MA 109 Recap Week 1

Recap Slides

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Week 1

Sequences

A sequence in a set X is a function from the natural numbers to X, i.e., a function $f : \mathbb{N} \to X$.

Monotonic Sequences

A sequence is said to be monotonically increasing sequence if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$

A sequence is said to be monotonically decreasing sequence if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$

Week 1

Limit of a sequence

A sequence a_n tends to a limit L, if for any $\epsilon > 0$, there exists $n_o \in \mathbb{N}$ such that

$$|a_n - L| < \epsilon \text{ for all } n > n_o$$

we represent this by,

$$\lim_{n\to\infty} a_n = L$$

or concurrently we say that $\{a_n\}_{n=1}^{\infty}$ converges to a limit L

Finding the limit L with this definition is not easy. First, guess the limit L and then prove that it satisfies the definition.

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Basic formulae

If a_n and b_n are two convergent sequences, then

1. $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm, \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
3. $\lim_{n\to\infty}(a_n / b_n) = \lim_{n\to\infty} a_n / \lim_{n\to\infty} b_n$ provided $\lim_{n\to\infty} b_n \neq 0$

Note that the constant sequence $a_n = c$ has limit c, so as a special case of (2) above, we have

$$\lim_{n\to\infty}(c \cdot a_n) = c \cdot \lim_{n\to\infty} a_n$$

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Sandwich Theorem

If a_n, b_n and c_n are convergent sequences such that $a_n \leq b_n \leq c_n$ for all n, then

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n \leq \lim_{n\to\infty} c_n$$

If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$ and b_n is a sequence satisfying $a_n \leq b_n \leq c_n$ for all n, then b_n converges and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n$$

Note that we haven't assumed that b_n converges in the second part. The theorem obtains the convergence of b_n.

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Bounded Sequences

A sequence a_n is said to be bounded if there is a real number $M > 0$ such that $|a_n| \leq M$ for every $n \in \mathbb{N}$

A sequence that is not bounded is called unbounded

Lemma

Every convergent sequence is bounded.

The above lemma is used to prove the product rule of limits (given in basic formulae on page 4

Week 1

Theorem

Let $\{x_n\}$ be a sequence of real numbers such that $x_n > 0$ for all n. Let $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lambda$. Then

1. If $\lambda < 1$, then $\{x_n\}$ is convergent and converges to 0.
2. If $\lambda > 1$, then $\{x_n\}$ is divergent.

Note - The result is inconclusive if $\lambda = 1$. Consider the following two sequences $x_n = n$ and $x_n = \frac{1}{n}$; one is divergent, whereas the other is convergent. But $\lambda = 1$ in both cases.

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Definition

A sequence a_n is said to be bounded above (resp. bounded below) if $a_n < M$ (resp. $a_n > M$) for some $M \in \mathbb{R}$.

A bounded sequence is bounded both above and below.

Theorem

A monotonically increasing (resp. decreasing) sequence bounded above (resp. below) converges.

Remark - If we change finitely many terms of a sequence, it does not affect the convergence and boundedness properties of a sequence.

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Supremum or Least Upper Bound (LUB)

The limit of a monotonically increasing sequence a_n bounded above.
Properties -

1. $a_n \leq M \ \forall \ n$
2. If M_1 is such that $a_n < M_1 \ \forall \ n$, then $M \leq M_1$.

Infinum or Greatest Lower Bound (GLB)

The limit of a monotonically decreasing sequence a_n bounded below.
Properties -

1. $a_n \geq M \ \forall \ n$
2. If M_1 is such that $a_n > M_1 \ \forall \ n$, then $M \geq M_1$.

Week 1

Cauchy Sequence

A sequence a_n in $\mathbb{R}$ is said to be a Cauchy Sequence if for every $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that

$$|a_n - a_m| < \epsilon \; \forall \; m, n > N.$$

Theorem

Every Cauchy sequence in $\mathbb{R}$ converges.

Theorem

Every convergent sequence is Cauchy.