbola coincides with the foci of the ellipse $\frac{x^2}{25}-\frac{y^2}{9}=1$. *Find the equation of the hyperbola if its eccentricity is 2.*

Sol. *The given ellipse is* $\frac{x^2}{25}+\frac{y^2}{9}=1$ or $\frac{x^2}{5^2}+\frac{y^2}{3^2}=1$ ... (1)

Here $5 > 3$ $\therefore a = 5$ and $b = 3$

$\therefore$ *The foci of the ellipse are on the x-axis.*

$\therefore$ *Eccentricity of the ellipse,* $e = \sqrt{1-\frac{b^2}{a^2}} = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$

$\therefore$ *Foci of the ellipse are* $(\pm ae, 0)$ or $(\pm 4, 0)$

Let the equation of the hyperbola be $\frac{x^2}{A^2}-\frac{y^2}{B^2}=1$.

$\therefore$ *Foci are* $(\pm Ae, 0)$ or $(\pm 2A, 0)$ $[\because e = 2]$

$\therefore$ $2A = 4$ i.e., $A = 2$ and $B = A\sqrt{e^2-1} = 2\sqrt{(2)^2-1} = 2\sqrt{3}$.

$\therefore$ *The equation of the hyperbola is* $\frac{x^2}{(2)^2}-\frac{y^2}{(2\sqrt{3})^2}=1$ or $\frac{x^2}{4}-\frac{y^2}{12}=1$

or $3x^2 - y^2 = 12.$

Illustration :

The equation $16x^2 - 3y^2 - 32x + 12y - 44 = 0$ *represents a hyperbola*

(A) the length of whose transverse axis is $2\sqrt{3}$ *(B) the length of whose conjugate axis is 8.*

(C) whose centre is (1, 2) *(D) whose eccentricity is* $\frac{\sqrt{19}}{3}$

Sol. *We have,* $16(x^2 - 2x) - 3(y^2 - 4y) = 44 \Rightarrow 16(x-1)^2 - 3(y-2)^2 = 48$

$$\Rightarrow \frac{(x-1)^2}{3} - \frac{(y-2)^2}{16} = 1,$$

centre is $x - 1 = 0$ & $y - 2 = 0 \Rightarrow (1, 2)$

$a^2 = 3 \Rightarrow a = \sqrt{3}$, $b^2 = 16 \Rightarrow b = 4.$

$\therefore$ $l(TA) = 2a = 2\sqrt{3}$ *and* $l(CA) = 2b = 2.4 = 8.$

This equation represents a hyperbola with eccentricity given

$$e = \sqrt{1 + \left(\frac{\text{Conjugate axis}}{\text{Transverse axis}}\right)^2} = \sqrt{1 + \left(\frac{4}{\sqrt{3}}\right)^2} = \sqrt{\frac{19}{3}}$$

Ans.[A, B, C]

Illustration :

The equation $9x^2 - 16y^2 - 18x + 32y - 151 = 0$ *represent a hyperbola -*

(A) The length of the transverse axes is 4 *(B) Length of latus rectum is 9*

(C) Equation of directrix is $x = \frac{21}{5}$ *and* $x = -\frac{11}{5}$ *(D) None of these*

Sol. *We have* $9x^2 - 16y^2 - 18x + 32y - 151 = 0$

$9(x^2 - 2x) - 16(y^2 - 2y) = 151$

$9(x^2 - 2x + 1) - 16(y^2 - 2y + 1) = 144$

$9(x - 1)^2 - 16(y - 1)^2 = 144$

$$\frac{(x-1)^2}{16} - \frac{(y-1)^2}{9} = 1$$

Shifting the origin at (1, 1) without rotating the axes

$$\frac{X^2}{16} - \frac{Y^2}{9} = 1$$

where $X = x - 1$ *and* $Y = y - 1$

This is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

where $a^2 = 16$ *and* $b^2 = 9$ *so*

The length of the transverse axes $= 2a = 8$; $l(CA) = 2b = 6.$

The length of the letus rectum $= \frac{2b^2}{a} = \frac{2 \cdot 9}{4} = \frac{9}{2}$ *and* $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{9}{16}} = \frac{5}{4}$

The equaiton of the directrix $X = \pm \frac{a}{e}$

$$x - 1 = \pm\frac{16}{5} \Rightarrow x = \pm\frac{16}{5} + 1$$

$$x = \frac{21}{5} \text{ and } x = -\frac{11}{5}$$

Equation of directrix is $x = \frac{21}{5}$ *and* $x = -\frac{11}{5}$ **Ans. [C]**

Illustration :

Show that the equation $7y^2 - 9x^2 + 54x - 28y - 116 = 0$ *represent a hyperbola. Find the co-ordinates of the centre, lengths of transverse and conjugate axes, eccentricity, latus-rectum, co-ordinate of foci, vertices and equations of the directrices of the hyperbola.*

Sol. *We have* $7y^2 - 9x^2 + 54x - 28y - 116 = 0$

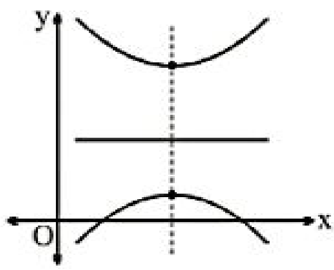
or $7(y^2 - 4y) - 9(x^2 - 6x) - 116 = 0$

or $7(y^2 - 4y + 4) - 9(x^2 - 6x + 9) = 116 + 28 - 81$

or $7(y - 2)^2 - 9(x - 3)^2 = 63$

or $$\frac{(y-2)^2}{9} - \frac{(x-3)^2}{7} = 1$$

or $$\frac{Y^2}{9} - \frac{X^2}{7} = 1.$$



where $X = x - 3$ *and* $Y = y - 2$.

This equation represents conjugate hyperbola. Comparing it with $\frac{Y^2}{b^2} - \frac{X^2}{a^2} = 1.$

We get $b^2 = 9$ *and* $a^2 = 7$

$\therefore b = 3$ *and* $a = \sqrt{7}$.

Centre : $X = 0, Y = 0$ *i.e.,* $x - 3 = 0, y - 2 = 0$

$\therefore$ *Centre is* $(3, 2)$

Length of transverse axis $= 2b = 6$

Length of conjugate axis $= 2a = 2\sqrt{7}$.

The eccentricity e is given by $e = \sqrt{1 + \frac{a^2}{b^2}} = \sqrt{1 + \frac{7}{9}} = \frac{4}{3}$.

The length of latus-rectum $= \frac{2a^2}{b} = \frac{2(7)}{3} = \frac{14}{3}$

The co-ordinates of foci $X = 0, Y = \pm be$

$\Rightarrow \quad x - 3 = 0, y - 2 = \pm 3 \times \frac{4}{3}$ *or* $(3, 2 \pm 4)$

i.e., $(3, -2)$ *and* $(3, 6)$

The co-ordinates of vertices are

or $X = 0, Y = \pm b$

or $(3, 2 \pm 3)$

or vertices are $(3, -1)$ and $(3, 5)$

The equation of directrices are $Y = \pm\dfrac{b}{e}$

$$\Rightarrow \quad y - 2 = \pm\frac{3}{4/3} \Rightarrow y = \left(2 \pm \frac{9}{4}\right)$$

i.e., $\quad y = \dfrac{17}{4}$ and $y = \dfrac{-1}{4}$.

Illustration :

Find the equation of the hyperbola having $e = \dfrac{3}{2}$ and foci at $(\pm 3, 0)$

Sol. The foci are at $(\pm 3, 0)$. These are on the x-axis.
Since centre of the hyperbola is the mid-point of the line segement joining the foci, therefore centre is $(0, 0)$.

Let the equation of the hyperbola be $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \qquad \ldots (1)$

where $a, b > 0$ and $b^2 = a^2 (e^2 - 1)$.

The foci of this hyperbola are $(\pm ae, 0)$. $\therefore ae = 3$ & $e = \dfrac{3}{2} \Rightarrow a = 2$

$$\therefore \quad b^2 = a^2 (e^2 - 1) = 4\left(\frac{9}{4} - 1\right) = 5.$$

$\therefore$ From (1), the requird equation of the hyperbola is $\dfrac{x^2}{4} - \dfrac{y^2}{5} = 1$.

