given C.D.F

$$F(x) = \begin{cases} 1 - \frac{4}{x^2}, & x > 2 \\ 0, & \text{if } x \leq 2. \end{cases}$$

$$\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{-n}{x^{n+1}}$$

$$\frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{-2}{x^3}.$$

The p.d.f of X is $f(x) = F'(x) = \begin{cases} -4\left(\frac{-2}{x^3}\right), & \text{if } x > 2 \\ 0, & \text{if } x \leq 2. \end{cases}$

$\therefore$ The pdf is $f(x) = \begin{cases} 8/x^3, & \text{if } x > 2 \\ 0, & \text{if } x \leq 2. \end{cases}$

PROBLEM 2 The cdf of a discrete random Variable X is given by

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{6} & \text{if } 0 \leq x < 2 \\ 1/2 & \text{if } 2 \leq x < 4 \\ 5/8 & \text{if } 4 \leq x < 6 \\ 1 & \text{if } x \geq 1. \end{cases}$$

Find the Probability distribution.

To find the Probability distribution of X, we have to find the probabilities at the Changing Points 0, 2, 4, 6.

$P(x=0) = \frac{1}{6}$, $P(x=2) = F(2) - F(0) = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$

$P(x=4) = F(4) - F(2) = \frac{5}{8} - \frac{1}{2} = \frac{1}{8}$

$P(x=6) = 1 - F(4) = 1 - \frac{5}{8} = \frac{3}{8}$.

The Probability distribution of X is

x	0	2	4	6
$P(x)$	1/6	1/3	1/8	3/8

MOMENTS AND MOMENT GENERATING FUNCTIONS

MOMENT The Expected Value of an Integral Power of a random Variable is called its moment.

Moments are classified into two types.

Moment about mean
(Central moments)

$$\boxed{\mu_r = E(x-\mu)^r}$$

Moment about any Point (a)

when $a=0$.

Moment about orgin
[Raw moments].

$$\boxed{\mu_r' = E(x-a)^r}$$

The four moments about the mean μ are

$$\mu_1 = E(x-\mu)' = E(x) - E(\mu) = \mu - \mu = 0 \quad \therefore \boxed{\mu_1 = 0}$$

$$\mu_2 = E(x-\mu)^2 = E(x^2) - [E(x)]^2 = \text{Variance of } x.$$

$$\mu_3 = E(x-\mu)^3$$

$$\mu_4 = E(x-\mu)^4.$$

MOMENT ABOUT ANY POINT

The r^{th} moment about any point (a) for a random variable X is defined as

$$\mu_r' = E(x-a)^r$$

First four moments about the point a

$$\mu_1' = E(x-a) = E(x) - a$$

$$\Rightarrow \mu_1' = \mu - a \Rightarrow \boxed{\mu = \mu_1' + a}$$

$$\mu_2' = E(x-a)^2$$

$$\mu_3' = E(x-a)^3$$

$$\mu_4' = E(x-a)^4$$

First four moments about the origin.

$$\mu_1' = E(x)$$

$$\mu_2' = E(x^2)$$

$$\mu_3' = E(x^3)$$

$$\mu_4' = E(x^4)$$

RELATION BETWEEN MOMENTS ABOUT THE MEAN AND MOMENTS ABOUT ANY POINT 'a'.

$$\mu_r = E[x-\mu]^r = E[x-a+a-\mu]^r$$

$$= E[(x-a)-(\mu-a)]^r \qquad \text{W.K.T} \quad \mu_1' = \mu - a$$

$$= E[(x-a) - \mu_1']^r$$

$$= E\left[(x-a)^r - rC_1 (x-a)^{r-1} \mu_1' + rC_2 (x-a)^{r-2} (\mu_1')^2 + \cdots + (-1)^r (\mu_1')^r\right]$$

$$= E(x-a)^r - rC_1 E(x-a)^{r-1} \mu_1' + rC_2 E(x-a)^{r-2} (\mu_1')^2 + \cdots + (-1)^r (\mu_1')^r.$$

$$\mu_r = \mu_r' - rC_1 \mu_{r-1}' \mu_1' + rC_2 \mu_{r-2}' (\mu_1')^2 + \cdots (-1)^r (\mu_1')^r$$

