Random Variable

Experiment: An operation or process which results in some well defined outcome and generates a set of data is called an Experiment.

Random Experiment: An experiment in which the outcome cannot be predicted with certainty is called a random experiment, even though all possible outcomes are known in advance.

Example: Tossing a fair coin is a random experiment. The possible outcomes are head or tail. But we cannot say with certainty that head will come up.

Similarly the experiments of throwing a die or taking a card from a pack of cards are random experiments.

Trials: Each performance of a random experiment is called a trial.

Example: Tossing a coin 6 times is 6 trials.

Sample space: The set of all possible outcomes of a random Experiment is called its sample space and it is denoted by S.

Example: (1) Toss a coin twice

Sample Space (S) = {HH, TT, TH, HT}

(2) Throw a die once

$S = \{1, 2, 3, 4, 5, 6\}$

Introduction: (Random Variable)

In many random experiments the outcomes may be numerical or non-numerical.

For example: The outcomes when a die is thrown are numerical whereas when we toss a coin the outcomes are non-numerical.

To deal with outcomes mathematically, we may assign numerical values to them.

ie, We can assign a real number x to every element s of the sample space.

Definition: A random variable (r.v) is a rule that assigns a real number to each possible outcome of a random experiment. (or)

Let S be the sample space associated with an experiment E. A r.v. X is a real valued function defined on S.

ie, $X : S \to R$ is a function.

Each $s \in S$ is associated with a real number $X(s) = x$.

<u>Example 1</u>: Toss a coin twice:

The sample space is $S = \{HH, HT, TH, TT\}$

Let X denote the 'number of heads' then X is a r.v.'s $X(TT) = 0$, $X(HH) = 2$, $X(HT) = 1$, $X(TH) = 1$.

ie, values of X are 0, 1, 2.

<u>Note</u>: (1) X is a 'variable' because it has different possible values. The term 'random' because it takes values with certain probabilities

(2) The terms 'stochastic variable' and 'change variable' are also used for 'random variable'.

Range Space: The set of all values taken by a r.v X is called its range space and is denoted by R_X.

Thus $R_X = \{X(s) / s \in S\}$

In the above example, the Range space is

$R_X = \{0, 1, 2\}$

Notation: If x is a real number and X is a r.v on S, we write $X = x$ for the event $\{s \in S / X(s) = x\}$. $P(X=x) = P\{s \in S / X(s) = x\}$

In example 1,

$P(X=0) = P\{TT\} = 1/4$

$P(X=1) = P\{(HT), (TH)\} = \frac{2}{4} = 1/2$

$P(X=2) = P\{HH\} = 1/4$

Types of random Variables:

a) Discrete r.v (b) Continuous r.v

a) **Discrete r.v**: A r.v X is said to be discrete if it takes a finite number of values or countably infinite number of values.

i.e., its range R_X is finite or countably infinite.

Note: The sample space on which a discrete r.v X is defined need not be discrete.

For example, If X is defined as the number of telephone calls to an office between 9 A.M and 11 A.M, then it is finite. But the sample space S is the time interval [9, 11] which is continuous.

A discrete r.v assumes each of its values with certain probability. These probabilities could be used to define a prob. function as below:

Prob. function (or) Prob. Mass function:

Let X be a discrete r.v which takes values $x_1, x_2, x_3, \ldots$. Let $P(X=x_i) = p(x_i) = p_i$ be prob. of x_i. Then the function p_i is called the prob. mass function of X if the number $p(x_i)$ satisfy the conditions

(i) $p(x_i) \geq 0 \ \forall \ i = 1, 2, 3, \ldots$ and

(ii) $\sum_{i=1}^{\infty} p(x_i) = 1$.

The set of ordered pairs of numbers $\{x_i, p(x_i)\}$ is called the prob. distribution of the r.v X.

The prob. distribution is usually displayed in the form of a table as given below.

x_i	x_1	x_2	x_3 ...
$p(x_i)$	$p(x_1)$	$p(x_2)$	$p(x_3)$ ...

Note: If $x_1 < x_2 < x_3 < \dots < x_i < \dots$, then

(i) $P(X \le x_i) = P(X = x_1) + P(X = x_2) + \dots + P(X = x_i)$

(ii) $P(X > x_i) = 1 - P(X \le x_i)$

(iii) $P(X \geq x_i) = 1 - P(X < x_i)$

(b) Continuous r.v: A r.v X is said to be Continuous if its range space R_X is an uncountable set of real numbers.

ie, the r.v assume values in an interval (a, b) or in an union of intervals.

Example: (1) If X denotes the lifetime of a transistor then X is a Continuous r.v.

(2) If X denotes the operating time between two failures of a machine, then X is a Continuous r.v.

Note: (i) To define a continuous r.v, the sample space should be continuous.

(ii) In the case of continuous r.v, we cannot talk of the first value, second value etc., becomes meaningless. So, we define prob. function of a continuous r.v as below.

Prob. density function: A function f, defined for all $x \in (-\infty, \infty)$ is called the pdf of a continuous r.v X If

(i) $f(x) \geq 0 \quad \forall \; x \in (-\infty, \infty)$

(ii) $\int_{-\infty}^{\infty} f(x)\, dx = 1$

Note: (i) $P(X < a) = \int_{-\infty}^{a} f(x)\, dx$, $P(X > a) = \int_{a}^{\infty} f(x)\, dx$

(ii) $P[a \leq X \leq b) = \int_{a}^{b} f(x)\, dx$

(iv) If $b = a$, $P(X = a) = \int_{a}^{a} f(x)\, dx = 0$

(v) For continuous r.v

$P(X \leq a) = P(X < a)$, $P(X > a) = P(X \geq a)$

$P(a < X \leq b) = P(a \leq X < b) = P(a \leq X \leq b) = P(a < X < b)$

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Now define another function for a r.v which will be suitable for both discrete and Continuous r.v.

Cumulative distribution function (CDF) or Distribution function:

If X is a r.v, discrete or continuous, then the function $F: \mathbb{R} \to [0,1]$ defined by $F(x) = P(X \leq x)$ is called the CDF of X.

(a) If X is a discrete r.v with values $x_1 < x_2 < x_3 \ldots < x_i < \ldots$ then $F(x) = \sum_{x_i \leq x} p(x_i)$

(b) If X is a continuous r.v with pdf $f(x)$ defined for all $x \in (-\infty, \infty)$, then $F(x) = \int_{-\infty}^{x} f(x)\,dx$.

Properties of distribution function $F(x)$

(1) $0 \leq F(x) \leq 1$, $-\infty < x < \infty$

(2) $F(x)$ is an increasing function of x

ie, $a < b \Rightarrow F(a) \leq F(b)$

(3) $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$ and

$F(\infty) = \lim_{x \to \infty} F(x) = 1$

(4) If X is a continuous r.v then $\frac{dF(x)}{dx} = f(x)$

Note: (1) If X is a discrete r.v with values $x_1 < x_2 < x_3 < \dots$ then $p(x_i) = F(x_i) - F(x_{i-1})$

Proof:

$$F(x_i) = p(x_1) + p(x_2) + p(x_3) + \dots + p(x_{i-1}) + p(x_i)$$

$$F(x_{i-1}) = p(x_1) + p(x_2) + p(x_3) + \dots + p(x_{i-1})$$

$$\therefore \; p(x_i) = F(x_i) - F(x_{i-1})$$

(2) If $a<b$, then $P(a < X \leq b) = F(b) - F(a)$

Proof: The event $X \leq b = (X \leq a) \cup (a < X \leq b)$

$$P(X \leq b) = P(X \leq a \cup a < X \leq b)$$

$= P(X \leq a) + P(a < X \leq b)$ since $X \leq a$ and $a < X \leq b$ are disjoint)

$$F(b) = F(a) + P(a < X \leq b)$$

$$\therefore \; P(a < X \leq b) = F(b) - F(a)$$

Problems: [Discrete r.v]

1) A r.v X has the following prob. distribution

x :	-2	-1	0	1	2	3
$p(x)$:	0.1	k	0.2	$2k$	0.3	$3k$

Find (i) k (ii) $P(X<2)$ (iii) $P(-2<X<2)$

Sol:

(i) $p(x) \geq 0$ for x. $\therefore k \geq 0$ and $\Sigma P(x) = 1$

$\Rightarrow 0.1 + k + 0.2 + 2k + 0.3 + 3k = 1$

$\Rightarrow 6k + 0.6 = 1 \Rightarrow k = \frac{0.4}{6}$ or $\frac{1}{15}$

(ii) $P(X<2) = 1 - P(X \geq 2) = 1 - [P(X=2) + P(X=3)]$

$= 1 - (0.3 + 3k) = 0.7 - 3 \cdot \frac{1}{15} = 0.5$

(iii) $P(-2<X<2) = P(X=-1) + P(X=0) + P(X=1)$

$= k + 0.2 + 2k = 0.2 + 3k$

$= 0.2 + 3 \cdot \frac{1}{15} = 0.4$

2) A r.v x has the following probability function

Value of $X=x$	0	1	2	3	4	5	6	7
$P(X=x)$	0	a	$2a$	$2a$	$3a$	a^2	$2a^2$	$7a^2+a$

Find (i) a, (ii) $P(X<4)$, (iii) $P(X \geq 4)$, (iv) $P(X \geq 6)$, (v) $P(X<6)$.

If $P(X \leq k) > 1/2$, then find the least value of k.

Find the CDF of X. Also find $P(1.5<X<4.5 / X>2)$,

$P(X<2)$, $P(X>3)$ and $P(1<X<5)$.

Sol:

(i) $p(x) \geq 0$ for all x $\therefore a \geq 0$ and $\Sigma p(x) = 1$

$\Rightarrow 0 + a + 2a + 2a + 3a + a^2 + 2a^2 + 7a^2 + a = 1$

$\Rightarrow 10a^2 + 9a - 1 = 0 \Rightarrow (10a-1)(a+1) = 0$

$\Rightarrow a = \frac{1}{10}$ or $a = -1$

Since $a \geq 0$, $a = \frac{1}{10}$

Probability distribution:

x	0	1	2	3	4	5	6	7
$p(x)$	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$

(ii) $P(X<4) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$= 0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} = \frac{5}{10}$ or $\frac{1}{2}$

(iii) $P(X \geq 4) = 1 - P(X<4) = 1 - \frac{1}{2} = \frac{1}{2}$

(iv) $P(X \geq 6) = P(X=6) + P(X=7) = \frac{2}{100} + \frac{17}{100} = \frac{19}{100}$

(v) $P(X<6) = 1 - P(X \geq 6) = 1 - \frac{19}{100} = \frac{81}{100}$

To find the least value of K

$P(X \leq 1) = \frac{1}{10}$, $P(X \leq 2) = \frac{3}{10}$, $P(X \leq 3) = \frac{5}{10} < \frac{1}{2}$

$P(X \leq 4) = \frac{8}{10} > \frac{1}{2}$

$\therefore$ The least value of $K = 4$.

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To find CDF of X

By defn $F(x) = P(X \le x)$

$F(0) = P(X \le 0) = 0$, $F(1) = P(X \le 1) = P(X \le 0) + P(X=1) = \frac{1}{10}$

$F(2) = P(X \le 2) = P(X \le 1) + P(X=2) = \frac{1}{10} + \frac{2}{10} = \frac{3}{10}$

$F(3) = P(X \le 3) = P(X \le 2) + P(X=3) = \frac{3}{10} + \frac{2}{10} = \frac{5}{10}$

Similarly

$F(4) = \frac{8}{10}$, $F(5) = \frac{81}{100}$, $F(6) = \frac{83}{100}$, $F(7) = 1$

$\therefore$ The CDF of $F(x)$ is

x	0	1	2	3	4	5	6	7
$F(x)$	0	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{5}{10}$	$\frac{8}{10}$	$\frac{81}{100}$	$\frac{83}{100}$	1

(or)

$$F(x) = \begin{cases} 0 & \text{for } x < 1 \\ \frac{1}{10} & \text{for } 1 \le x < 2 \\ \frac{3}{10} & \text{for } 2 \le x < 3 \\ \frac{5}{10} & \text{for } 3 \le x < 4 \\ \frac{8}{10} & \text{for } 4 \le x < 5 \\ \frac{81}{100} & \text{for } 5 \le x < 6 \\ \frac{83}{100} & \text{for } 6 \le x < 7 \\ 1 & \text{for } x \ge 7 \end{cases}$$

$P(1.5 < X < 4.5 / X > 2)$

$= \frac{P(1.5 < X < 4.5 \cap X > 2)}{P(X > 2)}$

$= \frac{P(2 < X < 4.5)}{1 - P(X \le 2)}$

$= \frac{P(X=3) + P(X=4)}{1 - P(X \le 2)}$

$= \frac{5/10}{7/10} = \frac{5}{7}$

$P(X < 2) = 0 + \frac{1}{10} = \frac{1}{10}$

$P(X > 3) = 1 - P(X \le 3)$

$= 1 - [0 + a + 2a + 2a]$

$= 1/2$

$P(1 < X < 5) = 2a + 2a + 3a = \frac{7}{10}$

If the discrete r.v x has the probability function given by the table.

x	1	2	3	4
$P(x)$	$\frac{K}{3}$	$\frac{K}{6}$	$\frac{K}{3}$	$\frac{K}{6}$

. Find K and the CDF of X.

Sol:

(i) $\Sigma p(x) = 1 \Rightarrow \frac{K}{3} + \frac{K}{6} + \frac{K}{3} + \frac{K}{6} = 1$

$\Rightarrow K = 1$

(ii)

x	1	2	3	4
$F(x)$	$\frac{1}{3}$	$\frac{3}{6}$	$\frac{5}{6}$	1

A discrete r.v X has the following prob. distribution

x	0	1	2	3	4	5	6	7	8
$p(x)$	a	3a	5a	7a	9a	11a	13a	15a	17a

(i) Find the value of a (ii) $P(0 < X < 3)$

(iii) $P(X \geq 3)$ (iv) Find the distribution fn of X.

Sol: (i) $\Sigma p(x) = 1$

$\Rightarrow a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$

$\Rightarrow 81a = 1 \Rightarrow a = \frac{1}{81}$

(ii) $P(0 < X < 3) = P(X=1) + P(X=2) = 3a + 5a = 8a = \frac{8}{81}$

(iii) $P(X \geq 3) = 1 - P(X < 3)$

$= 1 - [P(X=0) + P(X=1) + P(X=2)]$.