Illustration :

Find the equation of the hyperbola having eccentricity $e = \dfrac{4}{3}$ and vertices at $(0, \pm 7)$.

Sol. The vertices of the hyperbola are at $(0, \pm 7)$ and these are on the y-axis. Centre of the hyperbola will be the mid point of the vertex (focus also) $\Rightarrow C(0, 0)$

Let the equation of the hyperbola be $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$,

The vertices of this hyperbola are $(0, \pm b)$ $\therefore b = 7$

$$\text{Now, } a^2 = b^2 (e^2 - 1) = 7^2 \left[\left(\frac{4}{3}\right)^2 - 1\right] = 49 \left(\frac{16 - 9}{9}\right) = \frac{343}{9}$$

$\therefore$ From (1), the equaiton of this hyperbola is $\dfrac{y^2}{(7)^2} - \dfrac{x^2}{\frac{343}{9}} = 1$ or $\dfrac{y^2}{49} - \dfrac{9x^2}{343} = 1$

Illustration :

Find the equation of the hyperbola having vertices at (±5, 0) and foci at (±7, 0).

Sol. *The foci of the hyperbola are at (±7, 0). They are on the x-axis.*
Also centre of the hyperbola will be (0, 0).

Let the equaiton of the hypebola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ... (1)

where $a, b > 0$ *and* $b^2 = a^2 (e^2 - 1)$
The vertices of this hyperbola are $(\pm a, 0)$
But it is given that vertices are $(\pm 5, 0)$ $\therefore a = 5$
The foci of this hyperbola are $(\pm ae, 0)$ $\therefore ae = 7$
$\Rightarrow 5e = 7 \Rightarrow e = 7/5$

Now, $b^2 = a^2 (e^2 - 1) = 25\left(\frac{49}{25} - 1\right) = 24$

From (1), the required equation of the hyperbola is

$$\frac{x^2}{(5)^2} - \frac{y^2}{(\sqrt{24})^2} = 1 \quad or \quad \frac{x^2}{25} - \frac{y^2}{24} = 1$$

Illustration :

Find the equation of the hyperbola whose foci are $(0, \pm\sqrt{10})$ *and which passes through the point (2, 3).*

Sol. *The foci of the hyperbola are at* $(0, \pm\sqrt{10})$*. These are on the y-axis*
Clearly, centre of the hyperbola will be (0, 0)

Let the equaiton of the hyperbola be $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ ... (1)

where $a, b > 0$ *and* $a^2 = b^2 (e^2 - 1) \Rightarrow a^2 + b^2 = a^2e^2$ *i.e.* $a^2 + b^2 = 10$

$$\frac{9}{b^2} - \frac{4}{10 - b^2} = 1 \Rightarrow b^4 - 23b^2 + 90 = 0 \Rightarrow b^2 = 18 + 5$$

when $b^2 = 18$ *then* $a^2 = -8$ *not possible*
when $b^2 = 5 \Rightarrow a^2 = 5$ *possible.*

Hence the equaiton of hyperbola is $\frac{y^2}{5} - \frac{x^2}{5} = 1$ *or* $y^2 - x^2 = 5$

Illustration :

Find the equation of the hyperbola whose foci are (6, 4) and (–4, 4) and eccentricity is 2.

Sol. *The centre of the hyperbola is the mid-point of the line joining the two foci. So the co-ordinates of the centre are* $\left(\frac{6-4}{2}, \frac{4+4}{2}\right)$ *i.e., (1, 4)*

Let 2a and 2b be the lengths of transverse and conjugate axes and let e be the eccentricity.

Then equation of hyperbola is $\dfrac{(x-1)^2}{a^2} - \dfrac{(y-4)^2}{b^2} = 1$

$\because$ *Distance between the foci = 2ae*

$$\sqrt{(6+4)^2+(4-4)^2} = 2a \times 2 \quad \Rightarrow \quad 10 = 4a \; \Rightarrow a = \frac{5}{2}$$

$$\therefore\; b^2 = a^2\,(e^2-1) = \frac{25}{4}\,(4-1) = \frac{75}{4}$$

Thus the equation of the hyperbola is $\dfrac{(x-1)^2}{\left(\dfrac{25}{4}\right)} - \dfrac{(y-4)^2}{\left(\dfrac{75}{4}\right)} = 1.$

or $\quad 12\,(x-1)^2 - 4\,(y-4)^2 = 75$

or $\quad 12\,(x^2-2x+1) - 4\,(y^2-8y+16) = 75$

or $\quad 12x^2 - 4y^2 - 24x + 32y - 127 = 0.$

Illustration :

Obtain the equation of a hyperbola with co-ordinate axes as principal axes and given that the distances of one of its vertices from the foci are 9 and 1 units.

Sol. *Let equation of hyperbola is* $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1.$ ....*(1)*

If vertices are A (a, 0) and A′ (–a, 0) and foci are S (ae, 0) and S′ (–ae, 0)

Given $l\,(S'A) = 9$ *and* $l\,(SA) = 1$

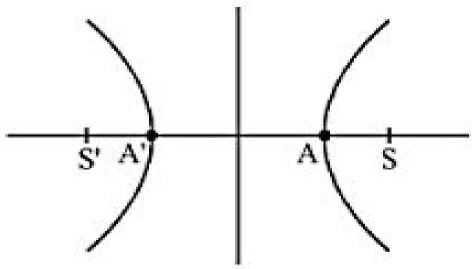
$\Rightarrow \quad a + ae = 9$ *and* $ae - a = 1$

or $\quad a\,(1+e) = 9$ *and* $a\,(e-1) = 1$

$$\therefore \quad \frac{a\,(1+e)}{a\,(e-1)} = \frac{9}{1} \Rightarrow 1 + e = 9e - 9 \Rightarrow e = \frac{5}{4}$$

$\because \quad a\,(1+e) = 9$

$$\therefore \quad a\left(1+\frac{5}{4}\right) = 9 \;\Rightarrow a = 4$$



$$b^2 = a^2\,(e^2-1) = 16\left(\frac{25}{16}-1\right) \;\Rightarrow b^2 = 9$$

From (1) equation of hyperbola is $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1.$

Practice Problem

Single correct question

Q.1 The eccentricity of the conic represented by $x^2 - y^2 - 4x + 4y + 16 = 0$ is

(A) 1 (B) $\frac{1}{2}$ (C) -1 (D) $\sqrt{2}$

Q.2 If e_1 and e_2 are the eccentricities of the conic sections $16x^2 + 9y^2 = 144$ and $9x^2 - 16y^2 = 144$, then

(A) $e_1^2 - e_2^2 = 1$ (B) $e_1^2 + e_2^2 < 3$ (C) $e_1^2 + e_2^2 = 3$ (D) $e_1^2 + e_2^2 > 3$

Q.3 Eccentricity of the hyperbola conjugate to the hyperbola $\frac{x^2}{4} - \frac{y^2}{12} = 1$ is

(A) $\frac{2}{\sqrt{3}}$ (B) 2 (C) $\sqrt{3}$ (D) $\frac{4}{3}$

Q.4 Locus of the centre of the circle which touches the two circles $x^2 + y^2 + 8x - 9 = 0$ and $x^2 + y^2 - 8x + 7 = 0$ externally, is

(A) $x^2 + y^2 = 16$ (B) $\frac{x^2}{1} + \frac{y^2}{15} = 1$ (C) $\frac{x^2}{1} - \frac{y^2}{12} = 1$ (D) $\frac{x^2}{1} - \frac{y^2}{15} = 1$

Q.5 If the eccentricity of the hyperbola $x^2 - y^2 \sec^2 \alpha = 5$ is $\sqrt{3}$ times the eccentricity of the ellipse $x^2 \sec^2 \alpha + y^2 = 25$, then a value of α is :

(A) $\pi/6$ (B) $\pi/4$ (C) $\pi/3$ (D) $\pi/2$

Q.6 The foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide. Then the value of b^2 is

(A) 5 (B) 7 (C) 9 (D) 4

Q.7 The focal length of the hyperbola $x^2 - 3y^2 - 4x - 6y - 11 = 0$, is

(A) 4 (B) 6 (C) 8 (D) 10

Q.8 The equation $\frac{x^2}{29 - p} + \frac{y^2}{4 - p} = 1$ $(p \neq 4, 29)$ represents

(A) an ellipse if p is any constant greater than 4.

