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# RANDOM VARIABLES.

A Random Variable (R.V) is a function that assigns a real number $X(s)$ to every element $s \in S$, where $S$ is the Sample Space Corresponding to a random ~~Variable~~ Experiment E.

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* The Outcome of random Experiments may be Numerical (or) non- numerical in nature.

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**Range Space:** The set of all values taken by a random Variable $X$ is called its range space and is denoted by $R_X$. Thus $R_X = \{X(s) \mid s \in S\}$

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**Example** Two coins are tossed.

Sample Space $S = \{HH, HT, TH, TT\}$

Let $X$ denotes 'no of Heads'.

Then
$$\left.\begin{aligned} X(HH) &= 2 \\ X(HT) &= 1 \\ X(TH) &= 1 \\ X(TT) &= 0 \end{aligned}\right\} \rightarrow$$

Values of $X$ is $0, 1, 2$.

$R_X = \{0, 1, 2\}$. $\boxed{R_X \text{ is a finite set.}}$

Random Variable 

Discrete Random Variable → Probability mass function (PMF)

Continuous random Variable → Probability Density function (PDF).

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## DISCRETE RANDOM VARIABLE.

A random Variable $X$ is said to be discrete if it takes a finite number of values (or) Countably infinite number of Values.

ie. its Range $R_X$ is finite (or) Countably Infinite.

Example

1. The number of Priniting mistakes in a book.
2. The number of telephones recieved in a office.

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A discrete Random Variable assumes each of its Value with Certain Probability.

These Probabilites could be used to define a Probability function (see below).

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## PROBABILITY FUNCTION (or) PROBABILITY MASS FUNCTION.

Let $X$ be a discrete random Variable which takes Values $x_1, x_2, x_3 \dots$ Let $P(X=x_i) = p(x_i)$ be Probability of $x_i$. Then ~~(...)~~ the function $p$ is called the Probability Mass function of $X$. If the numbers $P(x_i)$ satisfy the Conditions.

(i) $p(x_i) \geq 0 \quad \forall \; i = 1,2,3, \ldots$

(ii) $\sum_{i=1}^{\infty} P(x_i) = 1.$

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## PROBABILITY DISTRIBUTION:

The set of ordered Pairs of numbers $(x_i, P(x_i))$ is called the Probability distribution of the R.V.X.

The Probability distribution is usually displayed in the table form.

| $x_i$ | $x_1$ | $x_2$ | $x_3$ | ... |
|---|---|---|---|---|
| $P(x_i)$ | $P(x_1)$ | $P(x_2)$ | $P(x_3)$ | ... |

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## CUMMULATIVE DISTRIBUTION FUNCTION. (CDF).
## (or) DISTRIBUTION FUNCTION OF A RANDOM VARIABLE.

If $X$ is a random Variable, (discrete); then the function $F: R \to [0,1]$ defined by $F(x) = P(X \leq x)$ is called the Cummulative distribution function of $X$.

If $X$ is a discrete random Variable

$x_1 < x_2 < x_3 \ldots x_i$ then

$$F(x) = \sum_{x_i \leq x} P(x_i)$$

## Properties of distribution function $F(x)$

1) $0 \le F(x) \le 1$ , $-\infty < x < \infty$

- $F(-\infty) = 0$ , $F(\infty) = 1$.

2) If $x_1 < x_2 \Rightarrow F(x_1) < F(x_2)$.

3) If $X$ is discrete RV. with values $x_1 < x_2 < x_3 \cdots$
then $p(x_i) = F(x_i) - F(x_{i-1})$.

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## Some Results

* $P(X > x) = 1 - P(X \le x)$
* $P(X \ge x) = 1 - P(X < x)$
* $P(X \le x) = 1 - P(X > x)$
* $P(X < x) = 1 - P(X \ge x)$.
* $P(\bar{A}) = 1 - P(A)$.

## Conditional Probability

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

If $A$ & $B$ are dependent events.

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## EXPECTATION:- Expectation of a discrete random Variable.

$$E(X) = \sum_{i=1}^{\infty} x_i \, P(x_i).$$

[ Expectation of $X$ (or) Mean Value of $X$ ]

Let $X$ be a discrete random Variable taking values

| $x_i$ | $x_1$ | $x_2$ | $x_3$ | .... |
|---|---|---|---|---|
| $P(x_i)$ | $P(x_1)$ | $P(x_2)$ | $P(x_3)$ | .... |

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Properties :

1) $C$ is constant then $E(C) = C$

2) If $a, b$ are constants, then
$$E(aX + b) = aE(X) + b.$$

3) $E(aX) = aE(X)$, where $a$ is constant

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# VARIANCE OF X

$$Var(X) = V(X) = E(X^2) - [E(X)]^2$$

$$\sqrt{Var(X)} = \text{Standard Deviation of } X.$$

Properties of Variance:

1) $Var(X) \geq 0$ 2) $Var(a) = 0$, where $a$ is Constant

3) $V(aX \pm b) = a^2 Var(X) + Var(b) = a^2 Var(X)$ //

where $a$ and $b$ are Constants.

