Bayesian Time Series Analysis of US Unemployment Rates

STAT 447C Final Project - Project Report | Rohan Joseph (67089839)

Project GitHub URL: https://github.com/RohanUBC/STAT-447C-Final-Project

Introduction

The forecasting of US unemployment rates presents a complex and significant challenge for US policy makers. Accurate predictions of these rates enables the US government and businesses to make informed economic decisions, adapt to economic shifts, and implement strategies to foster stability and growth in the US economy. Furthermore, the unpredictability of labor markets, influenced by factors ranging from global economic conditions to domestic policy changes, necessitates the use of sophisticated methods not only provide a framework for forecasting but also allow for the quantification of uncertainty in those forecasts. Understanding and predicting unemployment rates is thus not only central to economic analysis but also to the well-being of the US economy.

The focus of this project is to model and forecast unemployment rates in the United States (US), utilizing a real-world dataset that has documented the monthly unemployment rate (from 01/1948 to 02/2024) in the US. A further aim is to quantify the uncertainty of the forecast. This is done to ensure that the forecasts are not only precise, but are also accompanied by reliable uncertainty measures. This approach aims to assess potential risks effectively, thereby facilitating more informed and prudent economic decision making. The dataset that is being used is from the US Federal Reserve on monthly US unemployment rates.

URL: https://fred.stlouisfed.org/series/UNRATE

We will adopt a Bayesian time series analysis approach, utilizing a Bayesian Structural Time Series (BSTS) model to capture the dynamics of the US labor market based on the provided data. This involves uncovering and modeling historical trends and seasonal patterns within the data, providing probabilistic forecasts of future unemployment rates. We opt for BSTS due to its flexibility in modeling complex time series data with trend and seasonality components. BSTS allows for the estimation of latent states, such as trend and seasonal effects, and provides a framework for forecasting future states while quantifying uncertainty. Please see Appendix C on a general overview on the BSTS model.

This project will face challenges in applying Bayesian time series analysis to economic data, particularly in modeling the seasonal patterns and long-term trends inherent in unemployment rates. The primary challenge lies in identifying the specific type of model, as well its specific components. This would require diving into the inherent structure of the data, while also validating the model's performance through error metrics and posterior predictive checks.

Literature Review

- Summarize key findings from existing literature on Bayesian forecasting and unemployment rate analysis.
- Highlight gaps your project aims to fill or how it builds upon previous work.
- Discussion of uncertainty quantification in economic forecasting.
- Calibration of probabilistic forecasts in existing literature.

[To be added]

Data Analysis

Framework of Model

As established in Appendix B, the general BSTS model consists of two sets of equations: the observational (response variable) and the state-space equations (how the parameters change over time). In order to do this, we would need to analyse a decomposition of the time series, the autocorrelation (ACF) plot, and the partial autocorrelation (PACF) plot, of the time series. This would also help us consider want components of a time series should be included in the model (for e.g., trend, seasonality, auto-regressive, etc.). All of this will be done on the training set and all code is available in Appendix B.

NOTE: We will be using all the data up until 02/2023 as the training set, and the remaining data (from 02/2023 to 02/2024) as the testing set (see Appendix B - Section 1).

Key Definitions:

Decomposition of a Time Series: This involves separating a time series into its constituent components, such as trend, seasonality, and noise, to better understand its underlying patterns and behaviors.

Autocorrelation: This is the correlation between a time series and its own lagged values, revealing how past observations relate to present or future values within the same series.

Autocorrelation Function (ACF) Plot: This is a visual representation of the correlation between a time series and its lagged values at different lags, which helps to identify patterns of autocorrelation and inform the selection of appropriate models for time series analysis.

Partial Autocorrelation Function (ACF) Plot: This shows the correlation between observations at different time lags, accounting for the influence of shorter lags, helping identify the direct effects of specific lags on the current observation in a time series.

Summary of the Decomposition of the Time Series

Additive Decomposition (Refer to Figure 3 in Appendix C) - The additive decomposition a seasonal component with constant amplitude, aligning with the typical assumptions or an additive model where seasonal effects do not scale with the level of the time series. The random (residual) component is fairly homoscedastic, indicating the potential effectiveness of the additive model in encompassing most fluctuations within the time series.

Multiplicative Decomposition (Refer to Figure 4 in Appendix C) - The multiplicative decomposition displays the seasonal component's amplitude slightly changing in proportion to the trend, which is characteristic of multiplicative models. This suggests that as the time series level changes, the seasonal effect scales accordingly.

While both decompositions indicates elements of their respective counterparts: the additive model shows some periods where amplitude changes slightly, while the multiplicative model demonstrates periods where amplitude remains constant. However, the additive model consistently presents a clearer pattern in unemployment rates across economic cycles with a constant seasonal amplitude. However, considering that the seasonal component is a crucial part of the BSTS model's structure. we plan to test both additive and multiplicative models in order to select the best one.

Trend Component (Refer to Section 5 in Appendix C - After decomposing the trend component and constructing a general linear regression of the trend, we observe that the he intercept and slope values are statistically significant at the 95% confidence interval as they both have p-values of $< 2 \times 10^{-16}$. However, given the the low R-squared value of 0.1133, this may suggest that the trend component may not be a reliable indicator for making future predictions. To address this, we will testing models with a local-linear trend, semi-local-linear trend, and even no trend.

AR(p) Process (Refer to Sections 6 and 7, Figures 5 and 6 in Appendix C) - Given a After looking at the ACF and PACF plots for the training data, we can see that there is significant autocorrelation at multiple lags, so much so that it would not make sense to simply isolate a few. However, the `bsts` library allows the data to set the lag values through its `AutoAr()` function, which can be added to the state-space for any models being constructed. By doing this, we are acknowledging the broader range of autocorrelation patterns revealed by the ACF plot. Hence, all models will have an auto AR process.

Model Selection Strategy

Based on the decomposition of the time series, we will be testing 6 unique combinations of model type (additive/multiplicative) and the presence of a trend component (none/semi-local linear/local linear), before selecting the best 2 out 6 to then further evaluate using posterior predictive checks. We will employ a hybrid model selection strategy that combines frequentest and

Bayesian validation methods, ensuring a robust framework for forecasting unemployment rates, as well as a comprehensive measures of the uncertainty in any forecasts. We will be adding both a seasonal and auto-regressive components to all models.

Models being Tested: "Additive with no trend", "Additive with semi-local linear trend", "Additive with local linear trend", "Multiplicative with no trend", "Multiplicative with semi-local linear trend", "Multiplicative with local linear trend".

Initial Model Selection

In the initial phase of models selection, each model was evaluated based on a series of metrics, after which a composite score was calculated in order to facilitate an objective comparison. The criteria used included a blend of statistical information and error metrics. Please see Appendix D - Section 1 to see code used to build the different models.

Criteria Used for Evaluation

AIC (Akaike Information Criterion): The AIC measures the quality of a model by assessing the trade-off between the goodness of fit and the complexity of the model, penalizing excessive complexity to avoid overfitting.

BIC (Bayesian Information Criterion): The BIC also evaluates model quality, but with a stricter penalty for complexity, often favoring simpler models, especially as the sample size increases.

MAE (Mean Absolute Error), RMSE (Root Mean Squared Error), and MAPE (Mean Absolute Percentage Error): These error metrics provide insights into the average error magnitude, error variance, and error proportionality respectively, contributing to a comprehensive understanding of model accuracy.

Results of the Initial Model Selection

The initial selection process ranked the six models based on the computed scores, Please see Appendix D - Section 2 to see how the model's and their associated statistics were built and compiled, and Appendix D - Section 3 to see the code used for the decision matrix and the table of composite scores.

The model with an "additive" model type and a "semi-local linear" trend component scored the highest. This model achieved the best balance between complexity and fit with the lowest MAE, MAPE, and RMSE, suggesting high accuracy and efficiency in forecasts. In addition, the "additive" model with a "local linear" trend component has the second highest composite score. The other four models performed well as seen by their good performance across the error metrics, However, the multiplicative models were ranked lower primarily due to their higher model complexity (AIC and BIC scores).

Both the "additive" and "multiplicative" models without a trend component showed the highest error metrics and the worst information criteria scores. This implied that there was significant under fitting, and also highlights their inability in capturing the underlying patterns in the unemployment data.

