Course code: MT401

Numerical Methods & its Applications

Books recommended:

1. Introductory Methods of Numerical Analysis - S.S. Sastry
2. Numerical methods in Engineering and science - B.S. Grewal
3. Applied Numerical Analysis - Curtis F. Gerald & Patric O. Wheatley

Introduction to Numerical Analysis:

As we are aware with the word "Analysis" in mathematics which usually means to solve problems via equations. In order to obtain answer, we obtain the procedures of algebra, calculus, etc. Numerical analysis also used to seek answer to a problem but with the caveat that only procedures now are used are addition, subtraction, multiplication, division and comparison.

As these operations are exactly what computers can do, hence, computers and numerical analysis are related intimately.

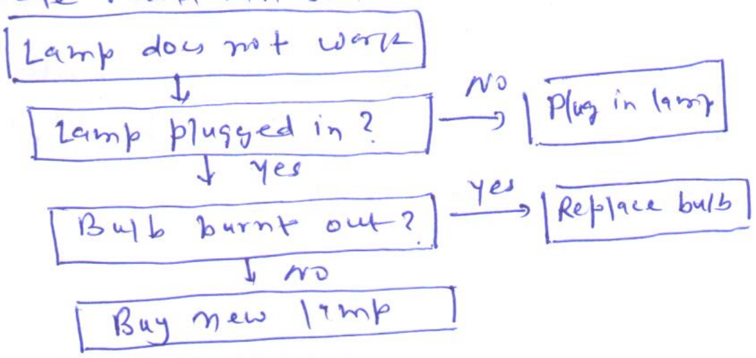
More formally, we can say, numerical analysis is study of algorithms for the problems of continuous mathematics.

Algorithm:

It is finite sequence of instructions, an explicit, step-by-step procedure for solving problem, often used for calculation and data processing. It is formally a type of effective method in which a lot of well defined instructions for completing a task, will when given an initial state, proceeds through a well defined series of successive states, eventually terminating in an end-state.

The transition from one state to the next state is not necessarily determinstic, some algorithms, known as probabilistic algorithms, incorporate randomness.

An example:



An example is finding numerical approximation of $\sqrt{2}$. In ancient Babylon it was tried and solved.

Much like the problem, we presented above, modern numerical analysis does not seek exact answers because most often they are impossible to find out. Instead, much of the numerical analysis is concerned with obtaining approximate solutions while maintaining reasonable bounds on errors.

In other words, aim of numerical analysis is to provide efficient method for numerical answers to such problems.

Numerical analysis finds applications in all fields of engineering and science.

Numerical Analysis and computers

Because numerical methods require tedious and repetitive arithmetic operations, therefore, ~~computer~~ use of computers is a practical way to solve problems. Chances of committing mistakes by a human is quite high and hence little confidence in the obtained result. Besides, cost of manpower ~~could~~ could be more than what could normally be afforded.

prior to 1950, manual calculations were only means to solve these problems. However, with the advent of computers, we have an efficient tool to solve problems. These are cost effective and fast.

A computer cannot do calculations on its own. In that way, it is simply a dumb box. In order to perform task, it needs to be given detailed and complete instructions: step-by-step. These instructions are known as computer program. Some important programming languages are: BASIC, FORTRAN, C, C++, Java, Pascal, etc. In recent times, many computer algebra systems have come up like Mathematica, MATLAB, Maple, etc.

Error in numerical analysis

Since, numerical data, most often, used are only approximate, being true to two, three or more figures. Sometimes, the method used are also approximate and, thus, the error in a computed result may be due to errors in data, or in method or both.

Numbers and their accuracy

There are two kinds of numbers, one called, exact - E.g. 1, 2, 3, ..., $\frac{1}{2}$, $\frac{3}{2}$, ..., e, etc. or approximate numbers like 3.1416, 3.14159265 (better approximation).

The digits which are used to express numbers are called significant digits or significant figures. For instance, 3.1416, 0.22225, 6.0985 and 5.1092 all have 5 significant figure. The numbers 0.00092 and 0.00041 have two significant digits because zeros only fix the position of decimal point. In case of ambiguity scientific notations should be used. An example, 23,900 has uncertain significant figures, but 2.39×10^4, 2.390×10^4, 2.3900×10^4 have 3, 4 and 5 significant figures, respectively.

Frequently, we come across numbers with large number of digits and will be necessary to truncate them to a usable number of figures. This is known as rounding off.

Examples: 3.9582 to 3.958
29.078 to 29.08
3.14159 to 3.142

In addition, we may come across another type of error known as truncating error. These are caused by using approximate formula for computation - such as when a function f(x) is evaluated from an infinite series for x, after truncating it at certain stage.

