Lagrangian Mechanics

DISCLAIMER:

This PDF contains most of the important formulae and derivations of Lagrangian Mechanics, but not all.

Content presented here might be concise, but it's complete.

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- **Important Relations between 2 Coordinate Systems:**
  Consider two coordinate systems Q and X. Coordinate transformation $X \to Q$ and inverse transformation $Q \to X$ can be written as

$$q^i = q^i(x^j(t), t) \qquad \& \qquad x^j = x^j(q^i(t), t)$$

Note that, if we forget about transformation, then all the coordinates are themselves function of time, i.e., $q^i = q^i(t)$ and $x^j = x^j(t)$. Also, the coordinate transformations are independent of velocities, i.e.,

$$\frac{dq^i}{d\dot{x}^j} = 0 \qquad \& \qquad \frac{dx^j}{d\dot{q}^i} = 0$$

- Now, we'll derive one important relation, which is $\frac{\partial \dot{q}^i}{\partial \dot{x}^j} = \frac{\partial q^i}{\partial x^j}$. Lets write expression for $\dot{q}^i$ using chain rule:

$$\dot{q}^i = \frac{dq^i}{dt} = \sum_k \frac{\partial q^i}{\partial x^k}\frac{dx^k}{dt} + \frac{\partial q^i}{\partial t}$$

Now, differentiate this w.r.t. $\dot{x}^j$

$$\begin{aligned}
\frac{\partial \dot{q}^i}{\partial \dot{x}^j} &= \frac{\partial}{\partial \dot{x}^j}\left(\sum_k \frac{\partial q^i}{\partial x^k}\dot{x}^k\right) + \frac{\partial}{\partial \dot{x}^j}\frac{\partial q^i}{\partial t} \\
&= \sum_k \left(\dot{x}^k \frac{\partial}{\partial \dot{x}^j}\frac{\partial q^i}{\partial x^k} + \frac{\partial q^i}{\partial x^k}\frac{\partial \dot{x}^k}{\partial \dot{x}^j}\right) + \frac{\partial}{\partial t}\frac{\partial q^i}{\partial \dot{x}^j} \\
&= \sum_k \left(\dot{x}^k \frac{\partial}{\partial x^k}\frac{\partial q^i}{\partial \dot{x}^j} + \frac{\partial q^i}{\partial x^k}\delta_{jk}\right) \\
&= \sum_k \frac{\partial q^i}{\partial x^k}\delta_{jk} \\
&= \frac{\partial q^i}{\partial x^j}
\end{aligned}$$

- Now, we'll derive second important relation, which is $\frac{d}{dt}\frac{\partial x^j}{\partial q^i} = \frac{\partial \dot{x}^j}{\partial q^i}$. Now, the partial derivative $\frac{\partial \dot{x}^j}{\partial q^i}$ itself can be expressed as a function $f(q^i(t), t)$, so we can expand

$$\begin{aligned}
\frac{d}{dt}\frac{\partial x^j}{\partial q^i} &= \sum_k \left[\frac{\partial}{\partial q^k}\left(\frac{\partial x^j}{\partial q^i}\right)\frac{dq^k}{dt}\right] + \frac{\partial}{\partial t}\frac{\partial x^j}{\partial q^i} \\
&= \sum_k \left[\frac{\partial}{\partial q^i}\left(\frac{\partial x^j}{\partial q^k}\right)\frac{dq^k}{dt}\right] + \frac{\partial}{\partial q^i}\frac{\partial x^j}{\partial t} \\
&= \sum_k \left[\frac{\partial}{\partial q^i}\left(\frac{\partial x^j}{\partial q^k}\frac{dq^k}{dt}\right) - \frac{\partial x^j}{\partial q^k}\left(\frac{\partial}{\partial q^i}\frac{dq^k}{dt}\right)\right] + \frac{\partial}{\partial q^i}\frac{\partial x^j}{\partial t} \\
&= \sum_k \left[\frac{\partial}{\partial q^i}\left(\frac{\partial x^j}{\partial q^k}\frac{dq^k}{dt}\right)\right] + \frac{\partial}{\partial q^i}\frac{\partial x^j}{\partial t} \\
&= \frac{\partial}{\partial q^i}\left[\sum_k \left(\frac{\partial x^j}{\partial q^k}\frac{dq^k}{dt}\right) + \frac{\partial x^j}{\partial t}\right] \\
&= \frac{\partial}{\partial q^i}\frac{dx^j}{dt}
\end{aligned}$$

