Unit - 3

Application of Partial Differential Equation

I. Classification of P.D.E of the second order.

$$A(x,y)\frac{d^2u}{dx^2} + B(x,y)\cdot\frac{d^2u}{dxdy} + C(x,y)\frac{d^2u}{dy^2} + f'\left(x,y,u,\frac{du}{dx},\frac{du}{dy}\right) = 0$$

Equation (1) is said to be linear (or) Quasi - linear, according as $f\left(x,y,u,\frac{du}{dx},\frac{du}{dy}\right)$ is linear or non- linear respectively.

Classification

The equation (1) is :

1) Elliptic if $B^2 - 4AC < 0$
2) Parabolic if $B^2 - 4AC = 0$
3) Hyperbolic if $B^2 - 4AC > 0$.

Problem :

1) ~~Classify the following PDE~~ :

Examples :-

1) Elliptic Type

i) $\frac{d^2 u}{dx^2} + \frac{d^2 u}{dy^2} = 0$ (Laplace Equation)

ii) $\frac{d^2 u}{dx^2} + \frac{d^2 u}{dy^2} = f(x,y)$ (Poisson equation)

2) Parabolic Type :

$\frac{d^2 u}{dx^2} = a \frac{du}{dt}$

(1-dim. heat equation)

3. Hyperbolic Type :

$\frac{d^2 u}{dx^2} = a \frac{d^2 u}{dt^2}$

(1-dim. Wave Equation)

Problem :

Classify the P.D.E of the following

1) $u_{xx} - 2u_{xy} + u_{yy} = 0$

(or)

$\frac{d^2 u}{dx^2} - 2\frac{du}{dxdy} + \frac{d^2 u}{dy^2} = 0$

$A = 1$, $B = -2$ and $C = 1$

$B^2 - 4AC = 4 - 4 = 0$, Parabolic

2) $x f_{xx} + y f_{yy} = 0$, $x > 0$, $y > 0$

$A = x$, $B = 0$ and $C = y$

$B^2 - 4AC = -4xy < 0$

Elliptic

3) $f_{xx} - 2 f_{xy} = 0$, $x > 0$, $y > 0$

$A = 1$, $B = -2$, $C = 0$

$B^2 - 4AC = 4 > 0$ Hyperbolic

4) $x^2 f_{xx} + (1 - y^2) f_{yy} = 0$

$A = x^2$, $B = 0$, $C = 1 - y^2$

$$\begin{aligned} B^2 - 4AC &= 0 - 4x^2(1 - y^2) \\ &= -4x^2 + 4x^2 y^2 \\ &= 4x^2(y^2 - 1) = -4x^2(1 - y^2) \end{aligned}$$

$x = 0$, parabolic

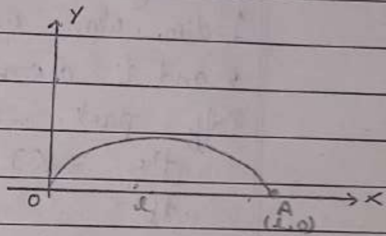
$x \neq 0$, $y = 1$, parabolic

$y < -1$ and $y > 1$, $(1 - y^2) < 0$, $B^2 - 4AC > 0 \rightarrow$ Hyperbolic

$-1 < y < 1$, $(1 - y^2) > 0 \rightarrow B^2 - 4AC < 0 \rightarrow$ Elliptic

→ Tranverse Vibrations of a streched Elastic String

Let us consider small transverse vibrations of an elastic string of length l, which is streched and then fixed at its two ends. Our aim is to study the displacement of the traverse vibration of the string when no external forces act on it. Take an end of the string as the Origin and the string in the equilibrium position as the x-axis and the line through the origin and perpendicular to the x-axis as y-axis.



To derive the D.E we make the following assumptions :

1> The tension T caused by steching the string before fixing it at the end points is constant at all points of the deflected string at all times.

2> T is so large that other external forces such as weight of the string and friction may be considered as negligible.

3> The string is homogeneous (ie. the mass of the string per unit length is constant) and perfectly elastic and so does not offer resistance to bending.

4> The slope of the deflection curve is small at all points and at all times.

The displacement $y(x,t)$ of the string at any time $t>0$ and at any point x is given by the equation

$$\frac{d^2y}{dt^2} = a^2 \frac{d^2y}{dx^2} \quad \text{--- (i), where } a^2 = \frac{T}{m} \text{ (positive)}$$

which is called as 1-dim wave equation.

→ Solution of the wave equation

Let $y(x,t) = X(x) \cdot T(t)$ be a solution of the 1-dim. wave eqn. Where X & T are functions of x and t alone respectively

Diff. part. w.r.to t and x twice,

$$\frac{d^2y}{dt^2} = XT'', \quad \frac{d^2y}{dx^2} = X''T.$$

$$\Rightarrow \quad XT'' = a^2X''T$$

$$\frac{X''}{X} = \frac{T''}{a^2T} = k, \text{ where } k \text{ is a constant}$$

$$X'' - kX = 0, \quad T'' - a^2kT = 0$$

The nature of the soln depends on the nature of the values of k

Case (1) k in positive let $k = p^2$

$$(D^2 - p^2)X = 0, \quad (D'^2 - a^2p^2)T = 0$$

$$m^2 - p^2 = 0 \qquad m' = \pm ap$$

$$m = \pm p$$

$$T = Ce^{apt} + D^{-apt}$$

$$X = Ae^{px} + Be^{-px}$$

Case (2) k is negative let $k = -p^2$

$(D^2 + p^2)X = 0$ $\qquad$ $(D'^2 + a^2p^2)T = 0$

$m^2 + p^2 = 0$ $\qquad$ $m' = \pm iap$

$X = A\cos px + B\sin px$ $\qquad$ $T = C\cos apt + D\sin apt$

Case (3). k is zero let $k = 0$

$D^2X = 0$ $\qquad$ $D^2T = 0$

$\frac{d^2X}{dx^2} = 0$ $\qquad$ $\frac{d^2T}{dt^2} = 0$

$\frac{dX}{dx} = A$ $\qquad$ $\frac{dT}{dt} = C$

$X = Ax + B$ $\qquad$ $T = Ct + D$

Since $y(x,t) = X, T$ is the solution of the wave equation, we have three possible solutions