(B) a hyperbola if p is any constant between 4 and 29.

(C) a rectangular hyperbola if p is any constant greater than 29.

(D) no real curve if p is less than 29.

Q.9 If $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$ represents family of hyperbolas where 'α' varies then
(A) distance between the foci is constant
(B) distance between the two directrices is constant
(C) distance between the vertices is constant
(D) distances between focus and the corresponding directrix is constant

Q.10 Let the major axis of a standard ellipse equals the transverse axis of a standard hyperbola and their director circles have radius equal to 2R and R respectively. If e_1 and e_2 are the eccentricities of the ellipse and hyperbola then the correct relation is
(A) $4e_1^2 - e_2^2 = 6$ (B) $e_1^2 - 4e_2^2 = 2$ (C) $4e_2^2 - e_1^2 = 6$ (D) $2e_1^2 - e_2^2 = 4$

Q.11 The equation to the chord joining two points (x_1, y_1) and (x_2, y_2) on the rectangular hyperbola $xy = c^2$ is

(A) $\frac{x}{x_1 + x_2} + \frac{y}{y_1 + y_2} = 1$ (B) $\frac{x}{x_1 - x_2} + \frac{y}{y_1 - y_2} = 1$

(C) $\frac{x}{y_1 + y_2} + \frac{y}{x_1 + x_2} = 1$ (D) $\frac{x}{y_1 - y_2} + \frac{y}{x_1 - x_2} = 1$

More than one

Q.12 Which of the following equations in parametric form can represent a hyperbolic profile, where 't' is a parameter.

(A) $x = \frac{a}{2}\left(t + \frac{1}{t}\right)$ & $y = \frac{b}{2}\left(t - \frac{1}{t}\right)$ (B) $\frac{tx}{a} - \frac{y}{b} + t = 0$ & $\frac{x}{a} + \frac{ty}{b} - 1 = 0$

(C) $x = e^t + e^{-t}$ & $y = e^t - e^{-t}$ (D) $x^2 - 6 = 2\cos t$ & $y^2 + 2 = 4\cos^2\frac{t}{2}$

Q.13 Let p and q be non-zero real numbers. Then the equation $(px^2 + qy^2 + r)(4x^2 + 4y^2 - 8x - 4) = 0$ represents
(A) two straight lines and a circle, when $r = 0$ and p, q are of the opposite sign.
(B) two circles, when $p = q$ and r is of sign opposite to that of p.
(C) a hyperbola and a circle, when p and q are of opposite sign and $r \neq 0$.
(D) a circle and an ellipse, when p and q are unequal but of same sign and r is of sign opposite to that of p.

Q.14 Let $C_1 : 9x^2 - 16y^2 - 18x + 32y - 23 = 0$ and $C_2 : 25x^2 + 9y^2 - 50x - 18y + 33 = 0$ are two conics then

(A) eccentricity of C_1 is $\frac{5}{4}$.

(B) eccentricity of C_2 is $\frac{5}{3}$.

(C) area of the quadrilateral with vertices at the foci of the conics is $\frac{8}{9}$.

(D) latus rectum of C_1 is greater than latus rectum of C_2.

Integer type question

Q.15 Find the centre, eccentricity and length of the axes of the hyperbola

$$3x^2 - 5y^2 - 6x + 20y - 32 = 0$$

Q.16 Consider the hyperbola $9x^2 - 16y^2 + 72x - 32y - 16 = 0$. Find the following:

(a) centre
(b) eccentricity
(c) focii
(d) equation of directrix
(e) length of the latus rectum
(f) equation of auxilary circle
(g) equation of director circle

Answer key

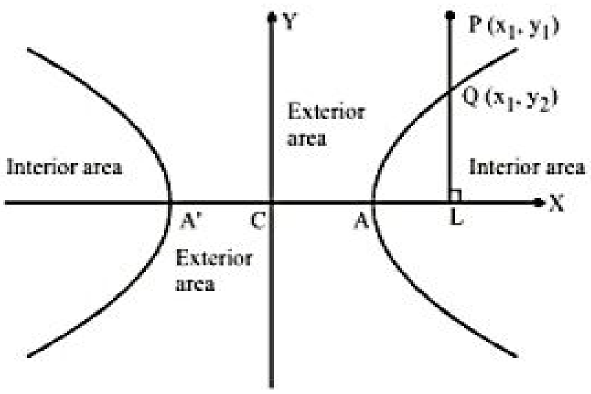
Q.1	D	Q.2	B	Q.3	A	Q.4	D	Q.5	B
Q.6	B	Q.7	C	Q.8	B	Q.9	A	Q.10	C
Q.11	A	Q.12	A, C, D	Q.13	A, B, C, D	Q.14	A, C, D		

Q.15 $(1, 2); 2\sqrt{\left(\frac{2}{5}\right)}; 2\sqrt{5}; 2\sqrt{3}$

Q.16 **(a)** $(-4, -1)$; **(b)** $\frac{5}{4}$; **(c)** $(1, -1), (-9, -1)$; **(d)** $5x + 4 = 0, 5x + 36 = 0$,

(e) $\frac{9}{2}$; **(f)** $(x+4)^2 + (y+1)^2 = 16$; **(g)** $(x+4)^2 + (y+1)^2 = 7$

POSITION OF A POINT WITH RESPECT TO HYPERBOLA :



Let $S(x, y) \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$ be the given hyperbola and $P(x_1, y_1)$ is the given point.

(i) If $S(x_1, y_1) > 0$ then $P(x_1, y_1)$ lie inside the ellipse.

(ii) If $S(x_1, y_1) < 0$ then $P(x_1, y_1)$ lie outside the ellipse.

(iii) If $S(x_1, y_1) = 0$ then $P(x_1, y_1)$ lie on the ellipse.

Illustration :

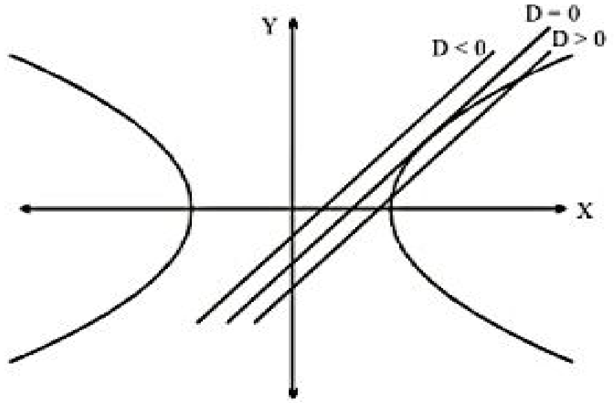
Find the position of the point (5, – 4) relative to the hyperbola $9x^2 - y^2 = 1$.

Sol. *Here* $S(x, y) \equiv 9x^2 - y^2 - 1$

and $S(5, -4) = 9(5)^2 - (-4)^2 - 1 = 255 - 16 - 1 = 208 > 0$

So the point (5, – 4) inside the hyperbola $9x^2 - y^2 = 1$.

INTERACTION OF A LINE AND A HYPERBOLA :



Let the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ....(1)

and the given line be $y = mx + c$ ....(2)

The point of intersection of line and hyperbola can be obtained by solving (1) and (2), therefore

Eliminating y from (1) and (2), we get $\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1$

$\Rightarrow b^2x^2 - a^2(mx + c)^2 = a^2 b^2 \Rightarrow (a^2m^2 - b^2) x^2 + 2mca^2x + a^2(b^2 + c^2) = 0$ ....(3)

Above equation is a quadratic in x and gives two values of x. It shows that every straight line will cut the hyperbola in two points, may be real, coincident or imaginary according as discriminant of (3) >, =, < 0.

i.e., $4m^2c^2a^4 - 4(a^2m^2 - b^2) a^2 (b^2 + c^2) >, =, < 0$

or $-a^2m^2 + b^2 + c^2 >, =, < 0$

or $c^2 >, =, < a^2m^2 - b^2$ ....(4)

EQUATIONS OF TANGENT :

(1) Point form :

The equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

Since point (x_1, y_1) lie on the hyperbola therefore we can use standard substitution to obtain the equation of tangent.