$= 1 - (a + 3a + 5a) = 1 - 9a = 1 - \frac{9}{81} = \frac{8}{9}$,

(iv) $F(x) = P(X \leq x)$

x	0	1	2	3	4	5	6	7	8
$F(x)$	$\frac{1}{81}$	$\frac{4}{81}$	$\frac{9}{81}$	$\frac{16}{81}$	$\frac{25}{81}$	$\frac{36}{81}$	$\frac{49}{81}$	$\frac{64}{81}$	1

A r.v X has the following prob. function

x :	0	1	2	3	4
$p(x)$:	k	$2k$	$5k$	$7k$	$9k$

Find k, $P(X \geq 3)$ and $P(0 < X < 4)$

Sol:

(i) $k + 2k + 5k + 7k + 9k = 1$

$\Rightarrow 24k = 1 \Rightarrow k = \frac{1}{24}$

(ii) $P(X \geq 3) = P(X=3) + P(X=4)$

$= 7k + 9k = 16k = \frac{16}{24}$ or $\frac{2}{3}$

(iii) $P(0 < X < 4) = P(X=1) + P(X=2) + P(X=3)$

$= 2k + 5k + 7k = 14k = 14 \cdot \frac{1}{24} = \frac{7}{12}$

6) The prob. function of a r.v X is given by
$P(X=x)=\frac{1}{2^x}$, $x=1,2,3\cdots$ Find (i) P(X is even),
(ii) P(X is odd), (iii) $P(X\geq 4)$ (iv) P(X is a multiple of 3)

Sol:

(i) $P(X \text{ is even}) = P(X=2)+P(X=4)+P(X=6)+\cdots$

$$= \frac{1}{2^2}+\frac{1}{2^4}+\frac{1}{2^6}+\frac{1}{2^8}+\cdots$$

$$= \frac{1}{2^2}\left[1+\frac{1}{2^2}+\frac{1}{2^4}+\cdots\right]$$

$$= \frac{1}{2^2}\left[1-\frac{1}{2^2}\right]^{-1} = \frac{1}{3}$$

(ii) $P(X \text{ is odd}) = 1-P(X \text{ is even}) = 1-\frac{1}{3} = 2/3$

(iii) $P(X\geq 4) = 1-P(X<4)$

$$= 1-[P(X=1)+P(X=2)+P(X=3)]$$

$$= 1-\left[\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}\right] = 1-\frac{7}{8} = \frac{1}{8}$$

(iv) $P(X \text{ is a multiple of } 3) = P(X=3)+P(X=6)+P(X=9)+\cdots$

$$= \frac{1}{2^3}+\frac{1}{2^6}+\frac{1}{2^9}+\frac{1}{2^{11}}+\cdots$$

$$= \frac{1}{2^3}\left[1+\frac{1}{2^3}+\frac{1}{(2^3)^2}+\cdots\right]$$

$$= \frac{1}{8}\left[1-\frac{1}{2^3}\right]^{-1} = \frac{1}{7}$$

Two Cards are drawn successively with replacement from a well shuffled deck of 52 Cards. Find the prob. dist. of number of kings.

Sol:

Let x denote the number of kings in

$\therefore$ The values of x are 0, 1, 2.

$P(\text{a king}) = \frac{4C_1}{52C_2} = \frac{4}{52}$ or $\frac{1}{13}$

$P(\text{not a king}) = 1 - \frac{1}{13} = \frac{12}{13}$

$P(x=0) = P(\text{both are not kings}) = \frac{12}{13} \cdot \frac{12}{13} = \frac{144}{169}$

$P(x=1) = P(\text{one king and one not a king})$

$= \left(\frac{1}{13} \times \frac{12}{13}\right) + \left(\frac{12}{13} \times \frac{1}{13}\right) = \frac{24}{169}$

$P(x=2) = P(\text{two kings}) = \frac{1}{13} \times \frac{1}{13} = \frac{1}{169}$

The prob. dist. of x is given below

x	0	1	2
$p(x)$	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

8) A shipment of 6 television sets contain 2 defective sets. A hotel makes a random purchase of 3 sets. If X is the number of defective sets purchased by the hotel, find the prob. dist. of X.

Sol:

Let X denotes the number of defective sets

Then the values of X are 0, 1, 2

$P(X=0) = \frac{4C_3}{6C_3} = \frac{1}{5}$, $P(X=1) = \frac{2C_1 \times 4C_2}{6C_3} = \frac{3}{5}$

$P(X=2) = \frac{2C_2 \times 4C_1}{6C_3} = \frac{1}{5}$

The prob. distribution of X is given below:

x	0	1	2
$p(x)$	$1/5$	$3/5$	$1/5$

9) A r.v X takes values 1, 2, 3, 4 such that,

$2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$. Find the prob. dist. function and hence find the CDF.

Sol: Let $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4) = K$

$\therefore P(X=1) = \frac{K}{2}$, $P(X=2) = \frac{K}{3}$, $P(X=3) = K$, $P(X=4) = \frac{K}{5}$

x	1	2	3	4
$p(x)$	$K/2$	$K/3$	K	$K/5$

$\Sigma p(x)=1 \Rightarrow \frac{k}{2}+\frac{k}{3}+k+\frac{k}{5}=1$

$\Rightarrow k=\frac{30}{61}$

$X=x$	1	2	3	4
prob. dist.	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$6/61$
CDF	$\frac{15}{61}$	$\frac{25}{61}$	$\frac{55}{61}$	1

10) Let X be a r.v such that $P(X=-2)=P(X=-1)=P(X=1)$ $=P(X=2)$ and $P(X<0)=P(X=0)=P(X>0)$. Determine the prob. mass function of X and dist. fn. of X.

Sol:

Let $P(X=-2)=P(X=-1)=P(X=1)=P(X=2)=a$

$\therefore P(X<0)=P(X=-2)+P(X=-1)=a+a=2a$

$\Rightarrow P(X<0)=P(X=0)=P(X>0)=2a$

X	-2	-1	0	1	2
$P(X=x)$	a	a	$2a$	a	a

Since $\Sigma p(x)=1 \Rightarrow a+a+2a+a+a=1$

$\Rightarrow a=\frac{1}{6}$

X	-2	-1	0	1	2
$P(X=x)$	$1/6$	$1/6$	$1/3$	$1/6$	$1/6$
$F(x)$	$1/6$	$2/6$	$4/6$	$5/6$	1

11) Let X be a discrete r.v whose CDF $F(x)$ is given by

$$F(x) = \begin{cases} 0, & x < -3 \\ 1/6, & -3 \le x < 6 \\ 1/2, & 6 \le x < 10 \\ 1, & 10 \le x \end{cases}$$

Find (i) $P(X \le 4)$ (ii) $P(-5 < X \le 4)$ (iii) the prob. dist.

Sol:

(i) $P(X \le 4) = F(4) = \frac{1}{6}$

(ii) $P(-5 < X \le +4) = F(4) - F(-5)$ $\because P(a < X \le b) = F(b) - F(a)$

$= \frac{1}{6} - 0 = \frac{1}{6}$

(iii) Given X is a discrete r.v and X take the values are $-3, 6, 10$.

$X = x$	-3	6	10
$P(X=x)$	$\frac{1}{6}$	$1/3$	$1/2$

Since $P(X \le -3) = P(X = 3) + P(X < -3)$

$\Rightarrow P(X = 3) = P(X \le -3) - P(X < -3)$

$= \frac{1}{6} - 0 = 1/6$

$P(X \le 6) = P(X = 6) + P(X < 6)$

$\Rightarrow P(X = 6) = P(X \le 6) - P(X < 6) = \frac{1}{2} - \frac{1}{6} = \frac{2}{6}$ or $\frac{1}{3}$

and $P(X \le 10) = P(X = 10) + P(X < 10)$

$\Rightarrow P(X = 10) = P(X \le 10) - P(X < 10) = 1 - \frac{1}{2} = 1/2$

Problems: [Continuous r.v]

1.) If the density function of a continuous r.v X is given by

$$f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ 3a - ax, & 2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$$

(i) Find a (ii) Find the CDF of X.

Sol:

(i) Given $f(x)$ is the pdf of a continuous r.v X.

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a - ax)\,dx = 1$$

$$\Rightarrow a\left[\frac{x^2}{2}\right]_0^1 + a[x]_1^2 + \left[3a \cdot x - a \cdot \frac{x^2}{2}\right]_2^3 = 1$$

$$\Rightarrow a\left(\tfrac{1}{2} - 0\right) + a(2-1) + \left[3a \cdot 3 - a \cdot \frac{9}{2} - 3a \cdot 2 + a \cdot \frac{4}{2}\right] = 1$$

$$\Rightarrow \frac{a}{2} + a + \cancel{a}\left[3a - \frac{5a}{2}\right] = 1 \Rightarrow a = \frac{1}{2}$$

$$\therefore f(x) = \begin{cases} \frac{x}{2}, & 0 \le x \le 1 \\ \frac{1}{2}, & 1 \le x \le 2 \\ \frac{1}{2}(3-x), & 2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$$

(ii) To find the CDF of X

W.K.T $F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$

If $x < 0$, then $F(x) = 0$.

If $0 \le x < 1$, then $F(x) = \int_{0}^{x} f(x)\,dx$

$$\Rightarrow F(x) = \int_{0}^{x} x\,dx = \frac{1}{2}\left[\frac{x^2}{2}\right]_0^x = \frac{x^2}{4}$$

If $1 \le x < 2$, then $F(x) = \int_{0}^{1} f(x)\,dx + \int_{1}^{x} f(x)\,dx$

$$= \int_{0}^{1} \frac{x}{2}\,dx + \int_{1}^{x} \frac{1}{2}\,dx.$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}\left[x\right]_1^x = \frac{1}{4} + \frac{1}{2}(x-1)$$

If $2 \le x < 3$, then $F(x) = \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{x} f(x)\,dx$

$$\Rightarrow F(x) = \int_{0}^{1} \frac{x}{2}\,dx + \int_{1}^{2} \frac{1}{2}\,dx + \int_{2}^{x} \frac{1}{2}(3-x)\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}\left[x\right]_1^2 + \frac{1}{2}\left[3x - \frac{x^2}{2}\right]_2^x$$

$$= \frac{1}{4}(-x^2 + 6x - 5)$$

If $x \ge 3$, then $F(x) = 1$

$$\therefore F(x) = \begin{cases} 0 & \text{If } x < 0 \\ x^2/4 & \text{If } 0 \le x < 1 \\ \frac{1}{4}(2x-1) & \text{If } 1 \le x < 2 \\ \frac{1}{4}(-x^2+6x-5) & \text{If } 2 \le x < 3 \\ 1 & \text{If } x \ge 3 \end{cases}$$

2.) If X is a continuous r.v whose pdf is given by

$$f(x)=\begin{cases} c(4x-2x^2), & 0<x<2 \\ 0, & \text{otherwise} \end{cases}$$. Find (i) c (ii) $P(X>1)$.

Sol:

(i) Given $f(x)$ is a pdf.

$\therefore f(x)\geq 0$ and $\int_{-\infty}^{\infty} f(x)dx=1$

$$\Rightarrow \int_0^2 c(4x-2x^2)dx=1 \Rightarrow c\left[\frac{4x^2}{2}-\frac{2x^3}{3}\right]_0^2=1$$

$$\Rightarrow c\left[2\cdot 4-\frac{2}{3}\cdot 8\right]=1$$

$$\Rightarrow 8c\left(1-\frac{2}{3}\right)=1 \Rightarrow c=\frac{3}{8}$$

(ii) $$P(X>1)=\int_1^2 f(x)dx=\frac{3}{8}\int_1^2(4x-2x^2)dx$$

$$=\frac{3}{8}\left[\frac{4x^2}{2}-\frac{2x^3}{3}\right]_1^2$$

$$=\frac{3}{8}\left[2\cdot 4-\frac{2}{3}\cdot 8-2+\frac{2}{3}\right]=\frac{1}{2}$$

3.) A r.v X has the pdf $f(x)=\begin{cases} 2x, & 0<x<1 \\ 0, & \text{elsewhere} \end{cases}$. Find

(i) $P(X<\frac{1}{2})$ (ii) $P(\frac{1}{4}<X<\frac{1}{2})$ (iii) $P(X>3/4 \mid X>1/2)$

(iv) $P(X<3/4 \mid X>1/2)$.

Sol:

(i) $$P(X<1/2)=\int_{-\infty}^{1/2} f(x)dx=\int_0^{1/2} 2x\,dx=2\left(\frac{x^2}{2}\right)_0^{1/2}=\frac{1}{4}$$

(ii) $P\left(\frac{1}{4} < x < \frac{1}{2}\right) = \int_{1/4}^{1/2} f(x)\,dx = \int_{1/4}^{1/2} 2x\,dx$

$$= 2\left(\frac{x^2}{2}\right)_{1/4}^{1/2} = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$$

(iii) $P(x > 3/4 \mid x > 1/2) = \dfrac{P(x > 3/4 \cap x > 1/2)}{P(x > 1/2)}$

$$= \frac{P(x > 3/4)}{P(x > 1/2)}$$

$$= \frac{\int_{3/4}^{1} 2x\,dx}{\int_{1/2}^{1} 2x\,dx} = \frac{(x^2)_{3/4}^{1}}{(x^2)_{1/2}^{1}}$$

$$= \frac{1 - \frac{9}{16}}{1 - 1/4} = \frac{\frac{7}{16}}{3/4} = \frac{7}{12}$$

(iv) $P(x < 3/4 \mid x > 1/2)$

$$= \frac{P(x < 3/4 \cap x > 1/2)}{P(x > 1/2)} = \frac{P\left(1/2 < x < \frac{3}{4}\right)}{P(x > 1/2)}$$

$$= \frac{\int_{1/2}^{3/4} 2x\,dx}{3/4} = \frac{(x^2)_{1/2}^{3/4}}{3/4} = \frac{\frac{9}{16} - \frac{1}{4}}{3/4}$$

$$= \frac{\frac{5}{16}}{3/4} = \frac{5}{12}$$

4) Verify whether f(x) is a pdf

(i) $f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty$ (ii) $f(x) = \begin{cases} |x|, & -1 \le x < 1 \\ 0, & \text{otherwise} \end{cases}$

Sol:

(i) To prove : $\int_{-\infty}^{\infty} f(x)\,dx = 1$ since f(x) is a pdf.

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{dx}{1+x^2} = 1 \Rightarrow \frac{1}{\pi}\left[\tan^{-1} x\right]_{-\infty}^{\infty} = 1$$

$$\Rightarrow \frac{1}{\pi}\left[\tan^{-1}\infty - \tan^{-1}(-\infty)\right] = 1$$

$$\Rightarrow \frac{1}{\pi}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = 1 \Rightarrow 1 = 1$$

∴ Given ~~pdf~~ function is a pdf.

(ii) $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} -x\,dx + \int_{0}^{1} x\,dx$

$$= -\left(\frac{x^2}{2}\right)_{-1}^{0} + \left(\frac{x^2}{2}\right)_{0}^{1} = \frac{1}{2} + \frac{1}{2} = 1$$

∴ Given function is a pdf.

5) A continuous r.v X has a p.d.f $f(x) = 6x(1-x)$, $0 \le x \le 1$. Determine b If $P(X<b) = P(X>b)$

Sol: Given $P(X<b) = P(X>b)$

$$= 1 - P(X \le b)$$

$$\Rightarrow P(X<b) + P(X \le b) = 1$$

$$\Rightarrow 2P(X<b) = 1 \Rightarrow P(X<b) = \frac{1}{2}$$

$$\Rightarrow \int_{0}^{b} f(x)\,dx = \frac{1}{2}$$

$$\Rightarrow \int_0^b 6x(1-x)\,dx = \frac{1}{2} \Rightarrow 6\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^b = \frac{1}{2}$$

$$\Rightarrow \left[3x^2-2x^3\right]_0^b = \frac{1}{2} \Rightarrow 3b^2-2b^3=\frac{1}{2}$$

$$\Rightarrow 4b^3-6b^2+1=0 \Rightarrow 4b^3-2b^2-4b^2+1=0$$

$$\Rightarrow 2b^2(2b-1)-(4b^2-1)=0$$

$$\Rightarrow 2b^2(2b-1)-(2b+1)(2b-1)=0$$

$$\Rightarrow (2b-1)(2b^2-2b-1)=0$$

$2b-1=0$ or $2b^2-2b-1=0$

$b=\frac{1}{2}$ $\qquad b=\frac{2\pm\sqrt{4+8}}{4}=\frac{1\pm\sqrt{3}}{2}$

$=\frac{1+\sqrt{3}}{2}>1$ or $\frac{1-\sqrt{3}}{2}<0$

These values are not possible.

