Before we start next section, some useful theorems are required to be stated here:

Theorem 1: If $f(x)$ is continuous in $a \leq x \leq b$ and if $f(a)$ and $f(b)$ are of opposite sign, then $f(\phi)=0$ for at least one number ϕ such that $a<\phi<b$.

Theorem 2: If $f(x)$ is continuous in $a \leq x \leq b$, $f'(x)$ exists in $a<x<b$ and $f(a)=f(b)=0$ then, there exists at least, one value of x, say ϕ, such that $f'(\phi)=0$, $a<\phi<b$.

This is known as Rolle's theorem.

Theorem 3: Let $f(x)$ be a function which is n times differentiable on $[a,b]$. If $f(x)$ vanishes at the $(n+1)$ distinct points $x_0, x_1, \ldots, x_n$ in (a,b), then there exists a number ϕ in (a,b) such that $f^n(\phi)=0$.

This is generalized Rolle's theorem.

Theorem 4: If $f(x)$ is continuous in $[a,b]$ and $f'(x)$ exists in (a,b), then there exists at least one value of x, say, ϕ between a and b such that

$$f'(\phi) = \frac{f(b)-f(a)}{b-a} \qquad a<\phi<b$$

If we set $b=a+h$, then

$$f(a+h) = f(a) + hf'(a+\theta h), \quad 0<\theta<1.$$

This is mean-value theorem for derivatives.

Solution of Nonlinear Equations

An important problem in science and engineering problems is to find root of a function, i.e., for a function $f(x) = 0$. In other words, root is that value of x for which $f(x)=0$.

For example, $$f(x) = x^2-6x+9$$

This function has root at 3 because $f(3) = 3^2-6\times3+9 = 0$

If a function is mapping from real numbers to real numbers, its zeros are points where the graph meets the x-axis. x-value of such point is called x-intercept. Hence, in this situation root can be called as an x-intercept.

If a function $f(x)$ is a quadratic, biquadratic or cubic, then algebric formulae are available for expressing the roots in terms of the coefficients

If $f(x)$ is a polynomial of higher degree or an expression involving transcendental functions, e.g., $1+\cos x - 5x$, etc., algebric methods are not available and recourse must be used to find roots by approximate methods.

(Transcendental function - a non algebric function, e.g.
$f(x) = \ln x^3 - 0.8$, $g(x) = e^{-0.3x} + 6x$, $h(x) = \cos x + x^2+3$, etc.)

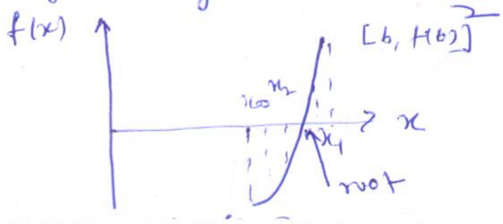
A substantial amount of mathematics has beem developed to find roots. Complex numbers were discovered to handle roots with negative discriminant, i.e., $\sqrt{-1}$.

Some of the methods, we describe here:

Bisection (Interval Halving) method

It is based on theorem 1 (described earlier).

Let $f(a)$ be negative and $f(b)$ be positive, then the root lies between a and b and let its approximate value is given by $x_0 = \frac{a+b}{2}$. If $f(x_0) = 0$, we conclude that x_0 is the root of equation $f(x) = 0$.



Otherwise, the root lies either between x_0 and b, or between x_0 and a depending on whether $f(x_0)$ is negative or positive, respectively.

Then, as before, we bisect (halve) the interval and repeat the process until the root is known with the desired accuracy.

For example, if $f(x_0) = +ive$, then root lies between x_0 and a.

On the other hand, if $f(x_0) = -ive$, then root lies between x_0 and b.

If more than one roots are there in an interval bisection method finds one of the roots.

It can be programmed via following steps:

1. Choose the real numbers a & b such that $f(a) f(b) < 0$
2. Set $x_r = \frac{a+b}{2}$
3. (a) If $f(a) f(x_r) < 0$ then root is in interval $[a, x_r]$. Then set $b = x_r$ and go to step 2.

   (b) If $f(a) f(x_r) > 0$ then root lies in interval $[x_r, b]$. Then set $a = x_r$ and goto step 2.

   (c). If $f(a) f(x_r) = 0$, x_r is the root and stop.

practically, we never observe condition (c), so we take convenient method of percentage error

$$\epsilon_r = \left[\frac{x_r' - x_r}{x_r'} \right] \times 100\% \qquad [x_r' \text{ is new value of } x_r]$$

The computation may be stopped when desired accuracy is reached.

In this method, main advantage is that is guarantee's to work for $f(x)$ if it is continuous and if values $x = a$ and $x = b$ actually brackets a root. Another advantage is that number of iterations (steps) are known in advance to achieve desired accuracy. Because the interval $[a, b]$ is divided by 2 each time, so we can say

$$\text{Error after } n \text{ iterations} < \left| \frac{b-a}{2^n} \right|$$

Example 1: Find the real root of the equation

$$f(x) = x^3 - x - 1 = 0$$

Soln: $f(1) = 1-1-1 = -1$ (-ive), $f(2) = 2^3-2-1 = 5$ (+ive)

So root lies between $x=1$ & 2.