The three possible solutions are

$y(x,t) = (Ae^{px} + Be^{-px})(Ce^{apt} + De^{-apt})$

$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat)$

$y(x,t) = (Ax + B)\ (Ct + D)$

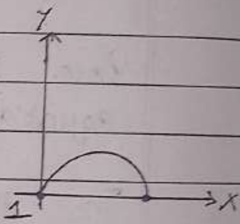
Since we deal with the vibration of an elastic string, $y(x,t)$, representing the displacement of the string at any point x, must be periodic in t.

Hence, the second solution which consists of periodic function in t is the proper solution of the problems on vibration of strings.

The arbitrary constants in the suitable solution are found out by using the boundary conditions of the problem.

→ Transverse vibration of the streched elastic body

$$\frac{d^2y}{dt^2} = a^2\frac{d^2y}{dx^2}, \quad \text{where} \quad a^2 = \frac{T}{M} \quad (\text{positive})$$



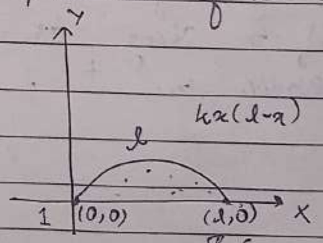
The possible solutions are :

$$y(x,t) = (Ae^{px} + Be^{-px})(Ce^{pat} + De^{-pat})$$

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat)$$

Type-I $\quad y(x,t) = (Ax + B)(Ct + D)$

Q no 1. A uniform string is streched and fastened to two points 'l' apart. Motion is started by displacing the string into the form of the curve $= kx(l-x)$ and then released from this position at time $t = 0$. Find the displacement of the string $y(x,t)$.



B C

I C

$\Rightarrow \dfrac{d^2y}{dt^2} = a^2 \dfrac{d^2y}{dx^2}, \quad a^2 = \dfrac{T}{m}$

The proper solution →

$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \quad - (1)$

i) $y(0,t) = 0$

ii) $y(l,t) = 0$

iii) $\dfrac{dy}{dt}(x,0) = 0$ (Initial velocity)

iv) $y(x,0) = kx(l-a)$ (Initial displacement)

BC (1) in (1)

$y(0,t) = A(C\cos pat + D\sin pat)$

$0 = A(C\cos pat + D\sin pat)$

$A = 0$

$y(x,t) = B\sin px(C\cos pat + D\sin pat)$

BC (ii) $y(l,t) = B\sin pl(C\cos pat + D\sin pat)$

$0 = B\sin pl(C\cos pat + D\sin pat)$

(Since $B \neq 0$) $\sin pl = 0 \longrightarrow pl = n\pi$

$p = \dfrac{n\pi}{l}, \quad n = 1, 2$

$y(x,t) = B\sin\left(\dfrac{n\pi x}{l}\right)\cdot\left(C\cos\dfrac{n\pi at}{l} + D\sin\dfrac{n\pi at}{l}\right)$

BC(iii) $\dfrac{dy}{dt}(x,t) = B\sin\left(\dfrac{n\pi x}{l}\right)\left[-C\sin\left(\dfrac{n\pi at}{l}\right)\cdot\left(\dfrac{n\pi a}{l}\right) + D\cos\left(\dfrac{n\pi at}{l}\right)\cdot\left(\dfrac{n\pi a}{l}\right)\right]$

$\dfrac{dy}{dt}(x,0) = B\sin\left(\dfrac{n\pi x}{l}\right)\cdot D\left(\dfrac{n\pi a}{l}\right)$

$$0 = B\sin\left(\frac{n\pi x}{l}\right)\cdot D\left(\frac{n\pi a}{l}\right)$$

$\Rightarrow$ $D = 0$

$$y(x,t) = BC\sin\left(\frac{n\pi x}{l}\right)\cdot\cos\left(\frac{n\pi at}{l}\right),\quad n=1,2,\dots$$

The general solution,

$$y(x,t) = \sum_{n=1}^{\infty}\lambda_n\sin\left(\frac{n\pi x}{l}\right)\cdot\cos\left(\frac{n\pi at}{l}\right)$$

BC (iv)

$$y(x,0) = \sum_{n=1}^{\infty}\lambda_n\sin\left(\frac{n\pi x}{l}\right),\quad 0<x<l$$

$$kx(l-x) = \sum_{n=1}^{\infty}\lambda_n\sin\left(\frac{n\pi x}{l}\right),\quad 0<x<l$$

By Half-Range sine series,

$$\lambda_n = \frac{2}{l}\int_0^l kx(l-x)\cdot\sin\left(\frac{n\pi x}{l}\right)\cdot dx$$

$$= \frac{2}{l}\left[kx(l-x)\cdot\left(\frac{-\cos\left(\frac{n\pi x}{l}\right)}{n\pi/l}\right) - k(l-2x)\left(\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right) + k(-2)\left(\frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}\right)\right]_0^l$$

$$\lambda_n = \frac{4kl^2}{n^3\pi^3}\left(1-(-1)^n\right).$$

The required solution is,

$$\therefore\ y(x,t) = \sum_{n=1}^{\infty}\frac{4kl^2}{n^3\pi^3}\left(1-(-1)^n\right)\cdot\sin\left(\frac{n\pi x}{l}\right)\cdot\cos\left(\frac{n\pi at}{l}\right)$$

② $f(x) = k\sin^3\left(\frac{\pi x}{l}\right)$.

