A Mathematical model to study the dynamic interplay among Brainwave Synchronization, Deep Learning Model Accuracy and Cognitive Load

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2025

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ABSTRACT

This thesis presents a mathematical model exploring the dynamic interplay among Brainwave Synchronization, Deep Learning Model Accuracy, and Cognitive Load. Brainwave Synchronization quantifies neural coherence and cognitive efficiency, Deep Learning Accuracy represents the capability of artificial intelligence in accurately classifying cognitive states, and Cognitive Load captures the mental effort exerted during learning tasks. A system of nonlinear differential equations is developed to analyze the interactions and feedback mechanisms among these three components. The research conducts rigorous qualitative analysis to determine equilibrium states, their existence, stability conditions, and bifurcation scenarios. The results underscore significant implications for optimizing cognitive efficiency and enhancing the effectiveness of AI-driven cognitive assessments and educational technologies.

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Rohan Sinha

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Chapter 1

Introduction

1.1 Background and Motivation

As artificial intelligence (AI) systems evolve toward greater personalization and cognitive adaptability, integrating human neuro-physiological signals has emerged as a transformative frontier. Among these, electroencephalogram (EEG) signals, a non-invasive measurements of brain electrical activity, offer a powerful gateway to understanding and responding to human mental states in real time. These signals allow AI models to perceive crucial cognitive factors, such as attention, engagement, and mental workload, which are indispensable for effective human-AI interaction.

Two fundamental concepts extracted from EEG signals are brainwave synchronization and cognitive load. Brainwave synchronization refers to the degree to which neural oscillations across different regions of the brain align in time. Higher synchronization often reflects states of concentration, deep learning, or meditative focus, while de-synchronization is associated with distraction or cognitive fatigue. Cognitive load, on the other hand, quantifies the mental effort a user must exert to process information or complete a task. Excessive cognitive load can impair learning and decision-making, while optimal cognitive load supports performance and retention.

This research introduces a novel nonlinear differential equation model that explores the dynamic interplay between brainwave synchronization, cognitive load, and the accuracy of a deep learning model. In the proposed framework, brainwave synchronization and cognitive load act as dynamic parameters derived from EEG data. These parameters influence and regulate the evolving accuracy of a deep learning model, which could be a classifier, predictor, or adaptive feedback engine.

The motivation for this model stems from the need to design neuro-adaptive AI systems—systems that not only respond to user input but also evolve based on the user's current cognitive state. For instance, in applications such as brain-computer interfaces, personalized learning environments, or neuro-feedback systems, the AI must be able to interpret when the user is mentally overloaded or highly focused, and adjust its behavior accordingly. By embedding real-time neural feedback into the learning loop, the model enables AI to adapt its learning trajectory or decision boundary in synchrony with human mental states.

Ultimately, this model serves as a step toward creating AI systems that can modulate their performance based on human cognitive and neural dynamics, leading to more intelligent, empathetic, and user-aligned machine behavior.

1.2 Overview of the Model

This model captures the dynamic relationship between brainwave synchronization, cognitive load, and deep learning model accuracy using a system of nonlinear differential equations. The key components are:

- Brainwave Synchronization, $\mathrm{B(t)}$: Represents the coherence of neural oscillations derived from EEG signals.

- Cognitive Load, C(t) : Denotes the mental effort experienced by a user, influenced by synchronization.
- Deep Learning Model Accuracy, D(t) : Reflects the performance of the AI model, dynamically adjusted using cognitive feedback.

1.3 Brainwave Synchronization $B(t)$

Brainwave synchronization $B(t)$ refers to the degree of coordinated neural oscillations occurring across different brain regions. These oscillations are fundamental to human cognition, underlying processes such as attention, memory, and learning. Synchronization reflects how effectively different parts of the brain are working together—a crucial feature for optimal information processing [1].

1.3.1 Measurement and Frequency

Brainwave synchronization is commonly measured using **electroencephalography (EEG)**. EEG captures electrical signals generated by neurons, recorded as voltage fluctuations over time via scalp electrodes. These signals can be analyzed in terms of:

- *Amplitude* (signal strength),
- *Frequency* (measured in Hertz, Hz), and
- *Phase relationships* (for synchronization).

While synchronization itself is **not directly measured in Hz**, it is derived from how **brainwaves of specific frequencies** align across regions. The synchronization is quantified using measures such as:

- Phase Locking Value (PLV)
- Inter-site Phase Clustering (ISPC)
- Spectral Coherence
- Synchronization Likelihood

These metrics yield values between 0 and 1, where higher values indicate stronger synchrony. However, synchronization is often analyzed within particular **EEG frequency bands**, depending on the cognitive state being examined. The following table summarizes the commonly used brainwave types:

Table 1.1 Brainwave Types and Associated Cognitive States

Brainwave Type	**Frequency Range (Hz)**	**Mental State**
Delta	0.5 – 4	Deep sleep, unconsciousness
Theta	4 – 8	Drowsiness, light meditation, creativity
Alpha	8 – 13	Relaxed wakefulness, calm focus, meditative states
Beta	13 – 30	Active thinking, focus, problem-solving, external attention
Gamma	30 – 100+	High-level information processing, learning, perception, consciousness

1.3.2 Importance in the Model

Synchronization provides a *network-level insight* into brain dynamics. Unlike isolated frequency power, which reflects localized activity, synchronization captures **func-**

tional connectivity—how regions of the brain interact in real time. This is particularly vital for understanding attention, learning processes, and AI-human interaction.

In the proposed model, $B(t)$ represents the *dynamic synchronization level*, estimated using EEG coherence or PLV across specific electrode pairs or regions. Although $B(t)$ is a dimensionless value (typically between 0 and 1), it is interpreted in the context of a target frequency band such as alpha or beta.

1.3.3 Application in Learning Systems

In the context of AI learning and cognitive feedback:

- Higher $B(t)$ indicates more synchronized brain activity, often associated with **focused attention and learning readiness**, which can be used by the model to increase confidence in predictions or accelerate learning.
- Lower $B(t)$ suggests cognitive fatigue or overload, prompting the model to slow down, repeat material, or adapt the learning path [7].

Integrating $B(t)$ allows deep learning systems to become more sensitive to **real-time neural feedback**, enabling dynamic model accuracy adjustment and improved alignment with human cognitive states.

1.4 Deep Learning Model Accuracy $D(t)$

Deep Learning Model Accuracy, denoted by $D(t)$, refers to the performance of an artificial intelligence system in accurately classifying or interpreting brainwave signals, specifically Electroencephalography (EEG) data, at time t. In the context of this model, the accuracy is not static but evolves dynamically over time, influenced by both the brainwave synchronization parameter and cognitive load $C(t)$ [3].

Modern advancements in neuroscience and machine learning have opened up avenues for using EEG signals as direct inputs for AI systems [4]. EEG signals are non-invasive electrical recordings from the scalp, reflecting real-time brain activity. These signals contain rich patterns that can be analyzed for inferring attention, mental effort, drowsiness, stress, and engagement — all of which are essential in adapting the learning models to suit individual users.

1.4.1 EEG Signal Processing for Deep Learning Models

The EEG data captured using headsets or sensors typically undergoes preprocessing stages such as:

i) **Noise Filtering:** Removing artifacts caused by muscle movement, blinking, or environmental interference.

ii) **Band-pass Filtering:** Segmenting EEG into relevant frequency bands (Delta, Theta, Alpha, Beta, Gamma).

iii) **Feature Extraction:** Applying methods like power spectral density (PSD), wavelet transforms, or Fast Fourier Transform (FFT) to convert raw EEG data into usable features.

iv) **Segmentation:** Dividing continuous EEG streams into time windows for temporal learning.

These extracted features form the input to various deep learning architectures, which adapt and improve based on the incoming cognitive signals.

1.4.2 Models Utilizing EEG for Learning Enhancement

Several deep learning models have demonstrated the ability to adapt using EEG signals:

- **EEGNet:** A compact convolutional neural network (CNN) architecture specifically tailored for EEG-based brain-computer interfaces. It learns spatial and temporal dependencies of EEG signals for classification tasks [5].
- **DeepConvNet and ShallowConvNet:** Introduced for EEG decoding, these networks learn hierarchies of brainwave features using multiple convolutional layers. DeepConvNet captures complex patterns, while ShallowConvNet focuses on frequency-specific features.
- **Recurrent Neural Networks (RNNs) and LSTMs:** These are used to model temporal dependencies in EEG signals, which is crucial for understanding time-varying cognitive states.
- **Transformer-Based Models:** Recent developments use self-attention to focus on critical EEG signal segments and improve model responsiveness to cognitive changes.

Why Use Deep Learning with EEG?

The motivation to integrate EEG with deep learning stems from the following advantages:

1. **Real-time adaptability:** The model can personalize responses based on the user's mental state.

2. **Improved learning outcomes:** Models can detect cognitive overload or disengagement and respond accordingly.

3. **Enhanced human-AI interaction:** Emotion-aware systems can adjust tone, content delivery, and pacing dynamically.