Similarly, we can derive $\frac{d}{dt}\frac{\partial q^i}{\partial x^j} = \frac{\partial \dot{q}^i}{\partial x^j}$. Also, we can write this relation, not just for coordinates, but for any function of form $\varphi(q^i, t)$, i.e.,

$$\frac{d}{dt}\frac{\partial \varphi}{\partial q^i} = \frac{\partial}{\partial q^i}\frac{d\varphi}{dt}$$

- **Virtual Displacement & Virtual Work:**

  A **virtual displacement** δx^j can be defined as an infinitesimal displacement in x_i carried out at a given instant, i.e., $dt = 0$. So, we can write difference between derivative and virtual displacement as

$$dx^j = \sum_i \frac{\partial x^j}{\partial q^i} dq^i + \frac{\partial x^j}{\partial t} dt, \qquad \text{while} \qquad \delta x^j = \sum_i \frac{\partial x^j}{\partial q^i} \delta q^i$$

  Now, lets consider that the components of force applied on a particle, in X and Q coordinate systems are given by F_j and G_i respectively. Then, the **virtual work** done δW is given by

$$\delta W = \sum_j F_j \, \delta x^j = \sum_j F_j \left(\sum_i \frac{\partial x^j}{\partial q^i} \delta q^i \right) = \sum_i \left(\sum_j \frac{\partial x^j}{\partial q^i} F_j \right) \delta q^i = \sum_i G_i \, \delta q^i$$

  from above equation, we get $G_i = \sum_j \frac{\partial x^j}{\partial q^i} F_j$, which means that the components of force are covariant.

- **Principle of Virtual Work & d'Alembert Principle:**

  Suppose that the system is in **static equilibrium**, i.e., the net force on each of the particles is zero ($\boldsymbol{F_r} = \boldsymbol{0}$)[1], so there is no unbalanced force. Now the virtual work done due to any arbitrary virtual displacement in the coordinates that is consistence with the constraints is written as

$$\sum_r \boldsymbol{F}_r \cdot \delta \boldsymbol{x}_r = 0$$

  (Since, the net force is zero, then definitely the work done is going to be zero.) Now, we can write the net force $\boldsymbol{F}_r$ as vector sum of *applied force* and *constraint force*, i.e.,

$$\boldsymbol{F}_r = \boldsymbol{F}_r^{(a)} + \boldsymbol{F}_r^{(c)}$$

  Constraint force is actually defined as

$$\boldsymbol{F}_r^{(c)} := \boldsymbol{F}_r - \boldsymbol{F}_r^{(a)}$$

  Now, the net constraint force on the system by definition should be zero ($\sum_r \boldsymbol{F}_r^{(c)} = \boldsymbol{0}$), otherwise the particle will accelerate out of the constraint surface. And in case of static equilibrium, the net applied force on the system has to be zero ($\sum_r \boldsymbol{F}_r^{(a)} = \boldsymbol{0}$), otherwise it won't be an equilibrium. If we displace the system virtually, the net virtual work done by the constraint forces is 0, i.e.,

$$\sum_r \boldsymbol{F}_r^{(c)} \cdot \delta \boldsymbol{x}_r = 0$$

  Note: If we would have considered the real displacement instead of virtual, then the same could have hold true, except the case when the constraint forces changes with time. So we freeze the time and talk about virtual displacement.

