Unit - 2

Theoritical Distribution

- Discreate
  - Binomial Distribution
  - Poisson Distribution
  - Geometric Distribution
- Continous
  - Uniform Distribution
  - Exponential Distribution
  - Normal Distribution.

I. Binomial Distribution :-

Definition :- Let A be some event associated with a random Experiment E, such that $P(A) = p$ & $P(\bar{A}) = q = 1-p$
Assuming that p remains the same for all repetition (if there are n repetition in total). If X denotes the number of times the event A has occurred, then X is called a binomial random variable with parameter p and n / we can say X follows a binomial distribution.

The probability mass function of Binomial R.V.

$$\boxed{P(X = r) = {}^nC_r\, p^r q^{n-r}} \quad r = 1, 2, 3, \ldots n \;\; \& \;\boxed{p+q = 1}.$$

Note :- 1. $\sum\limits_{r=0}^{n} P(X=r) = \sum\limits_{r=0}^{n} {}^nC_r\, q^{n-r} p^r$

$$= (q+p)^n = 1$$

$\therefore$ Binomial Distribution is a legitimate Probability Distribution.

2. Name is given so because the probabilities are the successive terms of expansion of binomial expression $(q+p)^n$.

Mean & Variance of Binomial Distribution :-

$$* \; E[x] = \sum_{r} x_r p_r$$

$$= \sum_{r=0}^{n} r \cdot {}^nC_r p^r q^{n-r}$$

$$= \sum_{r=0}^{n} r \cdot \frac{n!}{r!(n-r)!} p^r q^{n-r} \quad \text{---} (i)$$

$$= \sum_{r=0}^{n} \frac{\cancel{r} \cdot n(n-1)! \cdot p \cdot p^{r-1} \cdot q^{[n-1-(r-1)]}}{\cancel{r} \cdot (r-1)! [n-1-(r-1)]!}$$

$$= np \sum_{r=0}^{n} \frac{(n-1)!}{(r-1)!(n-1-(r-1))!} p^{r-1} q^{[n-1-(r-1)]}$$

$$= np \sum_{r=0}^{n} {}^{n-1}C_{r-1} p^{r-1} q^{(n-1-(r-1))}$$

$$= np \sum_{r=1} (q+p)^{n-1}$$

$$= np \cdot 1$$

$$\Rightarrow \boxed{E[x] = np} \leftarrow \text{Mean}$$

$$* \; E[x^2] = \sum_{r=0}^{n} (x_r)^2 p_r$$

$$= \sum_{r=0}^{n} r^2 p_r$$

$$= \sum_{r=0}^{n} r \cdot r \cdot {}^nC_r p^r q^{n-r}$$

$$= \sum_{r=0}^{n} [r(r-1) + r] \frac{n!}{r!(n-r)!} p^r q^{n-r}$$

$$= np + \sum_{r=2} \frac{r(r-1) n(n-1)(n-2)! \cdot p^2 \cdot p^{r-2} \cdot q^{n-2-(r-2)}}{r(r-1)(r-2)! [n-2-(r-2)]!}$$

$$\Rightarrow n(n-1)p^2 \sum_{r=2}^{n} {}^{n-2}C_{r-2}\, p^{r-2} q^{n-2-r-2} + np$$

$$\Rightarrow n(n-1)p^2 (p+q)^{n-2} + np$$

$$\Rightarrow n(n-1)p^2 + np$$

$$E[x^2] = n^2p^2 - np^2 + np$$

$$\text{Variance} = E[x^2] - (E[x])^2$$

$$= n^2p^2 - np^2 + np - (np)^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$= np(1-p)$$

$$\boxed{\text{Variance} = npq}$$

Moment Generating function of Binomial Distribution

$$M(t) = \cancel{\Sigma}\, E[e^{tx}]$$

$$= \sum_r e^{tx_r} p_r$$

$$= \sum_{r=0}^{n} e^{tr}\, {}^nC_r\, p^r q^{n-r}$$

$$= \sum_{r=0}^{n} (pe^t)^r \cdot {}^nC_r \cdot q^{n-r}$$

$$M(t) = (q + pe^t)^n$$

$\Rightarrow$ Moment Generating function of Binomial Dist.

$$\boxed{M(t) = (q + pe^t)^n}$$

Scanned by TapScanner

$$M(t) = (q + pe^t)^n \quad \text{---} ①$$

On Differentiating ① w.r.t.

$$M'(t) = np \cdot (q + pe^t)^{n-1}$$

$$M'(t)_{t=0} = np \cdot (q + pe^0)^{n-1}$$

$$= np \cdot (q+p)^{n-1}$$

$$= np.$$

$$\therefore \boxed{E[x] = np} \leftarrow \text{Mean}$$

$$M''(t) = np \cdot (n-1) \cdot p (q + pe^t)^{n-2} + (q+pe^t)^{n-1} \cdot np$$

$$M''(t)_{t=0} = np^2 (n-1) (q+p)^{n-2} + (q+p)^{n-1} \cdot np.$$

$$M''(t)_{t=0} = n^2p^2 - np^2 + np.$$

$$\therefore E[x^2] = n^2p^2 - np^2 + np$$

$$\text{Variance} = E[x^2] - (E[x])^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{(np)^2}$$

$$= np(1-p)$$

$$\boxed{\text{Variance} = npq}$$

Scanned by TapScanner

Q. If the mean and variance of a binomial distribution are 4 and 4/3 respectively. Find $P(x \geq 1)$.

Sol → Mean = 4

$\Rightarrow np = 4$

Variance = 4/3

$\Rightarrow npq = 4/3$

$\Rightarrow 4q = 4/3$

$\Rightarrow q = \frac{1}{3}$

We know, $p + q = 1$

$\Rightarrow \cancel{q = 1 - p}$ $p = 1 - q$

$\Rightarrow p = 1 - \frac{1}{3} = \frac{2}{3}$.

Putting this value in (i)

$np = 4$

$\Rightarrow \frac{2}{3} n = 4$

$\Rightarrow n = \frac{12}{2} = 6$.

$P(x = r) = {}^6C_r \left(\frac{2}{3}\right)^r \left(\frac{1}{3}\right)^{6-r}$

$P(x \geq 1) = 1 - P(x < 1)$.

$= 1 - P(x = 0)$

$= 1 - {}^6C_0 \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^{6-0}$

$= 1 - 1 \cdot 1 \cdot \left(\frac{1}{3}\right)^6$

$= 1 - \left(\frac{1}{3}\right)^6$ Ans

Binomial frequency Distribution.

$$\boxed{f(r) = N * P(r)}$$

frequency

$$\boxed{\frac{P(r+1)}{P(r)} = \frac{{}^nC_{r+1}\, p^{r+1} q^{n-r-1}}{{}^nC_r\, p^r q^{n-r}}}$$

$$= \frac{\frac{n!}{(r+1)!\,(n-r-1)!}\, p^{r+1} \cdot q^{n-r-1}}{\frac{n!}{r!\,(n-r)!}\, p^r q^{n-r}}$$

$$= \frac{\frac{\cancel{n!}}{(r+1)\,\cancel{r!}\,(n-r\cancel{-1})!}\, p \cdot \cancel{p^r} \cdot \cancel{q^{n-r-1}}}{\frac{\cancel{n!}}{\cancel{r!}\,(n-r)\,(n-r\cancel{-1})!}\, \cancel{p^r} \cdot q \cancel{q^{n-r-1}}}$$

$$= \frac{p}{r+1} \cdot \frac{n-r}{q}$$

$$\frac{p(r+1)}{p(r)} = \frac{n-r}{r+1} \cdot \frac{p}{q}$$

$$\Rightarrow \boxed{P(r+1) = \frac{n-r}{r+1} \cdot \frac{p}{q} \cdot P(r)}$$

Recurrence relation for probability of Binomial Distribution (fitting of binomial Distribution).

* $P(0) = {}^nC_0 \, p^0 \, q^{n-0}$

$$\boxed{P(0) = q^n}$$

* $P(1) = [P(r+1)]_{r=0} = \left[\frac{n-r}{r+1} \cdot \frac{p}{q}\right]_{r=0} * P(0)$

* $P(2) = [P(r+1)]_{r=1} = \left[\frac{n-r}{r+1} \cdot \frac{p}{q}\right]_{r=1} * P(1)$

and so on

Q Fit a binomial distribution for the following data:

x	0	1	2	3	4	5	6	Total
f	5	18	28	12	7	6	4	80

Sol→ Here, $N = 80$

$n = 6$

$$\bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{0*5 + 1*18 + 2*28 + 3*12 + 4*7 + 5*6 + 6*4}{80}$$

$$= \frac{192}{80} = 2.4.$$

We know, Mean $= np = 2.4$.

$6p = 2.4$

$p = 0.4$.

$q = 1 - p = 1 - 0.4 = 0.6$

$P(0) = q^n = (0.6)^6 = 0.046656$

$f(0) = N * P(0)$
$= 80 \times 0.046656$
$= 3.73248$

$P(1) = \left(\frac{n-r}{r+1} \cdot \frac{p}{q}\right)_{r=0} \cdot P(0)$

$= \left(\frac{6}{1} \cdot \frac{0.4}{0.6}\right) \cdot 0.046656$

$= 0.186624$

$f(1) = N * P(1)$
$= 80 * 0.186624$
$= 14.92$

Similarly,

$f(2) = 24.88$
$f(3) = 22.12$
$f(4) = 11.06$
$f(5) = 2.95$
$f(6) = 0.33$

Converting these value in whole numbers with condition that total frequency should be 80.

x	0	1	2	3	4	5	6
f	4	15	25	22	11	3	0

Scanned by TapScanner

Poisson Distribution :-

If X is a discreate random variable that can assume the values 0, 1, 2, - - - such that its probability ~~d~~ mass function is given by

$$\boxed{P(x=r) = \frac{e^{-\lambda}\lambda^{r}}{r!}, \quad r = 0, 1, 2 - - - \quad \lambda > 0}$$

Then, X is said to be Poisson distribution with parameter λ.

