Binomial No-Arbitrage Pricing Model

Contents

1 One-Period Binomial Model

The binomial asset-pricing model provides a powerful tool to understand arbitrage pricing theory and probability.

1.1 Model Setup

For the general one-period model, we call the beginning of the period time zero and the end of the period time one. At time zero, we have a stock whose price per share we denote by S_0, a positive quantity known at time zero. At time one, the price per share of this stock will be one of two positive values, which we denote $S_1(H)$ and $S_1(T)$, the H and T standing for head and tail, respectively.

We do not assume this coin is fair (i.e., the probability of head need not be one-half). We assume only that the probability of head, which we call p, is positive, and the probability of tail, which is $q = 1 - p$, is also positive. The outcome of the coin toss, and hence the value which the stock price will take at time one, is known at time one but not at time zero.

We introduce the two positive numbers:

$$d = \frac{S_1(T)}{S_0}, \quad u = \frac{S_1(H)}{S_0} \tag{1.1.1}$$

with the assumption $0 < d < u$.

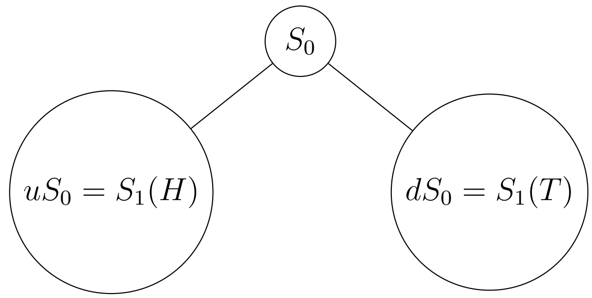

Figure 1: One-Period Binomial Tree

We introduce also an interest rate r. One dollar invested in the money market at time zero will yield $1 + r$ dollars at time one. Conversely, one dollar borrowed from the money market at time zero will result in a debt of $1 + r$ at time one. The mathematics requires only $r > -1$.

Important Note

An essential feature of efficient market is that if a trading strategy can turn nothing into something, then it must also run the risk of loss.

Otherwise there would be an arbitrage-defined as a trading strategy that begins with no money, has zero probability of losing money, and has a positive probability of making money.

Real markets sometimes exhibit arbitrage, but this is fleeting, as soon as someone discovers it, trading takes places and removes it.

To avoid arbitrage:

$$0 < d < 1 + r < u \tag{1.1.2}$$

So, here $d > 0$ for +ve stock.
If $d \geq 1 + r$, buy the stock by borrowing: even in the worst case, stock covers debt. If $u \leq 1 + r$, short the stock and invest: even in best case, gain from money market covers repurchase. Either scenario offers arbitrage. And, when $u > d \geq 1 + r$ we get arbitrage.
It is also common to have $d = \frac{1}{u}$ in most cases.
This model is far from real life scenario. Yet, we have to know it to know in depth about what arbitrage actually is.

1.2 European Call Option

Confers owner the right but not obligation to buy one share of stock at time one for strike price K. We assume $S_1(T) < K < S_1(H)$, so tail is worthless expire of option. If head, option can be exercised and yields a profit of $S_1(H) - K$.
We summarize by saying option at time one is worth $(S_1(H) - K)^+$ where $(\cdots)^+$ means we take maximum of expression in parentheses and zero.

Important Note

Arbitrage pricing Theory approach to option pricing problem is to replicate option by trading in stock and money market.

Check out the example 1.1.1 from Steve Shreve's Stochastic Calculus textbook given in GitHub Link. The tree diagram for this case is :

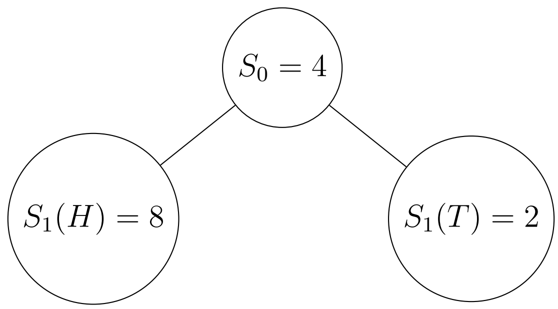

Figure 2: Tree diagram for Example 1.1.1

The reasoning in this example relies on several key assumptions:

- Divisible Shares: Stock can be bought or sold in fractions (not just whole shares).
- Equal Interest Rates: The rate for borrowing money is the same as the rate for investing.
- No Bid-Ask Spread: The buying and selling prices for the stock are identical (no transaction costs).
- Binary Price Movement: The stock can only move to one of two possible values in the next time period.