(2) Parametric form :

The equation of tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at $(a\sec\theta, b\tan\theta)$ is $\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$.

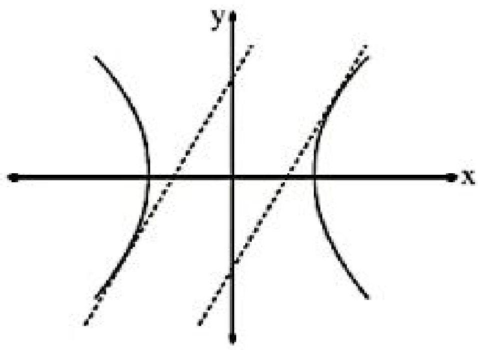
Note : Point of intersection of the tangents at θ_1 & θ_2 is $x=a\dfrac{\cos\frac{\theta_1-\theta_2}{2}}{\cos\frac{\theta_1+\theta_2}{2}}, y=b\dfrac{\sin\frac{\theta_1+\theta_2}{2}}{\cos\frac{\theta_1+\theta_2}{2}}$

(3) Slope form :

Given hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ......(1)

and the given line $y=mx+c$ touches hyperbola then solve (1) and (2), we get

$\frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2}=1$ or $b^2x^2-a^2(mx+c)^2=a^2b^2$



or $(b^2-a^2m^2)x^2-2a^2mcx-a^2(c^2+b^2)=0$ ......(3)

Since line (2) will be tangent to hyperbola (1)

if roots of equation (3) are equal i.e. $D=0$

$4a^4m^2c^2+4a^2(b^2-a^2m^2)(c^2+b^2)=0$

or $a^2m^2c^2+b^2c^2-a^2c^2m^2+b^4-a^2b^2m^2=0$

or $b^2c^2+b^4-a^2b^2m^2=0$

or $c^2+b^2-a^2m^2=0$

or $c^2=a^2m^2-b^2$ or $c=\pm\sqrt{a^2m^2-b^2}$. This is the required **condition of tangency**

Note :

(i) Equation of any tangent to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ may be taken as $y=mx+\sqrt{a^2m^2-b^2}$ or $y=mx-\sqrt{a^2m^2-b^2}$. The co-ordinates of the points of contact are $\left(\pm\frac{a^2m}{\sqrt{(a^2m^2-b^2)}},\pm\frac{b^2}{\sqrt{(a^2m^2-b^2)}}\right)$.

(ii) The equation of any tangent to the hyperbola $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

may be taken as $(y-k)=m(x-h)\pm\sqrt{a^2m^2-b^2}$.

Tangents drawn from outside point :

$y = mx \pm \sqrt{a^2m^2 - b^2}$ is a tangent to the standard hyperbola. ....(1)

If above tangent passes through (h, k) then

$(k - mh)^2 = a^2m^2 - b^2$

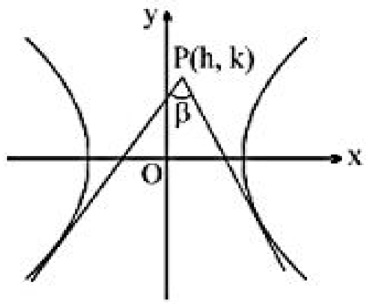
$(h^2 - a^2)m^2 - 2kmh + k^2 + b^2 = 0$ ....(2)

Above equation is quadratic in m therefore it has two roots m_1 and m_2.

Hence passing through a given point (h, k) there is a maximum of two tangents can be drawn to the hyperbola

therefore $m_1 + m_2 = \dfrac{2kh}{h^2 - a^2}$ ....(3)

$m_1 m_2 = \dfrac{k^2 + b^2}{h^2 - a^2}$ ....(4)



Equations (3) and (4) are used to find the locus of the point of intersection of a pair of tangents which enclose an angle β.

Now $\tan^2\beta = \dfrac{(m_1 + m_2)^2 - 4m_1m_2}{(1 + m_1m_2)^2}$

(substituting the values of $m_1 + m_2$ and m_1m_2 to get the locus)

If $\beta = 90°$ then $m_1m_2 = -1$, hence from (4)

$k^2 + b^2 = a^2 - h^2$

$x^2 + y^2 = a^2 - b^2$ which is the equation of director circle, of the given hyperbola.

Illustration :

Find the equation of the tangent to the hyperbola $x^2 - 4y^2 = 36$ which is perpendicular to the line $x - y + 4 = 0$.

Sol. *Let m be the slope of the tangent. Since the tangent is perpendicular to the line $x - y + 4 = 0$.*

$m \times 1 = -1 \Rightarrow m = -1$

Given hyperbola $x^2 - 4y^2 = 36$ *or* $\dfrac{x^2}{36} - \dfrac{y^2}{9} = 1$, $a^2 = 36$, $b^2 = 9$

$\therefore$ *tangent to above hyperbola is* $y = mx \pm \sqrt{36m^2 - 9}$

here $m = -1$.

equation of tangent is $y = -x \pm \sqrt{27}$ *or* $x + y \pm 3\sqrt{3} = 0$.

Illustration :

For what value of λ does the line $y = 2x + \lambda$ touches the hyperbola $16x^2 - 9y^2 = 144$?

Sol. *Equation of hyperbola is $16x^2 - 9y^2 = 144$ or $\frac{x^2}{9} - \frac{y^2}{16} = 1$*

Comparing this with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ we get $a^2 = 9$, $b^2 = 16$

and comparing this line $y = 2x + \lambda$ with $y = mx + c$.

$\therefore$ $m = 2$ and $c = \lambda$

If the line $y = 2x + \lambda$ touches the hyperbola $16x^2 - 9y^2 = 144$ then $c^2 = a^2m^2 - b^2$

$\Rightarrow \lambda^2 = 9(2)^2 - 16 = 36 - 16 = 20$

$\therefore \lambda = \pm\, 2\sqrt{5}$.

Illustration :

Find the locus of the mid points of the chords of the circle $x^2 + y^2 = 16$, which are tangent to the hyperbola $9x^2 - 16y^2 = 144$.

Sol. *Any tangent to hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ is $y = mx + \sqrt{(16m^2 - 9)}$* ...(1)

Let (x_1, y_1) be the mid- point of the chord of the circle $x^2 + y^2 = 16$, then equation of the chord is $(T = S_1)$

$$xx_1 + yy_1 - (x_1^2 + y_1^2) = 0 \qquad ...(2)$$

Since (1) and (2) are same, therefore

we get $\frac{m}{x_1} = -\frac{1}{y_1} = \frac{\sqrt{16m^2 - 9}}{-(x_1^2 + y_1^2)}$ $\Rightarrow m = -\frac{x_1}{y_1}$ *and* $(x_1^2 + y_1^2)^2 = y_1^2(16m^2 - 9)$

Eliminating m and replacing (x_1, y_1) by (x, y), we get the required locus is $(x^2 + y^2)^2 = 16x^2 - 9y^2$.

Ans.

Illustration :

Find the area of a triangle formed by the lines $x - y = 0$, $x + y = 0$ and any tangent to the hyperbola $x^2 - y^2 = a^2$.