Selected Models for Further Analysis

Additive Model with Local Linear Trend: This model offers the best balance between complexity and accuracy, making it the primary candidate for further validation using Bayesian posterior predictive checks.

Additive Model with Semi-Local Linear Trend: This model also provides reasonable balance between complexity and accuracy, while providing a valuable point of comparison to our initial model choice.

Model Validation and Comparison

In order to analyse and compare the two selected model, we will begin our analysis by generating the plots of the posterior predictive distribution for each model (see Appendix E). Visualizing the distributions serves to provide insights into the models' ability to capture the structure of the data and predict future values. Following this, we will conduct a cross-validation study to obtain a more rigerous statistical evaluation of the models' predictive performance (see Appendix F). The cross-validation metrics includes the Mean Absolute Error (MAE), Mean Absolute Percentage Error (MAPE), Root Mean Square Error (RMSE), and the Average Credible Intervals Width (Avg. CI Width).

Posterior Prediction Distribution and Credible Intervals Visual Analysis Results

The Additive Semi-Local Linear Model displayed a more faithful following of the actual unemployment trend, as indicated by the overlap of the predicted and actual lines in the plot (see Figures 7 and 8 in Appendix E). Its credible intervals were also consistently narrower than those of the Additive Local Linear Model, denoting higher precision.

Cross Validation Study Results

The results of the cross-validation study revealed that the Additive Semi-Local Linear Model outperformed the Additive Local Linear Model across all the metrics, as seen by its lower MAE, MAPE, and RMSE values, suggesting that its predictions are closer to the actual values and generally more accurate. Furthermore, it presented a smaller Average Credible Intervals Width, which signifies a higher average precision of the model's forecasts. Furthermore, the plots of the posterior predictive distribution supported the results of the study, with the Additive Semi-Local Linear Model showing narrower credible intervals and the predicted values tracking the actual data more closely than the Additive Local Linear Model.

Final Model

Based on the combined evidence from the posterior predictive distribution analysis and the cross-validation study, we select the *Additive Semi-Local Linear Model* as the best model for forecasting US Unemployment rates. This model not only offers greater accuracy but also provides more precise predictions, which is critical for reliable forecasting. Furthermore, the narrower credible intervals suggest that the forecasts made by the Semi-Local Linear Model carry less uncertainty, a valuable attribute for making informed decisions in uncertain environments.

Quantifying Uncertainty

In order to quantify the uncertainty of our forecasts, we achieved this by analyzing a variety of error metrics and the calculation of credible intervals. They served as important pieces of evidence in both the initial model selection, and the model validation and comparison phase. By carefully considering both the expected error metrics and the credible range of the forecasts, we are able to provide a holistic view of the forecast uncertainty, allowing for more informed decision-making processes, while taking into account both the precision and the potential variability of future estimates.

BSTS Model for US Unemployment Rates

We will now create a mathematical representation of the additive semi-local linear trend model using the Bayesian framework.

Observation Model: The observed unemployment rate at time t, y_t, is modeled as:

$$y_t \sim \text{Normal}(\mu_t + s_t + X_t, \sigma_y^2)$$

where μ_t represents the trend component, s_t denotes the seasonal component, X_t denotes the autoregressive process, and σ_y^2 is the observation variance.

Semi-Local Level (Trend) Component:

The semi-local component μ_t and its slope δ_t evolves over time as:

$$\mu_t \sim \text{Normal}(\mu_{t-1} + \delta_t, \sigma_\mu^2), \qquad \text{where } \sigma_\mu^2 \sim \text{Inverse-Gamma}(2, 1)$$

$$\delta_t \sim \text{Normal}(D + \rho(\delta_{t-1} - D), \sigma_\delta^2), \qquad \text{where } D = 0.0013537,\ \rho \sim \text{Uniform}(0, 1),\ \text{and } \sigma_\delta^2 \sim \text{Inverse-Gamma}(2, 1)$$

where σ_μ^2 is the variance of the semi-local component, D is the long-term slope (as calculated in Appendix C - Section 5), ρ is the autoregressive behavior of the trend, δ_{t-1} is the trend slope at the previous time point, and σ_δ^2 is the variance of the trend slope.

Seasonal Component:

The seasonal component s_t evolves over time as:

$$s_t \sim \text{Normal}\left(-\frac{1}{S-1}\sum_{i=1}^{S-1} s_{t-i}, \sigma_s^2\right), \qquad \text{where } s_1, s_2, ..., s_{12} \sim \text{Normal}(0, \sigma_s^2),\ \text{and } \sigma_\mu^2 \sim \text{Inverse-Gamma}(2, 1)$$

where S is the seasonal period (in this case $S = 12$) for monthly, s_{t-i} is the seasonal component as the previous time point, σ_s^2 is the variance of the seasonal component, and $s_1, s_2, ..., s_{12}$ are the initial seasonal component values (associated with every month of a year). We have the factor $\frac{1}{S-1}$ to ensure that the seasonal effects average out to zero over one full cycle.

Autoregressive Component:

The autoregressive process X_t evolves over time as:

$$X_t = \sum_{i=1}^{p} \phi_i \cdot X_{t-i} + \epsilon_t, \qquad \text{where } \epsilon_t \sim \text{Normal}(0, \sigma_X^2), \text{ and } \sigma_X^2 \sim \text{Inverse-Gamma}(2, 1)$$

where ϕ_i are the coefficients for the lags $i = 1, 2, ..., p$ and ϵ_t denotes the white noise of the AR(p) term. The order p of the autoregressive process is not pre-specified but is determined by the `AutoAr()` function, allowing the data itself to identify the most appropriate lags

Justification of Hyperparameters and Hyperpriors

This model uses weakly informative priors and data-driven specifications to robustly forecast US Unemployment rates, while providing a quantification of its forecast uncertainty. The model components are tailored to evolve based on historical data patterns, ensuring that predictions are both stable and sensitive to underlying trends. Furthermore, by adopting a flexible approach to the autoregressive order and the slope's persistence, the model remains adaptable to various economic scenarios.

Discussion

TODO

References

TODO

Appendix

Appendix A - Preliminary Data Analysis

Reading in the Dataset and Data Cleaning

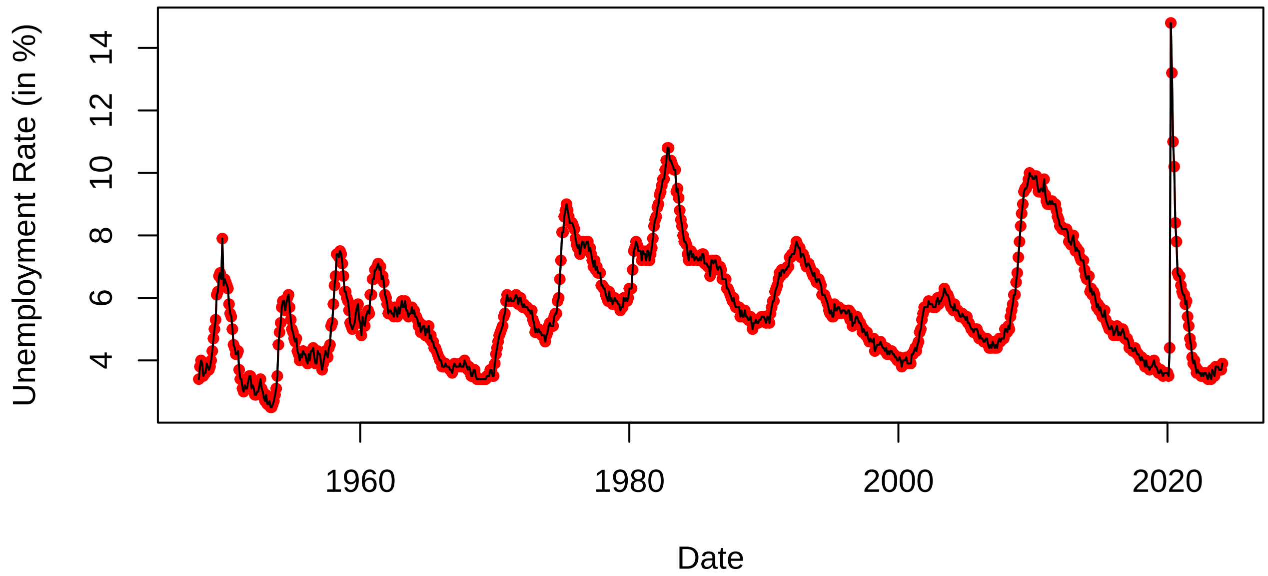
```
dat <- read_csv("UNRATE.csv")
dat$DATE <- as.Date(dat$DATE)
head(dat)
```

```
## # A tibble: 6 x 2
##   DATE       UNRATE
##   <date>      <dbl>
## 1 1948-01-01    3.4
## 2 1948-02-01    3.8
## 3 1948-03-01    4
## 4 1948-04-01    3.9
## 5 1948-05-01    3.5
## 6 1948-06-01    3.6
```