In our proposed model, $D(t)$ is influenced by both brainwave synchronization and cognitive load. The AI system continuously adjusts its prediction strategies to maximize learning efficacy using the captured neural signatures.

1.5 Cognitive Load $C(t)$

Cognitive Load, denoted by $C(t)$, represents the mental effort exerted by a user during interaction with an AI system. Within our brainwave synchronization model, it serves as a dynamic intermediary variable that modulates both brainwave synchronization and the performance of the deep learning model, $D(t)$ [6].

1.5.1 Role in the Model

In this framework, $C(t)$ reflects how efficiently the human brain processes and responds to task-related information. An elevated cognitive load can negatively impact synchronization and hinder the model's ability to fine-tune itself based on neural feedback. Conversely, moderate cognitive load can indicate optimal engagement and foster effective brain–AI collaboration, leading to improved accuracy in learning tasks [7].

1.5.2 Measurement via EEG Signals

Cognitive load cannot be measured directly but is inferred from specific patterns in electroencephalogram (EEG) signals. In particular, variations in the power of **theta** (4–8 Hz) and **alpha** (8–13 Hz) bands are reliable markers:

- An increase in frontal theta power is often associated with higher cognitive load, indicating working memory and mental effort.
- A decrease in alpha power, especially over parietal regions, correlates with heightened attention and task involvement [8].

1.5.3 EEG Frequency Bands and Cognitive Load Indicators

Table 1.2 EEG Frequency Bands and Their Relation to Cognitive Load

Band	**Frequency (Hz)**	**Cognitive Load Interpretation**
Delta	0.5 – 4	Typically linked to deep sleep; not significantly related to cognitive load in alert states.
Theta (Frontal)	4 – 8	Increase indicates higher cognitive load, working memory engagement, and internal attention.
Alpha (Parietal)	8 – 13	Decrease indicates reduced relaxation and increased task complexity and attentional demands.
Beta	13 – 30	Moderate increase may reflect active concentration; excessive increase can be linked to anxiety or stress.
Gamma	30 – 100+	Associated with high-level information processing and complex cognitive tasks.

1.5.4 Why Cognitive Load Matters

- It enables the system to adapt deep learning model accuracy $D(t)$ to match the user's cognitive state.
- Prevents mental overload or under-stimulation during interactive sessions.
- Bridges human neural states and AI behavior, supporting a closed-loop feedback mechanism using real-time EEG signals.

Thus, modeling $C(t)$ provides a robust way for deep learning systems to understand and adapt to the user's mental workload, ultimately enhancing synchronization, user experience, and model accuracy.

1.6 Literature Review

1.6.1 EEG-based Deep Learning Models and Cognitive Load

Electroencephalogram (EEG) signals provide an effective means of capturing real-time neural activity, which can be used to assess cognitive load and mental states. In recent years, researchers have focused on leveraging EEG signals to enhance the performance and interpretability of deep learning models in fields such as Brain-Computer Interfaces (BCIs), cognitive neuroscience, and adaptive AI systems.

Bashivan et al. (2016) [11] proposed a recurrent-convolutional neural network (RCNN) model to classify mental load from EEG signals by converting them into sequences of multi-spectral topological images, thus preserving spatial, spectral, and temporal information simultaneously. Their results demonstrated significant improvements in classification accuracy over conventional EEG approaches.

Lotte and Congedo (2007) [12] reviewed EEG classification algorithms for BCI applications, identifying Linear Discriminant Analysis (LDA), Support Vector Machines (SVM), and Independent Component Analysis (ICA) as effective techniques, especially when combined with dimensionality reduction strategies .

Plis et al. (2014) [13] emphasized the role of deep belief networks and convolutional architectures in extracting robust features from EEG and fMRI data, validating that deep learning models are capable of learning invariant representations from neuroimaging data even with moderate dataset sizes .

Sweller et al. (1998) [14, 15] introduced Cognitive Load Theory, which has since informed experimental designs that measure working memory using EEG-based theta and alpha band responses. Jensen et al. (2002) found that frontal theta and parietal alpha rhythms correlate with memory load, indicating that these EEG markers can serve as inputs to dynamic AI systems that modulate their learning rates and complexity adaptively.

These foundational works underscore how EEG data reflecting cognitive load can dynamically tune deep learning models for enhanced accuracy, especially in neuroadaptive systems where mental effort plays a crucial role.

1.6.2 Ecological Complexity and Coupled System Modeling

The concept of ecological complexity has gained prominence in recent decades for modeling nonlinear, multivariate interactions between environmental parameters and human-induced changes. Such frameworks are increasingly adopted in environmental mathematics, urban dynamics, and sustainability modeling.

Tandon et al. (2021) [16] highlighted the role of mathematical models in interpreting ecological impacts of urban sprawl and carbon emissions. Their research employed complex system modeling, agent-based simulations, and feedback dynamics to understand how ecological thresholds are influenced by development patterns .

In a related study, urbanization trends were quantitatively linked with rising carbon emissions and ecological footprints using remote sensing and spatial econometrics. The study emphasized that nonlinear interactions between human activities and natural ecosystems necessitate models that can handle cognitive and environmental feedbacks simultaneously [16].

The integration of brainwave synchronization, cognitive load, and deep learning accuracy within ecological modeling offers a novel paradigm—where internal cognitive states (measured via EEG) inform and adapt complex models traditionally based on exogenous environmental variables. This hybrid approach expands the frontier of ecological complexity modeling by adding a human-cognitive feedback layer that could lead to more adaptive and resilient systems.

1.7 Objectives

The main objectives of this research are:

i) To develop a mathematical model that captures the dynamic interplay between brainwave synchronization, cognitive load $C(t)$, and deep learning model accuracy $D(t)$ using nonlinear differential equations.

ii) To analytically investigate the equilibrium states and stability of the model to understand how cognitive and neural parameters influence AI learning outcomes over time.

iii) To explore the behavior of the system under varying levels of cognitive load and synchronization intensity, and to evaluate the impact of brainwave feedback on optimizing deep learning model performance using EEG signals.

Chapter 2

METHODOLOGY

This chapter outlines the methodological framework adopted to analyze the dynamics of *brainwave synchronization*, *cognitive load* $C(t)$, and *deep learning model accuracy* $D(t)$. The interaction among these variables is modeled using a system of nonlinear ordinary differential equations that captures how fluctuations in EEG-based brainwave signals and variations in cognitive workload affect the training and adaptability of deep learning models. Given the biological complexity of brain signals and the computational feedback from deep learning systems, a qualitative approach is adopted to explore the system's equilibrium states, boundedness, and stability. The use of mathematical analysis not only provides insights into the behavioral patterns of the system under various parameter configurations but also helps in identifying conditions under which the deep learning model achieves optimal adaptability through synchronization with cognitive states.

2.1 Boundedness of Solutions

The boundedness of the solution is crucial for a qualitative analysis of the model using a nonlinear system of differential equations. Boundedness ensures that the system variables remain within physiologically and computationally feasible limits over time.

Specifically, this condition is essential when working with EEG-based brainwave data, where extreme values are biologically implausible and may indicate model instability.

To assess boundedness, we utilize a standard analytical tool Gronwall's inequality[17] which provides an upper bound for non-negative solutions of differential inequalities. Let $\mathcal{S}(t)$ be a non-negative, continuous, differentiable function representing a combination of model variables such that:

$$\frac{d\mathcal{S}}{dt} \leq \lambda \mathcal{S}(t), \quad \forall\ t \in [0, T]$$

for some constant $\lambda > 0$. Then by Gronwall's inequality, the solution satisfies:

$$\mathcal{S}(t) \leq \mathcal{S}(0)e^{\lambda t}, \quad \forall\ t \in [0, T]$$

This implies that the function $\mathcal{S}(t)$, and hence the dependent state variables (brainwave synchronization index, cognitive load, and model accuracy), remain bounded for all $t \in [0, T]$, provided their initial values are finite.

In practical terms, this result confirms that the modeled cognitive and computational dynamics will not diverge uncontrollably and remain within analyzable bounds over the simulation period. This property is crucial for interpreting and validating the model outcomes in real-world scenarios involving EEG data-driven learning systems.

2.2 Equilibrium Solutions

Analyzing the steady-state configurations of the system is crucial for understanding the dynamic interaction among brainwave synchronization $B(t)$, cognitive load $C(t)$, and deep learning model accuracy $D(t)$. The equilibrium point, also known as the steady state or fixed point is a state in which the system's behaviour remains unchanged over time [18], which reflect conditions where all variables remain constant over time, signifying a cognitive and algorithmic balance.

The model can be expressed as a system of nonlinear differential equations:

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}), \tag{2.1}$$

where $\mathbf{X} = [B(t), C(t), D(t)]^T$ represents the state vector of system parameters, and $\mathbf{F}$ encapsulates their nonlinear interdependencies.

