`hw4`

Coding the Matrix, Summer 2013

Please fill out the stencil file named "`hw4.py`". While we encourage you to complete the Ungraded Problems, they do not require any entry into your stencil file.

Span of vectors over $\mathbb{R}$

Problem 1: Let $\mathcal{V} = \text{Span } \{[2,0,4,0],[0,1,0,1],[0,0,-1,-1]\}$. For each of the following vectors, show it belongs to $\mathcal{V}$ by writing it as a linear combination of the generators of $\mathcal{V}$.

(a) $[2,1,4,1]$

(b) $[1,1,1,0]$

(c) $[0,1,1,2]$

Problem 2: Let $\mathcal{V} = \text{Span } \{[0,0,1],[2,0,1],[4,1,2]\}$. For each of the following vectors, show it belongs to $\mathcal{V}$ by writing it as a linear combination of the generators of $\mathcal{V}$.

(a) $[2,1,4]$

(b) $[1,1,1]$

(c) $[5,4,3]$

(d) $[0,1,1]$

Span of Vectors over $GF(2)$

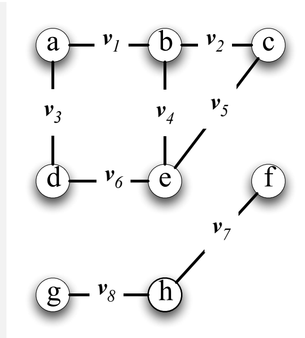
Problem 3: Let $\mathcal{V} = \text{Span } \{[0,1,0,1],[0,0,1,0],[1,0,0,1],[1,1,1,1]\}$ where the vectors are over $GF(2)$. For each of the following vectors over $GF(2)$, show it belongs to $\mathcal{V}$ by writing it as a linear combination of the generators of $\mathcal{V}$.

(a) $[1,1,0,0]$

(b) $[1,0,1,0]$

(c) $[1,0,0,0]$

Problem 4: The vectors over $GF(2)$ representing the graph



are

	a	b	c	d	e	f	g	h
$\boldsymbol{v}_1$	1	1						
$\boldsymbol{v}_2$		1	1					
$\boldsymbol{v}_3$	1			1				
$\boldsymbol{v}_4$		1			1			
$\boldsymbol{v}_5$			1		1			
$\boldsymbol{v}_6$				1	1			
$\boldsymbol{v}_7$						1		1
$\boldsymbol{v}_8$							1	1

For each of the following vectors over $GF(2)$, show it belongs to the span of the above vectors by writing it as a linear combination of the above vectors.

(a) $[0, 0, 1, 1, 0, 0, 0, 0]$

(b) $[0, 0, 0, 0, 0, 1, 1, 0]$

(c) $[1, 0, 0, 0, 1, 0, 0, 0]$

(d) $[0, 1, 0, 1, 0, 0, 0, 0]$

Linear Dependence over $\mathbb{R}$

Problem 5: For each of the parts below, show the given vectors over $\mathbb{R}$ are linearly dependent by writing the zero vector as a nontrivial linear combination of the vectors.

(a) $[1, 2, 0], [2, 4, 1], [0, 0, -1]$

(b) $[2, 4, 0], [8, 16, 4], [0, 0, 7]$

(c) $[0, 0, 5], [1, 34, 2], [123, 456, 789], [-3, -6, 0], [1, 2, 0.5]$

Problem 6: For each of the parts below, show the given vectors over $\mathbb{R}$ are linearly dependent by writing the zero vector as a nontrivial linear combination of the vectors.

(a) $[1, 2, 3], [4, 5, 6], [1, 1, 1]$

(b) $[0, -1, 0, -1], [\pi, \pi, \pi, \pi], [-\sqrt{2}, \sqrt{2}, -\sqrt{2}, \sqrt{2}]$

(c) $[1,-1,0,0,0],[0,1,-1,0,0],[0,0,1,-1,0],[0,0,0,1,-1],[-1,0,0,0,1]$

Problem 7: Show that one of the vectors is superfluous by expressing it as a linear combination of the other two.

$$\begin{aligned} \boldsymbol{u} &= [3,9,6,5,5] \\ \boldsymbol{v} &= [4,10,6,6,8] \\ \boldsymbol{w} &= [1,1,0,1,3] \end{aligned}$$

Problem 8: Give four vectors that are linearly dependent but such that any three are linearly independent.

