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Section : 02

Probability and Queuing Theory

Assignment - 1

1) Suppose the life in hours of a certain part of a radio tube is a continuous random variable X with PDF given by

$$f(x) = \begin{cases} \frac{100}{x^2}, & x \geq 100 \\ 0, & \text{elsewhere} \end{cases}$$

i) What is the probability that all of three such tubes in a given radio set will have to be replaced during the first 150 hours of operation? ii) What is the probability that none of three of the original tubes will have to be replaced during the first 150 hours of operation? iii) What is the probability that a tube will last less than 200 hours if it is known that the tube is still functioning after 150 hours of service?

⇒ i) A tube in the radio set will have to be replaced during the first 150 hours if its life is less than 150 hours. Hence the required probability is :

$$\therefore P(X \leq 150) = \int_{100}^{150} f(x)\,dx = \int_{100}^{150} \frac{100}{x^2}\,dx$$

$$= \left(\frac{-100}{x}\right)_{100}^{150} = \frac{-100}{150} + 1 = \frac{1}{3}$$

The probability that all three of the original tubes will have to be replaced during the first 150 hours is :

$$\left(\frac{1}{3}\right)^3 = \frac{1}{27}$$

ii) The probability that a tube is not replaced is given by $P(X > 150)$

$$= 1 - P(X \leq 150)$$

$$= 1 - \frac{1}{3}$$

$$= \frac{2}{3}$$

Hence, the probability that none of the three tubes will be replaced during the 150 hours of operation is $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$

iii) $P[(x<200) \mid (x>150)]$

$$= \frac{P(150<x<200)}{P(x>150)}$$

$$= \frac{\int_{150}^{200} \frac{100}{x^2}\,dx}{\int_{150}^{\infty} \frac{100}{x^2}\,dx} = \frac{\left[\frac{-100}{x}\right]_{150}^{200}}{\left[\frac{-100}{x}\right]_{150}^{\infty}}$$

$$= \frac{-\left[\frac{100}{200} - \frac{100}{150}\right]}{-\left[0 - \frac{100}{150}\right]} = \frac{\frac{1}{6}}{\frac{2}{3}} = \frac{1}{4}$$

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2) The following is the cumulative distribution function of a discrete random variable X :

x	-3	-1	0	1	2	3	5	8
$F(x)$	0.10	0.30	0.45	0.5	0.75	0.90	0.95	1.00

Find (i) the probability distribution of X ii) P(X is even)
iii) $P(X = -3 \mid X < 0)$ iv) P(|X| greater than or equal to 1)

⇒ i) From the given table,

$F(-3) = 0.10$, $F(-1) = 0.30$, $F(0) = 0.45$, $F(1) = 0.5$,
$F(2) = 0.75$, $F(3) = 0.90$, $F(5) = 0.95$, $F(8) = 1.00$

$P(X=-3) = F(-3) = 0.10$
$P(X=-1) = F(-1) - F(-3) = 0.30 - 0.10 = 0.20$
$P(X=0) = F(0) - F(-1) = 0.45 - 0.30 = 0.15$
$P(X=1) = F(1) - F(0) = 0.5 - 0.45 = 0.05$
$P(X=2) = F(2) - F(1) = 0.75 - 0.5 = 0.25$
$P(X=3) = F(3) - F(2) = 0.90 - 0.75 = 0.15$
$P(X=5) = F(5) - F(3) = 0.95 - 0.90 = 0.05$
$P(X=8) = F(8) - F(5) = 1.00 - 0.95 = 0.05$

$X=x$	-3	-1	0	1	2	3	5	8
$P(X=x)$	0.10	0.20	0.15	0.05	0.25	0.15	0.05	0.05

ii) $P(X \text{ is even})$

$= P(X = 0 \text{ or } X = 2 \text{ or } X = 8)$

$= P(X=0) + P(X=2) + P(X=8)$

$= 0.15 + 0.25 + 0.05$

$= 0.45$

iii) $P(X=-3 \mid X<0)$

$= \dfrac{P(X=-3) \cancel{\text{ or } X=-3}}{P(X=-3 \text{ or } X=-1)}$

$= \dfrac{0.10}{0.10+0.20}$

$= \dfrac{0.10}{0.30} = \dfrac{1}{3}$

iv) $P(|X| \text{ greater than or equal to } 1)$

$= P(|X| \geq 1)$

$= P(X=1 \text{ or } X=2 \text{ or } X=3 \text{ or } X=5 \text{ or } X=8)$

$= 0.05 + 0.25 + 0.15 + 0.05 + 0.05$

$= 0.55$

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3) A Random Variable X has a probability density function $f(x) = k(x-x^2)$, $0 \leq x \leq 1$. Find k, μ_r' and hence find the first four central moments.