Sol. *Any tangent to the hyperbola at $P(a \sec\theta, a \tan\theta)$ is*

$$x \sec\theta - y \tan\theta = a \qquad ...(i)$$

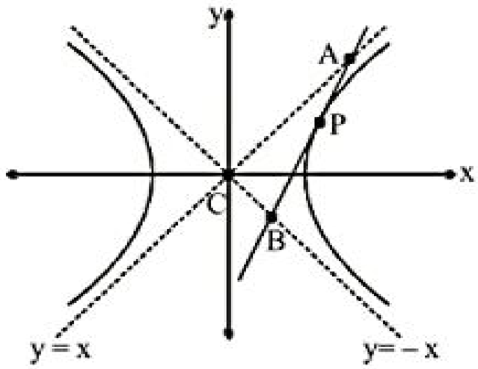
Also $x - y = 0$ ...(ii)

$x + y = 0$ ...(iii)

Solving the above three lines in pairs, we get the point A, B, C as

$$\left(\frac{a}{\sec\theta - \tan\theta}, \frac{a}{\sec\theta - \tan\theta}\right), \left(\frac{a}{\sec\theta + \tan\theta}, \frac{-a}{\sec\theta + \tan\theta}\right) \text{ and } (0, 0)$$

Since the one vertex is the origin therefore the area of the triangle ABC is



$$= \frac{1}{2} | (x_1y_2 - x_2y_1) |$$

$$= \left| \frac{a^2}{2}\left(\frac{-1}{sec^2\theta - tan^2\theta} - \frac{1}{sec^2\theta - tan^2\theta}\right)\right|$$

$$= \left| \frac{a^2}{2}(-2) \right| = a^2.$$

Illustration :

Find the equation and the length of the common tangents to hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ and } \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

Sol. *Tangent at $(a \sec \phi, b \tan \phi)$ on the 1st hyperbola is*

$$\frac{x}{a}\sec\phi - \frac{y}{b}\tan\phi = 1 \qquad ....(1)$$

Similarly tangent at any point $(b \tan \theta, a \sec \theta)$ on 2nd hyperbolas is

$$\frac{y}{a}\sec\theta - \frac{x}{b}\tan\theta = 1 \qquad ....(2)$$

If (1) and (2) are common tangents then they should be identical therefore

$$\frac{\frac{\sec\phi}{a}}{\frac{-\tan\theta}{b}} = \frac{\frac{-\tan\phi}{b}}{\frac{\sec\theta}{a}} = \frac{1}{1} \quad \Rightarrow \quad \frac{\sec\theta}{a} = -\frac{\tan\phi}{b}$$

or $\quad \sec\theta = -\frac{a}{b}\tan\phi \qquad ....(3)$

and $\quad -\frac{\tan\theta}{b} = \frac{\sec\phi}{a} \quad$ or $\quad \tan\theta = -\frac{b}{a}\sec\phi \qquad ....(4)$

$\because \quad sec^2\theta - tan^2\theta = 1$

$\Rightarrow \quad \frac{a^2}{b^2}\tan^2\phi - \frac{b^2}{a^2}\sec^2\phi = 1 \qquad$ {from (3) and (4)}

or $\quad \frac{a^2}{b^2}\tan^2\phi - \frac{b^2}{a^2}\tan^2\phi = 1 + \frac{b^2}{a^2} \quad$ or $\quad \left(\frac{a^2}{b^2} - \frac{b^2}{a^2}\right)\tan^2\phi = 1 + \frac{b^2}{a^2}$

$\therefore \quad \tan^2\phi = \frac{b^2}{a^2 - b^2} \quad$ and $\quad \sec^2\phi = 1 + \tan^2\phi = \frac{a^2}{a^2 - b^2}$

Hence the points of contact are

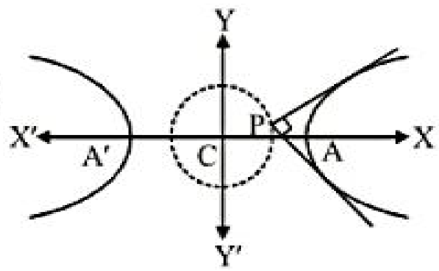
$$\left\{\pm\frac{a^2}{\sqrt{(a^2-b^2)}}, \pm\frac{b^2}{\sqrt{(a^2-b^2)}}\right\} \text{ and } \left\{\pm\frac{b^2}{\sqrt{(a^2-b^2)}}, \pm\frac{a^2}{\sqrt{(a^2-b^2)}}\right\} \quad \{from (3) and (4)\}$$

Length of common tangent i.e., the distance between the above points is $\sqrt{2}\,\frac{(a^2+b^2)}{\sqrt{(a^2+b^2)}}$ *and equation of common tangent on putting the values of* $\sec\theta$ *and* $\tan\theta$ *in (1) is*

$$\pm\frac{x}{\sqrt{(a^2+b^2)}} \mp \frac{y}{\sqrt{(a^2-b^2)}} = 1 \quad \text{or} \quad x \mp y = \pm\sqrt{(a^2-b^2)}. \quad \textbf{Ans.}$$

DIRECTOR CIRCLE :

The locus of the point of intersection of the tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, which are perpendicular to each other, is called the director circle.



Any tangent to hyperbola is $y = mx + \sqrt{a^2m^2 - b^2}$ ....(1)

If it passes through P(h, k), then $k - mh = \sqrt{a^2m^2-b^2}$

$k^2 + m^2h^2 - 2mk = a^2m^2 - b^2$

$m^2(h^2 - a^2) - 2mhk + (k^2 + b^2) = 0$

It is quadratic in m therefore it has two roots m_1 and m_2. Hence two tangents (real or imaginary) can be drawn from P(h, k).

If pair of perpendicular tangents are drawn from P(h, k) then $m_1 m_2 = \frac{k^2+b^2}{h^2-a^2} = -1.$

$\Rightarrow \quad h^2 + k^2 = a^2 - b^2$

$\therefore \quad$ locus of P(h, k) is $x^2 + y^2 = a^2 - b^2$.

Note :

(i) If $b^2 < a^2$ this circle is real

(ii) If $b^2 = a^2$ the radius of the circle is zero & it reduces to a point circle at the origin. In this case the centre is the only point from which the tangents at right angles can be drawn to the curve.

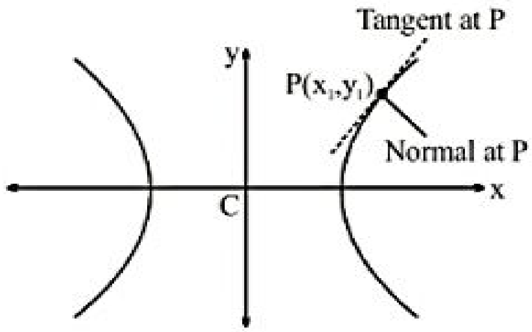
(iii) If $b^2 > a^2$, the radius of the circle is imaginary, so that there is no such circle & so no tangents at right angle can be drawn to the curve.

EQUATION OF NORMALS :

(1) Point form :

Equation of normal to a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at a point (x_1, y_1) is

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 + b^2$$



(2) Parametric form :

The equation of normal at $(a \sec\phi, b\tan\phi)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is i

$$ax\cos\phi + by\cot\phi = a^2 + b^2$$

Illustration :

Prove that the line $lx + my - n = 0$ will be a normal to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ if } \frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2+b^2)^2}{n^2}.$$

Sol. *The equation of any normal to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is*

$$ax\cos\phi + by\cot\phi = a^2 + b^2 \text{ or } ax\cos\phi + by\cot\phi - (a^2 + b^2) = 0 \qquad \text{.........(1)}$$

Since the straight line $lx + my - n = 0$ is given normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then (1) and $lx + my - n = 0$ represent the same line

$$\frac{a\cos\phi}{l} = \frac{b\cot\phi}{m} = \frac{(a^2+b^2)}{n}$$

or $\quad \sec\phi = \frac{na}{l(a^2+b^2)}$ *and* $\tan\phi = \frac{nb}{m(a^2+b^2)}$

$\because \quad \sec^2\phi - \tan^2\phi = 1$

$$\therefore \quad \frac{n^2a^2}{l^2(a^2+b^2)^2} - \frac{n^2b^2}{m^2(a^2+b^2)^2} = 1$$

$$\Rightarrow \quad \frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2+b^2)^2}{n^2}.$$

Illustration :