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**Problem: 1** A Probability distribution of a discrete random Variable $X$ is given by

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $P(x)$ | $a$ | $3a$ | $5a$ | $7a$ | $9a$ | $11a$ | $13a$ | $15a$ | $17a$ |

a) find the value of $a$ b) $P(0 < X < 3)$

c) $P(X \geq 3)$ d) find the distribution fn of $X$

Ans:

a) Since given $P(x)$ is a Probability mass function of $X$

$$\therefore \sum P(x) = 1, \quad P(x) \geq 0.$$

$$\therefore a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1.$$

$$81a = 1 \Rightarrow \boxed{a = \frac{1}{81}}$$

b) $P(0 < X < 3) = P(X=1) + P(X=2) = 3a + 5a = 8a$

$$= \frac{8}{81} \text{ //}$$

c) $P(x \geq 3) = P(x=3) + P(x=4) + \cdots + P(x=8)$

(or)

$P(x \geq 3) = 1 - P(X<3)$

$= 1 - \{P(x=0) + P(x=1) + P(x=2)\}$

$= 1 - \{a + 3a + 5a\} = 1 - 9a$

$= 1 - \frac{9}{81} = \frac{81-9}{81} = \frac{72}{81} = \frac{8}{9}$ //.

d) distribution function (or) cdf. Cummulative Dist. fn.

Cdf of X is $F(x) = P(X \leq x)$.

| $x$: | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $p(x)$: | $a$ | $3a$ | $5a$ | $7a$ | $9a$ | $11a$ | $13a$ | $15a$ | $17a$ |
| $F(x)$: | $a$ | $4a$ | $9a$ | $16a$ | $25a$ | $36a$ | $49a$ | $64a$ | $81a$. |
| $F(x)$ | $\frac{1}{81}$ | $\frac{4}{81}$ | $\frac{9}{81}$ | $\frac{16}{81}$ | $\frac{25}{81}$ | $\frac{36}{81}$ | $\frac{49}{81}$ | $\frac{64}{81}$ | $\frac{81}{81}=1$. |

$F(0) = P(X \leq 0) = P(x=0) = a$

$F(1) = P(X \leq 1) = P(x=0) + P(X=1) = a + 3a = 4a$

$F(2) = P(X \leq 2) = P(X=0) + P(X=1) + P(X=2) = 9a$

Simply

$F(3) = P(X \leq 3) = P(X=3) + P(X \leq 2) = 7a + 9a = 16a$

$F(4) = P(X \leq 4) = P(X \leq 3) + P(X=4) = 16a + 9a = 25a$

$F(5) = P(X \leq 5) = P(X \leq 4) + P(X=5) = 25a + 11a = 36a$

etc...

**Problem 2:** If $X$ is a discrete random Variable with the following Probability function.

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $P(x)$ | $a$ | $3a$ | $5a$ | $7a$ | $9a$ |

find 'a'.

a) find $P(x \geq 3)$ b) $P(0 < x < 4)$ c) mean

d) Variance e) $E(3x-4)$ f) $Var(3x-4)$.

Soln: Since $X$ is discrete R.V. $P(x_i) \geq 0$

$\therefore \sum P(x) = 1$

$\therefore a + 3a + 5a + 7a + 9a = 1 \Rightarrow \boxed{25a = 1}$

$\boxed{a = 1/25}$

a) $P(x \geq 3) = [1 - P(x<3)]$ or $[P(x=3) + P(x=4)]$

$= 7a + 9a = 16a = \boxed{\frac{16}{25}}.$

b) $P(0 < x < 4) = P(x=1) + P(x=2) + P(x=3)$

$= 3a + 5a + 7a = 15a = \boxed{\frac{15}{25}}$

c) mean of $X = E(X) = \sum x_i P(x_i)$

d) Variance of $X = E(X^2) - [E(X)]^2$.

| $x$ | $P(x)$ | $x \cdot P(x)$ | $x^2$ | $x^2 \cdot P(x)$ | |
|---|---|---|---|---|---|
| 0 | $1/25$ | 0 | 0 | 0 | |
| 1 | $3/25$ | $3/25$ | 1 | $3/25$ | |
| 2 | $5/25$ | $10/25$ | 4 | $20/25$ | |
| 3 | $7/25$ | $21/25$ | 9 | $63/25$ | |
| 4 | $9/25$ | $36/25$ | 16 | $144/25$ | |

c) Mean of $X = E(x) = \sum x P(x)$

$$= 0 + \frac{3}{25} + \frac{10}{25} + \frac{21}{25} + \frac{36}{25} = \boxed{\frac{70}{25}}$$

d) Variance of $X = E(x^2) - [E(x)]^2$

$$E(x^2) = \sum x^2 P(x)$$

$$= 0 + \frac{3}{25} + \frac{20}{25} + \frac{63}{25} + \frac{144}{25} = \frac{230}{25}.$$

$$Var(X) = \left(\frac{230}{25}\right) - \left(\frac{70}{25}\right)^2 = \frac{230}{25} - \frac{4900}{625}$$

$$= \frac{5750 - 4900}{625} = \boxed{\frac{850}{625}} // = \boxed{\frac{34}{25}}$$

e) $E(3x-4) = E(3x) - E(4) = 3E(x) - 4$

Since [ $E(a) = a$, $a$ is Constant, $E(ax) = aE(x)$ ]

$$= 3 * \frac{70}{25} - 4 = \frac{210 - 100}{25} = \boxed{\frac{110}{25}}$$

$$= \boxed{\frac{22}{5}}$$

f) $Var(3X-4) = 3^2 \, Var(x) - Var(4) = 9 \, Var(X)$



$$= 9 * \frac{850}{625} = \boxed{\frac{7650}{625}} = \frac{306}{25}$$

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**Problem: 3** A Random Variable X has the following Probability function.