Absolute, Relative and percentage error:

We always seek answers with least inaccuracy. Hence, it is of great importance.

$$\text{Absolute Error} = |\text{true value} - \text{approximate value}|$$

$$\text{Relative error} = \frac{\text{Absolute error}}{\text{True value}}$$

$$\text{percentage error} = \frac{\text{Absolute error}}{\text{True value}} \times 100\%$$

If $E_1, E_2, \ldots, E_n$ are absolute errors in 'n' numbers, then the magnitude of the absolute error in their sum is

$|E_1| + |E_2| + \cdots + |E_n|$

Important: When adding up several numbers with different absolute accuracies, the following could be adopted

(i). Isolate the number with greatest absolute error

(ii). Round-off all other numbers retaining in them one digit more than in the isolated number

(iii). Add up and (iv) Round-off sum by discarding one digit

If a quantity is like $y = ab$, then absolute error will be

$$E_{ab} = (a+E_1)(b+E_2) - ab \quad ; \quad E_1 \,\&\, E_2 \text{ are errors (absolute) in 'a' and 'b' respectively}$$

$$= aE_2 + bE_1 + E_1E_2 \simeq aE_2 + bE_1$$

If $y = a/b$, then

$$\frac{a+E_1}{b+E_2} - \frac{a}{b} = \frac{a}{b}\left(\frac{E_1}{a} - \frac{E_2}{b}\right)$$

Example 1: If the number X is rounded to r decimal places, then

$$\Delta X = \frac{10^{-r}}{2}$$

Appx. value, True value

$|X_1 - X| \le \Delta X$ — upper limit of absolute error and is said to be absolute accuracy

$$\frac{\Delta X}{|X|} = \frac{\Delta X}{|X_1|} \rightarrow \text{Relative accuracy}$$

If $X = 0.57$ and is correct to 2 decimal places, then

$$\Delta X = \frac{10^{-2}}{2} = 0.005$$

$$\text{Relative accuracy} = \frac{0.005}{0.57} \simeq 0.98 \text{ or}$$

98% (Percentage accuracy)

Example 2: Find the difference

$$\sqrt{6.37} - \sqrt{6.36}$$

to three significant figures.

Soln: $\sqrt{6.37} = 2.523885893$

& $\sqrt{6.36} = 2.521904043$

$\therefore \sqrt{6.37} - \sqrt{6.36} = 0.00198$, correct to 3 significant figures

A General Error Formula

More general formula for the error committed in using a certain formula or a functional relation

Let $u = f(x_1, x_2, \ldots, x_n)$ be a function of several variables and let each variable has Δx_i $(i=1,\ldots,n)$ error.

$\therefore$ Error in u is Δu given by

$$u + \Delta u = f(x_1 + \Delta x_1, x_2 + \Delta x_2, \ldots, x_n + \Delta x_n)$$

Expanding RHS by Taylor's series

$$f(x_1 + \Delta x_1, x_2 + \Delta x_2, \ldots) = f(x_1, x_2, \ldots, x_n) + \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + \cdots + \frac{\partial f}{\partial x_n}\Delta x_n$$
$$+ \frac{1}{2}\left[\frac{\partial^2 f}{\partial x_1^2}(\Delta x_1)^2 + \cdots + \frac{\partial^2 f}{\partial x_n^2}(\Delta x_n)^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}\Delta x_1 \Delta x_2 + \cdots\right] + \cdots$$

So, we get

$$u + \Delta u = f(x_1, x_2, \ldots, x_n) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\Delta x_i + \underbrace{\text{Term involving } (\Delta x_i)^2, \text{ etc.}}_{\text{Ignoring this as it is very small}}$$

$$\Delta u \simeq \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\Delta x_i$$
$$= \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + \cdots$$

$\therefore$ Relative error

$$E_R = \frac{\Delta u}{u} = \frac{\partial u}{\partial x_1}\frac{\Delta x_1}{u} + \frac{\partial u}{\partial x_2}\frac{\Delta x_2}{u} + \cdots$$

Example 3: If $u = 5xy^2/z^3$, find E_R at $x = y = z = 1$ and error in x, y, z is 0.001

Soln: $\partial u/\partial x = 5y^2/z^3$, $\frac{\partial u}{\partial y} = \frac{10xy}{z^3}$, $\frac{\partial u}{\partial z} = \frac{-15xy^2}{z^4}$

In general $\Delta x, \Delta y, \Delta z$ may be positive or negative. Hence, we take the absolute values.

