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PDE-based Image Restoration using Variational Denoising and Inpainting Models

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***Abstract*—A PDE-based image restoration model is proposed in this paper. It aims to restore degraded images that are affected by both noise and missing zones. The considered restoration approach is based on two PDE variational techniques. The first variational method performs an efficient noise reduction, while the second variational model provides the image reconstruction. By using both variational models, one achieves a much better enhancement of the degraded image.**

Keywords—image denoising; image restoration; PDE model; variational approach; inpainting model.

I. Introduction

The most important image pre-processing tasks, namely the image denoising and restoration, are performed by using some partial differential equation (PDE) based mathematical models [1]. The partial differential equations have been successful for solving various image processing and computer vision tasks since 1980s [1,2]. The variational and PDE-based approaches have been widely used and studied in these domains in the last decades, mainly because of their modeling flexibility and some advantages of their numerical implementation.

Image noise reduction with feature preservation is still a focus in the image processing field and a serious challenge for the researchers. An efficient denoising technique has to not only substantially reduce the quantity of image noise but also preserve the image boundaries and other characteristics [3]. The conventional smoothing models, such as the averaging, median, Wiener, or the classic 2D Gaussian filter succeed in noise reduction, but could also have undesired effects on edges or other image details and structures [4]. The PDE-based models provide efficient image filtering while preserving the features. The linear PDE-based denoising techniques are derived from the use of the Gaussian filter in multiscale image analysis [4]. The nonlinear PDE-based approaches are able to smooth images while preserving their edges, also avoiding the localization problems of linear filtering. Most popular nonlinear PDE denoising method is the influential nonlinear anisotropic diffusion scheme developed by P. Perona and J. Malik in 1987 [5]. Many smoothing methods derived from their algorithm have been proposed since then [6].

There are many ways to get the nonlinear PDEs. In image processing and computer vision it is very common to obtain them from some variational problems. The basic idea of any variational PDE technique is the minimization of a energy functional [1]. The variational approaches have important advantages in both theory and computation, compared with other methods. An influential variational denoising and restoration model was developed by Rudin, Osher and Fetami in 1992 [7]. Their technique, named Total Variation (TV) denoising, is based on the minimization of the TV norm [7] and is remarkably effective at simultaneously preserving boundaries whilst smoothing away the noise in flat regions. Because it suffers from the staircasing effect and its corresponding Euler- Lagrange equation is highly nonlinear and difficult to compute, many PDE approaches improving this classical variational model have been proposed in recent years [1,2].

Image reconstruction, which is known also as image inpainting, represents the computer vision process of restoring the missing areas of a damaged image as plausibly as possible from the known zones around them. The image reconstruction techniques are divided into the following categories: structural inpainting, textural inpainting, and combined approaches that perform simultaneous structure and texture inpainting. Texture-based inpainting is highly connected with the problem of texture synthesis. A lot of texture inpainting algorithms have been proposed since an influential texture synthesis model was developed by A. Efros and T. Leung [8]. In their approach texture is synthesized in a pixel by pixel way, by taking existing pixels with similar neighborhoods in a randomized fashion. Many other texture synthesis algorithms improving the speed and effectiveness of the Efros-Leung scheme have been elaborated in the last 15 years [9].

Structural inpainting uses PDE-based and variational reconstruction techniques. The PDE methods follow isophote directions in the image to perform the reconstruction. The first PDE-based inpainting model was introduced by Bertalmio et al. in [10]. Variational methods for image reconstruction have been introduced since 2001, when the Total Variation (TV) inpainting model was proposed by T. Chan and J. Shen [11]. Their variational scheme fills the missing image regions by minimizing the total variation, while keeping close to the original image in the known regions. It uses an Euler-

Lagrange equation and anisotropic diffusion based on the strength of the isophotes. TV inpainting model has been extended and considerably improved in other papers, too.

First we provide an effective variational PDE noise reduction technique, which is described in the next section. Then, a variational inpainting model is introduced in the third section. Some image enhancement experiments are provided in the fourth section. The paper finalizes with a conclusion's section and a list of references.