=> Since, $f(x)$ is a density function,

$\int_0^1 f(x)\,dx = 1$ or, $k\int_0^1 (x-x^2)\,dx = 1$

i.e. $k\left[\dfrac{x^2}{2} - \dfrac{x^3}{3}\right]_0^1$ or, $\dfrac{k}{6} = 1$ $\therefore k = 6$

$\therefore \quad \mu_r' = E[x^r] = \int_{-\infty}^{\infty} x^r f(x)\, dx$

$$= \int_0^1 x^r \cdot 6\,(x - x^2)\, dx$$

$$= 6\int_0^1 (x^{r+1} - x^{r+2})\, dx$$

$$= 6\left[\frac{x^{r+2}}{r+2} - \frac{\cancel{x}^{r+3}}{r+3}\right]_0^1$$

$$= 6\left[\left(\frac{1}{r+2} - \frac{1}{r+3}\right) - (0-0)\right]$$

$$= 6\;\frac{[r+3-r-2]}{(r+2)(r+3)}$$

$$= \frac{6}{(r+2)(r+3)}$$

Then,

$\therefore \quad \mu_1' = \dfrac{6}{(1+2)\,(1+3)} = \dfrac{1}{2}$

$\therefore \quad \mu_2' = \dfrac{6}{(2+2)\,(r+3)} = \dfrac{3}{10}$

$\therefore \quad \mu_3' = \dfrac{6}{(3+2)\,(3+3)} = \dfrac{1}{5}$

$\therefore \quad \mu_4' = \dfrac{6}{(4+2)(4+3)} = \dfrac{1}{7}$

Now,

$\therefore \quad \mu_1 = E(x - \bar{x})$

$$= E(x) - E(\bar{x})$$

$$= \bar{x} - \bar{x}$$

$$= 0$$

$\therefore \mu_2 = \mu_2' - (\mu_1')^2$

$= \frac{3}{10} - \left(\frac{1}{2}\right)^2$

$= \frac{1}{20}$

$\therefore \mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$

$= \frac{1}{5} - 3 \times \frac{3}{10} \times \frac{1}{2} + 2 \times \left(\frac{1}{2}\right)^3$

$= 0$

$\therefore \mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$= \frac{1}{7} - 4 \times \frac{1}{5} \times \frac{1}{2} + 6 \times \frac{3}{10} \times \left(\frac{1}{2}\right)^2 - 3 \times \left(\frac{1}{2}\right)^4$

$= \frac{3}{560}$

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4) Find the MGF of the random Variable X if its probability density function is given by $f(x) = \frac{1}{2} e^{-|x|}, -\infty < x < \infty$. Hence find the first four central moments.

$\Rightarrow M_x(t) = E(e^{xt})$

$= \int_{Rx} e^{xt} f(x)\, dx$

$= \int_{-\infty}^{\infty} e^{xt} \cdot \frac{1}{2} e^{-|x|}\, dx$

$= \frac{1}{2}\left[\int_{-\infty}^{0} e^{xt} \cdot e^{-(-x)}\, dx + \int_{0}^{\infty} e^{xt} \cdot e^{-(x)}\, dx\right]$

$= \frac{1}{2}\left[\int_{-\infty}^{0} e^{x(1+t)}\, dx + \int_{0}^{\infty} e^{x(t-1)}\, dx\right]$

$= \frac{1}{2}\left\{\left[\frac{e^{x(1+t)}}{1+t}\right]_{-\infty}^{0} + \left[\frac{e^{-x(1-t)}}{-(1-t)}\right]_{0}^{\infty}\right\}$

$= \frac{1}{2} \left\{\left(\frac{1}{1+t} - 0\right) + \left(0 + \frac{1}{1-t}\right)\right\}$

$$= \frac{1}{2}\left[\frac{1}{1+t} + \frac{1}{1-t}\right]$$

$$= \frac{1}{2}\left[\frac{1-t+1+t}{(1-t)(1+t)}\right]$$

$$= \frac{1}{2} \times \frac{2}{1-t^2}$$

$$= \frac{1}{1-t^2}$$

Then,

$$\therefore M'_x(t) = \frac{d}{dt}\left(\frac{1}{1-t^2}\right) = -1(1-t^2)^{-2} \cdot (-2t) = 2t(1-t^2)^{-2}$$

$$\therefore M''_x(t) = \frac{d}{dt}\left(\frac{2t}{(1-t^2)^2}\right) = \frac{2(1-t^2)^2 - 2(1-t^2)\cdot(-2t)\cdot 2t}{(1-t^2)^4}$$