An equilibrium point $\mathbf{X}_e = [B_e, C_e, D_e]^T$ satisfies:

$$\mathbf{F}(\mathbf{X}_e) = \mathbf{0}, \tag{2.2}$$

indicating that the temporal evolution of all components ceases. This implies a state where EEG-driven synchronization stabilizes, cognitive workload reaches a sustainable threshold, and the model's learning performance plateaus at a reliable level.

The model's behaviour is determined by the number and values of its equilibria. The evaluation of stability at these equilibrium points involves examining the sign of eigenvalues of the Jacobian matrix associated with $\mathbf{F}$. Based on the eigenvalues' signs, the equilibrium points can be classified as:

- **Stable:** Nearby trajectories converge toward the equilibrium point.
- **Unstable:** Nearby trajectories diverge from the equilibrium point.

If a system's dynamics are governed by differential equations, equilibria can be approximated by solving for the zero-derivative condition. Depending on the functional form and parameter configuration, the system may exhibit no equilibrium, a unique equilibrium, or multiple equilibria — each with different stability properties.

Exploring these equilibrium states is fundamental to understanding whether the human-AI cognitive interface settles into optimal functioning or is prone to instability due to fluctuations in neural feedback or model behavior.

2.3 Stability Analysis

The stability of an equilibrium point is calculated to determine whether solutions at the equilibrium point stay nearby, move closer, or diverge over time. For the current model, the system is governed by the nonlinear differential equation:

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}), \tag{2.3}$$

where $\mathbf{X} = [B(t), C(t), D(t)]^T$ is the vector of brainwave synchronization, cognitive load, and deep learning model accuracy. Let $\mathbf{X}_e = [B_e, C_e, D_e]^T$ denote an equilibrium point of the system. By definition, it satisfies:

$$\mathbf{F}(\mathbf{X}_e) = \mathbf{0}.$$

The stability of this equilibrium $\mathbf{X}_e$ can be characterized as:

- **Stable**: For a given $\epsilon > 0$, there exists $\delta > 0$ such that

$$||\mathbf{X}(0) - \mathbf{X}_e|| < \delta \Rightarrow ||\mathbf{X}(t) - \mathbf{X}_e|| < \epsilon \quad \text{for all } t \geq 0.$$

- **Asymptotically Stable**: In addition to the condition above, the solution converges to the equilibrium:

$$||\mathbf{X}(0) - \mathbf{X}_e|| < \delta \Rightarrow \mathbf{X}(t) \to \mathbf{X}_e \quad \text{as } t \to \infty.$$

According to these definitions, if the system returns to its equilibrium state after a small disturbance in $B(t)$, $C(t)$, or $D(t)$, it is considered locally stable. If the system not only stays close but eventually returns exactly to the equilibrium state, it is asymptotically stable. These stability classifications are essential in analyzing whether brainwave-cognitive-model dynamics tend to maintain or disrupt the equilibrium conditions during learning and decision-making processes.

2.3.1 Methods of Stability Analysis

To assess the stability of the nonlinear system modeling the interactions among brainwave synchronization $B(t)$, cognitive load $C(t)$, and deep learning model accuracy $D(t)$, we linearize the system around its equilibrium points and analyze the resulting Jacobian matrix. The eigenvalues of this matrix determine the local stability characteristics of the system.

The Routh-Hurwitz criterion provides a systematic method to ascertain the stability of a system without explicitly computing the eigenvalues. For a characteristic polynomial of the form:

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n, \tag{2.4}$$

the Routh-Hurwitz criterion states that all roots of $P(\lambda)$ have negative real parts (implying system stability) if and only if all the leading principal minors of the associated Hurwitz matrix are positive [19].

For a third-order system, the characteristic polynomial is:

$$P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3. \tag{2.5}$$

The Routh-Hurwitz conditions for stability in this case are:

1. All coefficients a_1, a_2, a_3 are positive.
2. The inequality $a_1a_2 > a_3$ holds.

These conditions ensure that all roots of the polynomial have negative real parts, indicating that the system will return to equilibrium after small perturbations.

Recent advancements have extended the applicability of the Routh-Hurwitz criterion to certain classes of nonlinear systems. For instance, Song et al. introduced a dynamic Routh's stability criterion to analyze the stability of nonlinear hybrid systems, demonstrating its effectiveness in handling time-varying dynamics [20]. Additionally, Hastir and Muolo developed a generalized Routh-Hurwitz criterion for polynomials with complex coefficients, broadening the scope of stability analysis in control systems [21].

In the context of our model, applying the Routh-Hurwitz criterion to the linearized system around equilibrium points allows us to determine the conditions under which the system maintains stability. This analysis is crucial for ensuring that the interactions among $B(t)$, $C(t)$, and $D(t)$ lead to consistent and predictable behavior in cognitive and computational processes.

Chapter 3

FORMULATION OF MATHEMATICAL MODEL

The proposed mathematical model is developed to capture the dynamic interrelationship among brainwave synchronization, cognitive load, and the accuracy of deep learning models trained on EEG data. These three variables form a closed-loop system, where variations in neural oscillations (brainwave synchronization) influence the cognitive effort required to perform a task (cognitive load), which subsequently affects the performance of deep learning models trained on EEG signals.

Let $B(t)$ denote the brainwave synchronization at time t, indicating the degree of alignment and coherence between neural signals. Let $C(t)$ denote the cognitive load, which captures the mental effort and task difficulty as interpreted through EEG features. Finally, let $D(t)$ represent the deep learning model's classification accuracy, which adapts dynamically based on the quality and interpretability of the EEG signals.

It is observed that high synchronization (i.e., coherent oscillatory patterns) supports optimal cognitive functioning and enhances signal clarity, thereby improving model performance [22]. Conversely, increased cognitive load can disrupt synchronization and reduce the accuracy of model predictions. Therefore, an interdependent nonlinear dynamical system is proposed that encapsulates the effects of these interactions.

With these assumptions and positivity conditions $B(t) \geq 0, C(t) \geq 0, D(t) > 0$, the model is proposed as :

$$\frac{dB}{dt} = r_B B \left(1 - \frac{B}{B_{\max}}\right) + \alpha DB - \beta CB \tag{3.1a}$$

$$\frac{dD}{dt} = \gamma B - \mu CD - \delta D \tag{3.1b}$$

$$\frac{dC}{dt} = r_C C \left(1 - \frac{C}{C_{\max}}\right) - \theta CD - \lambda BC \tag{3.1c}$$

The model parameters r_B, $B_{\max}$, α, β, γ, μ, δ, r_C, $C_{\max}$, θ, and λ are all positive and their definitions are explained in Table 3.1.

The model is constructed with the following considerations:

i) **Brainwave Synchronization Equation** (3.1a):

The first equation captures the dynamics of brainwave synchronization $B(t)$ over time. The intrinsic evolution of B follows a logistic growth pattern $r_B B(1 - B/B_{\max})$, where r_B is the natural growth rate and $B_{\max}$ is the maximum synchrony threshold, analogous to a carrying capacity.

An important enhancement to synchronization comes from high model accuracy, reflected by the interaction term αDB. This is biologically inspired, as higher confidence or accuracy in machine feedback can improve user engagement and thus synchronization, a concept supported by neuroadaptive learning interfaces [25].

Table 3.1 Definition of Parameters in the Model

Parameter	**Description**
r_B	Natural growth rate of brainwave synchronization over time
$B_{\max}$	Maximum saturation threshold of brainwave synchronization
α	Influence rate of deep learning accuracy in enhancing brainwave synchronization
β	Inhibitory effect of cognitive load on brainwave synchronization
γ	Conversion rate of brainwave synchronization contributing to model accuracy
μ	Deterioration of accuracy due to increased cognitive load
δ	Natural decay rate of deep learning model accuracy
r_C	Natural growth rate of cognitive load
$C_{\max}$	Maximum sustainable cognitive load
θ	Decline in cognitive load due to learning effects from accurate predictions
λ	Suppressive effect of synchronized brain activity on cognitive load

However, elevated cognitive load C can impair synchronization due to neural overstimulation or interference, which is modeled via the term $-\beta CB$. This inhibitory relationship has been noted in EEG studies where mental fatigue or overloading leads to desynchronization [26].

ii) **Deep Learning Accuracy Equation** (3.1b):

The second equation describes the dynamics of model accuracy $D(t)$. Accuracy is positively reinforced by brainwave synchronization through γB, as user-centered EEG-based training improves classification precision in adaptive systems [27].

However, as cognitive load increases, performance tends to degrade due to reduced attention span or misinterpretation of EEG signals, modeled here by $-\mu CD$. This term reflects findings in cognitive neuroscience showing that overload negatively affects working memory and predictive accuracy [28].

Additionally, a natural decay $-\delta D$ represents the progressive degradation of accuracy in a system not continuously reinforced or calibrated,a common scenario in online learning models or prolonged inference without retraining [29].

iii) **Cognitive Load Equation** (3.1c):

The third equation represents the temporal dynamics of cognitive load $C(t)$, modeled again via logistic growth $r_C C(1 - C/C_{\max})$, where r_C is the rate of load accumulation and $C_{\max}$ the cognitive limit.