II. VARIATIONAL IMAGE NOISE REMOVAL TECHNIQUE

The general variational framework used for image denoising is based on the following energy functional:

$$J[u]=\int_{\Omega}\psi\left(\|\nabla u\|^2\right)+\lambda(u-u_0)^2\,d\Omega,\ \lambda>0 \tag{1}$$

where the function ψ is the regularizer, or penalizer, of the smoothing term and α represents the regularization parameter or smoothness weight [7,12]. We model a robust smoothing component, based on a novel penalizer function and a proper value of the smoothness weight. So, we consider the following regularizer, $\psi:[0,\infty)\to[0,\infty)$:

$$\psi(s)=\eta\sqrt{\frac{K}{\beta}}\ln\left(s+\sqrt{s^2+\frac{\gamma}{\beta}}\right)+\delta\cdot s, \tag{2}$$
$$K>0,\ \eta,\beta,\gamma,\delta\in(0,1)$$

We use some proper values for the penalizer's parameters. Then, we compute a minimizer for the energy functional given by (1), using the regularizer function provided by (2):

$$\begin{aligned} u_{\min} &= \arg\min_{u\in U} J(u)= \\ &= \arg\min_{u\in U}\int_{\Omega}\psi\left(\|\nabla u\|^2\right)+\lambda(u-u_0)^2\,d\Omega \end{aligned} \tag{3}$$

The minimization result $u_{\min}$ represents the denoised image. The minimization process is performed by solving the following associated Euler-Lagrange equation [7,12]:

$$2\lambda(u-u_0)-2\,div\left(\psi'\left(\|\nabla u\|^2\right)\nabla u\right)=0 \tag{4}$$

which leads to the following PDE:

$$\frac{\partial u}{\partial t}=div\left(\psi'\left(\|\nabla u\|^2\right)\nabla u\right)-\lambda(u-u_0) \tag{5}$$

where $\psi'(s^2)=\dfrac{\delta(\beta s^2+\gamma)^{\frac{1}{2}}+\eta\sqrt{K}}{(\beta s^2+\gamma)^{\frac{1}{2}}}$. Therefore, the partial differential equation provided by (5) will take the following form:

$$\begin{cases} \dfrac{\partial u}{\partial t}=div\left(\dfrac{\delta\sqrt{\beta\|\nabla u\|^2+\gamma}+\eta\sqrt{K}}{\sqrt{\beta\|\nabla u\|^2+\gamma}}\cdot\nabla u\right)-\lambda(u-u_0) \\ u(0,x,y)=u_0 \end{cases} \tag{6}$$

One can demonstrate the PDE model given by (6) converges to a unique strong and stationary solution, $u^*=u_{\min}$ [13]. The numerical approximation of this model uses a 4-NN discretization of Laplacian operator. So, from (5) we get:

$$u(x,y,t+1)\cong u(x,y,t)div\left(\psi'\left(\|\nabla u\|^2\right)\nabla u\right)-\lambda(u-u_0) \tag{7}$$

which leads to the following iterative approximation:

$$u^{t+1}\cong u^t+\alpha\sum_{q\in N(p)}\psi'\left(\|\nabla u_{p,q}(t)\|^2\right)\nabla u_{p,q}(t)-\lambda(u-u_0) \tag{8}$$

where $N(p)=\{(x-1,y)(x+1,y)(x,y-1)(x.y+1)\}$, $p=(x,y)$, $\alpha\in(0,1)$, $t=1,\ldots,N$ and $\nabla u_{p,q}(t)=u(q,t)-u(p,t)$.

The iterative model developed by us converges fast to the solution $u^N\cong u_{\min}$, the parameter N taking quite low values. The effectiveness of the proposed denoising method is proved by the performed denoising experiments.

