SRM Institute of Science and Technology
Department of Mathematics
Booster Mathematics Class **Date:**

Subject Code	**21MAB201T**	**Faculty Name**	**Dr.S.Sangeetha**		
Subject Title	**Transforms and Boundary value problems**	**Faculty ID**	**100866**		
Name of the Student		**Student Register No.**		**Student Mobile No.**	
Objective: To solve one dimensional heat and one-dimensional wave problem and to find theFourier transform of the given function					

Topics Covered: 1D Heat, 1D Wave, Fourier and inverse Fourier transforms

Answer the following questions

1. i) A string is stretched and fastened to two points x = 0 and x = l apart. Motion is started by displacing the string into the form $y = k\,(lx - x^2)$ from which itis released at time t=0.Find the displacement of any point on the string at a distance of x from one end at time t. (10)

Answer:

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$\Rightarrow \dfrac{d^2y}{dt^2} = a^2 \dfrac{d^2y}{dx^2}, \quad a^2 = \dfrac{T}{m}$

The proper solution,

$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat)$ — (i)

i) $y(0,t) = 0$

ii) $y(l,t) = 0$

iii) $\dfrac{dy}{dt}(x,0) = 0$ (Initial velocity)

iv) $y(x,0) = kx(l-a)$ (Initial displacement)

BC (1) in (1)

$y(0,t) = A(C\cos pat + D\sin pat)$

$0 = A(C\cos pat + D\sin pat)$

$A = 0$

$y(x,t) = B\sin px\,(C\cos pat + D\sin pat)$

BC (ii) $y(l,t) = B\sin pl\,(C\cos pat + D\sin pat)$

$0 = B\sin pl\,(C\cos pat + D\sin pat)$

(Since $B \neq 0$) $\sin pl = 0 \Rightarrow pl = n\pi$

$p = \dfrac{n\pi}{l}, \quad n = 1, 2$

$y(x,t) = B\sin\left(\dfrac{n\pi x}{l}\right)\cdot\left(C\cos\dfrac{n\pi at}{l} + D\sin\dfrac{n\pi at}{l}\right)$

BC(iii) $\dfrac{dy}{dt}(x,t) = B\sin\left(\dfrac{n\pi x}{l}\right)\left[-C\sin\left(\dfrac{n\pi at}{l}\right)\cdot\left(\dfrac{n\pi a}{l}\right) + D\cos\left(\dfrac{n\pi at}{l}\right)\cdot\left(\dfrac{n\pi a}{l}\right)\right]$

$\dfrac{dy}{dt}(x,0) = B\sin\left(\dfrac{n\pi x}{l}\right)\cdot D\left(\dfrac{n\pi a}{l}\right)$

$$0 = B \sin\left(\frac{n\pi x}{l}\right) \cdot D\left(\frac{n\pi a}{l}\right)$$

$\Rightarrow$ $D = 0$

$$y(x,t) = BC \sin\left(\frac{n\pi x}{l}\right) \cdot \cos\left(\frac{n\pi a t}{l}\right), \quad n = 1, 2, \ldots$$

The general solution,

$$y(x,t) = \sum_{n=1}^{\infty} \lambda_n \sin\left(\frac{n\pi x}{l}\right) \cdot \cos\left(\frac{n\pi a t}{l}\right)$$

BC (iv)

$$y(x,0) = \sum_{n=1}^{\infty} \lambda_n \sin\left(\frac{n\pi x}{l}\right), \quad 0 < x < l$$

$$kx(l-x) = \sum_{n=1}^{\infty} \lambda_n \sin\left(\frac{n\pi x}{l}\right), \quad 0 < x < l$$

By Half-Range sine series,

$$\lambda_n = \frac{2}{l} \int_0^l kx(l-x) \cdot \sin\left(\frac{n\pi x}{l}\right) \cdot dx$$

$$= \frac{2}{l}\left[kx(l-x) \cdot \left(\frac{-\cos\left(\frac{n\pi x}{l}\right)}{n\pi/l}\right) - k(l-2x)\left(\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right) + k(-2)\left(\frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}\right)\right]_0^l$$

$$\lambda_n = \frac{4kl^2}{n^3\pi^3}\left(1 - (-1)^n\right).$$

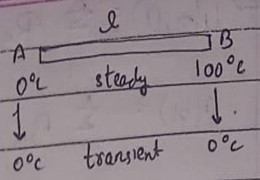
The required solution is,

$$\therefore\ y(x,t) = \sum_{n=1}^{\infty} \frac{4kl^2}{n^3\pi^3}\left(1-(-1)^n\right) \cdot \sin\left(\frac{n\pi x}{l}\right) \cdot \cos\left(\frac{n\pi a t}{l}\right)$$