$\therefore b=1/2$

6.) A continuous r.v X has a p.d.f. $f(x)=3x^2,\ 0\le x\le 1$. Find 'a' and 'b' such that $P(X\le a)=P(X>a)$ and $P(X>b)=0.05$.

Sol: Given $P(X\le a)=P(X>a)$

$=1-P(X\le a)$

$$\Rightarrow 2P(X\le a)=1 \Rightarrow P(X\le a)=1/2$$

$$\Rightarrow \int_0^a f(x)\,dx=\frac{1}{2} \Rightarrow \int_0^a 3x^2\,dx=\frac{1}{2}$$

$$\Rightarrow (x^3)_0^a=\frac{1}{2} \Rightarrow a=(1/2)^{1/3}$$

Given $P(X > b) = 0.05 \Rightarrow P(X>b) = \frac{5}{100} = \frac{1}{20}$

$$\Rightarrow \int_b^1 f(x)\,dx = \frac{1}{20} \Rightarrow \int_b^1 3x^2\,dx = \frac{1}{20}$$

$$\Rightarrow (x^3)_b^1 = \frac{1}{20} \Rightarrow 1 - b^3 = \frac{1}{20}$$

$$\Rightarrow b = \left(\frac{19}{20}\right)^{1/3}$$

7.) If $f(x) = k(1+x)$ in $2<x<5$ is the p.d.f of a Continuous r.v X, find (i) $P(X<4)$, (ii) $P(3<X<4)$.

Sol:

Given $f(x)$ is a p.d.f. $\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \int_2^5 k(1+x)\,dx = 1 \Rightarrow k\left[\frac{(1+x)^2}{2}\right]_2^5 = 1$$

$$\Rightarrow \frac{k}{2}(6^2 - 3^2) = 1 \Rightarrow \frac{k}{2}(36-9) = 1$$

$$\Rightarrow k = \frac{2}{27}$$

(i) $P(X<4) = \int_2^4 \frac{2}{27}(1+x)\,dx = \frac{2}{27}\left[\frac{(1+x)^2}{2}\right]_2^4$

$$= \frac{1}{27}(5^2 - 3^2) = \frac{16}{27}$$

(ii) $P(3<X<4) = \int_3^4 \frac{2}{27}(1+x)\,dx = \frac{2}{27}\left[\frac{(1+x)^2}{2}\right]_3^4$

$$= \frac{1}{27}(5^2 - 4^2) = \frac{1}{3}$$

8.) The pdf of a continuous r.v x is given by $f(x) = kx(2-x),\ 0<x<2$. Find the cumulative distribution function.

Sol: Given function $f(x)$ is the pdf.

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{0}^{2} kx(2-x)\,dx = 1$$

$$\Rightarrow k\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^2 = 1 \Rightarrow k\left[4 - \frac{8}{3}\right] = 1$$

$$\Rightarrow k\left(\frac{12-8}{3}\right) = 1 \Rightarrow k = \frac{3}{4}$$

To find CDF

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{0}^{x} \frac{3}{4}x(2-x)\,dx$$

$$= \frac{3}{4}\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^x = \frac{x^2}{4}(3-x)$$

$$\therefore F(x) = \begin{cases} 0, & x<0 \\ \frac{x^2}{4}(3-x), & 0 \le x < 2 \\ 1, & x \ge 2 \end{cases}$$

9.) If X is a continuous r.v with pdf

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ \frac{3}{2}(x-1)^2, & 1 \le x < 2 \\ 0, & \text{otherwise.} \end{cases}$$

find the CDF $F(x)$ and use it to find $P\left(\frac{3}{2} < X < \frac{5}{2}\right)$

If $x<0$ then $F(x)=0$

If $0\le x<1$, then $F(x)=\int_0^x x\,dx=\left[\frac{x^2}{2}\right]_0^x=\frac{x^2}{2}$

If $1\le x<2$ then $F(x)=\int_0^1 f(x)dx+\int_1^x f(x)dx$

$\Rightarrow F(x)=\int_0^1 x\,dx+\int_1^x \frac{3}{2}(x-1)^2 dx$

$=\left[\frac{x^2}{2}\right]_0^1+\frac{3}{2}\left[\frac{(x-1)^3}{3}\right]_1^x$

$=\frac{1}{2}+\frac{1}{2}(x-1)^3$

If $x\geq 2$ then $F(x)=1$

$$\therefore F(x)=\begin{cases} 0 & \text{If } x<0 \\ x^2/2 & \text{If } 0\le x<1 \\ \frac{1}{2}+\frac{1}{2}(x-1)^3 & \text{If } 1\le x<2 \\ 1 & \text{If } x\geq 2 \end{cases}$$

Now

$P\left(\frac{3}{2}<x<\frac{5}{2}\right)=F\left(\frac{5}{2}\right)-F\left(\frac{3}{2}\right)$

$=1-\left[\frac{1}{2}+\frac{1}{2}\left(\frac{3}{2}-1\right)^3\right]=\frac{7}{16}$

10.) If $f(x) = \begin{cases} x e^{-x^2/2} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$ then (i) show that

$f(x)$ is a pdf (ii) find $F(x)$.

Sol:

(i) To prove $f(x)$ is the pdf, we have to prove $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \int_{0}^{\infty} x e^{-x^2/2} \, dx$$

put $\frac{x^2}{2} = t \Rightarrow x^2 = 2t$

$2x\,dx = 2dt \Rightarrow x\,dx = dt$

when $x = 0 \Rightarrow t = 0$, $x = \infty \Rightarrow t = \infty$

$$\therefore \int_{0}^{\infty} x e^{-x^2/2} \, dx = \int_{0}^{\infty} e^{-t} \, dt$$

$$= \left[-e^{-t}\right]_0^{\infty} = 1$$

$\Rightarrow f(x)$ is a pdf

(ii) To find CDF $(F(x))$:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$$

If $x < 0$ then $F(x) = 0$

If $x \geq 0$ then $F(x) = \int_{0}^{x} x e^{-x^2/2} \, dx$

$$= \int_{0}^{x^2/2} e^{-t} \, dt = \left[-e^{-t}\right]_0^{x^2/2} = 1 - e^{-x^2/2}$$

11. The pdf of a continuous r.v is $f(x) = ce^{-|x|}$, $-\infty < x < \infty$, find value of c, the CDF of x.

Sol:

(i) $\int_{-\infty}^{\infty} ce^{-|x|} dx = 1$

$\Rightarrow c\left[\int_{-\infty}^{0} e^{x} dx + \int_{0}^{\infty} e^{-x} dx\right] = 1$

$\Rightarrow c\left[(e^{x})_{-\infty}^{0} + (-e^{-x})_{0}^{\infty}\right] = 1$

$\Rightarrow 2c = 1 \Rightarrow c = \frac{1}{2}$

(ii) To find CDF:

$F(x) = \int_{-\infty}^{x} f(x) dx$ where $f(x) = \begin{cases} \frac{1}{2}e^{x} & \text{if } x < 0 \\ \frac{1}{2}e^{-x} & \text{if } x \geq 0 \end{cases}$

If $x < 0$ then $F(x) = \int_{-\infty}^{x} \frac{1}{2} e^{x} dx$

$= \frac{1}{2}\left[e^{x}\right]_{-\infty}^{x} = \frac{e^{x}}{2}$

If $x \geq 0$ then $F(x) = \int_{-\infty}^{0} f(x) dx + \int_{0}^{x} f(x) dx$

$= \int_{-\infty}^{0} \frac{1}{2} e^{x} dx + \int_{0}^{x} \frac{1}{2} e^{-x} dx$

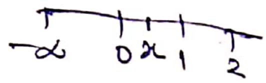
$= \frac{1}{2}\left[e^{x}\right]_{-\infty}^{0} + \frac{1}{2}\left[-e^{-x}\right]_{0}^{x}$

$= \frac{1}{2} - \frac{1}{2}(e^{-x} - 1) = 1 - \frac{e^{-x}}{2}$

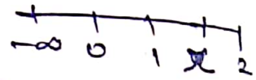
Sol:
$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

If $x < 0$ then $F(x) = 0$

If $0 \le x < 1$ then $F(x) = \int_{0}^{x} x\,dx$



$$\Rightarrow F(x) = \left[\frac{x^2}{2}\right]_0^x = \frac{x^2}{2}$$



If $1 \le x < 2$ then $F(x) = \int_{0}^{1} x\,dx + \int_{1}^{x} (2-x)\,dx$

$$\Rightarrow F(x) = \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^x$$
$$= \frac{1}{2} + 2x - \frac{x^2}{2} - \left(2 - \frac{1}{2}\right)$$
$$= 2x - \frac{x^2}{2} - 1$$

$$\therefore F(x) = \begin{cases} 0, & \text{If } x < 0 \\ x^2/2 & \text{If } 0 \le x < 1 \\ 2x - \frac{x^2}{2} - 1 & \text{If } 1 \le x < 2 \\ 1 & \text{If } x \geq 2 \end{cases}$$

A Continuous variable X follows the probability law $f(x) = Ax^2$, $0 \le x \le 1$, determine A and find the probability that X lies between 0.2 and 0.5 and $F(x)$.

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Sol: (i) $\int_0^1 Ax^2\,dx = 1 \Rightarrow A\left[\frac{x^3}{3}\right]_0^1 = 1$

$\Rightarrow A = 3$

$P(0.2 < X < 0.5) = \int_{0.2}^{0.5} 3x^2\,dx = 3\left[\frac{x^3}{3}\right]_{0.2}^{0.5}$

$= (0.5)^3 - (0.2)^3 = 0.117$

(iii) $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$

If $x < 0$ then $F(x) = 0$

If $0 \leq x < 1$ then $F(x) = \int_0^x 3x^2\,dx = x^3$

If $x \geq 1$ then $F(x) = 1$

$$\therefore F(x) = \begin{cases} 0 & \text{If } x < 0 \\ x^3 & \text{If } 0 \leq x < 1 \\ 1 & \text{If } x \geq 1 \end{cases}$$

If the r.v X has the following cumulative distribution function $F(x) = 1-(1+x)e^{-x}$ for $x \geq 0$, then find the probability that it assume the value less than 1.

Sol:

$f(x) = \frac{dF}{dx} = 0 - 1.e^{-x} - (1+x)(-e^{-x})$

$= -e^{-x} + e^{-x} + xe^{-x} = xe^{-x}$

$P(X<1) = \int_0^1 xe^{-x}\,dx = \left[-xe^{-x} - e^{-x}\right]_0^1 = \frac{e-2}{e}$

16.) If the cumulative distribution function of a r.v X is given by $F(x)=\begin{cases}1-\frac{4}{x^2} & \text{If } x>2 \\ 0 & \text{If } x\le 2.\end{cases}$ Find (i) $P(X<3)$

(ii) $P(4<X<5)$ (iii) $P(X\geq 3)$

sol:

$$f(x)=\frac{dF(x)}{dx}=\begin{cases}0 & \text{If } x\le 2 \\ \frac{8}{x^3} & \text{If } x>2\end{cases}$$

(i) $P(X<3)=\int_2^3 \frac{8}{x^3}\,dx = 8\left[\frac{x^{-2}}{-2}\right]_2^3 = -4\left[\frac{1}{9}-\frac{1}{4}\right]=\frac{5}{9}$

(ii) $P(4<X<5)=\int_4^5 \frac{8}{x^3}\,dx = 8\left[\frac{x^{-2}}{-2}\right]_4^5 = -4\left[\frac{1}{5^2}-\frac{1}{4^2}\right]=\frac{9}{100}$

(iii) $P(X\geq 3)=1-P(X<3)=1-\frac{5}{9}=\frac{4}{9}$

17.) The diameter of an electric cable X is a continuous random variable with pdf $f(x)=kx(1-x),\ 0\le x\le 1$. Find the value of k; $F(x)$ and $P(X\le 1/2 \mid 1/3<X<2/3)$

sol:

$$\int_0^1 kx(1-x)\,dx=1 \Rightarrow k\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1=1$$

$$\Rightarrow k=6.$$

$\therefore f(x)=6x(1-x)$

$$F(x)=P(X\le x) = \int_0^x 6x(1-x)\,dx = 6\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^x$$

$$=3x^2-2x^3$$

$$\therefore F(x) = \begin{cases} 0 & \text{for } x < 0 \\ x^2(3-2x) & \text{for } 0 \le x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

$$P(X < 1/2 \mid 1/3 < X < 2/3) = \frac{P(X < 1/2 \cap 1/3 < X < 2/3)}{P(1/3 < X < 2/3)}$$

$$= \frac{P(1/3 < X < 1/2)}{P(1/3 < X < 2/3)}$$

$$= \frac{\int_{1/3}^{1/2} 6x(1-x)\,dx}{\int_{1/3}^{2/3} 6x(1-x)\,dx} = \frac{13/54}{13/27} = 1/2$$

18.) For a continuous r.v X, the CDF is given by

$$F(x) = \begin{cases} 0 & \text{if } x < 2 \\ K(x-2) & \text{if } 2 \le x < 6 \\ 1 & \text{if } x \geq 6 \end{cases}$$

Find (i) the pdf of X (ii) the value of K (iii) $P(X > 4)$ (iv) $P(3 < X < 5)$.

Sol:

(i) $f(x) = \frac{d}{dx} F(x) = \begin{cases} \cancel{K}, & 2 \le x < 6 \\ 0, & \text{otherwise} \end{cases}$

(ii) $\int_2^6 f(x)\,dx = 1 \Rightarrow \int_2^6 K\,dx = 1 \Rightarrow K(x)_2^6 = 1$

$\Rightarrow K = 1/4$

(iii) $P(X > 4) = \int_4^6 \frac{1}{4}\,dx = \frac{1}{4}(x)_4^6 = 1/2$

(iv) $P(3 < X \le 5) = \int_3^5 \frac{1}{4}\,dx = \frac{1}{4}(x)_3^5 = 1/2$

—x—

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Moments

The expected value of an integral power of a random variable is called its moment.

Moments about the origin or raw moments.

