PROGRESSION & SERIES

1. INTRODUCTION :

A succession of numbers $t_1, t_2, \ldots t_n$ formed according to the some definite rule is called sequence. "A sequence is a function of natural numbers with codomain as the set of Real numbers or complex numbers"

Domain of sequence $= N$
if Range of sequence $\subseteq R \Rightarrow$ Real sequence
if Range of sequence $\subseteq C \Rightarrow$ Complex sequence

Sequence is called finite or infinite depending upon its having number of terms as finite or infinite respectively.
For example: 2, 3, 5, 7, 11, ... is a sequence of prime numbers. It is an infinite sequence.
A progression is a sequence having its terms in a definite pattern e.g.: 1, 4, 9, 16, ... is a progression as each successive term is obtained by squaring the next natural number.
However a sequence may not always have an explicit formula of n^{th} term.
Series is constructed by adding or subtracting the terms of a sequence e.g., $2 + 4 + 6 + 8 + \ldots$.. is a series. The term at n^{th} place is denoted by T_n and is called general term of a sequence or

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2. ARITHMETIC PROGRESSION (A.P.) :

2.1 Definition and Standard Appearance of A.P. :

It is sequence in which the difference between any term and its just preceding term remains constant throughout. This constant is called the "common difference" of the A.P. and is denoted by 'd' generally.
Standard appearance of an A.P. is

$$a, (a + d), (a + 2d) \ldots\ldots\ldots (a + \overline{n-1}\,d)$$

where 'a' denotes the first term of the AP

2.2 General term/n^{th} term/Last term of A.P. :

It is given by $T_n = a + (n - 1)d$
where a = first term, d = common difference and n = position of the term which we require.

Note : If $d > 0 \Rightarrow$ increasing A.P.
If $d < 0 \Rightarrow$ decreasing A.P.
If $d = 0 \Rightarrow$ all the terms remain same

Illustration :

If 5^{th} and 6^{th} terms of an A.P. are respectively 6 and 5. Find the 11^{th} term of the A.P.

Sol. $T_5 = 6, T_6 = 5$

$a + 4d = 6$

$a + 5d = 5$

$d = -1$

$a = 10$

$T_{11} = a + 10d$

$= 10 - 10 = 0.$ [Ans. 0]

Illustration :

In an A.P. if $a_2 + a_5 - a_3 = 10$ and $a_2 + a_9 = 17$ then find the 1^{st} term and the common difference.

Sol. *In an A.P.*

Let a_1 = first term & d = common difference

$a_2 + a_5 - a_3 = 10$

$a_1 + d + a_1 + 4d - (a_1 + 2d_1) = 10$

$a_1 + 3d_1 = 10$ ...(i)

$a_2 + a_9 = 17$

$a_1 + d + a_1 + 8d_1 = 17$

$a_1 + 3d_1 = 10$ ...(i)

$a_2 + a_9 = 17$

$a_1 + d + a_1 + 8d_1 = 17$

$2a_1 + 9d_1 = 17$ ...(ii)

On solving (i) & (ii) we get

$a_1 = 13, \quad d = -1$ [Ans. $a_1 = 13$ and $d = -1$]

Illustration :

If p^{th}, q^{th} and r^{th} term of an A.P. are respectively a, b, and c then find the value of

$$a(q - r) + b(r - p) + c(p - q)$$

Sol. $T_p = a, T_q = b, T_r = c$

Let first term be α and common difference be d.

$\alpha + (p - 1)d = a, \quad \alpha + (q - 1)d = b, \quad \alpha + (r - 1)d = c$

$(p - q)d = a - b$

$(q - r)d = b - c$

$(r - p)d = c - a$

$a(q - r) + b(r - p) + c(p - q)$

$$a\left(\frac{b-c}{d}\right)+b\left(\frac{c-a}{d}\right)+c\left(\frac{a-b}{d}\right)$$

$$= \frac{1}{d}(ab - ac + bc - ba + ca - cb)$$

$= 0.$ [Ans. 0]

2.3 Sum of n terms of an A.P. :

$$S_n = a \quad + (a+d) \quad + (a+2d) + \ldots\ldots\ldots\ldots + (a + \overline{n-1}\,d)$$
$$S_n = (a + \overline{n-1}\,d) + (a + \overline{n-2}\,d) + (a+d) + \ldots\ldots\ldots\ldots + a$$

$$2S_n = n\,[2a + (n-1)d]$$
$$S_n = \frac{n}{2}\,[2a + \overline{n-1}\,d]$$

or

$$S_n = \frac{n}{2}\left[a + a + (n-1)d\right] \Rightarrow S_n = \frac{n}{2}(a+l) \quad \text{where last term, } l = a + \overline{n-1}\,d$$

Illustration :

The first term of an A.P. is 5, the last is 45, and the sum 400. Find the number of terms and the common difference.

Sol. $a = 5,\ \ell = 45$

Let common difference = d

$a + (n-1)\,d = 45$

$S_n = 400$

$$\frac{n}{2}[a + \ell] = 400$$

$$\frac{n}{2}[5 + 45] = 400$$

$n = 16$

$a + (n-1)d = 45$

$5 + 15d = 45$

$d = \dfrac{8}{3}$. *[Ans. $n = 16,\ d = \dfrac{8}{3}$]*

Illustration :

Solve the equation $(x + 1) + (x + 4) + (x + 7) + \ldots\ldots + (x + 28) = 155$.

Sol. *We have,*

$(x + 1) + (x + 4) + \ldots\ldots + (x + 28) = 155$

Let n be the number of terms in the A.P. on L.H.S. Then,

$(x + 1) + (x + 4) + \ldots\ldots + (x + 28) = 155$

$\Rightarrow \quad \dfrac{10}{2}[(x + 1) + (x + 28)] = 155$

$\Rightarrow \quad x = 1.$ *[Ans. 1]*

Illustration :

How many terms of the sequence, $20 + 19\frac{1}{3} + 18\frac{2}{3} + ...$ *must be taken so that there sum is 300. Explain the reason of double answer*

Sol. $20 + 19\frac{1}{3} + 18\frac{2}{3} + ...$

$$a = 20, d = \frac{-2}{3}$$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$300 = \frac{n}{2}\left[2 \times 20 + (n-1)\left(\frac{-2}{3}\right)\right]$$

$$300 = 20\,n - \frac{n(n-1)}{3}$$

$$n^2 - 61n + 900 = 0$$

$$n = 25, 36$$

we got two values of n because when n = 25 all terms will be positive, but when n = 36, terms will become negative. *[Ans. 36 or 25]*

we got two values of n because when n = 25 all terms will be positive, but when n = 36, terms will become negative. *[Ans. 36 or 25]*

Illustration :

The sum of n terms of two A.P.'s are in the ratio of $7n + 1 : 4n + 27$*, find the ratio of their* 11^{th} *terms*

Sol. *Let the two series be* $a_1, a_1 + d_1, a + 2d_1 \ldots\ldots$ & $a_2\, a_2 + d_2, a_2 + 2d_2, \ldots\ldots$

$$S_1 = \frac{n}{2}[2a_1 + (n-1)d_1]$$

$$S_2 = \frac{n}{2}[2a_2 + (n-1)d_2]$$

$$\frac{S_1}{S_2} = \frac{7n+1}{4n+27}$$

$$\frac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2} = \frac{7n+1}{4n+27}$$

$$\frac{a_1 + \left(\frac{n-1}{2}\right)d_1}{a_2 + \left(\frac{n-1}{2}\right)d_2} = \frac{7n+1}{4n+27}$$

$T_{11} = a + 10d$

So to find ratio of 11^{th} term we should

Put $\frac{(n-1)}{2} = 10$

$n = 21$

$$\frac{a_1 + 10d_1}{a_2 + 10d_2} = \frac{7(21)+1}{4(21)+27} = \frac{4}{3}.$$ [Ans. 4/3]

Illustration :

In an A.P. t_n denotes n^{th} term and s_n denotes sum of first n terms.

If $t_7 = \frac{1}{9}$ and $t_9 = \frac{1}{7}$ then find the value of S_{63}.

Sol. *Let a and d are first term and common difference of the A.P.*

$$t_7 = \frac{1}{9} \text{ and } t_9 = \frac{1}{7}$$

$$a + 6d = \frac{1}{9} \qquad \text{........(i)}$$

$$a + 8d = \frac{1}{7} \qquad \text{.......(ii)}$$

,

$$a + 8d = \frac{1}{7} \qquad \text{.......(ii)}$$

Solving (i) and (ii), we get

$$a = \frac{1}{63},\ d = \frac{1}{63}$$

$$S_{63} = \frac{63}{2}\left(\frac{2}{63} + 62 \times \frac{1}{63}\right) = 32.$$ [Ans. 32]

Practice Problem

Q.1 In an A.P. if p^{th} term is q and q^{th} term is p then find its r^{th} term.

Q.2 Find the first negative term of the sequence $20, 19\frac{1}{4}, 18\frac{1}{2}, 17\frac{3}{4}, \ldots\ldots$

Q.3 In an A.P. if $S_p = q$ and $S_q = p$ then find S_{p+q}. (S_n denotes the sum of first n term of the A.P.)

Answer key

Q.1 $p + q - r$ Q.2 28^{th} Q.3 $-(p + q)$

2.4 Highlights of an A.P. :

(i) If each term of an A.P. is increased, decreased, multiplied or divided by the same non zero number, then the resulting sequence is also an AP.

(ii) Three numbers in AP can be taken as $a-d$, a, $a+d$; four numbers in AP can be taken as $a-3d$, $a-d$, $a+d$, $a+3d$; five numbers in AP are $a-2d$, $a-d$, a, $a+d$, $a+2d$ & six terms in AP are $a-5d$, $a-3d$, $a-d$, $a+d$, $a+3d$, $a+5d$ etc.

(iii) The common difference can be zero, positive or negative.

(iv) The sum of the two terms of an AP equidistant from the beginning & end is constant and equal to the sum of first & last terms.

(v) For any series, $T_n = S_n - S_{n-1}$. In a series if S_n is a quadratic function of n or T_n is a linear function of n, then the series is an A.P.

(vi) If a, b, c are in A.P. $\Rightarrow$ $2b = a + c$.

Illustration :

Illustration :

The sum of first 3 terms of an A.P. is 27 and the sum of their squares is 293. Find absolute value of S_{15}

Sol. *Let the terms be*

$a-d,\ a,\ a+d$

$3a = 27$

$a = 9$

$(a-d)^2 + a^2 + (a+d)^2 = 293$

$3a^2 + 2d^2 = 293$

$2d^2 = 50$

$d = \pm 5$

when $d = 5$ then A.P. is 4, 9, 14, ...

$$\therefore \quad S_{15} = \frac{15}{2}\,[2 \times 4 + 14 \times 5] = 585$$

when $d = -5$ then A.P. is 14, 9, 4,..........

$$\therefore \quad S_{15} = \frac{15}{2}\,[2 \times 14 + 14 \times (-5)] = -315$$

$\therefore \quad |S_{15}| = 315.$ *[Ans. 315 or 585]*

Illustration :

Find the nature and 30th term of the sequence whose sum to n terms is $5n^2 + 2n$

Sol. $S_n = 5n^2 + 2n$

$S_{n-1} = 5(n-1)^2 + 2(n-1)$

$T_n = S_n - S_{n-1} = 5(2n-1) + 2$

$= 10n - 3$

Series = 7, 17, 27, 37 ....

Series is A.P.

$T_{30} = 10 \times 30 - 3 = 297.$ [Ans. A.P. 7, 17, 27, 37 ...., $T_{30} = 297$]

Illustration :

If a^2, b^2, c^2 are in A.P. then prove that

(a) $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P. (b) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in A.P.

Sol. (a) Let $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

Sol. (a) Let $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

$$\frac{2}{c+a} = \frac{1}{a+b} + \frac{1}{b+c}$$

$$2(a+b)(b+c) = (c+a)(2b+a+c)$$

$$2b^2 = a^2 + c^2$$

hence, a^2, b^2, c^2 are in A.P.

so if a^2, b^2, c^2 are in A.P., then

$\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

(b) If a^2, b^2, c^2 are in A.P. then $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P. (Proved earlier)

Multiply each term with $a + b + c$

$$\frac{a+b+c}{b+c}, \frac{a+b+c}{c+a}, \frac{a+b+c}{a+b}$$

Subtract 1 from each term

$\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in A.P.

Illustration :

Find the condition that the roots of the equation $x^3 - px^2 + qx - r = 0$ may be in A.P. and hence solve the equation $x^3 - 12x^2 + 39x - 28 = 0$.

Sol. $x^3 - px^2 + qx - r = 0$

Let the three roots be $a - d, a, a + d$

$$3a = p \quad \Rightarrow \quad a = \frac{p}{3}$$

$$(a - d)\, a\, (a + d) = r \quad \Rightarrow \quad a\,(a^2 - d^2) = r$$

$$a^2 - d^2 = \frac{\mathrm{r}}{\mathrm{a}} = \frac{3r}{p}$$

$$a\,(a - d) + a\,(a + d) + (a - d)\,(a + d) = q$$

$$3a^2 - d^2 = q \quad \Rightarrow \quad 2a^2 + \frac{3r}{p} = q$$

$$2\left(\frac{p}{3}\right)^2 + \frac{3r}{p} = q$$

$$x^3 - 12x^2 + 39x - 28 = 0$$

Let the root be $a - d, a, a + d$

$$3a = 12 \quad \Rightarrow \quad a = 4$$

$$4(16 - d^2) = 28$$

$$16 - d^2 = 7$$

$$d^2 = 9$$

$$d = \pm 3$$

hence roots are 1, 4, 7. *[Ans. roots are 1, 4, 7]*

Illustration :

(a) If $\log_3 2$, $\log_3 (2^x - 5)$ & $\log_3\left(2^x - \frac{7}{2}\right)$ are in AP, determine x.

