ASSIGNMENT FIRST | 2023

Differential Equation

Rohit Raj Karki
CE-2020(21)
Kathmandu University

1) Solve the following first order differential equations, Initial value problems (Show the details of the work):

a. $2xydx + x^2dy = 0$

Solution:

$$\begin{aligned}
&2xydx + x^2dy = 0\\
\Rightarrow &2xydx = -x^2dy\\
\Rightarrow &\frac{2dx}{x} = \frac{dy}{y}\\
&\text{Integrating on both sides}\\
\Rightarrow &\int \frac{2dx}{x} = -\int \frac{dy}{y})\\
\Rightarrow &2ln|x| = -ln|y| + c^*\\
\Rightarrow &ln|x^2| + ln|y| = c^*\\
\Rightarrow &ln|x^2y| = c*\\
\Rightarrow &x^2y = e^{c*}\\
\Rightarrow &x^2y = c\\
&\text{where } c = \pm e^{c*}
\end{aligned}$$

□

b. $(x^2y - 2xy^2)dx = (x^3 - 3x^2y)dy$

Solution:

$$\begin{aligned}
&(x^2y - 2xy^2)dx = (x^3 - 3x^2y)dy\\
\Rightarrow &\frac{dy}{dx} = \frac{(x^2y - 2xy^2)}{(x^3 - 3x^2y}
\end{aligned}$$

$$\frac{dy}{dx} = \frac{\frac{y}{x} - 2(\frac{y}{x})^2}{1 - 3\frac{y}{x}} \tag{1}$$

$$\begin{aligned}
&\text{Put } \frac{y}{x} = v\\
\Rightarrow &y = vx\\
\Rightarrow &\frac{dy}{dx} = v + x\frac{dv}{dx}
\end{aligned}$$

Now equation 1 becomes

$$\begin{aligned}
&v + x\frac{dv}{dx} = \frac{v - 2v^2}{1 - 3v}\\
\Rightarrow &x\frac{dv}{dx} = \frac{v - 2v^2 - v + 3v^2}{1 - 3v}\\
\Rightarrow &x\frac{dv}{dx} = \frac{v^2}{1 - 3v}\\
\Rightarrow &\frac{1 - 3v}{v^2}dv = \frac{1}{x}dx\\
\Rightarrow &\frac{1}{v^2} - \frac{3}{v} = \frac{1}{x}dx
\end{aligned}$$

Integrating on both sides

$$\int \frac{1}{v^2}dv - \int \frac{3}{v}dv = \int \frac{1}{x}dx$$
$$\Rightarrow -\frac{1}{v} - 3ln|v| = ln|x| + c^*$$
$$\Rightarrow -v^{-1} - 3ln|v| = ln|x| + c^*$$
$$\Rightarrow -e^{\frac{x}{y}} - 3\frac{y}{x} = x + e^{c*}$$
$$\Rightarrow e^{\frac{x}{y}} + 3\frac{y}{x} + x = c$$
$$\text{where } c = -e^{c*}$$

□

c. $xdx + ydy + \frac{xdy - ydx}{x^2+y^2} = 0$
Solution:

$$xdx + ydy + \frac{xdy - ydx}{x^2 + y^2} = 0$$
$$\Rightarrow xdx + ydy + \frac{xdy - ydx}{x^2(1 + \frac{y^2}{x^2})} = 0$$
$$\Rightarrow xdx + ydy + \frac{d(y/x)}{1 + \frac{y^2}{x^2}} = 0$$

let $y/x = z$

$$xdx + ydy + \frac{dz}{1 + z^2} = 0$$

Integrating above equation

$$\int xdx + \int ydy + \int \frac{1}{1 + z^2} = \int 0$$
$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + tan^{-1}z = c$$
$$\Rightarrow x^2 + y^2 + tan^{-1}z = c_1 \text{is the required equation}$$

□

e. $(2cosy + 4x2)dx = xsinydy$
Solution:

$$(2cosy + 4x^2)dx = xsinydy$$
$$\Rightarrow (2cosy + 4x^3)dx - xsinydy = 0$$

Comparing eqns with $Mdx + Ndy = 0$ or $Pdx + Qdy = 0$
we get,

$$P = 2cosy + 4x^3 \tag{2}$$
$$Q = -xsiny \tag{3}$$

Here,

$$\frac{dP}{dx} = -2siny, \frac{dQ}{dy} = -siny$$

$$\frac{dP}{dx} \neq \frac{dQ}{dy}$$

So, not exact.

$$\begin{aligned} R &= \frac{1}{Q}(\frac{dp}{dy} - \frac{dQ}{dx}) \\ &= \frac{1}{-xsiny}(-2siny - siny) \\ &= \frac{1}{x} \end{aligned}$$

Integrating factor,

$$\begin{aligned} I.F &= e^{\int Rdx} \\ &= e^{\int \frac{1}{x}dx} \\ &= e^{lnx} \\ &= x \end{aligned}$$

so the exact differential equation becomes ,

$$\begin{aligned} & x(2cosy + 4x^2)dx - x(xsiny)dy = 0 \\ \Rightarrow & (2xcosy + 4x^3)dx - x^2sinydy = 0 \\ & M = 2xcosy + 4x^3, N = -x^2siny \end{aligned}$$