  We can now state the **principle of virtual work** as, 'when the system is in static equilibrium, the work done in displacing the system virtually (virtual work), which is consistence with constraint force done by applied force is zero'.

$$\sum_r \boldsymbol{F}_r^{(a)} \cdot \delta \boldsymbol{x}_r = 0$$

  Now, let's consider the case when the system is not in static equilibrium. Then by Newton's law, $\boldsymbol{F}_r = \dot{\boldsymbol{p}}_r$. So if we consider the force $\boldsymbol{f}_r = \boldsymbol{F}_r - \dot{\boldsymbol{p}}_r$, then this force can keep the system in equilibrium. So, from principle of virtual work, we get

$$\sum_r (\boldsymbol{F}_r - \dot{\boldsymbol{p}}_r) \cdot \delta \boldsymbol{x}_r = 0$$

[1] Index r represents the different particles in a system

Now, separate the applied and constraint force from this equation

$$\sum_r (\boldsymbol{F}_r^{(a)} - \dot{\boldsymbol{p}}_r) \cdot \delta \boldsymbol{x}_r + \sum_r \boldsymbol{F}_r^{(c)} \cdot \delta \boldsymbol{x}_r = 0$$

Since, the virtual work done by constraint forces is zero, so from this, we get the **d'Alembert principle**:

$$\sum_r (\boldsymbol{F}_r^{(a)} - \dot{\boldsymbol{p}}_r) \cdot \delta \boldsymbol{x}_r = 0$$

- **Central Lagrangian Equation & Lagrangian Equation of Motion:**
  Lets expand the force F_j in virtual work expression in terms of momentum $m\dot{x}^j$:

$$\begin{aligned}
\delta W = \sum_i G_i \, \delta q^i &= \sum_i \left[\sum_j \frac{\partial x^j}{\partial q^i} F_j \right] \delta q^i \\
&= \sum_i \left[\sum_j \frac{\partial x^j}{\partial q^i} \frac{d}{dt} m\dot{x}^j \right] \delta q^i \\
&= \sum_i \left[\sum_j \left(\frac{d}{dt}\left(m\dot{x}^j \frac{\partial x^j}{\partial q^i} \right) - m\dot{x}^j \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^i} \right) \right) \right] \delta q^i \\
&= \sum_i \left[\sum_j \left(\frac{d}{dt}\left(m\dot{x}^j \frac{\partial \dot{x}^j}{\partial \dot{q}^i} \right) - m\dot{x}^j \frac{\partial \dot{x}^j}{\partial q^i} \right) \right] \delta q^i \\
&= \sum_i \left[\frac{d}{dt} \frac{\partial}{\partial \dot{q}^i} \left(\sum_j \frac{1}{2} m\dot{x}^{j\,2} \right) - \frac{\partial}{\partial q^i} \left(\sum_j \frac{1}{2} m\dot{x}^{j\,2} \right) \right] \delta q^i \\
&= \sum_i \left[\frac{d}{dt} \frac{\partial T}{\partial \dot{q}^i} - \frac{\partial T}{\partial q^i} \right] \delta q^i
\end{aligned}$$

So, we get an important relation here, which is called **Central Lagrangian Equation**:

$$G_i = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^i} - \frac{\partial T}{\partial q^i}$$

T here is the kinetic energy of system. Now, let's say that a force can be written as negative gradient of some *scalar potential* $U(q^i)$ (conservative force), i.e.,

$$G_i = -\frac{\partial U}{\partial q^i}$$

Now, substituting this in central Lagrangian equation will give

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}^i} - \frac{\partial T}{\partial q^i} + \frac{\partial U}{\partial q^i} = 0$$

Now, let's define the **Langrangian** function as $\mathcal{L} := T(\dot{q}^i) - U(q^i)$, then we can write the above relation as **Lagrangian-d'Alembert Equation**:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \frac{\partial \mathcal{L}}{\partial q^i} = 0$$

Since $\dfrac{\partial U}{\partial \dot{q}^i} = 0$, $\dfrac{\partial \mathcal{L}}{\partial \dot{q}^i} = \dfrac{\partial T}{\partial \dot{q}^i}$. This equation is also called *Lagrangian Equation of Motion* or *Euler-Lagrangian Equation* (when derived using *principle of least action*).

