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Limiting Behaviour of solutions of Hamilton-Jacobi equations.

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0.1 Abstract

Using Legendre transform, the 'n' Euler-Lagrange equations associated with the variational problem can be modified into '2n' first order ordinary differential equations called the Hamilton's equations. The special property being that they are uncoupled and all those equations can be derived from a single scalar function called the *Hamiltonian*.

One of the primary variational problem would be to minimize the 'action' (functional) using Lagrangian formulation and one of the solutions to the minimization of the function that takes the form **u(x,t)** is by Hopf-Lax formula. Here we consider the particular case where the equation is U_t+$|U_x|^{1+1/n}$= 0. The solution of this equation is by using Hopf-Lax theory. We then study the limiting behavior of the solutions of the above equation as n approaches ∞. In the future, we plan to work on the relation between limiting solution and the equation U_t+$|U_x|$= 0.

0.2 Hamiltonian Formulation and Equations:

0.2.1 Legendre Transform:

One of the most important contact transformation is the Legendre transformation, used widely in calculus of variations, acting as a bridge between Euler- Lagrange equations and the Hamiltonian formulation.

The transformation is given by:

Considering a smooth function (called the Lagrangian) L: $\mathbb{R}^n$->$\mathbb{R}$ satisfying the following conditions:

1- The mapping q->|L(q) is convex.

2- $\lim_{q->\infty} \frac{L(q)}{|q|} \to \infty$

The Legendre Transform of L is then defined as:

$$L^*(p) = sup\ \{p.q - L(q)\} \tag{1}$$

where p$\in \mathbb{R}^n$. The transformation is called an involution as the inverse of the transform would produce the original equation, and here $L^*(p)$ represents the Hamiltonian given by *H(p)*.

In particular, a generalized relationship would be:

$$-H(t,q,\dot{q}) = \sum_{k=1}^{n} \dot{q}_k\, p_k + L(t,q,\dot{q})$$

The coordinates *(t,q,p)* are called the generalized coordinates, and the set of points described by the pairs **(q,p)** form the phase space. In mechanics, p_k are known as the generalized momenta, frequently encountered in the energy conservation problems in classical regime. Hamiltonian can be considered as the a function describing the total (Kinetic + Potential) energy of the system and the Lagrangian would be equal to the difference between kinetic and potential energy. For example, the Hamiltonian of a free particle moving would be:

$$H=\frac{1}{2}m|\dot{q}|^2 + \phi(t,q)$$

where $\phi(t,q)$:$\mathbb{R}^n$->$\mathbb{R}$ denotes the potential energy of the particle as a function of time and distance. (Generalized position coordinates.)

0.2.2 Hamilton's equations:

The functional or the action (classical mechanics) is defined as:

$$J(q) = \int_0^t L(t,q,\dot{q})dt \tag{2}$$

Where q=($q_1, q_2, ...$) are the generalized position coordinates and **L** is a smooth function satisfying the conditions stated above. 'q' can be the extremal of the function *J(q)* if it satisfies the conditions:

$$\frac{dp_k}{dt} + \frac{\partial L}{\partial q_k} = 0 \tag{3}$$

where

$$p_k = \frac{\partial L}{\partial \dot{q_k}} \tag{4}$$

and

$$\dot{q_k} = \frac{\partial H}{\partial q_k} \tag{5}$$

Now, t and q are passive variables in this transformation,

hence $\frac{\partial H}{\partial t}$ = - $\frac{\partial L}{\partial t}$ and $\frac{\partial H}{\partial q_k}$ = $\frac{\partial L}{\partial q_k}$, now this equation along with the one already pointed out would give us:

$$\dot{p_k} = -\frac{\partial H}{\partial q_k} \tag{6}$$

The solutions **q** to the Euler-Lagrange equations are thus mapped to solutions **(q,p)** to the remaining equations under the Legendre transformation. This set of 2n uncoupled ODE form the Hamilton's equations. Or the equations of the Hamiltonian system.