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $P(x)$ | 0 | $a$ | $2a$ | $2a$ | $3a$ | $a^2$ | $2a^2$ | $7a^2+a$ |

find (i) $a$ (ii) $P(X<4)$ (iii) $P(X \geq 4)$ (iv) $P(X \geq 6)$

(v) $P(X<6)$ (vi) $P\left(\frac{3}{2} < X \; \frac{9}{2} \,/\, X > 2\right)$

vii) $P(1 \leq X \leq 7)$ viii) find cdf.

ix) If $P(X \leq k) > 1/2$, find the least Value of K.

Soln: $\Sigma P(x) = 1$.

(i) $0 + a + 2a + 2a + 3a + a^2 + 2a^2 + 7a^2 + a = 1$

$$10a^2 + 9a - 1 = 0.$$

$$10a^2 + 9a + a - a - 1 = 0$$

$$10a^2 + 10a - a - 1 = 0$$

$$10a(a+1) - (a+1) = 0$$

$$(10a-1)(a+1) = 0.$$

$\therefore$ $10a - 1 = 0$ (or) $a + 1 = 0$

$10a = 1$ (or) $\boxed{a = -1}$

$\boxed{a = \frac{1}{10}}$

Since $a \geq 0$ (By Condition Pmf) $\therefore$ $\boxed{a = \frac{1}{10}}$

(ii) $P(x<4) = P(x=0)+P(x=1)+P(x=2)+P(x=3)$

$= 0+a+2a+2a$

$P(x<4) = 5a = \frac{5}{10} = \frac{1}{2}$  w.k.t $a = \frac{1}{10}$

(iii) $P(x\geq 4) = P(x=4)+P(x=5)+P(x=6)+P(x=7)$

(or)

$P(x\geq 4) = 1-P(x<4) = 1-\frac{5}{10}$ (from the above)

$$\boxed{P(x\geq 4) = \frac{5}{10} = \frac{1}{2}.}$$

iv) $P(x\geq 6) = P(x=6)+P(x=7) = 2a^2+7a^2+a$

$= 9a^2+a = 9\left(\frac{1}{10}\right)^2+\frac{1}{10} = \frac{9}{100}+\frac{1}{10}$.

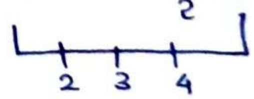
$= \frac{9+10}{100} = \frac{19}{100}$

v) $P(x<6) = 1-P(x\geq 6) = 1-\frac{19}{100} = \boxed{\frac{81}{100}}$

(vi) $P\left(\frac{3}{2}<x<\frac{9}{2} / x>2\right)$

$$\frac{P\left(\frac{3}{2}<x<\frac{9}{2} \cap x>2\right)}{P(x>2)}$$

$\frac{3}{2}=1.5$  $\frac{9}{2}=4.5$




$P(A/B) = \frac{P(A\cap B)}{P(B)}$

$P(x>2) = 1-P(x\leq 2)$

$= 1-\{P(x=0)+P(x=1)+P(x=2)\}$

$= 1-\{0+a+2a\}$

$= 1-3a = 1-3\left(\frac{1}{10}\right) = \frac{7}{10}$ //

$P(x>2)$

$= 1-P(X\leq 2)$

$= 1-$

$$P\left(\tfrac{3}{2} < x < \tfrac{9}{2} \cap x > 2\right) = P(1.5 < x < 4.5 \cap x > 2)$$

$$= P(2 < x < 4.5) = P(x=3) + P(x=4)$$

$$= 2a + 3a = 5a = \frac{5}{10}$$

$$P\left(\tfrac{3}{2} < x < \tfrac{9}{2} \,/\, x > 2\right) = \frac{P\left(\tfrac{3}{2} < x < 9/2 \cap x > 2\right)}{P(x > 2)}$$

$$= \frac{5/10}{7/10} = 5/7.$$

vii) $P(0 \le x \le 4) = P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4)$

$$= 0 + a + 2a + 2a + 3a = 8a.$$

$$= \frac{8}{10} = \boxed{\frac{4}{5}}$$

viii) cdf $F(x) = P(X \le x)$.

$F(0) = P(X \le 0) = P(x=0) = 0$

$F(1) = P(x \le 1) = P(x=0) + P(x=1) = 0 + a = \frac{1}{10}$

$F(2) = P(x \le 2) = P(x \le 1) + P(x=2) = a + 2a = 3a = \frac{3}{10}$

$F(3) = P(x \le 3) = P(x \le 2) + P(x=3) = 3a + 2a = 5a = \frac{5}{10}$

$F(4) = P(x \le 4) = P(x \le 3) + P(x=4) = 5a + 3a = 8a = \frac{8}{10}$

$F(5) = P(x \le 5) = P(x \le 4) + P(x=5) = 8a + a^2.$

$$= \frac{8}{10} + \frac{1}{100} = \frac{81}{100}$$

$$F(6) = P(X \le 6) = P(X \le 5) + P(X=6) = \frac{81}{100} + 2a^2$$

$$= \frac{81}{100} + \frac{2}{100} = \frac{83}{100}$$

$$F(7) = P(X \le 7) = P(X \le 6) + P(X=7)$$

$$= \frac{83}{100} + 7a^2 + a = \frac{83}{100} + \frac{7}{100} + \frac{1}{10}$$

$$= \frac{83+7+10}{100} = \frac{100}{100} = 1$$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $P(x)$ | 0 | $a$ | $2a$ | $2a$ | $3a$ | $a^2$ | $2a^2$ | $7a^2+a$ |
| $P(x)$ | 0 | $\frac{1}{10}$ | $\frac{2}{10}$ | $\frac{2}{10}$ | $\frac{3}{10}$ | $\frac{1}{100}$ | $\frac{2}{100}$ | $\frac{17}{100}$ |
| $F(x)$ | 0 | $\frac{1}{10}$ | $\frac{3}{10}$ | $\frac{5}{10}$ | $\frac{8}{10}$ | $\frac{81}{100}$ | $\frac{83}{100}$ | 1. |

ix) From the answer Part (viii)

$P(X \le 4) = \frac{8}{10} > \frac{1}{2}$

$P(X \le 5) = \frac{81}{100} > 1/2$

$P(X \le 6) = \frac{83}{100} > 1/2$

$P(X \le 7) = 1 > 1/2$

the minimum Value of 'k' for $P(X \le k) > 1/2$.