Note :- Poisson Distribution is a ~~Logarithmic~~ Legitimate probability distribution since

$$\sum_{r=0}^{\infty} P(x=r) = \sum_{r=0}^{\infty} \frac{e^{-\lambda}\lambda^{r}}{r!}$$

$$= e^{-\lambda} \sum_{r=0}^{\infty} \frac{\lambda^{r}}{r!}$$

$$= e^{-\lambda}\left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + - - \right]$$

$$= e^{-\lambda} \cdot e^{\lambda}$$

$$= 1$$

So, since $\sum_{r=0}^{\infty} P(x=r) = 1$

So, it is a legitimate Probability Distribution function

Note :- When an event occur rarely, the number of occurrence of such event may be assumed to follow poisson distribution. Following are some example of poisson distribution Events :-

1) No. of alpha particles emitted from a radioactive source in given time interval.
2) No. of printing errors in each page of a book
3) No. of road accident in a city per day.
4) ..etc.

Mean & Variance of Poisson Distribution

* $E[x]$ = Mean of x

$$= \sum_{r} x_r p_r$$

$$= \sum_{r=0}^{r} r \cdot \frac{e^{-\lambda} \lambda^r}{r!}$$

$$= \lambda e^{-\lambda} \sum_{r=1}^{\infty} \frac{\lambda^{r-1}}{(r-1)!}$$

$$= \lambda e^{-\lambda} \left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right]$$

$$= \lambda e^{-\lambda} e^{\lambda}$$

$$= \lambda$$

So, $\boxed{E[x] = \lambda}$

$$* \; E[x^2]_n = \sum_r x_r^2 \, p_r$$

$$= \sum_{r=0}^{\infty} r^2 \cdot \frac{e^{-\lambda}\lambda^r}{r!}$$

$$= \sum_{r=0}^{\infty} [r(r-1)+r] \frac{e^{-\lambda}\lambda^r}{r!}$$

$$= \sum_{r=0}^{\infty} r(r-1)\frac{e^{-\lambda}\lambda^r}{r!} + \sum_{r=0}^{\infty} r \cdot \frac{e^{-\lambda}\lambda^r}{r!}$$

$$= \sum_{r=0}^{\infty} \frac{r(r-1)e^{-\lambda}\lambda^2\lambda^{r-2}}{r(r-1)(r-2)!} + \underbrace{\sum_{r=0}^{\infty} r \cdot \frac{e^{-\lambda}\lambda^r}{r!}}$$

$$= \lambda^2 e^{-\lambda} \sum_{r=2}^{\infty} \frac{\lambda^{r-2}}{(r-2)!} + \lambda$$

$$= \lambda^2 e^{-\lambda}\left[1 + \frac{\lambda}{1} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right] + \lambda$$

$$= \lambda^2 e^{-\lambda} \cdot e^{\lambda} + \lambda$$

$$= \lambda^2 + \lambda$$

$$\text{Variance} = E[x^2] - (E[x])^2$$

$$= \lambda^2 + \lambda - (\lambda)^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$= \lambda$$

$$\Rightarrow \boxed{\text{Variance} = \lambda}$$

Scanned by TapScanner

Moment Generating function of Poisson Distribution

P.M.F

$\hookrightarrow P(X=r) = \frac{e^{-\lambda}\lambda^r}{r!}$

$$M(t) = E[e^{tx}].$$

$$= \sum_{r=0}^{\infty} e^{t\cancel{x}r} \cdot \frac{e^{-\lambda}\lambda^r}{r!}$$

$$= \sum_{r=0}^{\infty} e^{tr} \cdot \frac{e^{-\lambda}\lambda^r}{r!}$$

$$= \sum_{r=0}^{\infty} (\lambda e^t)^r \cdot \frac{e^{-\lambda}}{r!}$$

$$= e^{-\lambda} \sum_{r=0}^{\infty} \frac{(\lambda e^t)^r}{r!}$$

$$= e^{-\lambda}\left[1 + \frac{\lambda e^t}{1!} + \frac{(\lambda e^t)^2}{2!} + \frac{(\lambda e^t)^3}{3!} + \cdots\right]$$

$$= e^{-\lambda} \cdot e^{\lambda e^t}$$

or

$$\Rightarrow M(t) = e^{-\lambda + \lambda e^t}.$$

or

$$M(t) = e^{-\lambda(1-e^t)}.$$

or

$$\boxed{M(t) = e^{\lambda(e^t - 1)}}$$

Mean & Variance using MGF of Poisson

$$M(t) = e^{-\lambda} e^{\lambda e^t}$$

$$M'(t) = e^{-\lambda} e^{\lambda e^t} \cdot \lambda e^t$$

$$* \left[M'(t)\right]_{t=0} = e^{-\lambda} \cdot e^{\lambda} \cdot \lambda \cdot e^0$$

$$= \lambda$$

$$\Rightarrow \boxed{E[x] = \lambda}$$

$$\left[M''(t)\right] = \cancel{E[x^2]} = \lambda e^{-\lambda} \left[e^{\lambda e^t} \cdot e^t + e^t \cdot e^{\lambda e^t} \cdot \lambda e^t\right]$$

$$* \left[M''(t)\right]_{t=0} = E[x^2] = \lambda e^{-\lambda}\left[e^{\lambda} \cdot 1 + 1 \cdot e^{\lambda} \cdot \lambda\right].$$

$$= \lambda e^{-\lambda} e^{\lambda} + \lambda^2 e^{\lambda} e^{-\lambda}$$

$$\Rightarrow \lambda + \lambda^2$$

$$\Rightarrow E[x^2] = \lambda^2 + \lambda$$

$$* \text{ Variance} = E[x^2] - \left(E[x]\right)^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$= \lambda.$$

Q. The number of monthly breakdown of a computer is a R.V having a Poisson distribution with mean 1.8. Find Probability that their computer will function for a month

(a) without a breakdown
(b) with only one breakdown
(c) with atleast one breakdown.

Sol → Let X be the no. of breakdown of computer in a month.

X - follows Poisson distribution with mean 1.8.

$$\Rightarrow \lambda = 1.8$$

$$P(x=r) = \frac{e^{-\lambda}\lambda^r}{\underline{|r}} = \frac{e^{-1.8}(1.8)^r}{r!}$$

ⓐ $P(x=0) = \frac{e^{-1.8}(1.8)^0}{0!} = e^{-1.8}$

$= 0.1653$

ⓑ $P(x=1) = \frac{e^{-1.8}(1.8)^1}{1!} = 1.8\, e^{-1.8}$

$= 0.2975$

ⓒ $P(x \geq 1) = 1 - P(x<1)$

$= 1 - P(x=0)$

$\Rightarrow 1 - 0.1653$

$= 0.83847$

Q If x is a Poisson R.V such that $P(x=1)=0.3$ & $P(x=2)=0.2$. find $P(x=0)$.

Sol→ $P(x=r) = \frac{e^{-\lambda}\lambda^r}{r!}$

$P(x=1) = \frac{e^{-\lambda}\lambda^1}{1!} \Rightarrow e^{-\lambda}\lambda = 0.3$

$\Rightarrow e^{-\lambda} = \frac{0.3}{\lambda}$

$\Rightarrow e^{-\lambda} = \frac{3}{10\lambda}$ — - (i)

Scanned by TapScanner

$$P(x=2) = \frac{e^{-\lambda}\lambda^2}{2!} = 0.2$$

$$\Rightarrow e^{-\lambda} = \frac{0.2 \times 2}{\lambda^2}$$

$$\Rightarrow e^{-\lambda} = \frac{0.4}{\lambda^2} = \frac{4}{10\lambda^2} \quad \text{--- (2)}$$

from (i) & (ii)

$$\frac{3}{\cancel{10\lambda}} = \frac{4}{\cancel{10}\lambda^{\cancel{2}}}$$

$$\Rightarrow \lambda = \frac{4}{3}$$

$$P(x=0) = \frac{e^{-4/3}\left(\frac{4}{3}\right)^0}{0!}$$

$$P(x=0) = e^{-4/3} \quad \text{Ans}$$

Recurrence Relation for the probability of poisson distribution (Fitting of Poisson distribution) :-

$$P(x=r) = \frac{e^{-\lambda}\lambda^r}{r!} \quad r = 0, 1, 2 \ldots, \quad \lambda > 0.$$

frequency distribution is given by.