III. VARIATIONAL IMAGE INPAINTING APPROACH

We consider a robust variational PDE technique that reconstructs a degraded image that is observed in a number of points. Our approach uses a variational problem considered in the Sobolev distribution space whose Euler-Lagrange equation represents a nonlinear elliptic diffusion equation. The reconstructed image is determined from the following energy functional minimization:

$$\min\begin{cases} \int_{\Omega} g(u(x,y))dxdy+\frac{1}{2}\|u-u_0\|_{-1}^2; \\ u\in L^1(\Omega), u-u_0\in H^{-1}(\Omega) \end{cases} \tag{9}$$

where g: $R\to R$ represents a convex and lower semi-continuous function, u is the restored image and u_0 is the observed image, characterized by missing zones, on the bounded domain $\Omega\subseteq R^2$. The Euler–Lagrange optimality conditions from (9) are provided by next elliptic boundary value problem:

$$\begin{cases} -\Delta\beta(u)+u=u_0, \text{in } \Omega \\ \beta(u)=0, \text{on } \partial\Omega \end{cases} \tag{10}$$

where β constitutes the subdifferential of g defined as $\beta(r)=\{w\in R; w(r-s)\geq g(r)-g(s)\},\forall s\in R$. It represents a maximal monotone and multivalued function [14]. The minimization problem (9) has a unique solution u*, which satisfies the equation given by (10). The steady-state solution u* to the following evolution equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta\beta(u) + u \ni u_0, \text{in } (0,\infty)\times\Omega \\ u(0,x,y) = u_0(x,y), \quad (x,y)\in\Omega \\ u = 0, \qquad\qquad\qquad \text{on } (0,\infty)\times\partial\Omega \end{cases} \tag{11}$$

represents also the solution to (10) and, respectively to the minimization problem (9). The discrete version of the equation (11) is provided by the following steepest descent algorithm:

$$2u_{k+1} - \Delta\beta(u_{k+1}) = u_0 + u_k, k = 1,\ldots \tag{12}$$

Then, by using $Au = -\Delta\beta(u)$, we obtain the following implicit finite difference scheme:

$$u_{k+1} + hAu_{k+1} = u_k, k = 0,1,\ldots \tag{13}$$

that is next transformed into the explicit finite difference scheme:

$$u_{k+1} = -hA\beta(u^k) + u^k, k = 0,1,\ldots,N \tag{14}$$

where N represents the number of iterations, $t \in [0,T]$ and $Nh = T$. By replacing A with the discretized second order operator $AI \cong \frac{1}{4}\left(u^k_{i-1,j} + u^k_{i+1,j} + u^k_{i,j-1} + u^k_{i,j+1} - 4u^k_{i,j}\right)$, where $u^k_{i,j} = u^k(i,j)$, we get:

$$u^{k+1}_{i,j} \cong u^k_{i,j} + \frac{h}{4}\left(\beta(u^k_{i-1,j}) + \beta(u^k_{i+1,j}) + \beta(u^k_{i,j-1}) + \beta(u^k_{i,j+1}) - 4\beta(u^k_{i,j})\right) \tag{15}$$

If $\beta(r) = r^{\varepsilon+1}$, instead of (15) we consider the following explicit finite difference scheme:

$$u^{k+1}_{i,j} \cong u^k_{i,j} + \alpha\left(u^k_{i,j}\right)^{\varepsilon}\left(u^k_{i-1,j} + u^k_{i+1,j} + u^k_{i,j-1} + u^k_{i,j+1} - 4u^k_{i,j}\right) \tag{16}$$

where $\alpha \le 1$ and $\varepsilon \in (0,1)$, respectively. The image is reconstructed by applying the iterative algorithm given by (16) for $k = 0,1,\ldots,N - 1$. The degraded image $u^0 = u_0$ is transformed into the restored image u^N, that is closed to the original image, in several tens steps. The inpainting algorithm is applied to the image that has been previously smoothed through the described denoising method.

IV. Experiments

The described PDE-based variational approaches have been successfully applied on hundreds of images corrupted with various levels of Gaussian noise, which mean various values for mean and variance, and containing missing regions. The variational denoising algorithm has been applied by using several properly chosen parameters that provide optimal noise reduction results: $\lambda = 0.14, K = 25, \eta = 0.7, \beta = 0.66, \gamma = 0.5, \delta = 0.2, \alpha = 0.3, N = 15$.