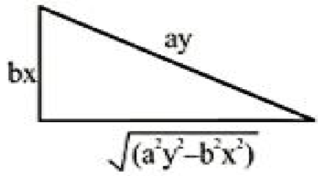
Find the locus of the foot of perpendicular from the centre upon any normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Sol. *Normal at* $P(a \sec \phi, b \tan \phi)$ *is*

$$ax \cos \phi + by \cot \phi = a^2 + b^2 \quad ....(1)$$

and equation of line perpendicular to (1) and passing through origin is

$$bx - ay \sin \phi = 0 \quad ....(2)$$

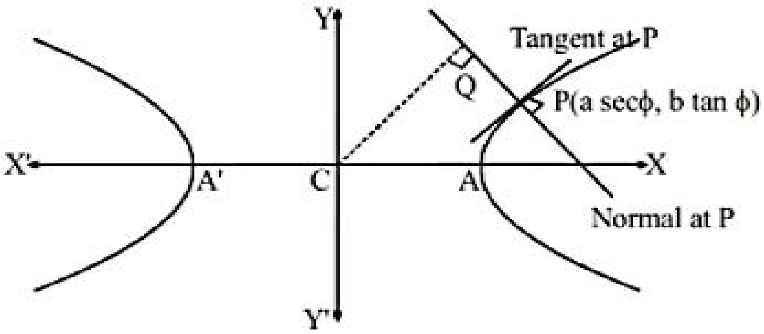


From (2)

$$\sin \phi = \frac{bx}{ay}$$

$$\therefore \quad \cos \phi = \frac{\sqrt{(a^2y^2 - b^2x^2)}}{ay}$$

$$\text{and} \quad \cot \phi = \frac{\sqrt{(a^2y^2 - b^2x^2)}}{bx}$$



Now by puttting value of $\cos \phi$ *and* $\cot \phi$ *in equation (1),* ϕ *will be elminated and get the locus*

$$\therefore \quad \text{from (1)}, \quad ax \times \frac{\sqrt{a^2y^2 - b^2x^2}}{ay} + by \times \frac{\sqrt{(a^2y^2 - b^2x^2)}}{bx} = a^2 + b^2$$

$$\Rightarrow \quad (x^2 + y^2)\sqrt{(a^2y^2 - b^2x^2)} = (a^2 + b^2)xy$$

or $\quad (x^2 + y^2)^2 (a^2y^2 - b^2x^2) = (a^2 + b^2)^2 x^2y^2$

which is required locus.

PROPERTIES :

(1) Normal and tangent at any point $P(x_1, y_1)$ meet the transverse axis at G and T respectively.

The equation of normal at P on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2 \quad ....(1)$$

The normal (1) meets the x-axis i.e. then put $y = 0$ in (1) we will get co-ordinates of G

i.e. $\quad G\left(\frac{(a^2 + b^2)}{a^2} x_1, 0\right) \quad$ or $\quad (e^2x_1, 0)$

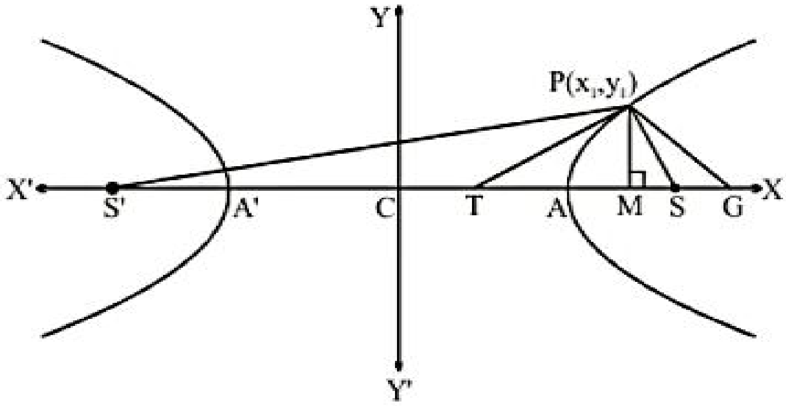
$$\therefore \quad CG = e^2x_1$$

Now,

$$SG = CG - CS = e^2x_1 - ae = e(ex_1 - a) = e \cdot SP$$

Similarly, $\quad S'G = e \cdot S'P$

$$\therefore \quad \frac{SG}{S'G} = \frac{SP}{S'P}$$

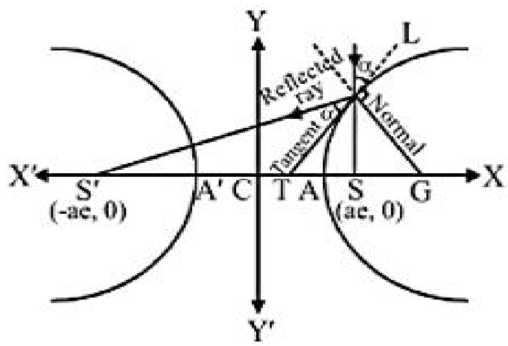


This relation shows that the normal PG is the external bisector of the angle SPS'. The tangent PT (perpendicular to PG) is therefore the internal bisector of the angle SPS'.

The tangent & normal at any point of a hyperbola bisect the angle between the focal radii.

Note : that the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{a^2 - k^2} - \frac{y^2}{k^2 - b^2} = 1 (a > k > b > 0)$ are confocal and therefore orthogonal.

(2) The tangent & normal at any point of a hyperbola bisect the angle between the focal radii. This spells the reflection property of the hyperbola as **"An incoming light ray "** aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus. It follows that if an ellipse and a hyperbola have the same foci, they cut at right angles at any of their common point.



(3) Locus of the feet of the perpendicular drawn from focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ upon any tangent is its auxiliary circle i.e. $x^2 + y^2 = a^2$ & the product of the feet of these perpendiculars is $b^2 \cdot$ i.e., square of semi-conjugate axis.

Similar property exists in ellipse also.

(4) The portion of the tangent between the point of contact & the directrix subtends a right angle at the corresponding focus.

(5) The foci of the hyperbola and the points P and Q in which any tangent meets the tangents at the vertices are concyclic with PQ as diameter of the circle.

PAIR OF TANGENTS :

The combined equation of the pair of tangents drawn from a point $P(x_1, y_1)$, lying outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1\right)^2 \text{ or } SS_1 = T^2$$

where $S = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$; $S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$ and $T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$.

CHORD OF CONTACT :

If the tangents from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ touch the hyperbola at Q and R, then the equation of the chord of contact QR is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

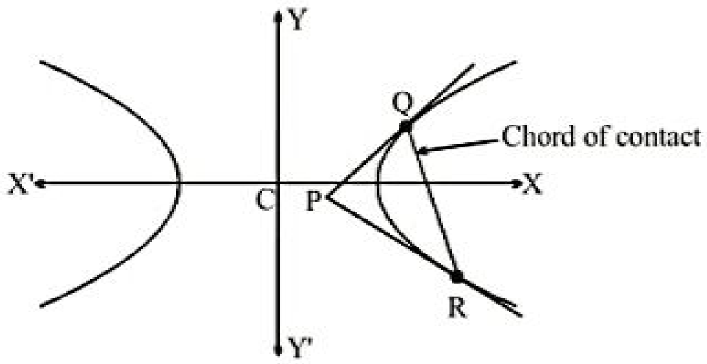


Illustration :

If tangent to the parabola $y^2 = 4ax$ intersect the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at A and B, then find the locus of point of inersection of tangents at A and B.

Sol. *Let $P = (h, k)$ be the point of intersection of tangents at A and B. Therefore, the equation of chord of contact AB is of hyperbola is*

$$\frac{xh}{a^2} - \frac{yk}{b^2} = 1$$

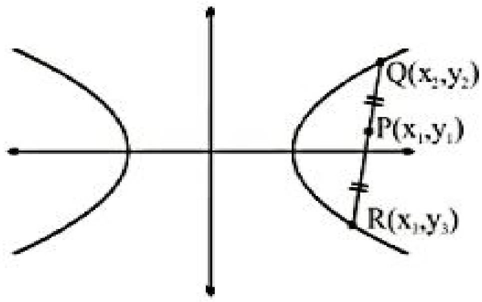
or $\quad y = \frac{xb^2h}{ka^2} - \frac{b^2}{k}$ *touches the given parabola*

Therefore $\quad -\frac{b^2}{k} = \frac{a}{\left(\frac{b^2h}{a^2k}\right)}$

$\Rightarrow$ *Locus of point P(h, k) is* $\quad y^2 = -\frac{b^4}{a^3}$

Equation of the chord bisected at a given point :

The equation of the chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, bisected at the point $P(x_1, y_1)$ is



$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$$

or $\quad T = S_1$, where $T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$ ans $S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$.

