A5(a)

Matrix A

We have

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}.$$

We perform row operations to find its rank:

$$r_2 \leftarrow r_2 - r_3 \quad \Longrightarrow \quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix},$$

$$r_3 \leftarrow r_3 - r_1 \quad \Longrightarrow \quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix},$$

$$r_3 \leftarrow r_3 - r_2 \quad \Longrightarrow \quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the last row is all zeroes, the rank of A is 2.

Matrix B

We have

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

We perform the following operations:

$$r_1 \leftrightarrow r_3 \quad \Longrightarrow \quad \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix},$$

$$r_2 \leftrightarrow r_3 \quad \Longrightarrow \quad \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix},$$

$$r_2 \leftarrow r_2 - r_1 \quad \Longrightarrow \quad \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix},$$

$$r_3 \leftarrow r_3 - r_1 \quad \Longrightarrow \quad \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix},$$

$$r_3 \leftarrow r_3 + r_2 \quad \Longrightarrow \quad \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, the rank of B is also 2.

A5(b)

For both matrices A and B, the pivot columns are the first two columns. Hence a minimal basis for the column space (of either matrix) is given by their first two original columns:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

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