⇒ Solution:

1) $y(0,t) = 0$
2) $y(l,t) = 0$ } (displacement at end points)

3) $\frac{dy}{dt}(x,0) = 0$ (Initial velocity)

4) $y(x,0) = k\sin^3\left(\frac{\pi x}{l}\right)$ (Initial displacement)

The proper solution,

$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat)$

(1) BC — $A = 0$

(2) — $p = \frac{n\pi}{l}$ $\left\{\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}\right\}$

(3) — $D = 0$

The general solution,

$y(x,t) = \sum_{n=1}^{\infty} \lambda_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right)$

(4) BC

$y(x,0) = \sum_{n=1}^{\infty} \lambda_n \sin\left(\frac{n\pi x}{l}\right)$

$k\sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} \lambda_n \sin\left(\frac{n\pi x}{l}\right)$

$\frac{k}{4}\left(3\sin\left(\frac{\pi x}{l}\right) - \sin\left(\frac{3\pi x}{l}\right)\right) = \sum_{n=1}^{\infty} \lambda_n \sin\left(\frac{n\pi x}{l}\right)$

$= \lambda_1 \sin\frac{\pi x}{l} + \lambda_2 \sin\frac{2\pi x}{l} + \lambda_3 \sin\frac{3\pi x}{l}$

Comparing the like terms on both sides,

$\lambda_2 = 0, \lambda_4, \lambda_5, \ldots = 0$

$\lambda_1 = \frac{3k}{4}, \lambda_3 = \frac{-k}{4}$

The required solution is,

$$y(x,t) = \frac{3k}{4} \sin\left(\frac{\pi x}{l}\right) \cdot \cos\left(\frac{\pi a t}{l}\right) - \frac{k}{4} \sin\left(\frac{3\pi x}{l}\right) \cdot \cos\left(\frac{3\pi a t}{l}\right)$$

Type - II

Non-zero initial velocity

Q A uniform string of length '$2l$' fastened at both ends is disturbed from its position of equilibrium by imparting to each of its points and initial velocity of magnitude $k(2lx - x^2)$. Find the displacement $y(x,t)$.

⇒ Solution :-

1) $y(0,t) = 0$

2) $y(2l, t) = 0$

3) $y(x, 0) = 0$

4) $\frac{dy}{dt}(x,0) = f(x)$

The proper solution,

$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat)$

(1) $\Rightarrow$ $y(0,t) = A\ (C\cos pat + D\sin pat)$

$0 = A$

$y(x,t) = B\cdot\sin px\ (C\cos pat + D\sin pat)$

(2) $\rightarrow$ $y(2l,t) = B\sin p(2l)\ (\quad)$

$0 = B\sin p(2l)\ (\quad)$

$0 = \sin p(2l)$

$p(2l) = n\pi,\ n = 1, 2, \ldots$

$p = \dfrac{n\pi}{2l}$

$$y(x,t) = B\sin\left(\frac{n\pi x}{2l}\right)\cdot\left(C\cos\frac{n\pi at}{2l} + D\sin\frac{n\pi at}{2l}\right)$$

(3) $y(x,0) = B\sin\left(\dfrac{n\pi x}{2l}\right)\cdot C$

$0 = B\sin\left(\dfrac{n\pi x}{2l}\right)\cdot C$

$C = 0$

$$y(x,t) = BD\sin\left(\frac{n\pi x}{2l}\right)\cdot \sin\left(\frac{n\pi at}{2l}\right),\ n = 1, 2, \ldots$$

The general solution,

$$y(x,t) = \sum_{n=1}^{\infty} \lambda_n \sin\left(\frac{n\pi x}{2l}\right)\cdot \sin\left(\frac{n\pi at}{2l}\right)$$

(4), $\frac{dy}{dt}(x,t) = \sum_{n=1}^{\infty} \lambda_n \sin\left(\frac{n\pi x}{2l}\right) \cdot \cos\left(\frac{n\pi at}{2l}\right) \cdot \left(\frac{n\pi a}{2l}\right)$

$$\frac{dy}{dt}(x,t) = \sum_{n=1}^{\infty} \lambda_n \left(\frac{n\pi a}{2l}\right) \sin\left(\frac{n\pi x}{2l}\right) \cdot \cos\left(\frac{n\pi at}{2l}\right)$$

$t=0$, $k(2lx - x^2) = \sum_{n=1}^{\infty} \lambda_n \left(\frac{n\pi a}{2l}\right) \cdot \sin\left(\frac{n\pi x}{2l}\right)$

$$f(x) := \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{2l}\right)$$

$$\lambda_n \left(\frac{n\pi a}{2l}\right) = B_n = \frac{2}{2l} \int_0^{2l} k(2lx - x^2) \cdot \sin\left(\frac{n\pi x}{2l}\right) dx$$

$$\lambda_n \left(\frac{n\pi a}{2l}\right) = \frac{1}{l}\Bigg[(2klx - kx^2) \cdot \left[-\cos\left(\frac{n\pi x}{2l}\right)\right] \cdot \frac{2l}{n\pi} - (2kl - 2kx) \cdot \left(-\sin\left(\frac{n\pi x}{2l}\right)\right)\left(\frac{2l}{n\pi}\right)^2 + (-2k) \cdot \left(\cos\left(\frac{n\pi x}{2l}\right)\left(\frac{2l}{n\pi}\right)^3\right)\Bigg]_0^{2l}$$

$$= \frac{1}{l}\left[\left(0 - 0 - 2k \cdot (-1)^n \cdot \frac{8l^3}{n^3\pi^3}\right) - \left(0 - 0 - 2k \cdot \left(\frac{8l^3}{n^3\pi^3}\right)\right)\right]$$

$$= \frac{1}{l}\left[-2k(-1)^n \cdot \frac{8l^3}{n^3\pi^3} + 2k \cdot \frac{8l^3}{n^3\pi^3}\right]$$

$$= \frac{8l^2}{n^3\pi^3}\left[2k - 2k(-1)^n\right]$$

$$= \frac{16kl^2}{n^3\pi^3}\left[1 - (-1)^n\right]$$

Or, $\lambda_n \left(\frac{n\pi a}{2l}\right) = \frac{16kl^2}{n^3\pi^3}\left[1 - (-1)^n\right]$

