Maths

Unit - 1

Impossible Event :- ϕ

Sure Event :- S

Mutually Exclusive Event :- A and B are said to be mutually Exclusive Event if $A \cap B = \phi$.

* Total number of function :- $|B|^{|A|} \Rightarrow m^n$

* $$P(A) = \frac{n(A)}{n(S)}$$

* P: (Power set of Sample Space) $\longrightarrow$ $[0 \quad 1]$

$\uparrow$ domain. $\qquad$ $\uparrow$ Range

Axiometric definition of Probability (Axiome of Probability) :-

Let S be Sample space and A be an event associated with random experiment. Then probability of event A, is defined as a real number satisfying following axiome :-

1) $0 \leq P(A) \leq 1$

2) $P(S) = 1$

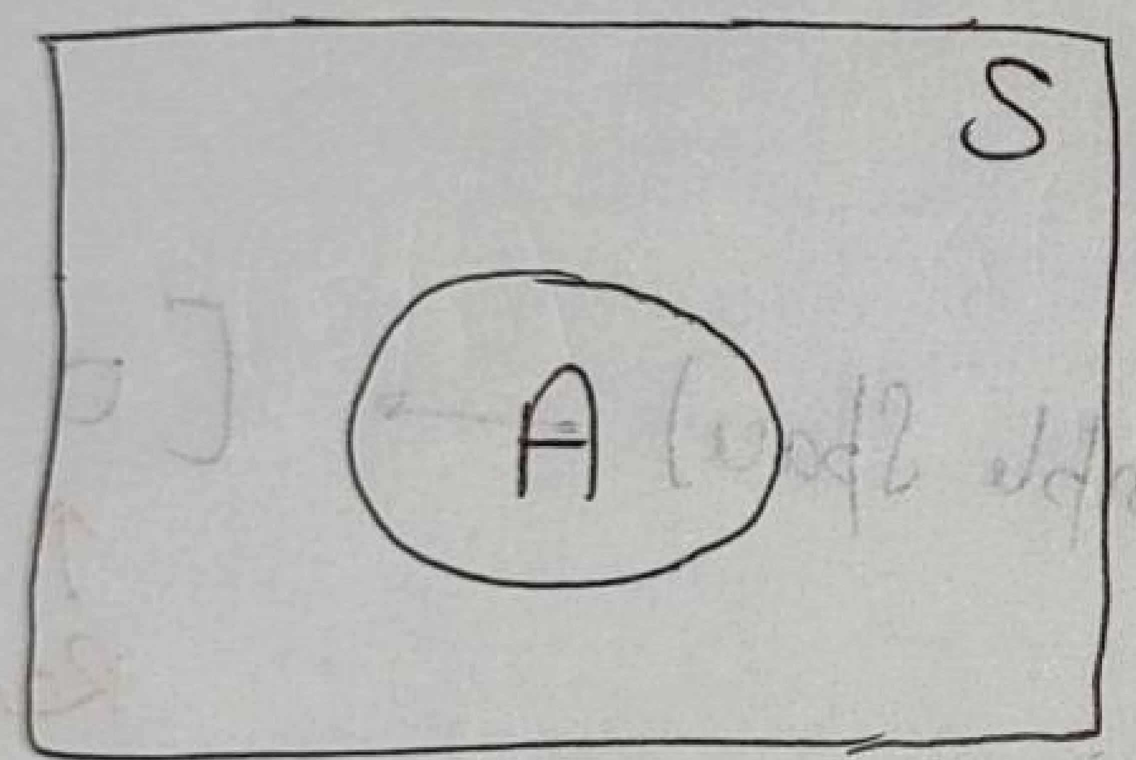
3) If A and B are mutually exclusive events, Then, $P(A \cup B) = P(A) + P(B)$.

4) If $A_1, A_2, A_3, A_4 - - A_n$ are set of mutually exclusive event, then, $P(A_1 \cup A_2 - - A_n) = P(A_1) + P(A_2) - - P(A_n)$.

* Simple Event → consisting of 1 singleton event.
Compound Event → more than 1 sigleton event.

Theorem 1 :- The probability of impossible event is 0.
i.e. $P(\phi) = 0$.

Proof :- Since ϕ and S are mutually exclusive event.
$$P(S \cup \phi) = P(S) + P(\phi)$$
$$\Rightarrow P(S) = P(S) + P(\phi)$$
$$\Rightarrow P(\phi) = 0$$



$$\Rightarrow \bar{A} = S - A$$
$$= (x \in S \;\&\&\; x \notin A)$$
$$\Rightarrow A \cup \bar{A} = S$$

* Complementary event :- Let S be sample space and A be an event then the complementary event of A is denoted by $\bar{A} = S - A$.

Theorem 2 :- Prove that $P(A) + P(\bar{A}) = 1$.

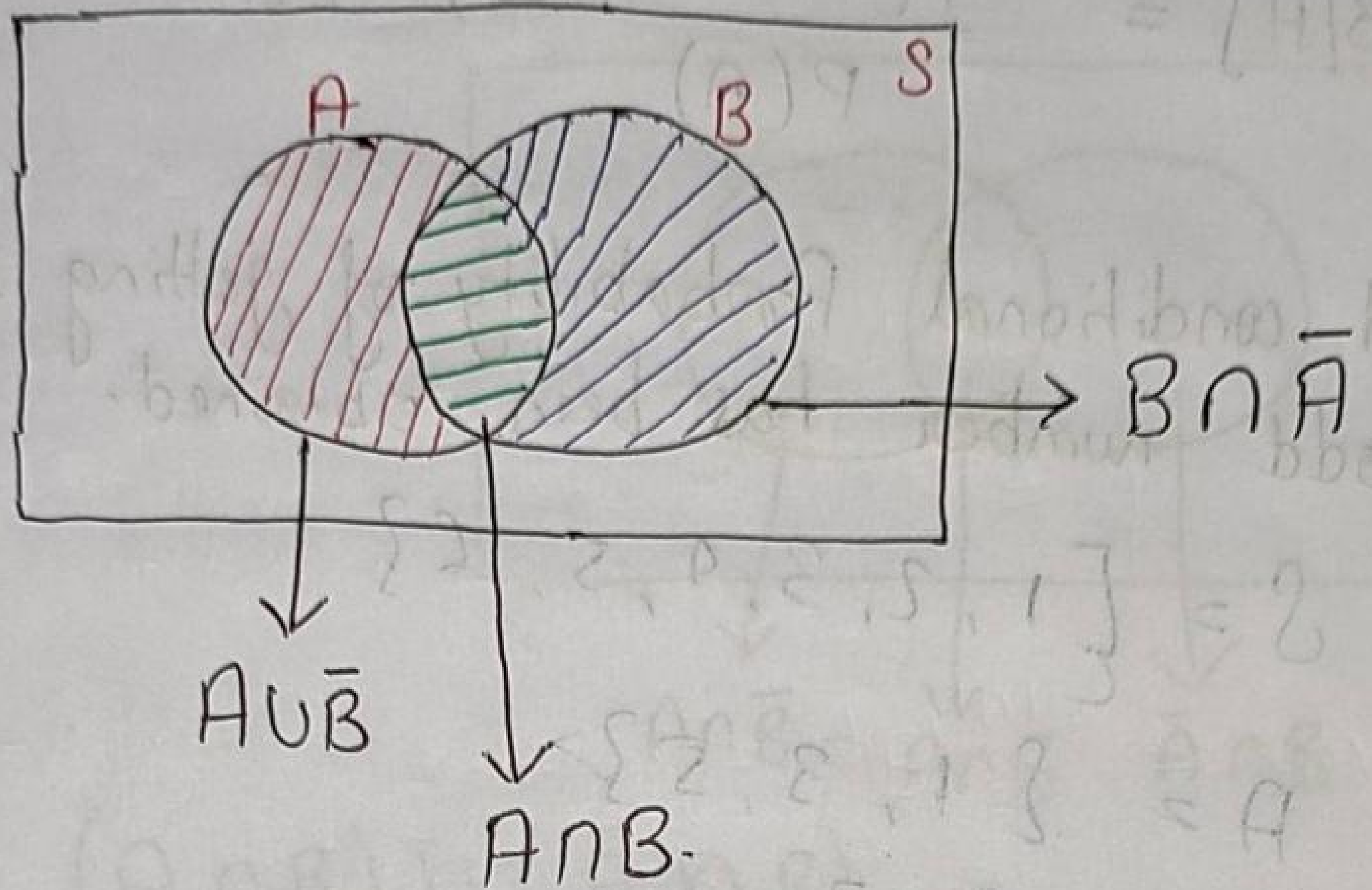
Proof :- A and $\bar{A}$ are mutually exclusive event.
$$\Rightarrow P(A \cup \bar{A}) = P(A) + P(\bar{A}).$$
$$P(S) = P(A) + P(\bar{A})$$
$$\Rightarrow P(A) + P(\bar{A}) = 1.$$