Historical US Unemployment Rates from 01/1948 to 02/2024

Figure 1: Historical US Unemployment Rate Over Time

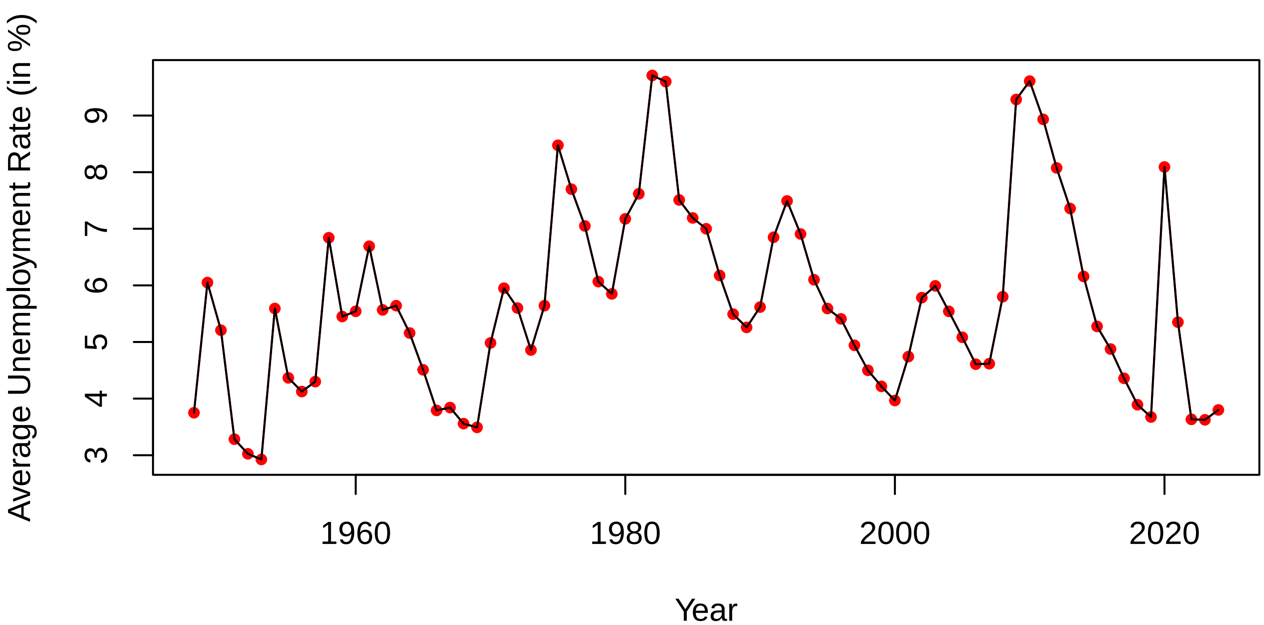
```
plot(dat$DATE, dat$UNRATE,
     type = "o", pch = 20,
     col = "red",
     xlab = "Date",
     ylab = "Unemployment Rate (in %)")
lines(dat$DATE, dat$UNRATE, col = "black")
```



Average Yearly US Unemployment Rates from 01/1948 to 02/2024

Figure 2: Average Yearly Unemployment Rate in the US

```
avg_yearly_unrate <- dat |>
  mutate(YEAR = format(DATE, "%Y")) |>
  group_by(YEAR) |>
  summarise(Average_Unemployment_Rate = mean(UNRATE, na.rm = TRUE))

avg_yearly_unrate$YEAR <- as.numeric(as.character(avg_yearly_unrate$YEAR))

plot(avg_yearly_unrate$YEAR, avg_yearly_unrate$Average_Unemployment_Rate,
     type = "o", pch = 20,
     col = "red",
     xlab = "Year",
     ylab = "Average Unemployment Rate (in %)")
lines(avg_yearly_unrate$YEAR, avg_yearly_unrate$Average_Unemployment_Rate, col = "black")
```



Appendix B - General Overview on the Bayesian Structural Time Series (BSTS) Model

The Bayesian Structural Time Series (BSTS) model is a statistical method that serves several applications such as feature selection, time series forecasting, and causal inference analysis. It operates on time series data to ascertain underlying patterns and forecast future data points.

BSTS models are composed of three primary elements:

1. *Kalman Filter:* A method for decomposing time series into components like trend and seasonality, allowing state variables to be modeled dynamically over time.
2. *Spike-and-Slab Prior:* A technique for feature selection that identifies which predictors in a regression are most informative.
3. *Bayesian Model Averaging (BMA):* A process where multiple models are averaged together to produce predictions or infer parameters, accounting for model uncertainty. In the BSTS framework, BMA is utilized extensively to generate samples from repeated Markov Chain Monte Carlo (MCMC) simulations into final model outputs, providing a comprehensive prediction that encompasses model variability.

A general BSTS model consists of twos set of equations,

1. *Observational equation:* Response variable as a function of predictors and/or latent variables.
2. *State-space equations:* How the parameters evolve over the time.

BSTS models are usually implemented using the `bsts` package in R, but can also be implemented using `rstan`. Stan's advanced MCMC algorithms enhance the flexibility and scalability of BSTS models, making it a powerful tool for time series analysis.

References

"Bayesian Structural Time Series." *SAP HANA Predictive Analysis Library (PAL)*, help.sap.com/docs/SAP_HANA_PLATFORM/2cfbc5cf2bc14f028cfbe2a2bba60a50/b9972576368640da9831d73a9d749c3b.html. Accessed 8 Apr. 2024.

Radtke, Tim. "Minimize Regret." *Minimize Regret - Rediscovering Bayesian Structural Time Series*, minimizeregret.com/post/2020/06/07/rediscovering-bayesian-structural-time-series/. Accessed 9 Apr. 2024.

Yabe, Taka. "Pystan - Causal Inference Using Bayesian Structural Time Series." *Takahiro Yabe*, 21 Feb. 2021, www.takayabe.net/post/pystan-bsts. Accessed 9 Apr. 2024.

Appendix C - Time Series Analysis

Section 1 - Setting up the Training and Testing Sets

We will be using all the data up until 02/2023 as the training set, and have the remaining data (from 03/2023 to 02/2024) as the testing set.

```
start_date <- as.Date("1948-01-01")
train_end_date <- as.Date("2023-02-01")
test_start_date <- as.Date("2023-03-01")
end_date <- as.Date("2024-02-01")

dat_train <- subset(dat, DATE <= train_end_date)
dat_test <- subset(dat, DATE >= test_start_date)

nrow(dat_train)
```

```
## [1] 902
```

```
nrow(dat_test)
```

```
## [1] 12
```