So $x_0 = \frac{1+2}{2} = \frac{3}{2}$, $\left(\frac{a+b}{2}\right)$

Check $f(x)$ at $x = x_0$

$f(x_0) = \frac{27}{8} - \frac{3}{2} - 1 = 7/8$ (+ive)

$\therefore$ Root lies between 1 & $\frac{3}{2}$. So now

$x_1 = \frac{1+1.5}{2} = 1.25$

$f(x_1) = (1.25)^3 - (1.25) - 1 = -0.2969$ (-ive)

So root lies between 1.25 & 1.5,

Next $x_2 = \frac{1.25+1.5}{2} = 1.375$

$f(x_2) = (1.375)^3 - 1.375 - 1 = 0.2246$ (+ive)

Root lies between 1.25 & 1.375

$x_3 = \frac{1.25+1.375}{2} = 1.3125$

$f(x_3) = -0.05$ (-ive)

Root lies between 1.3125 & 1.375

$x_4 = 1.34375$, $x_5 = 1.328125$

$x_6 = 1.33593$, etc.

Example 2: Find the positive root of equation $xe^x - 1 = 0$ to a tolerence of 0.05%

Soln: Let $f(x) = xe^x - 1 = 0$

$f(0) = -1$ (-ive) and $f(1) = e - 1 =$ +ive

So root lies between 0 and 1.

$\therefore$ $x_0 = \frac{0+1}{2} = 0.5$

$f(0.5) = -0.1756$ (-ive)

So root lies between 0.5 and 1.0

$x_1 = \frac{0.5+1.0}{2} = 0.75$

The error is

$$\epsilon_0 = \left|\frac{x_2 - x_1}{x_2}\right| \times 100 = \frac{0.25}{0.75} \times 100 = 33.33\%$$

$f(x_2) = 0.5878$ (+ve), the root lies between 0.5 and 0.75

$$x_3 = \frac{0.5 + 0.75}{2} = 0.625$$

$$\epsilon_1 = \left|\frac{0.625 - 0.75}{0.625}\right| \times 100 = 20\%$$

Proceeding this way, we can obtain

	Error		Error
$x_3 = 0.5625$	11.11%	$x_8 = 0.5684$	0.35%
$x_4 = 0.5938$	5.26%	$x_9 = 0.5674$	0.18%
$x_5 = 0.5781$	2.71%	$x_{10} = 0.5669$	0.09%
$x_6 = 0.5703$	1.37%	$x_{11} = 0.5671$	0.035%
$x_7 = 0.5664$	0.69%		

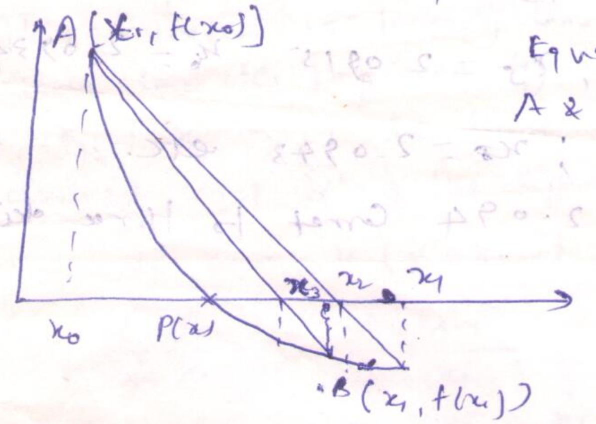
$\because$ 0.35% < 0.05% (as desired in the problem), the required root is 0.567, correct to three decimal places.

As is clear from the examples, it has slow convergence and this is major drawback of this method.

The ~~methods~~ method of false position

It is oldest method for finding real root of an equation, and closely resembles the bisection method.

Here, two chosen points x_0 & x_1 are ~~of opposite sign~~ such that $f(x_0)$ and $f(x_1)$ are of opposite signs. This indicates that root lies between x_0 & x_1.



Equation of chord joining two points A & B will be

$$\frac{y - f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Replacing curve AB by chord AB and taking point of intersection of the chord with the x-axis as an approximation to the root, so the abscissa of the point where chord cuts the x-axis $(y = 0)$ is given by

$$\frac{-f(x_0)}{x_2 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \qquad \left(\begin{matrix} x = x_2 \\ y = 0 \end{matrix}\right)$$

$$\therefore \quad x_2 - x_0 = -\frac{f(x_0)}{f(x_1) - f(x_0)}(x_1 - x_0)$$

$$\therefore \quad x_2 = x_0 - \frac{f(x_0)}{f(x_1) - f(x_0)}(x_1 - x_0) \qquad \text{--- (1)}$$

Now, if $f(x_0)$ & $f(x_2)$ are of opposite sign, then root lies between x_0 & x_2. So by replacing x_1 by x_2 in eqn (1),

Example. Find a real root of the equation $x^3 - 2x - 5 = 0$ by the method of false position to three decimal places.

Soln: Let $f(x) = x^3 - 2x - 5$

$$f(2) = -1 \quad \& \quad f(3) = 16$$

$\therefore$ A root lies between 2 & 3.

Taking $x_0 = 2$, $x_1 = 3$, $f(x_0) = -1$ & $f(x_1) = 16$

$$\therefore \quad x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$\therefore \quad x_2 = 2 - \frac{3 - 2}{16 - (-1)}(-1)$$

$$= 2 + \frac{1}{17} = 2.0588$$

Next

$$f(x_2) = f(2.0588) = (2.0588)^3 - 2(2.0588) - 5$$
$$= -0.3908$$

Root lies betn 2.0588 & 3.

$x_0 = 2.0588$ & $x_1 = 3$, $f(x_0) = -0.3908$

$f(x_1) = 16$

$\therefore x_3 = 2.0588 - \dfrac{0.9412}{16.3908}(-0.3908)$

$= 2.0813$

$x_4 = 2.0862$, $x_5 = 2.0915$, $x_6 = 2.0934$

$x_7 = 2.0941$; $x_8 = 2.0943$ etc.

Hence root is 2.094 Correct to three decimal places.