$\therefore$ $\lambda_n = \frac{32kl^3}{n^4\pi^4 a}\left[1 - (-1)^n\right]$

The required solution is,

$$\therefore y(x,t) = \sum_{n=1}^{\infty} \frac{32kl^3}{n^4\pi^4 a} (1-(-1)^n) \cdot \sin\left(\frac{n\pi x}{2l}\right) \cdot \sin\left(\frac{n\pi at}{2l}\right)$$

Q.no 2) A tightly stretched string of length 'l' fastened at both ends is disturbed from its equilibrium position with an ~~ex~~ initial velocity $v_0 \sin^3\left(\frac{\pi x}{l}\right)$.

Find its displacement.

$\Rightarrow$ (1) $y(0,t) = 0$

(2) $y(l,t) = 0$

(3) $y(x,0) = 0$

(4) $\frac{dy}{dt}(x,0) = v_0 \sin^3\left(\frac{\pi x}{l}\right)$.

The proper solution

$\therefore y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat)$

(1) $y(0,t) = A.(C\cos pat + D\sin pat)$

or, $0 = A$ (")

$\therefore A = 0$

$y(x,t) = B\sin px\,(C\cos pat + D\sin pat)$

(2) $y(l,t) = B\sin pl\,(C\cos pat + D\sin pat)$

or, $0 = B\sin pl$ (")

or, $0 = \sin pl$

or, $pl = n\pi$

$\therefore p = \frac{n\pi}{l}$

$\therefore y(x,t) = B\sin\left(\frac{n\pi x}{l}\right) \cdot \left(C\cos\left(\frac{n\pi at}{l}\right) + D\sin\left(\frac{n\pi at}{l}\right)\right)$

(3) $y(x,0) = B \sin\left(\frac{n\pi x}{l}\right) \cdot C$

$0 = C$

$\therefore y(x,t) = BD \sin\left(\frac{n\pi x}{l}\right) \cdot \sin\left(\frac{n\pi at}{l}\right)$, $n = 1, 2, \ldots$

$\therefore$ The general solution is

$$\therefore y(x,t) = \sum_{n=1}^{\infty} \lambda_n \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi at}{l}\right).$$

or, $$\frac{dy}{dt}(x,t) = \sum_{n=1}^{\infty} \lambda_n \sin\left(\frac{n\pi x}{l}\right) \cdot \cos\left(\frac{n\pi at}{l}\right) \cdot \left(\frac{n\pi a}{l}\right)$$

$$\therefore \frac{dy}{dt}(x,t) = \sum_{n=1}^{\infty} \lambda_n \left(\frac{n\pi a}{l}\right) \cdot \sin\left(\frac{n\pi x}{l}\right) \cdot \cos\left(\frac{n\pi at}{l}\right)$$

(4) $$\frac{dy}{dt}(x,0) = \sum_{n=1}^{\infty} \lambda_n \left(\frac{n\pi a}{l}\right) \cdot \sin\left(\frac{n\pi x}{l}\right)$$

or, $$v_0 \sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} \lambda_n \left(\frac{n\pi a}{l}\right) \cdot \sin\left(\frac{n\pi x}{l}\right)$$

$$\therefore f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\therefore \lambda_n\left(\frac{n\pi a}{l}\right) = B_n = \frac{2}{l}\int_0^l v_0 \sin^3\left(\frac{\pi x}{l}\right) dx$$

or $$\lambda_n\left(\frac{n\pi a}{l}\right) = \frac{2v_0}{4l}\int_0^l \left[3\sin\left(\frac{\pi x}{l}\right) - \sin\left(\frac{3\pi x}{l}\right)\right] dx$$

$$= \frac{v_0}{2l}\left[3\cdot\left(-\cos\left(\frac{\pi x}{l}\right)\right)\cdot\frac{l}{\pi} + \cos\left(\frac{3\pi x}{l}\right)\cdot\frac{l}{3\pi}\right]_0^l$$

$$= \frac{v_0}{2l}\left[\left(\frac{3l}{\pi} - \frac{l}{3\pi}\right) - \left(-\frac{3l}{\pi} + \frac{l}{3\pi}\right)\right]$$

$$= \frac{v_0}{2l} \cdot \left[\frac{2\cdot 3l}{\pi} - 2\frac{l}{3\pi}\right].$$

or, $\lambda_n\left(\frac{n\pi a}{l}\right) = v_0\left(\frac{8}{\pi} - \frac{1}{3\pi}\right)$

or $\lambda_n\left(\frac{n\pi a}{l}\right) = v_0\left(\frac{9-1}{3\pi}\right)$

$\therefore \lambda_n = \frac{8 v_0 l}{3 n \pi^2 a}$

The required solution is

$\therefore y(x,t) = \sum_{n=1}^{\infty} \frac{8 v_0 l}{3 n \pi^2 a} \sin\left(\frac{n\pi x}{l}\right) \cdot \sin\left(\frac{n\pi a t}{l}\right)$

$$\frac{v_0}{4}\left(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right) = B_1 \sin\left(\frac{\pi x}{l}\right) + B_2 \sin\left(\frac{2\pi x}{l}\right) + B_3 \sin\left(\frac{3\pi x}{l}\right) + \dots$$

Comparing the like terms on both sides,

$B_2 = 0, \; B_4, B_5 \dots = 0$

$\therefore B_1 = \frac{3v_0}{4}, \quad B_3 = -\frac{v_0}{4}$

~~The required solution is~~

$\therefore$ ~~$y(x,t)$~~

or, $\lambda_1\left(\frac{\pi a}{l}\right) = \frac{3v_0}{4}, \quad \lambda_3\left(\frac{3\pi a}{l}\right) = -\frac{v_0}{4}$

