Exercise 8.1. [6pts]

(a) Find $\langle(3,2)\rangle \in \mathbb{Z}_4 \times \mathbb{Z}_3$. Write multiples of $(3,2)$ one by one until all elements of $\langle(3,2)\rangle$ are exhausted.

(b) Find $\langle(3,2)\rangle \in U_5 \times \mathbb{Z}_3$. Write multiples of $(3,2)$ one by one until all elements of $\langle(3,2)\rangle$ are exhausted. I'd like to emphasize that the first group in the product is multiplicative.

Exercise 8.2. [2pts] Consider any ring R. Show that if its characteristic $\chi(R) \neq 0$, then for any $a \in R$ we have $n \cdot a = 0$.

Exercise 8.3. [2pts] Let F be a field and $f(x) \in F[x]$. Show that if $f(x)$ is divisible by a polynomial $g(x) = a_n x^n + \ldots$ of degree n, then it is divisible by some monic polynomial of degree n.

Exercise 8.3 is very useful when we want to show that $f(x)$ does not have divisors of degree n, it eliminates non-monic divisors from consideration. For instance, there are 20 linear polynomials in $\mathbb{Z}_5[x]$

$$x, x+1, x+2, x+3, x+4, 2x, 2x+1, \ldots, 4x+4,$$

and only 5 of them are monic. Now, say we need to check that a cubic $f(x) = 2x^3 + x + 2$ is irreducible.

$$\begin{aligned}
f(x) \text{ is NOT irreducible} \quad &\Leftrightarrow \quad f(x) = g(x)h(x), \quad \text{where } g(x), h(x) \text{ are non-constant} \\
&\Leftrightarrow \quad f(x) = g(x)h(x), \quad \text{where } g(x) \text{ or } h(x) \text{ is linear (because } \deg(f) = 3) \\
&\Leftrightarrow \quad f(x) \quad \text{has a linear factor} \\
&\Leftrightarrow \quad f(x) \quad \text{has a linear monic factor } x - \alpha \\
&\Leftrightarrow \quad f(\alpha) = 0 \quad \text{for some } \alpha \in \mathbb{Z}_5.
\end{aligned}$$

Now, $f(x) = 2x^3 + x + 2$ is not irreducible because $f(1) = 0$ and, hence, has a factor $x - 1$. This works for quadratic or linear f, because a quartic f can be a product of two quadratic polynomials.

Exercise 8.4. [5pts] Check if the following polynomials are irreducible or not.

(a) $f(x) = x^3 + 2x - 1 \in \mathbb{Z}_3[x]$

(b) $f(x) = x^3 + 2x^2 + 2x + 1 \in \mathbb{Z}_5[x]$

(c) To check if $f(x) = x^4 + x^3 + x^2 + x + 1 \in \mathbb{Z}_2[x]$ is irreducible you will need to consider linear factors and (irreducible) quadratic factors (which easy because $\mathbb{Z}_2[x]$ has a unique irreducible quadratic polynomial mentioned in class).

Exercise 8.5. [5pts] Find the remainder of division of $2x^6 + x^2 - 1$ by $x^2 + 3x + 2$ in $\mathbb{Z}_5[x]$.

Exercise 8.6. [10pts] For $f(x) = 4x^4 - x^3 + 3x^2 + x - 2$ and $g(x) = 4x^5 + x^3$ in $\mathbb{Z}_5[x]$ use the Euclidean algorithm to find

(a) $\gcd(f(x), g(x))$. [Hint. Do not forget that gcd must be monic].

(b) Polynomials $\alpha(x), \beta(x) \in \mathbb{Z}_5[x]$ satisfying $\gcd(f(x), g(x)) = \alpha(x)f(x) + \beta(x)g(x)$.

Definition 8.1. A **vector space** over a field F is a set V equipped with two operations:

- **(addition)** $+ : V \times V \to V$;
- **(scalar multiplication)** $\cdot : F \times V \to V$.

satisfying the following conditions for $a, b, c \in V$ and $\alpha, \beta \in F$:

- $a + b = b + a$ and $(a + b) + c = a + (b + c)$.
- $a + 0 = 0 + a$ and $a + (-a) = 0$.
- $\alpha(\beta a) = (\alpha\beta)a$ and $1a = a$.
- $(\alpha + \beta)a = \alpha a + \beta a$ and $\alpha(a + b) = \alpha a + \alpha b$.

Elements of V are called **vectors** and elements of F are called **scalars**.

Exercise 8.7. [+5pts] Show that the set of complex numbers $\mathbb{C}$ with standard complex addition and multiplication is a vector space over a field $\mathbb{R}$.

Exercise 8.8. [+5pts] Let F be a vector space. Show that $F^n = \{ (\alpha_1, \ldots, \alpha_n) \mid \alpha_1, \ldots, \alpha_n \in F \}$ with $+$ and $\cdot$ defined by

$$(\alpha_1, \ldots, \alpha_n) + (\beta_1, \ldots, \beta_n) = (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n),$$
$$c(\alpha_1, \ldots, \alpha_n) = (c\alpha_1, \ldots, c\alpha_n)$$

is a vector space over F.