- **Gauge Invariance of Lagrangian Equation of Motion:**
  Let $\mathcal{L}(q^i, \dot{q}^i, t)$ be a Lagrangian in some coordinate system with Lagrange equation of motion. Let

$$\tilde{\mathcal{L}} = \mathcal{L} + \frac{d}{dt}F(q^j, t)$$

where F is some arbitrary differentiable function. Then $\bar{\mathcal{L}}$ also satisfies the Lagrange equation of motion.

Proof: We can write

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial\tilde{\mathcal{L}}}{\partial\dot{q}^i}\right) - \frac{\partial\tilde{\mathcal{L}}}{\partial q^i} &= \frac{d}{dt}\left[\frac{\partial}{\partial\dot{q}^i}\left(\mathcal{L} + \frac{d}{dt}F(q^k, t)\right)\right] - \frac{\partial}{\partial q^i}\left(\mathcal{L} + \frac{d}{dt}F(q^k, t)\right) \\ &= \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}^i}\right) + \frac{d}{dt}\left[\frac{\partial}{\partial\dot{q}^i}\left(\frac{d}{dt}F(q^k, t)\right)\right] - \frac{\partial\mathcal{L}}{\partial q^i} - \frac{\partial}{\partial q^i}\frac{d}{dt}F(q^k, t)\end{aligned}$$

The terms above in red cancel and gives zero. Now, we can write the derivative of F as

$$\frac{d}{dt}F(q^k, t) = \sum_k \frac{\partial F}{\partial q^k}\dot{q}^k + \frac{\partial F}{\partial t}$$

Substituting this in equation above will give

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial\tilde{\mathcal{L}}}{\partial\dot{q}^i}\right) - \frac{\partial\tilde{\mathcal{L}}}{\partial q^i} &= \frac{d}{dt}\left[\frac{\partial}{\partial\dot{q}^i}\left(\sum_k \frac{\partial F}{\partial q^k}\dot{q}^k + \frac{\partial F}{\partial t}\right)\right] - \frac{\partial}{\partial q^i}\left(\sum_k \frac{\partial F}{\partial q^k}\dot{q}^k + \frac{\partial F}{\partial t}\right) \\ &= \frac{d}{dt}\left[\sum_k \frac{\partial}{\partial\dot{q}^i}\left(\frac{\partial F}{\partial q^k}\dot{q}^k\right) + \frac{\partial}{\partial\dot{q}^i}\frac{\partial F}{\partial t}\right] - \sum_k \frac{\partial}{\partial q^i}\left(\frac{\partial F}{\partial q^k}\dot{q}^k\right) - \frac{\partial}{\partial q^i}\frac{\partial F}{\partial t} \\ &= \frac{d}{dt}\left[\sum_k \frac{\partial}{\partial\dot{q}^i}\left(\frac{\partial F}{\partial q^k}\right)\dot{q}^k + \sum_k \frac{\partial F}{\partial q^k}\frac{\partial\dot{q}^k}{\partial\dot{q}^i} + \frac{\partial}{\partial t}\frac{\partial F}{\partial\dot{q}^i}\right] - \sum_k \frac{\partial}{\partial q^i}\left(\frac{\partial F}{\partial q^k}\dot{q}^k\right) - \frac{\partial}{\partial q^i}\frac{\partial F}{\partial t} \\ &= \frac{d}{dt}\left[\sum_k \frac{\partial}{\partial q^k}\frac{\partial F}{\partial\dot{q}^i}\dot{q}^k + \sum_k \frac{\partial F}{\partial q^k}\delta^k_i\right] - \sum_k \frac{\partial}{\partial q^i}\left(\frac{\partial F}{\partial q^k}\right)\dot{q}^k - \sum_k \frac{\partial F}{\partial q^k}\frac{\partial\dot{q}^k}{\partial q^i} - \frac{\partial}{\partial q^i}\frac{\partial F}{\partial t} \\ &= \frac{d}{dt}\left[\frac{\partial F}{\partial q^i}\right] - \sum_k \frac{\partial}{\partial q^i}\left(\frac{\partial F}{\partial q^k}\right)\dot{q}^k - \sum_k \frac{\partial F}{\partial q^k}\frac{\partial\dot{q}^k}{\partial q^i} - \frac{\partial}{\partial q^i}\frac{\partial F}{\partial t} \\ &= \sum_k \frac{\partial}{\partial q^k}\left(\frac{\partial F}{\partial q^i}\right)\dot{q}^k + \frac{\partial}{\partial t}\frac{\partial F}{\partial q^i} - \sum_k \frac{\partial}{\partial q^i}\left(\frac{\partial F}{\partial q^k}\right)\dot{q}^k - \frac{\partial}{\partial q^i}\frac{\partial F}{\partial t}\end{aligned}$$