* Addition Theorem of Probability :-

Theorem 3 :- If A and B are any two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Proof :-



$$A = (A \cap \bar{B}) \cup (A \cap B).$$

$$P(A) = P[(A \cap \bar{B}) \cup (A \cap B)]$$

$$\Rightarrow P(A) = P(A \cap \bar{B}) + P(A \cap B) \quad \text{---} \quad ①$$

Similarly, $P(B) = P(A \cap B) + P(\bar{A} \cap B)$. — ②

Adding ① and ②, we get

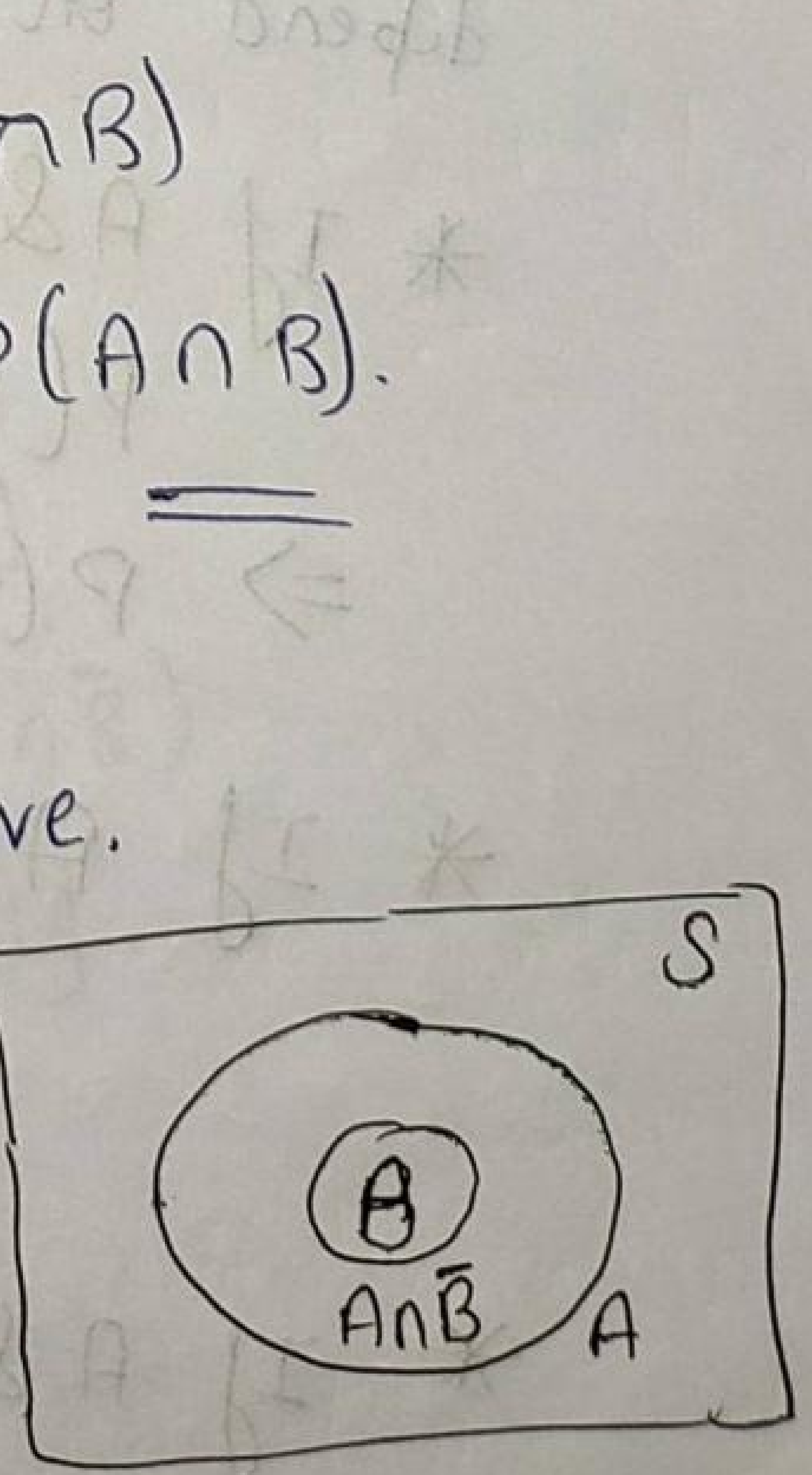
$$P(A) + P(B) = P(A \cap \bar{B}) + P(A \cap B) + P(\bar{A} \cap B) + P(A \cap B)$$

$$P(A) + P(B) = P(A \cup B) + P(A \cap B)$$

$$\Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Theorem 4 :- If $B \subset A$, $P(B) \leq P(A)$.

Proof :- B and $A \cap \bar{B}$ are mutually exclusive.



$$B \cup A \cap \bar{B} = A.$$

$$P(B \cup (A \cap \bar{B})) = P(A)$$

$$\Rightarrow P(B) + P(A \cap \bar{B}) = P(A).$$

$$\therefore P(B) \leq P(A).$$

Conditional Probability :-

The conditional Probability of an event B assuming that event A has happened is denoted by.

$$P(B/A) = \frac{P(A \cap B)}{P(A)} \quad ; \; P(A) \neq 0.$$

Q Find conditional Probability of getting 1 given that an odd number has been obtained.

Sol → $S = \{1, 2, 3, 4, 5, 6\}$

$A = \{1, 3, 5\}$

$B = \{1\}$

$$P(B/A) = \frac{n(A \cap B)}{n(A)} = \frac{1}{3} \text{ Ans}$$

$$\boxed{* \; P(A \cap B) = P(B/A) \times P(A)}$$ → Product theorem of Probability.

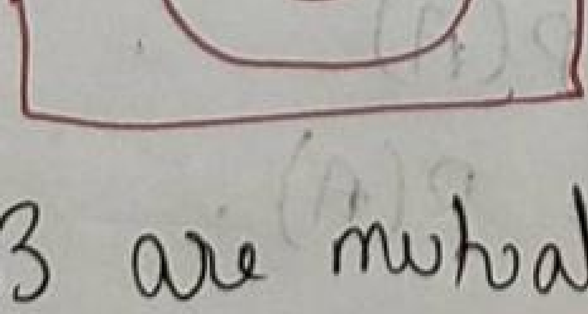
* Independent event :- A set of event is said to be independent if occurrence of any one of them does not depend on the occurrence ~~of~~ or non-occurrence of others.

* If A & B are independent, then.

$P(B/A) = P(B)$.

$\Rightarrow P(A \cap B) = P(B) * P(A)$.

