Report on Mine Depth Investigation:

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I: Introduction

The purpose of this report is to summarize various investigations into the trajectory of a 1 kg test mass while falling in a mineshaft of varying depths. We determine if the depth of the mineshaft can be measured by dropping a test mass and measuring the time taken to hit the bottom. We also consider how both the density and density profile of the Earth (and Moon) affect the time taken to fall down mineshafts that span the Earth.

Before going into the details of our findings, we must first review the essential elements of the physical models behind the simulations. In order to accurately simulate a falling object, we need to determine the main forces acting on the object as it falls. A falling object is subject to three major forces: **gravity, air drag,** and the **Coriolis force.**

The force of gravity on our test mass, as a function of depth (r) is given by:

$$F_g(r) = \frac{-Gm_{test}M_{earth,enclosed}(r)}{r^2}$$

where m_{test} is the test mass (1 kg), $M_{earth,enclosed}(r)$ is the total mass of the Earth **below** r, and G is the gravitational constant. The force is negative, since objects are attracted to the Earth.

Air drag can be written as: $F_{drag} = \alpha|\dot{r}|^{\gamma}$

where α is the drag coefficient, $\dot{r}$ is the speed of the object, and γ controls the relevant power of speed in the force (higher gamma suppresses small speeds). For our simulations, we chose $\gamma = 2$, which is appropriate for high speeds. One can determine $\alpha = -0.003924$ by setting the net force on the mass to be zero at terminal velocity (the largest mass speed); for this investigation the terminal velocity was 50 m/s.

Finally, the Coriolis force can be written as a vector equation: $\vec{F}_{Coriolis} = -2m(\vec{\Omega} \times \vec{v})$

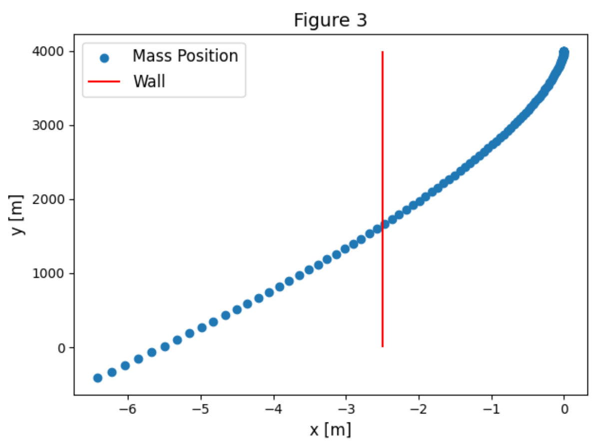
where $\vec{\Omega}$ is the rotation rate (and direction) of the Earth, $\vec{v}$ is the velocity vector of the test mass, and m is the mass of the test mass. The Coriolis force arises from a difference in perspective between a rotating observer and a nonrotating mass; the difference causes the observer to see a deflection in the mass trajectory, which can be treated as a force.

These forces can be combined, using $F_{net} = ma$, into a differential equation which can be solved numerically to determine the mass trajectory.

Finally, it is important to mention that while the density of the Earth is typically treated as constant, in practice it is not. In our investigation, we modeled the density of the Earth as: $\rho(r) = \rho_n\left(1 - \frac{r^2}{r_{earth}^2}\right)^n$, where n is a parameter we chose, ρ_n normalizes the density to keep the total Earth mass constant, and r_{earth} is the radius of the Earth. The enclosed mass, as a function of depth, was then calculated by integrating the density out to the depth.

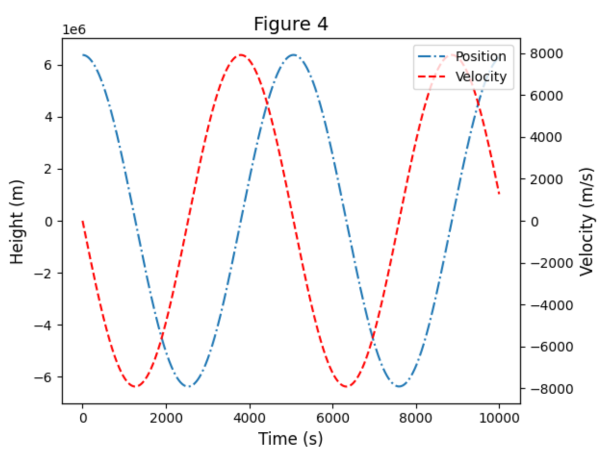
II: Calculation of Fall Time

To determine the fall times, we solved the differential equation using solve_ivp in scipy, and noted the time when the mass hit the bottom of the mine. We examined three scenarios; one with a constant gravitational acceleration (g), one where g varied as a function of depth, and the last where both drag and a depth-dependent g were considered. The investigation found that the constant g fall time was 28.6 s, the depth-dependent g fall time was also 28.6 s, and the air drag fall time was 83.5 s, which differed from the previous two fall times by 55.0 s. From this, it is evident that including depth-dependence in g is negligible, but the effect of air drag increased the falling time by almost a factor of 3, which is significant. Notably, the first fall time can be calculated using kinematic formulae, giving a fall time of 28.6 s, in agreement with the first two measurements.