Linear Dependence over $GF(2)$

Problem 9: For each of the parts below, show the given vectors over $GF(2)$ are linearly dependent by writing the zero vector as a nontrivial linear combination of the vectors.

(a) $[1,1,1,1],[1,0,1,0],[0,1,1,0],[0,1,0,1]$

(b) $[0,0,0,1],[0,0,1,0],[1,1,0,1],[1,1,1,1]$

(c) $[1,1,0,1,1],[0,0,1,0,0],[0,0,1,1,1],[1,0,1,1,1],[1,1,1,1,1]$

Problem 10: Each of the parts below specifies some of the vectors over $GF(2)$ specified in Problem 4. Show that these vectors are linearly dependent by giving a subset of those vectors whose sum is the zero vector. (Hint: Looking at the graph will help.)

(a) $\{\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3,\boldsymbol{v}_4,\boldsymbol{v}_5\}$

(b) $\{\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3,\boldsymbol{v}_4,\boldsymbol{v}_5,\boldsymbol{v}_7,\boldsymbol{v}_8\}$

(c) $\{\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3,\boldsymbol{v}_4,\boldsymbol{v}_6\}$

(d) $\{\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3,\boldsymbol{v}_5,\boldsymbol{v}_6,\boldsymbol{v}_7,\boldsymbol{v}_8\}$

Exchange Lemma for Vectors over $\mathbb{R}$

Problem 11: Let $S = \{[1,0,0,0,0],[0,1,0,0,0],[0,0,1,0,0],[0,0,0,1,0],[0,0,0,0,1]\}$, and let $A = \{[1,0,0,0,0],[0,1,0,0,0]\}$. For each of the following vectors $\boldsymbol{z}$, find a vector $\boldsymbol{w}$ in $S - A$ such that Span S = Span $(S \cup \{\boldsymbol{z}\} - \{\boldsymbol{w}\})$.

(a) $\boldsymbol{z} = [1,1,1,1,1]$

(b) $\boldsymbol{z} = [0,1,0,1,0]$

(c) $\boldsymbol{z} = [1,0,1,0,1]$

Exchange Lemma for Vectors over GF(2)

Problem 12: Let

	a	b	c	d	e	f	g	h
$\boldsymbol{v}_1$	1	1						
$\boldsymbol{v}_2$		1	1					
$\boldsymbol{v}_3$	1			1				
$\boldsymbol{v}_4$		1			1			

Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$. Each of the following parts specifies a subset A of S and a vector $\boldsymbol{z}$ such that $A \cup \{\boldsymbol{z}\}$ is linearly independent. For each part, specify a vector $\boldsymbol{w}$ in $S - A$ such that Span $S =$ Span $(S \cup \{\boldsymbol{z}\} - \{\boldsymbol{w}\})$. (Hint: Drawing subgraphs of the graph will help.)

(a) $A = \{\boldsymbol{v}_1, \boldsymbol{v}_4\}$ and $\boldsymbol{z}$ is

a	b	c	d	e	f	g	h
			1	1			

(b) $A = \{\boldsymbol{v}_2, \boldsymbol{v}_3\}$ and $\boldsymbol{z}$ is

a	b	c	d	e	f	g	h
		1	1				

(c) $A = \{\boldsymbol{v}_2, \boldsymbol{v}_3\}$ and $\boldsymbol{z}$ is

a	b	c	d	e	f	g	h
1				1			

Problem 13: Write and test a procedure `rep2vec(u, veclist)` with the following spec:

- *input:* a vector $\boldsymbol{u}$ and a list *veclist* of `Vecs` $[\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{n-1}]$. The domain of $\boldsymbol{u}$ should be $\{0, 1, 2, n-1\}$ where n is the length of `veclist`.
- *output:* the vector $\boldsymbol{v}$ such that $\boldsymbol{u}$ is the coordinate representation of $\boldsymbol{v}$ with respect to $\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{n-1}$, where entry i of $\boldsymbol{u}$ is the coefficient of $\boldsymbol{a}_i$ for $i = 0, 1, 2 \ldots, n-1$.