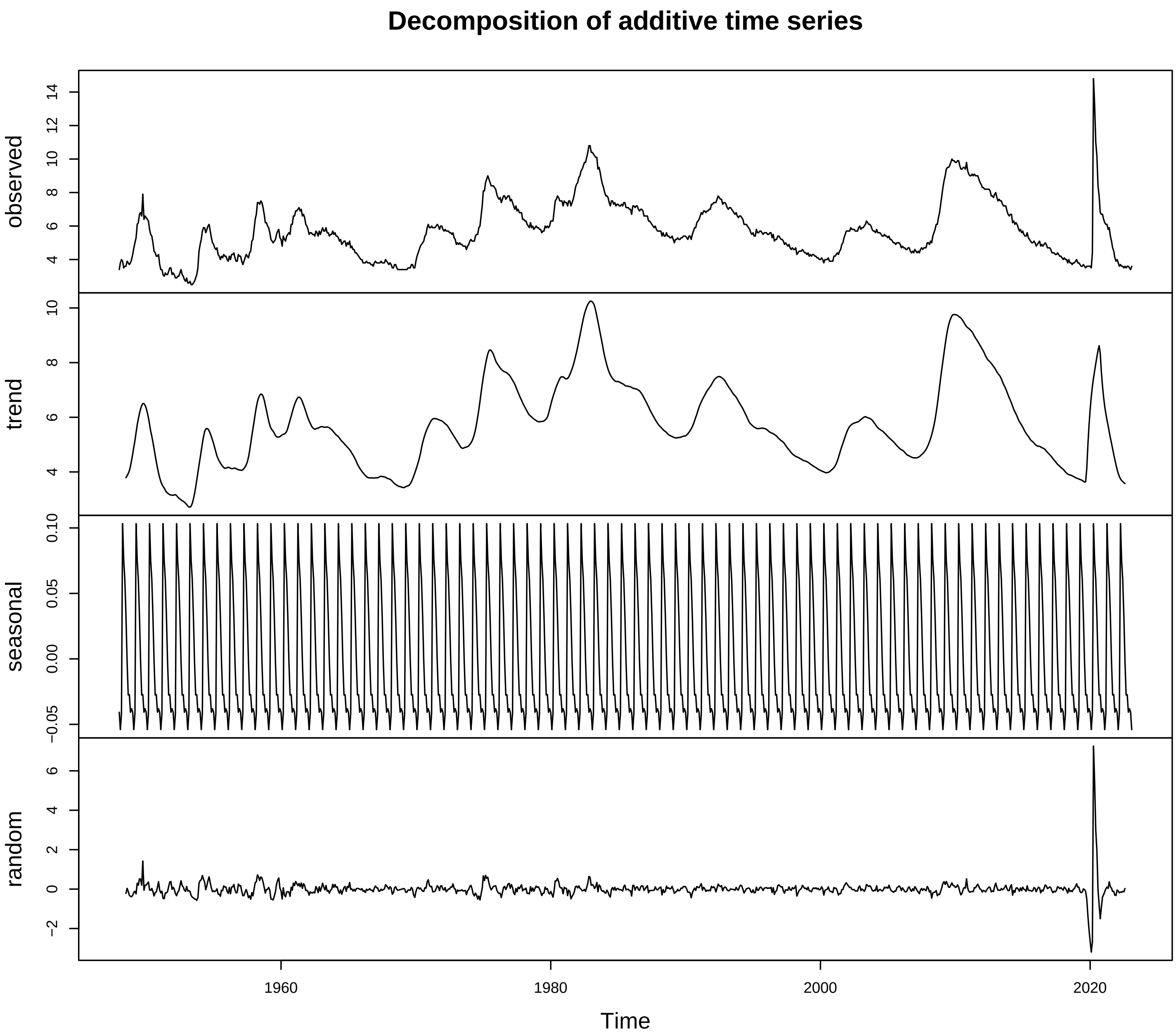
Section 2 - Creating a Time Series Object for the Training and Testing Sets

```
start_train <- c(1948, 1)
end_train <- c(2023, 2)
start_test <- c(2023, 3)
end_test <- c(2024, 2)

ts_unrate <- ts(dat, start = start_train, end = end_test, frequency = 12)

ts_unrate_train <-
  ts(dat_train$UNRATE, start = start_train, end = end_train, frequency = 12)

ts_unrate_test <-
  ts(dat_test$UNRATE, start = start_test, end = end_test, frequency = 12)
```

Section 3 - Additive Decomposition of the Time Series

Figure 3: Additive Decomposition of the Time Series of US Unemployment Rates

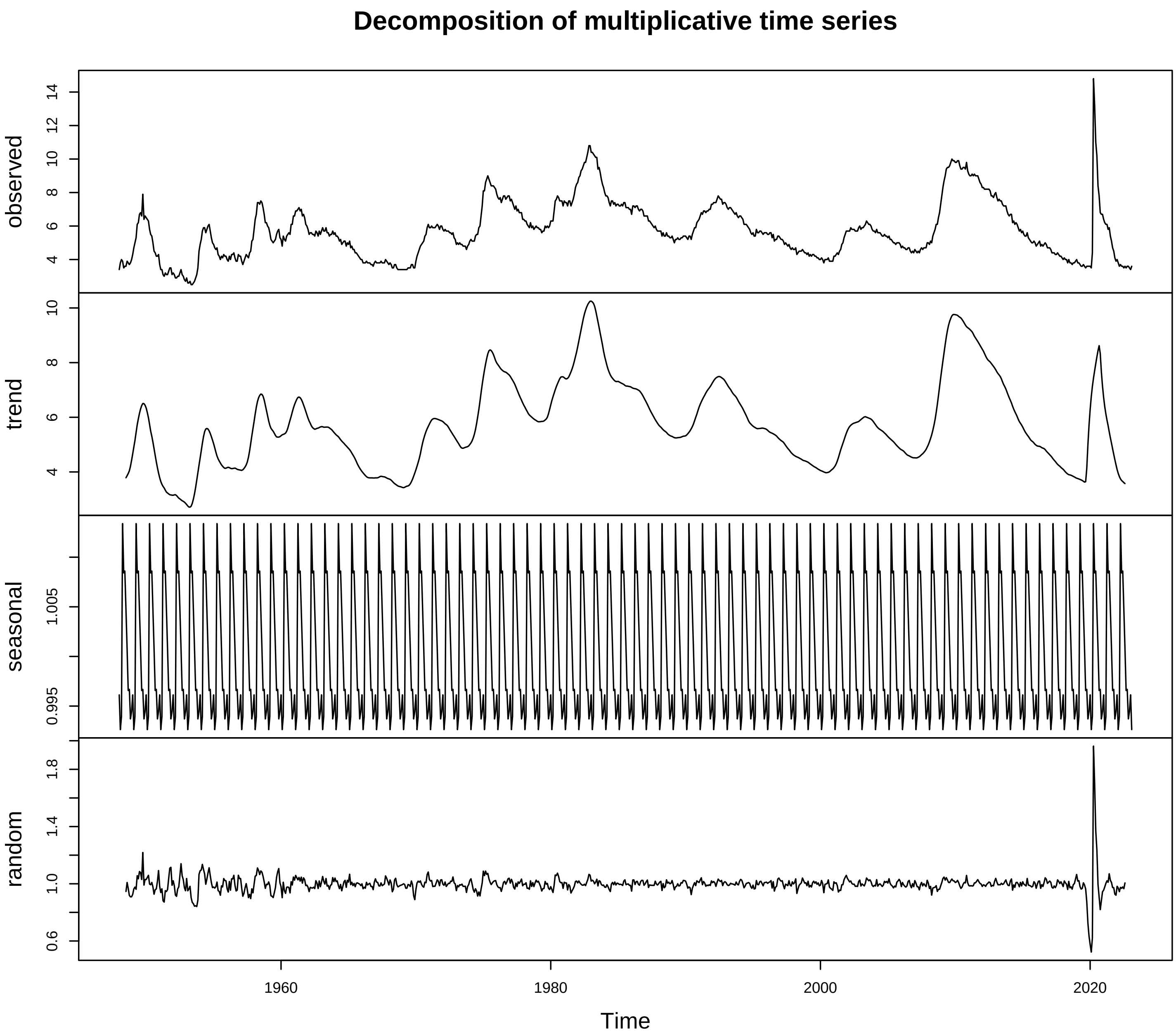
```
decompose_train_additive <- decompose(ts_unrate_train, type = "additive")
plot(decompose_train_additive)
```



Section 4 - Multiplicative Decomposition of the Time Series

Figure 4: Multiplicative Decomposition of the Time Series of US Unemployment Rates

```
decompose_train_multiplicative <- decompose(ts_unrate_train, type = "multiplicative")
plot(decompose_train_multiplicative)
```



Section 5 - Trend Component

```
decompose_train <- stl(ts_unrate_train, s.window = "periodic")
trend <- decompose_train$time.series[, "trend"]
time <- c(1:length(trend))

lm_trend <- lm(trend ~ time)
summary(lm_trend)
```

```
##
## Call:
## lm(formula = trend ~ time)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -2.8314 -1.2338 -0.1731  1.0934  4.5906
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) 5.1160463  0.1039631  49.210  < 2e-16 ***
## time        0.0013537  0.0001995   6.787 2.08e-11 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.56 on 900 degrees of freedom
## Multiple R-squared:  0.04869,    Adjusted R-squared:  0.04763
## F-statistic: 46.06 on 1 and 900 DF,  p-value: 2.079e-11
```

$$Trend = 5.1160463 + 0.0013537 \times Time$$

The intercept and slope values are statistically significant at the 95% confidence interval as they both have p-values of $< 2 \times 10^{-16}$. However, given the the low R-squared value of 0.04869, this may suggest that the trend component may not be a reliable indicator for making future predictions.