$$\boxed{f(r) = N \cdot P(r)}$$

$$\frac{P(r+1)}{P(r)} = \frac{\frac{\cancel{e^{-\lambda}}\lambda^{r+1}}{(r+1)!}}{\frac{\cancel{e^{-\lambda}}\lambda^r}{r!}} = \frac{\frac{\lambda \cdot \cancel{\lambda^r}}{(r+1)\cancel{r!}}}{\frac{\cancel{\lambda^r}}{\cancel{r!}}} = \frac{\lambda}{r+1}.$$

$$\therefore \boxed{P(r+1) = \frac{\lambda}{r+1} \cdot P(r)}$$

$$* \ P(0) = \frac{e^{-\lambda}\lambda^0}{0!} = e^{-\lambda}$$

$$* \ P(1) = \left(P[r+1]\right)_{r=0} = \left(\frac{\lambda}{r+1}\right)_{r=0} P(0).$$

$$* \ P(2) = \left[P(r+1)\right]_{r=1} = \left(\frac{\lambda}{r+1}\right)_{r=1} \times P(1)$$

and so on

Q Fit the poisson distribution for the following distribution.

x	0	1	2	3	4	5	Total
f	142	156	69	27	5	1	400

Sol → $N = 400$

$$\bar{x} = \frac{\sum fx}{\sum f}$$

$$= \frac{0 + 156 + 2\times 69 + 3\times 27 + 4\times 5 + 5\times 1}{400}$$

$$= \frac{400}{400} = 1$$

$$\Rightarrow \lambda = 1$$

$$P(0) = e^{-\lambda} = e^{-1} = \frac{1}{2.718} = 0.36791757$$

$f(0) = N * P(0)$

$= 400 \times 0.36791759$

$= 147.17$

$$P(1) = \left(\frac{\lambda}{r+1}\right)_{r=0} \cdot P(0)$$

$$= \left(\frac{1}{1}\right) \times 0.36791759$$

$$= 0.36791759$$

$f(1) = N * P(0)$

$= 400 \times 0.36791759$

$= 147.17$

Similarly. $f(2) = 73.58$

$f(3) = 24.53$

$f(4) = 6.13$

$f(5) = 1.23$.

x	0	1	2	3	4	5
f	147.17	147.17	73.58	24.53	6.13	1.23

Converting it into whole number such that the total frequency will remain same i.e. N = 400. we get following poisson distribution which fit the given distribution.

x	0	1	2	3	4	5	Total
f	147	147	74	25	6	1	400

Scanned by TapScanner

Geometric Distribution :-

Def. :- Let X be a discrete R.V denoting the number of trials of a random experiment required to obtain the first success.

$X = 1, 2, 3 \ldots \infty$

Now, $X = r$, if and only if there is failure in first $(r-1)$ trial and r^{th} trial result in success.

$$\Rightarrow \boxed{P(X=r) = q^{r-1} p}$$

where $p(A) = p$, $p(\bar{A}) = q$.

* If X is discrete R.V that can assume the value $1, 2, 3, \ldots \infty$ such that its probability mass function is given by.

* $P(X=r) = q^{r-1} p$, $r = 1, 2, 3 \ldots$

$\boxed{p+q=1}$

Then, X is called a geometric distribution.

Not :-

$$\sum_{r=1}^{\infty} P(X=r) = \sum_{r=1}^{\infty} q^{r-1} p = p \sum_{r=1}^{\infty} q^{r-1}$$

$$= p(1 + q + q^2 + q^3 + \ldots)$$

$$= p \cdot \left[\frac{1}{1-q}\right]$$

$$= p \cdot \frac{1}{p} \quad [\because 1-q = p]$$

$$= 1$$

So, Geometric distribution is a legitimate Probability Distribution.

Mean & Variance of Geometric distribution :-

$$E[x] = \sum_{r=1}^{\infty} x_r p_r$$

$$= \sum_{r=1}^{\infty} r \cdot pq^{r-1}$$

$$= p\sum_{r=1}^{\infty} r \cdot q^{r-1}$$

$$= p\left[1 + 2q + 3q^2 + 4q^3 + \cdots \infty\right]$$

$$= p(1-q)^{-2}$$

$$= \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}.$$

So, $\boxed{\text{Mean} = E[x] = \frac{1}{p}}$

$$E[x^2] = \sum_{r=1}^{\infty} x_r^2 p_r$$

$$= \sum_{r=1}^{\infty} r^2 \cdot pq^{r-1}$$

$$= \sum_{r=1}^{\infty} [r(r+1) - r]\, p q^{r-1}$$

$$= p\sum_{r=1}^{\infty} r(r+1) p q^{r-1} - p\sum_{r=1}^{\infty} r \cdot pq^{r-1}$$

$$= p\sum_{r=1}^{\infty} r(r+1) p q^{r-1} - \frac{1}{p}.$$

$$= p\left[1\times2 + 2\times3q + 3\times4q^2 + 4\times5q^3 + \cdots\right] - \frac{1}{p}.$$

$$= 2p\left[1 + 3q + 6q^2 + 10q^3 + \cdots\right] - \frac{1}{p}$$

$$= 2p(1-q)^{-3} - \frac{1}{p} = \frac{2p}{(1-q)^3} - \frac{1}{p}$$

Scanned by TapScanner

$$= \frac{2p}{p^3} - \frac{1}{p}.$$

$$= \frac{2}{p^2} - \frac{1}{p}.$$

$$= \frac{2-p}{p^2} = \frac{2-(1-q)}{p^2}$$

$$* \quad E[x^2] = \frac{1+q}{p^2}.$$

$$\text{Variance} = E[x^2] - (E[x])^2$$

$$= \frac{1+q}{p^2} - \left(\frac{1}{p}\right)^2$$

$$= \frac{1}{p^2}(1+q-1)$$

$$\boxed{\text{Variance} = \frac{q}{p^2}}$$

Moment Generating function of Geometric distribution :-

$$P(x=r) = pq^{r-1}$$

$$M(t) = E[e^{tx_r}]$$

$$= \sum_{r=1}^{\infty} e^{tr} \cdot pq^{r-1}$$

$$= p \sum_{r=1}^{\infty} \frac{e^{tr} \cdot pq^r}{q}$$

$$= p \sum_{r=1}^{\infty} \frac{(qe^t)^r p}{q}$$

$$= \frac{p}{q} \sum_{r=1}^{\infty} (e^t q)^r = \frac{p}{q} \sum_{r=1}^{\infty} (e^t q)^r \cdot \frac{e^t q}{e^t q}$$

$$= \frac{p e^t q}{q} \sum_{r=1}^{\infty} (q e^t)^{r-1}$$

$$= p e^t \sum_{r=1}^{\infty} (q e^t)^{r-1}$$

$$= p e^t \left[1 + q e^t + (q e^t)^2 + (q e^t)^3 + \cdots \right]$$

$$= p e^t \cdot \frac{1}{1 - q e^t} = p e^t (1 - q e^t)^{-1}$$

So, $\boxed{M(t) = \frac{p e^t}{(1 - q e^t)}}$

$$M'(t) = \frac{(1 - q e^t) \cdot p e^t - p e^t \cdot (-q e^t)}{(1 - q e^t)^2}$$

$$= \frac{p e^t \left[1 - q e^t + q e^t \right]}{(1 - q e^t)^2} = \frac{p e^t}{(1 - q e^t)^2}$$

$$\left[M'(t) \right]_{t \to 0} = \frac{p e^0}{(1 - q e^0)^2} = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

$$\Rightarrow \text{Mean} = \frac{1}{p}$$

$$M''(t) = \frac{(1 - q e^t)^2 p e^t - p e^t \left[2(1 - q e^t) \cdot (-q e^t) \right]}{(1 - q e^t)^4}$$

$$= \frac{p e^t (1 - q e^t) \left[1 - q e^t + 2 q e^t \right]}{(1 - q e^t)^4}$$

$$\left[M''(t)\right]_{t\to 0} = \frac{pe^0(1-qe^0)\left[1-qe^0+2qe^0\right]}{(1-qe^0)^4}$$

$$= \frac{p(1-q)\left[1-q+2q\right]}{(1-q)^4}$$

$$= \frac{p\times p\times\left[p+2q\right]}{p^4} = \frac{p^3+2p^2(1-p)}{p^4}$$

$$= \frac{p^3+2p^2-2p^3}{p^4}$$

$$= \frac{2p^2}{p^4} - \frac{2p^3}{p^4}$$

$$\left[M''(t)\right]_{t\to 0} = E[x^2] = \frac{2}{p^2} - \frac{1}{p}$$

$$\text{Variance} = E[x^2] - \left(E[x]\right)^2$$

$$= \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2}$$

$$= \frac{2-1-p}{p^2} = \frac{1-p}{p^2} = \frac{q}{p^2}$$

So, $\boxed{\text{Variance} = \frac{q}{p^2} \quad \text{Mean} = \frac{1}{p}}$

State & Prove Memoryless Property of Geometric Distribution

If X is a discreat R.V following a geometric distribution then $P(X>m+n/X>m) = P(X>n)$. where m and n are any two positive integer.

Proof :- Since X follows Geometric distribution

$$P(x=r) = pq^{r-1}$$

$$p(x>k) = \sum_{r=k+1}^{\infty} pq^{r-1}$$

$$= p(q^{k} + q^{k+1} + q^{k+2} + \cdots \infty)$$

$$= pq^{k}(1+q+q^{2}+q^{3}+ \cdots \infty)$$

$$= pq^{k}\left[\frac{1}{1-q}\right]$$

$$= \frac{pq^{k}}{p} = q^{k}$$

$$\Rightarrow \boxed{P(x>k) = q^{k}} \quad \text{------(i)}$$

Remember for MCQ

Now, $P(x>m+n/x>m) = \dfrac{P\{(x>m+n)\cap(x>m)\}}{P(x>m)}$

$$= \frac{P(x>m+n)}{P(x>m)} = \frac{q^{m+n}}{q^{m}} = q^{n}$$

$$= P(x>n)$$

So, $P(x>m+n/x>m) = P(x>n)$

Q If the probability that an applicant for a driver's licence will pass the test at any given trial is 0.8. What is the probability that he will finally pass the test (a) On the fourth trial
(b) In fewer than 4 trial.