The r^{th} moment of a r.v X about the origin is defined as $E(X^r)$ and is denoted by μ_r'.

$$\therefore \mu_r' = E(X^r)$$

The first four moments about the origin are

$$\mu_1' = E(X) = \text{Mean}$$

$$\mu_2' = E(X^2), \quad \mu_3' = E(X^3), \quad \mu_4' = E(X^4)$$

Moments about the mean or central moments

The r^{th} moment of a r.v X about the mean μ is defined as $E[X-\mu]^r$ and is denoted by μ_r

$$\therefore \mu_r = E(X-\mu)^r$$

The first four moments about the mean are

(i) $\mu_1 = E(X-\mu) = E(X) - \mu = \mu - \mu = 0$

$\therefore$ For every distribution discrete or continuous, the first moment about its mean is zero

(ii) $\mu_2 = E(X-\mu)^2 = \text{Variance of } X$

(iii) $\mu_3 = E(X-\mu)^3$ (iv) $E(X-\mu)^4 = \mu_4$

The moments about any point 'a'

The r^{th} moment of a r.v X about a point 'a' is defined as $E(X-a)^r$ and is also denoted by μ_r'

$$\therefore \mu_r' = E(X-a)^r$$

The first four moments about the point 'a' are

$$\mu_1' = E(X-a)$$
$$= E(X) - a$$
$$\Rightarrow \mu_1' = \mu - a$$
$$\therefore \mu = \mu_1' + a$$

$$\mu_2' = E(X-a)^2, \quad \mu_3' = E(X-a)^3, \quad \mu_4' = E(X-a)^4$$

Relation between moments about the mean and moments abouts any point 'a'

$$\mu_1 = 0$$
$$\mu_2 = \mu_2' - \mu_1'^2$$
$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$
$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

1.) The first four moments of a distribution about $X=4$ are $1, 4, 10, 45$. Show that the mean is 5, Variance is 3, $\mu_3 = 0$, $\mu_4 = 26$.

Sol: Let $\mu_1', \mu_2', \mu_3', \mu_4'$ be the first four moments about $x=4$.

Given $\mu_1'=1$, $\mu_2'=4$, $\mu_3'=10$, $\mu_4'=45$ about $x=4$.

$\mu_1' = E(X-4)$

$\Rightarrow 1 = E(X)-4 \Rightarrow E(X)=5$

Variance $(\mu_2) = \mu_2' - \mu_1'^2$

$= 4-1 = 3$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = 10-3(4)(1)+2(1) = 0$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$= 45 - (4\times 10\times 1) + (6\times 4\times 1) - (3\times 1) = 26$

2) The first two moments about 3 are 1 and 8. Find the mean and variance.

Sol! Let μ_1', μ_2' be the first two moments of x about 3.

Given $\mu_1'=1$, $\mu_2'=8$.

Mean $= \mu_1' + a = 1+3 = 4$

$\mu_2 = \mu_2' - \mu_1'^2 = 8-1 = 7$

$\therefore$ variance $= \mu_2 = 7$

3) The first three moments about the origin are 5, 26, 78. Show that the first three moments about the value $x=3$ are 2, 5, -48.

Sol: Given $E(X)=5$, $E(X^2)=26$, $E(X^3)=78$

Let μ_1', μ_2', μ_3' be the first three moments about $x=3$.

$\therefore \mu_1' = E(X-3) = E(X)-3 = 5-3 = 2$

$\mu_2' = E(X-3)^2 = E(X^2-6X+9)$

$= E(X^2) - 6E(X) + 9 = 26 - (6\times5)+9 = 5$

$\mu_3' = E(X-3)^3 = E(X^3 - 9X^2 + 27X - 27)$

$= E(X^3) - 9E(X^2) + 27E(X) - 27$

$= 78 - (9\times26) + (27\times5) - 27 = -48$

4) A continuous r.v X has the p.d.f $f(x) = kx^2e^{-x}$, $x \geq 0$. Find the r^{th} moment of X about the origin. Hence find the mean and variance.

Sol:

W.K.T. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\Rightarrow \int_{0}^{\infty} kx^2 e^{-x}\,dx = 1$

$k\left[x^2(-e^{-x}) - 2x(e^{-x}) + 2(-e^{-x})\right]_0^{\infty} = 1$

$\Rightarrow k = 1/2$

$\mu_r' = r^{th}$ moment about the origin

$$= E(x^r) = \int_{-\infty}^{\infty} x^r f(x)\,dx$$

$$= \int_0^{\infty} x^r \cdot \frac{1}{2} x^2 e^{-x}\,dx = \frac{1}{2}\int_0^{\infty} x^{r+2} e^{-x}\,dx$$

$$= \frac{1}{2}\int_0^{\infty} e^{-x} x^{r+3-1}\,dx \qquad \because \Gamma n = \int_0^{\infty} e^{-x} x^{n-1}\,dx$$

$$= \frac{1}{2}\Gamma(r+3) = \frac{1}{2}(r+2)! \qquad \because \Gamma n = (n-1)!$$

$$\text{Mean} = \mu_1' = \frac{1}{2} \cdot 3! = 3$$

$$\mu_2' = \frac{1}{2} \cdot 4! = 12$$

$$\text{Variance} = \mu_2' - \mu_1'^2 = 12 - 9 = 3$$

5) A continuous r.v x has pdf $f(x) = k(1-x)$ for $0<x<1$. Find the r^{th} moment about the origin. Hence find mean and variance.

Sol: W.K.T $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \int_0^1 k(1-x)\,dx = 1 \Rightarrow k\left(x - \frac{x^2}{2}\right)_0^1 = 1 \Rightarrow \boxed{k=2}$$

$\mu_r' = r^{th}$ moment about the origin

$$= E(x^r) = \int_{-\infty}^{\infty} x^r f(x)\,dx$$

$$= \int_0^1 x^r \cdot 2(1-x)\,dx$$

$$= 2\left[\frac{x^{r+1}}{r+1} - \frac{x^{r+2}}{r+2}\right]_0^1 = \frac{2}{(r+1)(r+2)}$$

$$\mu_1' = \text{Mean} = \frac{2}{2\times 3} = \frac{1}{3}$$

$$\mu_2' = \frac{2}{3\times 4} = \frac{1}{6}$$

$$\therefore \mu_2 = \mu_2' - \mu_1'^2 = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$$

6.) Find the first three moments of X if X has the following distribution:

x	-2	1	3
$p(x)$	$1/2$	$1/4$	$1/4$

Sol:

Let μ_1', μ_2', μ_3' be the first three moments about the origin.

$$\therefore \mu_1' = E(X) = \Sigma x\,p(x)$$

$$= \left((-2)\times \tfrac{1}{2}\right) + \left(1\times \tfrac{1}{4}\right) + \left(3\times \tfrac{1}{4}\right) = 0$$

$$\mu_2' = E(X^2) = \Sigma x^2 p(x)$$

$$= \left((-2)^2\times \tfrac{1}{2}\right) + \left(1\times \tfrac{1}{4}\right) + \left(3^2\times \tfrac{1}{4}\right) = \tfrac{9}{2}$$

$$\mu_3' = E(X^3) = \Sigma x^3 p(x)$$

$$= \left((-2)^3\times \tfrac{1}{2}\right) + \left(1^3\times \tfrac{1}{4}\right) + \left(3^3\times \tfrac{1}{4}\right) = 3$$

7.) Find the first four moments about the origin for a r.v X having the pdf

$$f(x) = \begin{cases} \dfrac{4x(9-x^2)}{81}, & 0 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$$

Sol:

Let μ_r' be the r^{th} moment about the origin.

$$\therefore \mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\,dx$$

$$= \int_0^3 x^r \cdot \frac{4x(9-x^2)}{81}\,dx$$

$$= \frac{4}{81} \int_0^3 (9x^{r+1} - x^{r+3})\,dx$$

$$= \frac{4}{81} \left[9 \cdot \frac{x^{r+2}}{r+2} - \frac{x^{r+3}}{r+4} \right]_0^3$$

$$= \frac{4}{81} \cdot 3^{r+4} \left[\frac{1}{r+2} - \frac{1}{r+4} \right]$$

If $r = 1, 2, 3, 4$ we get the first moments about the origin.

$$\therefore \mu_1' = \frac{4}{81} \cdot 3^5 \left[\frac{1}{3} - \frac{1}{5} \right] = \frac{8}{5}$$

$$\mu_2' = \frac{4}{81} \cdot 3^6 \left[\frac{1}{4} - \frac{1}{6} \right] = 3$$

$$\mu_3' = \frac{4}{81} \cdot 3^7 \left(\frac{1}{5} - \frac{1}{7} \right) = 6.17$$

$$\mu_4' = \frac{4}{81} \cdot 3^8 \left(\frac{1}{6} - \frac{1}{8} \right) = 13.5$$

H.W

8) Find the first four raw moments and central moments of $f(x) = kx(2-x)$, $0 \le x \le 2$.

Ans: $K = 3/4$, $\mu_1' = 1$, $\mu_2' = \frac{6}{5}$, $\mu_3' = \frac{8}{5}$, $\mu_4' = \frac{16}{7}$, $\mu_1 = 0$, $\mu_2 = 1/5$, $\mu_3 = 0$, $\mu_4 = 3/35$

9.) If $f(x) = ce^{-ax}$, $a > 0$, $x \geq 0$ is a pdf then find c and the first three moments about mean.

Sol:

W.K.T $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\Rightarrow \int_0^{\infty} ce^{-ax}\,dx = 1 \Rightarrow c\left[\frac{e^{-ax}}{-a}\right]_0^{\infty} = 1$

$\Rightarrow c\left(\frac{1}{a}\right) = 1 \Rightarrow c = a$

$\mu_1' = \int_0^{\infty} axe^{-ax}\,dx = a\left[x\left(\frac{-e^{-ax}}{a}\right) - 1\cdot\left(\frac{e^{-ax}}{a^2}\right)\right]_0^{\infty} = \frac{1}{a}$

$\mu_2' = a\int_0^{\infty} x^2 e^{-ax}\,dx = a\left[x^2\left(\frac{e^{-ax}}{-a}\right) - 2x\left(\frac{e^{-ax}}{a^2}\right) + 2\left(\frac{e^{-ax}}{-a^3}\right)\right]_0^{\infty}$

$= 2/a^2$

Similarly

$\mu_3' = 6/a^3$

$\therefore \mu_1 = 0$, $\mu_2 = \mu_2' - {\mu_1'}^2 = \frac{2}{a^2} - \frac{1}{a^2} = \frac{1}{a^2}$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2{\mu_1'}^3 = \frac{6}{a^3} - 3\cdot\frac{2}{a^3} + \frac{2}{a^3} = \frac{2}{a^3}$

—x—

Expectation and Variance

An average of a prob. dist. of a r.v X is called the expetation or the expected value or Mathematical expectation of X and is denoted by E(X).

Discrete r.v: Let X be a discrete r.v taking values $x_1, x_2, \dots x_n, \dots$ with probabilities $p(x_1), p(x_2) \dots p(x_n)$, then the expected value of X is defined as

$$E(X) = \sum_{i=1}^{\infty} x_i \, p(x_i)$$

Continuous r.v: Let X be a continuous r.v with pdf f defined in $(-\infty, \infty)$, then the expected value of X is defined as

$$E(X) = \int_{-\infty}^{\infty} x f(x) \, dx$$

Note: (1) E(X) is called the mean of the distribution or mean of X and is denoted by $\bar{X}$ or μ.

(2) $E(X^2) = \sum x_i^2 \, p(x_i)$ If X is discrete

$= \int_{-\infty}^{\infty} x^2 f(x) \, dx$ If X is continuous

Properties of Expectation:

(i) If c is constant, then $E(c) = c$

(ii) If a, b are constants, then $E(aX+b) = aE(X)+b$

Variance: Let X be a r.v with mean $E(X)$, then the variance of X is defined as

$$E[X-E(X)]^2.$$

It is denoted by $Var(X)$ or σ_X^2

i.e., $Var(X) = E[X-E(X)]^2$

For Discrete r.v: If X is a discrete r.v taking values $x_1, x_2, x_3 \ldots$ with probabilities $p(x_1), p(x_2), p(x_3)\ldots$ and $E(X) = \mu$ then

$$Var(X) = \sum_i (x_i - \mu)^2 p(x_i)$$

For Continuous r.v: If X is a Continuous r.v with pdf $f(x)$, $x \in (-\infty, \infty)$ then

$$Var(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

Prove that $Var(X) = E(X^2) - [E(X)]^2$

Proof: Let X be a r.v then mean of $X = E(X) = \mu$ (say)

$$\therefore Var(X) = E[X-E(X)]^2 = E[X-\mu]^2$$

$$= E[X^2 - 2\mu X + \mu^2]$$

$$= E(X^2) - 2\mu E(X) + \mu^2$$

$$= E(X^2) - 2\mu^2 + \mu^2 = E(X^2) - \mu^2$$

$$= E(X^2) - [E(X)]^2$$

Properties of variance

If a and b are constants, then

(i) $Var(aX) = a^2 Var(X)$ (ii) $Var(aX+b) = a^2 Var(X)$

(iii) $Var(aX-b) = a^2 Var(X)$

Note: $Var(b) = 0$

1. If X has the probability distribution

X	-1	0	1	2
$P(x)$	0.3	0.1	0.4	0.2

Find $E(X)$, $E(X^2)$, $Var(X)$, $E(2X+1)$, $Var(2X+1)$.

Sol:

(i) $E(X) = \Sigma x_i p(x_i) = (-1)\times 0.3 + (0\times 0.1) + (1\times 0.4) + (2\times 0.2)$

$= -0.3 + 0.4 + 0.4 = 0.5$

(ii) $E(X^2) = \Sigma x_i^2 p(x_i)$

$= [(-1)^2 \times 0.3] + [0\times 0.1] + [1^2 \times 0.4] + [2^2 \times 0.2]$

$= 0.3 + 0.4 + 0.8 = 1.5$

(iii) $Var(X) = E(X^2) - [E(X)]^2 = 1.5 - (0.5)^2 = 1.25$

(iv) $E(2X+1) = 2E(X) + 1 = (2\times 0.5) + 1 = 2$ $\quad \because E(aX+b) = aE(X)+b$

(v) $Var(2X+1) = 4\,Var(X) = 4 \times 1.25 = 5$ $\quad \because Var(aX+b) = a^2 Var(X)$

2) If X has the probability distribution

X	0	1	2	3	4
P(X)	1/25	3/25	5/25	7/25	9/25

. Find mean, Variance, $E(3X-4)$ and $Var(3X-4)$.

<u>Sol</u>:

(i) $\text{Mean} = E(X) = \sum_i x_i\, p(x_i)$

$$= \left(0\times\frac{1}{25}\right)+\left(1\times\frac{3}{25}\right)+\left(2\times\frac{5}{25}\right)+\left(3\times\frac{7}{25}\right)+\left(4\times\frac{9}{25}\right)$$

$$= \frac{70}{25} = \frac{14}{5}$$

(ii) $\text{Variance} = E(X^2) - [E(X)]^2$

Now $E(X^2) = \sum x_i^2\, p(x_i)$

$$= \left(0\times\frac{1}{25}\right)+\left(1^2\times\frac{3}{25}\right)+\left(2^2\times\frac{5}{25}\right)+\left(3^2\times\frac{7}{25}\right)+\left(4^2\times\frac{9}{25}\right)$$

$$= \frac{230}{25} = \frac{46}{5}$$

$\therefore\ Var(X) = \frac{46}{5} - \left(\frac{14}{5}\right)^2 = \frac{230-196}{25} = \frac{34}{25}$

(iii) $E(3X-4) = 3E(X) - 4 = 3\times\frac{14}{5} - 4 = \frac{22}{5}$

(iv) $Var(3X-4) = 9\,Var(X) = 9\times\frac{34}{25} = \frac{306}{25}$

3) In a gambling game of tossing 4 coins once, a man gets Rs. 10 If he gets all heads or all tails and loses Rs. 5 If he gets 1 or 2 or 3 heads.