* If $A \subset B$, then, $P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)}{P(A)} = 1$



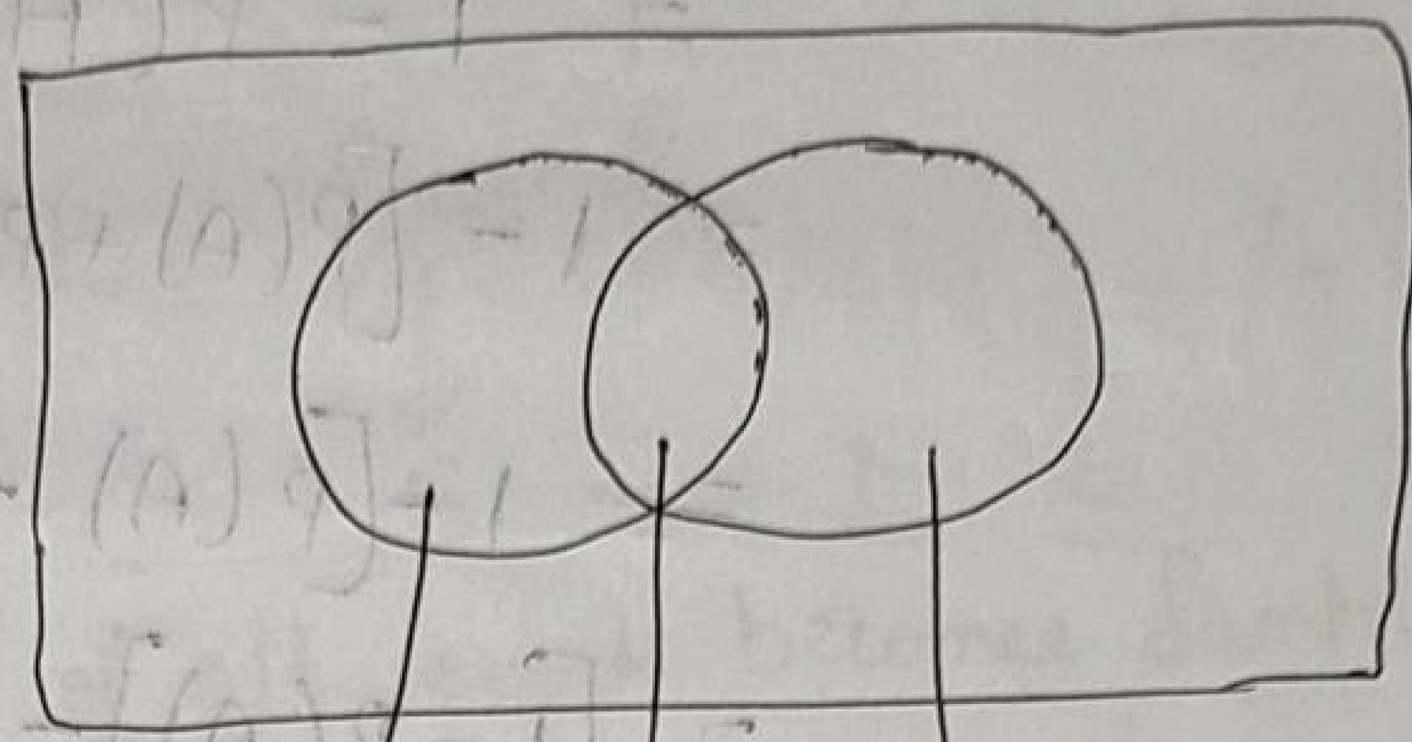
* If A & B are mutually exclusive event.

$P(B/A) = 0 \quad [\because P(A \cap B) = 0]$

Theorm 1 :- If A & B are independent events, then $\bar{A}$ & B are also independent.

Proof :- A and B are independent event

$\Rightarrow P(A \cap B) = P(A) \cdot P(B)$



$B = (A \cap B) \cup (\bar{A} \cap B)$

$P(B) = P(A \cap B) + P(\bar{A} \cap B)$.

$\Rightarrow P(B) = P(A) \cdot P(B) + P(\bar{A} \cap B)$.

$\Rightarrow P(\bar{A} \cap B) = P(B) - P(A) \cdot P(B)$

$\Rightarrow P(\bar{A} \cap B) = P(B) [1 - P(A)]$

$\Rightarrow P(\bar{A} \cap B) = P(\bar{A}) \cdot P(B)$.

Theorem 2 :- If A & B are independent events, then A and $\bar{B}$ are also independent.

Proof :- $A = (A \cap B) \cup (A \cap \bar{B})$

$P(A) = P(A \cap B) + P(A \cap \bar{B})$

$\Rightarrow P(A) = P(A) \cdot P(B) + P(A \cap \bar{B})$

$\Rightarrow P(A \cap \bar{B}) = P(A) [1 - P(B)]$

$\Rightarrow P(A \cap \bar{B}) = P(A) \cdot P(\bar{B})$

Theorem 3 :- If A and B are independent event, then $\bar{A}$ & $\bar{B}$ are also independent.

Proof :- $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B})$.

$$= 1 - P(A \cup B).$$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - [P(A) + P(B) - P(A) \cdot P(B)]$$

$$= [1 - P(A)] - P(B) + P(A) \cdot P(B)$$

$$= [1 - P(A)] - P(B)[1 - P(A)]$$

$$= [1 - P(A)] \cdot [1 - P(B)]$$

$$P(\bar{A} \cap \bar{B}) = P(\bar{A}) \cdot P(\bar{B}).$$

Q. If $P(A) = 0.4$, $P(B) = 0.7$, $P(A \cap B) = 0.3$.
Find $P(\bar{A} \cap \bar{B})$.

Sol→ $P(\bar{A} \cap \bar{B}) = 1 - [P(A) + P(B) - P(A \cap B)]$

$$= 1 - [0.4 + 0.7 - 0.3].$$

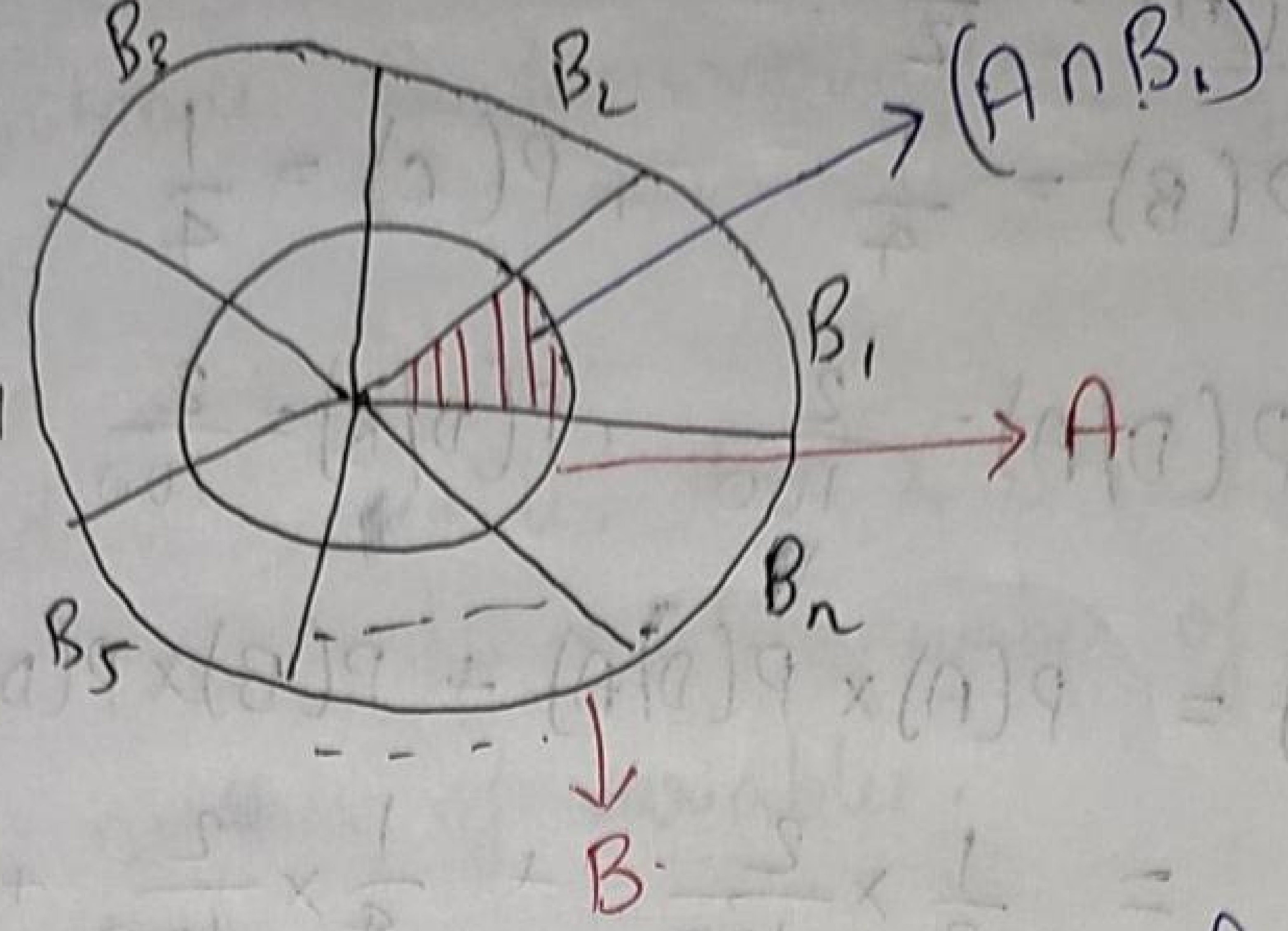
$$= 1 - 0.8$$

$$= 0.2 \text{ Ans}$$

Theorem of Total Probability :-

If $B_1, B_2, B_3 - - B_n$ are exhaustive and mutually exclusive event and A is another event associated with B_i ($i = 1$ to n), then

$$P(A) = \sum_{i=1}^{n} P(B_i) \cdot P(A/B_i).$$



* Exhaustive :– Union of all event becomes Sample Space.