0.2.3 Canonical Transformation

A Canonical Transformation generates a different set of Hamilton equations from the existing one and takes the form:

Given that

$$\dot{p_k} = -\frac{\partial H}{\partial q_k}$$

and that

$$\dot{q_k} = \frac{\partial H}{\partial q_k}$$

for k=1,2,....n. A symplectic (greek word for 'intertwined', another term for canonical transformation) map would then be:

$$\dot{Q_k} = \dot{Q_k}(\text{t,q,}\dot{q_k}),\ \dot{P_k} = \dot{P_k}(\text{t,q,}\dot{q_k}),$$

Then the Hamiltonian system transforms into another set of Hamiltonian given by:

$$\dot{Q_k} = \frac{\partial \hat{H}}{\partial P_k} \text{ and}$$
$$\dot{P_k} = -\frac{\partial \hat{H}}{\partial Q_k}$$

where $\hat{H}$ is a function of t, **Q** and **P**. ln short, a symplectic map is a transformation on the generalized coördinates that preserves the Hamiltonian structure.

We know that the Hamiltonian system of equations can be represented as a functional using Lagrangian:

$$\mathbf{J(q)} = \int_0^t L(t,q,\dot{q})\ \text{dt and } \mathbf{J(Q)} = \int_0^t \hat{L}(t,q,\dot{Q})\ \text{dt}$$

where:

$$L(t,\boldsymbol{q},\dot{q}) = \sum_{k=1}^{n} \dot{q}_k\, p_k - H(t,\boldsymbol{q},\dot{q}) \text{ and, } L(t,\boldsymbol{Q},\dot{Q}) = \sum_{k=1}^{n} \dot{q}_k\, p_k - H(t,\boldsymbol{Q},\dot{Q})$$

lf we regard **q** and **p** as independent variables and $\dot{q}$ as the derivative of **q**, then the Euler-Lagrange equations for the functional:

$$\mathbf{J(q)} = \int_0^t \sum_{k=1}^{n} \dot{q}_k p_k - H(t,\boldsymbol{q},\dot{q})dt \tag{7}$$

is exactly Hamilton's equations and the solution to the equations corresponding to the extremal (action) **J**.

One method for constructing a canonical transformation involves the introduction of a *generating function*.The method is based on the observation that two functionals are variationally equivalent if their integrands differ by a perfect differential.

Consider a smooth function $\boldsymbol{u}$: $\mathbb{R}^{\boldsymbol{n}}$->$\mathbb{R}$ such that:

$$\textstyle\sum_{k=1}^{n} \dot{q}_k p_k - H(t,\boldsymbol{q},\dot{q}) = \sum_{k=1}^{n} \dot{Q}_k P_k - H(t,\boldsymbol{Q},\dot{Q}) + \frac{du(t,q,\dot{q})}{dt}$$

The corresponding functionals are variationally equivalent and the transformation is symplectic. The function *u* can then be written as:

$$\frac{du(t,q,\dot{q})}{dt} = \sum_{k=1}^{n}(\frac{\partial u}{\partial q_k}\dot{q}_k + \frac{\partial u}{\partial Q_k}\dot{Q}_k) + \frac{\partial u}{\partial t} \tag{8}$$

where

$$p_k = \frac{\partial \hat{u}}{\partial q_k} \text{ and, } P_k = -\frac{\partial \hat{u}}{\partial Q_k}$$

We then get,

$$\hat{H}(t,q,\dot{q}) = H(t,q,\dot{q}) + \frac{\partial u}{\partial t} \tag{9}$$

This is the transformed system of Hamiltonian equations from the existing ones using the map. This step gets us closer the the Hamilton-Jacobi equation that solves the original problem.

