A7

(a)

We expand $f(x, y)$ as:

$$f(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j + \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij} y_i x_j + c.$$

(b)

The gradient of $f(x, y)$ with respect to x is given by:

$$\nabla_x f(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} = 2Ax + B^\top y.$$

In summation form, the k-th component of the gradient is:

$$\frac{\partial f}{\partial x_k} = 2 \sum_{j=1}^{n} A_{kj} x_j + \sum_{i=1}^{n} B_{ik} y_i.$$

This is derived as follows:

$$\text{For the first term:} \quad x^\top Ax = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}x_ix_j.$$

When we take $\dfrac{\partial}{\partial x_k}$, only terms where $i = k$ or $j = k$ contribute.

If $i = k$, the term becomes $A_{kj}x_kx_j$, and $\dfrac{\partial}{\partial x_k}(A_{kj}x_kx_j) = A_{kj}x_j$.

If $j = k$, the term becomes $A_{ik}x_ix_k$, and $\dfrac{\partial}{\partial x_k}(A_{ik}x_ix_k) = A_{ik}x_i$.

Adding these contributions gives: $2\displaystyle\sum_{j=1}^{n} A_{kj}x_j$.

$$\text{For the second term:} \quad y^\top Bx = \sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij}y_ix_j.$$

When we take $\dfrac{\partial}{\partial x_k}$, only terms where $j = k$ contribute.

If $j = k$, the term becomes $B_{ik}y_ix_k$, and $\dfrac{\partial}{\partial x_k}(B_{ik}y_ix_k) = B_{ik}y_i$.

This gives: $\displaystyle\sum_{i=1}^{n} B_{ik}y_i$.

$$\text{For the third term:} \quad c \text{ is a constant, so } \frac{\partial}{\partial x_k}c = 0.$$

$$\text{Combining these results:} \quad \frac{\partial f}{\partial x_k} = 2\sum_{j=1}^{n} A_{kj}x_j + \sum_{i=1}^{n} B_{ik}y_i.$$

(c)

The gradient of $f(x, y)$ with respect to y is given by:

$$\nabla_y f(x, y) = \begin{bmatrix} \frac{\partial f}{\partial y_1} \\ \frac{\partial f}{\partial y_2} \\ \vdots \\ \frac{\partial f}{\partial y_n} \end{bmatrix} = Bx.$$

In summation form, the k-th component of the gradient is:

$$\frac{\partial f}{\partial y_k} = \sum_{j=1}^{n} B_{kj}x_j.$$

This is derived as follows:

$$\begin{aligned}
\text{For the first term:} \quad & x^\top A x = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j. \\
& \text{This term does not involve } y, \text{ so } \frac{\partial}{\partial y_k}(x^\top A x) = 0. \\[1em]
\text{For the second term:} \quad & y^\top B x = \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij} y_i x_j. \\
& \text{When we take } \frac{\partial}{\partial y_k}, \text{ only terms where } i = k \text{ contribute.} \\
& \text{If } i = k, \text{ the term becomes } B_{kj} y_k x_j, \text{ and } \frac{\partial}{\partial y_k}(B_{kj} y_k x_j) = B_{kj} x_j. \\
& \text{This gives: } \sum_{j=1}^{n} B_{kj} x_j. \\[1em]
\text{For the third term:} \quad & c \text{ is a constant, so } \frac{\partial}{\partial y_k} c = 0. \\[1em]
\text{Combining these results:} \quad & \frac{\partial f}{\partial y_k} = \sum_{j=1}^{n} B_{kj} x_j.
\end{aligned}$$