What is his expected gain?

Sol: Let X denote the amount gained.

The possible values of X are 10, −5

$P(X=10) = P$ [getting 4 heads or 4 tails]

$= \frac{1}{16} + \frac{1}{16} = \frac{1}{8}$

$P(X=-5) = P$(1H and 3T or 2H and 2T or 3H and 1T)

$= 4C_1 \times \frac{1}{16} + 4C_2 \times \frac{1}{16} + 4C_3 \times \frac{1}{16}$

$= \frac{4}{16} + \frac{6}{16} + \frac{4}{16} = \frac{7}{8}$

The prob. dist. of X is

x	−5	10
$p(x)$	1/8	7/8

$E(x) = \sum_i x_i p(x_i) = \left(-5 \times \frac{1}{8}\right) + \left(10 \times \frac{7}{8}\right) = \frac{25}{8}$

4.) When a die is thrown find the expectation of the number on the die.

Sol: Let X denote the number on a die.

∴ The possible values of X are 1, 2, 3, 4, 5, 6

The prob. dist. of X is

x	1	2	3	4	5	6
$p(x)$	1/6	1/6	1/6	1/6	1/6	1/6

$\therefore E(x) = \sum x_i p(x_i)$

$= \left(1\times\frac{1}{6}\right)+\left(2\times\frac{1}{6}\right)+\left(3\times\frac{1}{6}\right)+\left(4\times\frac{1}{6}\right)+\left(5\times\frac{1}{6}\right)+\left(6\times\frac{1}{6}\right)$

$= 7/2$

5) A coin is tossed until a head appears. What is the expected value of the number of tosses?

Sol:

Let x denote the number of tosses required to get the first success (head).

$\therefore$ The head can appear in the first toss or second toss or third toss and so on.

ie, the events are $\{H, TH, TTH, TTTH, \ldots\}$

$\therefore$ The probabilities are $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots$

$E(x) = \sum x_i p(x_i)$

$= \frac{1}{2} + 2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + \ldots$ [$\because$ x takes the values $1, 2, 3, \ldots$]

$= \frac{1}{2}\left[1 + 2\cdot\frac{1}{2} + 3\cdot\left(\frac{1}{2}\right)^2 + \ldots\right]$

$= \frac{1}{2}\left(1-\frac{1}{2}\right)^{-2}$ [$\because (1-x)^{-2} = 1+2x+3x^2+\ldots$]

$= 2$

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A man draws 3 balls from an urn containing 5 white and 7 black balls. He gets Rs. 10 for each white ball and Rs. 5 for each black ball. Find his expectation.

Sol:

Let X denote the amount gained.

3 balls drawn may be

(i) 3 black (ii) 2 black & 1 white

(iii) 1 black and 2 white (iv) 3 white

$\therefore$ the values of X are 15, 20, 25, 30

$P(X=15) = P(3 \text{ black balls}) = \frac{7C_3}{12C_3} = \frac{7}{44}$

$P(X=20) = P(2 \text{ black and } 1 \text{ white}) = \frac{7C_2 \times 5C_1}{12C_3} = \frac{21}{44}$

$P(X=25) = P(1 \text{ black and } 2 \text{ white}) = \frac{7C_1 \times 5C_2}{12C_3} = \frac{14}{44}$

$P(X=30) = P(3 \text{ white balls}) = \frac{5C_3}{12C_3} = \frac{2}{44}$

The prob. distribution of X is

x	15	20	25	30
$p(x)$	$\frac{7}{44}$	$\frac{21}{44}$	$\frac{14}{44}$	$\frac{2}{44}$

$E(x) = \left(15 \times \frac{7}{44}\right) + \left(20 \times \frac{21}{44}\right) + \left(25 \times \frac{14}{44}\right) + \left(30 \times \frac{2}{44}\right)$

$= \frac{935}{44}.$

7.) Find the expectation of the number of failures proceeding the first success in an infinite series of independent trials with constant probability p of success in each trial.

Sol:

Let x denote the number of failures proceding first success.

Then the events are $\{S, FS, FFS, FFFS, \ldots\}$

Hence x takes the values $0, 1, 2, 3, \ldots$

and the corresponding probabilities are

$$p, qp, q^2p, q^3p, \ldots$$

$$E(x) = 0 \cdot p + 1 \cdot pq + 2 \cdot q^2 p + 3 \cdot q^3 p + \cdots$$

$$= qp[1 + 2q + 3q^2 + 4q^3 + \cdots]$$

$$= qp(1-q)^{-2} = qp \cdot \frac{1}{p^2} = \frac{q}{p} \quad \because 1-p = q$$

8.) The monthly demand for Allwyn watches is known to have the following distribution

Demand	1	2	3	4	5	6	7	8
Probability	0.08	0.12	0.19	0.24	0.16	0.10	0.07	0.04

Determine the expected demand for watches. Also compute the variance.

Sol: Let X denote the demand for Allwyn watches

$E(X) = \sum_i x_i p(x_i)$

$= (1\times0.08) + (2\times0.12) + (3\times0.19) + (4\times0.24) + (5\times0.16)$

$+ (6\times0.10) + (7\times0.07) + (8\times0.04) = 4.06$

$E(X^2) = \sum_i x_i^2 p(x_i)$

$= (1\times0.08) + (4\times0.12) + (9\times0.19) + (16\times0.24) + (25\times0.16)$

$+ (36\times0.10) + (49\times0.07) + (64\times0.04) = 19.7$

$Var(X) = E(X^2) - [E(X)]^2 = 19.7 - (4.06)^2 = 3.2164$

Given the following prob. dist. of X compute

(i) $E(X)$ (ii) $E(X^2)$ (iii) $E[2X\pm3]$ (iv) $Var(2X\pm3)$

$X = x$	-3	-2	-1	0	1	2	3
$P(X=x)$	0.05	0.10	0.30	0	0.30	0.15	0.1

Sol:

(i) $E(X) = \sum_i x_i p(x_i)$

$= (-3\times0.05) + (-2\times0.10) + (-1\times0.30) + 0 + (1\times0.3)$

$+ (2\times0.15) + (3\times0.1) = 0.25$

(ii) $E(X^2) = \sum_i x_i^2 p(x_i)$

$= (9\times0.05) + (4\times0.10) + (1\times0.30) + 0 + (1\times0.3)$

$+ (4\times0.15) + (9\times0.1) = 2.95$

(iii) $E(2X\pm3) = 2E(X)\pm3 = 2(0.25)\pm3 = -2.5, 3.5$

(iv) $Var(2X \pm 3) = 2^2 Var(X) = 4 Var(X)$

Now $Var(X) = E(X^2) - [E(X)]^2$

$= 2.95 - (0.25)^2 = 2.8875$

$\therefore Var(2X \pm 3) = 4(2.8875) = 11.55$

10) Find the mean and variance of the following density function $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 \le x < 2 \\ 0, & \text{otherwise.} \end{cases}$

Sol: $E(x) = \int_{-\infty}^{\infty} x f(x) dx$

$\Rightarrow E(x) = \int_0^1 x \cdot x \, dx + \int_1^2 x(2-x) dx$

$= \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_1^2$

$= \frac{1}{3} + \left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right) = 1$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$

$\Rightarrow E(X^2) = \int_0^1 x^3 dx + \int_1^2 x^2(2-x) dx$

$= \left(\frac{x^4}{4}\right)_0^1 + \left(\frac{2x^3}{3} - \frac{x^4}{4}\right)_1^2$

$= \frac{1}{4} + \left(\frac{2^4}{3} - \frac{2^4}{4}\right) - \left(\frac{2}{3} - \frac{1}{4}\right) = \frac{1}{4} + 2^4\left(\frac{1}{3} - \frac{1}{4}\right) - \left(\frac{8-3}{12}\right)$

$= \frac{1}{4} + \frac{2^4}{12} - \frac{5}{12} = \frac{14}{12}$ or $\frac{7}{6}$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{7}{6} - 1 = \frac{1}{6}$$

11.) The density function of a continuous r.v X is given by $f(x) = \begin{cases} x/2, & \text{If } 0<x<2 \\ 0 & \text{otherwise} \end{cases}$. Find (i) the expectation of X (ii) Variance of X (iii) $E(3X^2 - 2X)$.

<u>Sol:</u>

(i) $E(X) = \int_{-\infty}^{\infty} x f(x) dx$

$$= \int_0^2 x \cdot \frac{x}{2} dx = \frac{1}{2}\left(\frac{x^3}{3}\right)_0^2 = \frac{4}{3}$$

(ii) $Var(X) = E(X^2) - [E(X)]^2$

Now $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$

$$= \int_0^2 x^2 \cdot \frac{x}{2} dx = \frac{1}{2}\left(\frac{x^4}{4}\right)_0^2 = 2$$

$$\therefore Var(X) = 2 - \left(\frac{4}{3}\right)^2 = \frac{18-16}{9} = 2/9$$

(iii) $E(3X^2 - 2X) = 3E(X^2) - 2E(X)$

$$= 6 - \frac{8}{3} = \frac{10}{3}$$

12.) Find the expectation and Variance of the r.v X whose pdf is given by $f(x) = \begin{cases} 2e^{-2x}, & \text{If } x>0 \\ 0, & \text{otherwise} \end{cases}$

<u>Sol:</u>

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(i) $E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$

$= \int_0^{\infty} x \cdot 2e^{-2x}\, dx = 2\int_0^{\infty} x e^{-2x}\, dx$

W.K.T $\int u\,dv = uv - u'v_1 + u''v_2 - \cdots$

$u = x \qquad dv = e^{-2x}\, dx$

$u' = 1 \qquad v = \frac{e^{-2x}}{-2}$

$v_1 = \frac{e^{-2x}}{2^2}$

$E(X) = 2\left[-\frac{xe^{-2x}}{2} - \frac{e^{-2x}}{2^2}\right]_0^{\infty} = 2\cdot\frac{1}{2^2} = \frac{1}{2}$

(ii) $Var(X) = E(X^2) - [E(X)]^2$

Now $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$

$= 2\int_0^{\infty} x^2 e^{-2x}\, dx$

$E(X^2) = 2\left[x^2\cdot\left(\frac{e^{-2x}}{-2}\right) - (2x)\left(\frac{e^{-2x}}{2^2}\right) + 2\cdot\left(\frac{e^{-2x}}{-2^3}\right)\right]_0^{\infty}$

$= 2\cdot\frac{1}{2^2} = \frac{1}{2}$

$\therefore \quad Var(X) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

13) Find mean and standard deviation of the distributio[n]

$f(x) = \begin{cases} Kx(2-x), & 0 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}$

Sol:

W.K.T $\int_{-\infty}^{\infty} f(x)dx = 1$

$\Rightarrow \int_0^2 kx(2-x)dx = 1 \Rightarrow k\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^2 = 1$

$\Rightarrow k\left[4 - \frac{8}{3}\right] = 1 \Rightarrow k = 3/4$

Mean:

$E(X) = \int_{-\infty}^{\infty} x f(x)dx = \int_0^2 x \cdot \frac{3}{4}x(2-x)dx$

$= \frac{3}{4}\left[2\frac{x^3}{3} - \frac{x^4}{4}\right]_0^2 = \frac{3}{4} \cdot 2^4\left(\frac{1}{3} - \frac{1}{4}\right)$

$= \frac{3}{4} \cdot 2^4 \cdot \frac{1}{12} = 1$

Now $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^2 x^2 \cdot \frac{3}{4}x(2-x)dx$

$= \frac{3}{4}\left[2\frac{x^4}{4} - \frac{x^5}{5}\right]_0^2 = \frac{3}{4} \cdot 2^5\left(\frac{1}{4} - \frac{1}{5}\right) = \frac{6}{5}$

$\therefore Var(X) = E(X^2) - [E(X)]^2 = \frac{6}{5} - 1 = 1/5$

$\Rightarrow S.D = \sqrt{\sigma} = \frac{1}{\sqrt{5}}$

A test engineer found that the dist. function of the life time X in years of an equipment is given by

$$F(x) = \begin{cases} 0 & \text{If } x < 0 \\ 1 - e^{-x/5} & \text{If } x \geq 0 \end{cases}$$

Find (i) the pdf of X

(ii) the expected life time of the equipment

(iii) the variance of the life time " "

Sol:

(i) $f(x) = \frac{d}{dx} F(x)$ or $F'(x)$

$$= \begin{cases} \frac{1}{5} e^{-x/5}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(ii) $E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \frac{1}{5}\int_{0}^{\infty} x e^{-x/5}\, dx$

$$= \frac{1}{5}\left[x \cdot \frac{e^{-x/5}}{-1/5} - 1 \cdot \frac{e^{-x/5}}{1/5^2}\right]_0^{\infty}$$

$$= \frac{1}{5}\left[\frac{1}{1/5^2}\right] = 5$$

(iii) $Var(X) = E(X^2) - [E(X)]^2$

Now $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$

$$= \frac{1}{5}\int_{0}^{\infty} x^2 e^{-x/5}\, dx$$

$$= \frac{1}{5}\left[x^2 \cdot \frac{e^{-x/5}}{-1/5} - 2x \cdot \frac{e^{-x/5}}{1/5^2} + 2 \cdot \frac{e^{-x/5}}{-1/5^3}\right]_0^{\infty}$$

$$= \frac{1}{5}\left[2 \cdot \frac{1}{1/5^3}\right] = 50$$

$\therefore Var(X) = 50 - 25 = 25$

A continuous r.v has pdf $f(x) = \begin{cases} a+bx, & 0 \leq x \leq 1 \\ 0, & \text{otherwise.} \end{cases}$

If the mean of the distribution is $\frac{1}{2}$ then find a, b. Also Evaluate $Var(X)$.

Sol: W.K.T $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \int_0^1 (a+bx)\,dx = 1 \Rightarrow \left(ax + \frac{bx^2}{2}\right)_0^1 = 1$$

$$\Rightarrow 2a+b=2 \longrightarrow (1)$$

Given $E(X) = 2$

$$\Rightarrow \int_{-\infty}^{\infty} x f(x)\,dx = \frac{1}{2}$$

$$\Rightarrow \int_0^1 x(a+bx)\,dx = \frac{1}{2}$$

$$\Rightarrow \left[\frac{ax^2}{2} + \frac{bx^3}{3}\right]_0^1 = \frac{1}{2}$$

$$\Rightarrow 3a+2b=3 \longrightarrow (2)$$

Solving (1) and (2), we get $a=1,\ b=0$

Now $E(X^2) = \int_0^1 x^2 f(x)\,dx$

$$= \int_0^1 x^2(a+bx)\,dx = \int_0^1 x^2\,dx$$

$$= \left(\frac{x^3}{3}\right)_0^1 = \frac{1}{3}$$

$$\therefore Var(X) = E(X^2) - [E(X)]^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

H.W

16) Find the value of (i) k (ii) mean of the following dist.

a) $f(x) = \begin{cases} k(x-x^2), & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$ (b) $f(x) = \begin{cases} ke^{-x}, & 0<x<\infty \\ 0, & \text{otherwise} \end{cases}$

Ans: (a) $k=6$, $E(X)=\frac{1}{2}$ (b) $k=1$, $E(X)=1$

H.W

17.) A Continuous r.v X has the pdf $f(x) = \begin{cases} \frac{1}{2}(x+1), & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

Find the mean and Variance of X. Ans: $\frac{1}{3}, \frac{2}{9}$

18.) Let X be a r.v with $E(X)=10$ and $V(X)=25$. Find the +ve values of 'a' and 'b' such that $Y=aX-b$ has expectation 0 and variance 1.