0.3 Hamilton-Jacobi Equation:

Suppose there exists a generating function *u*: $\mathbb{R}^n$->$\mathbb{R}$ such that the transformed Hamiltonian: $\hat{H} = 0$, then the equation reduces to:

$$H(t, q, \dot{q}) + \frac{\partial u}{\partial t} = 0 \tag{10}$$

This is called the Hamilton-jacobi equation and can be expressed as:

$$H(t, q_1, q_2...., \tfrac{\partial u}{\partial q_1}, \tfrac{\partial u}{\partial q_2}....) + \tfrac{\partial u}{\partial t} = 0$$

Or an equivalent 'easier' notation would be:

$$U_t + H(Du, u, t) = 0 \tag{11}$$

Where

$$U_t = \tfrac{\partial u}{\partial t} \text{ and } Du = \tfrac{\partial u}{\partial q_i}$$

It can be observed that q does not explicitly occur in the equation, instead its derivatives do. Our aim is to solve the Hamilton-Jacobi problem with an initial condition (Initial value problem- IVP).

The initial value problem is given by:

$$\begin{cases} Ut + H(Du, u, t) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{in } \mathbb{R}^n \times (t = \infty) \end{cases} \tag{12}$$

where *u*: $\mathbb{R}^n \times [0, \infty)$->$\mathbb{R}$ is the unknown. $u = u(x, t)$ *and* $Du = (u_{x1}, u_{x2}....u_{xn})$.
We are given the Hamiltonian *H*: $\mathbb{R}^n$->$\mathbb{R}$ and the initial function *g*: $\mathbb{R}^n$->$\mathbb{R}$

Our goal would be to construct a weak solution which would exist at all times $t > 0$ even when the method of characteristics fail. One of the ways would be (if the function satisfies certain conditions) is to use Hopf-Lax formula.

0.4 Hopf-Lax Formula

Since the minimization of action through Lagrangian led us to the Hamiltonian, and that they form eventually the Hamilton-Jacobi PDE, there seems to be an intrinsic correlation between calculus of variations and the Hamilton PDE, which can be interpreted as follows:
So if $x \in \mathbb{R}^n$ and $t > 0$ is given, then we should try to minimize the action:

$$\text{J(q)= } \int_0^t L(t,q,\dot{q}) \text{ dt}$$

the functions q belonging the the admissible class:

$$\mathcal{A}\text{= } \{\ q \in C^2(0,t,\mathbb{R}^n) | q(0) = y, q(t) = x\ \}$$

Thus a C^2 point lies on $\mathcal{A}$ if it starts from y at $t = 0$ and then reaches x at t. Goal is to minimize the action or the curve that traces the path between those points.

Now to start with, the initial step is to find out the value of $q(0)$ and as it is not always known, it is compensated in the equation for the extremal:

$$\text{J(q)= } \int_0^t L(t,q,\dot{q}) \text{ dt + } g(q(0))$$

Hence we can construct a function to be a solution of the form:

$$u(x,t) = inf\{\int_0^t L(t,q,\dot{q})dt + g(q(0)) | q(0) = y, q(t) = x\} \tag{13}$$

The infimum taken over all C^1 functions of 'q'. We now solve the initial value problem defined as:

$$\begin{cases} Ut + H(Du,u,t) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{in } \mathbb{R}^n \times (t = \infty) \end{cases}$$

Assuming both H and L to be smooth, convex and super-linear. Also assuming that g:$\mathbb{R}^n$->$\mathbb{R}$ satisfies Lipschitz continuity given by :

$$Lip(g) := sup_{x,y \in R^n} \left\{ \frac{|g(x) - g(y)|}{|x - y|} \right\} < \infty \tag{14}$$

So, if $x \in \mathbb{R}^n$ and t >0 then the solution to the minimization problem is given by:

$$u(x,t) = min_{y \in R^n} \left\{ tL(\frac{x-y}{t}) + g(y) \right\} \tag{15}$$

This expression is called the Hopf-Lax formula. Suppose $x \in \mathbb{R}^n$ and t >0 and 'u' defined by the Hopf-Lax formula is differentiable on the points $(x,t) \in \mathbb{R}^n \times (0,\infty)$ then:

$$U_t + H(Du(x,t)) = 0$$

Function of this form- i.e., the Hamilton-Jacobi equation can be solved using the formula just defined.