$$(\Delta u)_{max} = \left|\frac{5y^2}{z^3}\Delta x\right| + \left|\frac{10xy}{z^3}\Delta y\right| + \left|\frac{15xy^2}{z^4}\Delta z\right|$$

$\because \Delta x = \Delta y = \Delta z = 0.001$ and $x = y = z = 1$, thus,

$$(E_R)_{max} = (\Delta u)_{max}/u = \frac{0.03}{5} = 0.006$$

Ex 4: If $z = \frac{1}{8}xy^3$, find percentage error in z when $x = 3.14 \pm 0.0016$ and $y = 4.5 \pm 0.05$

Solⁿ: Following above example, we get

$$\frac{\Delta z}{z} = \left|\frac{\Delta x}{x}\right| + 3\left|\frac{\Delta y}{y}\right| = 0.03$$

or $\underline{3\%}$ (percentage error)

Error in a Series approximation:

The error committed in a series expansion can be evaluated by using the remainder after 'n' terms

Taylor's series for $f(x)$ at $x = a$ is given by

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{\underline{|2}} f''(a) + \cdots$$

$$\cdots + \frac{(x-a)^{n-1}}{\underline{|n-1}} f^{n-1}(a) + R_n(x)$$

where $R_n(x) = \frac{(x-a)^n}{\underline{|n}} f^n(\phi)$, $a < \phi < x$

we see, as $n \to \infty$, $R_n \to 0$

Thus, if we approximate $f(x)$ by the first 'n' terms of a series, then the maximum error committed in this series will be given by remainder term.

Conversly, if the accuracy is known beforehand, then it would be possible to find n, the number of terms, that yield specified accuracy.

Below is shown an example to improve accuracy step by step.

Ex.5: Evaluate $f(1)$ using Taylor's series, where

$$f(x) = x^3 - 3x^2 + 5x - 10$$

Solⁿ: For $x = 1$, $f(1) = -7$ (Actual value), but let us see how increasing order from 0 to 3, increases accuracy of $f(1)$ gradually.

Let $h=1$, $x_i=0$ and $x_{i+1}=1$, we require to compute $f(x_{i+1})$.

Now, derivatives $f'(x) = 3x^2 - 6x + 5$

$f''(x) = 6x-6$, $f'''(x) = 6$

$f^{iv}(x)$ and higher order derivatives are zero.

$f'(x_i) = f'(0) = 5$, $f''(x_i) = f''(0) = -6$, $f'''(0) = 6$

$f(x_i) = f(0) = -10$

$$\therefore f(x_{i+1}) = f(x_i) + hf'(x_i) + \frac{h^2}{\lfloor 2} f''(x_i) + \frac{h^3}{\lfloor 3} f'''(x_i)$$

$$\therefore f(x_{i+1}) \simeq f(x_i) + O(h) \quad \text{[zero order approximation]}$$

$$f(1) = f(0) + O(h) = -10$$

Error with accurate value $-7+10$ i.e. $=3$

Now take first approximation

$$f(x_{i+1}) = f(x_i) + hf'(x_i) + O(h^2)$$

$$f(1) = f(0) + hf'(0) + O(h^2)$$

$$= -10 + 1 \times (5) + O(h^2) = -5$$

Error is now $-7+5 = -2$

Second order Taylor approximation

$$f(x_{i+1}) = f(x_i) + hf'(x_i) + \frac{h^2}{2} f''(x_i) + O(h^3)$$

$$f(1) = -10 + 5 + \tfrac{1}{2}(-6) + O(h^3) = -8$$

Error now is $-7+8 = 1$

Finally, $f(x_{i+1}) = f(x_i) + hf'(x_i) + \frac{h^2}{2} f''(x_i) + \frac{h^3}{6} f'''(x_i)$

$$f(1) = -10 + 5 + \tfrac{1}{2}(-6) + \tfrac{1}{6}(6)$$

$$= -7$$

This is exact value.

The following example shows, finding the number of terms that yield required accuracy.

Ex. 6. e^x can be expanded as

$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots + \frac{x^{n-1}}{\lfloor n-1} + \frac{x^n}{\lfloor n} e^{\phi}, \quad 0 < \phi < x$$

~~calculate~~ find number of terms such that their sum gives the value of e^x correct to 8 decimal places at $x=1$.

Soln: The remainder term is error term and is given by $\frac{x^n}{\lfloor n} e^{\phi}$, so that $\phi = x$ which gives the maximum absolute error.

For 8 decimal accuracy at $x=1$, we must have

$$\frac{1}{\lfloor n} < \frac{10^{-8}}{2} \quad \text{(Recall Example 1)}$$

Solving it, we get $n = 12$.

So 12 terms are required.