Sol:

Given $E(X)=10$, $V(X)=25$, $E(Y)=0$, $V(Y)=1$

Now $E(Y)=E(aX-b)=aE(X)-b$

$\Rightarrow 0=10a-b \longrightarrow (1)$

and $Var(Y) = var(aX+b) = a^2 var(X)$

$\Rightarrow 1=a^2 . 25 \Rightarrow a^2 = \frac{1}{25} \longrightarrow (2)$

Solving (1) and (2), we get

$a = \frac{1}{5}, \; b=2$

19) If X has the distribution function

$$F(x) = \begin{cases} 0 & \text{for } x<1 \\ \frac{1}{3} & \text{for } 1 \le x < 4 \\ \frac{1}{2} & \text{for } 4 \le x < 6 \\ \frac{5}{6} & \text{for } 6 \le x \le 10 \\ 1 & \text{for } x \ge 10 \end{cases}$$

Find (i) the probability distribution of x

(ii) $P(2<x<6)$ (iii) Mean of x (iv) Variance of X.

Sol:

(i) W.K.T $P(X=x_i) = F(x_i) - F(x_{i-1})$

$P(X=1) = F(1) - F(0) = 1/3 - 0 = 1/3$

$P(X=4) = F(4) - F(1) = \frac{1}{2} - \frac{1}{3} = 1/6$

$P(X=6) = F(6) - F(4) = \frac{5}{6} - \frac{1}{2} = 1/3$

$P(X=10) = F(10) - F(6) = 1 - 5/6 = 1/6$

$\therefore$ the prob. dist. of X is

x	1	4	6	10
$p(x)$	1/3	1/6	1/3	1/6

(ii) $P(2<X<6) = P(X=4) = \frac{1}{3}$

(iii) Mean of $X = E(X) = \Sigma x\, p(x)$

$= \left(1\times\frac{1}{3}\right) + \left(4\times\frac{1}{6}\right) + \left(6\times\frac{1}{3}\right) + \left(10\times\frac{1}{6}\right) = \frac{14}{3}$

(iv) $Var(X) = E(X^2) - [E(X)]^2$

Now $E(X^2) = \Sigma x^2 p(x)$

$= \left(1\times\frac{1}{3}\right) + \left(16\times\frac{1}{6}\right) + \left(36\times\frac{1}{3}\right) + \left(100\times\frac{1}{6}\right) = \frac{95}{3}$

$\therefore Var(X) = \frac{95}{3} - \left(\frac{14}{3}\right)^2 = \frac{89}{9}$

20) The cumulative distribution function of a r.v x is $F(x) = 1-(1+x)e^{-x}$, $x>0$. Find the probability density function of x, mean and variance of x.

Sol:

$$\text{pdf } f(x) = \frac{d}{dx}F(x) = \frac{d}{dx}\left[1-(1+x)e^{-x}\right]$$

$$= \frac{d}{dx}\left[1-e^{-x}-xe^{-x}\right]$$

$$= 0+e^{-x}-\left[-xe^{-x}+e^{-x}\right] = xe^{-x}, \quad x>0$$

$$\text{Mean} = E(X) = \int_0^\infty x f(x)\,dx = \int_0^\infty x^2 e^{-x}\,dx$$

$$= \left[x^2\left(\frac{e^{-x}}{-1}\right) - 2x\left(e^{-x}\right) + 2\left(\frac{e^{-x}}{-1}\right)\right]_0^\infty = 2$$

$$E(X^2) = \int_0^\infty x^2 \cdot xe^{-x}\,dx = \int_0^\infty x^3 e^{-x}\,dx$$

$$= \left[x^3\left(\frac{e^{-x}}{-1}\right) - 3x^2\left(e^{-x}\right) + 6x\left(\frac{e^{-x}}{-1}\right) - 6\left(e^{-x}\right)\right]_0^\infty = 6$$

$$\therefore \text{Var}(X) = E(X^2) - [E(X)]^2 = 6-4 = 2$$

Home work

1) Find the mean and variance of the discrete r.v X with the following prob. function

$$p(x) = \begin{cases} 2/5 & \text{for } x=3 \\ 3/5 & \text{for } x=6 \end{cases}$$

Ans: $\frac{24}{5}$, $\frac{54}{25}$

2) If the r.v X has the pdf $f(x) = ax^3$, $0<x<1$ then find (i) a (ii) $E(X)$ (iii) $Var(X)$ (iv) the value of m so that $P(X \le m) = 1/2$.

Ans: $a=4$, $E(X) = \frac{4}{5}$, $Var(X) = \frac{2}{75}$, $m = \sqrt[4]{1/2}$

3) A test engineer proposed, after a series of tests, that the life time X of component is a r.v with pdf
$$f(x) = \begin{cases} \frac{x}{100} e^{-x/10}, & x \ge 10 \\ 0, & \text{otherwise} \end{cases}$$
Find (i) $P(X>20)$ (ii) $E(X)$ (iii) $Var(X)$.

Ans: $P(X>20) = 0.406$, $E(X) = 20$, $Var(X) = 200$.

4) If X has the distribution function
$$F(x) = \begin{cases} 0 & \text{If } x<0 \\ 1/6 & \text{If } x \le x < 2 \\ 1/2 & \text{If } 2 \le x < 4 \\ 5/8 & \text{If } 4 \le x < 6 \\ 1 & \text{If } x \ge 6 \end{cases}$$
Find (i) the prob. dist. of X (ii) $P(1<X<5)$ (iii) mean and variance of X.

Ans: (i)

x	0	2	4	6
$P(x)$	$1/6$	$1/3$	$1/8$	$3/8$

(ii) $\frac{11}{24}$ (iii) $E(X) = \frac{37}{12}$, $E(X^2) = \frac{101}{6}$

$Var(X) = 7.32$

Moments Generating Function : (MGF)

MGF of a r.v X is defined as $E(e^{tX})$ where 't' is a real variable. It is denoted by $M_X(t)$ or $M(t)$.

i.e., $M_X(t) = E(e^{tX})$

If X is discrete, then $M_X(t) = \sum_r e^{tx_r} p_r$

where X takes the values $x_1, x_2, x_3 \ldots$ with probabilities $p_1, p_2, p_3 \ldots$

If X is a continuous rv with density function $f(x)$, then $M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$.

Properties of MGF

(1) Let X be a RV with MGF $M_X(t)$ and c is a constant. Then (a) $M_{cX}(t) = M_X(ct)$ (b) $M_{X+c}(t) = e^{ct} M_X(t)$.

(2) If X and Y are two independent r.v's then $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$.

(3) If the MGF of X is $M_X(t)$ and If $Y = aX+b$, then $M_Y(t) = e^{bt} M_X(at)$

(4) MGF of a r.v X generates all the moments about the origin.

~~Proof:~~ (5) $E(X^r) = \mu_r'$ is the coefficient of $\frac{t^n}{n!}$ in the expansion of $M_X(t)$ in series of powers of t

(6) $\mu_r' = E(X^r) = \left[\frac{d^r}{dt^r} M_X(t)\right]_{t=0}$

Property 4,5: (proof)

$M_X(t) = E(e^{tX})$

$= E\left[1 + \frac{(tX)}{1!} + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + \dots + \frac{(tX)^r}{r!} + \dots\right]$

$= 1 + \frac{t}{1!} E(X) + \frac{t^2}{2!} E(X^2) + \frac{t^3}{3!} E(X^3) + \dots + \frac{t^r}{r!} E(X^r) + \dots$

$= 1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \dots + \frac{t^r}{r!}\mu_r' + \dots \rightarrow (1)$

$\therefore \mu_1' =$ Co-eff. of t in the expansion of $M_X(t)$

$\mu_2' =$ " $\frac{t^2}{2!}$ " " "

$\vdots$

$\mu_r' =$ " $\frac{t^r}{r!}$ " " "

$\therefore M_X(t)$ generates all moments about the origin and hence we call it as MGF.

property 6: (proof)

Equation (1), Diff. w. r. to 't' we get

$M_X'(t) = \mu_1' + t\mu_2' + \dots + \frac{r t^{r-1}}{r!}\mu_r' + \dots$

$$M_X''(t) = \mu_2' + t\mu_3' + \dots + \frac{r(r-1)t^{r-2}}{r!}\mu_r' + \dots$$

$$M_X'''(t) = \mu_3' + t\mu_4' + \dots + \frac{r(r-1)(r-2)t^{r-3}}{r!}\mu_r' + \dots$$

$\vdots$

putting $t=0$, we get

$$M_X'(0) = \mu_1', \quad M_X''(0) = \mu_2', \quad M_X'''(0) = \mu_3' \dots$$

In general $\mu_r' = M_X^{(r)}(0)$

A coin is tossed three times. If X denotes the number of heads that appear, find the MGF of X and hence find the mean and variance.

Sol:

$S = \{HHH, HHT, THT, HTH, HTT, THH, TTH, TTT\}$

X is a discrete r.v and no. of heads. X taking values $0, 1, 2, 3$

$X=x$	0	1	2	3
$P(X=x)$	$1/8$	$3/8$	$3/8$	$1/8$

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{3} e^{tx} p(x)$$

$$= \frac{1}{8}\left[1 + 3e^t + 3e^{2t} + e^{3t}\right]$$

$$M_X'(t) = \frac{1}{8}\left[3e^t + 6e^{2t} + 3e^{3t}\right]$$

$$\therefore \text{Mean} = E(X) = M_X'(0) = \frac{1}{8}(3+6+3) = \frac{12}{8} = 3/2$$

$$M_X''(t) = \frac{3}{8}\left[e^t + 4e^{2t} + 3e^{3t}\right]$$

$$M_X''(0) = E(X^2) = \frac{3}{8}(1+4+3) = 3$$

$$\therefore Var(X) = E(X^2) - [E(X)]^2 = 3/4$$

Find the MGF of a r.v with the probability function $P(X=x) = q^{x-1}p$, $x = 1, 2, 3 \ldots$ Find the mean and variance.

Sol:

$$M_X(t) = E(e^{tX})$$

$$= \sum_{x=1}^{\infty} e^{tx} q^{x-1} p = \frac{p}{q}\sum_{x=1}^{\infty}(e^t q)^x$$

$$= \frac{p}{q}\left[(qe^t) + (qe^t)^2 + (qe^t)^3 + (qe^t)^4 + \cdots\right]$$

$$= \frac{p}{q} \cdot qe^t \left[1 + qe^t + (qe^t)^2 + (qe^t)^3 + \cdots\right]$$

$$= pe^t(1-qe^t)^{-1}$$

Diff. w.r. to 't', we get

$$M_X'(t) = p\left\{e^t(1-qe^t)^{-1} + e^t(-1)(1-qe^t)^{-2}(-qe^t)\right\}$$

$$= pe^t(1-qe^t)^{-1} \cdot \left[1 + qe^t(1-qe^t)^{-1}\right]$$

$$= pe^t(1-qe^t)^{-1}\left[\frac{1-qe^t+qe^t}{1-qe^t}\right]$$

$$= pe^t(1-qe^t)^{-2}$$

$$\mu_1' = M_X'(0) = p(1-q)^{-2} = \frac{p}{p^2} = \frac{1}{p} \qquad \because p+q=1$$

$$q = 1-p$$

$$\therefore \text{Mean} = 1/p \qquad \text{or } p = 1-q$$

Again diff. w.r. to 't', we get

$$M_X''(t) = p\{e^t(1-qe^t)^{-2} + e^t(-2)(1-qe^t)^{-3}(-qe^t)\}$$

$$= pe^t(1-qe^t)^{-2}[1+2qe^t(1-qe^t)^{-1}]$$

$$\mu_2' = M_X''(0) = p(1-q)^{-2}(1+2q(1-q)^{-1})$$

$$= \frac{p}{p^2}\left(1+\frac{2q}{p}\right)$$

$$\therefore \mu_2 = \text{Variance} = \mu_2' - \mu_1'^2$$

$$= \frac{1}{p^2}(p+2q) - \frac{1}{p^2}$$

$$= \frac{p+2q-1}{p^2} = \frac{2q-(1-p)}{p^2} = \frac{q}{p^2}$$

3) Find the MGF for the distribution where

$$f(x) = \begin{cases} 2/3, & \text{at } x=1 \\ 1/3, & \text{at } x=2 \\ 0; & \text{otherwise} \end{cases}$$

Sol: $M_X(t) = E(e^{tX}) = \sum_{x=1}^{2} e^{tx} f(x) = \frac{2}{3}e^t + \frac{1}{3}e^{2t}$

4) Find the MGF of the r.v whose moments are $\mu_r' = (r+1)!\,2^r$.

Sol: W.K.T $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!}(r+1)!\,2^r$$

$$M_X(t) = \sum_{r=0}^{\infty} (r+1)(2t)^r = 1+2(2t)+3(2t)^2 + \cdots$$

$$= (1-2t)^{-2} \qquad [\because (1-x)^{-2} = 1+2x+3x^2+\cdots$$

If a r.v 'x' has the MGF $M_X(t) = \frac{3}{3-t}$, find the S.D.

Sol: Method : 1

$$M_X(t) = \frac{3}{3-t} = 3(3-t)^{-1}$$

$$M_X'(t) = 3(3-t)^{-2}, \quad M_X''(t) = 6(3-t)^{-3}$$

$$\Rightarrow E(X) = M_X'(0) = 1/3, \quad E(X^2) = M_X''(0) = 2/9$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{2}{9} - 1/9 = 1/9$$

$$\therefore S.D = \sqrt{Var X} = \frac{1}{3}$$

Method : 2

$$M_X(t) = (1-t/3)^{-1} = 1 + \frac{t}{3} + \frac{t^2}{9} + \cdots$$

$$E(X) = \text{Co-eff. of } \frac{t}{1!} = 1/3$$

$$E(X^2) = \text{Co-eff. of } t^2/2! = \frac{2}{9}$$

$$\therefore Var(X) = E(X^2) - [E(X)]^2 = \frac{2}{9} - \frac{1}{9} = 1/9$$

Find the MGF of a r.v 'x' whose prob. function is $P(x) = \frac{1}{2^x}$, $x = 1, 2, 3 \ldots$. Hence find its mean.