We can easily see that that first term cancels thirds term, and second term cancels last term. Thus

$$\frac{d}{dt}\left(\frac{\partial\tilde{\mathcal{L}}}{\partial\dot{q}^i}\right) - \frac{\partial\tilde{\mathcal{L}}}{\partial q^i} = 0$$

- **Point Transformation in Lagrangian Mechanics:**
  Let $\mathcal{L}(q^i, \dot{q}^i, t)$ be a Lagrangian in some coordinate system with Lagrange equation of motion, i.e.,

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}^i} - \frac{\partial\mathcal{L}}{\partial q^i} = 0$$

Consider a coordinate transformation $q^i = q^i(x^j, t)$ (such transformations are called **point transformations**). If we express the Lagrangian as $\mathcal{L}(x^j, \dot{x}^j, t)$, then this Lagrangian satisfies Lagrange equation of motion w.r.t. coordinate system X.

Proof: First, we write the partial derivative

$$\frac{\partial\mathcal{L}}{\partial\dot{x}^j} = \sum_i \frac{\partial\mathcal{L}}{\partial q^i}\frac{\partial q^i}{\partial\dot{x}^j} + \sum_i \frac{\partial\mathcal{L}}{\partial\dot{q}^i}\frac{\partial\dot{q}^i}{\partial\dot{x}^j} + \frac{\partial\mathcal{L}}{\partial t}\frac{\partial t}{\partial\dot{x}^j}$$

The terms $\dfrac{\partial t}{\partial \dot{x}^j}$ and $\dfrac{\partial q^i}{\partial \dot{x}^j}$ are identically equal to zero because of their functional form. We can replace $\dfrac{\partial \dot{q}^i}{\partial \dot{x}^j} = \dfrac{\partial q^i}{\partial x^j}$. So we are left with

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^j} = \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \frac{\partial q^i}{\partial x^j}$$

Next, we write the derivative

$$\frac{\partial \mathcal{L}}{\partial x^j} = \sum_i \frac{\partial \mathcal{L}}{\partial q^i} \frac{\partial q^i}{\partial x^j} + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \frac{\partial \dot{q}^i}{\partial x^j} + \frac{\partial \mathcal{L}}{\partial t} \frac{\partial t}{\partial x^j}$$

Again, the last term vanishes because of its functional form. So, we are left with only first and second terms. Now, we can plug this equations into following expression

$$\begin{aligned}\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}^i} - \frac{\partial \mathcal{L}}{\partial x^i} &= \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \frac{d}{dt}\left(\frac{\partial q^i}{\partial x^j}\right) + \sum_i \frac{\partial q^i}{\partial x^j} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\right) - \sum_i \frac{\partial \mathcal{L}}{\partial q^i} \frac{\partial q^i}{\partial x^j} - \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \frac{\partial \dot{q}^i}{\partial x^j} \\ &= \sum_i \frac{\partial q^i}{\partial x^j}\left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\right) - \frac{\partial \mathcal{L}}{\partial q^i}\right] = 0\end{aligned}$$

So we get

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}^i} - \frac{\partial \mathcal{L}}{\partial x^i} = 0$$

That is, 'Lagrange Equation of Motion is invariant under coordinate transformation'.