(b) Solve the equation $\frac{x-1}{x} + \frac{x-2}{x} + \ldots\ldots + \frac{1}{x} = 3$

Sol. (a) $2\log_3 (2^x - 5) = \log_3 2 + \log_3\left(2^x - \frac{7}{2}\right)$

$$\log_3 (2^x - 5)^2 = \log_3 (2.2x - 7)$$

$$(2^x - 5)^2 = 2.2^x - 7$$

Put $2^x = t$

$$(t - 5)^2 = 2t - 7$$

$$t^2 - 10t + 25 = 2t - 7$$

$$t^2 - 12t + 32 = 0$$

$$t = 4, 8$$

$$2^x = 4, 8$$

$$x = 2, 3$$

but at $x = 2$, $2^x - 5$ is negative hence $x = 3$ is only solution. *[Ans: x = 3]*

(b) $= \dfrac{(x-1)+(x-2)+(x-3)+....+3+2+1}{x} = \dfrac{1+2+3+....+(x-1)}{x}$

$= \dfrac{\frac{(x-1)(x)}{2}}{x} = \dfrac{x-1}{2} = 3$

$\Rightarrow x = 7$ [Ans. x = 7]

Illustration :

If the sum of the roots of the equation $ax^2 + bx + c = 0$ is equal to the sum of the squares of their reciprocals, then show that bc^2, ca^2, ab^2 are in AP.

Sol. $ax^2 + bx + c = 0$ Let α, β be the roots

$$\alpha + \beta = \frac{1}{\alpha^2} + \frac{1}{\beta^2}$$

$$\alpha^2 \beta^2 (\alpha + \beta) = (\alpha + \beta)^2 - 2\alpha\beta$$

$$\frac{c^2}{a^2} \cdot \left(\frac{-b}{a}\right) = \frac{b^2}{a^2} - \frac{2c}{a}$$

$$2a^2 c = bc^2 + ab^2$$

hence bc^2, a^2c, ab^2 are in A.P.

Illustration :

Illustration :

Given $a_1, a_2, a_3.....a_n$ in A.P and

if $\dfrac{1}{a_1a_n} + \dfrac{1}{a_2a_{n-1}} + \dfrac{1}{a_3a_{n-1}} + ... + \dfrac{1}{a_na_1} = \dfrac{\lambda}{a_1+a_n}\left[\dfrac{1}{a_1} + \dfrac{1}{a_2} + ... + \dfrac{1}{a_n}\right]$ then find the value of λ.

Sol. $S = \dfrac{1}{a_1a_n} + \dfrac{1}{a_2a_{n-1}} + \dfrac{1}{a_3a_{n-1}} + ... + \dfrac{1}{a_na_1}$

$$S = \frac{1}{(a_1+a_n)}\left[\frac{a_1+a_n}{a_1a_n} + \frac{a_1+a_n}{a_2a_{n-1}} + \frac{a_1+a_n}{a_3a_{n-2}} + ... + \frac{a_1+a_n}{a_na_1}\right]$$

but $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = ...$

Hence

$$S = \frac{1}{(a_1+a_n)}\left[\frac{a_1+a_n}{a_1a_n} + \frac{a_2+a_{n-1}}{a_2a_{n-1}} + \frac{a_3+a_{n-2}}{a_3a_{n-2}} + ... + \frac{a_1+a_n}{a_na_1}\right]$$

$$= \frac{1}{(a_1+a_n)}\left[\frac{1}{a_n} + \frac{1}{a_1} + \frac{1}{a_{n-1}} + \frac{1}{a_2} + ... + \frac{1}{a_n} + \frac{1}{a_1}\right]$$

$$= \frac{2}{a_1+a_n}\left[\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right]$$

Hence $\lambda = 2$. [Ans. 2]

2.5 Arithmetic Mean (A.M.) :

2.5.1 Definition :

When three quantities are in A.P. then the middle one is called the *Arithmetic Mean* of the other two. *e.g.* a, b, c are in A.P. then 'b' is the arithmetic mean between 'a' and 'c' and $a + c = 2b$.
It is to be noted that between two given quantities it is always possible to insert any number of terms such that the whole series thus formed shall be in A.P. and the terms thus inserted are called the *arithmetic means*.

2.5.2 To insert 'n' AM's between a and b :

Let A_1, A_2, A_3 .... A_n are the n means between a and b.
Hence $a, A_1A_2,$ .... A_nb is an A.P. and b is the $(n + 2)^{th}$ terms.

Hence $b = a + (n + 1)d \quad \Rightarrow \quad d = \dfrac{b-a}{n+1}$

Now
$$A_1 = a + d$$
$$A_2 = a + 2d$$
$$\vdots$$
$$A_n = a + nd$$

$$\sum_{i=1}^{n} A_i = na + (1 + 2 + 3 + ... + n)d$$

$$= na + \frac{n(n+1)}{2}d = na + \frac{n(n+1)}{2}\cdot\frac{b-a}{n+1}$$

$$= \frac{n}{2}[2a + b - a] = n\left(\frac{a+b}{2}\right) = na$$

Hence the sum of n AM's inserted between a and b is equal to n times a single AM between them.

Illustration :

Insert 20 AM's between 4 and 67.

Sol. $a = 4,\ b = 67,\ n = 20\ \ d = \dfrac{b-a}{n+1} = \dfrac{67-4}{21} = 3$

$$A_1 = a + d = 4 + 3 = 7$$
$$A_2 = a + 2d = 4 + 6 = 13$$
$$A_3 = 19$$
$$A_4 = 25$$
$$A_{20} = a + 20d = 4 + 20 \times 3 = 64.$$

[Ans. 7, 13, 19, ..., 64]

Illustration :

If p arithmetic means are inserted between 5 and 41 so that the ratio $\dfrac{A_3}{A_{p-1}} = \dfrac{2}{5}$ *then find the value of p.*

Sol. $a = 5,\ b = 41$, *number of airthmetic means = p*

$$d = \frac{b-a}{n+1} = \frac{36}{p+1}$$

$$A_3 = a + 3d = 5 + 3\left(\frac{36}{p+1}\right) = \frac{5p+5+108}{p+1}$$

$$A_{p-1} = 41 - d = 41 - 2\left(\frac{36}{p+1}\right) = \frac{41p-31}{p+1} = \frac{5p+113}{41p-31} = \frac{2}{5}$$

$25p + 565 = 82p - 62 \Rightarrow 57p = 627 \quad \Rightarrow \quad p = 11.$ [Ans. $p = 11$]

Illustration :

If eleven A.M.'s are inserted between 28 and 10, then find the number of integral A.M.'s.

Sol. *Assume* $A_1, A_2, A_3 \ldots\ldots, A_{11}$ *be the eleven A.M.'s between 28 and 10, so* $28, A_1, A_2, \ldots\ldots A_{11}, 10$ *are in A.P. Let d be the common difference of the A.P. The number of terms is 13. Now,*

$$10 = T_{13} = T_1 + 12d = 28 + 12d$$

Sol. *Assume* $A_1, A_2, A_3 \ldots\ldots, A_{11}$ *be the eleven A.M.'s between 28 and 10, so* $28, A_1, A_2, \ldots\ldots A_{11}, 10$ *are in A.P. Let d be the common difference of the A.P. The number of terms is 13. Now,*

$$10 = T_{13} = T_1 + 12d = 28 + 12d$$

$$\Rightarrow \quad d = \frac{10-28}{12} = -\frac{18}{12} = -\frac{3}{2}$$

Hence, the number of integral A.M.'s is 5. [Ans. 5]

Practice Problem

Q.1 If the first 3 terms of an increasing A.P. are the roots of the cubic

$$4x^3 - 24x^2 + 23x + 18 = 0, \text{ then find } S_{19}.$$

Q.2 Divide 32 into four parts which are in A.P. such that the ratio of the product of extremes to the product of mean is 7: 15.

Q.3 If a, b, c are in A.P. then prove that

(a) $b + c\ ;\ c + a\ ;\ a + b$ are in A.P.

(b) $(b + c)^2 - a^2\ ;\ (c + a)^2 - b^2\ ;\ (a + b)^2 - c^2$ are in A.P.

Q.4 If n arithmetic means are inserted between 2 and 38, then the sum of the resulting series is obtained as 200. Then find the value of n.

Answer key

Q.1	418	Q.2	2, 6, 10, 14	Q.4	10

3. GEOMETRIC PROGRESSION (G.P.) :

3.1 Definition and Standard Appearance of G.P. :

In a sequence if each term (except the first non zero term) bears the same constant ratio with its immediately preceding term then the series is called a G.P. and the constant ratio is called the common ratio.
Standard appearance of a G.P. is

$$a, ar, ar^2, ar^3, ..., ar^{n-1} \ .$$

3.2 General term/n^{th} term/Last term of G.P. :

It is given by $T_n = a \cdot r^{n-1}$
where a = first term, r = common ratio and n = position of the term which we required.

Illustration :

In a G.P. if $t_3 = 2$ and $t_6 = -\frac{1}{4}$ find t_{10}.

Sol. $t_3 = 2, \ t_6 = -\frac{1}{4}$

$$ar^2 = 2 \ , \ ar^5 = \frac{1}{4}$$

$$1 \qquad\qquad 1$$

$$\frac{1}{r^3} = -8 \qquad \Rightarrow \qquad \frac{1}{r} = -2$$

$$r = \frac{-1}{2} \qquad a = 8$$

$$t_{10} = ar^9 = 8\left(\frac{-1}{2}\right)^9 = \frac{-1}{64} \qquad [Ans. -\frac{1}{64}]$$

Illustration :

Find the four successive terms of a G.P. of which the 2^{nd} term is smaller than the first by 35 and the 3^{rd} term is larger than the 4^{th} by 560.

Sol. *Let the four terms be*

$$a_1, ar_1, ar^2, ar^3$$
$$a - ar = 35$$
$$ar^2 - ar^3 = 560$$
$$\frac{a(1-r)}{ar^2(1-r)} = \frac{35}{560}$$
$$r^2 = 16$$
$$r = \pm 4$$

Since 2^{nd} term is less than 1^{st} so $r = -4$

$$a = 7$$

terms are 7, –28, 112, –448 *[Ans. 7, – 28, 112, – 448]*

Illustration :

If p^{th}, q^{th} and r^{th} terms of a G.P. are x, y and z respectively then find the value of $x^{q-r} \cdot y^{r-p} \cdot z^{p-q}$

Sol. *Let a and k be the first term and common ratio of the G.P. respectively.*

$$x = a\,k^{p-1}$$
$$y = a\,k^{q-1}$$
$$z = a\,k^{r-1}$$
$$x^{q-r} \cdot y^{r-p} \cdot z^{p-q}$$
$$= \left(\frac{x}{z}\right)^q \cdot \left(\frac{y}{x}\right)^r \cdot \left(\frac{z}{y}\right)^p$$
$$= (k^{p-r})^q\ (k^{q-p})^r (k^{r-q})^p$$
$$= k^{pq-qr\,+\,qr-rp+rp-pq}$$
$$= k^0 = 1.$$

[Ans. 1]

3.3 Sum of n terms of a G.P. :

$$S = a + ar + ar^2 + \ldots\ldots\ldots + ar^{n-1}$$
$$Sr = \quad + ar\ + ar^2 + \ldots\ldots\ldots\ldots\ldots\ldots + ar^n$$

subtract $\quad - \quad - \quad\quad\quad\quad -$

$$S(1-r) = a - ar^n = a\,(1 - r^n)$$

$$S(1-r) = a - ar^n = a\,(1 - r^n)$$

$$S = \frac{a(1-r^n)}{1-r}, \text{ where } r \neq 1, (\text{if } r = 1 \text{ then } S = na)$$

3.4 Sum of infinite terms of a G.P. :

If $|r| < 1$ and $n \to \infty$ then $r^n \to 0$ and in this case geometric series will be summable upto infinity and its sum is given by

$$S_\infty = \frac{a}{1-r}$$

Illustration :

Determine the number of terms in a G.P. if $a_1 = 3$, $a_n = 96$ and $S_n = 189$.

Sol. $a_1 = 3$

$$a_1\, r^{n-1} = 96$$
$$r^{n-1} = 32$$
$$\frac{a(1-r^n)}{1-r} = 189 \qquad \Rightarrow \qquad \frac{3(1-32r)}{1-r} = 189$$
$$93r = 186$$
$$r = 2$$
$$2^{n-1} = 32$$
$$n = 6.$$

[Ans. 6]

Illustration :

Find the least value of n ($n \in N$) for which sum of the series $1 + 3 + 3^2 + 3^3 + \ldots\ldots$ upto n terms exceeds 9000.

Sol. $S_n = 1 + 3 + 3^2 + \ldots\ldots + n$ *term*

$$S_n = \frac{1\cdot\left(3^{n-1}\right)}{3-1} > 9000$$

$$3^n - 1 > 18000$$

$$3^n > 18001$$

$$n \geq 9$$

least value of n = 9. [Ans. 9]

Illustration :

Find the sum of n terms of a sequence $\left(x+\frac{1}{x}\right)^2, \left(x^2+\frac{1}{x^2}\right)^2, \left(x^3+\frac{1}{x^3}\right)^2, \ldots\ldots$.

