DD2437 – Artificial Neural Networks and Deep Architectures (annda)

Lecture 8: **Hopfield networks**

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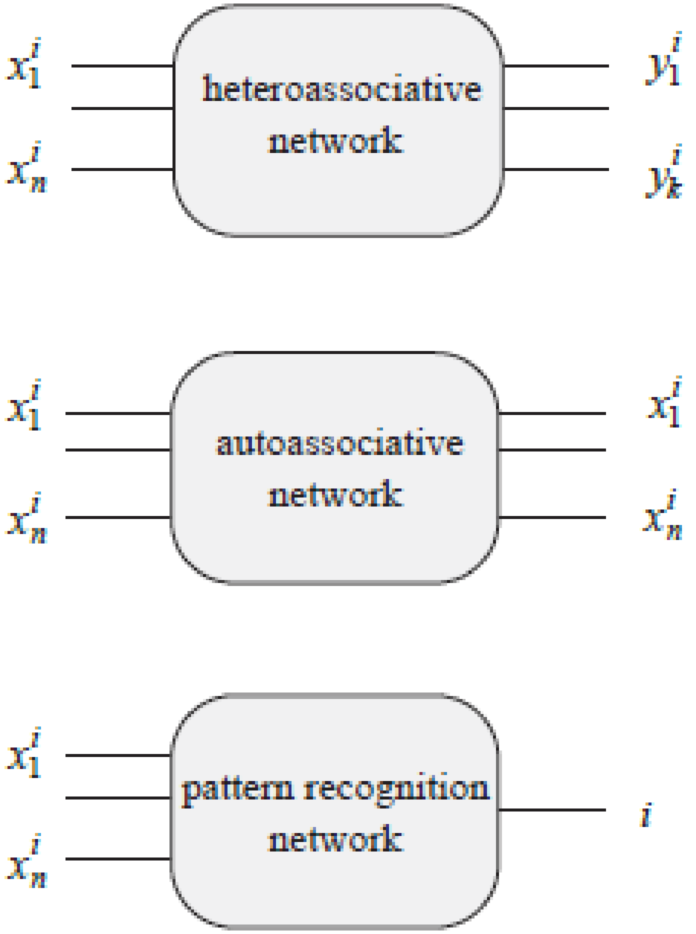
February 2019

