MA1101 - Functions

BIRMD

- Let $X, Y \neq \emptyset$. A relation $R \subseteq X \times Y$ between X and Y is said to be a *function* if for every $x \in X$, there exist a unique $y_x \in Y$ such that $(x, y_x) \in R$.
- For $X, Y \neq \emptyset$, $f : X \to Y$ be a function, then:
  1. X is said to be a *Domain*
  2. Y is said to be *Co-Domain*
  3. $f(x) = y_x$ is said to be *image* of x in Y.
  4. $f(X) := \{f(x) \in Y \mid x \in X\}$ is called *Range* of function.
- **Equality of Function:** For $X, Y, A, B \neq \emptyset$, $f : X \to Y$ and $g : A \to B$ are two functions. Then $f = g$ if
  1. $X = A$;
  2. $Y = B$;
  3. $f(x) = g(x)$ for all $x \in X$ (or A).
- For $X, Y \neq \emptyset$, $f : X \to Y$ be a function, then:
  1. **Injection:** f is *one-one/injective* iff for $x_1, x_2 \in X$, $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.
  2. **Surjection:** f is *onto/surjective* iff $\forall\, y \in Y$, $\exists\, x \in X \mid f(x) = y$.
  3. **Bijection:** f is *bijective* iff f is both *surjective* and *injective*.
- **Composition of Function:** Let $X, Y, Z \neq \emptyset$, and let $f : X \to Y$ and $g : Y \to Z$ be two functions, then the composition function $g \circ f : X \to Z$ is defined as

$$(g \circ f)_{(x)} := g(f(x)), \quad \forall\, x \in X$$

- **Inverse of Function:** Let $X, Y \neq \emptyset$ and $f : X \to Y$ be a *bijection*. The *inverse* of f is the function $f^{-1} : X \to Y$ defined as

$$f^{-1}(y) = x, \quad \text{where } y = f(x),\ \forall\, y \in Y$$

- **Identity Function:** $\mathrm{Id}_X : X \to X$ is called *identity* function, if $\mathrm{Id}_X(x) = x$ for all $x \in X$.
- **Theorem:** Let $X, Y \neq \emptyset$ and $f : X \to Y$ be a *bijection*. Then, $f^{-1} \circ f = \mathrm{Id}_X$ and $f \circ f^{-1} = \mathrm{Id}_Y$
- **Theorem:** Let $X, Y \neq \emptyset$ and $f : X \to Y$. Then, if
  1. $\exists\, g : Y \to X \mid g \circ f = \mathrm{Id}_X$, then f is *injective*.
  2. $\exists\, h : Y \to X \mid f \circ h = \mathrm{Id}_Y$, then f is *surjective*.

  If both such g, h exists together, then f is *bijective* and $g = h = f^{-1}$
- **Image and Inverse Image:** Let $X, Y \neq \emptyset$ and $f : X \to Y$. Let $A \subseteq X$ and $B \subseteq Y$, then we define
  1. the *image* of A as $f(A) := \{f(x) \mid x \in A\}$
  2. the *inverse image* of B as $f^{-1}(B) := \{x \in X \mid f(x) \in B\}$
- **Theorem:** Let $X, Y \neq \emptyset$ and $f : X \to Y$. Let $A, B \subseteq X$ and $C, D \subseteq Y$, then
  1. $f(A \cup B) = f(A) \cup f(B)$; $\quad f(A \cap B) = f(A) \cap f(B)$
  2. $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$; $\quad f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$