W.K.T $\boxed{\mu_0' = 1}$ $\quad \mu_0' = E(x^0) = 1.$

$$\mu_1 = \mu_1' - \mu_0 \mu_1'$$

$$\therefore \mu_1 = \mu_1' - \mu_1' = 0 \quad \therefore \boxed{\mu_1 = 0}$$

$$\mu_2 = \mu_2' - 2C_1 \mu_1' \mu_1' + 2C_2 \mu_0' (\mu_1')^2$$

$$\mu_2 = \mu_2' - 2(\mu_1')^2 + (\mu_1')^2 (1) \quad [\because \mu_0 = 1]$$

$$\boxed{\mu_2 = \mu_2' - (\mu_1')^2} = E(x^2) - [E(x)]^2 = \text{VARIANCE}(x).$$

$\therefore$ Similarly $\boxed{\mu_3 = \mu_3' - 3\mu_2' \mu_1' + 2(\mu_1')^3}.$

$$\boxed{\mu_4 = \mu_4' - 4\mu_3' \mu_1 + 6\mu_2' (\mu_1')^2 - 3(\mu_1')^4.}$$

Since moments of a random Variable are important in characterising or determining its distribution, it will be (helpful) useful if a function could be found that would give all the moments.

Such a function is called a $\boxed{\text{moment generating function.}}$

MOMENT GENERATING FUNCTION (MGF)

The moment generating function of a random Variable X is defined as $E(e^{tX})$ for all $x \in (-\infty, \infty)$

It is denoted by $M(t)$ (or) $M_X(t)$.

$$M(t) \text{ (or) } M_X(t) = \begin{cases} \sum e^{tx_i} p(x_i) & \rightarrow \text{D.R.V} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\, dx & \rightarrow \text{C.R.V.} \end{cases}$$

FINDING MOMENTS FROM THE EXPANSION OF MGF.

$$M_x(t) = E(e^{tx})$$

$$= E\left[1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \cdots + \frac{(tx)^r}{r!} + \cdots\right]$$

$$= 1 + \frac{E(tx)}{1!} + \frac{t^2}{2!}E(x^2) + \frac{t^3}{3!}E(x^3) + \cdots + \frac{t^r}{r!}E(x^r)$$

$$= 1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \cdots + \frac{t^r}{r!}\mu_r' + \cdots$$

$\mu_1' =$ Coefficient of $\frac{t}{1!}$

$\mu_2' =$ Coefficient of $\frac{t^2}{2!}$

$\vdots \qquad \vdots$

$\mu_r' =$ coefficient of $\frac{t^r}{r!}$

FINDING RAW MOMENTS (FROM) BY DIFFERENTIATION OF MGF without EXPANSION.

$$\mu_r' = M_x^{(r)}(0)$$

$$\mu_1' = \left[M_x'(t)\right]_{t=0} = \left[\frac{d}{dt}M_x(t)\right]_{t=0} = E(x)$$

$$\mu_2' = \left[M_x''(t)\right]_{t=0} = \left[\frac{d^2}{dt^2}M_x(t)\right]_{t=0} = E(x^2)$$

$\vdots \qquad \vdots$

$$\mu_r' = E(x^r) = \left[M_x^{\prime\prime\cdots r \text{ times}}(t)\right]_{t=0}$$

$$= \left[\frac{d^r}{dt^r} M_x(t)\right]_{t=0}.$$

<u>Proof</u>

$$M_x(t) = 1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \cdots \frac{t^r}{r!}\mu_r' + \cdots$$

$$M_x'(t) = \mu_1' + \frac{2t}{2}\mu_2' + \frac{3t^2}{3!}\mu_3' + \cdots + \frac{r\, t^{r-1}}{r!}\mu_r' + \cdots$$

$$M_x''(t) = \mu_2' + \frac{3\times 2t}{3!}\mu_3' + \cdots + \frac{r(r-1)}{r!} t^{r-2} \mu_r' + \cdots$$

$$\vdots$$

$$M_x^{(r)}(t) = \mu_r' + \text{terms containing } t.$$

When $t=0$. we get.

$$\left[M_x'(t)\right]_{t=0} = \mu_1' = M_x'(0)$$

$$\left[M_x''(t)\right]_{t=0} = \mu_2' = M_x''(0)$$

$$\vdots \quad = \quad \vdots \quad = \quad \vdots$$

$$\left[M_x^{(r)}(t)\right]_{t=0} = \mu_r' = M_x^{(r)}(0) //.$$

PROPERTIES MOMENT GENERATING FUNCTION (MGF).

