MAT137Y Tutorial 19 worksheet

Markus Johnson, Max Chen, Rohan Regi, Rishit Dagli

TOTAL POINTS

2 / 2

QUESTION 1

1 Q1+Q2 **2 / 2**

✓ **- 0 pts** *Complete and correct*

- 1 pts Some errors or incomplete

- 2 pts Completely incorrect, empty, or missing

signature

- 2 pts Fake Signature

MAT 137
Tutorial #19– Improper Integrals
Mar 7/Mar 8, 2023
Due on Thursday, Mar 9 by 11:59pm via GradeScope

- **Submissions are only accepted by Gradescope**. Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
- **Submit your polished solutions using only this template PDF.** You will submit a single PDF with your full written solutions. If your solution is not written using this template PDF (scanned print or digital) then you will receive zero. Do not submit rough work. Organize your work neatly in the space provided.
- **Show your work and justify your steps** on every question, unless otherwise indicated. Put your final answer in the box provided, if necessary.

We recommend you write draft solutions on separate pages and afterwards write your polished solutions here. You must fill out and sign the academic integrity statement below; otherwise, you will receive zero.

Academic integrity statement

I confirm that:

- I have read and followed the policies described in the Policies and FAQ.
- I understand that the collaboration of this worksheet is only among this group. I have not violated this rule while writing this worksheet.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while writing this assessment.

By signing this document, I agree that the statements above are true.

First Name	Last Name	UofT email	signature
Rishit	Dagli	rishit.dagli@mail.utronto.ca	Rishit Dagli
Max	Chen	maximum.chen@mail.utoronto.ca	[illegible]
Rohan	Regi	rohan.regi@mail.utoronto.ca	[illegible]
markus	johnson	markus.johnson@mail.utoronto.v	[illegible]

TA name:________________ TA signature: Elijah French

1. Determine whether the following integrals are convergent or divergent directly from the definition of improper integral. If they are convergent, calculate their value.

   (a) $\displaystyle\int_0^1 \ln x \, dx$

   We first use Integration by Parts to compute $\int \ln x \, dx$, by letting $u = \ln x$ and $dv = dx$ then we have, $du = (1/x)dx$,$v = x$,:

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx = x \ln x - \int 1 \, dx = x \ln x - x + C$$

   Now by definition of improper integrals, we have:

$$\int_0^1 \ln x \, dx = \lim_{R \to 0^+} \int_R^1 \ln x \, dx = \lim_{R \to 0^+} (x \ln x - x)\Big|_{x=R}^{x=1} = -1 - \lim_{R \to 0^+} (R \ln R) + \lim_{R \to 0^+} R$$

   We know $\lim_{R \to 0^+} R$ thus we now want to compute $\lim_{R \to 0^+}(R \ln R)$:

$$\lim_{x \to 0^+} (R \ln R) = \lim_{R \to 0^+} \frac{\ln R}{1/R}$$

   Since this is of the indeterminate form ∞/∞, the numerator, and denominator are differentiable and the limit of the quotient of the derivatives exists, so the preconditions for L'Hôpital's rule are satisfied:

$$\lim_{R \to 0^+} (R \ln R) = \lim_{R \to 0^+} \frac{\ln R}{1/R} = \lim_{R \to 0^+} \frac{1/R}{-1/R^2} = \lim_{R \to 0^+} -\frac{R^2}{R} = \lim_{R \to 0^+} (-R) = 0$$

   The limit exists and is a finite number, thus, the integral is convergent, and $\int_0^1 \ln x \, dx = -1$

   (b) $\displaystyle\int_2^\infty \frac{1}{x^2 - 1} \, dx$

   First, we will solve the indefinite integral $\int \frac{1}{x^2-1}$ using Partial Fractions:

$$\int \frac{1}{x^2 - 1} = \int \frac{1}{(x-1)(x+1)} = \frac{1}{2} \int \frac{1}{x-1} - \frac{1}{2} \int \frac{1}{x+1} = \frac{\ln(|x-1|)}{2} - \frac{\ln(|x+1|)}{2}$$

   By the definition of the improper integral, we can now write this as :

$$\int_2^\infty \frac{1}{x^2 - 1} \, dx = \lim_{R \to \infty} \int_2^R \frac{1}{x^2 - 1} \, dx = \lim_{R \to \infty} \frac{\ln(|x-1|)}{2} - \frac{\ln(|x+1|)}{2}\Big|_{x=2}^{x=R}$$
$$= \lim_{R \to \infty} \ln\left(\frac{|x-1|}{|x+1|}\right) = \lim_{R \to \infty} \ln(1) - \ln\left(\frac{2-1}{2+1}\right) = \ln(3)$$

   Thus, we have that $\displaystyle\int_2^\infty \frac{1}{x^2 - 1} \, dx$ is convergent and $\displaystyle\int_2^\infty \frac{1}{x^2 - 1} \, dx = \ln(3)$

Hint: For Question 1b, write $\dfrac{1}{x^2 - 1} = \dfrac{A}{x-1} + \dfrac{B}{x+1}$.