Section 6 - ACF and PACF Plots

Figure 5: Autocorrelation (ACF) Plot of a Time Series of US Unemployment Rates

```r
acf_ts_train <- acf(ts_unrate_train)
```

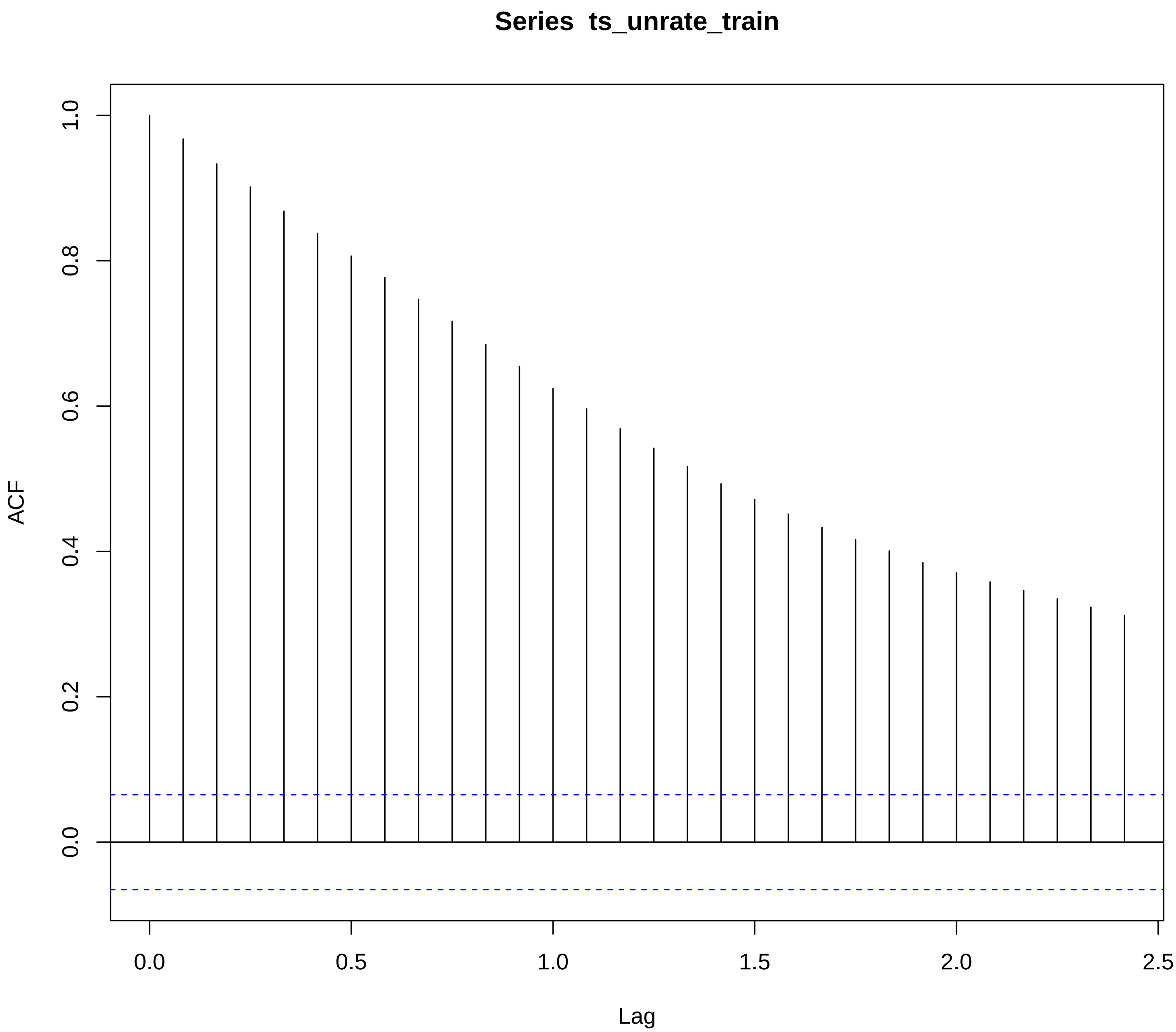
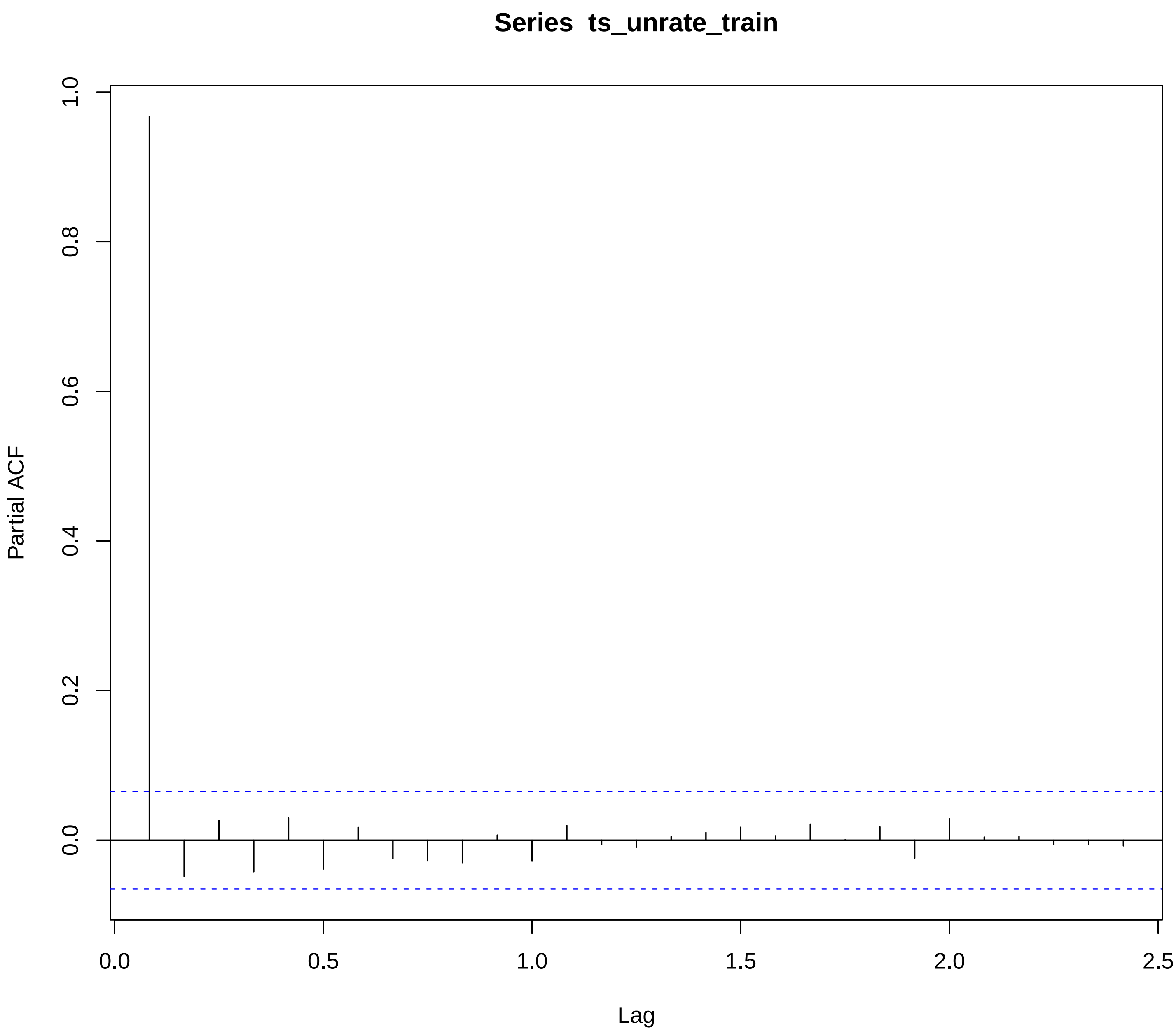