Let X be a Random Variable with MGF $M_x(t)$

and c is a Constant then

(i) $M_{cx}(t) = M_x(ct)$

(ii) $M_{x+c}(t) = e^{ct} M_x(t)$.

(iii) $M_{ax+b}(t) = e^{bt} M_x(at)$.

(iv) If X and Y are independent random Variables then

$$M_{x+y}(t) = M_x(t) \cdot M_y(t).$$

PROBLEM: 1 The first four moments of a distribution about $x=4$. are 1, 4, 10, 45. Show that the mean is 5, Variance 3, $\mu_3 = 0$, $\mu_4 = 0$.

Let $\left.\begin{array}{l} \mu_1' = 1 \\ \mu_2' = 4 \\ \mu_3' = 10 \\ \mu_4' = 25 \end{array}\right\}$ are the first four moments. (given).

Moment about $x=4$.

$$E(x-a)^r = \mu_r'$$

$\mu_1' = E(x-a)^1 \Rightarrow 1 = E(x-4)$

$1 = E(x) - 4 \Rightarrow \boxed{E(x) = 5}$

$\therefore$ Mean $= E(x) = 5$.

Variance $\mu_2 = \mu_2' - (\mu_1')^2 = 4 - (1)^2 = 4 - 1 = 3$ //.

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$

$\mu_3 = 10 - 3(4)(1) + 2(1) = 10 - 12 + 2 = 0$ //.

$\mu_4 = \mu_4' - 4\mu_3'\mu_1 + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$

$= 45 - 4(10)(1) + 6(4)(1) - 3(1)$

$= 45 - 40 + 24 - 3 = 26$ //.

PROBLEM : 2 The first two moments about 3 are 1 and 8. Find the mean and Variance.

Let μ_1', μ_2' be the first two moments of X about 3.

$\mu_1' = 1, \; \mu_2' = 8., \; a = 3$

$E(x-a)^r = \mu_r' \Rightarrow \mu_1' = E(x-3) = E(x) - 3$

$\therefore \mu_1' = E(x) - 3. \Rightarrow 1 = E(x) - 3 \Rightarrow \boxed{E(x) = 4}$

$\mu_2 = \mu_2' - (\mu_1')^2 = 8 - (1)^2 = 7$ //.

Problem: 3 A Continuous random Variable X has Pdf $f(x) = k(1-x)$ for $0 < x < 1$. Find the r^{th} moment about the origin. Hence find mean and Variance.

Given P.d.f of X is $f(x) = k(1-x), \quad 0 < x < 1$.

$$\int_{-\infty}^{\infty} f(x)\, dx = 1, \quad f(x) \geq 0.$$

$$\int_{-\infty}^{\infty} k(1-x)\, dx = 1 \quad [\Rightarrow \text{given } 0 < x < 1.]$$

$$\therefore \int_{0}^{1} k(1-x)\, dx = 1 \Rightarrow k\left[x - \frac{x^2}{2}\right]_0^1 = 1$$

$$\Rightarrow k\left[\left(1 - \tfrac{1}{2}\right) - 0\right] = 1$$

$$\Rightarrow \boxed{k = 2}$$

$$\therefore f(x) = \begin{cases} 2(1-x), & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

$\mu_r' = r^{th}$ moment about the origin.