- **The Stationary Action Principle and Euler-Lagrange Equation:**

For a given Lagrangian $\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ (may not be of form $T - U$), we define the **action** along a path or curve $\boldsymbol{q}(t)$ as

$$\mathcal{A}[\boldsymbol{q}] = \int_{t_1}^{t_2} \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)\, dt$$

The stationary action principle (also known as principle of least action) say that, if we fix the end points of path $\boldsymbol{q}(t_1)$ and $\boldsymbol{q}(t_2)$, then the particle will take the path or trajectory, which has 'stationary' (mostly 'least') action.

Now, lets model the situation mathematically and see the outcome of stationary action principle. First, we need a way to represent all possible paths between two fixed point $\boldsymbol{q}(t_1)$ and $\boldsymbol{q}(t_2)$. For that, lets first fix one path between these two points and call it the **unvaried path**, denoted by $\boldsymbol{q}_0(t)$.

Note: $\boldsymbol{q}_0(t_1) = \boldsymbol{q}(t_1)$ and $\boldsymbol{q}_0(t_2) = \boldsymbol{q}(t_2)$

Now we can describe all other possible paths between these two end points using another parameter, call it α, as

$$\boldsymbol{q}(t, \alpha) = \boldsymbol{q}_0(t) + \alpha \boldsymbol{\eta}(t)$$

Where $\boldsymbol{\eta}(t)$ is any arbitrary shape function.

Note: $\boldsymbol{q}(t, 0) = \boldsymbol{q}_0(t)$, and $\boldsymbol{\eta}(t_1) = \boldsymbol{0} = \boldsymbol{\eta}(t_2)$ as all path $\boldsymbol{q}(t, \alpha)$ has same end points (Dirichlet boundary condition).

The variation of path from unvaried path is the quantity $\delta \boldsymbol{q}(t)$ defined as

$$\delta \boldsymbol{q}(t) = \boldsymbol{q}(t, \alpha) - \boldsymbol{q}_0(t) = \alpha \boldsymbol{\eta}(t)$$

Further, we can also define the velocity along any path as

$$\dot{\boldsymbol{q}}(t, \alpha) = \dot{\boldsymbol{q}}_0(t) + \alpha \dot{\boldsymbol{\eta}}(t)$$

Now, we can write the expression for variation in action as

$$\delta \mathcal{A}[\boldsymbol{q}(t, \alpha)] = \mathcal{A}[\boldsymbol{q} + \delta \boldsymbol{q}] - \mathcal{A}[\boldsymbol{q}] = \frac{d\mathcal{A}}{d\alpha} = \int_{t_1}^{t_2} \sum_{i=0}^{n} \left(\frac{\partial \mathcal{L}}{\partial q^i} \frac{\partial q^i}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \frac{\partial \dot{q}^i}{\partial \alpha}\right) dt$$

We can find $\dfrac{\partial q^i}{\partial \alpha} = \eta^i(t)$ & $\dfrac{\partial \dot{q}^i}{\partial \alpha} = \dot{\eta}^i(t)$. Thus

$$\begin{aligned}\frac{d\mathcal{A}}{d\alpha} &= \sum_{i=0}^{n} \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial q^i}\eta^i(t) + \frac{\partial \mathcal{L}}{\partial \dot{q}^i}\dot{\eta}^i(t) \right) dt \\ &= \sum_{i=0}^{n} \int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial q^i}\eta^i(t)\, dt + \left[\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\eta^i \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \eta^i(t)\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) dt \\ &= \int_{t_1}^{t_2} \sum_{i=0}^{n} \left[\frac{\partial \mathcal{L}}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) \right] \eta^i(t)\, dt\end{aligned}$$