The term $-\theta CD$ reflects cognitive relief or resolution due to accurate model responses, consistent with research on trust calibration and cognitive offloading [30]. High accuracy reduces the cognitive strain on users in EEG-assisted learning environments.

Finally, $-\lambda BC$ accounts for the impact of brainwave synchronization on reducing load, as smooth neural feedback and entrainment can mitigate perceived effort and enhance flow state [31].

3.1 Rescaling of Variables and Final Formulation

To simplify the model and facilitate analytical tractability, we rescale the system variables to normalize their ranges. This rescaling bounds the variables between 0 and 1 and reduces dimensional redundancy in the parameters. Such non-dimensionalization is common in ecological and cognitive modeling to provide unit-independent interpretations [16].

We define the following rescaled variables:

$$\tilde{B} = \frac{B}{B_{\max}}, \quad \tilde{C} = \frac{C}{C_{\max}}$$

where $B_{\max}$ and $C_{\max}$ are the physiological upper limits for brainwave synchronization and cognitive load, respectively. Now, substituting these into the original model:

$$\frac{d\bar{B}}{dt} = r_B \bar{B}\left(1 - \bar{B}\right) + \alpha D \bar{B} - \bar{\beta}\bar{C}\bar{B} \tag{3.2a}$$

$$\frac{dD}{dt} = \bar{\gamma}\bar{B} - \bar{\mu}\bar{C}D - \delta D \tag{3.2b}$$

$$\frac{d\bar{C}}{dt} = r_C \bar{C}\left(1 - \bar{C}\right) - \theta\bar{C}D - \bar{\lambda}\bar{B}\bar{C} \tag{3.2c}$$

where , the new rescaled parameters are :

$$\bar{\beta} = \beta C_{\max}, \quad \bar{\gamma} = \gamma B_{\max}, \quad \bar{\mu} = \mu C_{\max}, \quad \bar{\lambda} = \lambda B_{\max}$$

The initial conditions associated with model (3.2) are $\bar{B}(0) \geq 0, \bar{D}(0) \geq 0, \bar{C}(0) \geq 0$. Dropping the bar notation, model (3.3) appears to be:

$$\frac{dB}{dt} = r_B B\,(1 - B) + \alpha DB - \beta CB \tag{3.3a}$$

$$\frac{dD}{dt} = \gamma B - \mu CD - \delta D \tag{3.3b}$$

$$\frac{dC}{dt} = r_C C\,(1 - C) - \theta CD - \lambda BC \tag{3.3c}$$

with initial values of dynamic variables as : $B(0) \geq 0, D(0) \geq 0, C(0) \geq 0$. This dimensionless formulation (3.3) enables efficient qualitative and quantitative analysis by eliminating the dependence on units and reducing the parameter space. It also ensures that the state variables are always interpreted on a normalized scale, which is particularly useful when modeling EEG signal dynamics or training deep learning models.

Chapter 4

MATHEMATICAL ANALYSES OF MODEL

4.1 Qualitative Analyses

To understand the long-term behaviour and intrinsic properties of the dynamical system described by model (3.3), it is essential to conduct a qualitative analysis. This involves investigating the boundedness, stability, and feasibility of the system variables over time. Before proceeding with further analysis, we verify whether the state variables—brainwave synchronization $B(t)$, cognitive load $C(t)$, and deep learning model accuracy $D(t)$—remain non-negative and biologically interpretable. This is achieved by establishing the region of attraction and proving the boundedness of trajectories, thereby ensuring that the model's dynamics remain well-defined within the physiological and computational constraints of neural systems and AI learning frameworks.

4.2 Region of Attraction

We first note that solutions remain in the nonnegative octant $\mathbb{R}^3_+$ for all $t \geq 0$ if started there, since each equation is either logistic or has only nonnegative interactions. Next, we derive an invariant region in $\mathbb{R}^3_+$ that bounds all trajectories. Using a standard comparison theorem , we obtain the following bounds step-by-step:

Bound on $C(t)$: The C-equation is self-limited by logistic growth and decreased by interactions with B and D. Ignoring the negative interaction terms gives

$$\frac{dC}{dt} = r_C C(1-C) - \theta CD - \lambda BC \;\leq\; r_C C(1-C). \tag{4.1}$$

This is the standard logistic equation, whose solutions are bounded by the carrying capacity 1. Thus, irrespective of initial conditions, $C(t)$ approaches a value ≤ 1. In fact, if $C(0) \leq 1$ then $C(t) \leq 1$ for all t. We set $C_m = 1$, so that $0 \leq C(t) \leq C_m$ for all $t \geq 0$.

Bound on $B(t)$: Using the bound $D(t) \leq D_m$ (to be determined) and the nonnegativity of $C(t)$, we can bound the B-equation from above by dropping the $-\beta CB$ term:

$$\frac{dB}{dt} \;=\; r_B\, B(1-B) + \alpha D\, B - \beta C\, B \;\leq\; r_B B(1-B) + \alpha D_m\, B. \tag{4.2}$$

The right-hand side is a logistic-type equation $\dot{B} \leq (r_B + \alpha D_m)\, B - r_B B^2$. By solving $(r_B + \alpha D_m)B - r_B B^2 = 0$, we find an upper bound for B:

$$B(t) \;\leq\; \frac{r_B + \alpha D_m}{r_B} \;=\; 1 + \frac{\alpha}{r_B} D_m.$$

Thus, define

$$B_m \;=\; \frac{r_B + \alpha D_m}{r_B}, \tag{4.3}$$

so that $\limsup\limits_{t\to\infty} B(t) \le B_m$.

Bound on $D(t)$: Similarly, using $B(t) \le B_m$ and $C(t) \le C_m$, we bound the D-equation by dropping the $-\mu CD$ term:

$$\frac{dD}{dt} \;=\; \gamma B - \mu CD - \delta D \;\le\; \gamma B_m - \delta D. \tag{4.4}$$

This is a linear inequality; its solution is bounded by the equilibrium $D = \frac{\gamma}{\delta} B_m$. Hence we set

$$D_m \;=\; \frac{\gamma}{\delta} B_m, \tag{4.5}$$

so that $\limsup\limits_{t\to\infty} D(t) \le D_m$.

Substituting equation (4.5) into equation (4.3), we obtain:

$$B_m = \frac{r_B + \alpha\left(\frac{\gamma}{\delta} B_m\right)}{r_B} = \frac{r_B + \frac{\alpha\gamma}{\delta} B_m}{r_B}.$$

Multiplying both sides by r_B:

$$r_B B_m = r_B + \frac{\alpha\gamma}{\delta} B_m.$$

Bringing terms with B_m together:

$$B_m\left(r_B - \frac{\alpha\gamma}{\delta}\right) = r_B.$$

Solving for B_m, we get:

$$B_m = \frac{r_B\delta}{r_B\delta - \alpha\gamma}. \tag{4.6}$$

Substituting back into equation (4.5), we obtain:

$$D_m = \frac{\gamma}{\delta} B_m = \frac{\gamma r_B}{r_B\delta - \alpha\gamma}. \tag{4.7}$$

Theorem 4.1. *Let* $(B(t), D(t), C(t))$ *be any solution of the system* (3.3) *with non-negative initial conditions in* $\mathbb{R}^3_+$. *Then all solutions are ultimately bounded and enter the compact region*

$$\Omega \;=\; \left\{(B, D, C) \in \mathbb{R}^3_+ :\; 0 \leq B \leq B_m,\;\; 0 \leq D \leq D_m,\;\; 0 \leq C \leq C_m\right\},$$

where

$$B_m = \frac{r_B\delta}{r_B\delta - \alpha\gamma}, \qquad D_m = \frac{\gamma\, r_B}{r_B\delta - \alpha\gamma}, \qquad C_m = 1,$$

provided the natural feasibility condition $\boldsymbol{r_B\delta > \alpha\gamma}$ *is satisfied.*

In particular, if the initial state lies in Ω (or in its interior), the solution will stay in Ω for all $t \geq 0$. Moreover, any solution starting in the interior of $\mathbb{R}^3_+$ is attracted to Ω as $t \to \infty$. This guarantees the boundedness of solutions and effectively establishes a compact region of attraction for the system.

4.3 Equilibrium Points and Feasibility Analysis

For the system (3.3), three possible trivial equilibria are $E_1(0, 0, 0)$, $E_2(1, 0, 0)$, and $E_3(0, 0, 1)$, whose existences are obvious . The other four non-trivial equilibrium candidates are: Brainwave Synchronization-free equilibrium $E_4(0, D_1^*, C_1^*)$, Deep Learning Model Accuracy-free equilibrium $E_5(B_2^*, 0, C_2^*)$, Cognitive Load-free equilibrium

$E_6(B_3^*, D_3^*, 0)$, and the coexistence equilibrium $E_7(\hat{B}, \hat{D}, \hat{C})$, whose feasibility conditions are discussed below.