The solution is :

$$\begin{aligned} u &= \int (2xcosy + 4x^3)dx + k(y) \\ &= \int 2xcosydx + \int 4x^3dx + k(y) \end{aligned}$$

$$u = x^2cosy + x^4 + k(y) \tag{4}$$

Partial differentiating w.r.t y

$$\begin{aligned} & \frac{\partial u}{\partial y} = \frac{\partial x^2cosy}{\partial y} + \frac{\partial x^4}{\partial y} + \frac{\partial k(y)}{\partial y} \\ & N = -x^2siny + \frac{dk(y)}{dy} \\ & -x^2siny = -x^2siny + \frac{dk(y)}{dy} \\ & \frac{dk(y)}{dy} = 0 \end{aligned}$$

Integrating,
k(y)=c From equation 4

$$u = x^2cosy + x^4 + c$$

Hence, $x^2cosy + x^4 + c$ is the required equation.

□

e. $y' + xy = xy^{-1}$
Solution:

$$y' + xy = xy^{-1}$$

Comparing with $y' + p(x)y = g(x)y^a$
a = -1
p(x) = x
g(x) = x

$$\begin{aligned} I.F &= e^{\int (1-a)pdx} \\ &= e^{\int (1+1)xdx} \\ &= e^{x^2} \end{aligned}$$

Now the solution is

$$\begin{aligned} & y^{1-a}e^{x^2} = \int (1-a)g(x)e^{x^2} \\ \Rightarrow & y^{1-a}e^{x^2} = \int 2xdx \\ \Rightarrow & y^{1-a}e^{x^2} = e^{x^2} + c \\ \therefore & y^2 = e^{-x^2}c + 1 \end{aligned}$$

□

f. $2xtanydx + sec^2ydy = 0$
Solution:

$$\begin{aligned} & 2xtanydx + sec^2ydy = 0 \\ \Rightarrow & 2xdx = -\frac{sec^2y}{tany}dy \end{aligned}$$

Integrating on both sides

$$\begin{aligned} & -\int 2xdx = \int \frac{sec^2y}{tany}dy \\ \Rightarrow & -x^2 + c_1 = ln|tany| \\ \Rightarrow & e^{-x^2+c_1} = tany \\ \therefore & e^{-x^2}c = tany \end{aligned}$$

□

g. $y' + \frac{y}{3} = \frac{1}{3}(1-2x)y^4$
Solution:
Comparing with $y' + p(x)y = g(x)y^a$
$a = 4$
$p(x) = \frac{1}{3}$
$g(x) = \frac{1}{3}(1-2x)$

$$\begin{aligned} I.F &= e^{\int (1-a)pdx} \\ &= e^{\int (1-4)\frac{1}{3}} \\ &= e^{\int -1dx} \\ &= e^{-x} \end{aligned}$$

The solution is

$$
\begin{aligned}
u \cdot \text{I.F} &= \int (1-a) g(x) \cdot \text{I.F} dx \\
\Rightarrow y^{1-a} e^{-x} &= \int (1-4) \frac{1}{3} (1-2x) e^{-x} dx \\
\Rightarrow y^{1-4} e^{-x} &= \int -3 \frac{1}{3} (1-2x) e^{-x} dx \\
\Rightarrow y^{-3} e^{-x} &= \int (-1+2x) e^{-x} dx \\
\Rightarrow y^{-3} e^{-x} &= (-1+2x) \int e^{-x} dx - \int \left[\frac{d(-1+2x)}{dx} \int e^{-x} dx \right] dx + c \\
\Rightarrow y^{-3} e^{-x} &= (1-2x) e^{-x} - \int -2e^{-x} dx + c \\
\Rightarrow y^{-3} e^{-x} &= (1-2x) e^{-x} - 2e^{-x} + c \\
\Rightarrow y^{-3} e^{-x} &= e^{-x} [1 - 2x - 2 + ce^{x}] \\
\Rightarrow y^{-3} &= -2x - 1 + ce^{x}
\end{aligned}
$$

Hence the equation is

$$y^{-3} = -2x - 1 + ce^{x}$$

□

i. $y' + x^2 = x^2 e^{3y}$

Solution:

$$
\begin{aligned}
\frac{dy}{dx} + x^2 &= x^2 e^{3y} \\
\Rightarrow \frac{dy}{dx} &= x^2(-1 + e^{3y}) \\
\Rightarrow \frac{dy}{-1 + e^{3y}} &= x^2 dx
\end{aligned}
$$

Integrating on both sides

$$
\begin{aligned}
\int \frac{dy}{-1 + e^{3y})} &= \int x^2 dx \\
\Rightarrow \frac{1}{3} \int \frac{3e^{-3y}}{-1 + e^{3y}} dy &= \frac{x^3}{3} + c \\
\Rightarrow \frac{1}{3} ln|1 - e^{-3y}| &= \frac{x^3}{3} + c
\end{aligned}
$$

□

j. $xy' + y = y^2 logx$

Solution: Dividing on both sides by x

$$
\begin{aligned}
y' + \frac{y}{x} &= y^2 \frac{logx}{x} \\
y' + \frac{1}{x} y &= \frac{logx}{x} y^2
\end{aligned}
$$