Sol. $\left(x+\frac{1}{x}\right)^2 \left(x^2+\frac{1}{x^2}\right)^2 \left(x^3+\frac{1}{x^3}\right)^2 \quad \left(x^n+\frac{1}{x^n}\right)^2$

Sol. $\left(x+\frac{1}{x}\right)^2, \left(x^2+\frac{1}{x^2}\right)^2, \left(x^3+\frac{1}{x^3}\right)^2 \ldots\ldots \left(x^n+\frac{1}{x^n}\right)^2$

$$x^2 + \frac{1}{x^2} + 2,\ x^4 + \frac{1}{x^4} + 2,\ x^6 + \frac{1}{x^6} + 2 + \ldots\ldots + x^{2n} + \frac{1}{x^{2n}} + 2$$

$$S = (x^2 + x^4 + \ldots\ldots + x^{2n}) + \left(\frac{1}{x^2}+\frac{1}{x^4}+\ldots\ldots+\frac{1}{x^{2n}}\right) + 2n$$

$$= x^2\left(\frac{x^{2n}-1}{x^2-1}\right) + \frac{1}{x^2}\frac{\left(1-\frac{1}{x}2n\right)}{\left(1-\frac{1}{x^2}\right)} + 2n$$

$$= x^2\frac{(x^{2n}-1)}{x^2-1} + \left(\frac{x^{2n}-1}{x^2-1}\right)\frac{1}{x^{2n}} + 2n$$

$$= \left(\frac{x^{2n}-1}{x^2-1}\right)\left(x^2+\frac{1}{x^{2n}}\right) + 2n.$$

[Ans. $\left(\frac{x^{2n}-1}{x^2-1}\right)\left(x^2+\frac{1}{x^{2n}}\right) + 2n$]

Illustration :

The sum of an infinite number of terms of a G.P. is 15 and the sum of their squares is 45. Find the series.

Sol. $a + ar + ar^2 + ar^2 + \ldots\ldots = 15$

$$\frac{a}{1-r} = 15 \qquad \ldots(i)$$

$a^2 + (ar)^2 + (ar^2)^2 + \ldots\ldots = 45$

$$\frac{a^2}{1-r^2} = 45 \qquad \ldots(ii)$$

$$\frac{a^2}{(1-r^2)} = 15 \qquad \ldots(iii)$$

$$\frac{(1-r)^2}{1-r^2} = \frac{45}{225} \quad \Rightarrow \quad \frac{1-r}{1+r} = \frac{1}{5}$$

$5 - 5r = 1 + r$

$r = 2/3$

$$a = 15\left(1-\frac{2}{3}\right) = 5$$

$$5, \frac{10}{3}, \frac{20}{9}, \frac{40}{27} \ldots\ldots \qquad \left[Ans.\ 5, \frac{10}{3}, \frac{20}{9}, \ldots\ldots\ldots\right]$$

Illustration :

If $x = a + a/r + a/r^2 + \ldots\ldots\infty$, $y = b - b/r + b/r^2 - \ldots\ldots\ \infty$ and $z = c + c/r^2 + c/r^4 + \ldots\ldots\ \infty$, then prove that $xy/z = ab/c$.

Sol. We have,

$$x = \frac{a}{1-\frac{1}{r}} = \frac{ar}{r-1}$$

$$y = \frac{b}{1-\left(-\frac{1}{r}\right)} = \frac{br}{1+r}$$

$$z = \frac{c}{1-\frac{1}{r^2}} = \frac{cr^2}{r^2-1}$$

$$\therefore \quad xy = \left(\frac{ar}{r-1}\right)\left(\frac{br}{r+1}\right) = \frac{abr^2}{r^2-1}$$

$$\therefore \quad \frac{xy}{z} = \left[\frac{\frac{abr^2}{r^2-1}}{\frac{cr^2}{r^2-1}}\right] = \frac{ab}{c}.$$

Practice Problem

Q.1 Which term of the G.P. $2, 1, \frac{1}{2}, \frac{1}{4}, ...$ is $\frac{1}{28}$?

Q.2 The first term of a G.P. is 1. The sum of the third and fifth terms is 90. Find the common ratio the G.P.

Q.3 Fifth term of a G.P. is 2. Find the product of its first nine terms.

Q.4 Prove that $6^{\frac{1}{2}} \times 6^{\frac{1}{4}} \times 6^{\frac{1}{8}} .....\infty = 6$.

Q.5 Find the sum of the following series :

(a) $\left(\sqrt{2}+1\right)+1+\left(\sqrt{2}-1\right)+......\infty$ (b) $\frac{1}{2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^4}+\frac{1}{2^5}+\frac{1}{3^6}+.....\infty$

Q.6 Sum of infinite number of terms in G.P. is 20 and sum of their squares is 100.
Then find the common ratio of G.P.

Answer key

Q.1 9^{th} Q.2 ± 3 Q.3 512 Q.5 (a) $\sqrt{2}-1$ (b) $\frac{19}{24}$ Q.6 $\frac{3}{5}$

3.5 Highlights of G.P. :

(i) If each term of a GP be multiplied or divided by the same non-zero quantity, the resulting sequence is also a GP.

(ii) If each term of a G.P. raised the same power then resulting sequence is also a G.P.

(iii) Any 3 consecutive terms of a GP can be taken as a/r, a, ar ; any 4 consecutive terms of a GP can be taken as a/r^3, a/r, ar, ar^3 & so on.

(iv) In a finite G.P. the product of the terms equidistant from the begining and the end are equal.
$a_1a_n = a_2a_{n-1} = a_3a_{n-2} = ...$

(v) If a, b, c are in GP $\Rightarrow$ $b^2 = ac$.

Illustration :

The sum of first 3 consecutive terms of a G.P. is 19 and their product is 216. Find S_{20}, also compute s_∞ if it exist.

Sol. *Let the 3 consecutive terms be*

$$\frac{a}{r}, a, ar$$

$$\frac{a}{r} + a + ar = 19$$

$$\frac{a}{r} \cdot a \cdot ar = 216$$
$$a^3 = 216$$
$$a = 6$$

$$\frac{6}{r} + 6 + 6r = 19$$
$$6(1 + r + r^2) = 19r$$
$$6r^2 - 13r + 6 = 0$$
$$(3r - 2)(2r - 3) = 0$$
$$r = \frac{2}{3}, \frac{3}{2}$$

when $r = \frac{3}{2}$, $S_{20} = 8\left[\left(\frac{3}{2}\right)^{20} - 1\right]$ and S_∞ does not exist.

when $r = \frac{2}{3}$, $S_{20} = 27\left(1 - \left(\frac{2}{3}\right)^{20}\right)$ and $S_\infty = \dfrac{9}{1-\frac{2}{3}} = 2$

[Ans. $S_{20} = 8\left(\left(\frac{3}{2}\right)^{20} - 1\right)$, S_∞ does not exist ; $S_{20} = 27\left(1 - \left(\frac{2}{3}\right)^{20}\right)$, $S_\infty = 27$]

Illustration :

If $a_n = n^{th}$ term of G.P. and

$a_1 + a_2 + a_3 = 13$ and $a_1^2 + a_2^2 + a_3^2 = 91$ then find a_{50}.

Sol. Given

$a_1 + a_2 + a_3 = 13$, Let $a_1 = \frac{a}{r}$, $a_2 = a$, $a_3 = ar$

$$\frac{a}{r} + a + ar = 13 \quad \Rightarrow \quad a\left(\frac{1}{r} + 1 + r\right) = 13$$

$$r + \frac{1}{r} = \frac{13}{a} - 1$$

$$r + \frac{1}{r} = \frac{13-a}{a}$$

$$r^2 + \frac{1}{r^2} + 2 = \left(\frac{13-a}{a}\right)^2$$

Now given

$$a_1^2 + a_2^2 + a_3^2 = 91$$

$$a^2\left(\frac{1}{r^2} + 1 + r^2\right) = 91$$

$$a^2\left(\frac{(13-a)^2}{a^2}-1\right) = 91$$

$(13 - a)^2 - a^2 = 91 \quad \Rightarrow \quad 13^2 - 26a = 91$

$26a = 169 - 91 \quad \Rightarrow \quad 26a = 78$

$a = 3$

$3r^2 + 3 = 10r \quad \Rightarrow \quad 3r^2 - 19r + 3 = r$

$3r^2 - 9r - r + 3 = 0$

$r = 3, \frac{1}{3}$

When $a = 3$ *and* $r = 3$ *then* $a_{50} = 3 \cdot 3^{49} = 3^{50}$.

and when $a = 3$ *and* $r = \frac{1}{3}$ *then* $a_{50} = 3 \cdot \left(\frac{1}{3}\right)^{49} = \left(\frac{1}{3}\right)^{48}$. [*Ans.* 3^{50} *or* $\left(\frac{1}{3}\right)^{48}$]

Illustration :

If a, b, c, d are in G.P., then prove that $(a^3 + b^3)^{-1}, (b^3 + c^3)^{-1}, (c^3 + d^3)^{-1}$ *are also in G.P.*

Sol. *Let* $b = ar$, $c = ar^2$ *and* $d = ar^3$. *Then*

$$\frac{1}{a^3+b^3} = \frac{1}{a^3(1+r^3)}$$

$$\frac{1}{\quad} = \frac{1}{\quad}$$

$$\frac{1}{b^3+c^3} = \frac{1}{a^3r^3(1+r^3)}$$

$$\frac{1}{c^3+d^3} = \frac{1}{a^3r^6(1+r^3)}$$

Clearly, $(a^3 + b^3)^{-1}$, $(b^3 + c^3)^{-1}$ *and* $(c^3 + d^3)^{-1}$ *are in G.P. with common ratio* $1/r^3$.

3.6 Sequences convertible to G.P. :

Illustration :

Use infinite series to compute the rational number corresponding to $0.4\overline{23}$

Sol.

$$x = 0.4\overline{23}$$

$$= 0.4 + 0.023 + 0.00023 + \ldots\ldots$$

$$= 0.4 + \frac{23}{10^3}+\frac{23}{10^5}+\frac{23}{10^7}+\ldots\ldots$$

$$= \frac{4}{10}+\frac{23}{10^3}\left(1+\frac{1}{10^2}+\frac{1}{10^4}+\ldots\ldots\right)$$

$$= \frac{4}{10}+\frac{23}{10^3}\left(\frac{1}{1-1/100}\right)$$

$$x = \frac{4}{10}+\frac{23}{990} = \frac{419}{990}$$

Illustration :

(a) If $9 + 99 + 999 + \ldots\ldots + \text{upto } 49 \text{ terms} = 10\dfrac{(10^{\lambda} - 1)}{\mu} - 49$, where $\lambda, \mu \in N$ then find the value of $\lambda + \mu$

(b) $0.9 + 0.99 + 0.999 + \ldots\ldots\text{ up to } 51 \text{ terms} = 51 - \dfrac{1}{p}\left(1 - \dfrac{1}{10^{q}}\right)$ where $p, q \in N$ then find the value of $p + q$.

Sol. (a) $S = 9 + 99 + 999 + \ldots\ldots + \text{upto } 49 \text{ terms}$

$S = 10 - 1 + 10^2 - 1 + 10^3 - 1 + \ldots\ldots + 10^{49} - 1$

$= (10 + 10^2 + 10^3 + \ldots\ldots 10^{49}) - 49$

$$S = 10 \cdot \left(\frac{10^{49} - 1}{9}\right) - 49$$

$\lambda + \mu = 49 + 9 = 58$

(b) $S = 0.9 + 0.99 + 0.999 + \ldots\ldots\text{up to } 51 \text{ terms}$

$$= \frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + \ldots\ldots \text{up to } 51 \text{ terms}$$

$$= 1 - \frac{1}{10} + 1 - \frac{1}{10^2} + 1 - \frac{1}{10^3} + \ldots\ldots + 1 - \frac{1}{10^{51}}$$

$$= 1 - \frac{1}{10} + 1 - \frac{1}{10^2} + 1 - \frac{1}{10^3} + \ldots\ldots + 1 - \frac{1}{10^{51}}$$

$$= 51 - \left(\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \ldots\ldots + \frac{1}{10^{51}}\right)$$

$$= 51 - \frac{\frac{1}{10}\left(1 - \frac{1}{10^{51}}\right)}{1 - \frac{1}{10}} = 51 - \frac{1}{9}\left(1 - \frac{1}{10^{51}}\right).$$

$\therefore \quad p + q = 60$ [Ans. (a) 58 (b) 60]

Illustration :

Find the sum $S = (x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + \ldots\ldots n$ terms.

Sol. It is easy to observe that

$$\frac{x^2 - y^2}{x - y} = x + y, \quad \frac{x^3 - y^3}{x - y} = x^2 + xy + y^2, \quad \frac{x^n - y^n}{x - y} = x^{n-1} + x^{n-2}y + \ldots\ldots + xy^{n-2} + y^{n-1}$$

$$S = \frac{1}{x - y}[(x^2 - y^2) + (x^3 - y^3) + \ldots\ldots n \text{ terms}]$$

$$= \frac{1}{x - y}\left[\frac{x^2(1 - x^n)}{1 - x} - \frac{y^2(1 - y^n)}{1 - y}\right].$$

[Ans. $\dfrac{1}{x - y}\left[\dfrac{x^2(1 - x^n)}{1 - x} - \dfrac{y^2(1 - y^n)}{1 - y}\right]$]

Illustration :

Find the sum of series $\frac{3}{19}+\frac{33}{19^2}+\frac{333}{19^3}+\frac{3333}{19^4}+..........\infty$

Sol. $S=\frac{3}{19}+\frac{33}{19^2}+\frac{333}{19^3}+\frac{3333}{19^4}+..........\infty$

or $S=\frac{3}{9}\left[\frac{9}{19}+\frac{99}{19^2}+\frac{999}{19^3}+......\infty\right]$

$$\Rightarrow \quad \frac{3}{9}\left[\frac{10-1}{19}+\frac{10^2-1}{19^2}+\frac{10^3-1}{19^3}+......\infty\right]$$

$$\Rightarrow \quad \frac{3}{9}\left[\left(\left(\frac{10}{19}\right)+\left(\frac{10}{19}\right)^2+\left(\frac{10}{19}\right)^3+......\infty\right)-\left(\frac{1}{19}+\frac{1}{19^2}+......\infty\right)\right]$$

$$S=\frac{3}{9}\left[\frac{10/19}{1-10/19}-\left(\frac{1/19}{1-1/19}\right)\right]$$

$$S=\frac{3}{9}\left[\frac{10/19}{9/19}-\frac{1}{18}\right]$$

$$=\frac{3}{9}\left[\frac{19}{18}\right]\Rightarrow\frac{19}{54}. \qquad [Ans.\ \frac{19}{54}]$$

$$=\frac{3}{9}\left[\frac{19}{18}\right]\Rightarrow\frac{19}{54}. \qquad [Ans.\ \frac{19}{54}]$$

Practice Problem

Q.1 If the continued product of three numbers in a G.P. is 216 and the sum of their products in pairs is 156, find the numbers.