Figure 6: Partial Autocorrelation (PACF) Plot of a Time Series of US Unemployment Rates

```
pacf_ts_train <- pacf(ts_unrate_train)
```



Section 7: Significant ACF and PACF Values

```
n <- length(ts_unrate_train)
significance_level <- 1.96 / sqrt(n)

significant_lags_acf <- which(abs(acf_ts_train$acf[-1]) > significance_level) - 1
significant_lags_pacf <- which(abs(pacf_ts_train$acf[-1]) > significance_level) - 1

cat("Significant lags for ACF: \n", significant_lags_acf, "\n\n")
```

```
## Significant lags for ACF:
##  0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28
```

```
cat("Significant lags for PACF: \n", significant_lags_pacf, "\n")
```

```
## Significant lags for PACF:
##
```

Appendix D - Building Models and Initial Model Selection

Section 1 - Functions for Building BSTS Models and the Initial Model Selection

```
# BSTS Model Building Function
build_bsts_model <- function(trend, n_iter, ts_unrate_train, model_type) {
  if (model_type == "multiplicative") {
    ts_unrate_train <- log(ts_unrate_train)
  }

  state_space <- list()
  state_space <- AddSeasonal(state_space, y = ts_unrate_train, nseasons = 12)
  state_space <- AddAutoAr(state_space, y = ts_unrate_train)

  if (trend == "local_linear") {
    state_space <- AddLocalLinearTrend(state_space, y = ts_unrate_train)
  }

  if (trend == "semi_local_linear") {
    state_space <- AddSemilocalLinearTrend(state_space, y = ts_unrate_train)
  }

  bsts_model <- bsts(ts_unrate_train, state.specification = state_space,
                     niter = n_iter, ping = 0, seed = 123)

  return(bsts_model)
}

# Initial Model Selection
calculate_log_likelihood <- function(model, data) {
    predicted_means <- predict(model)$mean
    residuals <- data - predicted_means
    estimated_sigma <- sd(residuals)
    log_likelihood <- sum(dnorm(residuals, mean = 0, sd = estimated_sigma, log = TRUE))
    return(log_likelihood)
}

calculate_aic <- function(model, data, k) {
    n <- length(data)
    log_likelihood <- calculate_log_likelihood(model, data)
    aic <- 2 * k - 2 * log_likelihood
    return(aic)
}

calculate_bic <- function(model, data, k) {
    n <- length(data)
    log_likelihood <- calculate_log_likelihood(model, data)
    bic <- log(n) * k - 2 * log_likelihood
    return(bic)
}

calculate_mape <- function(actuals, pred_values) {
  epsilon <- 1e-10
  actuals[actuals == 0] <- epsilon
  pred_values[pred_values == 0] <- epsilon

  ape <- abs((actuals - pred_values) / actuals)
  mape <- mean(ape, na.rm = TRUE) * 100
  return(mape)
```

```
}

model_selection <- function(bsts_model, ts_unrate_test, model_type, trend) {
  k_seasonal <- 11                              # 12 months - 1
  k_trend <- switch(trend,                      # Corrected variable name here
                    "none" = 0,
                    "local_linear" = 2,          # Level + Slope
                    "semi_local_linear" = 2)    # Assuming standard 2 parameters
  k <- k_seasonal + k_trend

  burn <- SuggestBurn(0.1, bsts_model)
  pred <- predict(bsts_model, horizon = length(ts_unrate_test),
                  burn = burn, quantiles = c(0.025, 0.975))

  if (model_type == "multiplicative") {
    pred_values <- exp(as.numeric(pred$mean))
  } else {
    pred_values <- as.numeric(pred$mean)
  }

  actuals <- as.numeric(ts_unrate_test)

  # Calculate metrics
  mae <- mean(abs(actuals - pred_values))
  rmse <- sqrt(mean((actuals - pred_values)^2))
  mape <- calculate_mape(actuals, pred_values)

  # Extract AIC and BIC from the model
  aic <- calculate_aic(bsts_model, ts_unrate_test, k)
  bic <- calculate_bic(bsts_model, ts_unrate_test, k)

  return(tibble("MAE" = mae,"RMSE" = rmse,"MAPE" = mape,
                "AIC" = aic, "BIC" = bic))
}
```

Section 2 - Code to Loop over the Initial 6 Models and Generate Statistics

```r
# Set seed for reproducibility
set.seed(123)

# Initialize the results tibble with proper column names
results <- tibble(
  "Model Type" = character(),
  "Include Trend" = character(),
  "MAE" = double(),
  "MAPE" = double(),
  "RMSE" = double(),
  "AIC" = double(),
  "BIC" = double()
)

model_types <- c("additive", "multiplicative")
trend_types <- c("none", "local_linear", "semi_local_linear")
n_iter <- 1000

model_dict <- list()

for (model_type in model_types) {
  for (trend in trend_types) {
    cat(sprintf("Running model: %s with trend: %s\n", model_type, trend))
    bsts_model <- build_bsts_model(trend, n_iter, ts_unrate_train, model_type)
    metrics <- model_selection(bsts_model, ts_unrate_test, model_type, trend)

    key <- paste(model_type, trend, sep = "-")
    model_dict[[key]] <- list(bsts_model = bsts_model, metrics = metrics)

    new_row <- tibble("Model Type" = model_type,
                      "Include Trend" = trend) %>%
      bind_cols(metrics)
    results <- bind_rows(results, new_row)
  }
}
```

```
## Running model: additive with trend: none
## Running model: additive with trend: local_linear
## Running model: additive with trend: semi_local_linear
## Running model: multiplicative with trend: none
## Running model: multiplicative with trend: local_linear
## Running model: multiplicative with trend: semi_local_linear
```

```r
results
```

```
## # A tibble: 6 x 7
##   `Model Type`   `Include Trend`     MAE  MAPE  RMSE    AIC    BIC
##   <chr>          <chr>             <dbl> <dbl> <dbl>  <dbl>  <dbl>
## 1 additive       none              0.205  5.57 0.246   29.1   34.5
## 2 additive       local_linear      0.160  4.36 0.188   25.1   25.9
## 3 additive       semi_local_linear 0.150  4.21 0.206   23.5   30.7
## 4 multiplicative none              0.169  4.58 0.194 3153.  3160.
## 5 multiplicative local_linear      0.208  5.60 0.240 3162.  3170.
## 6 multiplicative semi_local_linear 0.108  2.98 0.140 3130.  3145.
```

Section 3 - Decision Matrix

```r
# Assigning weights to each metric
# Note: Negative weights are used for metrics where lower values are better
# (AIC, BIC, MAE, RMSE, MAPE)
weights <- c("AIC" = -1, "BIC" = -1, "MAE" = -1, "RMSE" = -1, "MAPE" = -1)

# Calculate a comprehensive score for each model configuration considering all metrics
# Multiplying each metric by its corresponding weight and summing the results
# to compute a total score
results <- results %>%
  mutate(score = (AIC * weights["AIC"] +
                  BIC * weights["BIC"] +
                  MAE * weights["MAE"] +
                  RMSE * weights["RMSE"] +
                  MAPE * weights["MAPE"]))

# Sort the models by the scores from the decision matrix
model_selection_df <- results %>% arrange(desc(score))
print(model_selection_df)
```

```
## # A tibble: 6 x 8
##   `Model Type`   `Include Trend`     MAE  MAPE  RMSE    AIC    BIC   score
##   <chr>          <chr>             <dbl> <dbl> <dbl>  <dbl>  <dbl>   <dbl>
## 1 additive       local_linear      0.160  4.36 0.188   25.1   25.9   -55.7
## 2 additive       semi_local_linear 0.150  4.21 0.206   23.5   30.7   -58.7
## 3 additive       none              0.205  5.57 0.246   29.1   34.5   -69.6
## 4 multiplicative semi_local_linear 0.108  2.98 0.140 3130.  3145.  -6278.
## 5 multiplicative none              0.169  4.58 0.194 3153.  3160.  -6317.
## 6 multiplicative local_linear      0.208  5.60 0.240 3162.  3170.  -6337.
```