$\frac{8}{10} = 0.8$, $\frac{81}{100} = 0.81$

$\frac{83}{100} = 0.83$, 1

$\therefore$ k has minimum Value is $\boxed{4}$

# Continuous Random Variable:



**Definition** A random Variable X is Said to be Continuous if its Range Space $R_X$ is an Un countable set of real numbers. i.e. the random Variable assumes values in the an interval $(a,b)$ (or) in an union of intervals.

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EXAMPLE:

(i) X denotes the life time of a transistor.

(ii) X denotes the Operating time between two failures of a machine.

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* To define a Continuous random Variable, the Sample Space Should be Continuous.

* In the Case of Continuous Random Variable, we cannot Speak about first value, second value, and thus $P(x_1)$, $P(x_2)$ etc., becomes meaningless. So the Probability function of a Continuous random Variable is defined as below.

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## Probability Density Function of a Continuous Random Variable.

A function 'f', defined for all $x \in (-\infty, \infty)$ is called the Probability density function of a C.R.V 'X' if.

(i) $f(x) \geq 0$ (ii) $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

for all $\forall\ x \in (-\infty, \infty)$. Dr. E. Suresh, SRMIST

**Note:** 1) $P(X<a) = \int_{-\infty}^{a} f(x)\, dx$.

2) $P(X>a) = \int_{a}^{\infty} f(x)\, dx$.

3) $P(a \leq X \leq b) = \int_{a}^{b} f(x)\, dx$.

If $b=a$, $\int_{a}^{a} f(x)\, dx = 0$.

Since $X$ is a Continuous Random Variable.

$$P(X \leq a) = P(X<a), \quad P(X>a) = P(X \geq a)$$

$$P(a \leq X \leq b) = P(a<X \leq b) = P(a \leq X<b)$$
$$= P(a<X<b).$$

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## CUMMULATIVE DISTRIBUTION FUNCTION (CDF) (or) DISTRIBUTION FUNCTION of a random Variable

If $X$ is a Random Variable, discrete or Continuous, then the function $F: R \to [0,1]$ defined by $F(x) = P(X \leq x)$ is called the Cummulative distribution function of $X$.

If $X$ is a Continuous random Variable with 

P.d.f $f(x)$ defined for all $x \in (-\infty, \infty)$

then

$$\boxed{F(x) = \int_{-\infty}^{x} f(x)\,dx.}$$

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Properties of distribution function $F(x)$.

(i) $0 \leq F(x) \leq 1$. $\quad -\infty < x < \infty$

(ii) $F(x)$ is an increasing function of $x$.

$$a < b \Rightarrow F(a) < F(b).$$

(iii) $\lim_{x \to -\infty} F(x) = 0$ $\quad$ (iv) $\lim_{x \to \infty} F(x) = 1.$

$\therefore F(-\infty) = 0$ $\qquad F(\infty) = 1.$

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Problem 1 If $X$ is a Continuous random Variable with

Pdf
$$f(x) = \begin{cases} x & 0 \leq x < 1 \\ \frac{3}{2}(x-1)^2 & 1 \leq x < 2 \\ 0 & \text{otherwise} \end{cases}$$

find the Cummulative distribution function $F(x)$ of $X$ and use it to find $P\left(\frac{3}{2} < X < \frac{5}{2}\right)$.

The c.d.f of $X$ is $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx.$

If $x<0$ then $f(x)=0$ $\therefore F(x)=0$.

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If $0 \le x < 1$ then $F(x) = \int_0^x f(x)\,dx = \int_0^x x\,dx = \left[\frac{x^2}{2}\right]_0^x = \frac{x^2}{2}$

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If $1 \le x < 2$ then $F(x) = \int_0^x f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx.$

$$= \int_0^1 x\,dx + \int_1^x \frac{3}{2}(x-1)^2\,dx.$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \frac{3}{2}\left[\frac{(x-1)^3}{3}\right]_1^x = \frac{1}{2} + \frac{3}{2}\left[\frac{(x-1)^3}{3}\right]$$

$$= \frac{1}{2} + \frac{1}{2}(x-1)^3.$$

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If $x \ge 2$, then $F(x) = 1.$

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$\therefore$ The c.d.f of X is $F(x) = \begin{cases} 0 & \text{if } x<0 \\ \frac{x^2}{2} & \text{if } 0 \le x < 1 \\ \frac{1}{2}+\frac{1}{2}(x-1)^3 & \text{if } 1 \le x < 2 \\ 1. & x \ge 2 \end{cases}$

$$P\left(\frac{3}{2} < X < \frac{5}{2}\right) = \int_{3/2}^{5/2} f(x)\,dx = F\left(\frac{5}{2}\right) - F\left(\frac{3}{2}\right)$$

$$= 1 - \left[\frac{1}{2} + \frac{1}{2}\left(\frac{3}{2}-1\right)^3\right] = 1 - \left[\frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right)^3\right]$$

$$= 1 - \left[\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{8}\right] = 1 - \left[\frac{8+1}{16}\right] = \frac{16-9}{16} = \frac{7}{16}$$