Q.2 Three number are in G.P. whose sum is 70. If the extremes be each multiplied by 4 and the means by 5, they will be in A.P. Find the numbers.

Q.3 Using infinite G.P. express the number $2.3\overline{57}$ in rational form.

Q.4 Find the sum of the following series.
(i) $5+55+555+......$ to n terms
(ii) $0.3+0.33+0.333+......$ to n terms

Answer key

Q.1 2, 6, 18 or 18, 6, 2 Q.2 10, 20, 40 Q.3 $\frac{389}{165}$

Q.4 (i) $\frac{5}{9}\left[10\left(\frac{10^n-1}{9}\right)-n\right]$; (ii) $\frac{1}{3}\left[n-\frac{1}{9}\left(1-\frac{1}{10^n}\right)\right]$

3.7 Geometric Mean (G.M.) :

3.7.1 Definition :

If a, b, c are three positive numbers in G.P. then b is called the geometrical mean between a and c and $b^2 = ac$. If a and b are two positive real numbers and G is the G.M. between them, then

$$G^2 = ab$$

3.7.2 To insert 'n' GM's between a and b :

Let a and b are two positive numbers are $G_1, G_2, ...G_n$ are 'n' GM's then

$a\ G_1\ G_2\ ....G_n\ b$ is a G.P. with 'b' as its $(n+2)^{th}$ term.

Hence $\quad b = ar^{n+1}$

$$\therefore \quad r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

Now $\quad G_1 = ar,\ G_2 = ar^2,\ ....G_n = ar^n$

hence $\displaystyle\prod_{i=1}^{n} G_i = a^n \quad r^{1+2+....+n} = a^n \cdot r^{\frac{n(n+1)}{2}} = a^n\left[\left(\frac{b}{a}\right)^{\frac{1}{n+1}}\right]^{\frac{n(n+1)}{2}}$

$$= a^n \cdot \frac{b^{n/2}}{a^{n/2}} = a^{n/2} \cdot b^{n/2} = \left(\sqrt{ab}\right)^n = G^n$$

where G is the angle GM between a and b.

Hence product of n GM's inserted between of a and b is equal to the n^{th} power of a single GM between them.

Illustration :

Insert 4 GM's between 5 and 160.

Sol. Four GM between 5 & 160

$$r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}} = \left(\frac{160}{5}\right)^{\frac{1}{5}}$$

$$= (32)^{1/5} = 2$$

$G_1 = ar = 10$

$G_2 = 20$

$G_3 = 40$

$G_4 = 80.$ [Ans. 10, 20, 40, 80]

Illustration :

If AM between two positive numbers a and b is 15 and GM between a and b is 9. Find the numbers.

Sol. $\frac{a+b}{2} = 15$

$$a + b = 30$$

$$\sqrt{ab} = 9$$

$$\sqrt{a(30-a)} = 9$$

$$a(30 - a) = 81$$

$$a^2 = 30a + 81 = 0$$

$$a = 3, 27$$

Hence two nos. are 3, 27. *[Ans. 3, 27]*

Illustration :

If sum of two numbers a and b (a > b) is 3 times their GM and given that $a : b = \left(p+\sqrt{q}\right) : \left(p-\sqrt{q}\right)$ *where p and q are prime numbers. Find p + q*

Sol. $a + b = 3\sqrt{ab}$

$$\frac{a+b}{\ } = \frac{3}{\ }$$

Sol. $a + b = 3\sqrt{ab}$

$$\frac{a+b}{\sqrt{ab}} = \frac{3}{1}$$

$$\frac{a+b}{2\sqrt{ab}} = \frac{3}{2}$$

Applying componendo and dividendo

$$\frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \frac{3+2}{3-2}$$

$$\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}\right)^2 = \frac{5}{1}$$

$$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{5}}{1}$$

Again, applying componendo - dividendo.

$$\frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{5}+1}{\sqrt{5}-1}$$

$$\frac{a}{b} = \frac{6+2\sqrt{5}}{6-2\sqrt{5}} = \frac{3+\sqrt{5}}{3-\sqrt{5}}$$

$\therefore$ $a : b = 3+\sqrt{5} : 3-\sqrt{5}$

$\Rightarrow$ $p + q = 3 + 5 = 8.$ *[Ans. 8]*

Illustration :

If a, b, c are in G.P. and x, y are the AM's between a, b and b, c respectively then prove that

$$\frac{1}{x} + \frac{1}{y} = \frac{2}{b} \quad \text{and} \quad \frac{a}{x} + \frac{c}{y} = 2$$

Sol. *a, b, c are in G.P.* $\Rightarrow b^2 = ac$

Now

$$x = \frac{a+b}{2}, \qquad y = \frac{b+e}{2}$$

$$\frac{1}{x} + \frac{1}{y} = \frac{2}{a+b} + \frac{2}{b+c}$$

$$= 2\left[\frac{b+c+a+b}{(a+b)(b+c)}\right] = 2\left[\frac{2b+a+c}{ab+ac+b^2+bc}\right]$$

$$= 2\left[\frac{2b+a+c}{2b^2+ab+bc}\right] \quad (ac = b^2)$$

$$= \frac{2\,(2b+a+c)}{b\,(2b+a+c)} = \frac{2}{b}$$

$$\frac{a}{x} + \frac{c}{y} = \frac{2a}{a+b} + \frac{2c}{b+c}$$

$$\frac{a}{x} + \frac{c}{y} = \frac{2a}{a+b} + \frac{2c}{b+c}$$

$$= 2\left[\frac{ab+ac+ac+bc}{2b^2+ab+bc}\right] \qquad (\text{since } ac = b^2)$$

$$= 2\left[\frac{ab+bc+2b^2}{2b^2+ab+bc}\right]$$

$$= 2$$

3.8 Relation between A.M. and G.M. :

A.M. ≥ G.M

Let A and G be the AM and GM between two positive numbers a and b then $A \geq G$

(Sign of equality holds when $a = b$)

$$\text{AM between a \& b, } A = \frac{a+b}{2}$$

$$\text{GM between a \& b, } G = \sqrt{ab}$$

$$A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{1}{2}\left(\sqrt{a} - \sqrt{b}\right)^2$$

$\Rightarrow$ $A - G > 0 \Rightarrow A \geq G$

(If $a = b$ then $A - G = 0 \Rightarrow A = G$)

Illustration :

If $x > 0,\ y > 0, z > 0$ *then prove that* $(x + y)(y + z)(z + x) \geq 8xyz$

Sol. $(x + y)\ (y + z)\ (z + x)$

$$\frac{x+y}{2} \geq \sqrt{xy} \qquad (A.M. \geq G.M.)$$

$$\frac{y+z}{2} \geq \sqrt{yz}$$

$$\frac{z+x}{2} \geq \sqrt{zx}$$

$$\frac{(x+y)(y+z)(z+x)}{8} \geq xyz$$

$$(x + y)\ (y + z)\ (z + x) \geq 8xyz$$

Illustration :

Prove that a $\Delta\, ABC$ *is equilateral if and only if*

$$tanA + tanB + tanC = 3\sqrt{3}$$

Sol. $\dfrac{\tan A + \tan B + \tan C}{3} \geq (\tan A \tan B \tan C)^{1/3}$

since $A + B + C = \pi$

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$\dfrac{\tan A + \tan B + \tan C}{3}\ {}^{3}\ (\tan A + \tan B + \tan C)^{1/3}$

$(\tan A + \tan B + \tan C)^3 \geq 27\ (\tan A + \tan B + \tan C)$

$(\tan A + \tan B + \tan C)^2 \geq 27$

$\tan A + \tan B + \tan C \geq 3\sqrt{3}$

Illustration :

If $a + b + c = 3$ *and* a, b, c *are positive then prove that* $a^2b^3c^2 \leq \dfrac{3^{10} \cdot 2^4}{7^7}$

Sol. $a + b + c = 3$

We can write it as

$$\frac{a}{2} + \frac{a}{2} + \frac{b}{3} + \frac{b}{3} + \frac{b}{3} + \frac{c}{2} + \frac{c}{2} = 3$$

Now A.M. ≥ G.M.

$$\frac{\frac{a}{2}+\frac{a}{2}+\frac{b}{3}+\frac{b}{3}+\frac{c}{2}+\frac{c}{2}}{7} \ge \left(\frac{a^2}{4}\frac{b^3}{27}\frac{c^2}{4}\right)^{1/7}$$

$$\frac{3}{7} \ge \left(\frac{a^2b^3c^2}{2^4 \times 3^3}\right)^{1/7}$$

$$a^2\, b^3\, c^2 \le \frac{3^{10}\cdot 2^4}{7^7}$$

Illustration :

If a, b, c are positive real number then prove that $\frac{a^3}{4b}+\frac{b}{8c^2}+\frac{1+c}{2a} \ge \frac{5}{4}$

Sol. $\frac{a^3}{4b}, \frac{b}{8c^2}, \frac{1}{2a}, \frac{c}{4a}, \frac{c}{4a}$

Applying A.M. ≥ G.M.

$$\frac{\frac{a^3}{4b}+\frac{b}{8c^2}+\frac{1}{2a}+\frac{c}{4a}+\frac{c}{4a}}{5} \ge \left(\frac{a^3}{4b}\cdot\frac{b}{8c^2}\cdot\frac{1}{2a}\cdot\left(\frac{c}{4a}\right)^2\right)^{1/5}$$

$$\frac{a^3}{4b}+\frac{b}{8c^2}+\frac{1}{2a}+\frac{c}{2a} \ge \frac{5}{4}$$

Practice Problem

Q.1 Find the product of three geometric means between 4 and 1/4.

Q.2 Find two numbers whose arithmetic mean is 34 and the geometric mean is 16.

Q.3 If x, y, z are positive numbers then prove that $(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \ge 9$

Q.4 If x & y are positive number such that $\log_2 x + \log_2 y \ge 6$ then find the least value of $x + y$

Q.5 If $a + b + c + d = s$ where a, b, c, d are distinct positive numbers then show that
$(s-a)(s-b)(s-c)(s-d) > 81\, abcd$

Answer key

Q.1	1	Q.2	64 and 4	Q.4	16

4. ARITHMETIC GEOMETRIC PROGRESSION (A.G.P.) :

4.1 Standard appearance of an A.G.P. :

$$S = a + (a + d)r + (a + 2d)r^2 + (a + 3d)r^3 + \text{..........}$$

Here each term is the product of corresponding terms in an arithmetic and geometric series.

4.2 n^{th} term of A.G.P. :

$T_n = [a + (n - 1) d]\, r^{n-1}$

Where a = first term, d = common difference, r = common ratio and n = position of the term which we require.

4.3 Sum of n terms and infinite terms of an A.G.P. :

Let

$$S = a + (a + d)r + (a + 2d)r^2 + (a + 3d)r^3 + \text{..........} + \left(a + \overline{n-1}\,d\right)r^{n-1}$$

$$Sr = \quad + ar \quad + (a + d)r^2 + \text{..............................} + \left(a + \overline{n-2}\,d\right)r^{n-1} + \left(a + \overline{n-1}\,d\right)r^n$$

$$S(1 - r) = a + dr + dr^2 + \text{..} + dr^{n-1} - [a + (n - 1) d]\, r^n$$

$$= a + dr\left(\frac{1 - r^{n-1}}{1 - r}\right) - [a + (n - 1) d]\, r^n$$

$$S = \frac{a}{1-r} + dr\left(\frac{1 - r^{n-1}}{(1-r)^2}\right) - \frac{[a + (n-1)d]\, r^n}{1-r}$$

If $0 < |r| < 1$ and $n \to \infty$ then

$r^n, r^{n-1} \to 0$

$$\therefore \qquad s_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

Students are suggested not to learn the formula, process should be keep in mind. See the illustrations –

Illustration :

Find the sum to n terms and also S_∞.