$$A = AB_1 \cup AB_2 \cup AB_3 - - \cup AB_n$$

$$P(A) = P(AB_1) + P(AB_2) - - + P(AB_n)$$

$$= \sum_{i=0}^{n} P(AB_i)$$

$$P(A/B_i) = \frac{P(A \cap B_i)}{P(B_i)} \Rightarrow P(A \cap B_i) = P(A/B_i) \cdot P(B_i)$$

$$\Rightarrow P(A) = \sum_{i=0}^{n} P(A/B_i) \cdot P(B_i).$$

Q. A bolt is manufactured by 3 machine A, B, C. A turns out twice as many items as B and B and C produce equal number of items. 2% of bolt produced by A and B are defective. 4% of bolt produced by C is defective. All bolt are put into stock pile and 1 bulb is choosen at random. What is probability that it is defective.

Sol → Let A denote event of item produced by machine A.
Let B denote event of item produced by machine B.
Let C denote event of item produced by machine C.
D: event of item being defective

$P(A) = \frac{1}{2}$.

$P(B) = \frac{1}{4}$, $P(C) = \frac{1}{4}$

$P(D/A) = \frac{2}{100}$, $P(D/B) = \frac{2}{100}$, $P(D/C) = \frac{4}{100}$.

$$P(D) = P(A) \times P(D/A) + P(B) \times P(D/B) + P(C) \times P(D/C)$$

$$= \frac{1}{2} \times \frac{2}{100} + \frac{1}{4} \times \frac{2}{100} + \frac{4}{4 \times 100}$$

$$= \frac{10}{400}$$

$$= \frac{1}{40} \quad \underline{\underline{\text{Ans}}}$$

Random Variables :-

$$X : S \longrightarrow R.$$

→ A random variable is a function which assign each element of $s \in S$ to a real number.

Example :- $S = [HH, HT, TH, TT]$

$X \to$ Number of heads.

$X : S \to R$

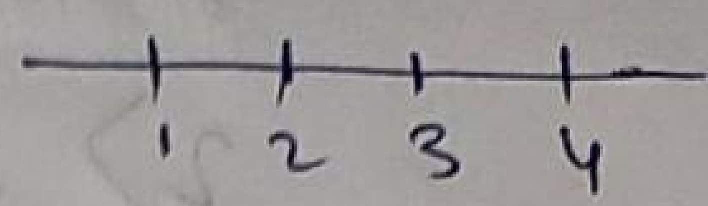
$X(HH) = 2$

$X(HT) = 1$

$X(TH) = 1$

$X(TT) = 0$.

Types :- (a) Discreat Random variable $\dot{1}\ \dot{2}\ \dot{3}\ \dot{4}$
(b) Conitnous Random variable



Discreat Random Variable :-

If X is a random variable, then X is said to be discreat if X can take only finit number of values or countably infinit number of variables.

Ex 1 :- The number shown when a die is thrown is a discreat R.V

$$X : S \to R.$$

$$[1, 2, 3, 4, 5, 6].$$

$$x(1) = 1,\ x(2) = 2.\ \ x(3) = 3,\ \ -\ -\ \ x(6) = 6.$$

Ex. 2 :- The square of number of point when die is thrown.

$$X = [1^2, 2^2 - - - 6^2]$$

$$X = [1, 4, 9, 16, 25, 36]$$

$\Rightarrow$ Discreat R.V.

Continous Random Variable :-

If X is a random variable which can take all values in an interval is called continous Random variable.

Probability function :-

If X is a discreat Random variable, which can take the values $x_1, x_2 - x_n$ such that probability of X

$P(X = x_i) = P_i$, then P_i is called the probability mass function (pmd) provided $P_i\ (i = 1, 2, - -)$ satisfies the following :-

1> All $P_i \geq 0$

2> $\sum_{i=0}^{n} P_i = 1$

$$X : x_1 \quad x_2 \quad x_3 \quad — \quad x_n$$

$$P_i : P(x_1) \quad P(x_2) \quad P(x_3 \quad — \quad P(x_n$$

↑

This table is known as Probability Distribution ~~function~~ (table)

Probability Density function :-

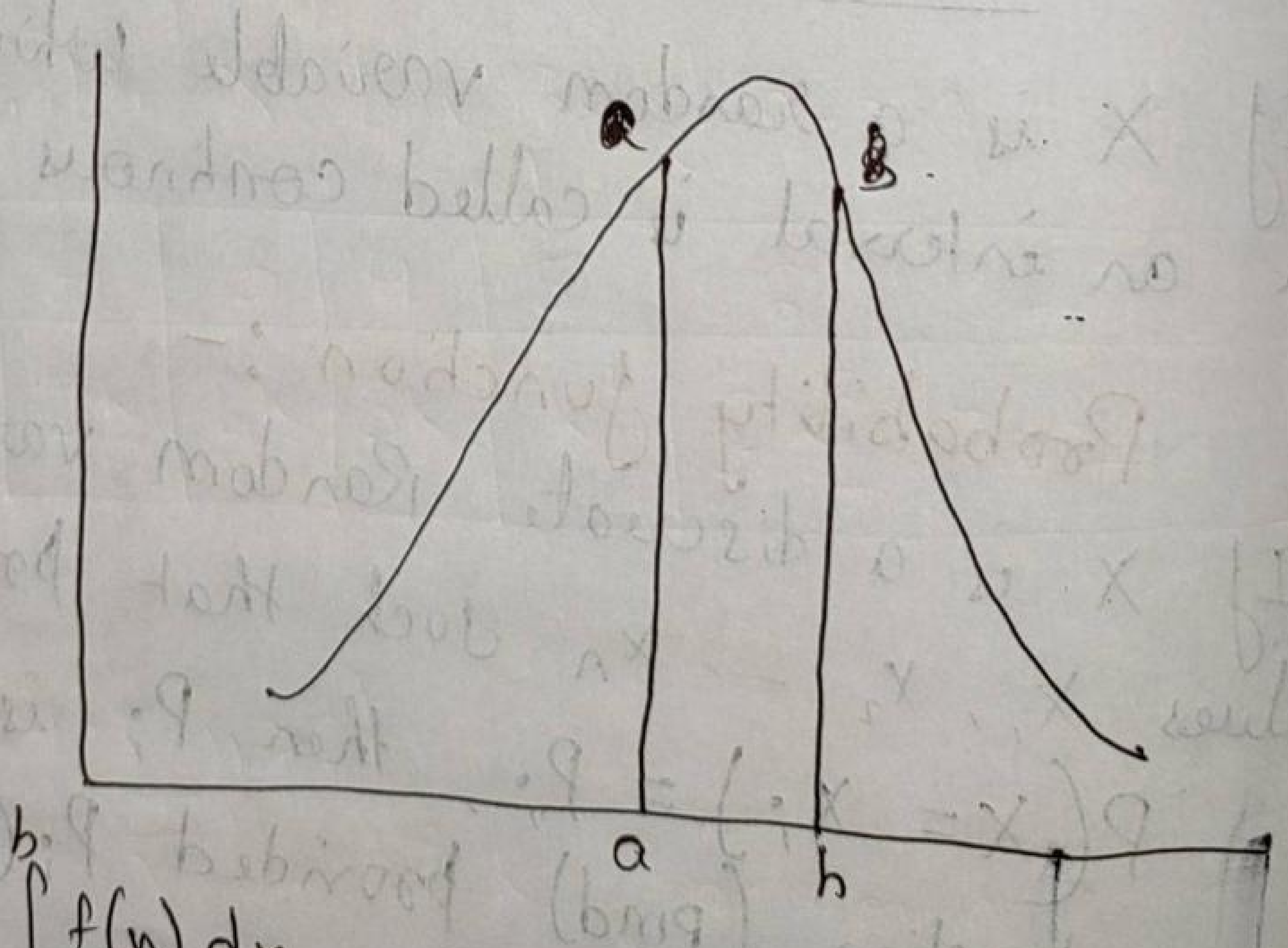
If X is continous RV such that

$$P\left\{x - \frac{1}{2}dx \leq X \leq x + \frac{1}{2}dx\right\} = f(x)\,dx$$

then, $f(x)$ is called the probability density function of X; provided $f(x)$ satisfies the following condition :-