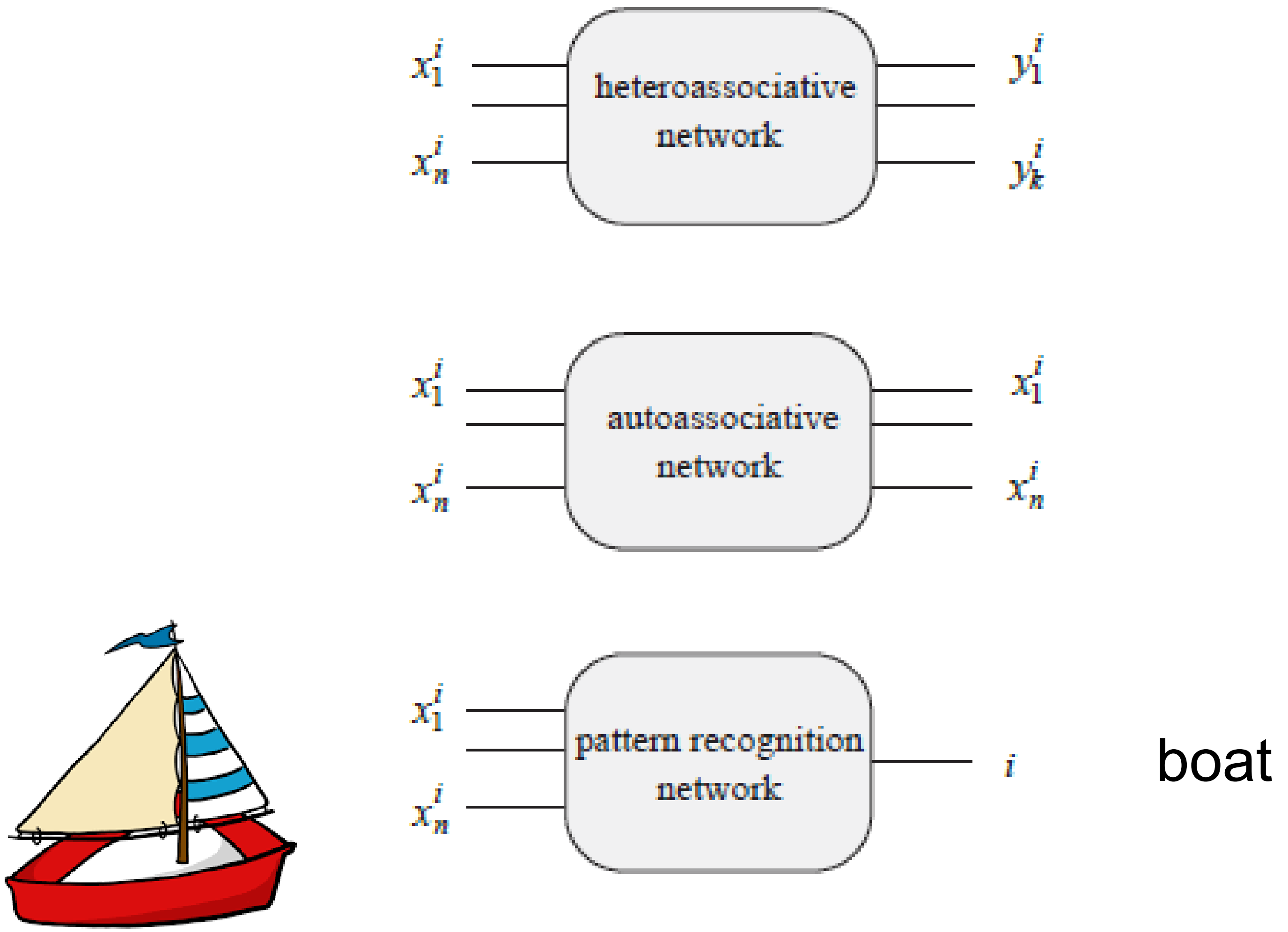
Lecture overview

- Associative memory, learning
- Hopfield networks
- Storage capacity
- Optimisation with Hopfield networks

Associative pattern recognition



Associative pattern recognition



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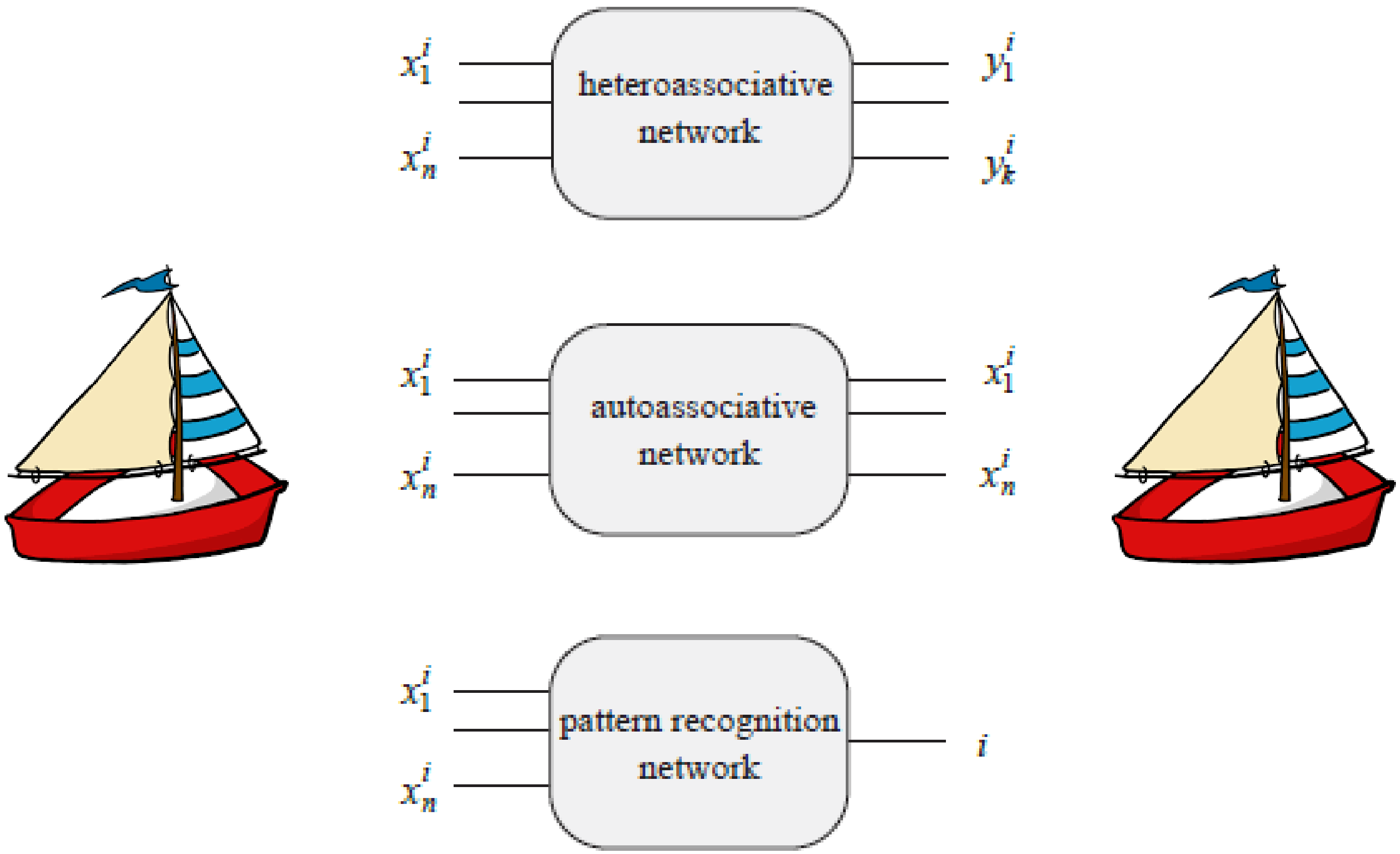
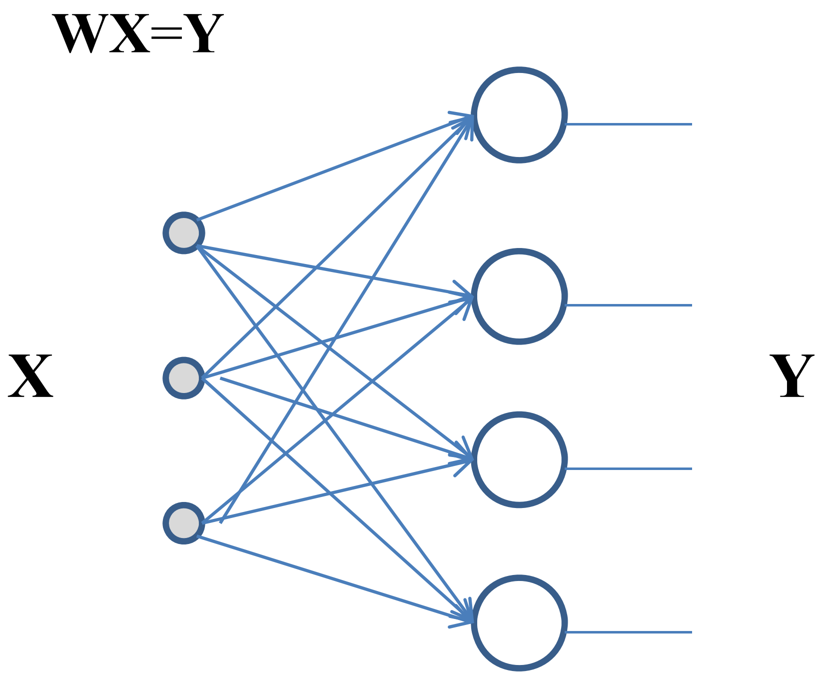