All of these assumptions - except the last one (the two-price restriction)—are also foundational to the Black-Scholes-Merton option-pricing model.

1.3 Replication Strategy and Risk-Neutral Pricing

Let the derivative security pay an amount $V_1(H)$ at time one if the outcome is head and $V_1(T)$ if the outcome is tail. A European call is a specific type of such a security. Another example is the European put option, which pays $(K - S_1)^+$ at time one, and a third example is a forward contract, with a payoff $S_1 - K$.
To determine the price V_0 at time zero, we replicate it as done in Example 1.1.1. Suppose an investor begins with wealth X_0 and purchases Δ_0 shares of the stock. The remaining wealth is put into the money market account. So the cash position becomes $X_0 - \Delta_0 S_0$.
At time one, the value of this portfolio becomes:

$$X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0) = (1+r)X_0 + \Delta_0(S_1 - (1+r)S_0) \tag{1.1.3}$$

We want this portfolio to match the derivative payoff:

$$X_1(H) = V_1(H), \quad X_1(T) = V_1(T) \tag{1.1.4}$$

That leads to the system of equations:

$$X_0 + \Delta_0 \left(\frac{S_1(H) - (1+r)S_0}{1+r} \right) = \frac{V_1(H)}{1+r} \tag{1.1.5}$$

$$X_0 + \Delta_0 \left(\frac{S_1(T) - (1+r)S_0}{1+r} \right) = \frac{V_1(T)}{1+r} \tag{1.1.6}$$

To solve, multiply the first equation by p and the second by $q = 1 - p$ and add:

$$X_0 + \Delta_0 \left(\frac{pS_1(H) + qS_1(T) - (1+r)S_0}{1+r} \right) = \frac{pV_1(H) + qV_1(T)}{1+r} \tag{1.1.7}$$

To eliminate the term involving Δ_0, we choose p such that:

$$S_0 = \frac{1}{1+r}[pS_1(H) + (1-p)S_1(T)] \tag{1.1.8}$$

Solving for X_0 gives:

$$X_0 = \frac{1}{1+r}[pV_1(H) + (1-p)V_1(T)] \tag{1.1.9}$$

To solve for p directly:

$$S_0 = \frac{1}{1+r}[puS_0 + (1-p)dS_0] = \frac{S_0}{1+r}[(u-d)p + d] \tag{1.1.10}$$

Solving for p:

$$p = \frac{1+r-d}{u-d}, \quad q = 1 - p = \frac{u-(1+r)}{u-d} \tag{1.1.11}$$

These are the **risk-neutral probabilities**.
To find Δ_0, subtract (1.1.6) from (1.1.5):

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} \tag{1.1.12}$$

This equation (1.1.12) is the **Delta Hedging formula**.

Hence, if the agent starts with wealth X_0 as given in (1.1.9) and buys Δ_0 shares (from 1.1.12), the portfolio value at time one will match the derivative's value in either outcome. This means the agent has hedged a short position in the derivative.

So, the no-arbitrage price of the derivative at time zero is:

$$V_0 = \frac{1}{1+r}[pV_1(H) + qV_1(T)] \tag{1.1.13}$$

Interpretation

This price allows perfect hedging and avoids arbitrage. Any other time-zero price introduces arbitrage opportunities.

1.4 Risk-Neutral Probability Interpretation

Although derived through replication and hedging, equation (1.1.13) is often interpreted probabilistically. The quantities p and q from equation (1.1.11) are not the true probabilities of head and tail, but rather **risk-neutral probabilities**.

In real markets, the expected rate of return on a stock typically exceeds the risk-free rate. That is:

$$S_0 < \frac{1}{1+r}[pS_1(H) + qS_1(T)] \tag{1.1.14}$$

However, under the risk-neutral measure, we artificially assume the stock grows at the risk-free rate so that:

$$S_0 = \frac{1}{1+r}[pS_1(H) + qS_1(T)] \tag{1.1.15}$$

This simplifies pricing since the **mean rate of return under risk-neutral probabilities** is the risk-free rate.

Important Conclusion

Prices of derivative securities in the binomial model depend only on the structure of possible price paths and not on their real-world probabilities. This carries over to continuous-time models, where prices depend on volatility, not drift.