$$\frac{3}{5} + \frac{5}{15} + \frac{7}{45} + \frac{9}{135} + \text{......}$$

Sol. $S = \dfrac{3}{5} + \dfrac{5}{15} + \dfrac{7}{45} + \dfrac{9}{135} + \text{......} \dfrac{(2n+1)}{5 \cdot 3^{n-1}}$

$$S = \frac{3}{5}+\frac{5}{5\cdot 3}+\frac{7}{5\cdot 3^2}+\frac{9}{5\cdot 3^3}+\ldots\ldots\frac{(2n+1)}{5\cdot 3^{n-1}} \qquad \ldots\ldots.(1)$$

$$\frac{1}{3}S = \frac{3}{5\cdot 3}+\frac{5}{5\cdot 3^2}+\frac{7}{5\cdot 3^3}+\ldots\ldots+\frac{2n-1}{5\cdot 3^{n-1}}+\frac{(2n+1)}{5\cdot 3^n} \qquad \ldots\ldots.(2)$$

$$\frac{2}{3}S = \frac{3}{5}+\frac{2}{5\cdot 3}+\frac{2}{5\cdot 3^2}+\frac{2}{5\cdot 3^3}+\ldots\ldots+\frac{2}{5\cdot 3^{n-1}}-\frac{(2n+1)}{5\cdot 3^n}$$

$$= \frac{3}{5}+\frac{2}{5\cdot 3}\left(1+\frac{1}{3}+\frac{1}{3^2}+\ldots\ldots\frac{1}{3^{n-2}}\right)-\frac{2n+1}{5\cdot 3^n}$$

$$= \frac{3}{5}+\frac{2}{15}\left(\frac{1-\frac{1}{3^{n-1}}}{1-\frac{1}{3}}\right)-\frac{(2n+1)}{5\cdot 3^n} = \frac{3}{5}+\frac{2}{15}\cdot\frac{3}{2}\left(1-\frac{1}{3^{n-1}}\right)-\frac{(2n+1)}{5\cdot 3^n}$$

$$= \frac{3}{5}+\frac{1}{5}-\frac{1}{5\cdot 3^{n-1}}-\frac{2n+1}{5\cdot 3^n} = \frac{4}{5}-\frac{3}{5\cdot 3^n}-\frac{2n+1}{5\cdot 3^n} = \frac{4}{5}-\frac{2n+4}{5\cdot 3^n} = \frac{2}{5}\left(2-\frac{n+2}{3^n}\right)$$

$$= \frac{3}{5}+\frac{1}{5}-\frac{1}{5\cdot 3^{n-1}}-\frac{2n+1}{5\cdot 3^n} = \frac{4}{5}-\frac{3}{5\cdot 3^n}-\frac{2n+1}{5\cdot 3^n} = \frac{4}{5}-\frac{2n+4}{5\cdot 3^n} = \frac{2}{5}\left(2-\frac{n+2}{3^n}\right)$$

$$\therefore\ S = \frac{3}{5}\left(2-\frac{n+2}{3^n}\right)$$

and $S_\infty = \frac{6}{5}$. [*Ans.* $S = \frac{3}{5}\left(2-\frac{n+2}{3^n}\right)$ *and* $S_\infty = \frac{6}{5}$]

Illustration :

If the sum to infinity of the series $3 + (3+d)\frac{1}{4} + (3+2d)\frac{1}{4^2} + \ldots\ldots\ \infty$ *is* $\frac{44}{9}$, *then find d.*

Sol. $S = 3 + (3+d)\frac{1}{4} + (3+2d)\frac{1}{4^2} + \ldots\ldots\ \infty$

$$\Rightarrow \quad \frac{1}{4}S = (3)\frac{1}{4}+(3+d)\frac{1}{4^2}+\ldots\ldots\infty$$

Subtracting (2) from (1), we have

$$\frac{3}{4}S = 3+(d)\frac{1}{4}+(d)\frac{1}{4^2}+\ldots\ldots\infty$$

$$= 3 + \frac{\frac{d}{4}}{1 - \frac{1}{4}}$$

$$= 3 + \frac{d}{3}$$

$\Rightarrow \quad S = 4 + \frac{4d}{9}$

Given,

$$4 + \frac{4d}{9} = \frac{44}{9}$$

$\Rightarrow \quad \frac{4d}{9} = \frac{8}{9}$

$\Rightarrow \quad d = 2.$ *[Ans. 2]*

Illustration :

If $|x| < 1$ *then compute the sum*

(a) $1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\ldots \infty$

If $|x| < 1$ *then compute the sum*

(a) $1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\ldots \infty$

(b) $1 + 3x + 6x^2 + 10x^3 + \ldots\ldots\ldots\ldots \infty$

Sol. (a) $S = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots \infty$

$xS = x + 2x^2 + 3x^2 + \ldots\ldots \infty$

$S(1 - x) = 1 + x + x^2 + x^3 + \ldots\ldots$

$$S(1 - x) = \frac{1}{1 - x}$$

$$S = \frac{1}{(1 - x)^2}$$

(b) $S = 1 + 3x + 6x^2 + 10x^3 + \ldots\ldots\ldots\ldots \infty$

$xS = \quad x + 3x^2 + 6x^3 + \ldots\ldots\ldots\ldots \infty$

$(1 - x)S = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\ldots \infty$

or by equation (1)

$$(1 - x)S = \frac{1}{(1 - x)^2}$$

$$S = \frac{1}{(1 - x)^3}.$$

[Ans. (a) $\frac{1}{(1-x)^2}$ *(b)* $\frac{1}{(1-x)^3}$*]*

5. MISCELLANEOUS SEQUENCES

5.1 Type-1 :

Sequences dealing with $\sum n$; $\sum n^2$; $\sum n^3$

(1) $\sum n = \dfrac{n(n+1)}{2}$

(2) $^{*}\sum n^2 = \dfrac{n(n+1)(2n+1)}{6}$

Proof:

Sum of the squares of the first n natural numbers

$S = 1^2 + 2^2 + 3^2 + \ldots\ldots + n^2$.

We have, $n^3 - (n-1)^3 = 3n^2 - 3n + 1$; and by changing n to $n - 1$, we get

$(n-1)^3 - (n-2)^3 = 3(n-1)^2 - 3(n-1) + 1$

$(n-2)^3 - (n-3)^3 = 3(n-2)^2 - 3(n-2) + 1$

$\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots$

$\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots$

$3^3 - 2^3 = 3 \times 3^2 - 3 \times 3 + 1$

$2^3 - 1^2 = 3 \times 2^2 - 3 \times 2 + 1$

$1^2 - 0^2 = 3 \times 1^2 - 3 \times 1 + 1$

Hence, by adding

$n^3 = 3(1^2 + 2^2 + 3^2 + \ldots\ldots + n^2) - 3\,(1 + 2 + 3 + \ldots\ldots + n) + n$

$$= 3S - \frac{3n(n+1)}{2} + n$$

$$\Rightarrow \qquad 3S = n^2 - n + \frac{3n(n+1)}{2}$$

$$= n\,(n+1)\left(n - 1 + \frac{3}{2}\right)$$

$$\Rightarrow \qquad S = \frac{n(n+1)(2n+1)}{6}$$

(3) $^{**}\sum n^3 = \left[\dfrac{n(n+1)}{2}\right]^2 = \left(\sum n\right)^2$

For proof :

* Consider the identity $k^4 - (k-1)^4 = 4k^3 - 6k^2 + 4k - 1$

Note : (i) $\displaystyle\sum_{r=1}^{n}(a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$;

(ii) $\displaystyle\sum_{r=1}^{n} k\,a_r = k\sum_{r=1}^{n} a_r$

Important: (iii) $\displaystyle\sum_{r=1}^{n} k = k\sum_{r=1}^{n} = kn$

Illustration :

Compute the value of $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1$.

Sol. $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1 = \sum_{i=1}^{n}\sum_{j=1}^{i}(j) = \sum_{i=1}^{n}\frac{i(i+1)}{2} = \frac{1}{2}\left[\sum n^2 + \sum n\right]$

$$= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right] = \frac{n(n+1)(2n+4)}{12} = \frac{n(n+1)(n+2)}{6}.$$

[Ans. $\frac{n(n+1)(n+2)}{6}$]

Illustration :

Find the value(s) of the positive integer n for which the quadratic equation, $\sum_{k=1}^{n}(x+k-1)(x+k) = 10n$ *has solutions* α *and* $\alpha + 1$ *for some* α.

n

Sol. $\sum_{k=1}^{n}(x+k-1)(x+k) = 10n$ ...(1)

$\Rightarrow \quad \sum\left[x^2 + (2k-1)x + (k-1)k\right] = 10n$

$\Rightarrow \quad x^2 \cdot n + n^2 x + \frac{(n-1)n(n+1)}{3} - 10\,n = 0$ ...(2)

$\Rightarrow$ If roots are α and $\alpha + 1$ (p) then difference = 1 By equation (2)

$\therefore \quad (\alpha + \beta)^2 - 4\,\alpha\beta = 1$ $\qquad \alpha + \beta = n$

or $\quad n^2 - 4\left(\frac{(n-1)(n+1)}{3} - 10\right) = 1$ $\qquad \alpha\beta = \frac{(n-1)(n+1)}{3} - 10$

or $\quad n = 11.$ [Ans. 11]

Illustration :

Compute the sum $(31)^2 + (32)^2 + (33)^2 + ... + (50)^2$

Sol. $S = (31^2) + (32^2) + (33^2) + \ldots\ldots + (n + 30)^2 + \ldots\ldots (50)^2$

$S = \sum_{n=1}^{20}(n+30)^2 \quad \Rightarrow \quad n^2 + 60n + 900 \quad \Rightarrow \quad \frac{n(n+1)(2n+1)}{6} + \frac{60n(n+1)}{2} + 900n$

$\Rightarrow \quad \frac{20\times 21\times 41}{6} + 30 \times 20 \times 21 + 900 \times 20$

$\Rightarrow \quad 33470.$ [Ans. 33470]

Illustration :

Find the sum to n terms of the series $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \ldots\ldots$.

Sol. $T_n = n(n + 1)(n + 2)$

Let S_n denote the sum to n terms of the given series. Then,

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} k(k+1)(k+2)$$

$$= \sum_{k=1}^{n} (k^3 + 3k^2 + 2k)$$

$$= \left(\sum_{k=1}^{n} k^3\right) + 3\left(\sum_{k=1}^{n} k^2\right) + 2\left(\sum_{k=1}^{n} k\right)$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + \frac{3n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + (2n+1) + 2\right\}$$

$$= \frac{n(n+1)}{2}\{n^2 + n + 4n + 2 + 4\} \quad = \frac{n(n+1)}{2}(n^2 + 5n + 6)$$

$$= \frac{n(n+1)}{2}\{n^2 + n + 4n + 2 + 4\} \quad = \frac{n(n+1)}{2}(n^2 + 5n + 6)$$

$$= \frac{n(n+1)(n+2)(n+3)}{4}$$

[Ans. $\frac{n(n+1)(n+2)(n+3)}{4}$]

Illustration :

Find the sum of the series : $\frac{1^2}{1} + \frac{1^2+2^2}{1+2} + \frac{1^2+2^2+3^2}{1+2+3} + \ldots\ldots$ *upto 31 terms.*

Sol. $T(r) = \frac{1^2 + 2^2 + \ldots\ldots + r^2}{1 + 2 + \ldots\ldots + r}$

$$= \frac{r(r+1)(2r+1)2}{6r(r+1)}$$

$$= \frac{1}{3}(2r+1)$$

$$\Rightarrow \quad \sum_{r=1}^{n} T(r) = \left(\frac{2}{3}\sum_{r=1}^{n} r\right) + \frac{n}{3} \quad = \frac{1}{3}n(n+1) + \frac{n}{3}$$

$$S_n = \frac{n(n+2)}{3}$$

$$\therefore \quad S_{31} = \frac{31 \times 33}{3} = 341.$$

[Ans. 341]

Practice Problem

Q.1 Find the sum of $1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\ldots\ldots$ to n terms and to ∞.

Q.2 Find the sum of series –

$1.1 + 3.01 + 5.001 + 7.0001 + \ldots\ldots +$ to n terms

Q.3 Find the sum of the series

$1.n + 2.(n-1) + 3.(n-2) + \ldots\ldots n.1$

Q.4 Find the sum of n terms of the series

$1^2 + 3^2 + 5^2 + 7^2 + \ldots\ldots$

Answer key

Q.1 $S_n = \frac{35}{16} - \frac{(12n+7)}{16\cdot 5^{n-1}}$; $S_\infty = \frac{35}{16}$

Q.2 $n^2 + \left(\frac{1}{9}\right)\left(1-\frac{1}{10}n\right)$

Q.3 $\frac{n(n+1)(n+2)}{6}$

Q.4 $\frac{n(4n^2-1)}{3}$

5.2 TYPE-2 (Using method of difference) :

5.2 TYPE-2 (Using method of difference) :

If $T_1, T_2, T_3, \ldots\ldots$ are the terms of a sequence then the terms

$T_2 - T_1, T_3 - T_2, T_4 - T_3 \ldots\ldots\ldots$

some times are in A.P. and some times in G.P. For such series we first compute their n^{th} term and then compute the sum to n terms, using sigma notation.

Illustration :

Find the sum of series .

(i) $6 + 13 + 22 + 33 + \ldots\ldots\ldots\ldots$ *n terms*

(ii) $5 + 7 + 13 + 31 + 85 + \ldots\ldots\ldots$ *up to n terms.*

Sol. *(i) Let* $S = 6 + 13 + 22 + 33 + \ldots\ldots + T_n$

$S = \quad 6 + 13 + 22 + \ldots\ldots + \quad + T_n$

$O = 6 + 7 + 9 + 11 + \ldots\ldots \quad - T_n$

or $T_n = 6 + (7 + 9 + 11 + \ldots\ldots)$

$= 6 + \left[\frac{n-1}{2}[2\times 7 + (n-2)2]\right] \quad = 6 + \left[\frac{(n-1)}{2}(14+24-4)\right]$

$= 6 + (n-1)(n+5) \quad = 6 + n^2 + 4n - 5$

$T_n = n^2 + 4n + 1$

$\therefore \quad S_n = \sum_{n=1}^{n} T_n \quad \Rightarrow \quad \frac{n(n+1)(2n+1)}{6} + \frac{4(n)(n+1)}{2} + n$

$$\Rightarrow \quad \frac{n(n+1)}{2}\left[\frac{2n+1}{3}+4\right]+n \;\Rightarrow\; \frac{n(n+1)}{2}\left[\frac{2n+13}{3}\right]+n \;\Rightarrow\; \frac{n(n+1)(2n+13)}{6}+n$$

(ii) $S = 5 + 7 + 13 + 31 + 85 \ldots\ldots + T_n$

$S = \quad 5 + 7 + 13 + 31 \ldots\ldots - T_n$

$O = 5 + 2 + 6 + 18 + 54 \ldots\ldots - T_n$

$Tn = 5 + 2 + 6 + 18 + 54 + \ldots\ldots\ldots\ldots$

$$T_n = 5 + \frac{2[3^{n-1}-1]}{3-1} \quad \Rightarrow \quad T_n = 5 + (3^{n-1} - 1)$$

$$S_n = \sum_{n=1}^{n} T_n \;\Rightarrow S_n + \frac{3^n - 1}{2} - n$$

$$S_n = 4n + \frac{1}{2}(3^n - 1). \qquad [Ans.\ (i)\ \frac{n(n+1)(6n+13)}{6}+n\ (ii)\ 4n + \frac{1}{2}(3^n + 1)\ ]$$

5.3 TYPE -3 (Splitting the nth term as a difference of two) :

Here is a series in which each term is composed of the reciprocal of the product of r factors in A.P., the first factor of the several terms being in the same A.P.