For stationary action

$$\frac{d\mathcal{A}}{d\alpha} = \int_{t_1}^{t_2} \sum_{i=0}^{n} \left[\frac{\partial \mathcal{L}}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) \right] \eta^i(t)\, dt = 0$$

Now, $\eta^i(t)$ can be any arbitrary set of independent functions, with same boundary condition of course, and that we can take second derivative of. So, to satisfy the above equation for all such $\eta^i(t)$, then term in bracket should be identically equal to zero, i.e.,

$$\frac{\partial \mathcal{L}}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) = 0 \qquad\qquad \forall\, i \in \{1, \dots, n\}$$

This is known as **Euler-Lagrange Equation**, which is nothing but Lagrange equation of motion. But this is not just an equation of motion. This is much more powerful than that.

- **Newton's Law from Lagrange equation and Conjugate Momentum:**

  If we use Cartesian coordinate system, and define the Lagrangian as

$$\mathcal{L} = T(\dot{x}, \dot{y}, \dot{z}) - U(x, y, z) = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) - U(x, y, z)$$

  Newton's law follows directly from Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) = \frac{\partial \mathcal{L}}{\partial x} \qquad \Rightarrow \qquad \frac{d}{dt} m\dot{x} = -\frac{\partial U}{\partial x}$$

  Similar equation can be written for y and z coordinate. Note that the quantity $\dfrac{\partial \mathcal{L}}{\partial \dot{x}}$ gave momentum along x-axis. So, for any generalized coordinate system, we can the quantity $\dfrac{\partial \mathcal{L}}{\partial \dot{q}^i}$ as **conjugate momentum** (p_i) corresponding to coordinate q^i.

$$p_i \coloneqq \frac{\partial \mathcal{L}}{\partial \dot{q}^i}$$

- **Noether's Theorem and Cyclic Coordinate:**

  In general, Noether's theorem says that every differentiable symmetry of the action of a physical system with conservative forces has a corresponding conservation law. But what does 'symmetry' actually mean?

  Well, for classical mechanics, we can use the symmetry of Lagrangian to get certain conservation law. We have already seen that Lagrangian in general has the following functional form: $\mathcal{L}(q^i, \dot{q}^i, t)$, where q^i represents the set of coordinates. Suppose there is a coordinate q^n such that the Lagrangian is independent of this coordinate q^n, i.e.,

$$\frac{\partial \mathcal{L}}{\partial q^n} = 0$$

  then the corresponding conjugate momentum is conserved, i.e.,

$$\frac{\partial \mathcal{L}}{\partial \dot{q}^n} = \text{constant}$$

  Such a coordinate q^n is called the **cyclic coordinate**.

  Proof: Follows trivially from Lagrange equation.

- **Conservation of Linear and Angular Momentum** follows directly from conversation of conjugate momentum in Cartesian and Polar coordinate system respectively.

- **Conservation of Energy from Lagrangian:**

  Like independence of Lagrangian from any coordinate q^n leads to conservation of corresponding conjugate momentum. Similarly, does the explicit independence of Lagrangian from time t correspond to some conservation?

  The answer is YES! If the Lagrangian don't have any explicit dependence on time, i.e., it has functional form $\mathcal{L}(q^i, \dot{q}^i)$, then it'll lead to conservation of energy, or *Hamiltonian* more precisely.