① A Continuous random Variable 'X' has a pdf $f(x) = K$, $0 \le x \le 1$. Find 'K' and $P(X \le 1/4)$ and $P(|X| \le 1)$, $P(|X| > 1)$



W.K.T $\int_{-\infty}^{\infty} f(x)\,dx = 1$ | $\boxed{f(x) = K}$

Sum of Probabilities is $\boxed{1}$

$$\int_0^1 K\,dx = 1 \Rightarrow [Kx]_0^1 = 1 \Rightarrow \boxed{K = 1}$$

$\therefore \boxed{f(x) = 1}$

$$P(X \le 1/4) = \int_{-\infty}^{1/4} f(x)\,dx = \int_0^{1/4} 1\,dx = 1/4$$

(ii) $P(|X| \le 1)$

$|X| \le 1$ means. $\boxed{-1 \le X \le 1}$

$$P(|X| \le 1) = P(-1 \le x \le 1)$$

$$= \int_{-1}^{1} f(x)\,dx = \int_0^1 (1)\,dx.$$

Since $f(x) = 1$ has limits. $0 \le x \le 1$

$$P(|X| \le 1) = [x]_0^1 = 1$$

$\therefore P(|X| \le 1) = 1.$

(iii) $P(|X| > 1)$

$$P(|X| > 1) = 1 - P(|X| \le 1) = 1 - 1 = 0 //.$$

2) A Random Variable X has the PDF $f(x) = \begin{cases} 2x, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$

Find (i) $P(X < 1/2)$ (ii) $P\left(\frac{1}{4} < X < \frac{1}{2}\right)$

(iii) $P\left(X > \frac{3}{4} / X > 1/2\right)$ (iv) Mean (v) Variance

(i) $$P(X < 1/2) = \int_{-\infty}^{1/2} f(x)\,dx = \int_{0}^{1/2} 2x\,dx$$ $[0<x<1 \to 2x]$ otherwise $\to 0$

$$= 2\left[\frac{x^2}{2}\right]_0^{1/2} = (1/2)^2 - 0 = 1/4$$

(ii) $$P\left(\frac{1}{4} < X < \frac{1}{2}\right) = \int_{1/4}^{1/2} f(x)\,dx = \int_{1/4}^{1/2} 2x\,dx = 2\left[\frac{x^2}{2}\right]_{1/4}^{1/2}$$

$$= (1/2)^2 - (1/4)^2 = 1/4 - \frac{1}{16} = \frac{3}{16}$$ [use calc]

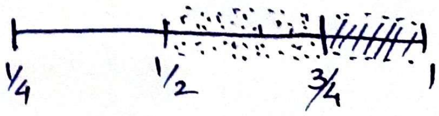
* $\boxed{ab/c}$ (or) $\boxed{\text{Shift}} + \boxed{ab/c}$ $\Rightarrow$ get fraction $\to$ $\boxed{3 \lrcorner 16}$ $\Rightarrow \frac{3}{16}.$

(iii) $P(X > 3/4 / X > 1/2)$

Conditional Probability $$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$A = X > 3/4$ , $B = X > 1/2$

$P(A \cap B) = P(X > 3/4 \cap X > 1/2)$




$$\boxed{P(A \cap B) = P(X > 3/4)}$$

$$\boxed{P(B) = P(X > 1/2)}$$

so $P(x>3/4) = \int_{3/4}^{1} f(x)\,dx = \int_{3/4}^{1} 2x\,dx = 2\left[\frac{x^2}{2}\right]_{3/4}^{1}$

$= \left[1-\left(\frac{3}{4}\right)^2\right] = 1-\frac{9}{16} = \frac{7}{16}$

$P(x>1/2) = 1-P(x\leq 1/2)$

In Continuous R.V $P(x\leq 1/2) = P(x<1/2)$

from (i) of Problem we get

$P(x>1/2) = 1-1/4 = \boxed{3/4}$

$P(A/B) = \frac{P(A\cap B)}{P(B)} = \frac{7/16}{3/4} = \boxed{\frac{7}{12}}$

(iv) Mean

Mean = Expectation = $E(x)$

$E(x) = \int_{-\infty}^{\infty} x\,f(x)\,dx = \int_0^1 x\cdot(2x)\,dx. = \int_0^1 2x^2 dx.$

$= 2\left[\frac{x^3}{3}\right]_0^1 = 2/3$

(v) Variance

Variance $= E(x^2) - [E(x)]^2$ , W.K.T $E(x) = 2/3$

$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2(2x)\,dx = \int_0^1 2x^3 dx$

$= 2\left[\frac{x^4}{4}\right]_0^1 = 1/2$

$$\text{Variance} = \left(\frac{2}{3}\right) - \left(\frac{1}{2}\right)^2 = E(x^2) - [E(x)]^2$$

$$\text{Variance} = \boxed{\frac{5}{12}}$$

(3) A Continuous random Variable X has a p.d.f $f(x) = 3x^2$, $0 \le x \le 1$. Find the value of k. (i) $P(X \le K) = P(X > K)$