(i) $f(x) \geq 0$ for all $x \in R$

(ii) $\int_{R_x} f(x)\,dx = 1$.



$$P(a \leq x \leq b) = \int_a^b f(x)\,dx.$$

Q A random variable n and the following probability distribution.

n	-2	-1	0	1	2	3
$f(n)$	0.1	k	0.2	$2k$	0.3	$3k$.

(i) find k (ii) $P(n<2)$ & $P(-2<n<2)$.

Sol → (i) $0.1+k+0.2+2k+0.3+3k = 1$

$\Rightarrow 6k = 1 - 0.6$

$\Rightarrow 6k = 0.4$

$k = \frac{4}{60} = \frac{1}{15}$ Ans

(ii) $P(n<2)$.

$= P(n=-2) + P(n=-1) + P(n=0) + P(n=1)$

$= \frac{1}{10} + \frac{1}{15} + \frac{2}{10} + \frac{2}{15}$

$= \frac{15}{30} = \frac{1}{2}$ Ans

$P(-2<n<2)$

$P(n=-1) + P(n=0) + P(n=1)$

$= \frac{1}{15} + \frac{2}{10} + \frac{2}{15}$

$= \frac{2}{5}$ Ans

Properties of the *cdf* $F(x)$

1. $F(x)$ is a non-decreasing function of x, i.e., if $x_1 < x_2$, then $F(x_1) \le F(x_2)$.
2. $F(-\infty) = 0$ and $F(\infty) = 1$.
3. If X is a discrete RV taking values $x_1, x_2, \ldots$, where $x_1 < x_2 < x_3 < \ldots < x_{i-1} < x_i < \ldots$, then $P(X = x_i) = F(x_i) - F(x_{i-1})$.
4. If X is a continuous RV, then $\frac{d}{dx} F(x) = f(x)$, at all points where $F(x)$ is differentiable.

Cummulative Distribution function :-

If x is a random variable (discreat or continous) then $P(X \le x)$ is called the cdf of X.

$$F(x) = \sum_{\substack{j \\ x_j \le x}} P_j$$

Q In previous Question. find cmd of different function.

$(x<-2)$ $x<-2 \Rightarrow F(x) = 0$

$(-2 \le x < -1)$ $x \le -2 \Rightarrow F(x) = 0.1$

$(-1 \le x < 0)$ $x \le -1 \Rightarrow F(x) = \frac{1}{10} + \frac{1}{15} = \frac{1}{6}$

$(0 \le x < 1)$ $x \le 0 \Rightarrow \frac{1}{6} + \frac{2}{10} = \frac{11}{30}$

$(1 \le x < 2)$ $x \le 1 \Rightarrow \frac{11}{30} + \frac{2}{15} = \frac{1}{2}$

$(2 \le x < 3)$ $x \le 2 \Rightarrow \frac{1}{2} + \frac{3}{10} = \frac{4}{5}$

$(3 \le x)$. $x \le 3 \Rightarrow \frac{4}{5} + \frac{3}{15} = 1$.

Q. A random variable X has the function distribution:-

X	0	1	2	3	4	5	6	7
P(X)	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2+k$

(i) find k

(ii) $P(1.5 < x < 4.5 / x > 2)$

(iii) The smallest value of k for which $P(x \le k) \ge \frac{1}{2}$.

Scanned by TapScanner

Sol → (i) $k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$\Rightarrow 9k + 10k^2 = 1$

$\Rightarrow 10k^2 + 10k - k - 1 = 0$

$\Rightarrow 10k(k+1) - 1(k+1) = 0$

$\Rightarrow (k+1)(10k-1) = 0$

$\Rightarrow k = -1$ (impossible)

$k = \frac{1}{10}$ Ans

(ii) $P(1.5 < x < 4.5 / x > 2) = \frac{P(1.5 < x < 4.5) \cap P(x > 2)}{P(x \geq 2)}$

$= \frac{P(x=3) + P(x=4)}{P(x=3) + P(x=4) + P(x=5) + P(x=6) + P(x=7)}$

$= \frac{\frac{2}{10} + \frac{3}{10}}{\frac{2}{10} + \frac{3}{10} + \frac{1}{100} + \frac{2}{100} + \frac{17}{100}}$

$= \frac{\frac{50}{100}}{\frac{70}{100}} = \frac{5}{7}$ Ans

(iii) $P(x \leq 0) = 0$

$P(x \leq 1) = \frac{1}{10}$

$P(x \leq 2) = \frac{3}{10}$

$P(x \leq 3) = \frac{5}{10}$

$P(x \leq 4) = \frac{8}{10} > \frac{1}{2}$

So, $k = 4$ Ans

Scanned by TapScanner

$\underline{\underline{Q}}$. If $p(x) = \begin{cases} xe^{-x^2/2} & x \geq 0 \\ 0 & x < 0 \end{cases}$

ⓐ Show that $p(x)$ is a ~~r.v.~~ pdf.

ⓑ Find c.d.f

Sol→ $p(x) \geq 0$ for every $x \geq 0$.

$$\int_0^\infty x e^{-x^2/2} dx = 1$$

$$= \int_0^\infty \frac{dt}{2} e^{-t/2} dt \quad \left[\begin{array}{l} x^2 = t \\ 2x\,dx = dt \Rightarrow x\,dx = \frac{dt}{2} \end{array} \right]$$

$$= \frac{1}{2} \int_0^\infty e^{-t/2} dt$$

$$= \frac{1}{2} \left[\frac{e^{-t/2}}{-1/2} \right]_0^\infty$$

$$= \frac{1}{2} \times 2 \left[0 + 1 \right]$$

$$= 1$$

Now, since $p(x) \geq 0$ &

$$\int_{R_x} p(x)\,dx = 1$$

So, $p(x)$ is a p.d.f

(ii) c.df :-

$$F(x) = P(X \leq x) = \int$$

for, $x \le 0 \Rightarrow F(x) = 0$.

$x \ge 0 \Rightarrow F(x) = P(-\infty \le X \le x)$

$$= P(-\infty \le x \le 0) + P(0 \le X \le x)$$

$$= \int_0^x x e^{-x^2/2} dx$$

$$= \int_0^{x^2} e^{-t/2} dx$$

$$= \frac{1}{2}\left[-e^{-t/2} \times 2\right]_0^{x^2}$$

$$= \frac{1}{2} \times 2 \left[-e^{-x^2/2} + 1\right]$$

$$= 1 - e^{-x^2/2} \quad \underline{\text{Ans}}$$

Q. A continous RV X such that can assume any value between $x = 2$ and $x = 5$ has a density function given by $f(x) = k(1 + x)$. Find $P(X < 4)$.