Sol → Let X denote the number of trials required to achieve the first success.
Then X follows geometric distribution.
p.m.f is given by.

$$P(X=r) = q^{r-1}p$$

Here, $p = 0.8 \Rightarrow q = 1 - 0.8$
$= 0.2$

(a) $P(X=4) = pq^3$
$= 0.8 \times (0.2)^3$
$= 0.0064$ Ans

(b) $P(X<4) = P(X=1) + P(X=2) + P(X=3)$
$= p + qp + q^2 p$
$= 0.8 + 0.8 \times 0.2 + (0.2)^2 \cdot 0.8$
$= 0.8(1 + 0.2 + 0.04)$
$= 0.992$ Ans

Q. A die is cast until 6 appears. What is the probability that it must be cast more than 5 times.

Sol → Let X denote the no. of trials required for first 6.
X follows geometric distribution.

$$P(X=r) = q^{r-1}p$$

Here, $p = \frac{1}{6}$; $q = 1 - \frac{1}{6} = \frac{5}{6}$.

Method 1

$$P(X>5) = 1 - P(X \le 5)$$

$$= 1 - [P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)]$$

$$= 1 - [p + qp + q^2p + q^3p + q^4p]$$

$$= 1 - p[1 + q + q^2 + q^3 + q^4].$$

$$= 1 - \frac{1}{6}\left[1 + \frac{5}{6} + \frac{25}{36} + \frac{125}{216} + \frac{625}{216 \times 6}\right]$$

$$= 0.401877$$

Method 2:-

$$P(X>5) = \sum_{r=6}^{\infty} pq^{r-1}$$

$$= p[q^5 + q^6 + q^7 + \cdots]$$

$$= pq^5\left[\frac{1}{1-q}\right] = q^5 = \left(\frac{5}{6}\right)^5$$

$$= 0.401877$$

For MCQ :- $P(X>5) = q^5$

$$= \left(\frac{5}{6}\right)^5 \text{ Ans}$$

Uniform Distribution
(Rectangular)

A Continous R.V is said to follow a uniform or rectangular distribution in any finite Interval if the pdf is constant in that interval.

If X is uniformly distributed in interval (a, b).
Then.. $f(x) = k$, where k is constant.

Lets find value of k.
We know for continous R.V $\int_a^b f(x) = 1$

$$\Rightarrow \int_a^b f(x)\,dx = 1 \Rightarrow \int_a^b k\,dx = 1$$

$$\Rightarrow k\int_a^b dx = 1$$

$$\Rightarrow k(b-a) = 1$$

$$\Rightarrow k = \frac{1}{b-a}$$

So, $\boxed{f(x) = \frac{1}{b-a}}$

Not :- 1) a & b are two parameters $(a<b)$. The distribution is called uniform because it assume a constant value in (a, b).

2) It is called rectangular distribution because the curve $y = f(x)$ describe a rectangle over X-axis and between ordinates $x=a$ to $x=b$.

Mean & Variance of Uniform Distribution

$$\text{Mean} = E[x] = \int_a^b x \cdot f(x)\,dx$$

$$= \int_a^b x \cdot \frac{1}{b-a}\,dx$$

$$= \frac{1}{(b-a)} \left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{(b-a)} \cdot \frac{b^2-a^2}{2}$$

$$\boxed{\text{Mean} = E[x] = \frac{b+a}{2}}$$

$$E[x^2] = \int_a^b x^2 \cdot f(x)\,dx$$

$$= \frac{1}{b-a}\int_a^b x^2\,dx = \frac{1}{b-a}\left|\frac{x^3}{3}\right|_a^b$$

$$= \frac{b^3-a^3}{3(b-a)} = \frac{(b-a)(b^2+ab+a^2)}{3(b-a)}$$

$$E[x^2] = \frac{b^2+ab+a^2}{3}$$

$$\text{Variance} = E[x^2] - (E[x])^2$$

$$= \frac{b^2+ab+a^2}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{4(b^2+ab+a^2) - 3(a^2+2ab+b^2)}{12}$$

$$= \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12}$$

$$\Rightarrow \boxed{\text{Variance} = \frac{(b-a)^2}{12}}$$

Moment Generating function of Uniform distribution :-

$$M(t) = E[e^{tx}]$$

$$= \int_a^b e^{tx} \cdot f(x)\, dx$$

$$= \int_a^b e^{tx} \cdot \frac{1}{b-a}\, dx$$

$$= \frac{1}{(b-a)} \left[\frac{e^{tx}}{t}\right]_a^b$$

$$= \frac{1}{t(b-a)} (e^{bt} - e^{at}).$$

$$\Rightarrow \boxed{M(t) = \frac{e^{bt} - e^{at}}{(b-a)t}}$$

Q. If X is uniformly distributed random variable with mean 1 and variance 4/3. find $P(X<0)$

Sol → Mean $= \frac{a+b}{2} = 1 \Rightarrow a+b = 2$ --- (1)

Variance $= \frac{(b-a)^2}{12} = \frac{4}{3} \Rightarrow (b-a)^2 = 16$

$\Rightarrow b-a = \pm 4.$

$a+b = 2$	$a+b = 2$
$b-a = 4.$	$b-a = -4.$
$\Rightarrow b = 3$	$2b = -2$
$a = -1$	$b = -1,\ a = 3.$
$\Rightarrow a < b$ (valid)	$b < a$ (Not valid)

$\Rightarrow a = -1, \ b = 3.$

$$f(n) = \frac{1}{b-a} = \frac{1}{3-(-1)} = \frac{1}{4}.$$

$$P(x<0) = P(-\infty < n < -1) + P(-1 < n < 0)$$

$$= 0 + \int_{-1}^{0} \frac{1}{4}\,dx$$

$$= \frac{1}{4}[n]_{-1}^{0} = \frac{1}{4}(0+1)$$

$$= \frac{1}{4} \ \underline{\text{Ans}}$$

Q. A R.V X has a uniform distribution over $(-3, 3)$
Find (a) $P(x<2)$ (b) $P(|n|<2)$
(c) $P(|n-2|<2)$ (d) Value of k for which $P(n>k) = \frac{1}{3}$.

Sol→ Pdf of Uniform distribution, $f(n) = \frac{1}{b-a}$

$$\Rightarrow f(n) = \frac{1}{3-(-3)} = \frac{1}{6}.$$

(a) $P(n<2) = \int_{-3}^{2} \frac{1}{6}\,dx = \frac{1}{6}[2+3]$

$$= \frac{5}{6} \ \underline{\text{Ans}}$$

(b) $P(|n|<2) = P(-2<n<2)$

$$= \int_{-2}^{2} \frac{1}{6}\,dx = \frac{1}{6}(2+2)$$

$$= \frac{2}{3} \ \underline{\text{Ans}}$$

(c) $P(|x-2|<2) = P(-2<x-2<2)$

$= P(-2+2<x-2+2<2+2)$

$= P(0<x<4)$.

$= P(0<x<3) + P(3<x<4)$

$= \int_0^3 \frac{1}{6}\,dx + 0$

$= \frac{1}{6}[x]_0^3$

$= \frac{3}{6} = \frac{1}{2}$. Ans

(d) $P(x>k) = \frac{1}{3} \Rightarrow P(k<x<3) = \frac{1}{3}$.

$\Rightarrow \int_k^3 \frac{1}{6}\,dx = \frac{1}{3}$.

$= \frac{1}{6}[x]_k^3 = \frac{1}{3}$

$\Rightarrow 3-k = \frac{6}{3}$

$\Rightarrow 3-k = 2$

$\Rightarrow k = 1$ Ans

Exponential distribution :-

A continous R.V X is said to follow an exponential distribution or negative exponential distribution with parameter $\lambda > 0$, if its pdf is given by.

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Not :- $\int_0^\infty f(x)\,dx = \int_0^\infty \lambda e^{-\lambda x}$

$$= \lambda \int_0^\infty e^{-\lambda x} = \lambda \left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^\infty$$

$$= \frac{\lambda}{\lambda}\left[0 + 1\right] = 1$$

So, it is a legitimat Probability distribution.