(ii) $P(X > b) = 0.05$

given $P(X \le K) = P(X > K)$

W.K.T $P(X \le K) + P(X > K) = 1$

[Sum of Probabilities = 1.] & given $P(X \le K) = P(X > K)$

$$\Rightarrow 2P(X \le K) = 1$$

$$\therefore P(X \le K) = 1/2.$$

$$\int_{-\infty}^{K} f(x)\,dx = 1/2 \Rightarrow \left[\text{Since } 0 \le x \le 1.\right]$$

$$\Rightarrow \int_0^K 3x^2\,dx = 1/2 \Rightarrow 3\left[\frac{x^3}{3}\right]_0^K = 1/2$$

$$K^3 = 1/2 \Rightarrow K = (1/2)^{1/3}$$

(ii) $P(X > b) = 0.05 \Rightarrow \int_b^{\infty} f(x)\,dx = 0.05 \Rightarrow \int_b^1 f(x)\,dx = 0.05.$

$$\Rightarrow \int_b^1 3x^2\,dx = 0.05 \Rightarrow 3\left[\frac{x^3}{3}\right]_b^1 = 0.05$$

$$b^3 = 0.05 \Rightarrow b = (0.05)^{1/3}.$$

(4) A Random Variable $X$ has a pdf, 

$f(x) = Kx^2e^{-x}$, $x \geq 0$, find $K$, mean, variance, and

$E(3x^2 - 2x)$

(i) given $f(x) = Kx^2e^{-x}$ , $x \geq 0$

Sum of Probabilities is 1.

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow K\int_0^{\infty} x^2e^{-x}\,dx = 1.$$

USE BERNOULLI.

$$\int u\,dv = uv - u'v_1 + u''v_2 \cdots\cdots$$

[diff] $u = x^2$ $\quad dv = e^{-x} \rightarrow$ [Integrate]

$u' = 2x \quad = -e^{-x}$

$u'' = 2 \quad = e^{-x}$

$u''' = 0 \quad v_2 = -e^{-x}.$

W.K.T $e^{-\infty} = 0$

$e^{0} = 1$

$$\therefore K\left[(x^2)(-e^{-x}) - (2x)(e^{-x}) + (2)(-e^{-x})\right]_0^{\infty} = 1$$

$$K\left[(0-0+0) - (0-0+2)\right] = 1 \Rightarrow \boxed{K = 1/2}$$

(ii) mean $E(x)$

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^{\infty} x \cdot K\, x^2e^{-x}\,dx$$

$$= \frac{1}{2}\int_0^{\infty} x^3e^{-x}\,dx.$$

$u = x^3 \qquad dv = e^{-x}$

$u' = 3x^2 \qquad v = -e^{-x}$

$u'' = 6x \qquad v_1 = +e^{-x}$

$u''' = 6 \qquad v_2 = -e^{-x}$

$\qquad\qquad v_3 = +e^{-x}$

$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 \cdots$

$$E(x) = \frac{1}{2}\left[x^3(e^{-x}) - (3x^2)(e^{-x}) + (6x)(-e^{-x}) - (6)(e^{-x})\right]_0^\infty$$

$$= \frac{1}{2}\left[(0 - 0 + 0 + 0) - (0 - 0 + 0 - 6)\right] = \frac{1}{2}[6] = 3.$$

$\therefore \boxed{E(X) = 3}$

Similarly $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^\infty x^2 k x^2 e^{-x}\,dx.$

$$= k\int_0^\infty x^4 e^{-x}\,dx. = 12$$

Variance $V(X) = E(X^2) - [E(X)]^2 = 12 - (3^2) = 12 - 9$

$\boxed{V(X) = 3}$

(iii) $E(3X^2 - 2X) = E(3X^2) - E(2X) = 3E(X^2) - 2E(X)$

$= 3(12) - 2(3) = 36 - 6 = \boxed{30}$

2) The amount of bread (in hundreds of kgs) that a certain bakery is able to sell in a day is a random variable $X$ with a pdf given by

$$f(x) = \begin{cases} Ax & \text{if } 0 \le x < 5 \\ A(10-x) & \text{if } 5 \le x < 10 \\ 0 & \text{otherwise.} \end{cases}$$

i) find the value of A

ii) Find the Probability of that in a day sales is

a) more than 500 kgs

b) Less than 500 kgs c) between 250 and 750 kgs

(iii) $P(X>5 / X<5)$, and $P(X>5 / 2.5 < X < 7.5)$

(iv) find c.d.f, v) mean vi) variance.

The Random Variable $X$ denotes the amount of sales in 100's of kgs. ∴ So, ∴ Sales 500 kgs mean

$\boxed{X=5}$

---

(i) find A. Since $f(x)$ is the p.d.f of $X$, ∴ $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_0^5 \overset{Ax}{f(x)}\,dx + \int_5^{10} A(10-x)\,dx = 1.$$

$$\left[A\frac{x^2}{2}\right]_0^5 + \left[A\left(10x - \frac{x^2}{2}\right)\right]_5^{10} = 1$$

$$A\left[\frac{5^2}{2}\right] + A\left[\left(10(10) - \frac{10^2}{2}\right) - \left(10(5) - \frac{5^2}{2}\right)\right] = 1$$

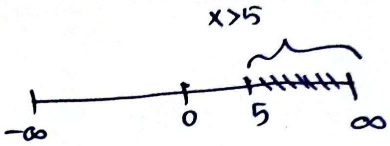
$$A\left[\frac{25}{2}\right] + A\left[\left(100 - \frac{100}{2}\right) - \left(50 - \frac{25}{2}\right)\right] = 1$$

$$A\left(\frac{25}{2}\right) + A\left[\frac{100}{2} - \frac{75}{2}\right] = 1 \Rightarrow \frac{25A}{2} + \frac{25A}{2} = 1$$

$$25A = 1 \Rightarrow \boxed{A = 1/25}$$

---

(ii) a) Probability more than 500 Kgs.