1. ii)	A rod of length l has its ends A and B kept at 0°C and 100°C until steady state condition prevails. If the temperature of B is reduced suddenly to 0°Cand kept so while that of A is maintained, find the temperature u (x, t) at a distance x from A and at time t.	(10)

Answer:

Q) For steady state,



$\therefore u(x) = Ax + B$

Boundary condition,

i) $u(0) = 0$

ii) $u(l) = 100$

(i) $\Rightarrow u(0) = B$

$\therefore B = 0$

(ii) $\Rightarrow u(l) = Al + B$

or, $100 = Al + 0$

$\therefore A = \dfrac{100}{l}$

$\therefore u(x) = \dfrac{100x}{l}$

Transient state,

i) $u(0,t) = 0$

ii) $u(l,t) = 0$

iii) $u(x,0) = \dfrac{100x}{l}$

The proper solution is:

$\therefore u(x,t) = (A\cos px + B\sin px)\cdot e^{-p^2a^2t}$

(i) $\Rightarrow u(0,t) = A\cdot e^{-p^2a^2t}$

or, $0 = A\, e^{-p^2a^2t}$

$\therefore A = 0$

$\therefore u(x,t) = B\sin px\cdot e^{-p^2a^2t}$

(ii) $\Rightarrow u(l,t) = B\sin pl\; e^{-p^2a^2t}$

or, $0 = B\sin pl\, e^{-p^2a^2t}$

or, $\sin pl = 0$

or, $pl = n\pi$

$\therefore p = \dfrac{n\pi}{l}$

$\therefore\ u(x,t) = B \sin\left(\frac{n\pi x}{l}\right) \cdot e^{-n^2\pi^2\alpha^2 t/l}$

The general solution is :

$\therefore\ u(x,t) = B \cdot \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) \cdot e^{-n^2\pi^2\alpha^2 t/l}$

(iii) $\rightarrow u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)$

or, $\frac{100x}{l} = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)$

$\therefore\ B_n = \frac{2}{l} \int_0^l \frac{100x}{l} \sin\left(\frac{n\pi x}{l}\right) dx$

$= \frac{200}{l^2} \left[x \cdot \left(-\cos\left(\frac{n\pi x}{l}\right)\right) \cdot \frac{l}{n\pi} - \left(-\sin\left(\frac{n\pi x}{l}\right)\right) \cdot \frac{l^2}{n^2\pi^2} \right]_0^l$

$= \frac{200}{l^2} \left[\left(\frac{-l^2 (-1)^n}{n\pi} - 0 \right) - (0 - 0) \right]$

$= -\frac{200(-1)^n}{n\pi}$

$= \frac{200(-1)^{n+1}}{n\pi}$

The required solution is :

$\therefore\ u(x,t) = \sum_{n=1}^{\infty} \frac{200(-1)^{n+1}}{n\pi} \cdot \sin\left(\frac{n\pi x}{l}\right) \cdot e^{-n^2\pi^2\alpha^2 t/l}$

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Find the Fourier transform of $f(x) = \{1 - x\,^\wedge 2, |x| \le 1$
 $0, |x| > 1$

Answer:

$$\Rightarrow F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2) \cdot e^{-isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2) \cdot (\cos sx + i \sin sx) \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \underbrace{\cos sx}_{\text{even}} + \underbrace{i \sin sx}_{\text{odd}} - \underbrace{x^2 \cos sx}_{\text{even}} - \underbrace{i x^2 \sin sx}_{\text{odd}} \, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{1} \cos sx - x^2 \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s} - \left(x^2 \frac{\sin sx}{s} - 2x \cdot \left(\frac{-\cos sx}{s^2} \right) + 2 \left(\frac{-\sin sx}{s^3} \right) \right) \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin s}{s} - \frac{\sin s}{s} - 2 \frac{\cos s}{s^2} + 2 \frac{\sin s}{s^3} \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{2 \sin s - 2s \cos s}{s^3} \right]$$

$$= \frac{4}{\sqrt{2\pi}} \left[\frac{\sin s - s \cos s}{s^3} \right]$$

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<td>(10)</td>
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Reference Text Book: T. Veerarajan, Transforms and Partial Differential Equations, Tata McGraw Hill, 2012.

Signature of the Student **Signature of the Staff** **Signature of HOD/MATHS**