A2

We are given the infinite-dimensional feature map

$$\phi(x) = \left[\frac{1}{\sqrt{0!}}e^{-x^2/2}x^0,\ \frac{1}{\sqrt{1!}}e^{-x^2/2}x^1,\ \frac{1}{\sqrt{2!}}e^{-x^2/2}x^2,\ \dots\right].$$

For inputs x and x', we have

$$\phi(x) = \left[\frac{e^{-x^2/2}x^0}{\sqrt{0!}},\ \frac{e^{-x^2/2}x^1}{\sqrt{1!}},\ \frac{e^{-x^2/2}x^2}{\sqrt{2!}},\ \dots\right],$$

$$\phi(x') = \left[\frac{e^{-x'^2/2}x'^0}{\sqrt{0!}},\ \frac{e^{-x'^2/2}x'^1}{\sqrt{1!}},\ \frac{e^{-x'^2/2}x'^2}{\sqrt{2!}},\ \dots\right].$$

The inner product is computed as

$$\langle \phi(x), \phi(x') \rangle = \sum_{n=0}^{\infty} \left(\frac{e^{-x^2/2}x^n}{\sqrt{n!}}\right)\left(\frac{e^{-x'^2/2}x'^n}{\sqrt{n!}}\right).$$

Simplifying:

$$\langle \phi(x), \phi(x') \rangle = e^{-x^2/2}e^{-x'^2/2}\sum_{n=0}^{\infty}\frac{(xx')^n}{n!}$$

After the expression

$$\sum_{n=0}^{\infty}\frac{(xx')^n}{n!} = e^{xx'},$$

(Taylor series expansion of e^x), we obtain

$$\langle \phi(x), \phi(x') \rangle = e^{-x^2/2}e^{-x'^2/2}e^{xx'} = e^{-\frac{x^2+x'^2}{2}+xx'}.$$

Writing xx' as a fraction with denominator 2, we have

$$-\frac{x^2}{2} - \frac{x'^2}{2} + \frac{2xx'}{2} = -\frac{x^2+x'^2-2xx'}{2}.$$

Noting that

$$x^2 + x'^2 - 2xx' = (x-x')^2,$$

we obtain

$$-\frac{x^2+x'^2}{2} + xx' = -\frac{(x-x')^2}{2}.$$

Hence,

$$\boxed{\langle \phi(x), \phi(x') \rangle = e^{-\frac{(x-x')^2}{2}}.}$$

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