Sol→ By property of p.d.f

$$\int_2^5 f(x)\,dx = 1$$

$$= \int_2^5 k(1 + x)\,dx = 1$$

$$k\left[x + \frac{x^2}{2}\right]_2^5 = 1$$

$$k\left[5 + \frac{25}{2} - 2 - 2\right] = 1$$

$$\Rightarrow k = \frac{2}{27}.$$

Scanned by TapScanner

Now,

$$P(x < 4) = P(2 \le n \le 4).$$

$$= \int_2^4 k(1+n)$$

$$= \int_2^4 \frac{2}{27}(1+x)\,dx$$

$$= \frac{2}{27}\left[n + \frac{n^2}{2}\right]_2^4$$

$$= \frac{2}{27}\left[4 + \frac{16}{2} - 2 - \frac{4}{2}\right]$$

$$= \frac{16}{27} \text{ Ans}$$

Mathematical Expectation :-

If n is a discreate random variable then the mathematical expectation of n or the arithemetic mean of n is defined as.

$$E[n] = \sum_i n_i p_i$$

If n is continous random variable then mathematical expectation of n or the arithmetic mean of n is defined as.

$$E[n] = \int_{R_n} n f(n)\,dx.$$

for example.

①

x	1	2	3	4	5	6
$P(x)$	1/6	1/6	1/6	1/6	1/6	1/6

find expectation of this distribution :-

$$E[x] = \sum_i x_i p_i$$

$$= 1\times\frac{1}{6} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + 4\times\frac{1}{6} + 5\times\frac{1}{6} + 6\times\frac{1}{6}$$

$$= \frac{1}{6}(1+2+3+4+5+6)$$

$$= \frac{21}{6} = 3.5$$

② $P(x) = \begin{cases} x e^{-x^2/2} & x \geq 0 \\ 0 & x \leq 0 \end{cases}$. find $E[x]$.

Sol→ $E[x] = \int_0^\infty x \cdot e^{-x^2/2} dx = 1$ Ans

* If $g(x)$ is a function of random variable x then the expected value of $g(x)$.

$$E[g(x)] = \sum_i g(x_i) p_i \longrightarrow \text{If } x \text{ is discreat}$$

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx \longrightarrow \text{If } x \text{ is continous.}$$

* from this page example ①, find $E[x^2]$.

$= g(x) = x^2$

$$\Rightarrow E[x^2] = \frac{(1)^2}{6} + \frac{(2)^2}{6} + \frac{(3)^2}{6} + \frac{(4)^2}{6} + \frac{(5)^2}{6} + \frac{(6)^2}{6}$$

$$= 91/6 \text{ Ans}$$

* $E[ax+b] = aE[x] + b$
* $E[ax] = aE[x]$.
* $E[b] = b(p_1 + p_2 + p_3 + \cdots)$
$$= b(1)$$
$$= b$$

So, $E[c] = c$

↑ constant

Moments

X :	1	2	3	4	5	6
X-1 :	0	1	2	3	4	5

$g(x) = (x-1)$

→ The rth order non-central moment about the value 'a' of a random variable X, denoted by μ_r' is defined as.

$$\mu_r' = E[(x-a)^r]$$

$$= \begin{cases} \sum_i (x_i - a)^r p_i \rightarrow \text{If } x \text{ is discrete} \\ \int_{-\infty}^{\infty} (x_i - a)^r f(x)\,dx \rightarrow \text{If } x \text{ is continous.} \end{cases}$$

* If $a = 0$, then $\mu_r' = E[x^r]$ is called as r^{th} order simple moment / raw moments.

If $a = \mu = E[x]$, then the r^{th} order central moment is defined as :

$$\mu_r = E\left[(x-\mu)^r\right].$$

↑ Remember there is no (') in this.

NOTE :- $r = 0$.

$$\mu_0' = E[1] = 1$$

$$\mu_1 = E[x - \mu]$$

$$\mu_1 = E[x] - \mu$$

$$= E[x] - E[x]$$

$$= 0$$

$$\Rightarrow \mu_1 = 0$$

$\Rightarrow 1^{st}$ order central moment $= 0$

* $\mu_2 = E\left[(x-\mu)^2\right] = \text{variance}$

$$\sigma_x^2 = \left(\sqrt{\mu_2}\right)^2 \Rightarrow \sigma_x^2 = \mu_2$$

↑ Variance

Relation Between Central & Non-Central Moment :-

$$\mu_r' = E\left[(x-a)^r\right]$$

$$\mu_1' = E[(x-a)]$$
$$= E[x] - a$$
$$\mu_1' = \mu - a$$
$$\mu_1' = d \quad (\text{say}).$$

$$\mu_n = E[(x-\mu)^n]$$
$$= E[(x-a) - (\mu-a)]^n$$
$$= E[(x-a) - d]^n$$

$$\mu_2 = \mu_2' - d^2$$
$$\mu_3 = \mu_3' - 3\mu_2' d + 2d^3$$
$$\mu_4 = \mu_4' - 4\mu_3' d + 6\mu_2' d^2 - 3d^4$$

Properties of variance :-

① $\mathrm{Var}(x) = E[x^2] - (E(x))^2$

Proof :- $\mathrm{Var}(x) = \mu_2 = \mu_2' - d^2$
$$= E[(x-a)^2] - (\mu-a)^2$$

Put $a = 0$

$$\Rightarrow \mathrm{Var}(x) = E[x^2] - \mu^2$$
$$= E[x^2] - (E[x])^2$$

Scanned by TapScanner

② $Var(ax+b) = a^2 \cdot Var(x)$

Q. If $Var(x) = 3$. find $Var(2x+3)$?

Sol → $Var(2x+3) = (2)^2 \cdot Var(x)$
$= 4 \cdot 3$
$= 12$.

Proof of $Var(ax+b) = a^2 Var(x)$.

$\Rightarrow Var(ax+b) = E[(ax+b) - E(ax+b)]^2$.
$= E[(ax+\cancel{b}) - aE[x] - \cancel{b}]^2$
$= E[a[x - E(x)]]^2$
$\Rightarrow a^2 \cdot E[x - E[x]]^2$
$\Rightarrow a^2 \cdot Var(x)$

Q.

X	-3	-2	-1	0	1	2	3
P(x)	0.05	0.10	0.30	0	0.30	0.15	0.10

(i) find $E[x]$ (iii) $E[2x+3]$
(ii) find $E[x^2]$ (iv) $Var(2x+3)$.

Sol → (i) $E[x] = \sum_i x_i p_i$
$= -3\times0.05 - 2\times0.10 - 1\times0.30 + 0 + 0.30 + 2\times0.15 + 3\times0.10$
$= 0.25$ Ans

(ii) $E[x^2] = \sum_i x_i^2 \cdot p_i$

$9\times0.05 + 4\times0.10 + 1\times0.30 + 0 + 1\times0.30 + 4\times0.15 + 9\times0.10$

$= 2.95$

(iii) $E[2x+3]$

$= 2\times E[x] + 3$

$= 2\times0.25 + 3$

$= 3.5$

(iv) $Var(2x+3)$

$= 4\times Var(x)$

$= 4\times[E(x^2) - [E(x)]^2]$

$= 4\times[2.95 - 0.0625]$

$= 4\times 2.8875$

$= 11.55$ Ans

Q. $F(x) = \begin{cases} \frac{1}{2}(x+1) & -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$

find find & variance of x.

Sol→ $E[x] = \int_{-1}^{1} x \cdot \frac{1}{2}(x+1)\,dx$

$= \frac{1}{2}\int_{-1}^{1}(x^2 + x)\,dx$

$$= \frac{1}{2} \times 2 \int_0^1 x^2$$

$$= \frac{1}{2} \times 2 \left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{3} \text{ Ans}$$

(ii) $\mu_2 = E[(x-\mu)^2]$

$$= E\left[\left(x - \frac{1}{3}\right)^2\right]$$

$$= \int_{-1}^{1} \left(x - \frac{1}{3}\right)^2 \frac{1}{2} \cdot (x+1)\, dx$$

$$= \int_{-1}^{1} \frac{(3x-1)^2}{9} \times \frac{1}{2} \times (x-1)\, dx$$

$$= \frac{1}{18} \int_{-1}^{1} (3x-1)^2 (x-1)\, dx$$

$$= \frac{1}{18} \int_{-1}^{1} 9x^3 + 9x^2 + x + 1 - 6x^2 - 6x\, dx$$

$$= \frac{1}{18} \times 2 \int_0^1 9x^2 - 6x^2 + 1\, dx$$

$$= \frac{2}{18} \left[3 \cdot \frac{x^3}{3} + x\right]_0^1$$

$$= \frac{2}{18} \times [1 + 1]$$

$$= \frac{2}{9} \text{ Ans}$$

Scanned by TapScanner

Q. The first three moment of a distribution about the value 2 of the R.V are 1, 16, −40. Find the coefficient of variation. Also find the first three ~~mean~~ moment about mean.