Illustration :

Find the locus of the mid - point of focal chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

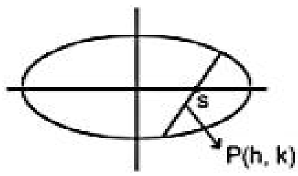
Sol. *Let $P = (h, k)$ be the mid-point*

$\therefore$ *equation of chord whose mid-point is given is $T = S_1$ i.e.* $\frac{xh}{a^2} - \frac{yk}{b^2} - 1 = \frac{h^2}{a^2} - \frac{k^2}{b^2} - 1$

since it is a focal chord,

$\therefore$ *it passes through focus, either $(ae, 0)$ or $(-ae, 0)$*

If it passes trhrough $(ae, 0)$



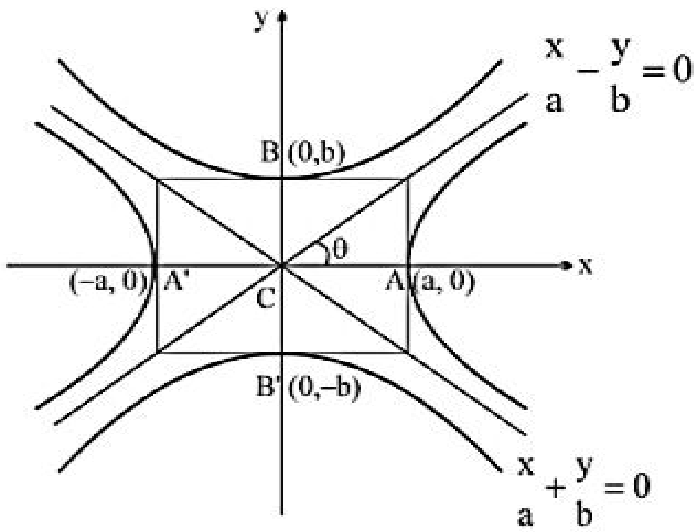
$\therefore$ *locus is* $\frac{ex}{a} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$.

ASYMPTOTES OF HYERPBOLA :

If the length of the perpendicular let fall from a point on a hyperbola to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola, then the straight line is called the Asymptote of the Hyperbola.

In short asymptote is tangent to hyperbola at infinity.

Let $y = mx + c$ is the asymptote of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.



Solving these two we get the quadratic as

$$(b^2 - a^2m^2)\,x^2 - 2a^2 mcx - a^2(b^2 + c^2) = 0 \qquad ....(1)$$

In order that $y = mx + c$ be an asymptote, both roots of equation (1) must approach at infinity, the conditions for which are

coeff of $x^2 = 0$ & coeff of $x = 0$.

$\Rightarrow \qquad b^2 - a^2m^2 = 0$ or $m = \pm\dfrac{b}{a}$ &

$a^2 mc = 0 \Rightarrow c = 0$.

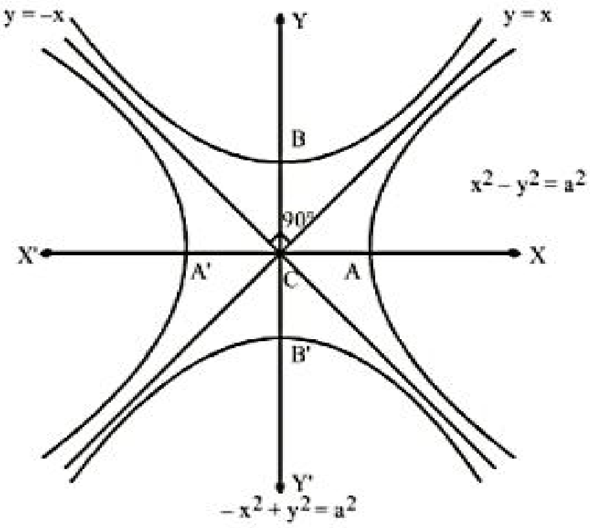
$\therefore$ equations of asymptote are $\dfrac{x}{a} + \dfrac{y}{b} = 0$ and $\dfrac{x}{a} - \dfrac{y}{b} = 0$.

combined equation to the asymptotes $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 0$.

As asymptotes of any hyperbola or a curve is a straight line which touches in it two points at infinity.

Note :

(i) If $b = a$, then $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ reduces $x^2 - y^2 = a^2$. The asymptotes of rectangular hyperbola $x^2 - y^2 = a^2$ are $y = \pm x$ which are at right angles.



(ii) A hyperbola and its conjugate have the same asymptote.

(iii) The asymptotes pass through the centre of the hyperbola & the bisectors of the angles between the asymptotes are the axes of the hyperbola.

(iv) Asymptotes are the tangent to the hyperbola from the centre.

(v) A simple method to find the coordinates of the centre of the hyperbola is expressed as a general equation of degree 2.

i.e. let $f(x, y) = 0$ represents a hyperbola.

Find $\dfrac{\partial f}{\partial x}$ & $\dfrac{\partial f}{\partial y}$. Then the point of intersection of $\dfrac{\partial f}{\partial x} = 0$ & $\dfrac{\partial f}{\partial y} = 0$.

gives the centre of the hyperbola.

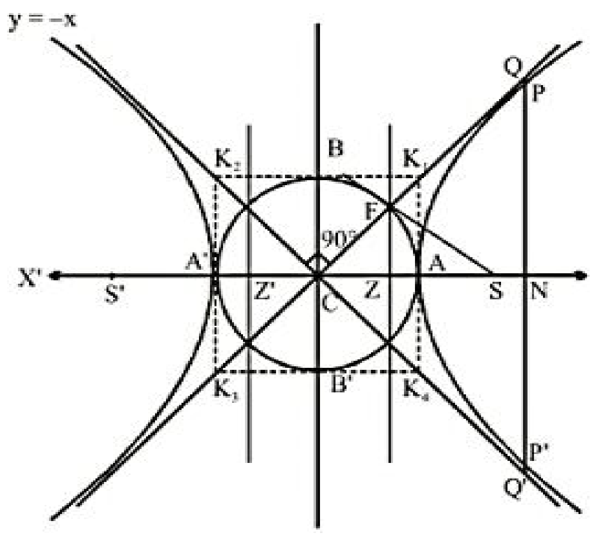
(vi) Let $H \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$; $A \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $C = \frac{x^2}{a^2} - \frac{y^2}{b^2} + 1 = 0$

be the equation of the hyperbola, asymptotes and the conjugate hyperbola respectively, then clearly $\mathbf{C + H = 2A}$ i.e., all the equation different only by constant term and the constant term of H, A and C are in A.P.

(vii) The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extremities of each axis parallel to the other axis. Here area of rectangle is 4ab.

PROPERTIES OF ASYMPTOTES :

(i) Perpendicular from the foci on either asymptote meet it in the same points as the corresponding directrix & the common points of intersection lie on the auxiliary circle.



(ii) The angle between the asymptotes of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $2\tan^{-1}\left(\frac{b}{a}\right)$.

and if the angle between the asymptote of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is 2θ then $e = \sec\theta$.

(iii) If from any point on the asymptote a straight line be drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted between the point & the curve is always equal to the square of the semi conjugate axis.

(iv) The tangent at any point P on a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with centre C, meets the asymptotes in Q and R and cuts off a Δ CQR of constant area equal to ab from the asymptotes & the portion of the tangent intercepted between the asymptote is bisected at the point of contact.

Illustration :

Find the asymptotes of the hyperbola $xy - 3y - 2x = 0$.

Sol. *Since equation of a hyperbola and its asymptotes differ in constant terms only.*

$\therefore$ *Pair of asymptotes is given by* $xy - 3y - 2x + \lambda = 0$ ......(1)

above equation represents two straight lines.

$$\therefore \quad abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\Rightarrow \quad 0 + 2 \times \frac{-3}{2} \times -1 \times \frac{1}{2} - 0 - 0 - \lambda\left(\frac{1}{2}\right)^2 = 0 \Rightarrow \lambda = 6.$$

From (1), the asymptotes of given hyperbola are given by

$$xy - 3y - 2x + 6 = 0 \text{ or } (y - 2)(x - 3) = 0.$$

$\therefore$ *Asymptotes are* $x - 3 = 0$ *and* $y - 2 = 0$.

Illustration :

The asymptotes of a hyperbola having centre at the point (1, 2) are parallel to the lines $2x + 3y = 0$ and $3x + 2y = 0$. If the hyperbola passes through the point (5, 3), show that its equation is $(2x + 3y - 8)(3x + 2y + 7) = 154$.