0.5 Solution method and Framework:

Considering the fact that the Hamiltonian function should be super-linear and convex, $U_t + |U_x| = 0$ is modified such that:

$$U_t + |Ux|^{1+1/n} = 0; u(x,0) = u_0(x). \tag{16}$$

where $u = u_0$ is in $\mathbb{R}^n \times (t = 0)$, and the admissible function is C^2 differentiable. The solution is based on finding out the Lagrangian's maximal value which is substituted in the Hopf-Lax equation and then the desired relationship is obtained.

The Lagrangian is given by:

$$L^*(p) = sup_{q \in \mathbb{R}} \{ pq - |q|^{1+1/n} \} \tag{17}$$

hence to find the maximum value, function inside the brackets can be written in the form:

$$g(q) = pq - |q|^{1+1/n} \tag{18}$$

Now differentiating with respect to q and equating the term to zero, we get

$$0 = g'(q) = p - (1 + 1/n)|q|^{1/n} sgn(q) \tag{19}$$

where $sgn(q)$ represents the Signum function in this case is:

$$sgn(x) = \frac{|x|}{x}$$

Hence considering two cases: when $q > 0$ and $q < 0$, we have

$$g(q) = pq - |q|^{1+1/n} = \begin{cases} pq - (-q)^{1+1/n} & \text{if q} < 0 \\ pq - (q)^{1+1/n} & \text{if q} > 0 \end{cases} \tag{20}$$

and hence:

$$g'(q) = \begin{cases} p + (1 + 1/n)(-q)^{1/n} & \text{if q} < 0 \\ p - (1 + 1/n)(q)^{1/n} & \text{if q} > 0 \end{cases} \tag{21}$$

Equating the derivative to zero,

$$\Rightarrow \begin{cases} p + (1 + 1/n)(-q)^{1/n} = 0 & \text{if q} < 0 \\ p - (1 + 1/n)(q)^{1/n} = 0 & \text{if q} > 0 \end{cases} \tag{22}$$

$$\Rightarrow \begin{cases} p = -(1 + 1/n)(-q)^{1/n} & \text{if q} < 0 \\ p = -(1 + 1/n)(q)^{1/n} & \text{if q} > 0 \end{cases} \tag{23}$$

$$\Rightarrow \begin{cases} (-q)^{1/n} = \frac{-p}{(1+\frac{1}{n})} & \text{if q} < 0 \\ (q)^{1/n} = \frac{p}{(1+\frac{1}{n})} & \text{if q} > 0 \end{cases} \tag{24}$$

$$\Rightarrow \begin{cases} q = -[\frac{-p}{(1+\frac{1}{n})}]^n & \text{if q} < 0 \\ q = [\frac{p}{(1+\frac{1}{n})}]^n & \text{if q} > 0 \end{cases} \tag{25}$$

Now, we observe that suppose n is an even integer, then for all $p \in (-\infty, \infty)$, the maximum among these two terms would be :

$$q = \left[\frac{p}{(1+\frac{1}{n})}\right]^n \tag{26}$$

and it turns out that this equation is the maximum for any value of $n \in \mathbb{N}$ and any $p \in \mathbb{R}$. (i.e., for p>0, p<0, and p=0). (The counterpart being equal in magnitude but with a negative sign, hence the minimum). This is due to the fact that the maximum term among the two possibilities is considered for any (all values) value of n and for all values of p, and the *maximum* can be only one of the terms which is strictly positive.

Now substituting this value of q in the equation for Lagrangian $L^*(p)$, we get,

$$L^*(p) = p[\frac{p}{(1+\frac{1}{n})}]^n - [\frac{p}{(1+\frac{1}{n})}]^{1+\frac{1}{n}}$$

$$\Rightarrow = \frac{p^{n+1}}{(1+\frac{1}{n})^n}[1 - \frac{1}{1+\frac{1}{n}}]$$

$$\Rightarrow = \frac{p^{n+1}}{(n+1)(1+\frac{1}{n})^n} \tag{27}$$

We observe from the set of equations that p and q both have the same sign, this implies that the product pq is strictly positive provided that $p, q > 0$, hence the Lagrangian is always positive, this motivates the final form given by:

$$\Rightarrow L^*(p) = \frac{|p|^{n+1}}{(n+1)(1+\frac{1}{n})^n} \tag{28}$$

We then substitute this value of Lagrangian in the Hopf-Lax formula and noting that p can be expressed as $(\frac{x-y}{t})$ where y refers to the initial value. We then get the form:

$$u(x,t) = min_{y \in R^n} \left\{ t(\frac{|x-y|^{n+1}}{n(1+\frac{1}{n})^{n+1}t^{n+1}}) + u_0(y) \right\} \tag{29}$$

