CLASS NOTES | 2020

Differential Equation

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Chapter 1

First Order Differential Equations

1.1 Linear Differential Equation

A differential equation of form

$$y' + p(x)y = r(x) \tag{1.1}$$

where

$$p, q \text{ are functions of x alone or constants.}$$

Characterstics Features

- If q = 0, the equation (1.1) becomes $y' + p(x)y = r(x)$ is homogeneous; otherwise it is non-homogeneous.

Question 1

Solve the following equation $y' + p(x)y = r(x)$.

Solution:

$$\begin{aligned}
&\Rightarrow y' = -p(x)y \\
&\Rightarrow \int \frac{dy}{y} = \int -pdx \\
&\Rightarrow ln|y| = \int -pdx + c \\
&\Rightarrow y = e^{\int pdx} \cdot e^{c_1} \\
&\Rightarrow y = c_2 \cdot e^{\int pdx}
\end{aligned}$$

□

Question 2

Solve the following equation $y' + py = r$

Solution:

$$\frac{dy}{dx} = py - r = 0$$
$$dy + (py - r)dx = 0$$
$$(py - r)dx + 1dy = 0 \tag{1.2}$$

Comparing with $Pdx + Qdy = 0$
P = py-r
Q = 1

$$\frac{\partial P}{\partial y} = P_y = p, \frac{\partial Q}{\partial x} = Q_y = 0$$

Eqn (1.2) is note exact
Let F = F(x) be the integrating factor of eqn (1.2)
Then $F = e^{\int R(x)dx}$

where

$$R(x) = \frac{1}{Q} \cdot (P_y - Q_x)$$

$$R(x) = p$$

$$F = e^{intpdx}$$

Multiplying eqn (1.1) with integrating factor

$$\Rightarrow e^{\int pdx}(y' + py) = re^{\int pdx}$$
$$\Rightarrow (ye^{\int pdx})' = re^{\int pdx}$$
$$\Rightarrow ye^{\int pdx} = \int re^{\int pdx} + C$$
$$\Rightarrow y = e^{\int -pdx}\left[\int re^{\int pdx} + C\right]$$

where

h = $\int pdx$
The equation is not exact.

□

Question 3

Solve the following equation $y' - y = e^{2x}$.

Solution:

$$y = e^{-x}\left[\int re^{x}dx + c\right]$$
$$= e^{-x}\left[re^{x} + c\right]$$

□

Question 4

Solve the following equation $x^3y' + 3x^2y = \frac{1}{x}$

Solution:

□

1.2 Bernoulli Equation

Definition 1.2.1

The equation of the following form is called Bernoulli equation

$$y' + p(x)y = r(x)y^a \tag{1.3}$$

If a = 0 or a = 1, equation (1.3) is linear otherwise equation is non linear.

Example 1.2.1
Consider the non linear equation

$$y' + py = ry^a \tag{1.4}$$

Let $u(x) = [y(x)]^{1-a}$

$$u' = (1-a)y^{-a} \cdot y' \tag{1.5}$$

1.3 Newton's Law of Cooling

Theorem 1.3.1
The temperature rate of change of the body over respective to time is

$$\frac{dT}{dt} \propto (T - T_0)$$

where

$$\text{T} = \text{f(t)}$$

$$\begin{aligned}
&\Rightarrow \frac{dT}{dt} = k(T - T_0) \\
&\Rightarrow \int \frac{dT}{T - T_0} = \int kdt \\
&\Rightarrow ln|T - T_0| = kt + c^* \\
&\Rightarrow T = T_0 + ce^{kt}
\end{aligned}$$

Question 5

A body temperature T is instantly 200°C is immersed in a liquid when temperature T_0 is constantly.

Solution: We have,

$$\begin{aligned}
&\frac{dT}{dt} = k(T - T_0) \\
&\text{initial temperature}(T_0) = 100^\circ C \\
&\text{at, t = 0, T = 200°C} \\
\Rightarrow &200 = 100 + ce^{kt} \\
\Rightarrow &ce^{kt} = 100 \\
&\therefore c = 100
\end{aligned}$$

Now,

$$\begin{aligned}
&\text{at t = 1 min, T = 100°C}\\
\Rightarrow &150 = 100 + ce^{kt}\\
\Rightarrow &50 = 100e^{kt}\\
\Rightarrow &e^{k} = \frac{1}{2}\\
\therefore &k = ln|\frac{1}{2}|
\end{aligned}$$

Now,

Temperature at 2 min is

$$\begin{aligned}
T &= T_0 + ce^{kt}\\
&= 100 + 100ce^{2ln|\frac{1}{2}|}\\
&= 100(1 + ce^{ln|\frac{1}{2}|^2})\\
&= 100(1 + \frac{1}{4})\\
&= 125°C
\end{aligned}$$

□

1.4 Electrical Circuits

- Ohm's Law

Definition 1.4.1

Voltage drop across resistor is directly proportional to current.

$$E_R \propto T$$

$$E_R = IR$$

- Henry Law

Definition 1.4.2

Voltage drop across inductor is directly proportional to rate of change of current.

$$V_L \propto \frac{dI}{dt}$$
$$V_L = L\frac{dI}{dt}$$

- Capcitor Law

Definition 1.4.3

$$E_e \propto charge(Q)$$
$$E_e = \frac{1}{C} * Q$$

- Kirchoff's Law

Definition 1.4.4

The algebraic sum of Voltage drop around the charge loop across the closed loop equals zero.

$$\sum IR = 0$$

Chapter 2

Second Order Differential Equation

2.1 Second Order Linear Differential Equation:

A differential equation of form

$$y'' + p(x)y' + q(x)y = r$$

$$\Rightarrow y'' + py' + qy = r \tag{2.1}$$

Note:-

It is called second order differential equation because it has maximum two degree and it has two solutions namely $y_1$andy_2.

Question 6

Show that the solutions of the following equation $y'' - y = 0$ are $y_1 = e^{-x}, y_2 = e^x$.

Solution: Let the solution of the equation is e^x then

$$y'' = e^x andy = e^x$$

$$\Rightarrow e^x - e^x = 0$$
$$\therefore 0 = 0(\text{True})$$

Let the solution of the equation is e^{-x} then

$$y'' = e^{-x} andy = e^{-x}$$

$$\Rightarrow e^{-x} - e^{-x} = 0$$
$$\therefore 0 = 0(\text{True})$$

□

Theorem 2.1.1

The fundamental theorem of Homogeneous Equation(Superposition Principle or Linearity Principle)

Statement: If y_1 and y_2 be the solutions of the differential equation $y'' + py' + qy = 0$, then $y = c_1y_1 + c_2y_2$.

Proof: Substituting $y = c_1y_1 + c_2y_2$, we find

$$y' = c_1y_1' + c_2y_2'$$
$$y'' = c_1y_1'' + c_2y_2''$$

Substituting y and its derivative in eqn

$$(c_1y_1'' + c_2y_2'') + p(c_1y_1' + c_2y_2') + q(c_1y_1 + c_2y_2) = 0$$
$$\Rightarrow c_1y'' + c_2y'' + pc_1y_1' + pc_2y_2' + qc_1y_1 + c_2y_2 = 0$$
$$\Rightarrow c_1(y_1'' + py_1' + qy_1) + c_2(y_2'' + py_2' + qy_2) = 0$$
$$\therefore 0 = 0(True)$$

Note:-

This theorem is only applicable to the homogeneous equation.