2. Using the Basic Comparison Test and/or the Limit Comparison Test, determine which ones of the following improper integrals are convergent or divergent. Do not calculate their value.

(a) $\displaystyle\int_{10}^{\infty} \frac{\sqrt{x-6}}{3x^2+5x+11}\,dx$

Th range of the integral is from 10 to ∞ and both the numerator and denominator are positive so all the terms in this integral are strictly positive.

We know that the $\lim_{n\to\infty} \frac{\sqrt{x}}{x^2}$ is convergent by the p-series test.

Now we calculate:

$$\begin{aligned}
\lim_{n\to\infty} \frac{x^2}{\sqrt{x}} \frac{\sqrt{x-6}}{3x^2+5x+11} &= \lim_{n\to\infty} \frac{x^2}{\sqrt{x}} \frac{\sqrt{x\left(1-\frac{6}{x}\right)}}{x^2\left(3+\frac{5}{x}+\frac{11}{x^2}\right)} \\
&= \lim_{n\to\infty} \frac{\sqrt{1-\frac{6}{x}}}{3+\frac{5}{x}+\frac{11}{x^2}} \\
&= \frac{\lim_{n\to\infty} \sqrt{1-\frac{6}{x}}}{\lim_{n\to\infty} 3+\frac{5}{x}+\frac{11}{x^2}} \qquad \text{By Limit laws} \\
&= \frac{1}{3}
\end{aligned}$$

Since $\frac{1}{3} > 0$ and finite then by the limit comaprision test and the fact that $\lim_{n\to\infty} \frac{\sqrt{x}}{x^2}$ is convergent, we have that $\displaystyle\int_{10}^{\infty} \frac{\sqrt{x-6}}{3x^2+5x+11}\,dx$ is convergent.

(b) $\displaystyle\int_{2}^{\infty} \frac{(\ln x)^{10}}{x^2}\,dx$ Hint: You may need to use the Big Theorem.

We first calculate:

$$\lim_{n\to\infty} \frac{\frac{(\ln x)^{10}}{x^2}}{\frac{1}{x^{\frac{3}{2}}}} = 0$$

Since we already know that $\lim_{n\to\infty} \frac{1}{x^{\frac{3}{2}}}$ converges thus we can say that $\lim_{n\to\infty} \frac{(\ln x)^{10}}{x^2}$ also converges.

Thus we have shown that $\displaystyle\int_{2}^{\infty} \frac{(\ln x)^{10}}{x^2}\,dx$ is convergent by the limit comparision test, p-test and Big Theorem.

Additional practice:

3. Determine whether the following integrals are convergent or divergent directly from the definition of improper integral. If they are convergent, calculate their value.

   (a) $\displaystyle\int_{-1}^{\infty} \frac{1}{x^2+1}\,dx$

   (b) $\displaystyle\int_{2}^{\infty} \frac{1}{x+2}\,dx$

   (c) $\displaystyle\int_{1}^{\infty} \ln x\,dx$

   (d) $\displaystyle\int_{0}^{\infty} \cos x\,dx$

   (e) $\displaystyle\int_{0}^{1} \frac{dx}{x^2}$

   (f) $\displaystyle\int_{0}^{1} \frac{dx}{\sqrt{x}}$

4. Using the Basic Comparison Test and/or the Limit Comparison Test, determine which ones of the following improper integrals are convergent or divergent. Do not calculate their value.

   (a) $\displaystyle\int_{1}^{\infty} \frac{\sin x + 2\cos x + 10}{x^2}\,dx$

   (b) $\displaystyle\int_{0}^{\infty} \frac{x-7}{x^2+x+5}\,dx$

   (c) $\displaystyle\int_{0}^{\infty} \frac{\arctan x}{x^{1.1}}\,dx$

   (d) $\displaystyle\int_{0}^{\infty} e^{-x^2}\,dx$