# Quantum Time and Quantum Spacetime

GEOMETRIC EVENT BASED
QUANTUM MECHANICS

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# Time in Quantum Mechanics - Introduction

- Time is just a *classical* parameter in the Schrödinger Equation. It indicates what is shown in the clock on the lab wall.
$$\imath\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$
But, classical systems does not exist. In a consistent theory of quantum mechanics, they are just a *limiting* situation. So, lets formulate the theory is non-classical way...

- We define time as "what is shown in the clock", like said earlier, but this time, lets use a quantum system as a clock. *e.g.,* a quantum particle on a line, etc. Then, time arises as *correlation between the system and the clock.*

- Given the Hilbert space of system, $\mathcal{H}_S$ and the Hilbert space of time (Hilbert space of quantum system used as clock) $\mathcal{H}_T$, then we can form a large generalized Hilbert space
$$\mathfrak{H} := \mathcal{H}_T \otimes \mathcal{H}_S$$

- Now, we can construct a constraint operator:
$$\hat{\mathbb{J}} := \hbar\hat{\Omega}_T \otimes \mathbb{1}_S + \mathbb{1}_T \otimes \hat{H}_S$$
All physical states satisfy the constraint:
$$\hat{\mathbb{J}}\,|\Psi\rangle\rangle = 0$$
(This is a Wheeler-DeWitt (WdW) equation. The double ket $|\Psi\rangle\rangle$ represents a bipartite state in $\mathfrak{H}$).

  > This equation means that for the physical states, the **Hamiltonian is the generator of time translation**.

  From this, we can get back the known quantum theory by **conditioning**.

- Projecting the bipartite state on time space:
$$|\psi(t)\rangle_S \;=\; {}_T\langle t\,|\,\Psi\rangle\rangle$$
And projecting the constraint will give the Schrödinger equation
$${}_T\langle t|\,(\hbar\hat{\Omega}_T + \hat{H}_S)\,|\Psi\rangle\rangle = 0 \quad\Leftrightarrow\quad \imath\hbar\frac{\partial}{\partial t}|\psi(t)\rangle_S \;=\; \hat{H}_S\,|\psi(t)\rangle_S$$

- Projecting the bipartite state on energy space:
$$|\psi(\omega)\rangle_S \;=\; {}_T\langle\omega\,|\,\Psi\rangle\rangle$$
And projecting the constraint will give the time independent Schrödinger equation
$${}_T\langle\omega|\,(\hbar\hat{\Omega}_T + \hat{H}_S)\,|\Psi\rangle\rangle = 0 \quad\Leftrightarrow\quad \hat{H}\;|\psi(\omega)\rangle_S \;=\; -\hbar\omega\;|\psi(\omega)\rangle_S$$

- All pure solution of WdW equation $\hat{\mathbb{J}}\,|\Psi\rangle\rangle = 0$ are of the form
$$|\Psi\rangle\rangle \;=\; \int dt\;|t\rangle_T \otimes |\psi(t)\rangle_S$$
which means that the conventional state of the system at time $t$ ($|\psi(t)\rangle_S \;=\; {}_T\langle t\,|\,\Psi\rangle\rangle$) is a **condition state**: The state given that the time was $t$.

- Is entanglement important? Could we do with classical correlation?
No! Without entanglement, we will get *either* time-dependent or time-independent Schrödinger equation, *but no both.*

- Now, time here is a quantum degree of freedom (with its own Hilbert space $\mathcal{H}_T$) and can be entangled
$$|\phi\rangle\rangle = \int dt\,\phi(t)\;|t\rangle_T \otimes |\psi(t)\rangle_S$$

- Quantization of time *does not* necessarily imply that *time is discrete.* It's a continuous quantum degree of freedom with choice $\mathcal{H} \equiv \mathcal{L}^2(\mathbb{R})$. We can form time operator as

$$\hat{T} = \int_{-\infty}^{\infty} dt \, |t\rangle \langle t|$$

- Up to now, the time Hilbert space is the Hilbert space is the Hilbert space of the clock that defines time. But, a physical interpretation of the time Hilbert space is not necessary. It can be seen as an **abstract purification space**.
- The reason for same treatment of time and space in Quantum mechanics is because earlier, time was a classical parameter; and also because relativity also give kind of similar treatment to time and space.
- It was easy to quantize the position in space but difficult to quantize the position in time. WHY?? Because we usually quantize "system" that are **extended in time and localized in space**.
  *e.g.,* Newton-Winger position operator: the position of a particle at time $t$.