Comparing with $y' + p(x)y = g(x)y^a$

$$p(x) = \frac{1}{x}, g(x) = log(x), a = 2$$

$$
\begin{aligned}
I.F. &= e^{\int (1-a)pdx} \\
&= e^{\int (1-2)\frac{1}{x}dx} \\
&= e^{\int -\frac{1}{x}dx} \\
&= e^{-log(x)} \\
&= e^{log(x^{-1})} \\
&= x^{-1}
\end{aligned}
$$

The solution is

$$
\begin{aligned}
&\Rightarrow u \cdot I.F = \int (1-a)g(x)I.Fdx \\
&\Rightarrow y^{1-a}x^{-1} = \int (1-2)\frac{log(x)}{x}x^{-1} \\
&\Rightarrow y^{1-2}x^{-1} = -\int \frac{log(x)}{x}x^{-1} \\
&\Rightarrow y^{-1}x^{-1} = -\int \frac{log(x)}{x^2} \\
&\Rightarrow \frac{1}{xy} = \big[-log(x)\frac{1}{x} - \int \frac{1}{x}\frac{-1}{x}dx\big] + c \\
&\Rightarrow x^{-1}y^{-1} = -1\big[\frac{log(x)}{-x} - \frac{1}{x}\big] + c \\
&\Rightarrow x^{-1}y^{-1} = \frac{log(x)}{x} + \frac{1}{x} + c \\
&\Rightarrow y^{-1} = log(x) + 1 + cx
\end{aligned}
$$

$$
\therefore y = \frac{1}{log(x) + 1 + cx}
$$

□

k. $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0$
Solution: Comparing with Pdx+Qdy =0
The above equation is not exact.
$P(x, y) = (xy^3 + y), Q(x, y) = 2(x^2y^2 + x + y^4)$

$$
\begin{aligned}
I.F &= e^{\int Rdx} \\
&= e^{\int \frac{1}{Q}\left[\frac{\partial P_y}{\partial y} - \frac{\partial Q_x}{\partial x}\right]dx} \\
&= e^{\int \frac{1}{Q}\left[\frac{\partial xy^3+y}{\partial y} - \frac{\partial 2(x^2y^2+x+y^4)}{\partial x}\right]dx} \\
&= e^{\int \frac{1}{Q}(3xy^2+1-4xy^2-2)dx} \\
&= e^{\int \frac{(-xy^2-1)}{2(x^2y^2+x+y^4)}}
\end{aligned}
$$

Here Integrating factor depends on both x and y.

Here,

$$\begin{aligned}
I.F &= e^{\int Rdx} \\
&= e^{\int \frac{1}{P}\left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right]dy} \\
&= e^{\int \frac{1}{(xy^3+y)}\left[\frac{\partial 2(x^2y^2+x+y^4)}{\partial x} - \frac{\partial (xy^3+y)}{\partial y}\right]dy} \\
&= e^{\int \frac{1}{(xy^3+y)}(4xy^2+2-3xy^2-1)dy} \\
&= e^{\int \frac{1}{(xy^3+y)}(xy^2+1)dy} \\
&= e^{\int \frac{1}{y}dy} \\
&= e^{\int log(y)} \\
&= y
\end{aligned}$$

Here R only depends on y.
So the differential equation becomes

□

l. $y' = 2(y-1)tanh2x, y(0) = 0$
Solution:

$$\begin{aligned}
&y' = 2ytanh2x - 2tanh2x \\
\Rightarrow &y' - (2tanh2x)y = -2tanh2x
\end{aligned}$$

Comparing with y'+p(x)y=r(x)
p(x) =- 2tanh 2x

$$\begin{aligned}
I.F &= e^{\int p(x)dx} \\
&= e^{\int -2tanh2xdx} \\
&= e^{-2\frac{ln(cos2x)}{2}} \\
&= e^{ln(cos2x)^{-1}} \\
&= \frac{1}{cosh2x}
\end{aligned}$$

The solution is

$$\begin{aligned}
&y'\frac{1}{cosh2x} - (2tanh2x)\frac{1}{cosh2x}y = -2tanh2x\frac{1}{cosh2x} \\
\Rightarrow &y'(cosh2x)^{-1} - 2\frac{sin2hx}{cosh2xcosh2x}y = -2\frac{sin2hx}{cosh2xcosh2x} \\
\Rightarrow &y'(cosh2x)^{-1} - 2sin2hx(cosh2x)^{-2}y = 2sin2hx(cosh2x)^{-2} \\
\Rightarrow &dy(cosh2x)^{-1} - 2sin2hx(cosh2x)^{-2}ydx = 2sin2hx(cosh2x)^{2} \\
\Rightarrow &dy(cosh2x)^{-1} = \frac{-2sin2hx}{(cosh2x)^{-2}}dx
\end{aligned}$$

Integrating on both sides

$$\begin{aligned}
y(cosh2x)^{-1} &= \int \frac{-2sin2hx}{(cosh2x)^2}dx \\
\Rightarrow y(cosh2x)^{-1} &= -2\int \frac{sin2hx}{(cosh2x)^2}dx \\
\Rightarrow y(cosh2x)^{-1} &= \frac{-2}{2}\frac{(cosh2x)^{-2+1}}{-2+1} + c \\
\Rightarrow y(cosh2x)^{-1} &= \frac{-2\cdot -1}{2}(cosh2x)^{-1} + c \\
\Rightarrow y(cosh2x)^{-1} &= (cosh2x)^{-1} + c \\
\Rightarrow y = 1 + \frac{c}{(cosh2x)^{-1}}
\end{aligned}$$

$$\begin{aligned}
y(0) &= 1 + \frac{c}{(cos0)^{-1}} \\
\Rightarrow 0 &= 1 + \frac{c}{(1)^{-1}} \\
\Rightarrow 0 &= 1 + c \\
\therefore c &= -1
\end{aligned}$$