$\therefore \lambda_1 = \frac{3v_0 l}{4\pi a}, \quad \lambda_3 = \frac{-v_0 l}{12\pi a}$

The required solution,

$$y(x,t) = \lambda_1 \sin\left(\frac{\pi x}{l}\right) \cdot \sin\left(\frac{\pi a t}{l}\right) + \lambda_3 \sin\left(\frac{3\pi x}{l}\right) \cdot \sin\left(\frac{3\pi a t}{l}\right)$$

$$= \frac{3v_0 l}{4\pi a} \sin\left(\frac{\pi x}{l}\right) \cdot \sin\left(\frac{\pi a t}{l}\right) + \frac{v_0 l}{12\pi a} \sin\left(\frac{3\pi x}{l}\right) \cdot \sin\left(\frac{3\pi a t}{l}\right)$$

Q. A string is stretched b/w two fixed pts at a distance of 60 cm and the points of the string are given initial velocities v, where

$$v = \frac{\lambda x}{30}, \text{ in } 0 < x < 30$$

$$= \frac{\lambda}{30}(60 - x), \text{ in } 30 < x < 60$$

x being the distance from an end point. Find the displ. of the string at any time.

$\Rightarrow$ $\frac{d^2y}{dt^2} = a^2 \frac{d^2y}{dx^2}$, $a^2 = \frac{T}{m}$

The proper solutions

$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat)$

i) $y(0,t) = 0$

ii) $y(60,t) = 0$

~~iii) $\frac{dy}{dt}(60,0) = 0$~~

iii) $y(x,0) = 0$

iv) $\frac{dy}{dt}(x,0) = f(x)$

(i) $\Rightarrow$ $y(0,t) = A(C\cos pat + D\sin pat)$

$\therefore 0 = A$

$y(x,t) = B\sin px(C\cos pat + D\sin pat)$

(ii) $\Rightarrow$ $y(60,t) = B\sin p(60)(C\cos pat + D\sin pat)$

or, $0 = B\sin p(60)($ " " $)$

or, $0 = \sin p(60)$

or, $p(60) = n\pi$

$\therefore p = \frac{n\pi}{60}$

$\therefore\; y(x,t) = B\sin\left(\frac{n\pi x}{60}\right)\left(C\cos pat + D\sin pat\right)$

(iii) $\rightarrow y(x,0) = B\sin\left(\frac{n\pi x}{60}\right)\cdot C$

or, $0 = B\sin\left(\frac{n\pi x}{60}\right)\cdot C$

$\therefore\; C = 0$

$\therefore\; y(x,t) = BD\sin\left(\frac{n\pi x}{60}\right)\cdot\sin\left(\frac{n\pi a t}{60}\right),\; n=1,2,\ldots$

The general solution,

$\therefore\; y(x,t) = \sum_{n=1}^{\infty}\lambda_n\sin\left(\frac{n\pi x}{60}\right)\cdot\sin\left(\frac{n\pi a t}{60}\right)$

$\frac{dy}{dt}(x,t) = \sum_{n=1}^{\infty}\lambda_n\sin\left(\frac{n\pi x}{60}\right)\cdot\cos\left(\frac{n\pi a t}{60}\right)\cdot\left(\frac{n\pi a}{60}\right)$

$\frac{dy}{dt}(x,t) = \sum_{n=1}^{\infty}\lambda_n\left(\frac{n\pi a}{60}\right)\cdot\sin\left(\frac{n\pi x}{60}\right)\cdot\cos\left(\frac{n\pi a t}{60}\right)$

$t=0$

(iv) $\rightarrow f(x) = \sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi x}{60}\right)$

$\therefore\; \lambda_n\left(\frac{n\pi a}{60}\right) = B_n = \frac{2}{60}\int_0^{60} f(x)\cdot\sin\left(\frac{n\pi x}{60}\right)\cdot dx$

$= \frac{1}{30}\left[\int_0^{30}\frac{\lambda x}{30}\cdot\sin\left(\frac{n\pi x}{60}\right)dx + \int_{30}^{60}\frac{\lambda}{30}(60-x)\cdot\sin\left(\frac{n\pi x}{60}\right)dx\right]$

$= \frac{\lambda}{900}\left[\int_0^{30}x\cdot\sin\left(\frac{n\pi x}{60}\right)dx + \int_{30}^{60}(60-x)\cdot\sin\left(\frac{n\pi x}{60}\right)dx\right]$

$= \frac{\lambda}{900}\left[x\cdot\left[-\cos\left(\frac{n\pi x}{60}\right)\right]\frac{60}{n\pi} - \left[0-\sin\left(\frac{n\pi x}{60}\right)\right]\cdot\left(\frac{60}{n\pi}\right)^2\right]_0^{30} +$

$\left[(60-x)\cdot\left(-\cos\left(\frac{n\pi x}{60}\right)\right)\cdot\frac{60}{n\pi} - (-1)\left(-\sin\left(\frac{n\pi x}{60}\right)\right)\cdot\left(\frac{60}{n\pi}\right)^2\right]_{30}^{60}$

$$= \frac{\lambda}{900}\left[\left\{\left(-30.\cos\left(\frac{n\pi}{2}\right).\frac{60}{n\pi} + \sin\left(\frac{n\pi}{2}\right).\frac{3600}{n^2\pi^2}\right) - (0-0)\right\}\right.$$
$$\left. + \left\{(0-0) - \left(-30.\cos\left(\frac{n\pi}{2}\right).\frac{60}{n\pi} - \sin\left(\frac{n\pi}{2}\right).\frac{3600}{n^2\pi^2}\right)\right\}\right]$$

$$= \frac{\lambda}{900}\left[\frac{-1800}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{3600}{n^2\pi^2}.\sin\left(\frac{n\pi}{2}\right) + \frac{1800}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{3600}{n^2\pi^2}.\sin\left(\frac{n\pi}{2}\right)\right]$$

or, $$\lambda_n\left(\frac{n\pi a}{60}\right) = \frac{\lambda}{900}\frac{7200}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$\therefore \lambda_n = \frac{480000}{n^3\pi^3 a}\sin\left(\frac{n\pi}{2}\right)$$