Illustration :

Illustration :

Find the sum of n terms of the series and also find S_∞.

$$\frac{1}{1\cdot2\cdot3\cdot4} + \frac{1}{2\cdot3\cdot4\cdot5} + \frac{1}{3\cdot4\cdot5\cdot6} + \ldots\ldots$$

Sol. $\frac{1}{1\cdot2\cdot3\cdot4} + \frac{1}{2\cdot3\cdot4\cdot5} + \ldots\ldots + \frac{1}{n(n+1)(n+2)(n+3)}$.

$$S = \frac{1}{3}\left[\frac{4-1}{1\cdot2\cdot3\cdot4}+\frac{5-2}{2\cdot3\cdot4\cdot5}+\ldots\ldots+\frac{(n+3)-n}{n(n+1)(n+2)(n+3)}\right]$$

$$T_1 = \frac{1}{3}\left(\frac{1}{1\cdot2\cdot3}-\frac{1}{2\cdot3\cdot4}\right)$$

$$T_2 = \left(\frac{1}{2\cdot3\cdot4}-\frac{1}{3\cdot4\cdot5}\right)$$

....

....

$$T_n = \frac{1}{3}\left(\frac{1}{n(n+1)(n+2)}-\frac{1}{(n+1)(n+2)(n+3)}\right)$$

$$S = T_1 + T_2 + \ldots\ldots + T_n = \frac{1}{3}\left[\frac{1}{1\cdot2\cdot3}-\frac{1}{(n+1)(n+2)(n+3)}\right]$$

$$[Ans.\ S_n = \frac{1}{18}-\frac{1}{3(n+1)(n+2)(n+3)}\ ;\ S_\infty=\frac{1}{18}]$$

Illustration :

Find the sum of n terms of the series $\frac{3}{1\cdot2\cdot4} + \frac{4}{2\cdot3\cdot5} + \frac{5}{3\cdot4\cdot6} + ...$ *and also find sum of infinite terms* (S_∞)

Sol. $\frac{3}{1\cdot2\cdot4} + \frac{4}{2\cdot3\cdot5} + \frac{5}{3\cdot4\cdot6} + ... \frac{(n+2)}{n(n+1)(n+3)}$

$$= \frac{3^2}{1\cdot2\cdot3\cdot4} + \frac{4^2}{2\cdot3\cdot4\cdot5} + ... + \frac{(n+2)^2}{n(n+1)(n+2)(n+3)}$$

$$T_n = \frac{n^2+4n+4}{n(n+1)(n+2)(n+3)} = \frac{n^3+4n+3}{n(n+1)(n+2)(n+3)} + \frac{1}{n(n+1)(n+2)(n+3)}$$

$$T_n = \frac{1}{n(n+2)} + \frac{1}{n(n+1)(n+2)(n+3)}$$

$$S_1 = \sum_{n=1}^{n}\frac{1}{n(n+2)},\ S_2 = \sum_{n=1}^{n}\frac{1}{n(n+1)(n+2)(n+3)}$$

$$S_1 = \frac{1}{1\cdot3} + \frac{1}{2\cdot4} + \frac{1}{3\cdot5} + ... \frac{1}{n(n+2)}$$

$$S_1 = \frac{1}{2}\left[\frac{3-1}{1\cdot3} + \frac{4-2}{2\cdot4} + \frac{5-3}{3\cdot5} + ... + \frac{(n+2)-n}{n(n+2)}\right] = \frac{1}{2}\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + ... + \frac{1}{n} - \frac{1}{n+2}\right]$$

$$S_1 = \frac{}{2}\left[\frac{}{1\cdot3} + \frac{}{2\cdot4} + \frac{}{3\cdot5} + ... + \frac{}{n(n+2)}\right] = \frac{}{2}\left[\frac{}{1} - \frac{}{3} + \frac{}{2} - \frac{}{4} + \frac{}{3} - \frac{}{5} + ... + \frac{}{n} - \frac{}{n+2}\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right]$$

$$S_2 = \frac{1}{3}\left[\frac{1}{1\cdot2\cdot3} - \frac{1}{(n+1)(n+2)(n+3)}\right]$$

$S = S_1 + S_2.$

Illustration :

Find sum of n terms (S_n) *for* $\frac{1}{2\cdot4} + \frac{1\cdot3}{2\cdot4\cdot6} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot8} + ...$

Sol. $S_n = \frac{1}{2\cdot4} + \frac{1\cdot3}{2\cdot4\cdot6} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot8} + ... + \frac{1\cdot3\cdot5......(2n-1)}{2\cdot4\cdot6\cdot8......(2n+2)}$

$$T_n = \frac{1\cdot3\cdot5......(2n-1)}{2\cdot4\cdot6\cdot8......(2n+2)}$$

$$= \frac{1\cdot3\cdot5...(2n-1)\cdot[(2n+2)-(2n+1)]}{2\cdot4\cdot6\cdot8......(2n+2)}$$

$$T_n = \frac{1\cdot3\cdot5...(2n-1)}{2\cdot4\cdot6\cdot8......2n} - \frac{1\cdot3\cdot5...(2n+1)}{2\cdot4\cdot6\cdot8......(2n+2)}$$

$$T_1 = \frac{1}{2} - \frac{1\cdot3}{2\cdot4}$$

$$T_2 = \frac{1 \cdot 3}{2 \cdot 4} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}$$

$$T_n = \frac{1 \cdot 3 \cdot 5 \ldots\ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots\ldots 2n} - \frac{1 \cdot 3 \cdot 5 \ldots\ldots (2n+1)}{2 \cdot 4 \cdot 6 \cdot 8 \ldots\ldots (2n+2)}$$

$$S_n = \frac{1}{2} - \frac{1 \cdot 3 \cdot 5 \ldots\ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots\ldots (2n+2)}.$$

$$\left[Ans. S_n = \frac{1}{2} - \frac{1 \cdot 3 \cdot 5 \ldots\ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots\ldots (2n+2)}\right]$$

for infinte terms – $S_\infty = \frac{1}{2} - \lim_{n \to \infty} \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right) \ldots\ldots \left(\frac{2n-1}{2n}\right)\left(\frac{1}{2n+2}\right)$

$$\therefore \quad S_\infty = \frac{1}{2}$$

5.4 TYPE-4:

Here is a series in which each terms is composed of r factor in A.P., the first factor of the several terms being in the same A.P.

Illustration :

$1 \cdot 2 \cdot 3 \cdot 4 + 2 \cdot 3 \cdot 4 \cdot 5 + 3 \cdot 4 \cdot 5 \cdot 6 + \ldots\ldots$ *up to n terms*

Sol. $T_n = n(n+1)(n+2)(n+3)$

$$T_n = \frac{1}{5} n (n+1)(n+2)(n+3)[(n+4) - (n-1)]$$

$$T_n = \frac{n(n+1)(n+2)(n+3)(n+4)}{5} - \frac{(n-1)(n)(n+1)(n+2)(n+3)}{5}$$

$$T_1 = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{5} - 0$$

$$T_2 = \frac{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}{5} - \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{5}$$

$$T_3 = \frac{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}{5} - \frac{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}{5}$$

... .

... .

$$T_n = \frac{n(n+1)(n+2)(n+3)(n+4)}{5} - \frac{(n-1)(n)(n+1)(n+2)(n+3)}{5}$$

$$S_n = T_1 + T_2 + T_3 + \ldots\ldots + T_n = \frac{n(n+1)(n+2)(n+3)(n+4)}{5}$$

$$\left[Ans.\ S_n = \frac{1}{5}[n(n+1)(n+2)(n+3)(n+4)]\ \right]$$

Practice Problem

Q.1 If $I(r) = r(r^2 - 1)$, then find $\sum_{r=2}^{\infty} \frac{1}{I(r)}$.

Q.2 If $S = \frac{1}{1 \cdot 3 \cdot 5} + \frac{1}{3 \cdot 5 \cdot 7} + \frac{1}{5 \cdot 7 \cdot 9} + ...$ to infinity, then find the vlaues of $[36S]$, where $[\cdot]$ represents the greatest integer function.

Q.3 If $\sum_{r=1}^{n} t_r = \frac{n}{8}(n+1)(n+2)(n+3)$, then $\sum_{r=1}^{n} \frac{1}{t_r}$.

Q.4 Find the sum of n terms of the series
$3 + 8 + 22 + 72 + 266 + 1036 + \ldots\ldots$

Q.5 Show that the sum of the n terms of the series
$$\frac{3}{1^2} + \frac{5}{1^2+2^2} + \frac{7}{1^2+2^2+3^2} + \frac{9}{1^2+2^2+3^2+4^2} + \ldots\ldots = \frac{6n}{n+1}$$

Answer key

Q.1 $\frac{1}{4}$ Q.2 3   Q.3 $\frac{1}{2} - \frac{1}{(n+1)(n+2)}$ Q.4 $\frac{1}{3}(4^n - 1) + n(n+1)$

6. HARMONIC PROGRESSION (H.P.) :

6.1 Definition and Standard Appearance of H.P. :

A sequence is said to be in H.P. if the reciprocals of its terms are in A.P.

e.g. if $a_1, a_2, a_3, ...$ are in H.P. then $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}$ .... are in A.P.

A standard H.P. is $\frac{1}{a} + \frac{1}{a+d} + \frac{1}{a+2d} + ... + \frac{1}{a+\overline{n-1}d}$

For every HP there is a corresponding A.P.
Terms of harmonic series are the outcomes of an A.P.

6.2 General term/nth term/last term of H.P. :

$$T_n = \frac{1}{a+(n-1)d}$$

where a and d are respectively the first term and the common difference of the corresponding A.P. and n = position of the term which we required.

Note:

(i) There is no general formula for finding the sum to n terms of H.P.

(ii) If a, b, c are in H.P. $\Rightarrow$ $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in **A.P**

$\therefore$ $\frac{2}{b} = \frac{1}{a} + \frac{1}{c}$ $\Rightarrow$ $b = \frac{2ac}{a+c}$ $\Rightarrow$ a, b, c are in HP

also $\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$ i.e. $\frac{a-b}{ab} = \frac{b-c}{bc}$ i.e. $\frac{a}{c} = \frac{a-b}{b-c}$

Illustration :

If the 3rd, 6th and last term of a H.P. are $\frac{1}{3}, \frac{1}{5}, \frac{3}{203}$, find the number of terms.

Sol. $T_3 = \frac{1}{3}$, $T_6 = \frac{1}{5}$, $T_n = \frac{3}{203}$

then 3rd, 6th and nth termof A.P. series are 3, 6, $\frac{203}{3}$.

$a + 2d = 3 \quad \Rightarrow \quad a = 5d = 5$

$d = \frac{2}{3},\ a = \frac{5}{3}$

$a + 2d = 3 \quad \Rightarrow \quad a = 5d = 5$

$d = \frac{2}{3},\ a = \frac{5}{3}$

$a + (n-1)d = \frac{203}{3} \quad \Rightarrow \quad \square\frac{5}{3} + (n-1)\,\frac{2}{3} = \frac{203}{3}$

$(n-1)^2 = 198$

$n = 100.$ *[Ans. 100]*

Illustration :

If m^{th} term of an H.P. is n, and n^{th} term is equal to m then prove that $(m+n)^{th}$ term is $\frac{mn}{m+n}$

Sol. m^{th} *term of A.P.* $= \frac{1}{n}$

n^{th} *term of A.P.* $= \frac{1}{m}$ $\Rightarrow$ $a + (m-1)d = \frac{1}{n}$

$a + (n-1)d = \frac{1}{m}$ $\Rightarrow$ $d = \frac{1}{mn},\ a = \frac{1}{mn}$

$T_{m+n} = a + (m+n-1)d = \frac{1}{mn} + (m-m-1)\,\frac{1}{mn}$

$T_{m+n} = \frac{mn}{m+n}.$

Illustration :

If a, b, c are in HP, find the value of $\dfrac{b+a}{b-a} + \dfrac{b+c}{b-c}$.

Sol. *a, b, c are in HP, then*

$$\frac{1}{a}, \frac{1}{b}, \frac{1}{c} \text{ are in A.P.}$$

$$S = \frac{b+a}{b-a} + \frac{b+c}{b-c} = \frac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a}-\frac{1}{b}} + \frac{\frac{1}{c}+\frac{1}{b}}{\frac{1}{c}-\frac{1}{b}}$$

Let $\dfrac{1}{a}-\dfrac{1}{b} = \dfrac{1}{b}-\dfrac{1}{c} = d$

$$S = \frac{\left(\frac{1}{a}+\frac{1}{b}\right)-\left(\frac{1}{c}+\frac{1}{b}\right)}{d} = \frac{\left(\frac{1}{a}-\frac{1}{c}\right)}{d} = \frac{2d}{d} = 2$$

[Ans. 2]

6.3 Harmonic Mean (H.M.) :

6.3 Harmonic Mean (H.M.) :

If a, b, c are in H.P. then middle term is called the harmonic mean between them. Hence if H is the harmonic mean (H.M.) between a and b then a, H, b are in H.P. and $H = \dfrac{2ab}{a+b}$.