So the solution is :

$$\begin{aligned}
y(cosh2x)^{-1} &= (cosh2x)^{-1} - 1 \\
\Rightarrow y &= 1 - \frac{1}{(cosh2x)^{-1}} \\
\therefore y &= 1 - cosh2x
\end{aligned}$$

□

m. $xy' = y + x^2sec(y/x), y(1) = \pi$

Solution:

$$y' = \frac{y}{x} + xsec(y/x) \tag{5}$$

let y =vx

$y' = v + x\frac{dv}{dx}$ From eqn 5 we get

$$\begin{aligned}
v + x\frac{dv}{dx} &= v + xsecv \\
\Rightarrow \frac{dv}{dx} &= secv \Rightarrow & dx = secvdv
\end{aligned}$$

Integrating on both sides

$$\begin{aligned}
\int dx &= \int \frac{1}{secvdv} \\
\int dx &= \int cosv \\
\Rightarrow x + c &= sinv \\
\Rightarrow x + c &= sin(\frac{y}{x})
\end{aligned}$$

for $y(1) = \pi$

$$\begin{aligned}
1 + c &= sin\left(\frac{\pi}{1}\right) \\
c &= -1
\end{aligned}$$

The solution is

$$x - 1 = sin\left(\frac{y}{x}\right)$$

□

n. $3y^2dx + xdy = 0, y(1) = 1/2$
Solution:

$$\frac{dx}{x} = \frac{-dy}{3y^2}$$

Integrating on both sides

$$-\int \frac{dx}{x} = \int \frac{dy}{3y^2}$$
$$-ln(x) + c = \frac{-1}{3y}$$
$$ln(x) + c = \frac{1}{3y}$$

For $y(1) = 1/2$

$$ln(1) + c = \frac{1}{3 * 1/2}$$
$$c = \frac{2}{3}$$

The solution is

$$ln(x) + \frac{2}{3} = \frac{1}{3y}$$
$$\Rightarrow 3ln(x) + 2 = 3\frac{1}{3y}$$
$$\Rightarrow 3ln(x) + 2 = \frac{1}{y}$$
$$\therefore y = \frac{1}{ln(x^3) + 2}$$

□

o. $e^x y' = 2(x + 1)y2, y(0) = 1/6$
Solution:

$$e^x y' = 2(x + 1)y^2$$
$$\Rightarrow e^x \frac{dy}{dx} = 2(x + 1)y^2$$
$$\Rightarrow \frac{dy}{y^2} = 2(x + 1)\frac{dx}{e^x}$$

Integrating on both sides

$$\begin{aligned}
&\int \frac{dy}{y^2} = \int 2(x+1)\frac{dx}{e^x} \\
\Rightarrow &\frac{-1}{y} = 2\int (x+1)e^{-x} + c \\
\Rightarrow &\frac{-1}{y} = 2\Big[\{-(x+1)e^{-x}\} - \int -e^{-x}dx\Big] + c \\
\Rightarrow &\frac{-1}{y} = -2\Big[\{(x+1)e^{-x}\} + e^{-x}\Big] - c \\
\Rightarrow &\frac{1}{y} = 2\Big[\{(x+1)e^{-x}\} + e^{-x}\Big] - c
\end{aligned}$$

For $y(0) = 1/6$

$$\begin{aligned}
&\frac{1}{1/6} = 2\Big[\{(0+1)e^0\} + e^0\Big] - c \\
\Rightarrow &c = -2
\end{aligned}$$

The solution is

$$\begin{aligned}
&\frac{1}{y} = 2\Big[\{(x+1)e^{-x}\} + e^{-x}\Big] + 2 \\
\therefore &\frac{1}{y} = 2e^{-x}(x+1) + 2
\end{aligned}$$

□

p. $2yy' + y^2 sinx = sinx, y(0) = \sqrt{2}$
Solution:

□

2. Mathematical Modeling (Develop a mathematical model and solve related problems):

a. (Exponential Growth): If relatively small populations are left undisturbed,then the time rate of growth is proportional to the population present. If in aculture of yeast the rate of growth y'(t) is proportional to the amount presenty(t) at time t, and if y(t) doubles in 1 day, how much can be expected after 3days at the same rate of growth? After 1 week?
Solution: For a exponential growth,

$$y = e^{kt} \tag{6}$$

Differentiating eqn i w.r.t t we get,

$$\begin{aligned}
\frac{dy}{dt} &= ke^{kt} \\
y' &- k \cdot y \\
\frac{y'}{y} &= k
\end{aligned}$$