  Proof: Assume the general function form of Lagrangian, i.e., $\mathcal{L}(q^i, \dot{q}^i, t)$, then the total time derivative of Lagrangian can be written as

$$\frac{d\mathcal{L}}{dt} = \sum_i \frac{\partial \mathcal{L}}{\partial q^i}\dot{q}^i + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}^i}\ddot{q}^i + \frac{\partial \mathcal{L}}{\partial t}$$

  From Lagrange equation, we can substitute $\frac{\partial \mathcal{L}}{\partial q^i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial q^i}\right)$. We can write $\frac{\partial \mathcal{L}}{\partial \dot{q}^i}$ as simply p_i. Thus we get

$$\frac{d\mathcal{L}}{dt} = \sum_i \frac{dp_i}{dt}\dot{q}^i + \sum_i p_i \frac{d\dot{q}^i}{dt} + \frac{\partial \mathcal{L}}{\partial t}$$

  From chain rule and linearity of derivatives, we can write

$$\frac{d\mathcal{L}}{dt} = \frac{d}{dt}\left(\sum_i p_i \dot{q}^i\right) + \frac{\partial \mathcal{L}}{\partial t}$$

  Now, we define the **Hamiltonian** as

$$\mathcal{H} := \sum_i p_i \dot{q}^i - \mathcal{L}$$

  Substituting this in above equation will give

$$\frac{d\mathcal{H}}{dt} = -\frac{\partial \mathcal{L}}{\partial t}$$

  Now, if Lagrangian has no explicit time dependence, i.e., $\frac{\partial \mathcal{L}}{\partial t} = 0$, then it implies that the Hamiltonian is conserved, i.e.,

$$\mathcal{H} = \text{constant}$$

  We can easily verify that for Cartesian coordinate system, the Hamiltonian is nothing but the total energy of system, i.e.,

$$\mathcal{H} = T(\dot{x}, \dot{y}, \dot{z}) + U(x, y, z) = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + U(x, y, z)$$

- **Lagrange Multiplier:** Suppose we have a function $f : U \to \mathbb{R}$, where U is a open set in $\mathbb{R}^n$. Now, we want to find the stationary points of this function along some constraint surface (n-surface) $g(\boldsymbol{x}) = 0$ ($\boldsymbol{x} \in \mathbb{R}^n$), or set of constraint surfaces $\{g_i(\boldsymbol{x}) = 0\}_{i=1}^m$. To find such a stationary point, we use Lagrange multiplier λ.

  ◇ **Single Constraint:** The given function is f and the only constraint is $g(\boldsymbol{x}) = 0$. Suppose $\boldsymbol{x}_0 \in \mathbb{R}^n$ is a stationary point of f on constraint surface, then clearly $g(\boldsymbol{x}_0) = 0$. Now, any allowed direction where one can move along the surface is perpendicular to the gradient ∇g. And when one starts moving from stationary point along the constraint surface, then the function f should not change, which implies that the direction of ∇f should be perpendicular to allowed movable direction. Since both ∇f and ∇g are perpendicular to allowed movable direction on surface, therefore the should be parallel (or anti-parallel), and we can write

$$\nabla f = \lambda \nabla g$$

where λ is called the *Lagrange Multiplier*. This equation along with constraint equation gives total $n+1$ equations (given below) to solve for $n+1$ variables (all x^i and λ). The set of equations are

$$\begin{aligned} \frac{\partial f}{\partial x^i} - \lambda \frac{\partial g}{\partial x_i} &= 0, && \text{(set of } n\text{-equations)} \\ g(\boldsymbol{x}) &= 0, && \text{(single equation)} \end{aligned}$$

◇ **Multiple Constraint:** The philosophy in case of multiple constraints is also same. We want to find point where movement in allowed direction is perpendicular to ∇f. For set of m-constraints $\{g_i(\boldsymbol{x}) = 0\}_{i=1}^m$, we will have m Lagrange multipliers $\{\lambda_j\}_{j=1}^m$, such that

$$\nabla f = \sum_{j=1}^m \lambda_j \nabla g_j$$

This along with the constraint equations will give $n+m$ equations (given below) to solve for $n+m$ (all x^i and λ_j). The set of equations are

$$\begin{aligned} \frac{\partial f}{\partial x^i} - \sum_{j=1}^m \lambda_j \frac{\partial g_j}{\partial x^i} &= 0, && \text{(set of } n\text{-equations)} \\ g_j(\boldsymbol{x}) &= 0, && \text{(set of } m\text{-equations)} \end{aligned}$$