(Recall that $AM = \dfrac{a+b}{2}$ and $GM = \sqrt{ab}$ if $a > 0, b > 0$)

6.3.1 To insert n HM between a and b :

Let $H_1, H_2 \ldots\ldots H_n$ are n HM's between a and b
hence $a, H_1, H_2, \ldots\ldots H_n$ b are in H.P.

$$\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2} \ldots\ldots \frac{1}{H_n}, \frac{1}{b} \text{ are in A.P.}$$

$$\frac{1}{b} = \frac{1}{a} + (n+1)d \quad ; \quad -\frac{1}{a} = (n+1)d \ ; \ d = \frac{a-b}{ab(n+1)}$$

$$\frac{1}{H_1} = \frac{1}{a} + d$$

$$\frac{1}{H_2} = \frac{1}{a} + 2d$$

$$\frac{1}{H_3} = \frac{1}{a} + 3d$$

$$\vdots$$

$$\frac{1}{H_n} = \frac{1}{a} + nd$$

$$\sum_{i=1}^{n} \frac{1}{H_i} = \frac{n}{a} + \frac{d(n)(n+1)}{2} = \frac{n}{a}s + \frac{n(n+1)}{2} \cdot \frac{(a-b)}{ab(n+1)}$$

$$= n\left[\frac{1}{a} + \frac{a-b}{2ab}\right] = \frac{n}{2ab}[2b + a - b] = \frac{n(a+b)}{2ab} = n \cdot \frac{1}{H}$$

Hence sum of the reciprocals of all the n HM's between a and b is equal to *n* times the reciprocal of single HM between a and b.

Note: For 3 numbers a, b, c MH is defined as reciprocals of a, b and c i.e. of reciprocals $\frac{1}{3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$.

$$\text{H.M.} = \frac{3}{\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)}.$$

a b c

If $A_1\ A_2$ ....A_n are n quantities then $\text{HM} = \dfrac{1}{\frac{1}{A_1} + \frac{1}{A_2} + \ldots\ldots + \frac{1}{A_n}}$.

Illustration :

If a^2, b^2, c^2 are in A.P. show that $b + c$, $c + a$, $a + b$ are in H.P.

Sol. *a^2, b^2, c^2 are in A.P.*

Let $b + c$, $c + a$, $a + b$ *are in H.P.*

then $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ *are in A.P.*

$$\frac{2}{c+a} = \frac{1}{b+c} + \frac{1}{a+b}$$

$$2(a + b)(b + c) = (2b + a + c)(a + c)$$

$$2b^2 = a^2 + c^2$$

hence a^2, b^2, c^2 *are in A.P.*

So *if a^2, b^2, c^2 are in A.P. then* $b + c$, $c + a$, $a + b$ *are in H.P.*

Illustration :

If $\dfrac{a-x}{px} = \dfrac{a-y}{qy} = \dfrac{a-z}{rz}$ *and p, q, r are in A.P. then prove that x, y, z are in H.P.*

Sol. *Let* $\dfrac{a-x}{px} = \dfrac{a-y}{qy} = \dfrac{a-z}{rz} = k$

$$P = \frac{a-x}{kx},\ q = \frac{a-y}{kx},\ r = \frac{a-z}{kz}$$

$$2\left(\frac{a-y}{ky}\right) = \frac{a-x}{kx} + \frac{a-z}{kz}$$

$$2\left(\frac{a}{y} - 1\right) = \frac{a}{x} - 1 + \frac{a}{z} - 1$$

$$\frac{2a}{y} = \frac{a}{x} + \frac{a}{z}$$

$$\frac{2}{\ } = \frac{1}{\ } + \frac{1}{\ }$$

$$\frac{2}{y} = \frac{1}{x} + \frac{1}{z}$$

Hence x, y, z are in H.P.

Illustration :

If the roots of the equation $x^3 - 11x^2 + 36x - 36 = 0$ *are in H.P. find the middle root.*

Sol. $x^3 - 11x^2 + 36x - 36 = 0$

If roots are in H.P. then roots of new equation

$$\frac{1}{x^3} - \frac{11}{x^2} + \frac{36}{x} - 36 = 0 \text{ are in A.P.}$$

$$-36x^3 + 36x^2 - 11x + 1 = 0$$

$$36x^3 - 36x^2 + 11x - 1 = 0$$

Let the roots be α, β, γ.

$$\alpha + \beta + \gamma = 1$$

$$3\beta = 1 \qquad (2\beta = \alpha + \gamma)$$

$$\beta = \frac{1}{3}$$

So middle root is 3. *[Ans. 6, 3, 2]*

6.4 Relation between A.M, G.M. and H.M :

If a and b are two positive numbers then $A \geq G \geq H$ and A, G, H are in G.P. i.e. $G^2 = AH$

Proof: We have $A = \dfrac{a+b}{2}$, $G = \sqrt{ab}$ and $H = \dfrac{2ab}{a+b}$

G
a A b
H

now $AH = ab = G^2$ $\Rightarrow$ A G H are in G.P.

also $\dfrac{A}{G} = \dfrac{G}{H}$; $\therefore$ $A \geq G \Rightarrow G \geq H$

Hence $A \geq G \geq H$ Infact $AM \geq GM \geq HM$

Illustration :

If 9 arithmetic and harmonic means be inserted between 2 and 3, prove that $A + 6/H = 5$ where A is any of the A.M.'s and H the corresponding H.M.

Sol. *Let A_i, H_i $(i = 1, 2, \ldots\ldots 9)$ denote the 9 A.M.'s and 9 H.M.'s between 2 and 3. If d denote the common difference of A.P. then*

$$3 = T_{11} = 2 + 10\,d \qquad or \qquad d = 1/10.$$

L et A denote the its mean, then

$$A = T_{i+1} = 2 + di = 2 + i/10$$

Again Let 2, H_1, H_2, ……, H_9, 3 be in H.P.

1 1 1 1 1

or $$\frac{1}{2}, \frac{1}{H_1}, \frac{1}{H_2}, \ldots\ldots, \frac{1}{H_9}, \frac{1}{3}$$

If d_1 is the common difference of this A.P., then

$$\frac{1}{3} = T_{11} = \frac{1}{2} + 10d_1 \qquad or \qquad d_1 = -\frac{1}{60}$$

If H is the ith H.m., then

$$\frac{1}{H} = \frac{1}{2} + d_1 i = \frac{1}{2} - \frac{i}{60}$$

Now $$A + \frac{6}{H} = \left(2 + \frac{i}{10}\right) + 6\left(\frac{1}{2} - \frac{i}{60}\right) = 5 + \frac{i}{10} - \frac{i}{10} = 5.$$

Illustration :

If a, b & c are in A.P. & a^2, b^2 & c^2 are in H.P., then prove that either $a = b = c$ or a, b & $-\dfrac{c}{2}$ are in G.P.

Sol. $2b = a + c$

$$\frac{1}{b^2} - \frac{1}{a^2} = \frac{1}{c^2} - \frac{1}{b^2}$$

$$\Rightarrow \qquad \frac{(a-b)(a+b)}{b^2a^2} = \frac{(b-c)(b+c)}{b^2c^2}$$

$\Rightarrow \quad ac^2 + bc^2 = a^2 b + a^2 c$

$ac(c-a) + b(c-a)(c+a) = 0$

$(c-a)(ab+bc+ca) = 0$

for, $\quad c = a, \quad a = b = c$

for $\quad ab + bc + ca = 0$

$(a+b) + ca = 0$

$2b^2 + ca = 0$

$$b^2 = -\frac{ac}{2}$$

$a, b, -\frac{c}{2}$ *are in G.P.*

Illustration :

If a, b, c are positive real number representing the sides of a triangle, prove that

$$ab + bc + ca < a^2 + b^2 + c^2 < 2(ab + bc + ca)$$

or $$1 < \frac{a^2+b^2+c^2}{ab+bc+ca} < 2$$

Hence prove that

$$3(ab + bc + ca) < (a + b + c)^2 < 4(ab + bc + ca)$$

or $$3 < \frac{(a+b+c)}{ab+bc+ca} < 4.$$

$$3(ab + bc + ca) < (a + b + c)^2 < 4(ab + bc + ca)$$

or $$3 < \frac{(a+b+c)}{ab+bc+ca} < 4.$$

Sol. $\frac{a^2+b^2}{2} > ab, \frac{b^2+c^2}{2} > bc$ *and* $\frac{c^2+a^2}{2} > ca \qquad [\because A.M. > G.M.]$

$\Rightarrow \quad a^2 + b^2 > 2ab, b^2 + c^2 > 2bc$ *and* $c^2 + a^2 > 2ca$

$\Rightarrow \quad a^2 + b^2 + b^2 + c^2 + c^2 + a^2 > 2(ab + bc + ca)$

$\Rightarrow \quad a^2 + b^2 + c^2 > ab + bc + ca$

$\Rightarrow \quad ab + bc + ca < a^2 + b^2 + c^2 \qquad$ ...(i)

In a triangle ABC with sides BC = a, CA = b, AB = c, we have $b^2 + c^2 - a^2 = 2bc \cos A$

$\Rightarrow \quad b^2 + c^2 - a^2 < 2bc \qquad [\because \cos A < 1]$

Similarly, we have $c^2 + a^2 - b^2 < 2ca$*and* $\quad a^2 + b^2 - c^2 < 2ab$

Adding these three, we get

$a^2 + b^2 + c^2 < 2(ab + bc + ca) \qquad$ ...(ii)

From (i) and (ii), we get

$ab + bc + ca < a^2 + b^2 + c^2 < 2(ab + bc + ca) \qquad$ ...(iii)

$\Rightarrow \quad 1 < \frac{a^2+b^2+c^2}{ab+bc+ca} < 2$

Adding 2(ab + bc + ca) throughout in (iii)

$$3(ab + bc + ca) < (a + b + c)^2 < 4(ab + bc + ca)$$

$3 < \frac{(a+b+c)}{ab+bc+ca} < 4. \qquad$ *Hence proveed*

Illustration :

If a, b, c d be four distinct positive quantities in H.P., then show that

(a) $a + d > b + c$ (b) $ad > bc$.

Sol. $\because$ a, b, c, d are in H.P.

(a) Then A.M. > H.M.

for first three terms $\therefore$ $\frac{a+c}{2} > b$ ...(i)

and for last three terms, $\frac{b+d}{2} > c$ ...(ii)

from (i) and (ii) $a + c + b + d > 2b + 2c$

$\Rightarrow$ $a + d > b + c$

(b) Again G.M. > H.M.

For first three terms, $\sqrt{ac} > b$

$\Rightarrow$ $ac > b^2$ ...(i)

and for last three terms $\sqrt{bd} > c$

$\Rightarrow$ $bd > c^2$ ...(ii)

from (i) and (ii) $(ac)(bd) > b^2c^2$

$\Rightarrow$ $ad > bc$.

Practice Problem

Practice Problem

Q.1 If p^{th} term of an HP is qr and q^{th} term is pr. Find its r^{th} terms.

Q.2 If 'a' is the A.M. of 'b' and 'c' ; 'b' the G.M. of 'c' and 'a', then prove that 'c' is the H.M. of 'a' and 'b'.

Q.3 If the roots of the equation $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ are equal then prove that a, b, c are in H.P.

Q.4 If $a^x = b^y = c^z = d^w = ...$ and a, b, c, d are in G.P. then prove that x, y, z, w... are in H.P.

Q.5 Let $n \in N$, $n > 25$. Let A, G, H denote the arithmetic mean, geometric mean & harmonic mean of 25 & n. The least value of n for which A, G, H $\in \{25, 26, ... n\}$

Q.6 Let a_1, a_2, a_3, a_4 and a_5 be such that a_1, a_2 & a_3 are in A.P. and a_3, a_4 and a_5 are in H.P. then $\log a_1$, $\log a_3$ & $\log a_5$ are in

(A) G.P. (B) A.P. (C) H.P. (D) None

Q.7 Let Show that if $a, b, c \in R^+$ then $\frac{bc}{b+c} + \frac{ac}{a+c} + \frac{ab}{a+b} \le \frac{1}{2}(a+b+c)$

Q.8 If a, b, c, d be four distinct positive quantities in G.P., then show that

(a) $a + d > b + c$ (b) $c^{-1}d^{-1} + a^{-1}b^{-1} > 2(b^{-1}d^{-1} + a^{-1}c^{-1} - a^{-1}d^{-1})$

Answer key

Q.1	pq	Q.5	225	Q.6	B

Solved Examples

Q.1 If sum of n, 2n, 3n terms of an A.P. are S_1, S_2, S_3, respectively then find the value of $\frac{S_3}{S_2 - S_1}$.

Sol. Let a be the first term and d be the common difference of the given A.P. Then, sumof n terms is

$$S_1 = \frac{n}{2}[2a + (n-1)d]$$

Sum of 2n terms,

$$S_2 = \text{Sum of 2n terms} = \frac{2n}{2}[2a + (2n-1)d]$$

Sum of 3n terms,

$$S_3 = \frac{3n}{2}[2a + (3n-1)d]$$

Now,

$$S_2 - S_1 = \frac{2n}{2}[2a + (2n-1)d] - \frac{n}{2}[2a + (n-1)d]$$

$$= \frac{n}{2}[2\{2a + (2n-1)d\} - \{2a + (n-1)\}d]$$

$$= \frac{n}{2}[2a + (3n-1)d]$$

$\therefore \quad 3(S_2 - S_1) = \frac{3n}{2}[2a + (3n-1)d] = S_3$ [Using (3)]

$\Rightarrow \quad \frac{S_3}{S_2 - S_1} = 3.$ [Ans. 3]

Q.2 If the 10th, 15th, 25th terms of an A.P. are in G.P then find the common ratio of the G.P.

