A3(a)

We know:

$$P(Z \leq z) = P(X + Y \leq z) = P(Y \leq z - X).$$

We can marginalize over X:

$$P(Z \leq z) = \int_{-\infty}^{\infty} P(Y \leq z - x) f(x)\, dx.$$

Now, let $G(y)$ denote the CDF of Y. Then:

$$P(Y \leq z - x) = G(z - x).$$

Substituting this into $P(Z \leq z)$:

$$P(Z \leq z) = \int_{-\infty}^{\infty} G(z - x) f(x)\, dx. \tag{1}$$

From the Fundamental Theorem of Calculus:

$$\frac{d}{dz} G(z - x) = g(z - x),$$

where $g(y)$ is the PDF of Y. This works because the derivative of the CDF $G(y)$ with respect to its variable yields the PDF $g(y)$, which represents the rate of change of the cumulative probability.

Differentiating $P(Z \leq z)$ using (1):

$$h(z) = \frac{d}{dz} P(Z \leq z) = \frac{d}{dz} \int_{-\infty}^{\infty} G(z - x) f(x)\, dx.$$

By the chain rule and the Fundamental Theorem of Calculus:

$$h(z) = \int_{-\infty}^{\infty} g(z - x) f(x)\, dx.$$

Thus:

$$h(z) = \int_{-\infty}^{\infty} f(x) g(z - x)\, dx.$$

$\blacksquare$