# MA 109 Recap Week 2

## Recap Slides

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# Week 2

## Limit of a function

A function $f : (a, b) \to \mathbb{R}$ is said to converge to a limit $L$ at a point $x_o \in [a, b]$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - L| < \epsilon$$

for all $x \in (a, b)$ such that $0 < |x - x_o| < \delta$
We write this as $\lim_{x \to x_o} f(x) = L$

What this definition says is that the distance between $f(x)$ and $L$ can be made as small as we want by making the distance between $x$ and $x_o$ sufficiently small.

Limit of a function can exist at a point on which function is not defined.

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## Formulae

If $\lim_{x \to x_o} f(x) = L_1$ and $\lim_{x \to x_o} g(x) = L_2$ then,

1. $\lim_{x \to x_o} f(x) \pm g(x) = L_1 \pm L_2$
2. $\lim_{x \to x_o} f(x)g(x) = L_1 L_2$
3. $\lim_{x \to x_o} f(x)/g(x) = L_1/L_2$ if $L_2 \neq 0$

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## Sandwich Theorem for functions

If $\lim_{x \to x_o} f(x) = L_1$, $\lim_{x \to x_o} g(x) = L_2$ and $\lim_{x \to x_o} h(x) = L_3$ for functions $f$, $g$ and $h$ on some interval $(a, b)$ such that $f(x) \le g(x) \le h(x)$ for all $x \in (a, b)$, then

$$L_1 \le L_2 \le L_3$$

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If $\lim_{x \to x_o} f(x) = \lim_{x \to x_o} h(x) = L$ and if $g(x)$ is a function satisfying $f(x) \le g(x) \le h(x)$ for all $x \in (a, b)$, then $g(x)$ tends to a limit as $x \to x_o$ and

$$\lim_{x \to x_o} g(x) = L$$

We haven't assumed convergence of $g(x)$ here. We get convergence of $g(x)$ for free.

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## Limits at Infinity

We say that $f : \mathbb{R} \to \mathbb{R}$ tends to a limit $L$ as $x \to \infty$ (resp. $x \to -\infty$) if for all $\epsilon > 0$ there exists $x_o \in \mathbb{R}$ such that

$$|f(x) - L| < \epsilon$$

whenever $x > x_o$ (resp. $x < x_o$) and we denote it as

$$\lim_{x\to\infty} f(x) = L \text{ or } \lim_{x\to-\infty} f(x) = L$$

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## Limits from the left

If $f : (a, b) \to \mathbb{R}$ is a function and $c \in (a, b)$, then we can define the limit of the function $f(x)$ as x approaches c from the left (if it exists) as a number $L^-$ such that for all $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L^-| < \epsilon$ whenever $|x - c| < \delta$ and $x \in (a, c)$.

## Limits from the right

If $f : (a, b) \to \mathbb{R}$ is a function and $c \in (a, b)$, then we can define the limit of the function $f(x)$ as x approaches c from the right (if it exists) as a number $L^+$ such that for all $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L^-| < \epsilon$ whenever $|x - c| < \delta$ and $x \in (c, b)$.

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## Continuity

If $f : [a, b] \to \mathbb{R}$ is a function and $c \in [a, b]$, then f is said to be continuous at the point $c$ if and only if

$$\lim_{x \to c} f(x) = f(c)$$

## Continuous Function

A function f on (a,b) is said to be continuous if and only if it is continuous at every point c in (a,b).

Note - You can change brackets from ( to [ to include a, b or both

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## Theorem

Let $f : (a, b) \to (c, d)$ and $g : (c, d) \to (e, f)$ be functions such that f is continuous at $x_0$ in (a; b) and g is continuous at $f(x_0) = y_0$ in $(c, d)$. Then the function $g(f(x))$ (also written as $g \circ f(x)$ sometimes) is continuous at $x_0$. So the composition of continuous functions is continuous.

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## Theorem - Intermediate Value Theorem

Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function. For every $u$ between $f(a)$ and $f(b)$ there exists $c \in [a, b]$ such that $f(c) = u$.

## Theorem

Every polynomial of odd degree has at least one real root. (Consequence of IVT)

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## Theorem

A continuous function on a closed bounded interval $[a, b]$ is bounded and attains its infimum and supremum, that is, there are points $x_1$ and $x_2$ in $[a, b]$ such that $f(x_1) = m$ and $f(x_2) = M$, where $m$ and $M$ denote the infimum and supremum respectively.

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## Sequential Continuity

A function f (x) is continuous at a point a if and only if for every sequence $x_n \to a$

$$\lim_{x_n \to a} f(x_n) = f(a)$$

Do mention limit of various functions briefly

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## Differentiability

A function $f : (a, b) \to \mathbb{R}$ is said to be differentiable at a point $c \in (a, b)$ if

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists. In this case value of the limit is denoted by $f'(c)$ and is called the derivative of $f$ at $c$.

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## Slope of the tangent

The derivative $f'(c)$ gives us the slope of the curve, that is, the slope of the tangent to the curve $y = f(x)$ at $(c, f(c))$. This becomes clear if we rewrite the derivative as

$$\lim_{y \to c} \frac{f(y) - f(c)}{y - c}$$