The required solution is,

$$\therefore y(x,t) = \sum_{n=1}^{\infty}.\frac{480000}{n^3\pi^3 a}.\sin\left(\frac{n\pi}{2}\right).\sin\left(\frac{n\pi x}{60}\right).\sin\left(\frac{n\pi a t}{60}\right)$$

Date ______
Page ______

One dimensional heat flow

$$\frac{du}{dt} = \alpha^2 \frac{d^2u}{dx^2}, \text{ where } \alpha^2 = \frac{k}{c\rho} \frac{cm^2}{sec} \text{ (positive)}$$

The possible solutions are,

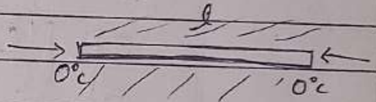
$u(x,t) = (Ae^{px} + Be^{-px}) e^{p^2\alpha^2 t}$

$u(x,t) = (A\cos px + B\sin px) e^{-p^2\alpha^2 t}$ (Transient state)

$u(x,t) = Ax + B$ (Steady state)

Qno1) A uniform bar of length 'l' through which heat flows is insulated at its sides. The ends are kept at 0 temperature, if the initial temperature at the interior points of the bar is given by $k(lx - x^2)$, $0 < x < l$. Find the temperature distribution of bar at time t.

⇒ ~~The proper solution is~~



1) $u(0,t) = 0$

2) $u(l,t) = 0$

3) $u(x,0) = k(lx - x^2)$ (initial temp)

The proper solution is:

$\therefore u(x,t) = (A\cos px + B\sin px) e^{-p^2\alpha^2 t}$

(1) ⇒ $u(0,t) = A \cdot e^{-p^2\alpha^2 t}$

or, $0 = A \cdot e^{-p^2\alpha^2 t}$

$\therefore A = 0$

$u(x,t) = B\sin px \, e^{-p^2\alpha^2 t}$

(2) ⇒ $u(l,t) = B\sin pl \, e^{-p^2\alpha^2 t}$

or, $0 = B\sin pl \, e^{-p^2\alpha^2 t}$

or, $\sin pl = 0$

or, $pl = n\pi$

$\therefore p = \frac{n\pi}{l}$

$$\therefore\ u(x,t) = B \sin\left(\frac{n\pi x}{l}\right) \cdot e^{\left(\frac{-n^2\pi^2\alpha^2 t}{l^2}\right)}, \quad n = 1, 2, 3 \ldots$$

The general solⁿ is:

$$\therefore\ u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) \cdot e^{\frac{-n^2\pi^2\alpha^2 t}{l^2}}$$

$$(3) \Longrightarrow u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) \cdot 1$$

or, $$k(lx - x^2) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\therefore\ B_n = \frac{2}{l} \int_0^l k(lx - x^2) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2k}{l}\left[(lx - x^2)\left(-\cos\left(\frac{n\pi x}{l}\right)\frac{l}{n\pi}\right) - (l - 2x)\cdot\left(-\sin\left(\frac{n\pi x}{l}\right)\cdot\frac{l^2}{n^2\pi^2}\right) + (-2)\cos\left(\frac{n\pi x}{l}\right)\cdot\frac{l^3}{n^3\pi^3}\right]_0^l$$

$$= \frac{2k}{l}\left[\left(0 - 0 - 2\cdot(-1)^n\cdot\frac{l^3}{n^3\pi^3}\right) - \left(0 - 0 - 2\,\frac{l^3}{n^3\pi^3}\right)\right]$$

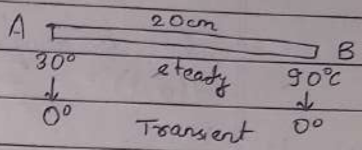
$$= \frac{2k}{l}\left[-\frac{2(-1)^n l^3}{n^3\pi^3} + \frac{2l^3}{n^3\pi^3}\right]$$

$$= \frac{4kl^2}{n^3\pi^3}\left[1 - (-1)^n\right]$$

The required solution is:

$$\therefore\ u(x,t) = \sum_{n=1}^{\infty} \frac{4kl^2}{n^3\pi^3}\left[1 - (-1)^n\right] \cdot \sin\left(\frac{n\pi x}{l}\right) \cdot e^{\left(\frac{-n^2\pi^2\alpha^2 t}{l^2}\right)}$$

Q.no. A rod of length 20 cm has its ends 'A' and 'B' kept at $30°c$ and $90°c$ respectively until, steady state conditions provide. If the temperature at each end is then suddenly reduced to $0°c$ and maintained so. Find the temperature distribution in the rods.



~~(1) $u(0,t)$~~ = For steady state,

$u(x) = Ax + B$

Boundary condition,

i) $u(0) = 30$

ii) $u(20) = 90$

$\Rightarrow$ $u(0) = B \Rightarrow 30 = B$

$\Rightarrow$ $u(20) = 20A + B$

or, $90 = 20A + 30$

$\therefore$ $A = 3$

$\therefore$ $u(x) = 3x + 30$

Transient state,

(1) $u(0,t) = 0$

(2) $u(20,t) = 0$

(3) $u(x,0) = 3x + 30$ (initial temp.)

