A4(c)

Mean

First, consider the expectation of $\hat{\mu}_n$:

$$\mathbb{E}[\hat{\mu}_n] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i].$$

Since $\mathbb{E}[X_i] = \mu$ for all i, this simplifies to:

$$\mathbb{E}[\hat{\mu}_n] = \frac{1}{n} \cdot n \cdot \mu = \mu.$$

Now, compute the expectation of $\sqrt{n}(\hat{\mu}_n - \mu)$:

$$\mathbb{E}[\sqrt{n}(\hat{\mu}_n - \mu)] = \sqrt{n} \cdot \mathbb{E}[\hat{\mu}_n - \mu].$$

Substituting $\mathbb{E}[\hat{\mu}_n] = \mu$, we get:

$$\mathbb{E}[\sqrt{n}(\hat{\mu}_n - \mu)] = \sqrt{n} \cdot (\mathbb{E}[\hat{\mu}_n] - \mu) = \sqrt{n} \cdot (\mu - \mu) = 0.$$

Variance

The variance of $\sqrt{n}(\hat{\mu}_n - \mu)$ is:

$$\mathrm{Var}\left(\sqrt{n}(\hat{\mu}_n - \mu)\right) = \mathrm{Var}\left(\sqrt{n} \cdot \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)\right).$$

Simplify the scaling factor $\sqrt{n} \cdot \frac{1}{n} = \frac{1}{\sqrt{n}}$:

$$\mathrm{Var}\left(\sqrt{n}(\hat{\mu}_n - \mu)\right) = \frac{1}{n} \cdot \mathrm{Var}\left(\sum_{i=1}^{n}(X_i - \mu)\right).$$

Since $X_1, X_2, \ldots, X_n$ are independent, the variance of the sum is the sum of variances:

$$\mathrm{Var}\left(\sum_{i=1}^{n}(X_i - \mu)\right) = \sum_{i=1}^{n}\mathrm{Var}(X_i - \mu).$$

Since $\mathrm{Var}(X_i - \mu) = \mathrm{Var}(X_i) = \sigma^2$ for all i, this becomes:

$$\mathrm{Var}\left(\sum_{i=1}^{n}(X_i - \mu)\right) = n \cdot \sigma^2.$$

Substitute this back:

$$\mathrm{Var}\left(\sqrt{n}(\hat{\mu}_n - \mu)\right) = \frac{1}{n} \cdot n \cdot \sigma^2 = \sigma^2.$$

Final Results

- **Mean:**
$$\mathbb{E}[\sqrt{n}(\hat{\mu}_n - \mu)] = 0.$$
- **Variance:**
$$\mathrm{Var}[\sqrt{n}(\hat{\mu}_n - \mu)] = \sigma^2.$$

$\blacksquare$