Integrating both sides w.r.t t

$$lny = kt + c \text{which is the general solution} \tag{7}$$

We have the conditions,
$Att = 0, y' = ky$ i.e
the general solution is

$$lny = kt + c \tag{8}$$

$$lny = k\cot 0 + c$$
$$lny = c$$

So the equation becomes lny = kt+lny If y(t) doubles in 1 day, At t =1 day,

$$ln2y = k + lny$$
$$\Rightarrow ln2y - lny = 2k$$
$$\Rightarrow ln[\frac{2y}{y}] = k$$
$$\Rightarrow ln2 = k$$
$$\therefore k = ln2$$

At t = 3 day,

$$lnay = k \cdot 3 + lny$$
$$\Rightarrow lnay = ln3 \cdot 3 + lny$$
$$\Rightarrow ln[\frac{ay}{y}] = ln2 \cdot 3$$
$$\Rightarrow lna = 2ln2$$
$$\Rightarrow a = e^{ln2^3}$$
$$\therefore a = 8$$

So the population is 8 times the initial population.
Hence the equation becomes,
$ln8y = 3k + lny$ which is ,
$y'(t) = 8y(t)$ At t=3
Again , For t = 7 let y(t) changes by be

$$lnby = k \cdot 70 + lny$$
$$\Rightarrow ln\left(\frac{by}{y}\right) = k \cdot 7$$
$$\Rightarrow lnb = 7k$$
$$\Rightarrow lnb = 7 * ln2$$
$$\Rightarrow lnb = ln(2)^7$$
$$\Rightarrow b = 2^7$$
$$\therefore b = 128$$

Hence the population is 128 times the initial population.
which is ,

$$y'(7) = 128 \cdot y(7)$$

□

b. (Airplane takeoff): An airplane taking off from a landing field has a run of 2 kilometers. If the plane starts with a speed of 10 meters/sec, moves with constant acceleration, and makes the run in 50 sec, with what speed does it take off? What happens if the acceleration is 1.5 meters/sec2?
Solution: Let y be the distance or a run of airplane

$y(0) = 0$ [At t =0 distance covered is 0]
$y'(0) = 10$ m/s [Given velocity initial velocity =10 m/s]
$y(50) = v$ m/s [Final velocity after t=50sec =v]
$y''(50) = a$ m/s2 [accelration to be determined]

Integrating above equation w.r.t t

$$y'(t) = at + c \tag{9}$$

At t=0

$$y'(0) = c$$
$$c = 10$$

so eqn 9 becomes

$$y'(t) = at + 10$$

Integrating above equation w.r.t t

$$y(t) = \frac{at^2}{2} + 10t + c$$

At t =0

$$y(0) = 0 + 0 + c$$
$$c = 0$$

So $y(t) = \frac{at^2}{2} + 10t$ At t=50

$$y(50) = \frac{a50^2}{2} + 10 * 50$$
$$\Rightarrow 2000 = a * 1250 + 500$$
$$\Rightarrow a = \frac{1500}{1250}$$
$$\therefore a = 1.2m/s^2$$

□

c. (Sugar Inversion): Experiments show that rate of inversion of cane sugar in dilute solution is proportional to the concentration y(t) of unaltered sugar. Let the concentration at t = 0 and at t = 4 hours. Find y(t). ***Solution:*** As Rate of inversion of cane sugar =initial concentration of unaltered sugar

$$\frac{dy(t)}{dt} \alpha y(t)$$
$$\Rightarrow \frac{dy(t)}{dt} = ky(t)$$
$$\Rightarrow \frac{dy(t)}{y(t)} = kdt$$

Integrating on both sides

$$\int \frac{dy(t)}{y(t)} = \int kdt$$
$$\therefore ln|y(t)| = kt + c \tag{10}$$

At $t = 0$,

$$ln|y(0)| = k * 0 + c$$
$$c = ln|y(0)| \text{ (Initial condition)}$$

So eqn 10 becomes ,

$$ln|y(t)| = kt + ln|y(0)|$$

At $t = 4$

$$
\begin{aligned}
&ln|y(4)| = k * 4 + ln|y(0)| \\
\Rightarrow &ln|\frac{y(t)}{y}| = 4k \\
\Rightarrow &\frac{y(t)}{y} = e^{4k} \\
\Rightarrow &y(t) = ye^{4k}
\end{aligned}
$$

$\square$

d. (Newton's Law of Cooling): Experiments show that the time rate of change of the temperature T of a body is proportional to the difference between T and the temperature TA of the surrounding medium. A thermometer, reading 5∘C , is brought into a room whose temperature is 22∘C. One minute later the thermometer reading is 12∘C. How long does it take until the reading is practically 22∘C, say, 21.9∘C.

Solution: According to Newton's law of cooling

$$
\frac{dT}{dt}\alpha(T - T_A)
$$

Temperature of the surrunding medium (i.e room) is 22C

$$
\begin{aligned}
&\frac{dT}{dt} = k(T - 22) \\
\Rightarrow &\int \frac{dt}{(T-22)} = \int kdt \\
\Rightarrow &\int \frac{dt}{(T-22)} = \int kdt \\
\Rightarrow &log(|T - 22|) = kt + C \\
\Rightarrow &T - 22 = e^{kt+c} \\
\Rightarrow &T = ce^{kt} + 22
\end{aligned}
$$

Now At $t = 0, T = 5$C

$$
\begin{aligned}
&T = 22 + ce^0 \\
\Rightarrow &5 = c + 22 \\
\Rightarrow &c = 5 - 22 \\
\Rightarrow &c = -17
\end{aligned}
$$