Sol:

$$M_X(t) = \sum_{x=1}^{\infty} e^{tx} P(x) = \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x$$

$$= \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \cdots$$

$$M_X(t) = \frac{e^t}{2}\left[1+\frac{e^t}{2}+\left(\frac{e^t}{2}\right)^2+\left(\frac{e^t}{2}\right)^3+\cdots\right]$$

$$= \frac{e^t}{2}\left(1-\frac{e^t}{2}\right)^{-1} = \frac{e^t}{2}\left(\frac{2-e^t}{2}\right)^{-1} = \frac{e^t}{2-e^t}$$

$$M_X'(t) = e^t(2-e^t)^{-1} + e^t(-1)(2-e^t)^{-2}(-e^t)$$

$$\cancel{M_X'(0) = e^0\, 2^{-1} + 2^{-2} = \frac{1}{2}+\frac{1}{4}} =$$

$$M_X'(0) = (2-1)^{-1} + (2-1)^{-2} = 2$$

$\therefore$ Mean $= 2$

H.W

7.) If a r.v X has the MGF $M_X(t) = \frac{2}{2-t}$, find the variance. Ans: $M_X'(0) = 1/2$, $M_X''(0) = 1/2$, $Var(X) = 1/2$

8.) If X represents the outcome, when a fair die is tossed. Find the MGF of X and hence find $E(X)$ and $Var(X)$

Sol:

X denotes the number on a die

$X=x$	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$M_X(t) = \sum_{x=1}^{6} e^{tx} P(X=x)$$

$$= \frac{1}{6}\left[e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}\right]$$

$$M_X'(t) = \frac{1}{6}\left[e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t}\right]$$

$$E(X) = M_X'(0) = \frac{1}{6}(1+2+3+4+5+6) = \frac{21}{6} = 7/2$$

$$M_x''(t) = \frac{1}{6}\left[e^{t} + 4e^{2t} + 9e^{3t} + 16e^{4t} + 25e^{5t} + 36e^{6t}\right]$$

$$E(X^2) = M_x''(0) = \frac{1}{6}\left[1+4+9+16+25+36\right] = \frac{91}{6}$$

$$Var(X) = E(X^2) - \left[E(X)\right]^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

9.) The RV X has MGF $M_X(t) = \frac{3}{3-t}$. Find first four moments about its origin and also find four moments about its mean.

Sol:

Given $M_X(t) = 3(3-t)^{-1}$

$\Rightarrow M_X'(t) = 3(3-t)^{-2}$, $M_X''(t) = 6(3-t)^{-3}$, $M_X'''(t) = 18(3-t)^{-4}$

and $M_X^{IV}(t) = 72(3-t)^{-5}$

$\mu_1' = M_X'(0) = \frac{1}{3}$, $\mu_2' = M_X''(0) = \frac{6}{27} = \frac{2}{9}$, $\mu_3' = \frac{18}{3^4} = \frac{2}{9}$, $\mu_4' = \frac{8}{27}$

To find four moments about its mean

$\mu_1 = 0$, $\mu_2 = \mu_2' - \mu_1' = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = \frac{2}{9} - 3\cdot\frac{2}{9}\cdot\frac{1}{3} + 2\cdot\frac{1}{27} = \frac{2}{27}$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$= \frac{8}{27} - 4\cdot\frac{2}{9}\cdot\frac{1}{3} + 6\cdot\frac{2}{9}\cdot\frac{1}{9} - 3\cdot\frac{1}{3^4} = \frac{1}{9}$

10.) If x is a discrete RV with prob. distribution $p(x) = \frac{1}{k^x}$, $x = 1, 2, \ldots$ k a constant, find the MGF of X and hence find its mean and variance.

Sol: $M_X(t) = E(e^{tX}) = \sum_{x=1}^{\infty} \left(\frac{e^t}{k}\right)^x = \frac{e^t}{k} + \left(\frac{e^t}{k}\right)^2 + \left(\frac{e^t}{k}\right)^3 + \cdots$

$$= \frac{e^t}{k}\left[1 + \frac{e^t}{k} + \frac{e^{2t}}{k^2} + \cdots\right] = \frac{e^t}{k}\left(1 - \frac{e^t}{k}\right)^{-1}$$

$$= \frac{e^t}{k - e^t}$$

$$M_X'(t) = e^t(k-e^t)^{-1} + e^t(-1)(k-e^t)^{-2}(-e^t)$$

$$= e^t(k-e^t)^{-1}\left[1 + \frac{e^t}{(k-e^t)}\right] = \frac{ke^t}{(k-e^t)^2}$$

$$M_X''(t) = k\left[e^t(k-e^t)^{-2} + e^t(-2)(k-e^t)^{-3}(-e^t)\right]$$

$$= ke^t(k-e^t)^{-2}\left[1 + \frac{2e^t}{k-e^t}\right]$$

$$= \frac{ke^t(k+e^t)}{(k-e^t)^3}$$

$\Rightarrow M_X'(0) = \frac{k}{(k-1)^2}$, $M_X''(0) = \frac{k(k+1)}{(k-1)^3}$

$\therefore$ Mean $= \mu_1' = \frac{k}{(k-1)^2}$, Variance $= \frac{k(k+1)}{(k-1)^3} - \frac{k^2}{(k-1)^4} = \frac{k(k^2-k-\cdots}{(k-1)^{\cdots}}$

$$= \frac{\cancel{k(k+1) - k(k-1)}}{(k-1)^3}$$

11) A r.v X has the pdf given by $f(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

Find a) MGF b) the first four moments about the origin.

Sol:

$$M_X(t) = 2\int_0^{\infty} e^{tx} e^{-2x} dx = 2\int_0^{\infty} e^{-(2-t)x} dx$$

$$= 2\left[\frac{e^{-(2-t)x}}{-(2-t)}\right]_0^{\infty} = \frac{2}{2-t}$$

$\Rightarrow M_X'(t) = 2(2-t)^{-2}$, $M_X''(t) = 4(2-t)^{-3}$, $M_X'''(t) = 12(2-t)^{-4}$

$M_X^{iv}(t) = 48(2-t)^{-5}$

$\therefore \mu_1' = M_X'(0) = \frac{1}{2}$, $\mu_2' = M_X''(0) = \frac{1}{2}$, $\mu_3' = \frac{3}{4}$, $\mu_4' = 3/2$

12) Find the MGF of the r.v 'x' having pdf

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Sol:

$$M_X(t) = \int_0^1 e^{tx} \cdot x\,dx + \int_1^2 e^{tx}(2-x)\,dx$$

$$\Rightarrow M_X(t) = \left[x \cdot \frac{e^{tx}}{t} - 1 \cdot \frac{e^{tx}}{t^2}\right]_0^1 + \left[(2-x)\frac{e^{tx}}{t} - (-1)\frac{e^{tx}}{t^2}\right]_1^2$$

$$= \cancel{\frac{e^t}{t}} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \cancel{\frac{e^t}{t}} - \frac{e^t}{t^2}$$

$$= \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{2e^t}{t^2} = \frac{(e^t-1)^2}{t^2}$$

13) Let X be a r.v with pdf $f(x) = \begin{cases} \frac{1}{3}e^{-x/3}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$

Find a) $P(X > 3)$ b) MGF of X (c) $E(X)$ and $Var(X)$.

Sol:

a) $$P(X > 3) = \frac{1}{3}\int_3^{\infty} e^{-x/3} dx = \frac{1}{3}\left[\frac{e^{-x/3}}{-1/3}\right]_3^{\infty} = e^{-1} = \frac{1}{e}$$

(b) $M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} \cdot \frac{1}{3} e^{-x/3} \, dx$

$= \frac{1}{3} \int_0^\infty e^{-(\frac{1}{3}-t)x} \, dx = \frac{1}{3} \left[\frac{e^{-(\frac{1}{3}-t)x}}{-(\frac{1}{3}-t)} \right]_0^\infty$

$= \frac{1}{3} \left[\frac{1}{(\frac{1}{3}-t)} \right] = \frac{1}{1-3t}$

(c) Mean $= M_X'(0)$, Variance $= M_X''(0) - [M_X'(0)]^2$

$= E(X)$ $\qquad\qquad = E(X^2) - [E(X)]^2$

$M_X'(t) = (-1)(1-3t)^{-2}(-3) = 3(1-3t)^{-2}$

$M_X''(t) = 3(-2)(1-3t)^{-3}(-3) = 18(1-3t)^{-3}$

$\Rightarrow M_X'(0) = 3, \quad M_X''(0) = 18$

$\therefore$ Mean $= 3$, Variance $= 18 - 9 = 9$

Find the MGF of the dist. given by $f(x) = \begin{cases} \lambda e^{-\lambda x} & ; x > 0 \\ 0 & ; \text{elsewhere} \end{cases}$ and hence find fourth moment.

Sol:

$M_X(t) = \int_0^\infty \lambda e^{-\lambda x} e^{tx} \, dx = \lambda \int_0^\infty e^{-(\lambda - t)x} \, dx$

$= \lambda \left[\frac{e^{-(\lambda - t)x}}{-(\lambda - t)} \right]_0^\infty = \frac{\lambda}{\lambda - t}$

$\Rightarrow M_X'(t) = \lambda(-1)(\lambda - t)^{-2}(-1) = \lambda(\lambda - t)^{-2}$

$M_X''(t) = \lambda(-2)(\lambda - t)^{-3}(-1) = 2\lambda(\lambda - t)^{-3}$, $M_X'''(t) = 6\lambda(\lambda - t)^{-4}$

$M_X^{IV}(t) = 24\lambda(\lambda - t)^{-5}$, $\therefore M_X^{IV}(0) = 24/\lambda^4$

Find the MGF of a r.v 'X' having the density function $f(x) = \begin{cases} x/2, & 0 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$ and also find first three moments about origin.

Sol:

$$M_X(t) = \int_0^2 e^{tx} \cdot \frac{x}{2} dx = \frac{1}{2}\left[x \cdot \frac{e^{tx}}{t} - 1 \cdot \frac{e^{tx}}{t^2}\right]_0^2$$

$$= \frac{1}{2}\left[\frac{2e^{2t}}{t} - \frac{e^{2t}}{t^2} + \frac{1}{t^2}\right] = \frac{1}{2t^2}\left[2te^{2t} - e^{2t} + 1\right]$$

$$E(X) = \int_0^2 x \cdot \frac{x}{2} dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 = \frac{4}{3}$$

$$E(X^2) = \int_0^2 x^2 \cdot \frac{x}{2} dx = \frac{1}{2}\left[\frac{x^4}{4}\right]_0^2 = 2$$

$$E(X^3) = \int_0^2 x^3 \cdot \frac{x}{2} dx = \frac{1}{2}\left[\frac{x^5}{5}\right]_0^2 = \frac{16}{5}$$

A r.v X has density function given by $f(x) = \begin{cases} 1/K, & \text{for } 0 < x < K \\ 0, & \text{otherwise} \end{cases}$. Find (i) MGF (ii) r^{th} moment (iii) Mean (iv) Variance.

Sol:

(i) $$M_X(t) = E(e^{tX}) = \int_0^K \frac{1}{K} e^{tx} dx = \frac{1}{K}\left(\frac{e^{tx}}{t}\right)_0^K = \frac{e^{Kt} - 1}{K}$$

(ii) $$\mu_r' = \int_0^K x^r \cdot \frac{1}{K} dx = \frac{1}{K}\left[\frac{x^{r+1}}{r+1}\right]_0^K = \frac{K^r}{r+1}$$

(iii) $r=1$ in (ii) $\Rightarrow \mu_1' = \frac{K}{2}$ $\quad \therefore$ Mean $= \frac{K}{2}$

(iv) $r=2$ in (iii) $\Rightarrow \mu_2' = \frac{K^2}{3}$

$$\text{Variance} = \mu_2' - {\mu_1'}^2 = \frac{K^2}{3} - \frac{K^2}{4}$$

$$= \frac{K^2}{12} //.$$

— x —

Note: 1) MGF does not exist. MGF is valid for t lying in an interval containing zero

2) $M_X(t) = \sum_x e^{tx} p(x)$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

t being the real parameter and it is being assumed that the RHS is absolutely convergent for some +ve number h such that $-h < t < h$.

Tchebycheff Inequality:

Introduction: If we know the prob. distribution of a r.v X, we may compute $E(X)$ and $Var(X)$. Conversely, If $E(X)$ and $Var(X)$ are known, we cannot construct the prob. distribution of X and hence compute quantities such as $P\{|X-E(X)| \le k\}$. Although we cannot evaluate such probabilities from a knowledge of $E(X)$ and $V(X)$, several approximation techniques have been developed to upper and/or lower bounds to such probabilities. The most important of such techniques is Tchebycheff inequality.

Statement: If X is a RV with $E(X)=\mu$ and $Var(X)=\sigma^2$, then $P\{|X-\mu| \geq c\} \le \frac{\sigma^2}{c^2}$ where $c>0$.

or $P\{|X-\mu| \le c\} \geq 1-\frac{\sigma^2}{c^2}$ where $c>0$.

Alternative forms: If we put $c=k\sigma$, where $k>0$, then Tchebycheff inequality takes the form

$$P\left\{\left|\frac{X-\mu}{k}\right| \geq \sigma\right\} \le \frac{1}{k^2}$$

(or) $$P\left\{\left|\frac{X-\mu}{k}\right| \le \sigma\right\} \geq 1-\frac{1}{k^2}$$

1) A r.v X has mean $\mu=12$ and variance $\sigma^2=9$ and an unknown prob. distribution. Find $P(6<X<18)$.

Sol:

By Tchebycheff's inequality

$$P\{|X-\mu|\le c\} \ge 1-\frac{\sigma^2}{c^2}$$

$$\Rightarrow P\{-c \le X-\mu \le c\} \ge 1-\frac{\sigma^2}{c^2}$$

$$\Rightarrow P\{\mu-c \le X \le \mu+c\} \ge 1-\frac{\sigma^2}{c^2}$$

~~Taking~~ Given $\mu=12$, $\sigma^2=9$

$$P\{12-c < X < 12+c\} \ge 1-\frac{9}{c^2}$$

putting $c=6$, we get

$$P\{6<X<18\} \ge 1-\frac{9}{36}$$

$$\Rightarrow P\{6<X<18\} \ge 3/4$$

2.) If the RV X is uniformly distributed over $(-\sqrt{3}, \sqrt{3})$, Compare $P\{|X-\mu| \ge 3\sigma/2\}$ and compare it with the upper bound obtained by Tchebycheff's inequality.

Sol:

$\left.\begin{array}{l}\text{Uniform distribution}\\ \text{pdf}\end{array}\right\}$: $f(x)=\frac{1}{b-a}$, $a<x<b$

Mean $= \frac{1}{2}(b+a)$

Variance $= \frac{1}{12}(b-a)^2$

Mean $= \mu = \frac{\sqrt{3}-\sqrt{3}}{2} = 0$, Variance $= \sigma^2 = \frac{1}{12}(\sqrt{3}+\sqrt{3})^2 = 1$

ie, $\mu = 0$, $\sigma^2 = 1$

$$\therefore P\left\{|X-\mu| \geq \frac{3\sigma}{2}\right\} = P\{|X| \geq 3/2\}$$

$$= 1 - P\{-3/2 < X < 3/2\}$$

$$= 1 - \int_{-3/2}^{3/2} \frac{1}{2\sqrt{3}}\,dx \qquad \because f(x) = \frac{1}{b-a},\ a<x<b$$

$$= 1 - \frac{1}{2\sqrt{3}}[x]_{-3/2}^{3/2}$$

$$= 1 - \frac{1}{2\sqrt{3}}\left[\frac{3}{2}+\frac{3}{2}\right]$$

$$= 1 - \frac{\sqrt{3}}{2} = 0.134$$

By Tchebycheff's inequality

$$P\{|X-\mu| \geq k\sigma\} \leq \frac{1}{k^2}$$

$$\therefore P\{|X-\mu| \geq \tfrac{3}{2}\sigma\} \leq \frac{4}{9} = 0.444$$

which is a poor upper bound.

3.) Can we find a RV X for which $P\{\mu-2\sigma \leq X \leq \mu+2\sigma\} = 0.6$?