Sol. Let the first term and common ratio of the AP be a and d respectively.

$T_{10} = a + 9d$, $T_{15} = a + 14d$, $T_{25} = a + 24d$.

$$R = \frac{T_{15}}{T_{10}} = \frac{T_{25}}{T_{15}}$$

$R = \frac{a+14d}{a+9d} = \frac{a+24d}{a+14d} \Rightarrow \frac{a+14d-a-24d}{a+9d-a-14d} = 2.$ [Ans. 2]

Q.3 The sum of the squares of three distinct real numbers, which are in G.P. is S^2. If their sum is α S, if $\alpha^2 \in (a, b) - \{c\}$ then find the value of $ab + c$.

Sol. Let the numbers be ar, a, a/r such that $a\left(r+1+\frac{1}{r}\right) = \alpha S$

and $$a^2\left(r^2+1+\frac{1}{r^2}\right) = S^2$$

Put $r + \frac{1}{r} = t$ $\quad\therefore\quad$ $r^2 + \frac{1}{r^2} = r^2 - 2$

$\therefore$ $a(t + 1) = \alpha S$ $\quad$ and $\quad$ $a^2(r^2 - 1) = S^2$

Eliminating S, we get $a^2(r^2 - 1) = \frac{a^2(r+1)^2}{\alpha^2}$

$\therefore$ $(t - 1)\alpha^2 = (t + 1)$

or $$t = \frac{\alpha^2+1}{\alpha^2-1}$$

Now $t = r + \frac{1}{r}$ $\quad\therefore\quad$ $r^2 - rt + 1 = 0$

Now $t = r + \frac{1}{r}$ $\quad\therefore\quad$ $r^2 - rt + 1 = 0$

For t to be real $r^2 - 4 > 0$ $\quad\therefore\quad$ $(t + 2)(t - 2) > 0$

$\therefore$ $t < -2$ or $t > 2$

Hence from (1), we get

$$\frac{\alpha^2+1}{\alpha^2-1} < -2 \quad \text{or} \quad \frac{\alpha^2+1}{\alpha^2-1} > 2$$

or $$\frac{\alpha^2+1}{\alpha^2-1} + 2 < 0 \quad \text{or} \quad \frac{\alpha^2+1}{\alpha^2-1} - 2 > 0$$

$(\alpha^2 - 1)$ is positive or negative

$$\therefore \quad \frac{3\left(\alpha^2-\frac{1}{3}\right)(\alpha^2-1)}{(\alpha^2-1)^2} < 0$$

$$\therefore \quad \alpha^2 \in \left(\frac{1}{3},\ 1\right) \cup (1,\ 3) \Rightarrow \alpha^2 \in \left(\frac{1}{3}, 3\right) - \{1\}$$

$\therefore$ $a = \frac{1}{3}$, $b = 3$, $c = 1$.

$ab + c = 2.$ [Ans. 2]

Q.4 An A.P. and a H.P., have the same first term, the same last term, and the same number of terms; prove that the product of the rth term from the beginning in one series and the rth term from the end in the other is independent of r.

Sol. T_r of A.P. $= a + (r-1)d$

where $b = a + (n-1)d$

$\therefore$ T_r of A.P. $= a + (r-1)\dfrac{b-a}{n-1}$

$$= \frac{a(n-r)+(r-1)}{n-1} \qquad \ldots(i)$$

T_r from end of H.P. a, …!, b (n terms)

$= $ Reciprocal of T_r from end of A.P. $\dfrac{1}{a} \cdots\cdots \dfrac{1}{b}$

$= $ Reciprocal to T_r from begining of A.P. $\dfrac{1}{b} - \dfrac{1}{a}$ (n terms)

Replace a by $\dfrac{1}{b}$ and b by $\dfrac{1}{a}$ in (i) then take reciprocal.

or reciprocal of $\dfrac{\frac{1}{b}(n-r)+\frac{1}{a}(r-1)}{n-1} = \dfrac{ab(n-1)}{a(n-r)+b(r-1)}$ …(ii)

or reciprocal of $\dfrac{\quad}{n-1} = \dfrac{\quad}{a(n-r)+b(r-1)}$ …(ii)

Multiplying (i) and (ii), we get the product = ab, which is independent of r.

Q.5(a) If $A_1, A_2 : G_1, G_2$; and H_1, H_2 be two A.M.'s and G.M.'s and H.M.'s between two quantities, then prove that $\dfrac{G_1 G_2}{H_1 H_2} = \dfrac{A_1 + A_2}{H_1 + H_2}$.

(b) If $A_1, A_2 : G_1, G_2$; and H_1, H_2 be two A.M.'s and G.M.'s and H.M.'s between two quantities 'a' and 'b' then $A_1H_2 = A_2H_1 = G_1G_2 = ab$.

Sol.(a) Sum of n A.M.s = n × single A.M.

$$A_1 + A_2 = 2\left(\frac{a+b}{2}\right) = a + b$$

Product of n G.M.s = (single G.M.)^n

$$G_1G_2 = (\sqrt{ab})^2 = ab$$

$\dfrac{1}{a}, \dfrac{1}{H_1}, \dfrac{1}{H_2}, \dfrac{1}{b}$ are in A.P.

$\therefore$ $\dfrac{1}{H_1} + \dfrac{1}{H_2} = \dfrac{1}{a} + \dfrac{1}{b} = \dfrac{a+b}{ab}$

or $\dfrac{H_1 + H_2}{H_1H_2} = \dfrac{A_1 + A_2}{G_1G_2}$

(b) a, A_1, A_2, b are in A.P. ...(i)

a, H_1, H_2, b are in H.P.

$\therefore \quad \frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{b}$ are in A.P.

Multiply by ab.

$\therefore \quad b, \frac{ab}{H_1}, \frac{ab}{H_2}, a$ are in A.P.

Take in reverse order.

or $\quad a, \frac{ab}{H_2}, \frac{ab}{H_1}, b$ are in A.P. ...(ii)

Compare (i) and (ii)

$\therefore \quad A_1 = \frac{ab}{H_2}$ and $A_2 = \frac{ab}{H_1}$

$\therefore \quad A_1H_2 = A_2H_1 = ab = G_1G_2$

Q.6 Let the sequence $a_1, a_2, \ldots\ldots a_n$ form an A.P. and let $a_1 = 0$, prove that

$$\frac{a_3}{a_2} + \frac{a_4}{a_3} + \frac{a_5}{a_4} + \ldots\ldots \frac{a_n}{a_{n-1}} - a_2\left(\frac{1}{a_2} + \frac{1}{a_3} + \ldots\ldots + \frac{1}{a_{n-2}}\right) = \frac{a_{n-1}}{a_2} + \frac{a_2}{a_{n-1}}.$$

Sol. Let d be the common difference of the given A.P.

Then since $a_1 = 0$, we have $a_2 = d$

$a_3 = 2d, \ldots\ldots a_n = (n-1)d.$

Hence $\text{L.H.S.} = \frac{2d}{d} + \frac{3d}{2d} + \frac{4d}{3d} + \ldots\ldots + \frac{(n-1)d}{(n-2)d} - d\left(\frac{1}{d} + \frac{1}{2d} + \frac{1}{3d} + \ldots\ldots + \frac{1}{(n-3)d}\right)$

$$= (1+1) + \left(1+\frac{1}{2}\right) + \left(1+\frac{1}{3}\right) + \ldots\ldots + \left(1+\frac{1}{n-3}\right) + \left(1+\frac{1}{n-2}\right) - \left(1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n-3}\right)$$

$$= 1 + 1 + 1 + \ldots\ldots \text{ to } (n-2) \text{ terms} + \frac{1}{n-2}$$

$$= (n-2) + \frac{1}{n-2} = \frac{a_{n-1}}{a_2} + \frac{a_2}{a_{n-1}}$$

Q.7 Prove that if the sum of n terms of the following series

$$\frac{2n+1}{2n-1}+3\left(\frac{2n+1}{2n-1}\right)^2+5\left(\frac{2n+1}{2n-1}\right)^2+\ldots\ldots$$

is 36 then n = 4.

Sol. We put $x=\frac{2n+1}{2n-1}$ ...(i)

$\therefore \quad 1-x=-\frac{2}{2n-1}$ ...(ii)

or $\quad \frac{x}{1-x}=-\frac{2n+1}{2}$ ...(iii)

Let $\quad S = x + 3x^2 + 5x^3 + \ldots\ldots (2n-1)\,x^n$.

$\therefore \quad xS = \quad x^2 + 3x^3 + \ldots\ldots (2n-3)\,x^n + (2n-1)\,x^{n+1}$

$\therefore \quad S\,(1-x) = x + [2x^2 + 2x^3 + \ldots\ldots (n-1) \text{ terms}] - (2n-1)x^{n+1}$

$\therefore \quad S=\frac{x}{1-x}\,[1-2n+1+(2n+1)\,x^{n-1}-(2n+1)\,x^{n-1}]$

$=-\frac{2n+1}{2}(-2n)=n\,(2n+1)=36$, given

$\therefore \quad n = 4.$

$\therefore \quad n = 4.$

Q.8 If $\sum_{r=1}^{n} I(r)=n(2n^2+9n+13)$, then find the sum $\sum_{r=1}^{n}\sqrt{I(r)}$.

Sol. $S_n=\sum_{r=1}^{n} I(r) = n(2n^2+9n+13)$

$\Rightarrow \quad I(r) = S_r - S_{r-1}$

$= r(2r^2 + 9r + 13) - (r-1)\,(2\,(r-1)^2 + 9(r-1) + 13)$

$= 6r^2 + 12r + 6 = 6(r+1)^2$

$\Rightarrow \quad \sqrt{I(r)}=\sqrt{6}(r+1)$

$\Rightarrow \quad \sum_{r=1}^{n}\sqrt{I(r)}=\sqrt{6}\sum_{r=1}^{n}(r+1)$

$$=\sqrt{6}\left(\frac{n^2+3n}{2}\right)$$

$$=\sqrt{\frac{3}{2}}(n^2+3n)$$

Q.9 Find the sum to n terms of the series $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \ldots\ldots$.

Sol. Clearly, n^{th} term of the given series is negative or positive accordingly as n is even or odd, respectively.

(a) n is even :

$$1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \ldots\ldots + (n-1)^2 - n^2$$

$$= (1^2 - 2^2) + (3^2 - 4^2) + (5^2 - 6^2) + \ldots\ldots + ((n-1)^2 - n^2)$$

$$= (1-2)(1+2) + (3-4)(3+4) + (5-6)(5+6) + \ldots\ldots + ((n-1)-(n))(n-1+n)$$

$$= -(1+2+3+4+\ldots\ldots+(n-1)+n)$$

$$= -\frac{n(n+1)}{2}$$

(b) n is odd :

$$(1^2 - 2^2) + (3^2 - 4^2) + \ldots\ldots + \{(n-2)^2 - (n-1)^2\} + n^2$$

$$= (1-2)(1+2) + (3-4)(3+4) + \ldots\ldots + [(n-2)-(n-1)][(n-2)+(n-1)] + n^2$$

$$= -(1+2+3+4+\ldots\ldots+(n-2)+(n-1)) + n^2$$

$$= -\frac{(n-1)(n-1+1)}{2} + n^2$$

$$= \frac{n(n+1)}{2}$$

Q.10 Find the greatest value of $(a+x)^3(a-x)^4$ for any real value of x numerically less than a.

Sol. Let $z = (a+x)^3(a-x)^4$

$$= 3^3 \cdot 4^4 \left(\frac{a+x}{3}\right)^3 \left(\frac{a-x}{4}\right)^4 \quad \ldots (1)$$

z will be maximum, when $\left(\frac{a+x}{3}\right)^3 \left(\frac{a-x}{4}\right)^4$ is maximum but $\left(\frac{a+x}{3}\right)^3 \left(\frac{a-x}{4}\right)^4$ is product of

$3 + 4 = 7$ factors.

The sum of which $= 3\left(\frac{a+x}{3}\right) + 4\left(\frac{a-x}{4}\right) = (a+x) + (a-x) = 2a.$

$\therefore$ $\left(\frac{a+x}{3}\right)^3 \left(\frac{a-x}{4}\right)^4$ will be maximum if all the factors are equal i.e., if $\frac{a+x}{3} = \frac{a-x}{4}$.

or $4a + 4x = 3a - 3x$ or $x = -\frac{a}{7}$

So from (1) maximum value of z

$$= 3^3 \cdot 4^4 \left[\frac{a-(a/7)}{3}\right]^3 \left[\frac{a-(a/7)}{4}\right]^4$$

$$= 3^3 \cdot 4^4 \left(\frac{6a}{3\times 7}\right)^3 \left(\frac{8a}{7\times 4}\right)^4$$

Q.11 Find the sum $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)(r+3)}$

Sol. $\frac{1}{r(r+1)(r+2)(r+3)}$

$$= \frac{r+3-r}{3[r(r+1)(r+2)(r+3)]}$$

$$= \frac{r+3-r}{3[r(r+1)(r+2)(r+3)]}$$

$$= -\frac{1}{3}\left[\frac{1}{(r+1)(r+2)(r+3)} - \frac{1}{r(r+1)(r+2)}\right]$$

$$= -\frac{1}{3}[V(r) - V(r-1)]$$

$\Rightarrow$ $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)(r+3)}$

$$= -\frac{1}{3}[V(n) - V(0)]$$

$$= \frac{1}{3}\left[\frac{1}{6} - \frac{1}{(n+1)(n+2)(n+3)}\right]$$