Mean and variance of exponential distribution :-

$$E[x] = \int_0^\infty x \cdot f(x)\,dx$$

$$= \int_0^\infty x \cdot \lambda e^{-\lambda x}\,dx = \lambda \int_0^\infty x \cdot e^{-\lambda x}\,dx$$

$$= \lambda \left[x \cdot \frac{e^{-\lambda x}}{-\lambda} - 1 \cdot \frac{e^{-\lambda x}}{\lambda^2} \right]_0^\infty$$

$$= \lambda \left[0 - 0 - 0 + \frac{1}{\lambda^2} \right] = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

$$\boxed{\therefore \text{Mean} = E[x] = \frac{1}{\lambda}}$$

$$E[x^2] = \int_0^\infty x^2 \lambda e^{-\lambda x} dx$$

$$= \lambda \left[x^2 \cdot \frac{e^{-\lambda x}}{-\lambda} - 2x \cdot \frac{e^{-\lambda x}}{\lambda^2} + 2 \cdot \frac{e^{-\lambda x}}{-\lambda^3} \right]_0^\infty$$

$$= \lambda \left[0 + 0 + 0 - 0 - 0 + 2 \cdot \frac{e^0}{\lambda^3} \right]$$

$$= \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2}.$$

$$\text{Variance} = E[x^2] - (E[x])^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2}. = \frac{1}{\lambda^2}.$$

$$\boxed{\therefore \text{Variance} = \frac{1}{\lambda^2}}$$

Moment Generating function of exponential distribution :-

$$M(t) = E[e^{tx}] = \int_0^\infty e^{tx} \cdot f(x)\, dx$$

$$= \int_0^\infty e^{tx} \cdot \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty e^{-x(\lambda - t)} dx$$

$$= \lambda \left[\frac{e^{-(\lambda - t)x}}{-(\lambda - t)} \right]_0^\infty$$

$$= \lambda \left[0 + \frac{1}{(\lambda - t)} \right] = \frac{\lambda}{\lambda - t}$$

$$\boxed{M(t) = \frac{\lambda}{\lambda - t}}$$

$$M(t) = \frac{\lambda}{\lambda - t}$$

$$= \frac{\lambda}{\lambda\left(1-\frac{t}{\lambda}\right)} = \left(1-\frac{t}{\lambda}\right)^{-1}$$

$$= 1 + \frac{t}{\lambda} + \left(\frac{t}{\lambda}\right)^2 + \left(\frac{t}{\lambda}\right)^3 + - - -$$

Mean $= \mu_1' = E[x] =$ Coefficient of $\frac{t}{1!} = \frac{1}{\lambda}$.

$\mu_2^2 = E[x^2] =$ Coefficient of $\frac{t^2}{2!} = \frac{2!}{\lambda^2} = \frac{2}{\lambda^2}$.

Variance $= E[x^2] - (E[x])^2$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$\Rightarrow \boxed{\begin{array}{l} \text{Mean} = \frac{1}{\lambda} \\ \text{Variance} = \frac{1}{\lambda^2} \end{array}}$$

Memoryless Property of exponential distribution :-

If X is exponentially distributed, then

$$P(X > s+t \mid X > s) = P(X > t) \text{ for any } s, t > 0$$

Proof :- $P(X > k) = \int_k^\infty \lambda e^{-\lambda x}\,dx = \lambda \left| \frac{e^{-\lambda x}}{-\lambda} \right|_0^\infty$

$$= \frac{\lambda\left[0 + e^{-k\lambda}\right]}{\lambda}$$

$$\boxed{P(X > k) = e^{-\lambda k}} \quad - - - (1)$$

$$P(x > s+t \mid x > s) = \frac{P[(x > s+t) \cap (x > s)]}{P(x > s)}$$

$$= \frac{P(x > s+t)}{P(x > s)}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$

$$= e^{-\lambda s - \lambda t + \lambda s}$$

$$= e^{-\lambda t}$$

$$= P(x > t)$$

$$\therefore P\left(\frac{x > s+t}{x > s}\right) = P(x > t).$$

Q. The time (in hours) required to repair a machine is exponentially distributed with parameter $\lambda = \frac{1}{2}$.

(a) What is the probability that the repair time exceeds 2h?

(b) What is the conditional probability that a repair takes atleast 10hr given that it exceeds 9hr.

Sol → If x represent the time to repair the machine the ~~machine~~ p.d.f of x is given by

$$f(x) = \lambda e^{-\lambda x} = \frac{1}{2} e^{-x/2} \quad \left[\because \lambda = \frac{1}{2}\right]$$

ⓐ $P(x>2) = P(2<x<\infty)$

$$= \int_2^\infty f(x)\,dx$$

$$= \int_2^\infty \frac{1}{2} e^{-x/2}\,dx$$

$$= \frac{1}{2}\left[\frac{e^{-x/2}}{-1/2}\right]_2^\infty$$

$$= \frac{1}{2}\left[0 + 2e^{-2/2}\right]$$

$$= \frac{1}{2} \times 2e^{-1} = \frac{1}{e}$$

$$= 0.3679 \quad \underline{\text{Ans}}$$

ⓑ $P(x \geq 10 / x>9) = P(x>1)$ [By memoryless property]

$$= P(1<x<\infty)$$

$$= \int_1^\infty f(x)\,dx$$

$$= \int_1^\infty \frac{1}{2} e^{-x/2}\,dx$$

$$= \frac{1}{2}\left[\frac{e^{-x/2}}{-1/2}\right]_1^\infty$$

$$= \frac{1}{2}\left[0 + 2e^{-1/2}\right]$$

$$= \frac{1}{2} \times 2 \times \frac{1}{e^{0.5}} = 0.6065 \quad \underline{\text{Ans}}$$

Q. The mileage which car owner get with a certain kind of radial tire is a R.V having an exponential distribution with mean 40,000 km.
find probability that of these tire will last
(a) atleast 20,000 km
(b) atmost 30,000 km.

Sol → Let x be milage obtained by tire, the pdf is given by:

$$f(x) = \lambda e^{-\lambda x}$$

$$= \frac{1}{40,000} \cdot e^{-\frac{x}{40,000}} \quad \left[\begin{array}{l} \because \text{Mean} = 40,000 \\ \Rightarrow \frac{1}{\lambda} = 40,000 \\ \Rightarrow \lambda = \frac{1}{40,000} \end{array}\right]$$

(a) $P(x \geq 20,000) = P(20,000 < x < \infty)$

$$= \int_{20,000}^{\infty} \frac{1}{40,000} e^{-\frac{x}{40,000}} dx$$

$$= \frac{1}{40000} \left[\frac{e^{-x/40000}}{-1/40000} \right]_{20000}^{\infty}$$

$$= \frac{1}{40000} \times 40,000 \left[0 + e^{-\frac{20000}{40000}} \right]$$

$$= 1 \times e^{-0.5}$$

$$= 0.6065 \text{ Ans}$$

(b) $P(x \leq 30,000) = P(0 < x < 30000)$

$$= \int_0^{30000} \frac{1}{40000} e^{-x/40000} dx.$$

$$= \frac{1}{40000}\left[\frac{e^{-x/40000}}{-1/40000}\right]_0^{30000}$$

$$= \frac{1}{40000} \times 40000 \left[-e^{-\frac{30000}{40000}} + e^{0}\right]$$

$$= 1\left(1 - e^{-3/4}\right).$$

$$= 1 - e^{-0.75}$$

$$= 0.5270 \text{ Ans}$$

Normal Distribution

A continous R.V X is said to follow a normal distribution with parameter μ & σ, if its probability density function is given by.

$$\boxed{f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}}, \quad \begin{array}{c} -\infty < x < \infty \\ -\infty < \mu < \infty \\ \sigma > 0 \end{array}$$

*Symbolically X-follows $N(\mu, \sigma)$.

NOTE :- $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx.$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

$$= \text{Let } t = \frac{x-\mu}{\sigma\sqrt{2}} \Rightarrow dt = \frac{dx}{\sigma\sqrt{2\pi}}.$$

$$\Rightarrow dx = \sigma\sqrt{2}\,dt$$

Scanned by TapScanner

$$\Rightarrow \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2} dt\, \sigma\sqrt{2}$$

$$\Rightarrow \frac{\sigma\sqrt{2}}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{t^2} dt$$

$$= \frac{\sigma\sqrt{2} \times \sqrt{\pi}}{\sigma\sqrt{2\pi}} \quad \left[\because \int_{-\infty}^{\infty} e^{t^2} dt = \sqrt{\pi}\right]$$

$$= 1$$

$\therefore$ $f(x)$ is a legitimate probability distribution.

Standard Normal Distribution :-

The Normal distribution $N(0,1)$ is called the standard normal distribution whose density function is given by :-

$$\boxed{\phi(z) = \frac{1}{\sqrt{2\pi}} \int e^{-z^2/2}} \quad -\infty < z < \infty$$

This is obtained by putting $\mu = 0$ & $\sigma = 1$ and by changing x and f to z and ϕ respectively.

If X is Normal Distribution with $N(\mu, \sigma)$ and if $\boxed{Z = \frac{x-\mu}{\sigma}}$ $\rightarrow$ Standard Normal Dist.

