MA 109 Recap Week 6

Recap Slides

Rohan Nafde
Hrishikesh Jedhe Deshmukh

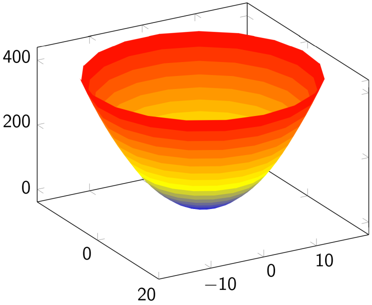
`https://github.com/RohanNafde`

Week 6

Level Curves and Contour lines

For a function $f : \mathbb{R}^2 \to \mathbb{R}$ level sets are the sets of the form $f(x, y) = c$, where c is a constant. Level sets are present in xy-plane.

When we plot $f(x, y) = c$ for some constant c, we get a curve. Such a curve is called a contour line.



Week 6

Limits

A function $f : U \to \mathbb{R}$ is said to tend to a limit L as $x = (x_1, x_2, ...x_n)$ approaches $c = (c_1, c_2, ...c_n)$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon$$

whenever $0 < ||x - c|| < \delta$

Note: $||x|| = \sqrt{x_1^2 + x_2^2 + ...x_n^2}$

For a function in $\mathbb{R}^2$ it is possible to approach a point from infinitely many directions. So the limit must exist when we approach the point from all these directions.

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Continuity

The function $f : U \to \mathbb{R}$ is said to be continuous at a point c if

$$\lim_{x \to c} f(x) = f(c)$$

Iterated limits

Let $\lim_{(x,y)\to(a,b)} f(x,y) = L$. Suppose that,

$$\lim_{x \to a} f(x,y) = g(y) \text{ and } \lim_{y \to b} f(x,y) = h(x)$$

Then, $\lim_{y\to b} g(y) = \lim_{x \to a} h(x) = L$

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Partial derivative

The partial derivative of $f : U \to \mathbb{R}$ with respect to x_1 at the point (a, b) is defined by,

$$\frac{\partial f}{\partial x_1}(a, b) := \lim_{x_1 \to a} \frac{f((x_1, b)) - f((a, b))}{x_1 - a}$$

Directional derivative

The directional derivative of f in the direction v at a point $x = (x_1, x_2)$ is defined as

$$\Delta_v = \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}$$

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Differentiability for functions of two variables

A function $f : U \to \mathbb{R}$ is said to be differentiable at a point (x_0, y_0) if $\frac{\partial f}{\partial x}(x_0, y_0)$ and $\frac{\partial f}{\partial y}(x_0, y_0)$ exist and

$$\lim_{(h,k)\to 0} \frac{f(x_0 + h, y_0 + k) - f(x_0, y_0) - \frac{\partial f}{\partial x}(x_0, y_0)h - \frac{\partial f}{\partial y}(x_0, y_0)k}{||(h, k)||} = 0$$

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Gradient

$$\nabla f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)\hat{i} + \frac{\partial f}{\partial y}(x_0, y_0)\hat{j}$$

Gradient and directional derivative

$$\nabla_v f = \nabla f \cdot v$$

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Chain rule

Let $x, y : I \to \mathbb{R}$ are differentiable functions. Now for function $z(t) = f(x(t), y(t))$ from I to $\mathbb{R}$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

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Functions from $\mathbb{R}^m$ to $\mathbb{R}^n$

Let U be a subset of $\mathbb{R}^{>}$ and let $f : U \to \mathbb{R}^n$ be a function. If $x \in U$, $f(x)$ will be a n-tuple where each coordinate is a function of x.

$$f(x) = (f_1(x), f_2(x), \ldots.f_n(x))$$

When $m = n$, vector valued functions are called vector fields.

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The derivative for $f : U \to \mathbb{R}^n$

The function f is said to be differentiable at a point x if there exists a $n \times m$ matrix $Df(x)$ such that

$$\lim_{h \to 0} \frac{||f(x+h) - f(x) - Df(x).h||}{||h||} = 0$$

$$Df(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_m} \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots & \frac{\partial f_n}{\partial x_m} \end{pmatrix}$$

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Local maxima and minima

We say that a function $f(x, y)$ attains local maxima at a point (x_0, y_0) if there is a disc

$$D_r(x_0, y_0) = \{(x, y) \mid ||(x, y) - (x_0, y_0)|| < r\}$$

of radius $r > 0$ around (x_0, y_0) such that $f(x, y) \leq f(x_0, y_0)$ for every point (x, y) in $D_r(x_0, y_0)$

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Critical Points

A point (x_0, y_0) is called a critical point of $f(x, y)$ if

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0$$

This means the tangent plane at point (x_0, y_0) is parallel to xy-plane

First derivative test

If (x_0, y_0) is a point of local extremum and if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist, then (x_0, y_0) is a critical point.

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Second derivative test

Define Hessian of f as

$$\begin{pmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{yx}(x_0, y_0) & f_{yy}(x_0, y_0) \end{pmatrix}$$

Let D be the determinant of Hessian.

$$D = f_{xx} f_{yy} - f_{xy} f_{yx}$$

Second derivative test contd.

- If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then (x_0, y_0) is a local minimum
- If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then (x_0, y_0) is a local maximum
- If $D < 0$, then (x_0, y_0) is a saddle point of f
- If $D = 0$ then further examination is required

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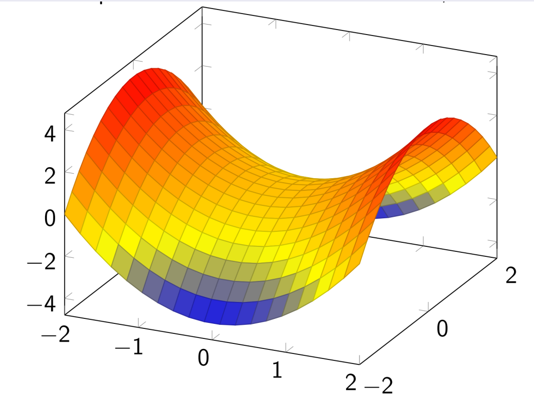
What's a saddle point?



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Taylor's theorem in two variables

If f is a C^2 function in a disc around (x_0, y_0), then

$$f(x_0+h, y_0+k) = f(x_0, y_0)+f_x h+f_y k+\frac{1}{2!}[f_{xx}h^2+2f_{xy}hk+f_{yy}k^2]+\widetilde{R}_2(h,k)$$

where $\widetilde{R}_2(h,k)/||(h,k)|| \to 0$ as $||(h,k)|| \to 0$