Sol →
$$\mu_1' = E[(x-2)] = 1$$
$$\Rightarrow E[x] - 2 = 1$$
$$\Rightarrow E[x] = 3$$

$$\mu_2' = E[(x-2)^2] = 16.$$
$$\mu_3' = E[(x-2)^3] = -40$$
$$\mu_1 = 0$$
$$\mu_2 = \mu_2' - d^2$$
$$= \mu_2' - \mu_1'^2$$
$$= 16 - 1$$
$$= 15$$
$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3.$$
$$= -40 - 3 \times 16 \times 1 + 2(1)^3$$
$$= -86$$

$$S.D = \sqrt{\mu_2} = \sqrt{15}$$
Mean = 3.

$$\text{Coefficient of variation} = \frac{SD}{Mean} \times 100$$

$$COV = \frac{\sqrt{15}}{3} \times 100$$
$$= 129.1\%.$$

Scanned by TapScanner

Q $f(u) = ku^2 e^{-u} \quad u > 0.$

Find mean & variance.

Sol → $\int_0^\infty k u^2 e^{-u} = 1$

$\Rightarrow k \int_0^\infty u^2 e^{-u}\, dx = 1$

$\Rightarrow k \left[u^2 \cdot \frac{e^{-u}}{-1} - 2x \cdot \frac{e^{-u}}{(-1)(-1)} + 2 \cdot \frac{e^{-u}}{(-1)(-1)(-1)} \right]_0^\infty = 1$

$\Rightarrow k \left[0 - 0 + 0 - 0 - 0 + 2 \cdot \frac{1}{-1} \right] = 1$

$\Rightarrow 2k = 1$

$\Rightarrow k = \frac{1}{2}$

$f(u) = \frac{1}{2} u^2 e^{-u}$

$E[u] = \int_0^\infty u \cdot \frac{1}{2} u^2 \cdot e^{-u}$

$= \frac{1}{2} \left[\left[u^3 \frac{e^{-u}}{-1} \right]_0^\infty - \int_0^\infty \frac{3u^2 e^{-u}}{-1}\, dx \right]$

$= \frac{1}{2} \left[0 + 3 \int_0^\infty u^2 e^{-u}\, dx \right]$

$= \frac{1}{2} \left[0 + 3 \times 2 \right]$

$= 3.$

$E[u^2] = \int_0^\infty u^2 \cdot \frac{1}{2} u^2 e^{-u^2}\, dx$

$= \frac{1}{2} \int_0^\infty u^4 e^{-u}\, dx$

Scanned by TapScanner

$$= \frac{1}{2}\left[\left[x^4 \frac{e^{-x}}{-1}\right]_0^\infty - \int_0^\infty 4x^3 e^{-x} dx\right]$$

$$= \frac{1}{2}\left[0 - 4\int_0^\infty x^3 e^{-x} dx\right]$$

$$= \frac{1}{2} \times 4 \times 6$$

$$= 12.$$

$$\text{Variance} = E[x^2] - (E[x])^2$$

$$= 12 - (3)^2$$

$$= 12 - 9$$

$$= 3 \quad \underline{\underline{\text{Ans}}}$$

Moment Generating function :-

If X is a random variable (discreate or continous) then moment generating function is defined as.

$$M(t) = E[e^{tx}].$$

$$= \begin{cases} \sum_{i=1} e^{tx_i} p_i & \text{if } x \text{ is discreat} \\ \int_{-\infty}^{\infty} e^{tx} f(x) & \text{if } x \text{ is continous} \end{cases}$$

$$M(t) = E[e^{tx}]$$

$$= E\left[1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \cdots + \frac{(tx)^r}{r!} + \cdots\right]$$

$\Rightarrow M[t] = 1 + \frac{t}{1!} E(x) +$

$$\Rightarrow M(t) = 1 + \frac{t}{1!} E[x] + \frac{t^2}{2!} E[x^2] + \frac{t^3}{3!} E[x^3] + \cdots$$

$$\cdots + \frac{t^r}{r!} E[x^r] + \cdots$$

$$1 + \frac{t}{1!} \mu_1' + \frac{t^2}{2!} \mu_2' + \cdots + \frac{t^r}{r!} \mu_r' + \cdots$$

$\mu_r' = E[x^r]$ = coefficient of $\frac{t^r}{r!}$ in the expansion of $M(t)$ in powers of t.

$$M'(t) = \frac{dm(t)}{dt} = \mu_1' + \frac{2t}{2!} \mu_2' + \frac{3t^2}{3!} \mu_3' + \cdots$$

Put $t = 0$ $M'(0) = \mu_1' = E[x]$

$$M''(t) = \mu_2' + \frac{6t}{3!} \mu_3' + \cdots$$

$$m''(0) = \mu_2'$$

$$\boxed{\left(\frac{d^r m(t)}{dt^r}\right)_{t=0} = \mu_r'}$$

Imp.

* NOTE:- If Moment generating function of x is $M_x(t)$ Then moment generating function of $y = ax+b$.

$$M_y(t) = E[e^{ty}]$$
$$= E[e^{t(ax+b)}]$$
$$= E[e^{tax}e^{tb}]$$
$$= e^{tb}E[e^{tax}]$$
$$\Rightarrow M_y(t) = e^{tb}M_x[at].$$

$\because M(t) = E[e^{tx}]$
$M(at) = E[e^{atx}].$

Q. If a random variable x has a m.g.f and $m(t) = \frac{3}{3-t}$ obtain the standard derivation of x.

Sol → $M(t) = \frac{3}{3-t} = \frac{3}{3(1-\frac{t}{3})} = \left(1-\frac{t}{3}\right)^{-1}$

$$\Rightarrow M(t) = 1 + \frac{t}{3} + \left(\frac{t}{3}\right)^2 + \left(\frac{t}{3}\right)^3 + - -$$

$\mu_r' =$ Coefficient of $\frac{t^r}{r!}$ in expansion of $M(t)$

$$\mu_1' = E[x] = \frac{1}{3}$$
$$\mu_2' = E[x^2] = \frac{2}{9}$$

Scanned by TapScanner

Standard deviation $= \sigma_n^2 = Var(x)$

$$= E[n^2] - (E[n])^2$$

$$= \frac{2}{9} - \frac{1}{9}$$

$$= \frac{1}{9}$$

$$\sigma = \sqrt{\frac{1}{9}} = \frac{1}{3}$$

Q If n represent the outcome when a fair dice is tossed. Find the m.g.f of n and hence find $E[n]$ and $Var(n)$.

Sol →

X	1	2	3	4	5	6
P(n)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$.

$$P(x=i) = \frac{1}{6} \quad i = 1, 2 - - - 6.$$

$$M(t) = E[e^{tx}]$$

$$M(t) = \sum_i e^{tx_i} p_i$$

$$= e^t \times \frac{1}{6} + e^{2t} \times \frac{1}{6} + e^{3t} \times \frac{1}{6} + e^{4t} \times \frac{1}{6} + e^{5t} \times \frac{1}{6} + \frac{e^{6t}}{6}.$$

$$M(t) = \frac{1}{6}\left(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}\right).$$

$$M'(t) = \frac{1}{6}\left[e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t}\right]$$

$$M'(t)_{t=0} = \frac{1}{6}[1+2+3+4+5+6]$$

$$= \frac{21}{6} = E[n]$$

Scanned by TapScanner

$$\mu_2^2 = M''(t)$$

$$= M'[M'(t)]$$

$$= \frac{1}{6}\left[e^t + 4e^{2t} + 9e^{3t} + 16e^{4t} + 25e^{5t} + 36e^{6t}\right]$$

$$\mu_2^2 = \frac{1}{6}[1+4+9+16+25+36]$$

$$= \frac{91}{6}.$$

$$\text{Variance} = E[x^2] - (E[x])^2$$

$$= \frac{91}{6} - \left(\frac{21}{6}\right)^2$$

$$= \frac{546 - 441}{36}$$

$$= \frac{105}{36} \text{ Ans}$$

Function of Random Variable

* How to find $f_y(y)$ when $f_x(x)$ is given?