then z has distribution $N(0,1)$

$$Z = \frac{x-\mu}{\sigma}$$

for. $x = \mu$, $Z = \frac{\mu-\mu}{\sigma} = 0$

$x = x_1$, $Z = \frac{x_1 - \mu}{\sigma} = z_1$

$$P(\mu < x \leq x_1) = P\left(\frac{\mu-\mu}{\sigma} < \frac{x-\mu}{\sigma} < \frac{x_1-\mu}{\sigma}\right)$$

$$= P(0 < z < z_1)$$

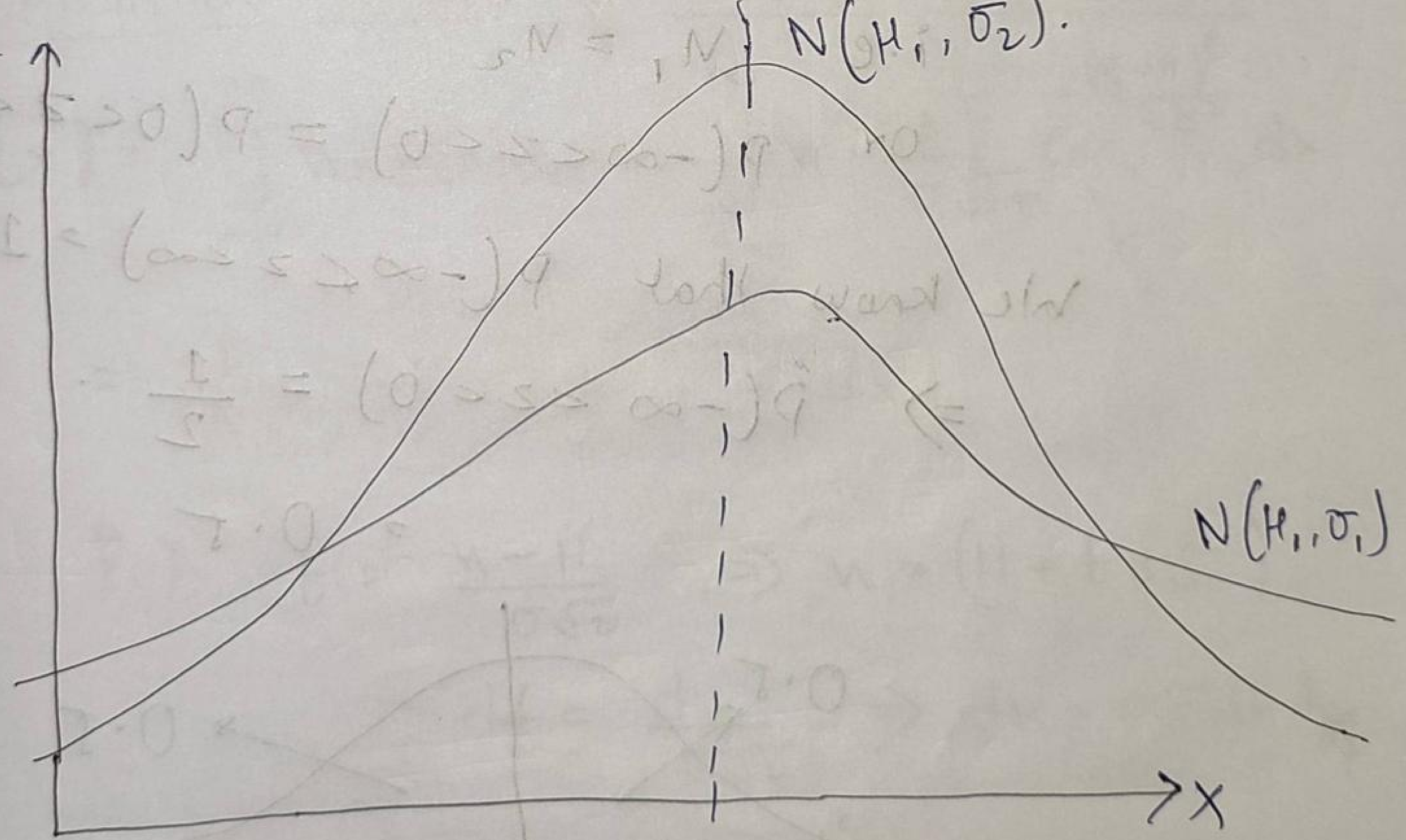
$$= \int_0^{z_1} \phi(z)\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{z_1} e^{-z^2/2}\, dz.$$

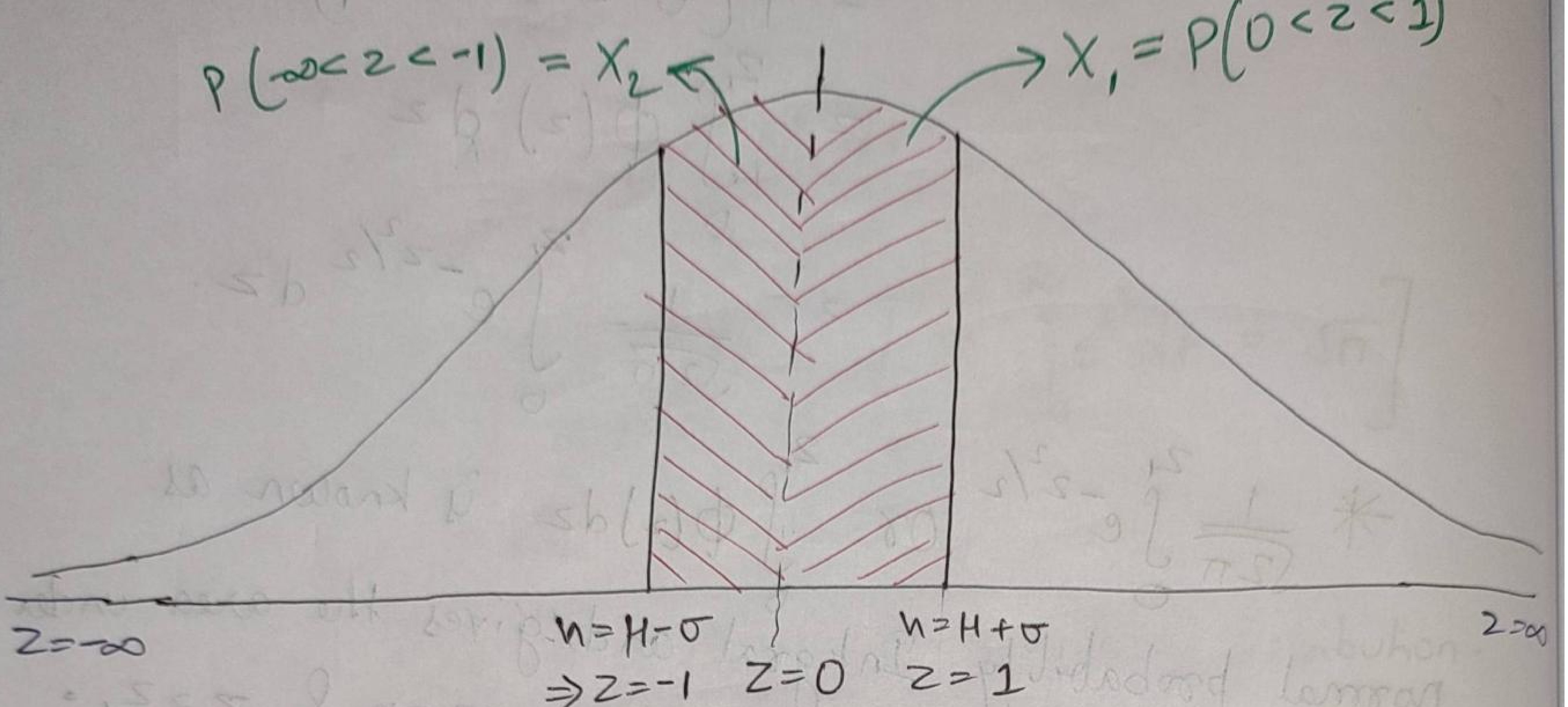
* $\frac{1}{\sqrt{2\pi}} \int_0^{z_1} e^{-z^2/2}$ or $\int_0^{z_1} \phi(z)dz$ is known as normal probability Integral and gives the area under normal probability curve between $z=0$ & $z=z_1$.

* All these values have been Tabulated.

Normal Probability Curve :-



The graph of $y = f(x)$ that is given for $\sigma = \sigma_1$ and $\sigma = \sigma_2$ is a well known well shaped curve and is called Normal probability curve.