Appendix E - Posterior Predictive Distributions

Section 1 - Code to Generate a Plot of the Posterior Predictive Distributions

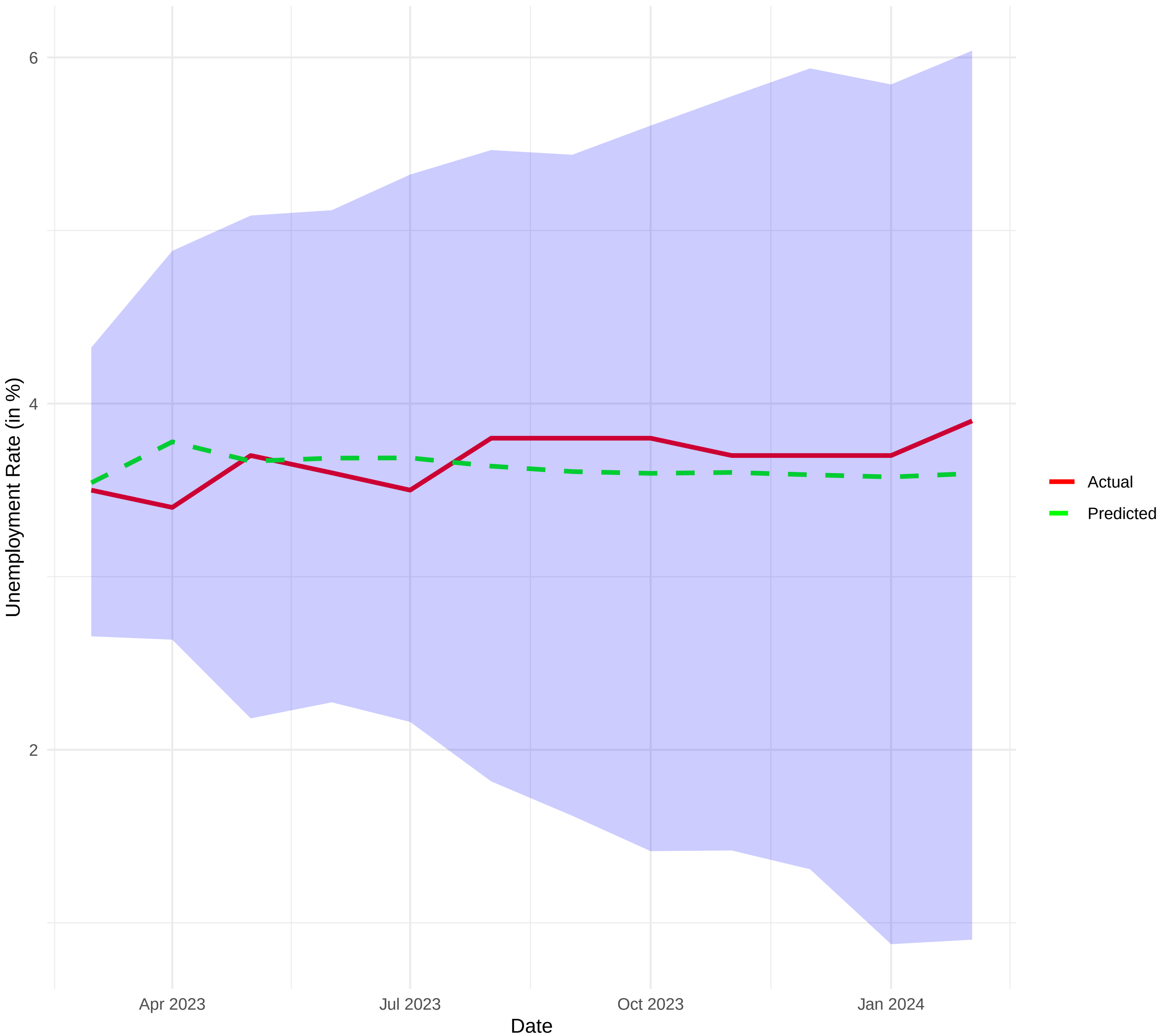
```
generate_plot_of_post_pred_dist <- function(bsts_model, ts_unrate_test) {
  burn <- SuggestBurn(0.1, bsts_model)
  predictions <- predict(bsts_model, horizon = length(ts_unrate_test),
                         burn = burn, quantiles = c(0.025, 0.975))

  lower_bound <- predictions$interval["2.5%",]
  upper_bound <- predictions$interval["97.5%",]

  credible_intervals <- data.frame(
    Time = seq(test_start_date, by = "month", length.out = length(ts_unrate_test)),
    Lower = lower_bound,
    Upper = upper_bound
  )

  credible_intervals_widths <- credible_intervals$Upper - credible_intervals$Lower
  avg_credible_intervals_widths <- mean(credible_intervals_widths)
  cat("Average Credible Intervals Width:", avg_credible_intervals_widths, "\n")

  test_dates <- seq(from = as.Date("2023-03-01"), by = "month",
                    length.out = length(ts_unrate_test))

  plot_data <- data.frame(
    Time = test_dates,
    Actual = as.numeric(ts_unrate_test),
    Predicted = predictions$mean,
    Lower = credible_intervals$Lower,
    Upper = credible_intervals$Upper
  )

  posterior_plot <-
    ggplot(plot_data, aes(x = Time)) +
    geom_line(aes(y = Actual, colour = "Actual"), size = 1.2) +
    geom_line(aes(y = Predicted, colour = "Predicted"),
              size = 1.2, linetype = "dashed") +
    geom_ribbon(aes(ymin = Lower, ymax = Upper), fill = "blue", alpha = 0.2) +
    scale_colour_manual(values = c("Actual" = "red", "Predicted" = "green")) +
    labs(x = "Date", y = "Unemployment Rate (in %)") +
    theme_minimal() +
    theme(legend.title = element_blank())

  return(posterior_plot)
}
```

Section 2 - Posterior Predictive Distribution for Additive Local Linear Model

Figure 7 - Posterior Predictive Distribution with Credible Intervals for the Additive Local Linear Model

```r
additive_local_linear <- model_dict[["additive-local_linear"]]
additive_local_linear_model <- additive_local_linear$bsts_model

generate_plot_of_post_pred_dist(additive_local_linear_model, ts_unrate_test)
```

```
## Average Credible Intervals Width: 3.63104
```