$$\cancel{= \frac{2-2t^2+8t^2-8t^4}{(1-t^2)^2}}$$

$$\cancel{= \frac{2+6t^2-8t^4}{(1-t^2)^2}}$$

$$= \frac{2(1-4t^2+2t^4+8t^2-8t^4}{(1-t^2)^4}$$

$$= \frac{2+4t^2-6t^4}{(1-t^2)^4} = \frac{2(1-t^2)(1+3t^2)}{(1-t^2)^4} = \frac{2+6t^2}{(1-t^2)^3}$$

$$\cancel{\therefore M'''_x(t) = \frac{d}{dt}\left(\frac{2+4t^2-6t^4}{(1-t^2)^4}\right)}$$

$$\cancel{= \frac{(8t-24t^3)(1-t^2)^4-(2+4t^2-6t^4)\cdot 4(1-t^2)^3\cdot(-2t)}{(1-t^2)^8}}$$

$$\cancel{= \frac{(1-t^2)^3(8t-8t^3-24t^3+24t^5+16t+32t^3-48t^5)}{}}$$

$$\therefore M\,x(t) = \frac{d}{dt}\left(\frac{2+6t^2}{(1-t^2)^3}\right)$$

$$= \frac{12t\,(1-t^2)^3 - (2+6t^2)\,3.(1-t^2)^2(-2t)}{(1-t^2)^6}$$

$$= \frac{(1-t^2)^2\,(12t-12t^3-12t+36t^2)}{(1-t^2)^6}$$

$$= \frac{36t^2-12t^3}{(1-t^2)^4}$$

$$\therefore M''''_x(t) = \frac{d}{dt}\left(\frac{36t^2-12t^3}{(1-t^2)^4}\right)$$

$$= \frac{(72t-36t^2)(1-t^2)^4 - (36t^2-12t^3)\,4(1-t^2)^3.(-2t)}{(1-t^2)^8}$$

$$= \frac{(1-t^2)^3\,(72t-72t^3-36t^2+36t^4+288t^3-96t^4)}{(1-t^2)^8}$$

$$= \frac{72t-36t^2+216t^3-60t^4}{(1-t^2)^5}$$

Again,

$\therefore \mu_1' = M_x'(0) = 0$

$\therefore \mu_2' = M_x''(0) = 2$

$\therefore \mu_3' = M_x'''(0) = 0$

$\therefore \mu_4' = M_x''''(0) = 0$

Now

$$\therefore \mu_1 = E(x-\mu)$$

$$= E(x) - \mu$$

$$= \mu - \mu$$

$$= 0$$

$\therefore \mu_2 = \mu_2' - (\mu_1')^2$

$= 2 - 0^2 = 2$

$\therefore \mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$

$= 0 - 3\times2\times0 + 2\times0$

$= 0$

$\therefore \mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$= 0 - 4\times0\times0 + 6\times2\times0 - 3\times0$

$= 0$

Hence, these are the four central moments.

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5. A random variable X takes the values $-1, 1, 3, 5$ with associated probabilities $\frac{1}{6}, \frac{1}{6}, \frac{1}{6}$, & $\frac{1}{2}$. Find by direct computation $P(|X-3| \geq 3)$. Also, find an upper bound to this probability by applying Tchebycheff's inequality.

$\Rightarrow$ $P(X=-1) = \frac{1}{6}$, $P(X=1) = \frac{1}{6}$, $P(X=3) = \frac{1}{6}$ and $P(X=5) = \frac{1}{2}$

$\therefore \mu = E(x) = -1\times\frac{1}{6} + 1\times\frac{1}{6} + 3\times\frac{1}{6} + 5\times\frac{1}{2} = 3$

$\therefore E(x^2) = (-1)^2\times\frac{1}{6} + 1^2\times\frac{1}{6} + 3^2\times\frac{1}{6} + 5^2\times\frac{1}{2} = \frac{43}{3}$

$\therefore Var(X) = \sigma^2 = E(X^2) - \{E(X)\}^2 = \frac{43}{3} - 3^2 = \frac{16}{3}$

Then,

$\therefore P(|X-3| \geq 3) = P(-\infty \leq X-3 \leq -3) + P(3 \leq X-3 \leq \infty)$

$= P(X \leq 0) + P(6 \leq X)$

$= P(X=-1)$

$= \frac{1}{6}$

From Tchebycheff's Inequality,

$$\therefore P(|X-3| \geq 3) \leq \frac{\sigma^2}{c^2}$$

$$= \frac{16}{3 \times (3)^2}$$

$$= \frac{16}{27}$$

The two values do not coincide.