$$\mu_r' = E(x^r) = \int_{-\infty}^{\infty} x^r f(x)\, dx. = \int_0^1 x^r 2(1-x)\, dx.$$

$$= 2\int_0^1 (x^r - x^{r+1})\, dx$$

$$= 2\left[\frac{x^{r+1}}{(r+1)} - \frac{x^{r+2}}{(r+2)}\right]_0^1$$

$$= 2\left[\frac{1}{r+1} - \frac{1}{r+2}\right] = 2\left[\frac{(r+2)-(r+1)}{(r+1)(r+2)}\right] = \frac{2}{(r+1)(r+2)}$$

$$\therefore \boxed{\mu_r' = \frac{2}{(r+1)(r+2)}}$$

$$\mu_1' = \frac{2}{2\times 3} = 1/3 = E(x) = \text{Mean}.$$

$$\mu_2' = \frac{2}{3\times 4} = \frac{1}{6}$$

$$\text{Variance} = E(x-\mu)^2 = \mu_2 = \mu_2' - (\mu_1')^2.$$

$$\mu_2 = \frac{1}{6} - (1/3)^2 = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}.$$

$$\therefore \text{ Mean of } x = \mu_1' = 1/3$$

$$\text{Variance of } x = \mu_2 = \frac{1}{18}.$$

Problem 4: Find the moment generating function of the RV X, whose Probability function $P(x) = \frac{1}{2^x}$, $x = 1, 2, 3, \ldots$ Hence find its mean and Variance. ⑥

Given X is a discrete R.V.

$$P(x) = \frac{1}{2^x}, \quad x = 1, 2, 3, \ldots$$

$$M_X(t) = E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx} P(x) = \sum_{x=1}^{\infty} e^{tx} \frac{1}{2^x}.$$

$$= e^t \frac{1}{2} + \frac{(e^t)^2}{2^2} + \frac{(e^t)^3}{2^3} + \ldots$$

$$= \frac{e^t}{2}\left[1 + \left(\frac{e^t}{2}\right) + \left(\frac{e^t}{2}\right)^3 + \left(\frac{e^t}{2}\right)^4 + \ldots\right]$$

$$= \frac{e^t}{2}\left[1 - \frac{e^t}{2}\right]^{-1} \qquad (1-x)^{-1} = 1 + x + x^2 + \ldots$$

$$= \cancel{\frac{e^t}{2}} / \times \cancel{(2-e^t)} = \frac{e^t}{2} \frac{1}{\left[1 - \frac{e^t}{2}\right]} = \frac{e^t}{2} * \frac{1}{\left[\frac{2-e^t}{2}\right]} = \frac{2e^t}{2(2-e^t)}.$$

$$\therefore M_X(t) = \frac{e^t}{2-e^t}.$$

$$\text{Now } [M_X'(t)] = \frac{d}{dt}\left[\frac{e^t}{2-e^t}\right] = \frac{(2-e^t)(e^t) - e^t(-e^t)}{(2-e^t)^2}$$

$$= \frac{[2-e^t+e^t]e^t}{(2-e^t)^2} = \frac{2e^t}{(2-e^t)^2}$$

$$\therefore M_X'(t) = \frac{2e^t}{(2-e^t)^2}.$$

$$M_X''(t) = 2\left[\frac{(2-e^t)^2\cdot(e^t) - e^t\, 2(2-e^t)\cdot(-e^t)}{(2-e^t)^4}\right]$$

$$= 2\left[\frac{(2-e^t)^2 e^t + 2e^t\cdot e^t(2-e^t)}{(2-e^t)^4}\right]$$

$$= \frac{2(2-e^t)e^t\left[(2-e^t)+2e^t\right]}{(2-e^t)^4}$$

$$M_X''(t) = \frac{2e^t\left[2+e^t\right]}{(2-e^t)^3}$$

$$\mu_2' = M_X''(0) = \frac{2\times 3}{(2-1)^3} = 6.$$

$$\mu_1' = M_X'(0) = \frac{2}{(2-1)^2} = 2.$$

Mean of $X = \mu_1' = 2$

Variance of $X = \mu_2' - (\mu_1')^2 = 6 - 2^2 = 2$ //.

PROBLEM : 5 Let the random Variable X have p.d.f $f(x) = \frac{1}{2} e^{-x/2}$, $x>0$. Find the moment generating fn and hence find the mean and variance of X.