$t = 1$min$, T = 12$C

$$
\begin{aligned}
&12 = 22 - 17e^k \\
\Rightarrow &12 = 22 - 17e^k \\
\Rightarrow &- 10 = -17e^k \\
\Rightarrow &k = ln|\frac{10}{17}| \\
\Rightarrow &k = -0.53062
\end{aligned}
$$

Hence at $T = 22$ t=?

$$
\begin{aligned}
&21.9 = 22 + -17e^{-0.53062t} \\
\Rightarrow &- 0.1 = -17e^{-0.53062t} \\
\Rightarrow &0.1/17 = e^{-0.53062t} \\
\Rightarrow &t = \frac{ln|\frac{0.1}{17}|}{-0.53062} \\
\therefore &t = 9.67 \text{ min}
\end{aligned}
$$

□

e. (Newton's Law of Cooling): The body of a murder victim was discovered at 11 : 00PM. The doctor took the temperature of the body at 11 : 30PM, which was 94.6∘F. He again took the temperature after 1 hour when it showed 93.4∘F, and noticed that the temperature of room was 70∘F. Estimate the time of murder (Normal temperature of human body is 98.6∘F).

Solution: According to Newton's law of cooling

$$\frac{dT}{dt}\alpha(T-70)$$

$$\begin{aligned}
&\frac{dT}{dt}=-\lambda(T-70)\\
&\Rightarrow\int dt(T-70)=\int-\lambda dt\\
&\Rightarrow\int dt(T-70)=\int-\lambda dt\\
&\Rightarrow log(|T-70|)=-\lambda t+C
\end{aligned}$$

Now $t=0, T=94.6$

$$\begin{aligned}
&log|94.6-70|=C\\
&\Rightarrow C=loge(24.6)
\end{aligned}$$

$t=1, T=93.4$

$$\begin{aligned}
&log|93.4-70|=-\lambda+C\\
&\Rightarrow log(23.4)=\lambda+C\\
&\Rightarrow \lambda=log(24.623.4)\\
&\Rightarrow \lambda=log(123117)\\
&\therefore \lambda=0.05
\end{aligned}$$

$t=t, T=98.6$

$$\begin{aligned}
&log|98.6-70|=-0.05t+log|24.6|\\
&\Rightarrow log|143123|=0.05t\\
&\Rightarrow 0.15066=-0.05t_1\\
&\therefore t_1=-3.0132
\end{aligned}$$

So, time of death was 3 hours before i.e., 8.30 pm(approx).

□

g. (Current): If a electromotive force of $160cos5t$ is impressed on a series circuit composed of a 20ohm resistor and a 10^{-1} H inductor, then find the steady state and transient current in the circuit.

Solution:

$$E_R+E_L=E \tag{11}$$

$E=160cos5t$
$R=20ohm$
$L=10^{-1}H$
From equation 11

$$\begin{aligned}
&IR+L\frac{dI}{dT}=E\\
&\Rightarrow\frac{R}{L}+\frac{dI}{dT}=\frac{E}{L}\\
&\Rightarrow\frac{20}{10^{-1}}+\frac{dI}{dT}=\frac{160cos5t}{10^{-1}}\\
&\Rightarrow I'+200I=1600cos5t
\end{aligned}$$

which looks like

$$y' + p(x)y = r(x)$$

$$\begin{aligned} I.F &= e^{\int p(x)dt} \\ &= e^{\int 200dt} \\ &= e^{200t} \end{aligned}$$

The solution is

$$\begin{aligned} I'e^{200t} + 200Ie^{200t} &= 1600cos5t \cdot e^{200t} \\ \Rightarrow (Ie^{200t})' &= 1600cos5t \cdot e^{200t} \end{aligned}$$

Integrating w.r.t t

$$\begin{aligned} Ie^{200t} &= \int 1600cos5t \cdot e^{200t} \\ \Rightarrow I &= e^{-200t} \cdot 1600 \int cos5t \cdot e^{200t} \\ \Rightarrow I & \end{aligned}$$

□

h. Find and solve the model for drug injection into the bloodstream if, beginning at t = 0, a constant amount A gm/min is injected and the drug is simultaneously removed at a rate proportional to the amount of the drug present at time t.

Solution: let the amoutn of the drug in the blood stream at any time be y.
Then according to the question is

$$\frac{dy}{dt} = \text{the rate of change of amoutn of drug} = -kt$$

-ve sign indicates removal
k = constant of proportionality

$$\begin{aligned} \frac{dy}{dt} &= -kt \\ \Rightarrow dy &= -ktdt \\ \therefore y(t) &= \frac{-kt^2}{2} + c \end{aligned}$$

At t = 0 ,y = A

$$\Rightarrow c = A$$

$$\Rightarrow y(t) = A + \frac{-kt^2}{2}$$

So at any time there is $A - \frac{kt^2}{2}$ g of drug in bloodstrea.
Total time ,$y = 0 \Rightarrow \frac{kt^2}{2} = A \Rightarrow t = \sqrt{\frac{2A}{k}}$

□

3. If $M(x,y)dx + N(x,y)dy = 0$ is any differential equation with $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ and $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M+N} = h(w)$ where $w = x - y$, then show that the integrating factor is $F(x,y) = e^{\int h(w)dw}$. Find the integrating factor and solve $(x+xy)dx + (y-xy)dy = 0$.