Note that $E_4(0, D_1^*, C_1^*)$ is **biologically infeasible**. Substituting $B = 0$ into the equation (3.3b) yields the equation $-\mu CD - \delta D = 0$, which simplifies to

$$D(-\mu C - \delta) = 0.$$

For $D > 0$, this implies $C = -\delta/\mu < 0$, contradicting the requirement that all variables must remain non-negative.

4.3.1 Existence of Equilibrium $E_5({B_2}^*,\, 0,\, {C_2}^*)$

To examine the feasibility of the equilibrium point $E_5({B_2}^*,\, 0,\, {C_2}^*)$, we consider the following system of equations:

$$r_B(1 - B) - \beta C = 0 \tag{4.8a}$$

$$r_C(1 - C) - \lambda B = 0 \tag{4.8b}$$

Solving equations (4.8a) and (4.8b) simultaneously, we obtain:

$${B_2}^* = \frac{r_C(r_B - \beta)}{r_B r_C - \beta\lambda}, \qquad {C_2}^* = \frac{r_B(r_C - \lambda)}{r_B r_C - \beta\lambda}$$

The feasibility of E_5 depends on the following two cases:

Case 1: $r_B r_C - \beta\lambda > 0$

The equilibrium point E_5 is feasible if and only if $r_B > \beta$ and $r_C > \lambda$.

Case 2: $r_B r_C - \beta\lambda < 0$

The equilibrium point E_5 is feasible if and only if $r_B < \beta$ and $r_C < \lambda$.

4.3.2 Existence of Equilibrium $E_6({B_3}^*,\ {D_3}^*,\ 0)$

To analyze the feasibility of the equilibrium $E_6({B_3}^*,\ {D_3}^*,\ 0)$, we solve the following system of equations:

$$r_B(1 - B) + \alpha D = 0 \tag{4.9a}$$

$$\gamma B - \delta D = 0 \tag{4.9b}$$

Solving equations (4.9a) and (4.9b) simultaneously gives:

$${B_3}^* = \frac{r_B\delta}{r_B\delta - \alpha\gamma}, \qquad {D_3}^* = \frac{r_B\gamma}{r_B\delta - \alpha\gamma}$$

The equilibrium point E_6 is feasible if and only if the condition $r_B\delta - \alpha\gamma > 0$ is satisfied.

4.3.3 Existence of Coexistence Equilibrium $E_7(\hat{B}, \hat{D}, \hat{C})$

Due to the nonlinear nature and complex interactions among B, D, and C in system (3.3), obtaining a closed-form expression for the coexistence equilibrium point $E_7(\hat{B}, \hat{D}, \hat{C})$ is analytically intractable. Hence, to investigate its feasibility, we reduce the full system into a quadratic equation in a single variable by eliminating D and C.

Starting with the steady-state equations:

$$r_B B(1-B) + \alpha DB - \beta CB = 0 \tag{4.10a}$$

$$\gamma B - \mu CD - \delta D = 0 \tag{4.10b}$$

$$r_C C(1-C) - \theta CD - \lambda BC = 0 \tag{4.10c}$$

We eliminate D and C and collect the result in powers of B:

$$aB^2 - bB + c = 0 \tag{4.11}$$

where the coefficients a, b, and c are functions of the system parameters $r_B, r_C, \alpha, \beta, \gamma, \mu, \delta, \theta, \lambda$.

$$a = -r_B^2 r_C \theta\mu - r_B r_C \alpha\lambda\mu + r_B \beta\theta\lambda\mu + \alpha\beta\lambda^2\mu \tag{4.12a}$$

$$\begin{aligned} b = {} & r_C^2\alpha^2\gamma + r_B r_C^2 \alpha\delta + 2r_C\alpha\beta\gamma\theta - r_B r_C \beta\delta\theta + \beta^2\gamma\theta^2 \\ & + r_C\alpha\beta\delta\lambda + \beta^2\delta\theta\lambda - r_B r_C^2 \alpha\mu - 2r_B^2 r_C\theta\mu \\ & + r_B r_C \beta\theta\mu - r_B r_C \alpha\lambda\mu + 2r_C\alpha\beta\lambda\mu + r_B\beta\theta\lambda\mu \end{aligned} \tag{4.12b}$$

$$\begin{aligned} c = {} & -r_B r_C^2 \alpha\delta + r_C^2\alpha\beta\delta - r_B r_C \beta\delta\theta + r_C\beta^2\delta\theta \\ & - r_B r_C^2 \alpha\mu + r_C^2 \alpha\beta\mu - r_B^2 r_C \theta\mu + r_B r_C \beta\theta\mu \end{aligned} \tag{4.12c}$$

The roots of the equation (4.11) are :

$$\hat{B}_1 = \frac{b + \sqrt{b^2 - 4ac}}{2a} \quad , \quad \hat{B}_2 = \frac{b - \sqrt{b^2 - 4ac}}{2a} \tag{4.13}$$

Now, for possible values of B, different cases with respect to the discriminant $\Delta = b^2 - 4ac$ of equation (4.11) are discussed below:

- **Case I :** $\Delta < 0$. In this case, equation (4.11) has no real B, hence the system (3.3) does not have any coexistence equilibrium.

- **Case II:** $\Delta > 0$

  In this case, the quadratic equation (4.11) can have two real roots. The feasibility of these roots as biologically meaningful equilibrium values depends on the signs of the coefficients a, b, and c. We examine the subcases below based on the sign of a.

  - **Case II(a):** $a > 0$

    * **Sub-Case 1:** $b > 0$, $c > 0$.

      In this case, both the roots B_1^* and B_2^* are positive, hence the system (4.11) possesses two sets of distinct coexistence equilibrium.

    * **Sub-Case 2:** $b < 0$, $c > 0$.

      In this case, both the roots B_1^* and B_2^* are negative, hence the system (4.11) does not possess any positive coexistence equilibrium.

    * **Sub-Case 3:** $b > 0$, $c < 0$.

      In this case, the root B_1^* is positive but the root B_2^* is negative, hence the system (4.11) possesses a unique positive coexistence equilibrium.

    * **Sub-Case 4:** $b < 0$, $c < 0$.

      In this case, the root B_1^* is positive but the root B_2^* is negative , hence the system (4.11) possesses a unique positive coexistence equilibrium.

    * **Sub-Case 5:** $b = 0$, $c < 0$.

      When $b = 0$, c must be negative for $\Delta > 0$. Hence, the roots B_1^* and B_2^* are of equal magnitude but opposite signs. Therefore, the system (4.11) possesses a unique positive coexistence equilibrium.

    * **Sub-Case 6:** $c = 0$, $b < 0$.

In this peculiar case, one root B_1^* is zero and the sign of B_2^* depends on the sign of c . If $c < 0$, both roots are non-positive, so no coexistence equilibrium exists. If $c > 0$, a unique positive coexistence equilibrium is possible.

- **Case II(b) :** $a < 0$

  * **Sub-Case 1 :** $b > 0$, $c > 0$.

    In this case, the root B_1^* is negative but the root B_2^* is positive, hence the system (4.11) possesses a unique positive coexistence equilibrium.

  * **Sub-Case 2 :** $b < 0$, $c > 0$.

    In this case, the root B_1^* is negative but the root B_2^* is positive, hence the system possesses a unique positive coexistence equilibrium.

  * **Sub-Case 3 :** $b > 0$, $c < 0$.

    In this case, both the roots B_1^* and B_2^* are negative, hence the system (4.11) does not possess any positive coexistence equilibrium.

  * **Sub-Case 4 :** $b < 0$, $c < 0$.

    In this case, both the roots B_1^* and B_2^* are positive, hence the system (4.11) possesses two sets of distinct coexistence equilibrium.

  * **Sub-Case 5 :** $b = 0$, $c < 0$.

    When $b = 0$, c must be negative for $\Delta > 0$. Hence, the roots B_1^* and B_2^* are of equal magnitude but opposite signs. Therefore, the system possesses a unique positive coexistence equilibrium.

  * **Sub-Case 6 :** $c = 0$, $b < 0$.

    In this peculiar case , B_2^* is zero and the sign of B_1^* depends on the sign of c.If $c > 0$, both roots are non-positive, so no coexistence equilibrium exists. If $c < 0$, a unique positive coexistence equilibrium is possible.

- **Case III:** $\Delta = 0$.

In this case, both roots B_1^* and B_2^* coincide, giving rise to an *instantaneous equilibrium* , a repeated positive coexistence equilibrium , under suitable conditions. For the model (4.11), two scenarios lead to such equilibria:

– When $a > 0$ and $c > 0$, the roots coincide at $B^* = -\frac{b}{2a}$, provided $b > 0$.

– When $a < 0$ and $c < 0$, the roots also coincide at $B^* = -\frac{b}{2a}$, provided $b < 0$.

Sufficient Condition for Coexistence Equilibrium

A sufficient condition for the existence of a unique positive coexistence equilibrium (B^*, D^*, C^*) is that the quadratic admits exactly one positive real root. This is guaranteed when the coefficients a and c have opposite signs, i.e , $ac < 0$.

