

A4(c)

Mean

First, consider the expectation of $\hat{\mu}_n$:

$$\mathbb{E}[\hat{\mu}_n] = \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n X_i\right] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[X_i].$$

Since $\mathbb{E}[X_i] = \mu$ for all i , this simplifies to:

$$\mathbb{E}[\hat{\mu}_n] = \frac{1}{n} \cdot n \cdot \mu = \mu.$$

Now, compute the expectation of $\sqrt{n}(\hat{\mu}_n - \mu)$:

$$\mathbb{E}[\sqrt{n}(\hat{\mu}_n - \mu)] = \sqrt{n} \cdot \mathbb{E}[\hat{\mu}_n - \mu].$$

Substituting $\mathbb{E}[\hat{\mu}_n] = \mu$, we get:

$$\mathbb{E}[\sqrt{n}(\hat{\mu}_n - \mu)] = \sqrt{n} \cdot (\mathbb{E}[\hat{\mu}_n] - \mu) = \sqrt{n} \cdot (\mu - \mu) = 0.$$

Variance

The variance of $\sqrt{n}(\hat{\mu}_n - \mu)$ is:

$$\text{Var}(\sqrt{n}(\hat{\mu}_n - \mu)) = \text{Var}\left(\sqrt{n} \cdot \frac{1}{n} \sum_{i=1}^n (X_i - \mu)\right).$$

Simplify the scaling factor $\sqrt{n} \cdot \frac{1}{n} = \frac{1}{\sqrt{n}}$:

$$\text{Var}(\sqrt{n}(\hat{\mu}_n - \mu)) = \frac{1}{n} \cdot \text{Var}\left(\sum_{i=1}^n (X_i - \mu)\right).$$

Since X_1, X_2, \dots, X_n are independent, the variance of the sum is the sum of variances:

$$\text{Var}\left(\sum_{i=1}^n (X_i - \mu)\right) = \sum_{i=1}^n \text{Var}(X_i - \mu).$$

Since $\text{Var}(X_i - \mu) = \text{Var}(X_i) = \sigma^2$ for all i , this becomes:

$$\text{Var}\left(\sum_{i=1}^n (X_i - \mu)\right) = n \cdot \sigma^2.$$

Substitute this back:

$$\text{Var}(\sqrt{n}(\hat{\mu}_n - \mu)) = \frac{1}{n} \cdot n \cdot \sigma^2 = \sigma^2.$$

Final Results

- **Mean:**

$$\mathbb{E}[\sqrt{n}(\hat{\mu}_n - \mu)] = 0.$$

- **Variance:**

$$\text{Var}[\sqrt{n}(\hat{\mu}_n - \mu)] = \sigma^2.$$

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