Sol. *Let the asymptotes be $2x + 3y + \lambda = 0$ and $3x + 2y + \mu = 0$. Since asymptotes passes through (1, 2), then $\lambda = -8$ and $\mu = -7$.*

Thus the equation of asymptotes are $2x + 3y - 8 = 0$ and $3x + 2y - 7 = 0$.

Let the equation of hyperbola be $(2x + 3y - 8)(3x + 2y - 7) + \lambda = 0$ ......(1)

Hyperbola passes through (5, 3) then $(10 + 9 - 8)(15 + 6 - 7) + \lambda = 0$

$\therefore \lambda = -154.$

Putting the value of λ in (1), we get $(2x + 3y - 8)(3x + 2y - 7) - 154 = 0$ which is the equation of required hyperbola.

The rectangular hyperbola $xy = c^2$

When the centre of any rectangular hyperbola be at the origin and its asymptotes coincide with the co-ordinate axes then equation of hyperbola is $xy = c^2$.

Here the equation of asymptotes is $xy = 0$ and the equation conjugate hyperbola is $xy = -c^2$.

Note :

(i) The equation of a rectangular hyperbola having co-ordinate axes as its asymptotes is $xy = c^2$. If the asymptotes of a rectangular hyperbola are $x = \alpha$, $y = \beta$ then its equation is $(x - \alpha)(y - \beta) = c^2$ or $xy - \alpha y - \beta x + \lambda = 0$.

(ii). Parametric equation of $xy = c^2$ is $x = ct$ and $y = \dfrac{c}{t}$.

$\therefore$ $(x, y) = \left(ct, \dfrac{c}{t}\right)$ $(t \neq 0)$ is called a 't' point on the rectangular hyperbola.

Properties of rectangular hyperbola $xy = c^2$.

(a) Equation of a chord joining the points $P(t_1)$ & $Q(t_2)$ is $x + t_1t_2y = c(t_1 + t_2)$ with slope $m = -\frac{1}{t_1t_2}$.

(b) Equation of the tangent at $P(x_1, y_1)$ is $\frac{x}{x_1} + \frac{y}{y_1} = 2$ & at $P(t)$ is $\frac{x}{t} + ty = 2c$.

(c) Point of intersection of tangents at 't_1' and 't_2' is $\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1+t_2}\right)$.

(d) Equation of normal is $y - \frac{c}{t} = t^2(x - ct)$ or $xt^3 - yt - ct^4 + c = 0$

(e) Equation of normal (x_1, y_1) is $xx_1 - yy_1 = x_1^2 - y_1^2$.

(f) Chord with a given middle point as (h, k) is $kx + hy = 2hk$.

Illustration :

If the normal at the point 't_1' to the rectangular hyperbola $xy = c^2$ meets it again at the point 't_2', prove that $t_1^3 t_2 = -1$.

Sol. *Since the equation of normal at* $\left(ct_1, \frac{c}{t_1}\right)$ *to the hyperbola* $xy = c^2$ *is* $xt_1^3 - yt_1 - ct_1^4 + c = 0$.

but this passes through $\left(ct_2, \frac{c}{t_2}\right)$ *then* $ct_2t_1^3 - \frac{c}{t_2}t_1 - ct_1^4 + c = 0$

$\Rightarrow t_2^2t_1^3 - t_1 - t_1^4t_2 + t_2 = 0 \Rightarrow t_2t_1^3(t_2 - t_1) + (t_2 - t_1) = 0$

$\Rightarrow (t_1^3t_2 + 1)(t_2 - t_1) = 0 \Rightarrow t_1^3t_2 + 1 = 0 \qquad [\because t_2 \neq t_1]$

$\therefore t_1^3t_2 = -1.$

Illustration :

A triangle has its vertices on a rectangular hyperbola. Prove that the orthocentre of the triangle also lies on the same hyperbola.

Sol. $\therefore$ *Co-ordinates of A, B and C are* $\left(ct_1, \frac{c}{t_1}\right), \left(ct_2, \frac{c}{t_2}\right)$ *and* $\left(ct_3, \frac{c}{t_3}\right)$ *respectively are the vertices of a triangle lie on the rectangular hyperbola* $xy = c^2$.

Now slope of BC is $\dfrac{\frac{c}{t_3}-\frac{c}{t_2}}{ct_3-ct_2} = \dfrac{-1}{t_2t_3}$

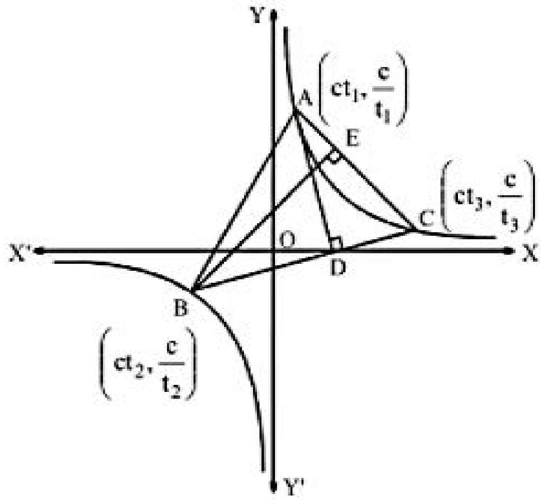
$\therefore$ Slope of AD is t_2t_3

Equation of altitude AD is $y-\dfrac{c}{t_1} = t_2t_3\,(x-ct_1)$

or $t_1y - c = xt_1t_2t_3 - ct_1^2\,t_2t_3$ ......(1)

Similarly equation of altitude BE is

$$t_2y - c = x\,t_1\,t_2\,t_3 - c\,t_1t_2^2\,t_3 \qquad ......(2)$$



Solving (1) and (2), we get the orthocentre $\left(\dfrac{-c}{t_1t_2t_3}, -ct_1t_2t_3\right)$ which also lies on $xy = c^2$.

Illustration :

If the normals at (x_i, y_i), $i = 1, 2, 3, 4$ on the rectangular hyperbola, $xy = c^2$, meet at the point (α, β) then show that

(i) $\Sigma x_i = \alpha$ (ii) $\Sigma y_i = \beta$ (iii) $\Pi x_i = \Pi y_i = -c^4$ (iv) $\Sigma x_i^2 = \alpha^2$ (v) $\Sigma y_i^2 = \beta^2$

Sol. Let $(x_i, y_i) = \left(ct_i, \dfrac{c}{t_i}\right)$, $i = 1, 2, 3, 4$ are the points on the rectangular hyperbola $xy = c^2$.

Equation of normal to the hyperbola $xy = c^2$ at $\left(ct, \dfrac{c}{t}\right)$ is $ct^4 - t^3x + ty - c = 0$

It passes through (α, β), then $ct^4 - t^3\alpha + t\beta - c = 0$

it is biquadratic equation in t. Let the roots of this equation are t_1, t_2, t_3, t_4 then $\Sigma t_1 = \dfrac{\alpha}{c}$

$$\Sigma t_1t_2 = 0,\ \Sigma t_1t_2t_3 = \frac{-\beta}{c},\ t_1t_2t_3t_4 = -1$$

Now (i) $\Sigma x_i = c\,\Sigma t_1 = \alpha$

(ii) $\Sigma y_i = c\left(\Sigma\dfrac{1}{t_i}\right) = \left(\dfrac{\Sigma\, t_1t_2t_3}{t_1t_2t_3t_4}\right) = \beta$

(iii) $\Pi\, x_i = c^4\,\Pi\, t_i = -c^4$

and $\Pi\, y_i = c^4\left(\dfrac{1}{\Pi\, t_i}\right) = -c^4$

(iv) $\Sigma x_i^2 = c^2\left(\Sigma\, t_i^2\right) = c^2\left((\Sigma\, t_1)^2 - 2\Sigma\, t_1t_2\right) = \alpha^2$

(v) $\Sigma y_i^2 = \left((\Sigma\, y_i)^2 - \Sigma\, y_1y_2\right) = b^2 - 2c^2\,\Sigma\left(\dfrac{1}{t_1t_2}\right) = b^2 - 2c\cdot\dfrac{\Sigma\, t_1t_2}{t_1t_2t_3t_4} = \beta^2.$