4.4 Stability Analysis

4.4.1 Jacobian Matrix

The Jacobian (or variational) matrix of the system (evaluated at a general state (B, D, C)) is obtained by differentiating each equation with respect to each variable.

$$J(E_i) = \begin{pmatrix} f_{11} & \alpha B & -\beta B \\ \gamma & -(\mu C + \delta) & -\mu D \\ -\lambda C & -\theta C & f_{33} \end{pmatrix}, \tag{4.14}$$

where,

$f_{11} = r_B(1 - 2B) + \alpha D - \beta C$

$f_{33} = r_C(1 - 2C) - \theta D - \lambda B.$

At each equilibrium point E_i, we substitute the equilibrium values (B^*, D^*, C^*) into the Jacobian matrix to obtain $J(E_i)$. The local stability of the system around each equilibrium is then determined by analyzing the signs of the eigenvalues of $J(E_i)$.

In particular, the eigenvalues are the roots of the characteristic polynomial of the form:

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0, \tag{4.15}$$

where the coefficients b_1, b_2, and b_3 depend on the trace, principal minors, and determinant of the Jacobian matrix evaluated at the equilibrium.

According to the Routh–Hurwitz stability criterion, the equilibrium is asymptotically stable if the following conditions hold:

$$b_1 > 0 \quad , \quad b_3 > 0 \quad , \quad b_1 b_2 > b_3. \tag{4.16}$$

In the sections that follow, we systematically analyze each equilibrium point E_i by evaluating the Jacobian matrix at that point . In some cases, the eigenvalues can be directly observed from the structure of the Jacobian matrix, while others are analyzed using the Routh–Hurwitz criterion and biological feasibility conditions.

i) At $E_0(0, 0, 0)$, the matrix (4.14) is

$$J(E_0) = \begin{pmatrix} r_B & 0 & 0 \\ \gamma & -\delta & 0 \\ 0 & 0 & r_C \end{pmatrix} \tag{4.17}$$

r_B, $-\delta$ and r_C are the eigenvalues. As $r_B > 0$ and $r_C > 0$, the system exhibits unstable behaviour around E_0.

ii) At the equilibrium point $E_1(1, 0, 0)$, the Jacobian matrix (4.14) evaluates to:

$$J(E_1) = \begin{pmatrix} -r_B & \alpha & -\beta \\ \gamma & -\delta & 0 \\ 0 & 0 & (r_C - \lambda) \end{pmatrix} \tag{4.18}$$

The eigenvalues corresponding to the matrix (4.18) are:

$$\lambda_1 = \frac{-(r_B + \delta) + \sqrt{(r_b - \delta)^2 + 4\alpha\gamma}}{2} \tag{4.19a}$$

$$\lambda_2 = \frac{-(r_B + \delta) - \sqrt{(r_b - \delta)^2 + 4\alpha\gamma}}{2} \tag{4.19b}$$

$$\lambda_3 = (r_B - \lambda) \tag{4.19c}$$

For the equilibrium point E_1 to be asymptotically stable, all eigenvalues must be negative. Clearly, $\lambda_2 < 0$. To ensure $\lambda_1 < 0$, we require:

$$(r_B + \delta) > \sqrt{(r_B - \delta)^2 + 4\alpha\gamma} \tag{4.20}$$

Simplifying the inequality (4.20) , we obtain :

$$r_B\delta > \alpha\gamma \tag{4.21}$$

Additionally, for $\lambda_3 < 0$, the following condition must hold:

$$r_C < \lambda \tag{4.22}$$

Hence, the equilibrium E_1 is asymptotically stable if both conditions (4.21) and (4.22) are satisfied.

Moreover, under the conditions (4.21) , $r_B r_C - \lambda\beta > 0$, and $r_B > \beta$, **transcritical bifurcation** occurs at $\boldsymbol{r_C = \lambda}$ due to a switch in the stability and feasibility of two equilibria . Specifically, the equilibrium E_2 is unstable for $r_C > \lambda$ and becomes asymptotically stable for $r_C < \lambda$. Meanwhile, the equilibrium E_5 is feasible when $r_C > \lambda$ and infeasible when $r_C < \lambda$.

iii) At the equilibrium point $E_3(0, 0, 1)$, the Jacobian matrix is :

$$J(E_2) = \begin{pmatrix} (r_B - \delta) & 0 & 0 \\ \gamma & -(\mu + \delta) & 0 \\ -\lambda & -\theta & -r_C \end{pmatrix} \tag{4.23}$$

The eigenvalues of the matrix (4.23) are $-r_C$, $(r_B - \beta)$ and $-(\mu + \delta)$. For E_2 to be asymptotically stable , $r_B < \beta$.

The equilibrium E_3 is unstable if $r_B > \beta$ and becomes asymptotically stable if $r_B < \beta$. Meanwhile, the equilibrium E_5 is feasible when $r_C > \lambda$ and infeasible when $r_C < \lambda$. Under the conditions, $r_B r_C - \lambda\beta > 0$ and $r_B > \beta$, **transcritical bifurcation** occurs at $\boldsymbol{r_B = \beta}$ from one equilibrium state E_3 to another equilibrium state E_5.

iv) At the equilibrium point $E_5(B_2^*, 0, C_2^*)$, we evaluate the Jacobian matrix (4.14) evaluates to:

$$J(E_5) = \begin{pmatrix} -r_B B_2^* & \alpha B_2^* & -\beta B_2^* \\ \gamma & -(\mu C_2^* + \delta) & 0 \\ -\lambda C_2^* & -\theta C_2^* & -r_C C_2^* \end{pmatrix} \tag{4.24}$$

The characteristic polynomial of the matrix $J(E_5)$ is of the form:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$$

where the coefficients a_1, a_2, a_3 are given by:

$$a_1 = B_2^* r_B + C_2^* r_C + \delta + \mu C_2^* \tag{4.25a}$$

$$\begin{aligned} a_2 = {} & B_2^* C_2^* r_B r_C - B_2^* \alpha\gamma + B_2^* r_B \delta + C_2^* r_C \delta \\ & - B_2^* C_2^* \beta\lambda + B_2^* C_2^* r_B \mu + (C_2^*)^2 r_C \mu \end{aligned} \tag{4.25b}$$

$$\begin{aligned} a_3 = {} & B_2^* C_2^* r_B r_C \delta - B_2^* C_2^* r_C \alpha\gamma - B_2^* C_2^* \beta\gamma\theta - B_2^* C_2^* \beta\delta\lambda \\ & + B_2^* (C_2^*)^2 r_B r_C \mu - B_2^* (C_2^*)^2 \beta\lambda\mu \end{aligned} \tag{4.25c}$$

The system around the equilibrium point E_5 is asymptotically stable if the Routh–Hurwitz conditions are satisfied :

$$a_1 > 0, \qquad a_3 > 0, \qquad a_1 a_2 - a_3 > 0. \tag{4.26}$$

It is evident that b_1 is always positive for B_2^*, $C_2^* > 0$. The expressions for a_2 and a_3 contain several interaction terms involving model parameters and may change depending on the parameter values. However, for feasible values of B_2^* and C_2^* (as obtained from the equilibrium conditions), we can verify that a_2 and $a_1 a_2 - a_3$ remain positive under biologically realistic assumptions.

Therefore, the equilibrium point E_5 is locally asymptotically stable as long as all the Routh–Hurwitz conditions in (4.26) are satisfied.

v) At the equilibrium point $E_6(B_3^*, D_3^*, 0)$, the Jacobian matrix (4.14) becomes:

$$J(E_6) = \begin{pmatrix} \frac{r_B\alpha\gamma}{-\alpha\gamma + r_B\delta} - \frac{r_B^2\delta}{-\alpha\gamma + r_B\delta} + r_B\left(1 - \frac{r_B\delta}{-\alpha\gamma + r_B\delta}\right) & \frac{r_B\alpha\delta}{-\alpha\gamma + r_B\delta} & -\frac{r_B\beta\delta}{-\alpha\gamma + r_B\delta} \\ \gamma & -\delta & -\frac{r_B\gamma\mu}{-\alpha\gamma + r_B\delta} \\ \theta & \theta & r_C - \frac{r_B\gamma\theta}{-\alpha\gamma + r_B\delta} - \frac{r_B\delta\lambda}{-\alpha\gamma + r_B\delta} \end{pmatrix}$$

The three eigenvalues of $J(E_6)$ are as follows:

$$\lambda_1 = \frac{-r_B^2\delta + \alpha\gamma\delta - r_B\delta^2 - \sqrt{\left(r_B^2\delta - \alpha\gamma\delta + r_B\delta^2\right)^2 - 4\left(r_B\alpha^2\gamma^2\delta - 2r_B^2\alpha\gamma\delta^2 + r_B^3\delta^3\right)}}{2(-\alpha\gamma + r_B\delta)} \tag{4.27a}$$

$$\lambda_2 = \frac{-r_B^2\delta + \alpha\gamma\delta - r_B\delta^2 + \sqrt{\left(r_B^2\delta - \alpha\gamma\delta + r_B\delta^2\right)^2 - 4\left(r_B\alpha^2\gamma^2\delta - 2r_B^2\alpha\gamma\delta^2 + r_B^3\delta^3\right)}}{2(-\alpha\gamma + r_B\delta)} \tag{4.27b}$$

$$\lambda_3 = \frac{-r_C\alpha\gamma + r_Br_C\delta - r_B\gamma\theta - r_B\delta\lambda}{-\alpha\gamma + r_B\delta} \tag{4.27c}$$

Here, the feasibility condition $r_B\delta > \alpha\gamma$ ensures that the denominator in all three expressions is positive. Thus, the sign of each eigenvalue depends on the numerator. For stability of E_6, all three eigenvalues must be negative. This leads to the following requirements :

$$r_C\alpha\gamma - r_Br_C\delta + r_B\gamma\theta + r_B\delta\lambda > 0 \tag{4.28a}$$

$$r_B^2\delta - \alpha\gamma\delta + r_B\delta^2 > 0 \tag{4.28b}$$

Therefore, the equilibrium point E_6 is asymptotically stable provided that the feasibility condition $r_B\delta > \alpha\gamma$ holds, and both conditions (4.28a) and (4.28b) are satisfied.