Sol:

$$P\{\mu-2\sigma \leq X \leq \mu+2\sigma\} = P\{|X-\mu| \leq 2\sigma\}$$

By Tchebycheff's inequality $P\{|X-\mu| \leq c\} \geq 1 - \frac{\sigma^2}{c^2}$

(or) $P\left\{\left|\frac{X-\mu}{k}\right| \leq \sigma\right\} \geq 1 - \frac{1}{k^2}$

$$\Rightarrow P\{\mu-2\sigma \leq X \leq \mu+2\sigma\} \geq 1 - \frac{1}{2^2} \qquad \because k = 2$$

$$\geq 0.75$$

$\therefore$ there does not exist a RV X satisfying the given condition.

A discrete RV X takes the values $-1, 0, 1$ with probabilities $\frac{1}{8}, \frac{3}{4}, \frac{1}{8}$ respectively. Evaluate $P\{|X-\mu| \geq 2\sigma\}$ and Compare it with the upper bound given by Tchebycheff's inequality.

Sol:

Given

X	-1	0	1
$P(X=x)$	$1/8$	$3/4$	$1/8$

$$E(X) = \left(-1 \times \frac{1}{8}\right) + \left(0 \times \frac{3}{4}\right) + \left(1 \times \frac{1}{8}\right) = 0$$

$$E(X^2) = \left((-1)^2 \times \frac{1}{8}\right) + \left(0 \times \frac{3}{8}\right) + \left(1^2 \times \frac{1}{8}\right) = \frac{2}{8} = \frac{1}{4}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{1}{4} - 0 = \frac{1}{4}$$

$$\Rightarrow P\{|X-\mu| \geq 2\sigma\} = P\{|X-0| \geq 2 \cdot \frac{1}{2}\}$$

$$= P\{|X| \geq 1\}$$

$$= 1 - P\{|X| < 1\}$$

$$= 1 - P\{-1 < X < 1\} = 1 - P\{X=0\}$$

$$= 1 - \frac{3}{4} = 1/4$$

By Tchebycheff's inequality

$$P\{|X-\mu| \geq K\sigma\} \leq \frac{1}{K^2}$$

$$\Rightarrow P\{|X-\mu| \geq 2\sigma\} \leq \frac{1}{2^2} = \frac{1}{4}$$

The two values Coincide.

5.) A fair die is tossed 720 times. Use Tchebycheff's inequality to find a lower bound for the prob. of getting 100 to 140 sixes.

Sol: Let X be the number of sixes

$p = P\{\text{getting '6' in a single toss}\} = 1/6$

$q = \frac{5}{6}$ and $n = 720$

X follows a binomial distribution

$\therefore$ Mean $= np$ and Variance $= npq$

$\Rightarrow$ mean $= 720 \times \frac{1}{6}$ and Variance $= 720 \times \frac{1}{6} \times \frac{5}{6} = 100$

i.e., $\mu = 120$ and $\sigma = 10$.

By Tchebycheff's inequality

$P\{|X-\mu| \le K\sigma\} \ge 1 - \frac{1}{K^2}$.

$\Rightarrow P\{|X-120| \le 10K\} \ge 1 - \frac{1}{K^2}$

$\Rightarrow P\{-10K \le X-120 \le 10K\} \ge 1 - 1/K^2$

$\Rightarrow P\{120-10K \le X \le 120+10K\} \ge 1 - \frac{1}{K^2}$

putting $K=2$, we get

$P\{100 \le X \le 140\} \ge 1 - \frac{1}{4} = 3/4$

$\therefore$ the lower bound for the prob. $= 0.75$

6) Use Tchebycheff's inequality to find how many times a fair coin must be tossed in order that the prob. that the ratio of the number of heads to the number of tosses will lie between 0.45 and 0.55 will be at least 0.95.

Sol: Given $P\left\{0.45 \le \frac{X}{n} \le 0.55\right\} \geq 0.95 \longrightarrow ①$

Let X be the number of heads

Then X follows a binomial distribution.

ie, $X \sim B(np, \sqrt{npq})$ where $p = q = 1/2$

$\therefore \frac{X}{n} \sim B\left(\frac{1}{2}, \frac{1}{2\sqrt{n}}\right)$

By Tchebycheff's inequality

$$P\left\{\left|\frac{X}{n} - \frac{1}{2}\right| \le c\right\} \geq 1 - \frac{\sigma^2}{c^2}$$

$$\Rightarrow P\left\{0.5 - c \le \frac{X}{n} \le 0.5 + c\right\} \geq 1 - \frac{1}{4nc^2} \quad \left[\because \sigma^2 = \frac{1}{4n}\right]$$

putting $c = 0.05$, we get

$$P\left\{0.45 \le \frac{X}{n} \le 0.55\right\} \geq 1 - \frac{100}{n} \longrightarrow ②$$

Comparing (1) and ②, we get

$$1 - \frac{100}{n} = 0.95$$

$$\Rightarrow 1 - 0.95 = \frac{100}{n}$$

$$\Rightarrow n = \frac{100}{0.05}$$

$$\boxed{n = 2000}$$

7.) A RV X is exponentially distributed with parameter 1. Use Tchebycheff's inequality to show that $P(-1 \le X \le 3) \geq 3/4$. Find the actual probability also.

Sol:

For Exponential distribution:

$$\text{pdf}: f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

$$\text{Mean} = \frac{1}{\lambda} \quad \text{and} \quad \text{variance} = \frac{1}{\lambda^2}$$

Here $\lambda = 1$

i.e., $\mu = 1$ and $\sigma = 1$

By Tchebycheff's inequality,

$$P\{|X-1| \le 2\} \geq 1 - 1/4$$

$$\Rightarrow P\{-1 \le X \le 3\} \geq 3/4$$

$\because P(-1 \le X \le 3) = P\{|X-\mu| \le k\sigma\} \geq 1 - \frac{1}{k^2}$

To find Actual probability:

$$P(-1 \le X \le 3) = \int_0^3 e^{-x}\, dx$$

$$= \left(-e^{-x}\right)_0^3 = 1 - e^{-3} = 0.9502$$

Function of One Random Variable

Let X be an RV with the associated Sample space S_x and a known probability distribution. Let g be a scalar function that maps each $x \in S_x$ into $y = g(x)$. The expression $Y = g(X)$ defines a new RV Y. For a given outcome; $X(s)$ is a number of x and $g[X(s)]$ is another number specified by $g(x)$. This number is the value of The RV Y. Ie, $Y(s) = y = g(x)$.

The sample space S_y of Y is the set

$$S_y = \{y = g(x) : x \in S_x\}$$

To Find $f_Y(y)$, when $f_X(x)$ is known

Let us now desire a procedure to find $f_Y(y)$, the pdf of Y, when $Y = g(X)$, where X is a continuous RV with pdf $f_X(x)$ and $g(x)$ is a strictly monotic function of x.

Case (i): $g(x)$ is a strictly increasing function of x.

$F_Y(y) = P(Y \leq y)$ where $F_Y(y)$ is the CDF of Y

$$= P[g(X) \leq y]$$

$$= P[X \leq g^{-1}(y)] = F_X(g^{-1}(y))$$

Diff. both sides w.r.to y, we get

$$f_Y(y) = f_X(x) \cdot \frac{dx}{dy} \quad \text{where } x = g^{-1}(y) \rightarrow ①$$

<u>Case</u> <u>(ii)</u> $g(x)$ is strictly decreasing function of x.

$$F_Y(y) = P(Y \leq y)$$
$$= P[g(X) \leq y]$$
$$= P[X \geq g^{-1}(y)]$$
$$= 1 - P[X \leq g^{-1}(y)]$$
$$= 1 - F_X[g^{-1}(y)]$$

$$\therefore f_Y(y) = -f_X(x) \cdot \frac{dx}{dy} \rightarrow ②$$

From ① & ②, we get

$$f_Y(y) = f_X(x) \left|\frac{dx}{dy}\right| \quad \text{or} \quad f_Y(y) = \frac{f_X(x)}{\left|\frac{dy}{dx}\right|}$$

$$\text{or} \quad f_Y(y) = f_X(x) / |g'(x)|$$

<u>Note</u>: 1) The above formula for $f_Y(y)$ can be used only when $x = g^{-1}(y)$ is single valued.

2) When $x = g^{-1}(y)$ takes finitely many values $x_1, x_2, \ldots x_n$ or $g(x) = y$ has finitely many values to find $f_Y(y)$ we use the extended formula:

$$f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|} + \cdots + \frac{f_X(x_n)}{|g'(x_n)|} \quad \text{or}$$

$$f_Y(y) = f_X(x_1)\left|\frac{dx_1}{dy}\right| + f_X(x_2)\left|\frac{dx_2}{dy}\right| + \dots + f_X(x_n)\left|\frac{dx_n}{dy}\right|$$

Procedure to find $f_Y(y)$

1) Write y in terms of x.
2) Write x as a function of y.
3.) Find $\frac{dx}{dy}$.
4.) Write $f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$.
5.) Using the range of x, find the range of Y.

1.) Let X be a continuous RV with pdf

$$f(x) = \begin{cases} \frac{x}{2}, & 1 < x < 5 \\ 0, & \text{otherwise} \end{cases}$$

Find the pdf of $Y = 2X - 3$.

Sol: Given $g(x) = y = 2x - 3$

$$\therefore x = \frac{y+3}{2} \Rightarrow \frac{dx}{dy} = \frac{1}{2}$$

$$f_Y(y) = f_X(x)\cdot\left|\frac{dx}{dy}\right| = \frac{x}{2}\cdot\frac{1}{2} = \frac{(y+3)}{8}$$

$$1 < x < 5 \Rightarrow 1 < \frac{y+3}{2} < 5 \Rightarrow 2 < y+3 < 10$$

$$\Rightarrow -1 < y < 7$$

$$\therefore f_Y(y) = \begin{cases} \frac{y+3}{8}, & -1 < y < 7 \\ 0, & \text{otherwise} \end{cases}$$

2.) If the pdf of a RV X is $f_X(x) = 2x;\ 0<x<1$, find the pdf of $y = e^{-x}$

Sol:

Given $g(x) = y = e^{-x}$

$$\frac{dy}{dx} = -e^{-x} = -y$$

Taking log on both sides

$\log y = e^{-x}$

or $x = -\log y$

$= \log \frac{1}{y}$

$$f_Y(y) = \frac{f_X(x)}{\left|\frac{dy}{dx}\right|} = \frac{2x}{|-y|} = \frac{2\log \frac{1}{y}}{y}$$

$$0<x<1 \Rightarrow -1<-x<0 \Rightarrow e^{-1}<e^{-x}<e^{0}$$

as e^x is an increasing function of x.

i.e., $\frac{1}{e}<y<1$

$$\therefore f_Y(y) = \frac{2}{y}\log\frac{1}{y}, \quad \frac{1}{e}<y<1.$$

3.) If X has an exponential distribution with parameter 1, find the pdf of $Y = \sqrt{X}$.

Sol:

$$Y = \sqrt{X} \Rightarrow y = \sqrt{x} \Rightarrow y^2 = x$$

$$2y\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$$

X follows an exponential distribution.

pdf: $f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$

$\therefore f(x) = e^{-x}$

$$f_Y(y) = \frac{f_X(x)}{\left|\frac{dy}{dx}\right|} = \frac{e^{-y^2}}{\left|\frac{1}{2y}\right|} = 2ye^{-y^2}$$

Since $x > 0 \Rightarrow y > 0$

$\therefore f_Y(y) = 2ye^{-y^2}, \quad y > 0$

4) If X is uniformly distributed RV in $(-\pi/2, \pi/2)$, find the pdf of $Y = \tan X$.

Sol:

$$y = \tan x \Rightarrow \frac{dy}{dx} = \sec^2 x$$
$$= 1 + \tan^2 x \qquad \because x = \tan^{-1} x$$
$$= 1 + y^2$$

X follows uniform distribution

pdf: $f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$

$$\therefore f(x) = \frac{1}{\frac{\pi}{2} + \frac{\pi}{2}} = \frac{1}{\pi}$$

$$f_Y(y) = f_X(x) \cdot \frac{dx}{dy} = \frac{1}{\pi} \cdot \frac{1}{1+y^2}$$

$$-\frac{\pi}{2} < x < \frac{\pi}{2} \Rightarrow \tan(-\pi/2) < \tan x < \tan \pi/2$$
$$\Rightarrow -\infty < y < \infty$$

$$\therefore f_Y(y) = \frac{1}{\pi(1+y^2)}, \quad -\infty < y < \infty$$

6) Given the RV X with pdf $f(x)=\begin{cases} 2x, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$

find the pdf of $Y=8X^3$.

Sol:

$g(x)=y=8x^3$ or $x=\frac{1}{2}y^{1/3}$

$8x^3=g(x)$ is strictly increasing in $(0,1)$

$\frac{dy}{dx}=24x^2=24\cdot\frac{1}{4}y^{2/3}=6y^{2/3}$

$f_Y(y)=\frac{2x}{6y^{2/3}}=\frac{y^{1/3}}{6y^{2/3}}=\frac{y^{-1/3}}{6}$

$0<x<1 \Rightarrow 0<x^3<1 \Rightarrow 0<8x^3<8$ or $0<y<8$

$\therefore f_Y(y)=\begin{cases} \frac{1}{6y^{1/3}}, & 0<y<8 \\ 0, & \text{otherwise.} \end{cases}$

7) If the continuous RV X has pdf $f_X(x)=\frac{2}{9}(x+1)$, in $-1<x<2$ and 0, elsewhere, find the pdf of $Y=X^2$.

Sol:

The transformation function $y=x^2$ is not monotonic in $(-1,2)$. So we divide the interval into two parts.

ie., $(-1,1)$ and $(1,2)$

$\because y=x^2$

$\frac{dy}{dx}=2x$

In $(-1,0)$, $x=-\sqrt{y}$

$f_Y(y)=\frac{f_X(x)}{\left|\frac{dy}{dx}\right|}=\frac{1}{2\sqrt{y}}\frac{2}{9}[1-\sqrt{y}]$

In $(0,1)$, $x = \sqrt{y}$

$$f_Y(y) = \frac{f_X(x)}{2\sqrt{y}} = \frac{1}{2\sqrt{y}} \cdot \frac{2}{9}[1+\sqrt{y}]$$

In $(-1,1)$, $f_Y(y) = \frac{1}{2\sqrt{y}}\left[\frac{2}{9}(1+\sqrt{y}) + \frac{2}{9}(1-\sqrt{y})\right]$

$$= \frac{2}{9\sqrt{y}}$$

In $-1 < x < 1 \Rightarrow 0 < x^2 < 1 \Rightarrow 0 < y < 1$

In $1 < x < 2$, $x = \sqrt{y}$

$$\therefore f_Y(y) = \frac{2}{9}[1+\sqrt{y}] \cdot \frac{1}{2\sqrt{y}} = \frac{1}{9}\left[1 + \frac{1}{\sqrt{y}}\right]$$

In $1 < x < 2$, $1 < x^2 < 4$ or $1 < y < 4$

$$\therefore f_Y(y) = \begin{cases} \frac{2}{9\sqrt{y}}, & 0 < y < 1 \\ \frac{1}{9}\left[1 + \frac{1}{\sqrt{y}}\right], & 1 < y < 4. \end{cases}$$

— x —