$$P(X > 5) = \int_5^{\infty} f(x)\,dx. = \int_5^{10} A(10-x)\,dx.$$

$$= -A\int_5^{10}(x-10)\,dx = -A\left[\frac{(x-10)^2}{2}\right]_5^{10}$$

$$= -\frac{A}{2}\left[0 - 5^2\right] = \frac{25A}{2} = \frac{25}{2}\left(\frac{1}{25}\right) = 1/2.$$

---

(iii) b) $P(X<5)$

$$P(X<5) = 1 - P(X \geq 5) = 1 - P(X>5)$$

Since C.R.V.

$$= 1 - 1/2 \quad \text{(from the above)}$$

$$P(X<5) = 1/2.$$

---

(iv) $P(X>5 / X<5)$

$$P(X>5/X<5) = \frac{P(X>5 \cap X<5)}{P(X<5)}$$

There no common 'x' between $X>5$ & $X<5$

$$P(X>5 \cap X<5) = 0 \,//.$$

$$\therefore P(X>5/X<5) = 0 \,/.$$

c) Probability of Sales between 250 to 750 kgs.

$$P(2.5 < X < 7.5) = \int_{2.5}^{7.5} f(x)\,dx = \int_{2.5}^{5} f(x)\,dx + \int_{5}^{7.5} f(x)\,dx.$$

$$= \int_{2.5}^{5} Ax\,dx + \int_{5}^{7.5} A(10-x)\,dx.$$

$$= A\left[\frac{x^2}{2}\right]_{2.5}^{5} + (-A)\int_{5}^{7.5} (x-10)\,dx.$$

$$= \frac{A}{2}\left[25 - 6.25\right] - A\left[\frac{(x-10)^2}{2}\right]_{5}^{7.5}$$

$$= \frac{A}{2}\left[25 - 6.25\right] - \frac{A}{2}\left[6.25 - 25\right]$$

$$= \frac{A}{2}\left[25 - 6.25 - 6.25 + 25\right] = \frac{A}{2}\left[50 - 12.5\right]$$

$$= \frac{A}{2}\left[50 - \frac{25}{2}\right] = \frac{A}{2}\left[\frac{75}{2}\right] = \frac{A}{4} \times 75 = \frac{75}{4}\left(\frac{1}{25}\right)$$

$$\therefore P(2.5 < X < 7.5) = 3/4.$$

---

iv) $P(X>5 / 2.5 < X < 7.5)$ $\qquad P(B/A) = \frac{P(A \cap B)}{P(A)}$

$$P(A \cap B) = P(X > 5 \cap 2.5 < X < 7.5) = P(5 < X < 7.5)$$

from the above.

$$P(5 < X < 7.5) = -A\int_{5}^{7.5} (x-10)\,dx = -\frac{A}{2}\left[6.25 - 25\right]$$

$$= -\frac{A}{2}\left[\frac{25}{4} - 25\right] = -\frac{A}{2}\left[-\frac{75}{4}\right] = \frac{75A}{8}$$

$$P(5 < X < 7.5) = \frac{75}{8} \times \frac{1}{25} = 3/8.$$

$$P(2.5 < X < 7.5) = 3/4 \quad \text{(from the above.)}$$

$$P(X>5 / 2.5<X<7.5) = \frac{3/8}{3/4} = \frac{3}{8} \times \frac{4}{3} = \frac{4}{8} = \frac{1}{2}.$$



iv) c.d.f

$$F(x) = \int_{-\infty}^{x} f(x)\,dx.$$

If $x<0$ $\quad F(x) = 0.$

If $x \in (0,5)$, then $F(x) = \int_0^x Ax\,dx = A\left[\frac{x^2}{2}\right]_0^x = \frac{Ax^2}{2}$

$$= \frac{x^2}{50} \;//.$$

If $x \in (5,10)$, then $F(x) = \int_{-\infty}^{x} f(x)\,dx.$

$$= \int_0^5 f(x)\,dx + \int_5^x f(x)\,dx.$$

$$= \int_0^5 Ax\,dx + \int_5^x A(10-x)\,dx$$

$$= A\left[\frac{x^2}{2}\right]_0^5 + (-A)\int_5^x (x-10)\,dx = \frac{25A}{2} - A\left[\frac{(x-10)^2}{2}\right]_5^x$$

$$= \frac{25}{2}\left(\frac{1}{25}\right) - \left(\frac{1}{25}\right)\left[\frac{(x-10)^2}{2} - \frac{5^2}{2}\right]$$

$$= \frac{1}{2} - \frac{1}{50}\left[(x-10)^2 - 25\right] = \frac{1}{2} - \frac{1}{50}(x-10)^2 + \frac{25}{50}$$

$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{50}(x-10)^2 = 1 - \frac{1}{50}(x-10)^2 \;//.$$

If $x \geq 10$ $\quad F(x) = 1.$

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^2/50 & \text{if } 0 \leq x < 5 \\ 1 - \frac{1}{50}(x-10)^2 & \text{if } 5 \leq x < 10 \\ 1 & \text{if } x \geq 10. \end{cases}$$

c.d.f.

Dr. E. Suresh, SRMIST

# RELATION BETWEEN CDF & PDF.

* If X is a Continuous R.V. with P.d.f then

$$F(x) = \int_{-\infty}^{x} f(x)\, dx, \quad \text{for each } x \in (-\infty, \infty).$$

By fundamental theorem of integral Calculus, we get.

$$F'(x) = f(x) \geq 0, \quad \forall x \text{ at the points where}$$

F is differentiable.