The proper solution,

$\therefore$ $u(x,t) = (A\cos px + B\sin px)\, e^{-p^2\alpha^2 t}$

(1) $\Rightarrow$ $u(0,t) = A \cdot e^{-p^2\alpha^2 t}$

or $0 = A \cdot e^{-p^2\alpha^2 t}$

$\therefore$ $A = 0$

$\therefore$ $u(x,t) = B\sin px\, e^{-p^2\alpha^2 t}$

(2) $\Rightarrow$ $u(20,t) = B \sin p20 \cdot e^{-p^2\alpha^2 t}$

or, $0 = B \sin p20 \, e^{-p^2\alpha^2 t}$

$\therefore \; p = \frac{n\pi}{20}$

$\therefore \; u(x,t) = B \sin \frac{n\pi x}{20} \cdot e^{\frac{-n^2\pi^2\alpha^2 t}{400}}, \; n = 1, 2, 3 \ldots$

The general sol^n is,

$\therefore \; u(x,t) = \sum_{n=1} B_n \sin\left(\frac{n\pi x}{20}\right) e^{\frac{-n^2\pi^2\alpha^2 t}{400}}$

(3) $\Rightarrow$ $u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{20}\right)$

or, $3x + 30 = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{20}\right)$

$$\therefore \; B_n = \frac{2}{20} \int_0^{20} (3x+30) \cdot \sin\left(\frac{n\pi x}{20}\right) dx$$

$$= \frac{1}{10} \left[(3x+30)\left(-\cos\left(\frac{n\pi x}{20}\right)\right) \cdot \frac{20}{n\pi} - 3 \cdot \left(-\sin\left(\frac{n\pi x}{20}\right)\right) \cdot \frac{400}{n^2\pi^2} \right]_0^{20}$$

$$= \frac{1}{10} \left[\left(90 \cdot -(-1)^n \cdot \frac{20}{n\pi} - 0 \right) - \left(\frac{30(-1)20}{n\pi} - 0 \right) \right]$$

$$= \frac{1}{10} \left[\frac{600}{n\pi} - \frac{1800(-1)^n}{n\pi} \right]$$

$$= \frac{1}{10} \cdot \frac{600}{n\pi} [1 - 3(-1)^n]$$

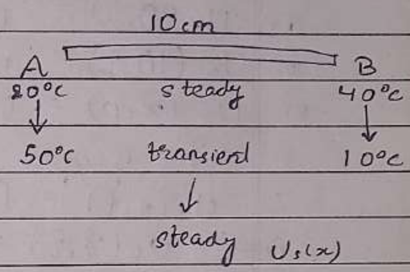
$$= \frac{60}{n\pi} [1 - 3(-1)^n]$$

The required sol^n is:

$$\therefore \; u(x,t) = \sum_{n=1}^{\infty} \frac{60}{n\pi} [1 - 3(-1)^n] \cdot \sin\left(\frac{n\pi x}{20}\right) \cdot e^{\frac{-n^2\pi^2\alpha^2 t}{400}}$$

Problems with non-zero boundary values.

Qn. A rod 10 cm long has its ends 'A' and 'B' kept at $20°$ and $40°$ respectively until steady state condn provide. The temperature at A is suddenly raised to $50°C$ and at the same instant B is lowered to $10°C$. Find the subsiquent temperature of the rod.



$\Rightarrow$ Steady state

$\therefore$ $u(x) = Ax + B$

i) $u(0) = 20$,

ii) $u(10) = 40$

$\therefore$ $u(0) = B$

$\Rightarrow$ $B = 20$

$\therefore$ $u(10) = 10A + B$

or, $40 = 10A + 20$

$\therefore$ $A = 2$

$\therefore$ $u(x) = 2x + 20$

$\Rightarrow$ Transient state

1) $u(0,t) = 50$

2) $u(10,t) = 10$

3) $u(x,0) = 2x + 20$

The soln.

$\therefore$ $u(x,t) = U_t(x,t) + U_s(x)$

$U_s(0) = 50$, $U_s(10) = 10$

$U_s(x) = Cx + D$

or $U_s(0) = D$

$\therefore D = 50$

or, $U_s(10) = 10C + D$

or, $10 = 10C + 50$

$\therefore C = -4$

$\therefore U_s(x) = -4x + 50$

$U_t(x,t) = U(x,t) - U_s(x)$

(1) $U_t(0,t) = U(0,t) - U_s(0) = 50 - 50 = 0$

(2) $U_t(10,t) = U(10,t) - U_s(10) = 10 - 10 = 0$

(3) $U_t(x,0) = U(x,0) - U_s(x) = 6x - 30$

The proper soln is:

$\therefore U_t(x,t) = (A\cos px + B\sin px)\, e^{-p^2\alpha^2 t}$

1) $\Rightarrow U_t(0,t) = A \cdot e^{-p^2\alpha^2 t}$

or, $0 = A \cdot e^{-p^2\alpha^2 t}$

$\therefore A = 0$

$\therefore U_t(x,t) = B\sin px\, e^{-p^2\alpha^2 t}$

2) $\Rightarrow U_t(10,t) = B\sin p(10) \cdot e^{-p^2\alpha^2 t}$

or, $0 = B\sin 10p \cdot e^{-p^2\alpha^2 t}$

or, $\sin 10p = 0$

or, $10p = n\pi$

$\therefore p = \frac{n\pi}{10}$

$\therefore U_t(x,t) = B\sin\left(\frac{n\pi x}{10}\right) \cdot e^{-\frac{n^2\pi^2\alpha^2 t}{100}}$

The general solution is:

$\therefore U_t(x,t) = \sum_{n=1}^{\infty} B_n \cdot \sin\left(\frac{n\pi x}{10}\right) \cdot e^{-\frac{n^2\pi^2\alpha^2 t}{100}}$

(3) $\Longrightarrow$ $u_t(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{10}\right) \cdot 1$

or, $6x-30 = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{10}\right)$

$\therefore$ $B_n = \frac{2}{10} \int_0^{10} (6x-30) \cdot \sin\left(\frac{n\pi x}{10}\right) dx$

$= \frac{1}{5} \left[(6x-30) \cdot \left(-\cos\left(\frac{n\pi x}{10}\right)\right) \cdot \frac{10}{n\pi} - 6 \left(-\sin\left(\frac{n\pi x}{10}\right)\right) \frac{100}{n^2\pi^2} \right]_0^{10}$

