MA 109 Recap Week 3

Recap Slides

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Week 3

Global Maxima and Minima

The function f is said to attain a global maximum (resp. global minimum) at a point $x_0 \in X$ if $f(x) \leq f(x_0)$ (resp. $f(x) \geq f(x_0)$) for all $x \in X$

Local Maxima and Minima

Let $f : X \to \mathbb{R}$ be a function and x_0 be in X. Suppose there is a sub-interval $x_0 \in (c, d) \subset X$ such that $f(x_0) \geq f(x)$ (resp. $f(x_0) \leq f(x)$) for all $x \in (c, d)$, then f is said to have a local maximum (resp. local minimum) at x_0

Note - In a closed bounded interval, a continuous function always attains a maximum and minimum.

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Rolle's Theorem

Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function which is differentiable in (a, b) and $f(a) = f(b)$. Then there is a point x_0 in (a, b) such that $f'(x_0) = 0$.

Mean Value Theorem

Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function and that f is differentiable in (a, b). Then there is a point x_0 in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(x_0)$$

Fermat's Theorem

If $f : [a, b] \to \mathbb{R}$ is differentiable and has a local minimum or maximum at a point $x_0 \in (a, b)$, then $f'(x_0) = 0$.

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Darboux Theorem

Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. If c, d $(c < d)$ are points in (a, b), then for every u between $f'(c)$ and $f'(d)$, there exists an x in $[c, d]$ such that $f'(x) = u$.

Stationary Points

A point x_0 in (a, b) such that $f'(x_0) = 0$ is called a stationary point

Second Derivative Test

If $f'(x_0) = 0 and : -$

1. If $f''(x_0) > 0$, the function has a local minimum at x_0
2. If $f''(x_0) < 0$, the function has a local maximum at x_0
3. If $f''(x_0) = 0$, nothing can be said. Check further.

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Concavity and Convexity

A function $f : I \to \mathbb{R}$ (for an interval I) is said to be concave (or concave downwards) if

$$f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2)$$

Replace the $\geq$ sign above with

- $\leq$ for convex or concave upwards
- $>$ for strictly concave
- $<$ for strictly convex

Theorem

A differentiable function is convex (resp. concave) if and only if its derivative is monotonically increasing (resp. decreasing)

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Theorem

A twice differentiable function on an interval will be convex if its second derivative is everywhere non-negative. If the second derivative is positive, the function will be strictly convex

Converse is not true

Point of Inflection

A point of inflection x_0 for a function f is a point where the function changes its behavior from concave to convex or vice-versa. At such a point $f''(x_0) = 0$

But this is only a necessary, not a sufficient condition.
Sufficient is lowest order non-zero derivative is odd.

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Some Notations

- The space $C^k(I)$, will denote the space of k times continuously differentiable functions on an interval I, for some fixed $k \in \mathbb{N}$, that is, the space of functions for which k derivatives exist and such that the k^{th} derivative is a continuous function
- The space $C^\infty(I)$ will consist of functions that lie in $C^k(I)$ for every $k \in N$. Such functions are called smooth or infinitely differentiable functions
- The k^{th} derivative of a function $f(x)$ is denoted by $f^{(k)}(x)$

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Taylor Polynomials

Given a function $f(x)$ which is n times differentiable at x_0
We define Taylor polynomials of degree k $(0 \leq k \leq n,\ k \in \mathbb{N})$ at x_0

$$P_k(x_0) = \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!}(x - x_0)^k$$

Remainder

Let $f \in C^n([a, b])$ and suppose that $f^{(n+1)}$ exists on (a, b). Then there exists $c \in (a, b)$ such that

$$f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(b - a)^{(n+1)}$$

and $R_n(x) = f(x) - P_n(b)$

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Taylor Series

Just make $n = \infty$ in Taylor Polynomials.
This means Taylor's Polynomials are just partial sums of Taylor Series

Power Series

If the Taylor Series converges in a given range, we call it Power Series
E.g. e^x and $\frac{1}{1-x}$

Approximation

We can approximate such functions upto n^{th} degree Taylor Polynomials and the error can be given by $R_n(x)$