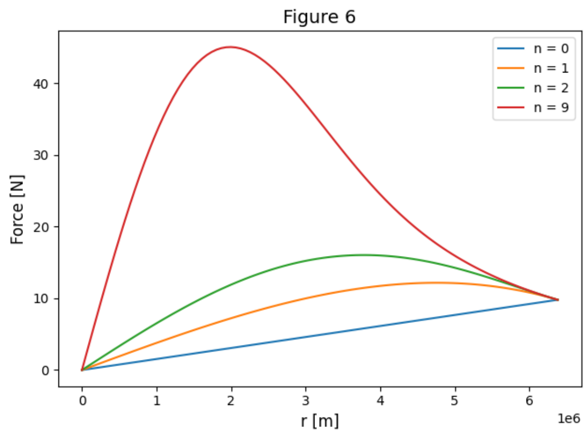
III: Feasibility of Depth Measurement Approach

In this analysis, we included the Coriolis force to the differential equation, allowing us to model the deflection of the mass in the shaft. **Figure 3** shows the position of the mass in the well at various points in time, without air drag. The mass hits the side of the well 1646.1 m above the bottom of the mineshaft, 21.9 s after being released. From this, we would not recommend proceeding with the proposed depth measurement technique, since the mass colliding with the side of the wall would affect the accuracy of the method. With air drag, the mass still collides with the wall 2697.5 m from the bottom of the well, at a time of 29.6 s. So, including air drag does not change the fact that the test mass will collide with the wall; rather, it makes the collision happen higher up.



IV: Calculation of Crossing Times

We now consider the case of a mineshaft running through the entire Earth (through the center), ignoring the Coriolis force as well as air drag. We first consider a uniform density for Earth. The trajectory and velocity of the mass is shown in **Figure 4**; notably it is periodic and sinusoidal. The test mass takes 1266.5 s to reach the center of the Earth, and has a velocity of 7910.8 m/s at this point. The ratio between this crossing time and the orbital period of a mass rotating around the Earth at a distance of the Earth's radius was determined; the result was 4.0. Thus, the crossing time is 4 times the orbital period, which makes sense since a mass would need to cover a distance of the Earth's radius 4 times to return back to the same point in the mineshaft (with the same velocity).



So the orbital period and the mineshaft "period" are exactly the same. We also considered cases where the density of the Earth was non-uniform, using the formula addressed in the introduction. **Figure 6** shows the gravitational force as a function of radius for differing n values in the density formula. While the forces are the same at the surface of the Earth (9.8 N), the n = 0 case increases linearly from 0 to this value, while the n = 9 case increases sharply, peaks at above 40 N, and then decreases until hitting 9.8 N. The other cases are "in-between" these two extremes. This density profile difference lends itself to differences in the crossing times: for n = 0, 1, 2, and 9, the crossing times were 1267.2 s, 1096.9 s, 1035.1 s, and 943.9 s, respectively. The n = 0 and n = 9 case differ by 323.3 s, which is a factor of 0.74 decrease in crossing time. We also exactly modeled the effect of density on crossing time, assuming uniform density. To do so, we first determined the crossing time of the moon using the methods detailed above, and found it to be 1624.9 s. Then, we conducted a pen-and-paper calculation to determine the relationship between density and crossing time to be: $T = \frac{1}{4}\rho^{-\frac{1}{2}}\sqrt{\frac{3\pi}{G}}$, where ρ is density and T is crossing time. Using this formula, we calculated the moon crossing time to be 1625.1 s, a difference of 0.2 s from the earlier method. We also applied this formula to the Earth, determining a crossing time of 1267.3 s, a difference of 0.1 s from the time of 1267.2 s determined earlier. So, the formula matches the earlier results.

V: Discussion and Future Work

In doing these calculations, we made several approximations; further work can improve these approximations to give more accurate results. For one, we only included $\gamma = 2$ in drag; at low speeds, this would mean that drag is suppressed, leading to a lower falling time than expected. Future work can include $\gamma = 1$ to better capture the overall drag force. Also, we assumed a particular form for the density profile of Earth in our analysis; this profile is not likely to capture all features of the real density profile. Future work could incorporate the actual density profile of Earth as a function of depth to determine the crossing time more accurately. Finally, our analysis assumed a spherical Earth; in reality, the Earth is slightly ellipsoidal (egg-shaped). This implies that the distribution of mass in the Earth is not spherically symmetric; a more careful analysis would treat the density as a function of depth, azimuth, and zenith, which would allow a better calculation of the crossing time.

To summarize, our investigations have concluded that the proposed method of determining the mineshaft depth **will not work**; the mass will contact the side of the well before hitting the bottom. We have also found that air drag dramatically increases the time the mass takes to hit the bottom. Our investigations into Earth's density found that, for large n, the crossing time is reduced, and we found an exact relation between density and crossing time for objects of uniform density.