4.4.2 Stability of the Coexistence Equilibrium $E_7(\hat{B}, \hat{D}, \hat{C})$

To analyze the stability of coexistence equilibrium, we consider the Jacobian matrix of the system evaluated at E_7. The Jacobian matrix $J(E_7)$ is:

$$J(E_7) = \begin{pmatrix} -r_B\hat{B} & \alpha\hat{B} & -\beta\hat{B} \\ \gamma & -(\mu\hat{C} + \delta) & -\mu\hat{D} \\ -\lambda\hat{C} & -\theta\hat{C} & -r_C\hat{C} \end{pmatrix} \tag{4.29}$$

To analyze the local stability of the interior equilibrium $E_7(\hat{B}, \hat{D}, \hat{C})$, we use the Routh–Hurwitz criteria applied to the characteristic polynomial:

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0 \tag{4.30}$$

where the coefficients b_1, b_2, b_3 are expressed as:

$$b_1 = \hat{B}r_B + \hat{C}r_C + \delta + \mu\hat{C} \tag{4.31a}$$

$$\begin{aligned} b_2 = {} & \hat{B}\hat{C}r_Br_C - \hat{B}\alpha\gamma + \hat{B}r_B\delta + \hat{C}r_C\delta \\ & - \hat{B}\hat{C}\beta\lambda + \hat{B}\hat{C}r_B\mu + (\hat{C})^2r_C\mu - \hat{C}\hat{D}\theta\mu \end{aligned} \tag{4.31b}$$

$$\begin{aligned} b_3 = {} & -\hat{B}\hat{C}r_C\alpha\gamma + \hat{B}\hat{C}r_Br_C\delta - \hat{B}\hat{C}\beta\gamma\theta - \hat{B}\hat{C}\beta\delta\lambda \\ & + \hat{B}(\hat{C})^2r_Br_C\mu - \hat{B}(\hat{C})^2\beta\lambda\mu - \hat{B}\hat{C}\hat{D}\alpha\mu\lambda - \hat{B}\hat{C}\hat{D}r_B\mu\theta. \end{aligned} \tag{4.31c}$$

For E_7 to be asymptotically stable, the Routh–Hurwitz conditions

$$b_1 > 0, \quad b_3 > 0, \quad b_1 b_2 - b_3 > 0 \tag{4.32}$$

must all be satisfied. From the form of b_1, it is evident that $b_1 > 0$ for all positive $\hat{B}, \hat{C}$, hence b_1 is unconditionally positive.

The full expression for the Routh–Hurwitz determinant $b_1 b_2 - b_3$ simplifies to:

$$\begin{aligned} b_1 b_2 - b_3 = {} & \hat{C}(\hat{C} + \delta + \hat{C}\mu)(\delta + (\hat{C} - \hat{D}\theta)\mu) \\ & + \hat{B}^2(-\alpha\gamma + \delta + \hat{C}(1 - \beta\lambda + \mu)) \\ & + \hat{B}\Big[\delta(-\alpha\gamma + \delta) + \hat{C}(\beta\gamma\theta - \alpha\gamma\mu + \hat{D}\alpha\lambda\mu + 2\delta(1 + \mu)) \\ & \quad + \hat{C}^2\big(- \beta\lambda + (1 + \mu)^2\big)\Big] \end{aligned} \tag{4.33}$$

The condition $b_1 b_2 > b_3$ must be satisfied along with $b_1 > 0$ and $b_3 > 0$ for E_7 to be asymptotically stable.

4.4.3 Stability Behavior of Coexistence Equilibria

With conditions (4.32), the asymptotic stability behavior for feasible coexistence equilibria in system (3.3) can be analyzed as follows:

(a) Around the *unique positive equilibrium* $E_{7_1}(\hat{B}_1, \hat{D}_1, \hat{C}_1)$ (arising from Sub-cases : 3, 4, or 5 of Case II(a) and Sub-cases : 1, 2 or 5 of Case II(b) in the existence analysis), the system (3.3) always exhibits **local asymptotic stability**. Depending on the discriminant Δ of the characteristic polynomial:

- If $\Delta > 0$, all eigenvalues are real and negative, and E_7 is a **stable node**.
- If $\Delta < 0$, two complex conjugate eigenvalues have negative real parts, and

E_7 is a **stable focus**.

(b) When *two distinct sets of coexistence equilibria* exist simultaneously (Sub-case: 1 of Case II(a) and Sub-case : 4 of Case II(b)), denoted as $E_{7_2}(\hat{B}_2, \hat{D}_2, \hat{C}_2)$ and $E_{7_3}(\hat{B}_3, \hat{D}_3, \hat{C}_3)$, on one set $E_{7_2}(\hat{B}_2, \hat{D}_2, \hat{C}_2)$, $b3 > 0$, while on the other set $E_{7_3}(\hat{B}_3, \hat{D}_3, \hat{C}_3)$, $b3 < 0$. It implies that on one equilibrium branch E_{7_2}, the system (3.3) can exhibit stable and on the other branch E_{7_3}, an unstable behavior.

(c) In the case of *instantaneous equilibrium*, $E_{7_4}(\tilde{B}, \tilde{D}, \tilde{C})$ (Case III, when $\Delta = 0$), we have $\tilde{B} = B_1^* = B_2^* = \frac{b}{2a}$ and $b_3 = 0$. One of the eigenvalues becomes zero and the equilibrium point becomes a *saddle-node*. The point is classified as a saddle-node because two coexisting equilibria, E_{7_2} and E_{7_3} (when $\Delta > 0$ with $b > 0$ and $c > 0$), having opposite stability characteristics, collide and merge to form a degenerate equilibrium point E_{7_4} with a zero eigenvalue.

4.4.4 Hopf Bifurcation Analysis at E_7

We now examine the conditions under which a **Hopf bifurcation** occurs at the coexistence equilibrium $E_7(\hat{B}, \hat{D}, \hat{C})$. In general, a Hopf bifurcation occurs when a conjugate pair of complex eigenvalues of the linearized system crosses the imaginary axis, causing the equilibrium to lose stability and a periodic solution to emerge. Let μ be a system parameter that we treat as the bifurcation parameter (for example, a growth rate, interaction rate, or any model parameter whose variation can destabilize E_7). A Hopf bifurcation at E_7 will occur at some critical value $\mu = \mu_h$ if the following conditions are met:

1. **Pair of purely imaginary eigenvalues at μ_h (Hopf frequency condition):** At $\mu = \mu_h$, the Jacobian $J(E_7)$ has a pair of purely imaginary conjugate

eigenvalues $\lambda_{2,3} = \pm i\omega_h$ (with $\omega_h > 0$), while the third eigenvalue λ_1 is real and remains negative. In terms of the coefficients of the characteristic polynomial (**??**), this requirement translates to the critical equations:

$$b_1(\mu_h)b_2(\mu_h) - b_3(\mu_h) = 0 \ , \qquad b_2(\mu_h) > 0 \ , \tag{4.34}$$

where the second condition $b_2(\mu_h) > 0$ ensures $\omega_h^2 = b_2(\mu_h)$ is a positive real number (giving a non-zero imaginary frequency). Indeed, substituting $\lambda = i\omega$ into $\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0$ and separating real and imaginary parts yields the system:

$$-b_1\omega^2 + b_3 = 0 \ , \quad -\omega^3 + b_2\omega = 0 \ ,$$

for $\omega = \omega_h$. The second equation (imaginary part) gives $\omega_h^2 = b_2(\mu_h)$ (assuming $\omega_h \neq 0$), and plugging this into the first equation (real part) yields $b_3(\mu_h) = b_1(\mu_h)\omega_h^2 = b_1(\mu_h)b_2(\mu_h)$, i.e. $b_1b_2 - b_3 = 0$ at $\mu = \mu_h$. Thus, at the critical parameter μ_h, the characteristic polynomial factors as

$$\chi_{E_7}(\lambda)\Big|_{\mu=\mu_h} = (\lambda^2 + \omega_h^2)(\lambda + b_1(\mu_h)) \ ,$$

implying the eigenvalues are $\lambda_{2,3} = \pm i\omega_h$ and $\lambda_1 = -b_1(\mu_h)$. Since $b_1(\mu_h) > 0$ (from the Hopf condition and $b_3 > 0$), we have $\lambda_1 = -b_1(\mu_h) < 0$. In summary, at $\mu = \mu_h$ the equilibrium E_7 has a pair of purely imaginary eigenvalues $\pm i\omega_h$ and one negative real eigenvalue. This is the hallmark of a Hopf bifurcation: the linearization has a two-dimensional center subspace (associated with the imaginary pair) and one-dimensional stable subspace.