Solution: The differential equation is,
$M(x,y)dx + N(x,y)dy = 0$
As the differential equation is not exact
Let F is the integrating factor so,

$$FM(x,y)dx + fN(x,y)dy = 0 \tag{12}$$

For the exact diff. eqn

$$\begin{aligned}
&\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \\
\Rightarrow &\frac{\partial FM}{\partial y} = \frac{\partial FN}{\partial x} \\
\Rightarrow &F_y M - F_x N = F(N_x - M_y)
\end{aligned}$$

$$M\frac{\partial f(x,y)}{\partial y} - N\frac{\partial F(x,y)}{\partial x} = F(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}) \tag{13}$$

From eqn 12 and 13

$$\begin{aligned}
&F_x N - F_y M = F_n(M+N)(x-y) \\
\Rightarrow &F_x N - F_y M = F_n(x-y)M + F_n(x-y)N
\end{aligned}$$

Equating corresponding elements,

$$F_x = F_n(x-y) \tag{14}$$

$$F_y = -F_n(x-y) \tag{15}$$

Now Considering F(x-y) = F,

$$\frac{\partial F(x-y)}{\partial x} = F'(x-y) \tag{16}$$

$$\frac{\partial F(x-y)}{\partial x} = F'(x-y) \tag{17}$$

Equating 16,17,15,14

$$F'(x-y) = F_n(x-y) \tag{18}$$

$$-F'(x-y) = F_n(x-y) \tag{19}$$

From eqn 18

$$\begin{aligned}
&\frac{\partial F(x-y)}{\partial (x-y)} = F_n(x-y) \\
\Rightarrow &\frac{1}{F}\partial F(x-y) = h(x-y)d(x-y)
\end{aligned}$$

Integrating on both sides

$$\begin{aligned}
&ln(F(x-y)) = \int h(w)dw \\
\Rightarrow &F(x-y) = e^{\int h(w)dw} \\
&\text{Hence,} I.F = e^{\int h(w)dw}
\end{aligned}$$

Again,

$$\begin{aligned}
&(x+xy)dx + (y-xy)dy = 0 \\
&P = (x+xy), A = y - yx \\
&P_y = x, Q_x = y
\end{aligned}$$

$P_y \neq Q_x$ (not exact) $\frac{\partial M/\partial y - \partial N/\partial x}{M+N} = \frac{x+y}{x+y+xy-xy} = \frac{x+y}{x+y} = 1$
$I.F = e^{int d(x-u)} = e^{x-u}$
Multiplying by I.F

$$\begin{aligned}
&e^{x-y}(x+xy)dx + e^{x-y}(y-xy)dy = 0\\
&M = e^{x-y}(x+xy), N = e^{x-y}(y-xy)\\
\Rightarrow&\frac{\partial M}{\partial y} = xe^{x-y}(-1) + x(e^{x-y} + ye^{x-y}(-1))\\
\Rightarrow&\frac{\partial M}{\partial y} = -xye^{x-y}\\
&\frac{\partial N}{\partial x} = ye^{x-y} - (e^{x-y} + xe^{x-y})y\\
&\frac{\partial N}{\partial x} = -xye^{x-y}\\
\Rightarrow&\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\text{which is exact}
\end{aligned}$$

So,

$$\begin{aligned}
M &= \frac{\partial y}{\partial x}\\
&= e^{x-y}(x+xy)\\
\partial u &= (e^{x-y}(x+xy))dx
\end{aligned}$$

Integrating on both sides,

$$\begin{aligned}
&u = (x+xy)\int e^{x-y}dx - \int (1-y)e^{-xy}dx\\
\Rightarrow&u = (x+xy)e^{x-y} - e^{x-y} - ye^{x-y} + f(y)\\
\Rightarrow&u = (x+xy)e^{x-y} - (1+y)e^{x-y} + f(y)\\
\Rightarrow&\frac{\partial u}{\partial y} = -xe^{x-y} + x(e^{x-y} - ye^{x-y}) + e^{x-y} - (e^{x-y} - ye^{x-y}) + f(y)\\
\Rightarrow&\frac{\partial u}{\partial y} = e^{x-y}(-x + x - xy) + e^{x-y}(1 - 1 + y) + f'(y)\\
\Rightarrow&e^{x-y}(y-xy) = -e^{x-y}xy + ye^{x-y}f'(y)\\
\Rightarrow&e^{x-y}(y-xy) = e^{x-y}(y-yx) + f'(y)
\end{aligned}$$

$f'(y) = 0$
$f(y) = c$
$\therefore u(x,y) = e^{x-y}[x + xy - 1 - y] + c$ is the required eqn

□

4. Find orthogonal trajectories for the following:

(a) $y = \sqrt{x+c}$

Solution:

$$\begin{aligned}
y =& \sqrt{x+c} \\
\Rightarrow & y^2 = x + c \\
\Rightarrow & y^2 - x - c = 0 \\
\Rightarrow & c = y^2 - x \\
\Rightarrow & y' = \frac{d}{dx}(x+c)^{-1/2} \\
\Rightarrow & y' = \frac{1}{2}\frac{1}{\sqrt{x+c}} \\
\Rightarrow & y' = \frac{1}{2}\frac{1}{\sqrt{x+y^2-x}} \\
\Rightarrow & y' = \frac{1}{2y} = f(x,y)
\end{aligned}$$