This can be finally reduced to the expression :

$$u(x,t) = min_{y \in R^n} \left\{ \left(\frac{|x-y|^{n+1}}{n(1+\frac{1}{n})^{n+1}t^{n}} \right) + u_0(y) \right\} \tag{30}$$

Now we observe the value the function converges to when $n \to \infty$, the constants in the expression on the right hand side reduces to only one term, $u_0(y)$. The function is the minima of the expression, hence to avoid blowing up of the expression as $n \to \infty$, there is a bound on $|\frac{x-y}{t}|$ given by:

$$|\frac{x-y}{t}| < 1$$

$$\Rightarrow -(x-y) < t < (x-y)$$

$$\Rightarrow x - t < y < x + t$$

This bound on the term and the fact that other expressions apart from n doesn't affect the expression much, the first term of the expression on the RHS tends to zero and the function as $n \to \infty$ reduces to:

$$u(x,t) = min_{y \in (x-t, x+t)} u_0(y) \tag{31}$$

0.6 Future Work: Vanishing viscosity solution approach

The method of Viscosity solution was first described by M. Crandall and P. Lions in the year 1983. In this case we consider not the generalized perspective, instead the application of the method on the Hamilton-Jacobi equation. It has been presented earlier and will be considered here in the same form.

Given the equation of the type:

$$\begin{cases} u_t + H(Du, x) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{in } \mathbb{R}^n \times (t = \infty) \end{cases} \tag{32}$$

The solution by this method avoids solving the equation above directly but it does instead by considering:

$$\begin{cases} u_t^\epsilon + H(Du^\epsilon, x) = \epsilon \Delta u^\epsilon & \text{in } \mathbb{R}^n \times (0, \infty) \\ u^\epsilon = g & \text{in } \mathbb{R}^n \times (t = \infty) \end{cases} \tag{33}$$

For $\epsilon > 0$ and Δu^ϵ represents the Laplacian of u and if the analysis is one dimensional as in this case, it can also be denoted as u^ϵ_{xx}.

The first set represents a completely non-linear PDE while the second set is a quasi-linear parabolic PDE which is comparatively easier (well behaved) to solve and of course, we would expect the equations to converge to the weak solution of the original form as $\epsilon \to 0$. This method is called the vanishing viscosity.

If $u \in C(\mathbb{R}^n \times [0, \infty))$ is a viscosity solution to the Hamilton-Jacobi equation of the type - $u_t + H(Du, x) = 0$ with initial data g then:

1. $u(x, 0) = g(x)$ for all $x \in \mathbb{R}^n$

2. For all $v \in C^\infty(\mathbb{R}^n \times (0, \infty))$

• If $u - v$ has a local maximum at a point $(x_0, t_0) \in (\mathbb{R}^n \times (0, \infty))$ then:

$$v_t(x_0, t_0) + H(Dv(x_0, t_0), x_0) \leq 0$$

• If $u - v$ has a local minimum at a point $(x_0, t_0) \in (\mathbb{R}^n \times (0, \infty))$ then:

$$v_t(x_0, t_0) + H(Dv(x_0, t_0), x_0) \geq 0$$

Hence it can be observed that u can be a viscosity solution even if it is not *differentiable*, it would require us to compare it with a series of test functions. It can also be proved that every classical solution is a viscosity solution. Our aim as we proceed would focus on proving that

$$u(x, t) = min_{y \in (x-t, x+t)} u_0(y)$$

would be a viscosity solution even if it is not a classical solution. (Solutions having a possibility of crossing characteristics).

0.7 Bibliography

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