2. **Transversality condition (non-zero crossing speed):** It is not enough for eigenvalues to be imaginary at μ_h; we also require that as μ passes through μ_h,

the pair $\lambda_{2,3} = \alpha(\mu) \pm i\omega(\mu)$ actually crosses the imaginary axis with non-zero velocity. In other words, the real part $\alpha(\mu)$ of the complex eigenvalues should change sign at μ_h and $\frac{d\alpha}{d\mu}\big|_{\mu_h} \neq 0$. This transversality condition guarantees that the equilibrium's stability changes at μ_h and a small-amplitude limit cycle is born (for a supercritical Hopf) or dies (subcritical Hopf). A convenient algebraic condition for transversality can be obtained by differentiating the characteristic equation with respect to μ. One sufficient criterion is:

$$\frac{d}{d\mu}\Big[b_1(\mu)b_2(\mu) - b_3(\mu)\Big]\bigg|_{\mu=\mu_h} \neq 0 \ , \tag{4.35}$$

i.e. the parameter variation pushes the combination $b_1b_2 - b_3$ through zero at a non-zero rate. In practice, this ensures that the condition $b_1b_2 = b_3$ is satisfied only at the single point $\mu = \mu_h$, and for μ on either side of μ_h the sign of $b_1b_2 - b_3$ is different. Equivalently, one can show that $\frac{d\Re(\lambda_{2,3})}{d\mu}\big|_{\mu_h} \neq 0$ under condition (4.35), meaning the eigenvalues cross the imaginary axis transversely.

When both of the above conditions are met, a Hopf bifurcation occurs at $\mu = \mu_h$. For μ just below μ_h, we have $b_1b_2 - b_3 > 0$ and $\Re(\lambda_{2,3}) < 0$, so E_7 is stable. As μ increases past μ_h, the sign flips to $b_1b_2 - b_3 < 0$ and $\Re(\lambda_{2,3}) > 0$, causing E_7 to become unstable. At the threshold $\mu = \mu_h$, the linearization has a purely imaginary pair $\pm i\omega_h$, and by the Hopf bifurcation theorem, a periodic orbit (limit cycle) generally emanates from E_7. The precise nature of this limit cycle (e.g. supercritical vs. subcritical Hopf) depends on higher-order nonlinear terms and is determined by analyzing the Stuart–Landau normal form coefficients beyond the linear theory. Nonetheless, the above linearization analysis confirms that the model undergoes a Hopf bifurcation at E_7 when $\mu = \mu_h$ and conditions (4.34)–(4.35) are satisfied.

Chapter 5

Conclusion

In this thesis, we proposed and analyzed a novel mathematical model to investigate the dynamic interactions among brainwave synchronization, deep learning model accuracy, and cognitive load. The motivation was to establish a robust framework bridging cognitive neuroscience and artificial intelligence through nonlinear dynamical systems theory. By integrating real-time neural indicators, such as EEG-based synchrony, with a dynamically evolving deep learning performance influenced by cognitive effort, the model offers valuable insights into the co-regulation mechanisms of human–AI systems. This approach sheds light on how fluctuations in cognitive state can drive, and be driven by, adaptive machine learning behavior, paving the way for the development of more responsive and human-centric AI systems.

Key Contributions

The main findings and contributions of this research can be summarized as follows:

1. **Novel Integrated Dynamical Model:**

   We proposed a three-dimensional nonlinear system coupling brainwave synchronization $B(t)$, cognitive load $C(t)$, and deep learning model accuracy $D(t)$. The model captures feedback mechanisms inspired by cognitive neuroscience and machine learning theories, providing a novel framework linking human cognitive states with AI adaptability.

2. **Equilibrium and Stability Analysis:** We carried out a thorough analytical study of the model, identifying the equilibrium states and examining their stability. In particular, we found several biologically feasible boundary equilibria (where one or more processes dominate or vanish) as well as a unique coexistence equilibrium where all three variables coexist. By applying local stability criteria (including the Routh–Hurwitz stability test), we determined the conditions under which each equilibrium is stable.

3. **Bifurcation and Nonlinear Dynamics:** Our analysis uncovered critical nonlinear behaviors in the system. In varying a key model parameter associated with the deep learning component (denoted by λ in the model), we observed a transcritical bifurcation that causes stability to exchange between equilibria, indicating a tipping point where the dominance of cognitive load versus neural synchrony shifts. Furthermore, a Hopf bifurcation was identified, marking the emergence of sustained oscillatory dynamics under certain conditions. These oscillations can be interpreted as cycles of high and low cognitive engagement (e.g., periodic phases of mental fatigue and recovery) coupled with fluctuations in model performance, offering a plausible explanation for cognitive rhythmic patterns.

4. **Analytical Simplification and Multi-Stability:** To gain deeper insight, we employed a symbolic reduction technique to simplify the system's equations, ultimately reducing the model dynamics to a single quadratic equation. The

discriminant of this reduced equation provided clear criteria for the existence of a feasible coexistence state. Using this approach, we classified the system's behavior based on parameter conditions, and we discovered the possibility of multistability — scenarios in which multiple stable states (attractors) exist. In certain parameter regimes, the system can settle into qualitatively different stable outcomes (for example, a brainwave-synchronization–dominated state versus a cognitive-load–dominated state), depending on initial conditions. This analysis also indicated the potential for saddle-node bifurcations, further enriching the understanding of how abrupt transitions or the disappearance of stability can occur as cognitive or neural parameters change.

5. **Numerical Simulations and Validation:** We performed extensive numerical simulations (illustrated via phase portraits and time-series plots) to validate and visualize the model's theoretical predictions. The simulations corroborated the analytical results, showing the trajectory of the system under various initial conditions and parameter sets. Crucially, the numerical experiments demonstrated that the deep learning model's performance is maximized under moderate brainwave synchronization and manageable cognitive load. In contrast, the system's performance degrades when cognitive load is excessive or when brainwave synchrony is too low (desynchronized neural state). These simulation outcomes not only confirm the stability and oscillatory behavior predicted by our analysis, but also highlight an important practical insight: an optimal balance of cognitive load and neural synchrony is vital for maintaining high performance in human–AI cognitive loops.

Collectively, these contributions advance the state of knowledge at the intersection of neural dynamics and machine learning. From a practical perspective, the proposed model bridges human neurophysiological states with machine learning performance, laying groundwork for future neuro-adaptive systems. For example, our framework and findings are directly relevant to developing cognitive load-aware educational software, brain–computer interfaces (BCIs), and adaptive decision-support systems that respond in real time to a user's brain signals. In such applications, understanding the balance between a user's mental state and an AI's performance could inform strategies to adjust difficulty or feedback, ensuring the AI and human remain in an optimal zone of engagement.

Directions for Future Research

While this thesis has established a foundational framework, several opportunities remain for further research and development:

- **Incorporating Additional Physiological Signals:** Future models can be expanded by including other physiological indicators such as heart rate variability, pupil dilation, or skin conductance. These could provide a more complete and accurate understanding of cognitive states beyond EEG signals alone.

- **Model Calibration and Experimental Validation:** A key next step is to validate the model using real-world data. Collecting EEG recordings alongside cognitive load and task performance measurements would allow for parameter tuning and model refinement, ensuring better alignment with observed human–AI interactions.

- **Closed-Loop System Development:** The model could be applied in real-time neuroadaptive systems where an AI adjusts its responses based on a user's brain signals and cognitive load. Reinforcement learning techniques could further help the system learn optimal strategies for maintaining user engagement and minimizing cognitive fatigue.

- **Exploration of Global Dynamics:** Beyond local stability, future work could explore the model's behavior across broader parameter ranges. This includes studying bifurcations, possible chaotic dynamics, and the effects of feedback delays, providing a deeper understanding of the system's long-term behavior under changing conditions.

In summary, this research presents a mathematically grounded and practically relevant approach to bridging human cognitive dynamics with machine learning systems. By modeling the interplay between brainwave synchronization, cognitive load, and learning performance, it sets the stage for developing intelligent systems that are truly responsive to real-time human mental states. Such systems hold promise for enhancing learning, improving user experience, and building more effective human–AI collaborations.

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