Features of this Curve :-

1. Both the side has equal as curve is symmetric about mean.

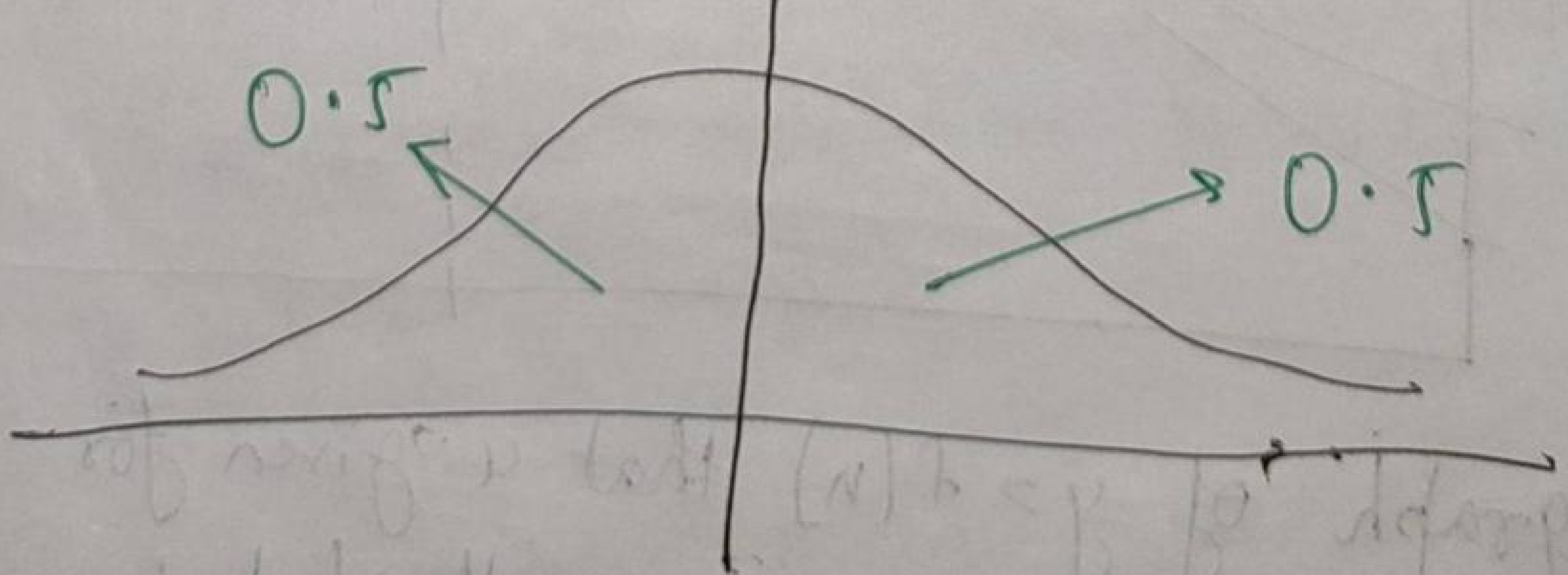
i.e. $x_1 = x_2$

or $P(-\infty < z < 0) = P(0 < z < \infty)$

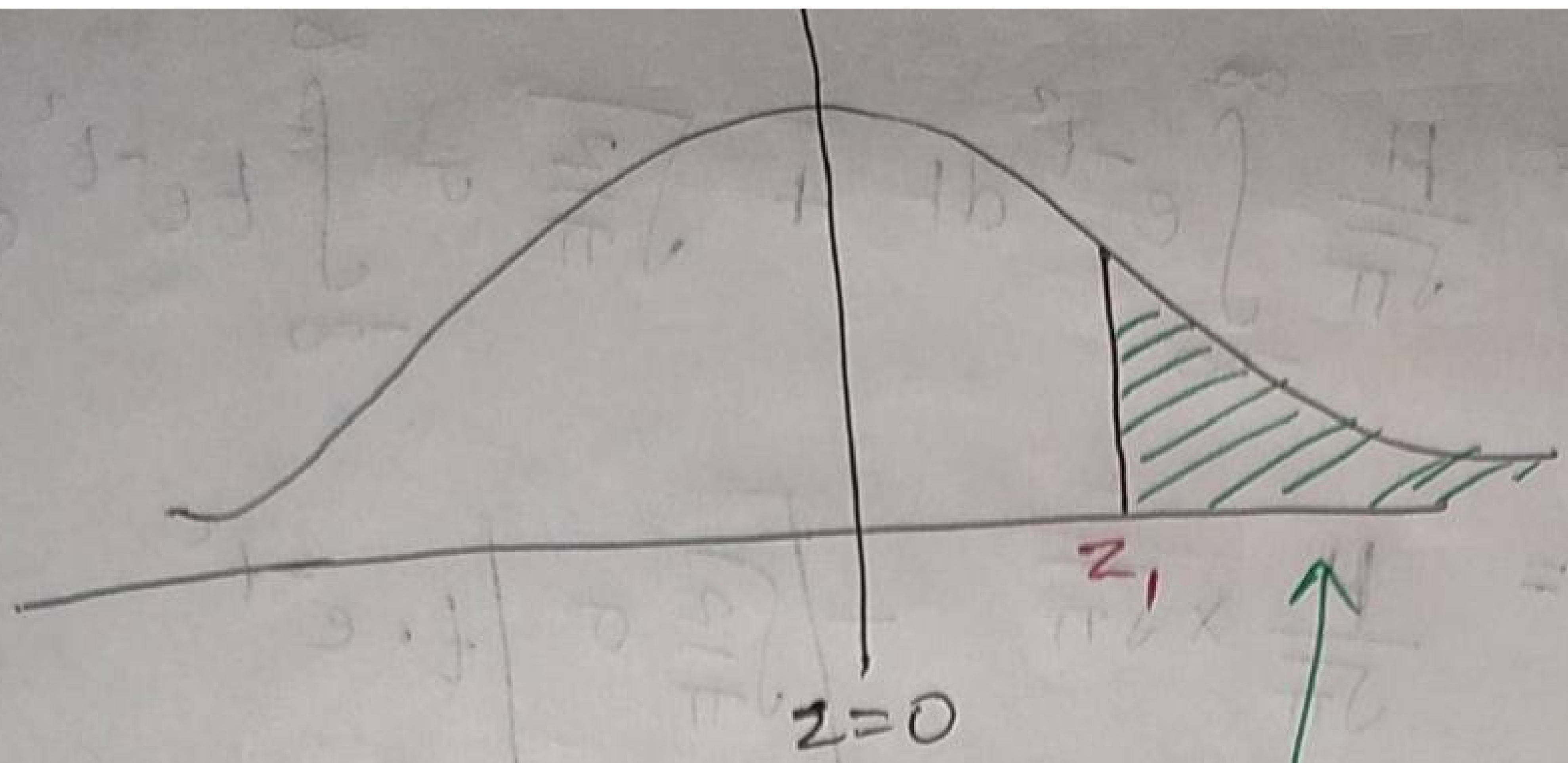
We know that $P(-\infty < z < \infty) = 1$

$$\Rightarrow P(-\infty < z < 0) = \frac{1}{2}$$

$$= 0.5$$



2.



Probability of this area
$P(z_1 < z < \infty)$.
$= 0.5 - P(0 < z < z_1)$.

* NOTE :-

Mean, Median, Mode of Normal Distribution is equal.

Mean & Variance of Normal Distribution :-

$$E[x] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

Put $t = \frac{x-\mu}{\sigma\sqrt{2}} \Rightarrow x = (\mu + t\sigma\sqrt{2})$

$dt = \frac{dx}{\sigma\sqrt{2}} \Rightarrow dx = \sigma\sqrt{2}\, dt$

$$\Rightarrow \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + t\sigma\sqrt{2}) \cdot e^{-t^2} \cdot \sigma\sqrt{2}\, dt$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (\mu + t\sigma\sqrt{2}) \cdot e^{-t^2} dt$$

$$= \frac{\mu}{\sqrt{\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-t^2} dt}_{\downarrow} + \sqrt{\frac{2}{\pi}}\, \sigma \underbrace{\int_{-\infty}^{\infty} t \cdot e^{-t^2} dt}_{\text{odd function}}$$

$$= \frac{\mu}{\sqrt{2\pi}} \times \sqrt{\pi} + \sqrt{\frac{2}{\pi}}\, \sigma \times 0$$

$$\boxed{\begin{array}{l} \text{How?} \\ t \to \text{odd} \\ e^{-t^2} \to \text{even} \\ \text{odd} \times \text{even} = \text{odd} \end{array}}$$

$$= \mu$$

$$\Rightarrow E[x] = \mu.$$

$$E[x^2] = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

$$= \frac{\sigma\sqrt{2}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sqrt{2}\sigma t)^2 e^{-t^2} dt \quad \left[\begin{array}{l} \text{Put } t = \frac{x-\mu}{\sqrt{2}\sigma} \\ \Rightarrow dx = dt \cdot \sqrt{2}\sigma \\ x = \mu + \sqrt{2}\sigma t \end{array}\right]$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (\mu + \sqrt{2}\sigma t)^2 e^{-t^2} dt$$

$$= \frac{1}{\sqrt{\pi}} \left[\mu \int_{-\infty}^{\infty} e^{-t^2} dt + 2\sqrt{2}\mu\sigma \int_{-\infty}^{\infty} t \cdot e^{-t^2} + 2\sigma^2 \int_{-\infty}^{\infty} t^2 e^{-t^2} dt \right]$$

$$= \frac{1}{\sqrt{\pi}} \left[\mu \cdot \sqrt{\pi} + 2\sqrt{2}\mu\sigma \times 0 + 2\sigma^2 \int_{-\infty}^{\infty} t^2 e^{-t^2} dt \right]$$

$$= \mu + 0 + \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^2 e^{-t^2} dt.$$

$$\text{Put } t^2 = u \Rightarrow 2t\,dt = du$$

$$\Rightarrow dt = \frac{du}{2t}$$

$$\Rightarrow \mu + 0 + \frac{2\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} 2 \cdot t^2 e^{-t^2}$$

$$= \mu + 0 + \frac{2\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} 2 \cdot u \cdot e^{-u} \cdot \frac{du}{2\sqrt{u}}$$

$$= \mu^2 + \frac{2\sigma^2}{\sqrt{\pi}} \int_0^\infty \sqrt{u}\, e^{-u}\, du.$$

$$= \mu^2 + \frac{2\sigma^2}{\sqrt{\pi}} \int_0^\infty u^{1/2}\, e^{-u}\, du$$

$$= \mu^2 + \frac{2\sigma^2}{\sqrt{\pi}} \int_0^\infty u^{3/2-1}\, e^{-u}\, du$$

$$= \mu^2 + \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) \quad \text{[By Gamma function]}$$

$$\Rightarrow \mu^2 + \frac{\cancel{2}\sigma^2}{\cancel{\sqrt{\pi}}} \times \frac{1}{\cancel{2}} \times \cancel{\sqrt{\pi}}$$

$$= \mu^2 + \sigma^2.$$

$$\text{Variance} = E[x^2] - \left(E[x]\right)^2$$

$$= \mu^2 + \sigma^2 - \mu^2$$

$$= \sigma^2$$

Moment Generating function of $N(0,1)$ and $N(\mu,\sigma)$:-

The moment Generating function of $N(0,1)$ is given by.

$$M_Z(t) = E\left[e^{tz}\right]$$

$$= \int_{-\infty}^{\infty} e^{tz}\, \phi(z)\, dz$$

Scanned by TapScanner

$$= \int_{-\infty}^{\infty} e^{tz} \cdot \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{z^2-2tz}{2}\right)} dz.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\frac{z^2-2tz}{2} + \frac{t^2}{2} - \frac{t^2}{2}\right]} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\frac{(z-t)^2 - t^2}{2}\right]} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z-t)^2}{2}} \cdot e^{t^2/2} dz$$

$$= \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{z-t}{\sqrt{2}}\right)^2} dz$$

Put $\frac{z-t}{\sqrt{2}} = u \Rightarrow du = \frac{dz}{\sqrt{2}}$

$\Rightarrow dz = \sqrt{2}\,du.$

$$\Rightarrow \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2} \sqrt{2}\, du.$$

$$= \frac{e^{t^2/2}}{\sqrt{2\pi}} \times \sqrt{2} \int_{-\infty}^{\infty} e^{-u^2} du$$

$$= \frac{e^{t^2/2}}{\sqrt{2\pi}} \times \sqrt{2} \times \sqrt{\pi}$$

$$= e^{t^2/2}$$

So, M.g.f of $N(0,1) = e^{t^2/2}$.