The MGF of X is $M_X(t)$

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \frac{1}{2}\int_0^{\infty} e^{-x/2} . e^{tx} . dx$$

$$= \frac{1}{2}\int_0^{\infty} e^{-\frac{1}{2}(1-t(2))x} dx \qquad \int e^{ax} dx = \frac{e^{ax}}{a}$$

$$= \frac{1}{2}\left[\frac{e^{-1/2(1-2t)x}}{-1/2(1-2t)}\right]_0^{\infty} \qquad \text{if } 1-2t>0 \Rightarrow t<1/2.$$

$$= \frac{1}{2}\left[0 - \frac{1}{-1/2(1-2t)}\right] = \frac{1}{2}\times 2\times\frac{1}{(1-2t)}$$

$$M_X(t) = \frac{1}{1-2t} \quad \text{if } t<1/2.$$

Mean and Variance:

$$M_X(t) = \frac{1}{1-2t} = (1-2t)^{-1}$$

W.K.T $(1-x)^{-1} = 1+x+x^2+\cdots$

$$(1-2t)^{-1} = 1+(2t)+(2t)^2+(2t)^3+\cdots$$

$$\therefore M_X(t) \quad 1+2t+(2t)^2+(2t)^3+\cdots \rightarrow (1)$$

Differentiating w.r.t to 't'.

$$M_x'(t) = 2 + 8t + 24t^2 + \cdots$$

$$M_x''(t) = 8 + 48t + \cdots$$

Therefore: Mean of $X = E(X) = \mu_1' = M_x'(0) = 2.$

$$\mu_2' = M_x''(0) = 8$$

Variance $V(x) = \mu_2' - (\mu_1')^2 = 8 - (2)^2 = 8 - 4 = 4$

PROBLEM 5.1 A PERFECT COIN IS TOSSED THREE TIMES. If X denotes the number of heads that appear, find the MGF of X and hence find the mean and Variance.

Coin is tossed three times.

Given X Represents the number of heads that appear in 3 Tosses.

Values of X are 0, 1, 2, 3.

$$S = \{\overset{3}{HHH}, \overset{2}{HHT}, \overset{2}{HTH}, \overset{2}{THH}, \overset{1}{HTT}, \overset{1}{THT}, \overset{1}{TTH}, \overset{0}{TTT}\}$$

$n(S) = 8.$

$$p(x) = \begin{cases} \frac{1}{8} & \text{for } x = 0 \\ \frac{3}{8} & \text{for } x = 1 \\ \frac{3}{8} & \text{for } x = 2 \\ \frac{1}{8} & \text{for } x = 3 \end{cases}$$

$$M_x(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx}\, p(x)$$

$$= \frac{1}{8} \times e^0 + \frac{3}{8} \times e^t + \frac{3}{8} e^{2t} + \frac{1}{8} e^{3t}$$

$$= \frac{1}{8}\left[1+3e^t+3e^{2t}+e^{3t}\right] = \frac{1}{8}(1+e^t)^3$$

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3.$$

Mean & Variance

$$M_x(t) = \frac{1}{8}\left[1+3e^t+3e^{2t}+e^{3t}\right]$$

$$M_x'(t) = \frac{1}{8}\left[3e^t+6e^{2t}+3e^{3t}\right]$$

$$M_x''(t) = \frac{1}{8}\left[3e^t+12e^{2t}+9e^{3t}\right]$$

$$\mu_1' = M_x'(0) = \frac{1}{8}\left[3+6+3\right] = \frac{12}{8} = 3/2.$$

$$\mu_2' = M_x''(0) = \frac{1}{8}\left[3+12+9\right] = \frac{24}{8} = 3.$$

$$\text{Mean of } X = \mu_1' = M_x'(0) = 3/2.$$

$$\text{Variance of } X = \mu_2' - (\mu_1')^2 = 3 - \left(\frac{3}{2}\right)^2 = 3 - \frac{9}{4}$$

$$\therefore \text{Mean} = \frac{3}{2}$$

$$\text{Variance} = \frac{3}{4}.$$

Tchebycheff Inequality

If X is a Random Variable with $E(x) = \mu$ and Variance $Var(x) = \sigma^2$, then

$$P\{|X-\mu| \geq c\} \leq \frac{\sigma^2}{c^2}; \quad \text{here } c>0.$$

Another form of Tchebycheff Inequality

$$P\{|X-\mu| < c\} \geq 1-\frac{\sigma^2}{c^2}, \quad \text{here } c>0$$

Alternative forms

If we put $c = k\sigma$, where $k>0$, then

Tchebycheff inequality takes the form

$$P\left\{\left|\frac{X-\mu}{k}\right| \geq \sigma\right\} \leq \frac{1}{k^2}$$

$$P\left\{\left|\frac{X-\mu}{k}\right| < c\right\} \geq 1-\frac{1}{k^2}.$$

(or)