* If Cdf is given $F(x) \Rightarrow$ To find p.d.f

$$\boxed{F'(x) = f(x)}$$

* If X is a discrete R.V. with Values. $x_1 < x_2 < x_3 < \cdots$, then.

$$P(x_i) = F(x_i) - F(x_{i-1}).$$

**PROBLEM: 1** Find the pdf for given cummulative distribution function of a R.V X is

$$F(x) = \begin{cases} 1 - \dfrac{4}{x^2}, & \text{if } x > 2 \\ 0, & \text{if } x \leq 2. \end{cases}$$

given C.D.F

$$\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{-n}{x^{n+1}}$$

$$\frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{-2}{x^3}.$$

$$F(x) = \begin{cases} 1 - \frac{4}{x^2}, & x > 2 \\ 0, & \text{if } x \leq 2. \end{cases}$$

The p.d.f of X is $f(x) = F'(x) = \begin{cases} -4\left(\frac{-2}{x^3}\right), & \text{if } x > 2 \\ 0, & \text{if } x \leq 2. \end{cases}$

$\therefore$ The pdf is $f(x) = \begin{cases} 8/x^3, & \text{if } x > 2 \\ 0, & \text{if } x \leq 2. \end{cases}$

PROBLEM 2 The cdf of a discrete random Variable X is given by

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{6} & \text{if } 0 \leq x < 2 \\ 1/2 & \text{if } 2 \leq x < 4 \\ 5/8 & \text{if } 4 \leq x < 6 \\ 1 & \text{if } x \geq 1. \end{cases}$$

Find the Probability distribution.

To find the Probability distribution of X, we have to find the probabilities at the changing points 0, 2, 4, 6.

$P(x=0) = \frac{1}{6}$, $P(x=2) = F(2) - F(0) = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$

$P(x=4) = F(4) - F(2) = \frac{5}{8} - \frac{1}{2} = \frac{1}{8}$

$P(x=6) = 1 - F(4) = 1 - \frac{5}{8} = \frac{3}{8}.$

The probability distribution of X is

| $x$ | 0 | 2 | 4 | 6 |
|---|---|---|---|---|
| $P(x)$ | 1/6 | 1/3 | 1/8 | 3/8 |

2) The cdf of a continuous R.V. X is given by

$$F(x) = \begin{cases} 0 & , \quad x < 0 \\ x^2 & , \quad 0 \le x < 1/2 \\ 1 - \frac{3}{25}(3-x)^2 & , \quad 1/2 \le x < 3 \\ 1 & , \quad x \ge 3. \end{cases}$$

find the P.d.f of X and evaluate $P(|X| \le 1)$ and $P(1/3 \le X < 4)$ using Pdf and cdf.

$F'(x) = f(x)$ (or) $\frac{d}{dx}[F(x)] = f(x)$

$$\therefore \quad f(x) = \begin{cases} 0 & , \quad x < 0 \\ 2x & , \quad 0 \le x < 1/2 \\ +\frac{6}{25}(3-x) & , \quad 1/2 \le x < 3 \\ 0 & , \quad x \ge 3. \end{cases}$$

Evaluate $P(|X| \le 1)$

P.d.f : $P(|X| \le 1) = P(-1 \le X \le 1) = \int_0^{1/2} 2x \, dx + \int_{1/2}^{1} \frac{6}{25}(3-x) dx$

$$= \left[x^2\right]_0^{1/2} + \frac{6}{25}\left[3x - x^2\right]_{1/2}^{1} = \frac{13}{25}.$$

cd.f : $P(|X| \le 1) = P(-1 \le X \le 1) = F(1) - F(-1)$

$F(x) = 1 - \frac{3}{25}(3-x)^2$ at $1/2 \le x < 3$ $\qquad F(1) = 1 - \frac{12}{25} = \frac{13}{25}$

So $F(-1) = 0$

$F(x) = 0 \quad , \quad x < 0$

$P(|X| \le 1) = F(1) - F(-1) = \frac{13}{25} - 0 // = \boxed{\frac{13}{25}}$

**Pdf:** $P\left(\frac{1}{3} \le x < 4\right) = \int_{1/3}^{4} f(x)\,dx = \int_{1/3}^{1/2} f(x)\,dx + \int_{1/2}^{3} f(x)\,dx$

$$= \int_{1/3}^{1/2} 2x\,dx + \int_{1/2}^{3} \frac{6}{25}(3-x)\,dx$$

$$= \left[x^2\right]_{1/3}^{1/2} + \frac{6}{25}\left[3x - x^2\right]_{1/2}^{3} = \boxed{\frac{8}{9}} \quad \left[\text{Use Calculator}\right]$$

$$= \left[\left(\tfrac{1}{2}\right)^2 - \left(\tfrac{1}{3}\right)^2\right] + \frac{6}{25}\left[(9-9) - \left(\tfrac{3}{2} - \tfrac{1}{4}\right)\right] = \boxed{\frac{8}{9}}$$

**cdf:** $P\left(1/3 \le x < 4\right) = F(4) - F(1/3)$

$F(x) = 1$ at $x \ge 3$. $\quad F(4) = 1$

$F(x) = x^2$ at $0 \le x < 1/2 \quad F(1/3) = (1/3)^2 = \frac{1}{9}$

$$P\left(1/3 \le x < 4\right) = F(4) - F(1/3) = 1 - \frac{1}{9} = \boxed{\frac{8}{9}}$$

Points to Remember.

* $P(a \le x \le b) = F(b) - F(a)$
* $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$
* $F(\infty) = \lim_{x \to \infty} F(x) = 1$
* $0 \le F(x) \le 1$