Your procedure should not use any loops or comprehensions but of course can use the operations on instances of `Mat` and `Vec` and can also use procedures from the `matutil` module. Note that the procedures `coldict2mat` and `rowdict2mat` (defined in `matutil`) can accept lists, not just dictionaries.

In the following problem and others to come, you should make use of the `solve` procedure of the `solver` module.

Problem 14: Write and test a procedure `vec2rep(veclist, v)` with the following spec:

- *input:* a list `veclist` of vectors $[\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{n-1}]$, and a vector $\boldsymbol{v}$ with the domain $\{0, 1, 2, \ldots, n-1\}$ where n is the length of `veclist`. You can assume $\boldsymbol{v}$ is in Span $\{\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{n-1}\}$.
- *output:* the vector $\boldsymbol{u}$ whose coordinate representation with respect to $\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{n-1}$ is $\boldsymbol{v}$.

As in Problem 13, your procedure should use no loops or comprehensions directly but can use procedures defined in `matutil`.

Problem 15: Write and test a procedure `is_superfluous(L, i)` with the following spec:

- *input:* a list L of vectors, and an integer i in $\{0, 1, \ldots, n-1\}$ where $n = \text{len}(L)$

- *output:* `True` if the span of the vectors in L equals the span of

$$L[0], L[1], \ldots, L[i-1], L[i+1], ..., L[n-1]$$

Your procedure should not use loops or comprehensions but can use procedures defined in the module `matutil` and can use the procedure `solve(A,b)` defined in `solver` module. Your procedure will most likely need a special case for the case where len(L) is 1.

Note that the `solve(A,b)` always returns a vector $\boldsymbol{u}$. It is up to you to check that $\boldsymbol{u}$ is in fact a solution to the equation $A\boldsymbol{x} = \boldsymbol{b}$. Moreover, over $\mathbb{R}$, even if a solution exists, the solution returned by `solve` is approximate due to roundoff error. To check whether the vector $\boldsymbol{u}$ returned is a solution, you should compute the residual $\boldsymbol{b} - A * \boldsymbol{u}$, and test if it is close to the zero vector:

```
>>> residual = b - A*u
>>> residual * residual
1.819555009546577e-25
```

If the sum of squares of the entries of the residual (the dot-product of the residual with itself) is less than, say 10^{-14}, it is pretty safe to conclude that $\boldsymbol{u}$ is indeed a solution.

Problem 16: Write and test a procedure `is_independent(L)` with the following spec:

- *input:* a list L of vectors
- *output:* `True` if the vectors form a linearly independent list.

Your algorithm for this procedure should be based on the Span Lemma. You can use as a subroutine any one of the following:

- the procedure `is_superfluous(L, b)` from Problem 15, or
- the `solve(A,b)` procedure from the `solver` module (but see the provisos in Problem 15).

You will need a loop or comprehension for this procedure.

Problem 17: Write and test a procedure `superset_basis(S, L)` with the following spec:

- *input:* a linearly independent list S of vectors, and a list L of vectors such that every vector in S is in the span of L.
- *output:* a linearly independent list T containing all vectors in S such that the span of T equals the span of L (i.e. T is a basis for the span of L).

Your procedure should be based on either a version of the Grow algorithm or a version of the Shrink algorithm. Think about each one to see which is easier for you. You will need a loop or comprehension for this procedure. You can use as a subroutine any one of the following:

- the procedure `is_superfluous(L, b)` from Problem 15, or
- the procedure `is_independent(L)` from Problem 16, or
- the procedure `solve(A,b)` from the `solver` module (but see the provisos in Problem 15).

Problem 18: Write and test a procedure `exchange(S, A, z)` with the following spec:

- *input:* A list S of vectors, a list A of vectors that are all in S (such that len$(A) <$ len(S)), and a vector z such that $A + [z]$ is linearly independent
- *output:* a vector $\boldsymbol{w}$ in S but not in A such that

$$\text{Span } S = \text{Span } (\{\boldsymbol{z}\} \cup S - \{\boldsymbol{w}\})$$

Your procedure should follow the proof of the Exchange Lemma. You should use the `solver` module or the procedure `vec2rep(veclist, u)` from Problem 14. You can test whether a vector is in a list using the expression `v in L`.