$$P\{|X-\mu| \geq k\sigma\} \leq 1/k^2 \quad \text{(upper bound)}$$

$$P\{|X-\mu| < k\sigma\} \geq 1-\frac{1}{k^2} \quad \text{(lower bound)}.$$

Tchebycheff Inequality

Problem 1: Let X be a random Variable with mean 12, and Variance 9. If the probability function is Not Known, find $P(6 < x < 18)$

given $\mu = 12$, $\sigma^2 = 9$, $\sigma = 3$.

By Tchebycheff's Inequality

$$P(|x-\mu| < K\sigma) \geq 1 - \frac{1}{K^2}$$

$$P(|x-12| < 3K) \geq 1 - \frac{1}{K^2}$$

$$P(-3K < x-12 < 3K) \geq 1 - \frac{1}{K^2}$$

$$P(-3K+12 < x < 3K+12) \geq 1 - \frac{1}{K^2}$$

To find $P(6 < x < 18)$

$-3K+12 = 6 \Rightarrow \boxed{K = 2.}$

$3K = 12-6$

$\therefore P(6 < x < 8) \geq 1 - \frac{1}{4} \Rightarrow P(6 < x < 8) \geq 3/4.$

The $P(6 < x < 8)$ has minimum 75%.

Problem 2:

If X denotes the sum of the numbers obtained when 2 dice are thrown, Obtain an upper bound for $P\{|X-7| \geq 3\}$. Compare with exact probability.

X	occurance.	$P(x)$
2	(1,1)	$1/36$
3	(1,2), (2,1)	$2/36$
4	(2,2), (1,3), (3,1)	$3/36$
5	(1,4), (4,1), (3,2) (2,3)	$4/36$
6	(1,5), (5,1), (2,4), (4,2) (3,3)	$5/36$
7	(1,6), (6,1), (2,5), (5,2), (4,3), (3,4)	$6/36$
8	(4,4), (2,6), (6,2), (5,3), (3,5)	$5/36$
9	(6,3), (3,6), (4,5), (5,4).	$4/36$
10	(4,6) (6,4) (5,5)	$3/36$
11	(6,5) (5,6)	$2/36$
12	(6,6).	$1/36$.

To find the Bound we need $E(x)$, $Var(x)$

$$E(x) = \sum_{x=2}^{12} x \cdot P(x)$$

$$= 2\left(\frac{1}{36}\right) + 3\left(\frac{2}{36}\right) + 4\left(\frac{3}{36}\right) + \cdots + 12\left(\frac{1}{36}\right)$$

$$E(x) = \frac{252}{36} = 7. = \mu.$$

$$\boxed{\text{Mean is } 7}$$

Variance $\sigma^2 = E(x^2) - [E(x)]^2$

$$E(x^2) = \sum_{x=2}^{12} x^2 P(x)$$

$$= 2^2\left(\frac{1}{36}\right) + 3^2\left(\frac{2}{36}\right) + 4^2\left(\frac{3}{36}\right) + \cdots + 12^2\left(\frac{1}{36}\right)$$

$$E(x^2) = \frac{1974}{36}$$

$$Var(x) = E(x^2) - [E(x)]^2 = \frac{1974}{36} - (7)^2 = \frac{105}{18.} = \frac{35}{6}$$

By chebychev's inequality

$$P\{|x-\mu| \geq K\sigma\} \leq 1/K^2$$

$$P\{|x-\mu| \geq c\} \leq \frac{\sigma^2}{c^2}$$

$$P\{|x-\mu| \geq c\} \leq \frac{35/6}{c^2}.$$

from the Problem

$C = 3$

$$P\{|x-\mu| \geq 3\} \leq \frac{35/6}{9}$$

$P(|X-7| \geq 3) \leq \frac{35}{54} \cong 0.6481$ (64.81 %)

Actual Probability

$P(|X-7| \geq 3) = 1 - P(|X-7| < 3)$

$= 1 - P(-3 < X-7 < 3)$

$= 1 - P(4 < X < 10)$

$= 1 - \{P(X=5) + P(X=6) + P(X=7) + P(X=8) + P(X=9)\}$

$= 1 - \frac{1}{36}[4+5+6+5+4] = 1 - \frac{24}{36} = 1/3 = 0.33$

33%

There is much differcence between the actual value and the upper bound given Tchebycheff inequality

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Problem 3: If X is a random Variable

A fair die is thrown 720 times, use Tchebycheff inequality to find a lower bound for Probability of getting 100 to 140 sixes.