According to the method comparison, our technique outperforms many other noise reduction algorithms. Its denoising performance has been assessed by using the *norm of the error image* measure.

TABLE I. Norm-of-the-error values for several models

This algorithm	5.1×10^{3}
Perona-Malik scheme	5.9×10^{3}
Quadratic model	6.1×10^{3}
2D Gaussian	7.2×10^{3}
Median	6×10^{3}
Average	6.3×10^{3}
Wiener	5.7×10^{3}

As one can see in TABLE I, the proposed denoising approach provides the lowest NE values. It achieves much better edge-preserving denoising results than non-PDE filters, such as the average, Gaussian, or median filters [4]. It also produces a better restoration and converges faster than other PDE variational schemes, like the quadratic variational model, characterized by the regularizer $\psi(s^2) = s^2$, or the Perona-Malik variational scheme, given by $\psi(s^2) = \lambda^2\left(\log\left(1 + \frac{s^2}{\lambda^2}\right)\right)$ [12]. The proposed PDE model also outperforms the well-known TV denoising algorithm [7], because it reduces considerably the staircasing effect [15].

The proposed reconstruction technique executes also quite fast, being characterized by a low time complexity. Its approximate running time is 1 second. The optimal inpainting results are obtained for a number of iterations $N = 40$. Method comparisons have also been performed. The performance of this restoration technique has been compared with performances of some other inpainting methods, such as TV inpainting [11] and those based on Gaussian processes, such as GPR models [16]. We have found that our restoration approach runs faster that many other algorithms, while it produces comparable good results.

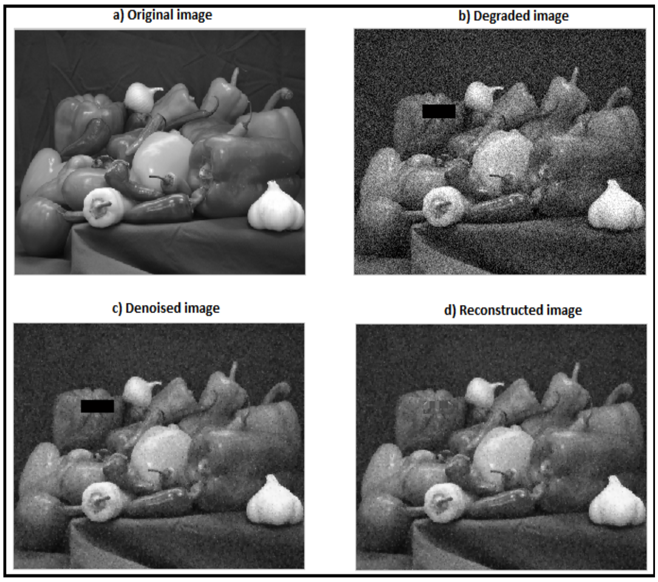

Fig. 1. Image restoration example

An image restoration example is displayed in Fig. 1. In (a) one can see the original *Peppers* image, while the degraded image is depicted in (b). The denoised image is represented in (c) and the reconstruction result is provided in (d). Our image noise removal and reconstruction experiments have been performed by using MATLAB.

V. Conclusions

We have proposed an effective image enhancement approach based on two robust variational PDE models, in this article. We have brought important contributions in both image denoising and inpainting domains.

The denoising method minimizes an energy functional that is based on a novel regularizer function. The discretization of the PDE model representing its Euler-Lagrange equation is another contribution of this paper. Our smoothing method outperforms other denoising schemes, has an edge preserving character and removes the staircasing effect.

The proposed variational reconstruction model is based on a functional energy minimization whose corresponding Euler-Lagrange equation constitutes a nonlinear elliptic diffusion equation. It works properly on images still affected by missing zones, which have already been denoised, as resulting from our successful restoration experiments.

While the described techniques use second-order diffusion equations, our future research in the PDE-based image restoration field will focus on developing some improved enhancement models based on fourth-order partial differential equations.

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