① If $g(x)$ is strictly increasing function then $f_y(y) = f_x(x)\frac{dx}{dy}$ --- ①

② If $g(x)$ is strictly decreasing function then $f_y(y) = -f_x(x)\frac{dx}{dy}$ --- ②

$$\boxed{f_y(y) = f_x(x) \cdot \left|\frac{dx}{dy}\right|}$$

NOTE :- $f_y(y) = f_x(x) \cdot \left| \frac{1}{g'(x)} \right|$

Q. Given R.V x with density function.

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

find pdf of $y = 8x^3$.

Sol → $f_y(y) = f_x(x) \left| \frac{dy}{dx} \right|$.

$$y = 8x^3$$

$$\Rightarrow x^3 = \frac{y}{8} \Rightarrow x = \left(\frac{y}{8}\right)^{1/3}$$

$$\Rightarrow x = \frac{y^{1/3}}{2} \qquad \frac{dx}{dy} = \frac{1}{2} \times \frac{1}{3} y^{-2/3}.$$

$$f_y(y) = f_x(x) \cdot \left| \frac{dy}{dx} \right|.$$

$$= 2x \cdot \left| \frac{dy}{dx} \right|$$

$$= 2 \cdot \frac{y^{1/3}}{2} \cdot \frac{1}{6} y^{-2/3}$$

$$= \frac{1}{6} y^{-1/3}, \quad 0 < y < 8$$ Ans

Question for homework :-

1) $f_x(x) = 2x \quad 0 < x < 1$. find the pdf of $y = 3x+1$

2) $f_x(x) = e^{-x} \quad x \geq 0$, find the pdf of $y = 2x+1$

Tchebycheff's Inequality :-

If X is a RV with $E(X) = \mu$ and $Var(X) = \sigma^2$

then $P(|X-\mu| \geq c) \leq \frac{\sigma^2}{c^2}$. where $c > 0$.

$$1 - P[|x-\mu| < c] \leq \frac{\sigma^2}{c^2}$$

$$\Rightarrow P(|x-\mu| < c] \geq 1 - \frac{\sigma^2}{c^2}$$

for $c = k\sigma$

$$P[|x-\mu| \geq k\sigma] \leq \frac{\sigma^2}{\sigma^2 k^2}$$

$$P[|x-\mu| \geq k\sigma] \leq \frac{1}{k^2}$$

Q. X is a RV with $\mu = 12$, $\sigma^2 = 9$. find probability $P(6 < x < 18)$.

Sol→ $P[|x-\mu| \geq c] \leq \frac{\sigma^2}{c^2}$

$$P[|x-12| \geq c] \leq \frac{9}{c^2}$$

$$1 - P[|x-12| < c] \leq \frac{9}{c^2}$$

$$P[|x-12| < c] \geq 1 - \frac{9}{c^2}$$

$$\therefore P[-c < x-12 \leq c] \geq 1 - \frac{9}{c^2}$$

$\Rightarrow P\left[-c+12 \leq x \leq 12+c\right] \geq 1-\frac{9}{c^2}$.

Put $c = 6$.

$P(6 < x < 18) \geq 1 - \frac{9}{36}$.

$P(6 < x < 18) \geq \frac{27}{36}$

$P(6 < x < 18) \geq \frac{3}{4}$

Q. If the R.V X is uniformly distributed over $(-\sqrt{3}, \sqrt{3})$, compute $P\left\{|x-\mu| \geq \frac{3\sigma}{2}\right\}$ and compare the result with the upper bound obtained by Tchebycheff's inequality.

Sol→ If X is uniformly distributed in the interval (a, b) then pdf of x.

$$f(x) = \frac{1}{b-a}$$

$$\text{Mean} = E[x] = \frac{a+b}{2}$$

$$\text{Variance} = \sigma^2 = \frac{1}{12}(b-a)^2$$

$$\therefore f(x) = \frac{1}{\sqrt{3}-(-\sqrt{3})} = \frac{1}{2\sqrt{3}}$$

$$\text{Mean} = \frac{\sqrt{3}-\sqrt{3}}{2} = 0$$

$$\text{Variance} = \frac{1}{12}(2\sqrt{3})^2 = 1$$

$$P\left[|u-\mu| \geq \frac{3\sigma}{2}\right]$$

$$\Rightarrow P\left[|u-0| \geq \frac{3 \cdot 1}{2}\right]$$

$$= P\left[|u| \geq \frac{3}{2}\right].$$

$$= 1 - P\left[|u| < \frac{3}{2}\right]$$

$$= 1 - P\left[-\frac{3}{2} < X < \frac{3}{2}\right].$$

$$= 1 - \int_{-3/2}^{3/2} \frac{1}{2\sqrt{3}}\, dx = 1 - \frac{1}{2\sqrt{3}}\left[\frac{3}{2} + \frac{3}{2}\right]$$

$$= 1 - \frac{\sqrt{3}}{2}.$$

$$= 0.134.$$

$$P(|u-\mu| \geq c) \leq \frac{\sigma^2}{c^2}.$$

$$c = k\sigma.$$

$$P(|u-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

$$P\left(|u-0| \geq \frac{3}{2}\sigma\right) \leq \frac{1}{\left(\frac{3}{2}\right)^2}.$$

$$\Rightarrow P\left(|u| \geq \frac{3}{2}\sigma\right) \leq \frac{4}{9}.$$

$$= 0.444.$$

Scanned by TapScanner

Q Can we find a R.V x for which $P(\mu-2\sigma < x < \mu+2\sigma) = 0.6$.

Sol→ $P[\mu-2\sigma \le x \le \mu+2\sigma] = P[|x-\mu| \le 2\sigma]$

$1 - P[|x-\mu| > 2\sigma] < \frac{1}{2^2}$

$P[|x-\mu| > 2\sigma] \ge 1 - \frac{1}{2^2}$

$P(|x-\mu| > 2\sigma] \ge 0.75$

$P(|x-\mu| \le 2\sigma) + P(|x-\mu| > 2\sigma) \le 1$

$P(|x-\mu| \le 2\sigma) + 0.75 \le 1$

$P(|x-\mu| \le 2\sigma) \le 0.25$

$\Rightarrow$ There does not exist a R.V X satisfying given condition.

Q. A discreat R.V. x takes the values $[-1, 0, 1]$ with probability $\frac{1}{8}, \frac{3}{4}, \frac{1}{8}$ respectively. Evaluat probability $P[|x-\mu| \ge 2\sigma]$ and compare it with upper bound of Tchebycheff's Inequality.

Sol→

X	-1	0	1
$P(x)$	$\frac{1}{8}$	$\frac{3}{4}$	$\frac{1}{8}$

$$\text{Mean} = -1\times\frac{1}{8} + 0\times\frac{3}{4} + 1\times\frac{1}{8} = 0$$

$$E[x^2] = (1)\times\frac{1}{8} + 0 + \frac{1}{8} = \frac{1}{4}.$$

$$\text{Variance} = 0.25 = \frac{1}{4}.$$

$$\sigma = \sqrt{\text{variance}} = \sqrt{\frac{1}{4}}$$

$$\sigma = \frac{1}{2}.$$

$$P[|x-\mu| \geq 2\sigma] = P[|x-0| \geq 2\times\frac{1}{2}]$$

$$= P[|x| \geq 1]$$

$$\Rightarrow 1 - P[|x| < 1].$$

$$\Rightarrow 1 - P(x=0)$$

$$\Rightarrow 1 - \frac{3}{4} = \frac{1}{4}$$

$$P(|x-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

$$P(|x-\mu| \geq 2\sigma) \leq \frac{1}{2^2}$$

$$P(x-\mu \geq 2\sigma) \leq \frac{1}{4}$$

So, the two values concides.

<u>Question</u> <u>for</u> <u>homework</u>:-

* Question 5 & 6 of page 190