P(Success) = $\boxed{1/6 = p}$, $p+q=1$, $q = 1-p$ $\boxed{q = 5/6}$

$n = 720$

mean and Variance $\boxed{\mu = 120}$ $\boxed{\sigma^2 = 100}$

mean $= np$, Variance $= npq$.

$M = 720 \times 1/6$ $V = 720 \times \frac{5}{6} \times 1/6$

To find lower bound

$$P(|x-\mu| < c) \geq 1 - \frac{\sigma^2}{c^2}$$

$$P(|x-120| < c) \geq 1 - \frac{100}{c^2}$$

$$P(-c < x-120 < c) \geq 1 - \frac{100}{c^2}$$

$$P(-c+120 < x < 120+c) \geq 1 - \frac{100}{c^2}$$

given $P(100 < x < 140)$

$\therefore \quad -c+120 = 100 \quad \Rightarrow \quad c = 120-100 \Rightarrow \boxed{c=20}$

$$P(100 < x < 140) \geq 1 - \frac{100}{(20)^2} = 1 - \frac{100}{400}$$

$$P(100 < x < 140) \geq \frac{3}{4}.$$

$\therefore$ The Probability of 100 to 140 sixes as has lower bound of at least 75%

Problem :4 A discrete R.v. X takes the values $-1, 0, 1$ with Probabilities $\frac{1}{8}, \frac{3}{4}, \frac{1}{8}$ respectively. Evaluate $P(|x-\mu| \geq 2\sigma)$ and Compare it with the upper bound given by Tchebycheff's inequality.

X	-1	0	1
$P(x)$	$1/8$	$3/4$	$1/8$

$\therefore \quad \sigma^2 = 1/4 \Rightarrow \boxed{\sigma = 1/2}$

$\boxed{\mu = 0}$

$$E(x) = \mu = (-1)\left(\frac{1}{8}\right) + (0)\left(\frac{3}{4}\right) + (1)\left(1/8\right) = 0$$

$$E(x^2) = (-1)^2\left(\frac{1}{8}\right) + (0)\left(\frac{3}{4}\right) + (1)^2\left(\frac{1}{8}\right) = \frac{1}{8} + \frac{1}{8} = 1/4$$

$$Var(x) = [E(x^2)] - [E(x)]^2 = \frac{1}{4} - 0 = 1/4 = \sigma^2$$

$$P(|x-\mu| \geq 2\sigma) = P(|x| \geq 1) = 1 - P(|x| < 1)$$

$$= 1 - \{P(x=0) + P(x=(-1))\}$$

$$= 1 - \left\{\frac{1}{8} + \frac{3}{4}\right\} = 1 - \left\{\frac{7}{8}\right\} = \frac{1}{8}$$

$$P(|x-\mu| \geq k\sigma) \leq 1/k^2$$

$$P(|x-\mu| \geq 2\sigma) \leq 1/4$$

FUNCTIONS OF A RANDOM VARIABLE.

In many Engg. Problems a mathematical model of the System of interest and the input signal will be given and we will be asked to find the characteristics of the output Signals. If the input to a System is random then the output of System will also be random in general.

If the input is set of random Variables then the output will be also a Set of random Variables.

FUNCTION OF ONE RANDOM VARIABLE:

Let X be a Continuous random Variable with pdf $f_X(x)$ and $Y = g(x)$ be a given Transformation of X, where g is a differentiable function of x.

Case (i): If g is Strictly monotonic i.e., g is a one-to-one function of X. and if $f_Y(y)$ is the pdf of Y then.

$$f_Y(y) = f_X(x) \cdot \left| \frac{dx}{dy} \right| , \quad \text{where } x = g^{-1}(y).$$

Case (ii) If g is not strictly monotonic, that is, g is not one-to-one function we find the sub intervals in which it is strictly increasing or strictly decreasing and in this intervals. find their inverse.