Scanned by TapScanner

Now, Mgf of $N(\mu, \sigma)$ is given by.

$M_x(t) = M_{\sigma z + \mu}(t)$. $\left[\because z = \frac{x - \mu}{\sigma} \quad x = \sigma z + \mu \right]$

$= e^{\mu t} M_z(\sigma t)$ $\left[\because \text{Mgf of } x \text{ is } M_x(t) \text{ and if } Y = ax + b \text{ Then, } M_y(t) = e^{bt} M_x(at) \right]$

$= e^{\mu t} M_z(\sigma t).$

$= e^{\mu t} \cdot e^{\sigma^2 t^2/2}$

$= e^{t(\mu + \frac{\sigma^2}{2} t)}.$

$= e^{t(\mu + \frac{\sigma^2}{2} t)}.$

$\therefore$ Mgf of $N(\mu, \sigma) = e^{t(\mu + \frac{\sigma^2}{2} t)}$

Now, $M_x(t) = e^{t(\mu + \frac{\sigma^2}{2} t)}$

$= 1 + \left(\mu + \frac{\sigma^2}{2} t\right) \frac{t}{1!} + \left(\mu + \frac{\sigma^2}{2} t\right)^2 \frac{t^2}{2!} + \cdots + \cdots \infty$

$= 1 + \mu \frac{t}{1!} + \frac{\sigma^2}{2} \frac{t^2}{1!} + \mu^2 \frac{t^2}{2!} + \frac{\sigma^2}{2} \times 2\mu \frac{t^3}{2!} + \cdots \infty$

$= 1 + \mu \frac{t}{1!} + \frac{\sigma^2}{2} \frac{t^2}{1!} + \mu^2 \frac{t^2}{2!} + \cdots \infty$

$E[x] =$ Coefficient of $\frac{t}{1!} = \mu.$

$E[x^2] =$ coefficient of $\frac{t^2}{2!} = \frac{\sigma^2}{2} \times 2 + \mu^2$

$= \sigma^2 + \mu^2$

Scanned by TapScanner

$$\text{Variance} = E[x^2] - (E[x])^2$$

$$= \mu^2 + \sigma^2 - \mu^2$$

$$\Rightarrow \sigma^2$$

So, Mean $= \mu$

Variance $= \sigma^2$.

Q. X is normally distributed and mean of X is 12 and S.D is 4. Find out the probability of following:

(i) $X \geq 20$ (ii) $X \leq 20$ (iii) $0 \leq x \leq 12$.

Sol → (i) $P(X \geq 20)$

$$= P(20 \leq x \leq \infty)$$

$$= P\left(\frac{20-\mu}{\sigma} \leq \frac{x-\mu}{\sigma} \leq \frac{\infty-\mu}{\sigma}\right).$$

$$= P\left(\frac{20-12}{4} \leq z \leq \infty\right).$$

$$= P(2 \leq z < \infty)$$

$$= 0.5 - P(0 \leq z \leq 2).$$

$$= 0.5 - 0.4772 \quad \text{[from table]}$$

$$= 0.0228 \quad \text{Ans}$$

(ii) $P(X \leq 20) = P(-\infty < X \leq 20)$

$$= P\left(-\infty \leq \frac{x-\mu}{\sigma} \leq \frac{20-\mu}{\sigma}\right)$$

$= P(-\infty < z \le 2)$

$= P(-\infty \le z < 0) + P(0 < z \le 2)$

$= 0.5 + P(0 < z \le 2)$

$= 0.5 + 0.4772$

$= 0.9772$ <u>Ans</u>

(iii) $P(0 \le X \le 12) = P\left(\frac{0-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{12-\mu}{\sigma}\right)$

$= P\left(\frac{0-12}{4} \le z \le 0\right)$

$= P(-3 \le z \le 0)$

$= P(0 \le z \le 3)$ [By Symmetric]

$= 0.4987$

Q. The marks obtained by a number of student in a certain subject are approximately normally distributed with mean 65 and S.D is 5. If 3 student is selected at random from the group. What is the probability that atleast one of them would have scored above 75.

Solution → Let X represent the marks obtained by the student. X follows the distribution $N(65, 5)$.

P(a student score is above 75) = P(X > 75)

$= P(75 < x < \infty)$

$= P\left(\frac{75-\mu}{\sigma} < \frac{x-\mu}{\sigma} < \frac{\infty-\mu}{\sigma}\right)$

$= P\left(\frac{75-65}{5} \leq z < \infty\right)$

$= P(2 \leq z < \infty)$

$= 0.5 - P(0 < z < 2)$

$= 0.5 - 0.4772$

$= 0.0228$ Ans

Let p = P(a student score above 75) = 0.0228.

$q = 1 - p = 1 - 0.0228$

$= 0.9772$

& $n = 3$.

Let Y denote the students score above 75 marks. then Y follows a binomial distribution.

P(at least 3 student score above 75)

$= P(Y \geq 1)$

$= 1 - P(Y = 0)$

$= 1 - {}^3C_0\, p^0 q^{3-0}$

$= 1 - 1 \cdot 1 \cdot q^3 = 1 - (0.9772)^3$

$= 0.667$ Ans

Q. If the actual amount of instant coffee which a filling machine puts into '6-ounce' jars is a R.V having a normal distribution with S.D = 0.05 ounce and if only 3% of the Jars are to contain less than ounce of coffee. what must be the mean fill of these jars.

Sol→ Let X be the actual amount of coffee put into the jars, then X follows $N(\mu, 0.05)$.

Given. $P(x \le 6) = 0.03$ $\left[3\% = \frac{3}{100}\right]$

$$P(-\infty < x < 6) = 0.03$$

$$= P\left(\frac{-\infty-\mu}{\sigma} < \frac{x-\mu}{\sigma} < \frac{6-\mu}{\sigma}\right) = 0.03$$

$$= P\left(-\infty < z < \frac{6-\mu}{0.05}\right) = 0.03$$

$$= P\left(\frac{6-\mu}{0.05} < z < \infty\right) = 0.03$$

$$= 0.5 - P\left(0 < z < \frac{6-\mu}{0.05}\right) = 0.03$$

$$= P\left(0 < z < \frac{6-\mu}{0.05}\right) = -0.3 + 0.5$$

$$= P\left(0 < z - \frac{6-\mu}{0.05}\right) = 0.47$$

From table of areas, we have.

$$P(0 < z < 1.808) = 0.47$$

On Comparing. $\frac{\mu - 6}{0.05} = 1.808$

$\Rightarrow \mu = 6.094$ ounces.

Additive Property of Normal Distribution :-

If X_i $(i = 1, 2, 3 \ldots \infty)$ be n independent normal R.V. with mean μ_i and variance σ_i^2.

Then, $\sum_{i=1}^{n} a_i x_i$ is also a normal R.V.

with Mean $\sum_{i=1}^{n} a_i \mu_i$

Variance $\sum_{i=1}^{n} a_i^2 \sigma_i^2$

Q. The marks obtained by the student in Maths, Physics and chemistry in an examination are normally distributed with mean 52, 50, and 48 and S.D 10, 8 and 6 respectively. Find the probability that a student selected at random has secured a total of (i) 180 or above (ii) 135 or less.

Sol → Let X, Y, Z denote the marks obtained by student in Maths, physics and chemistry respectively.

Given. X-follows N(52, 10)

Y-follows N(50, 8)

Z-follows N(48, 6)

By, additive property.

$T = X+Y+Z$ follows the distribution.

$$N\left(52+50+48,\ \sqrt{10^2+8^2+6^2}\right)$$

$$= N(150,\ 14.14).$$

(i) $P(T \geq 180) = P(180 < T < \infty)$

$$= P\left(\frac{180-\mu}{\sigma} < \frac{T-\mu}{\sigma} < \frac{\infty-\mu}{\sigma}\right)$$

$$= P\left(\frac{180-150}{14.14} < Z < \infty\right)$$

$$= 0.5 - P(0 < Z < 2.12)$$

$$= 0.5 - 0.4830$$

$$= 0.0170. \quad \underline{\text{Ans}}$$

(ii) $P(T \leq 135) = P(-\infty < T < 135)$

$$= P\left(-\infty < \frac{T-\mu}{\sigma} < \frac{135-150}{14.14}\right)$$

$$= P(-\infty < Z < -1.06)$$

$P(1.06 < Z < \infty)$ $= 0.5 - P(0 < Z < 1.06)$

$$= 0.5 - 0.3554.$$

$$= 0.1446 \quad \underline{\text{Ans}}$$