# Difficulty in unifying General Relativity and Quantum Mechanics:

- Quantum Mechanics talks about *Quantum System* (infinitely extended in time and can be either finite or infinite in space), where as General Relativity talks about *Events* (finite in space and time).
- Our two main theories talks about two different things.
- The question is: **Which one is most fundamental concept?** Event or Quantum Systems?
- There are two approaches:
  1 **General Relativistic theory of Quantum Mechanics:** Quantum systems are considered more fundamental and **events are what happened to a quantum system**.
  2 **Quantum theory of General Relativity:** Events are considered to be more fundamental and **quantum system are succession of events**.
- Quantum gravity approaches up to now mostly takes *quantum systems to be more fundamental.*
- Quantum mechanics uses time conditioned quantities, like states $|\psi(t)\rangle$ or observables $X(t)$. They can't be *relativistically covariant.* (Covariance: formula look the same in all reference frame)
- What about QFT? Isn't it relativistically covariant?
  QFT uses a couple of tricks to recover covariance:
  1. Uses operators/observables in the Heisenberg picture with covariant spacetime dependence. *e.g.,*

  $$a^{\dagger} e^{-\iota x^{\mu} p_{\mu}}$$

  2. Use a state that is invariant for Lorentz transform. *e.g.,* the vacuum $|0\rangle$.

# Geometric Event Based (GEB) Quantum Mechanics

- This approach consider **quantum events** as fundamental. **Quantum System** are derived as a quantum state for a *succession of events in quantum spacetime.*
- A quantum event has a **position or coordinate in spacetime**, but also an **energy-momentum**. Because without energy, nothing can happen. And without momentum, nothing can be localized.
- Basic observables:

$$\overline{X} := (X^0, X^1, X^2, X^3) \qquad\qquad \overline{P} := (P^0, P^1, P^2, P^3)$$

  And their canonical commutations:

$$[X^{\mu}, P^{\nu}] = -\iota \eta^{\mu\nu} \qquad \& \qquad [X^{\mu}, X^{\nu}] = 0 = [P^{\mu}, P^{\nu}]$$

  where $\eta^{\mu\nu}$ is Minkowski metric. The rationale of this choice is that, on one hand, it allows us to satisfy Poincaré algebra.
- Now, the Hilbert space for quantum events $\mathcal{H} \equiv \mathcal{L}^2(\mathbb{R}^4)$

## Outcomes of GEB Approach:

- **A universe with single event:**

$$|\phi\rangle = \int d^4x\, \phi(\bar{x})\, |\bar{x}\rangle = \int d^4p\, \tilde{\phi}(\bar{p})\, |\bar{p}\rangle$$

$\phi(\bar{x}) := \langle \bar{x} \mid \phi \rangle$ ; $\tilde{\phi}(\bar{p}) := \langle \bar{p} \mid \phi \rangle = \int d^4x\, \frac{1}{4\pi^2}\, e^{\iota \overline{x} \underline{p}}\, \phi(\bar{x})$ are amplitudes (wavefunctions).

- **Born Rule for Probability:**

$$P(\bar{x}|\phi) = |\phi(\bar{x})|^2 = |\langle \bar{x} \mid \phi \rangle|^2$$

**Unconditioned** probability that the event is in spacetime cooridnate $\bar{x}$. The Born rule in Quantum Mechanics is **conditioned**, *i.e.*, it give the probability that the particle is at position $\vec{x}$ [1] give that the time is $t$.

> Quantum mechanical pobabilities are not covariant but GEB probabilities are covariant.

- **Uncertainty Principle:** The canonical commutation relation $[X^\mu, P^\nu] = -\iota\eta^{\mu\nu}$ gives the Heisenberg uncertainty principle, but it also give Heisenberg-Robertson inequality

$$\sigma_{X^0}\sigma_{P^0} \geq \frac{\hbar}{2}$$

In Quantum mechanics, it is completely meaningless. But in GEB approach, it is not meaningless.

> An event cannot be localized unless it has an energy spread.

- **Lorentz Transformation:** Lorentz transformation in QFT is a Nightmare! Because we need to quantize from the start in the new reference frame: rerun the quantization procedure (equal time commutation relation, etc.)
But in GEB approach, *Lorentz transformation is just an unitary transformation on the GEB state* (Wigner's prescription on how to describe symmetries of a theory)

$$|\phi'\rangle = U_\Lambda\, |\phi\rangle$$

- **Multiple Events:** For multiple events, we use tensor products. If the number of events $n$ is fixed, then we can write the GEB state as

$$\left|\phi^{[n]}\right\rangle = \sum_{\sigma_1,\ldots,\sigma_n} \int d^4x_1 \cdots d^4x_n\; \phi^{[n]}(\bar{x}_1, \sigma_1; \ldots; \bar{x}_n, \sigma_n)\; |\bar{x}_1, \sigma_1; \ldots; \bar{x}_n, \sigma_n\rangle$$