$= \frac{1}{5} \left[\left(-30 \cdot (-1)^n \frac{10}{n\pi} - 0\right) - \left((-30)(-1) \cdot \frac{10}{n\pi} - 0\right) \right]$

$= \frac{1}{5} \left[\frac{-300 (-1)^n}{n\pi} - \frac{300}{n\pi} \right]$

$= \frac{-60}{n\pi} [1 + (-1)^n]$

$\therefore$ $u_t(x,t) = \sum_{n=1}^{\infty} \frac{-60}{n\pi} [1 + (-1)^n] \cdot \sin \frac{n\pi x}{10} e^{\frac{-n^2\pi^2 a^2 t}{100}}$

The required solⁿ is:

$\therefore$ $u(x,t) = -4x + 50 + \sum_{n=1}^{\infty} \frac{-60}{n\pi} [1 + (-1)^n] \cdot \sin\left(\frac{n\pi x}{10}\right) e^{-\frac{n^2\pi^2 a^2 t}{100}}$

One - Dimensional heat flow.

⇒ The following empirical laws are taken as the basis of investigation.

(1) Heat flows from a higher to lower temp.

(2) The amount of heat required to produce a given temp. change in a body is proportional to the mass of the body and to the temp. change. This constant of proportionality is known as the specific heat (c) of the conducting material.

(3) The rate at which heat flows through an area is proportional to the area and to the temp. gradient normal to the area. This constant of proportionality is known as the thermal conductivity (K) of the material.

$\frac{du}{dt} = \alpha^2 \frac{d^2u}{dx^2}$ is the heat energy (or) 1. dim. diffusion energy

where, $\alpha^2 = \frac{k}{c\rho}$ $\frac{cm^2}{sec}$ (positive)

$\frac{k}{c\rho}$ is called diffusivity $\left(\frac{cm^2}{sec}\right)$ of the substance

where, ρ - density of the material

c - specific heat

k - thermal conductivity of the material

The possible solutions are,

$\therefore$ $u(x,t) = (Ae^{px} + Be^{-px})e^{p^2\alpha^2 t} \longrightarrow (1)$

$\therefore$ $u(x,t) = (A\cos px + B\sin px)\, e^{-p^2\alpha^2 t} \longrightarrow (2)$ Transient state

$\therefore$ $u(x,t) = Ax + B \longrightarrow (3)$ Steady state

Here u is a decreasing funⁿ as the temp., flows from higher to lower temperature.

As we are dealing with heat conduction problem u decreases as t increases

$\therefore$ proper solⁿ is (2) (transient heat flow problems)

For steady state $\rightarrow$ temp. does not vary with time, the proper solⁿ is (3).

Date
Page

Q. Solve the wave equation
$u_{tt} = 4u_{xx}$; $0 < x < \pi$, $t > 0$
$u(0,t) = 0 = u(\pi, t)$, $t \geq 0$
$u(x,0) = \sin x - 2\sin 3x$; $u_t(x,0) = 0$, $0 \leq x < \pi$
Hence find $u\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\Rightarrow$ i) $u(0,t)$.
The proper solution is :
$\therefore$ $u(0,t) = 0$
or, $(A\cos px + B\sin px)(C\cos pat + D\sin pat) = 0$
or, $A(C\cos pat + D\sin pat)$
$\therefore$ $A = 0$
Now, required sol^n :
$\therefore$ $u(x,t) = B\sin px\,(C\cos pat + D\sin pat)$
ii) $u(\pi, t) = 0$
Now, $B\sin p\pi\,(C\cos pat + D\sin pat) = 0$
$\therefore$ $\sin p\pi = 0$
or $p = \frac{n\pi}{\pi}$

$\therefore$ $p = n$ $n = 1, 2, 3$
The required sol^n:
$B\sin nx\,(C\cos nat + D\sin nat)$
iii) $\frac{du}{dt}(x,0) = 0$
$\Rightarrow$ $B\sin nx\,[C\sin(nat)(na) + D\cos(nat)\cdot na] = 0$
or, $B\sin nx\, D\, na = 0$
or $BD\sin nx\,(na) = 0$
$\therefore$ $D = 0$

The req solⁿ :-

$B \sin nx \; C \cos(nat)$

$\therefore$ $BC \sin nx \cos(nat) = u(x,t)$

iv) $u(x,0) = \sin x - 2\sin 3x$

or, $\sin x - 2\sin 3x = \sum_{n=1}^{\infty} \lambda_n \sin x$

or, $\sin x - 2\sin 3x = \lambda_1 \sin x + \lambda_2 \sin 2x + \lambda_3 \sin 3x$

Now, $\lambda_1 = 1$; $\lambda_3 = -2$; $\lambda_2 = 0$

The reqⁿ solⁿ:

$\therefore$ $u(x,t) = \sin(nx)\cos(nat) - 2\sin(nx)\cos(nat)$

Now,

$\therefore$ ~~$u\left(\frac{\pi}{2},\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) + 2\sin\left(\frac{\pi}{2}\right)\cdot\cos\left(\frac{na\pi}{2}\right)$~~

~~$= -3$~~

$u(x,t) = \sum_{n=1}^{\infty} \lambda_n \sin nx \cdot \cos nat$, $\lambda_1 = 1$, $\lambda_3 = -2$

$u(x,t) = \sin x \cos at - 2\sin 3x \cos 3at$

$u_{tt} = 4u_{xx}$

$u_{tt} = a^2 u_{xx}$

$a^2 = 4$

$a = 2$

$u(x,t) = \sin x \cos 2t - 2\sin 3x \cos 6t$

$u\left(\frac{\pi}{2},\frac{\pi}{2}\right) = \sin\frac{\pi}{2}\cos\pi - 2\sin\frac{3\pi}{2}\cos 3\pi$

$= -3$