$x = g^{-1}(y)$ has values $x_1, x_2, \dots, x_n$

then the pdf f_Y of Y is given by

$$f_Y(y) = f_X(x_1)\left|\frac{dx_1}{dy}\right| + f_X(x_2)\left|\frac{dx_2}{dy}\right| + \dots + f_X(x_n)\left|\frac{dx_n}{dy}\right|$$

PROBLEM 1: Let X be a continuous random variable with pdf

$$f(x) = \begin{cases} \frac{x}{12}, & 1 < x < 5 \\ 0, & \text{otherwise} \end{cases}$$

Find the pdf of $Y = 2X - 3$.

given $f_X(x) = \frac{x}{12}$ $\quad 1 < x < 5$.

The transformation function $y = 2x - 3$ is a one-one fn

Since it is strictly increasing as $\frac{dy}{dx} = 2 > 0$.

we solve for x in terms of y $\quad \therefore 2x = y + 3$

$$\frac{dx}{dy} = \frac{1}{2}. \leftarrow \boxed{x = \frac{y+3}{2}}$$

limits. when $x=1$ $y=-1$

$x=5$ $y=7$. $\therefore -1 < y < 7$.

By the formula $f_y(y) = f_x(x)\left|\frac{dx}{dy}\right| = \frac{x}{12}\cdot\frac{1}{2} = \frac{x}{24}$

W.K.T $x = \frac{y+3}{2}$, the pdf of Y is

$$f_y(y) = \begin{cases} \frac{y+3}{48} & , -1 < y < 7. \\ 0 & , \text{otherwise}. \end{cases}$$

Problem: 2 If X has exponential distribution

$$f_x(x) = \begin{cases} \lambda e^{-\lambda x} & , \text{if } x \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

with Parameter λ. find the pdf of $Y = e^{-\lambda x}$

given $$f_x(x) = \begin{cases} \lambda e^{-\lambda x} & , \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The transformation function $y = e^{-\lambda x}$ is one - one

$$\frac{dy}{dx} = -\lambda e^{-\lambda x} < 0$$

So y is decreasing strictly.

$$y = e^{-\lambda x}$$

$$\log_e y = -\lambda x \Rightarrow x = -\frac{1}{\lambda} \log_e y$$

$$\frac{dx}{dy} = -\frac{1}{\lambda} \times \frac{1}{y}.$$

When $x = 0$, $y = 1$.

$x \to \infty \to y \to 0$.

$$\boxed{\therefore 0 \leq y \leq 1}$$

By formula $f_Y(y) = f_X(x) \left|\frac{dx}{dy}\right|$

$$= \lambda e^{-\lambda x} \left|-\frac{1}{\lambda} \cdot \frac{1}{y}\right| = \lambda \cdot y \times \frac{1}{\lambda} \times \frac{1}{y}$$

$$= 1.$$

$\therefore$ The P.d.f of Y is $f_Y(y) = \begin{cases} 1 & \text{if } 0 < y \leq 1 \\ 0, & \text{otherwise.} \end{cases}$

Problem 3: If X is uniformly distributed in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ find the pdf of $Y = \tan x$

Uniform distribution $f(x) = \frac{1}{b-a}; a \leq x \leq b$

Here $(a,b) = \left(-\pi/2, \pi/2\right)$ $f_X(x) = \frac{1}{\pi/2 - (-\pi/2)} = \frac{1}{\pi}.$

given $f_X(x) = \begin{cases} \frac{1}{\pi}, & -\pi/2 < x < \pi/2 \\ 0, & \text{otherwise} \end{cases}$

The transformation function

$y = \tan x \qquad -\pi/2 < x < \pi/2$ is One-one

$x = \tan^{-1} y \qquad -\infty < y < \infty$

$$\frac{dx}{dy} = \frac{1}{1+y^2}$$

The p.d.f of Y is $f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$

$$= \frac{1}{\pi} \left(\frac{1}{1+y^2} \right)$$

$$\therefore \; f_Y(y) = \frac{1}{\pi(1+y^2)}, \quad -\infty < y < \infty.$$