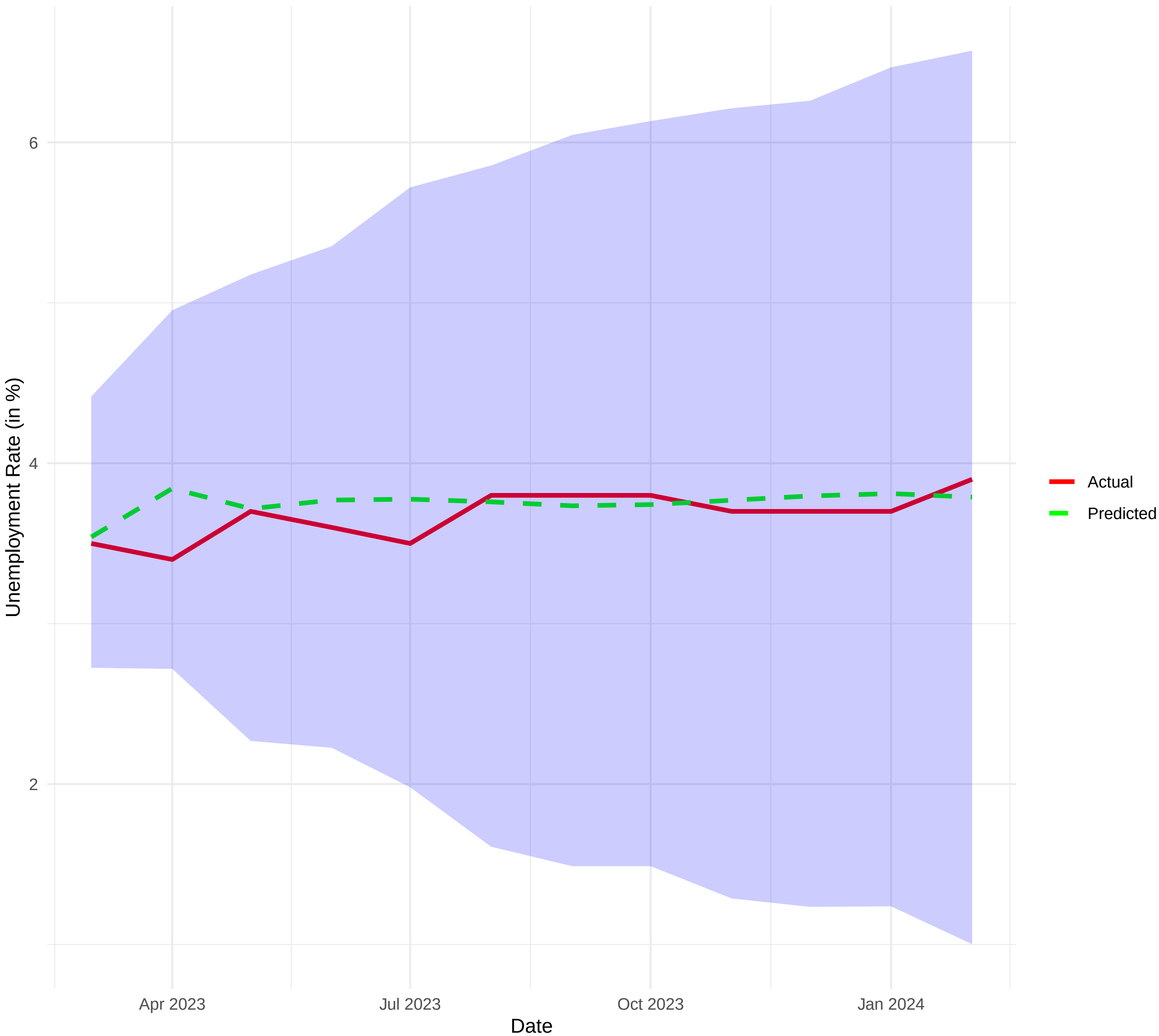


Section 3 - Posterior Predictive Distribution for Additive Semi-Local Linear Model

Figure 8 - Posterior Predictive Distribution with Credible Intervals for the Additive Semi-Local Linear Model

```
additive_semi_local_linear <- model_dict[["additive-semi_local_linear"]]
additive_semi_local_linear_model <- additive_semi_local_linear$bsts_model

generate_plot_of_post_pred_dist(additive_semi_local_linear_model, ts_unrate_test)
```

```
## Average Credible Intervals Width: 3.992042
```



Appendix F - Cross Validation

Section 1 - Function for General Cross-Validation of BSTS Model

```
cross_validate_bsts_model <- function(trend, ts_unrate_train, test_start_date) {
  # Set seed for reproducibility
  set.seed(123)

  n_iter <- 500
  forecast_horizon <- 12   # Forecast horizon (1 year)
  step_size <- 24          # Step size for the rolling window (2 years)

  error_metrics <- tibble(MAE = double(), MAPE = double(), RMSE = double(),
                          `Avg. Credible Intervals Width` = double())

  n_windows <- (length(ts_unrate_train) - forecast_horizon) / step_size

  # Perform rolling window cross-validation
  for (i in seq_len(n_windows)) {
    train_end <- i * step_size
    valid_start <- train_end + 1
    valid_end <- valid_start + forecast_horizon - 1

    train_set <- ts_unrate_train[1:train_end]
    valid_set <- ts_unrate_train[valid_start:valid_end]

    bsts_model <- build_bsts_model(trend, n_iter, train_set, "additive")

    # Forecast and calculate error metrics
    burn <- SuggestBurn(0.1, bsts_model)
    pred <- predict(bsts_model, horizon = length(valid_set), burn = burn)
    pred_values <- as.numeric(pred$mean)
    actuals <- as.numeric(valid_set)

    lower_bound <- pred$interval["2.5%",]
    upper_bound <- pred$interval["97.5%",]

    mae <- mean(abs(actuals - pred_values))
    rmse <- sqrt(mean((actuals - pred_values)^2))
    mape <- calculate_mape(actuals, pred_values)

    credible_intervals <- data.frame(
      Time = seq(test_start_date, by = "month", length.out = length(valid_set)),
      Lower = lower_bound,
      Upper = upper_bound
    )

    credible_intervals_widths <- credible_intervals$Upper - credible_intervals$Lower
    avg_credible_intervals_widths <- mean(credible_intervals_widths)

    new_row <- tibble(MAE = mae, RMSE = rmse, MAPE = mape,
                      `Avg. Credible Intervals Width` = avg_credible_intervals_widths)
    error_metrics <- bind_rows(error_metrics, new_row)
  }

  # Cross-validation Summary statistics
  summary_metrics <- error_metrics %>%
    summarise("Average MAE" = mean(`MAE`, na.rm = TRUE),
              "Average MAPE" = mean(MAPE, na.rm = TRUE),
```

```r
              "Average RMSE" = mean(RMSE, na.rm = TRUE),
              "Avg. CI Width" =
                mean(`Avg. Credible Intervals Width`, na.rm = TRUE))

  return(summary_metrics)
}
```

Section 2 - Cross-Validation for the Additive Local Linear Model and Additive Semi-Local Linear Model

```r
summary_metrics_local <-
  cross_validate_bsts_model("local_linear", ts_unrate_train, test_start_date)
summary_metrics_semi_local <-
  cross_validate_bsts_model("semi_local_linear", ts_unrate_train, test_start_date)

summary_metrics_local <- summary_metrics_local %>%
  mutate("Model Trend" = "local")
summary_metrics_semi_local <- summary_metrics_semi_local %>%
  mutate("Model Trend" = "semi-local")

combined_summary_metrics <-
  bind_rows(summary_metrics_local, summary_metrics_semi_local)

combined_summary_metrics
```

```
## # A tibble: 2 x 5
##   `Average MAE` `Average MAPE` `Average RMSE` `Avg. CI Width` `Model Trend`
##           <dbl>          <dbl>          <dbl>           <dbl> <chr>
## 1         0.639          10.6           0.764            4.32 local
## 2         0.593           9.77          0.696            3.61 semi-local
```

Appendix G - Estimation of Hyperparameters for BSTS Model

Before we can specify the BSTS model for the US Unemployment rates, we need to estimate the hyperparameters that will inform the prior distributions. This involves analyzing historical unemployment rate data to inform the choice of prior distributions for various model components, specifically the observation noise variance, trend variance, seasonal variance, and white noise variance. Note, that we are assuming that the white noise variance is a fraction of the observation noise.

```r
obs_noise_variance_estimate <- var(diff(ts_unrate_train), na.rm = TRUE)

trend_model <- lm(ts_unrate_train ~ time(ts_unrate_train))
trend_variance_estimate <- var(resid(trend_model), na.rm = TRUE)

seasonal_differences <- diff(ts_unrate_train, lag = 12)
seasonal_variance_estimate <- var(seasonal_differences, na.rm = TRUE)

ar_model <- ar(ts_unrate_train, order = 1, method = "mle")
sigma_X_empirical <- sd(residuals(ar_model), na.rm = TRUE)

alpha <- 2

beta_obs_noise <- obs_noise_variance_estimate * (alpha - 1)
beta_trend <- trend_variance_estimate * (alpha - 1)
beta_seasonal <- seasonal_variance_estimate * (alpha - 1)
beta_AR_process <- sigma_X_empirical^2 * (alpha - 1)

# Print final hyperparameters
cat("Inverse-Gamma alpha:", alpha, "\n")
```

```
## Inverse-Gamma alpha: 2
```

```r
cat("Inverse-Gamma beta for Observation Noise:", beta_obs_noise, "\n")
```

```
## Inverse-Gamma beta for Observation Noise: 0.1778888
```

```r
cat("Inverse-Gamma beta for Trend:", beta_trend, "\n")
```

```
## Inverse-Gamma beta for Trend: 2.784333
```

```r
cat("Inverse-Gamma beta for Seasonal:", beta_seasonal, "\n")
```

```
## Inverse-Gamma beta for Seasonal: 2.099184
```

```r
cat("Inverse-Gamma beta for AR(1) process:", beta_AR_process, "\n")
```

```
## Inverse-Gamma beta for AR(1) process: 0.1777053
```