Here, $\sigma_i$ stand for spins. If the number of events is not fixed, then we can instead use a **Fock space**:

$$\left|\phi^{[n]}\right\rangle = \sum_{\sigma_1,\ldots,\sigma_n} \int d^4x_1 \cdots d^4x_n\; \phi^{[n]}(\bar{x}_1, \sigma_1; \ldots; \bar{x}_n, \sigma_n)\; a^\dagger_{\overline{x}_1,\sigma_1} \cdots a^\dagger_{\overline{x}_n,\sigma_n}\; |0\rangle_4$$

Here, $a^\dagger_{\overline{x}_i,\sigma_i}$ are certain **creation operators**, which create an event at position $\overline{x}_i$. We can also introduce different commutator for these creation operators depending on whether we have Bosonic or Fermionic events. The usual commutation relations are

$$\begin{aligned} \text{Bosonic:} \quad & [a_{\overline{x},\sigma}, a^\dagger_{\overline{x}',\sigma'}] = \delta_{\sigma,\sigma'}\delta^{(4)}(\overline{x} - \overline{x}'),\;\; [a_{\overline{x},\sigma}, a_{\overline{x}',\sigma'}] = 0; \\ \text{Fermionic:} \quad & \{a_{\overline{x},\sigma}, a^\dagger_{\overline{x}',\sigma'}\} = \delta_{\sigma,\sigma'}\delta^{(4)}(\overline{x} - \overline{x}'),\;\; \{a_{\overline{x},\sigma}, a_{\overline{x}',\sigma'}\} = 0; \end{aligned}$$

Bosonic events give rise to Bosons, whereas Fermionic events give rise to Fermions.
Also, the 4D vacuum $|0\rangle_4$ is different from 3D vacuum $|0\rangle_3$. $|0\rangle_3$ means there are no particle at time $t$ in Heisenberg picture. We can get the 3D vacuum from 4D vacuum by considering an event state of **zero 4-momentum**: ground state of the field.

$$|0\rangle_3 = \text{foliate}(a^\dagger_{\overline{p}=0}\, |0\rangle_4)$$

[1]We use $\overline{a}$ to describe the contravariant 4-vector, and $\underline{a}$ for covariant 4-vector. Where as $\vec{a}$ represent simply 3D vector in space.

- **Joint Probability for Multiple Events:** The joint probability for the $n$ events to happen at $n$ spacetime coordinates is given by

$$P^{[n]}(\bar{x}_1, \sigma_1; \dots; \bar{x}_n, \sigma_n) = |\phi^{[n]}(\bar{x}_1, \sigma_1; \dots; \bar{x}_n, \sigma_n)|^2$$

  Dirac tried to do deal with such multiple events case using conventional quantum mechanics, and he got stuck at the case where each event has its own time, and he didn't know what to do with that. But with GEB approach, such a problem doesn't exist.

- **QFT from GEB:** We can get the QFT back from GEB QM using **constraints**:
  - ◇ Start from a single-particle wavefunction:

$$\Psi_{\mathrm{QM}}(\vec{x}, \sigma | t)$$

    (probability amplitude of finding particle in $\vec{x}$ at given time $t$)
  - ◇ Construct a GEB state from it

$$\Psi_{\mathrm{QM}}(\overline{x}, \sigma) := \Psi_{\mathrm{QM}}(\vec{x}, \sigma | t); \qquad |\Psi_{\mathrm{QM}}\rangle := \sum_{\sigma} \int d^4x \; \Psi_{\mathrm{QM}}(\overline{x}, \sigma) \; |\overline{x}, \sigma\rangle$$

    GEB state describe the whole dynamics of the **particle as a state of a sequence of events**.
    Note: Normalization of QM wavefunction doesn't imply normalization of GEB state.
  - ◇ Write it as an eigenstate of a constraint operator

$$K\,|\Psi_{\mathrm{QM}}\rangle = 0$$

  Though using constraints isn't that necessary for single particle system, but it become very necessary for more complex systems. And this same procedure works for more complex QFT systems.
  Example of some constraints are:
  1. Klein-Gordan Equation Constraint:

$$K_{\mathrm{KG}+} = \int d^4p \; \left(\Theta(p^0)\overline{p} \cdot \underline{p} - m^2\right)^2 \; |\overline{p}\rangle \langle \overline{p}|$$

  2. Dirac Equation Constraint:

$$\begin{aligned} J_D &:= \overline{\gamma} \cdot P - m \\ &= \sum_{\sigma, \sigma'} \int d^4p \; \left(\gamma_{\sigma,\sigma'} \cdot \underline{p} - m\,\delta_{\sigma,\sigma'}\right) \; |\overline{p}, \sigma\rangle \langle \overline{p}, \sigma'| \end{aligned}$$