Now the orthogonal trajectories is given by

$$\begin{aligned}
& y' = -\frac{1}{f(x,y)} \\
\Rightarrow & y' = -2y \\
\Rightarrow & \frac{y'}{y} = -2
\end{aligned}$$

Now Integrating we get

$$\begin{aligned}
& \int \frac{dy}{y} = \int -2dx \\
\Rightarrow & ln|y| = -2x + c \\
\therefore & ln|y| + 2x = c
\end{aligned}$$

$\square$

(b) $(x-c)^2 + y^2 = c^2$

Solution:

$$\begin{aligned}
& x^2 - 2cx + c^2 + y^2 = c^2 \\
\Rightarrow & x^2 - 2cx + y^2 = 0 \\
\Rightarrow & c = \frac{x^2 + y^2}{2x}
\end{aligned}$$

Also,

$$
\begin{aligned}
&x^2 - 2cx + y^2 = \\
\Rightarrow &y^2 = -x^2 + 2cx \\
\Rightarrow &y = \sqrt{2cx - x^2} \\
\Rightarrow &y' = \frac{d}{dx}(2cx - x^2)^{-1/2} \\
\Rightarrow &y' = -\frac{1}{2}\frac{1}{\sqrt{2cx - x^2}}(2c - 2x) \\
\Rightarrow &y' = \frac{1}{2}\frac{1}{\sqrt{2x\frac{x^2+y^2}{2x} - x^2}}\frac{2(x^2+y^2)}{2x} - 2x \\
\Rightarrow &y' = \frac{2y^2 - 2x^2}{4xy} \\
\therefore &y' = \frac{y^2 - x^2}{2xy}
\end{aligned}
$$

Now the orthogonal trajectories is given by

$$
\begin{aligned}
&y' = -\frac{1}{f(x,y)} \\
\Rightarrow &y' = \frac{2xy}{y^2 - x^2} \\
\Rightarrow &\frac{dy}{dx} = \frac{2xy}{y^2 - x^2} \\
\Rightarrow &-2xydx(x^2 - y^2)dy = 0
\end{aligned}
$$

Comparing above with $M(x,y)dx + N(x,y)dy = 0$

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{-2xy}{\partial y} = -2x \\
\frac{\partial N}{\partial x} &= \frac{x^2 - y^2}{\partial x} = 2x
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ So the D.E is not exact

$$
\begin{aligned}
I.F. &= Ce^{\int Rdx} \\
&= Ce^{\int \frac{1}{M}(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dx} \\
&= e^{\frac{1}{-2xy}(2x+2x)dy} \\
&= e^{\frac{1}{2xy}4xdy} \\
&= e^{\frac{2}{y}dy} \\
&= e^{-2ln|y|} \\
&= y^{-2} \\
&= \frac{1}{y^2}
\end{aligned}
$$

So,

$$
-\frac{2x}{y}dx + (\frac{x^2}{y^2} - 2)dy = 0
$$

The solution is

$$u = [\int M dx + k(y)]$$
$$\Rightarrow u = \int \frac{-2x}{y} dx + k(y)$$
$$\Rightarrow u = \frac{-x^2}{y} + k(y)$$

Diff. partially w.r.t y

$$\frac{\partial u}{\partial y} = \frac{x^2}{y} + k(y)$$
$$N = \frac{-x^2}{y} + \frac{dk}{dy}$$
$$(-1 + \frac{-x^2}{y}) = \frac{-x^2}{y} + \frac{dk}{dy}$$
$$\frac{dk}{dy} = -1 Integrating w.r.t y k = -y + c$$

Hence,

$$u = \frac{-x^2}{y} - y + c$$
$$\therefore u = \frac{-x^2 - y^2 + cy}{y} \text{is required solution}$$

□

(d) $\frac{x^2}{a} + \frac{y^2}{a-\lambda} = 1$
Solution:

$$\frac{x^2}{a} + \frac{y^2}{a-\lambda} = 1$$

Differentiating w.r.t x

$$\frac{2x}{a} + \frac{1}{(a-\lambda)} 2y \cdot \frac{dy}{dx}$$
$$\Rightarrow \frac{2y}{(a-\lambda)} \cdot \frac{dy}{dx} = \frac{-2x}{a}$$
$$\Rightarrow \frac{dy}{dx} = \frac{-2x}{a} \cdot \frac{a-\lambda}{2y}$$
$$\Rightarrow \frac{dy}{dx} = \frac{-x}{y} \cdot \frac{a-\lambda}{a}$$
$$\Rightarrow y' = \frac{x}{y} \frac{(\lambda - a)}{a} = f(x, y)$$

The orthogonal trajectories is given by

$$y' = -\frac{1}{f(x,y)}$$
$$\Rightarrow y' = \frac{y}{x} \frac{a}{\lambda - a}$$
$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} \frac{a}{\lambda - a}$$
$$\Rightarrow \frac{dy}{y} = \frac{dx}{x} \cdot \frac{a}{\lambda - a}$$

Integrating

$$\therefore lny = \frac{a}{\lambda - a} lnx + c \text{ is the required solution.}$$

$\square$