Linear associative memory networks

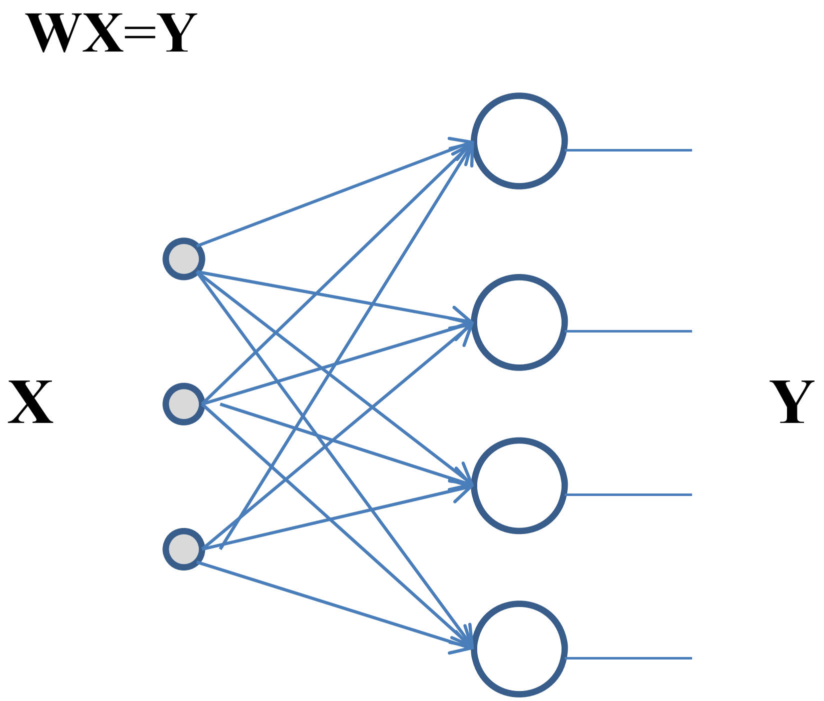
- Single layer networks (see lecture 2, correlation memory)

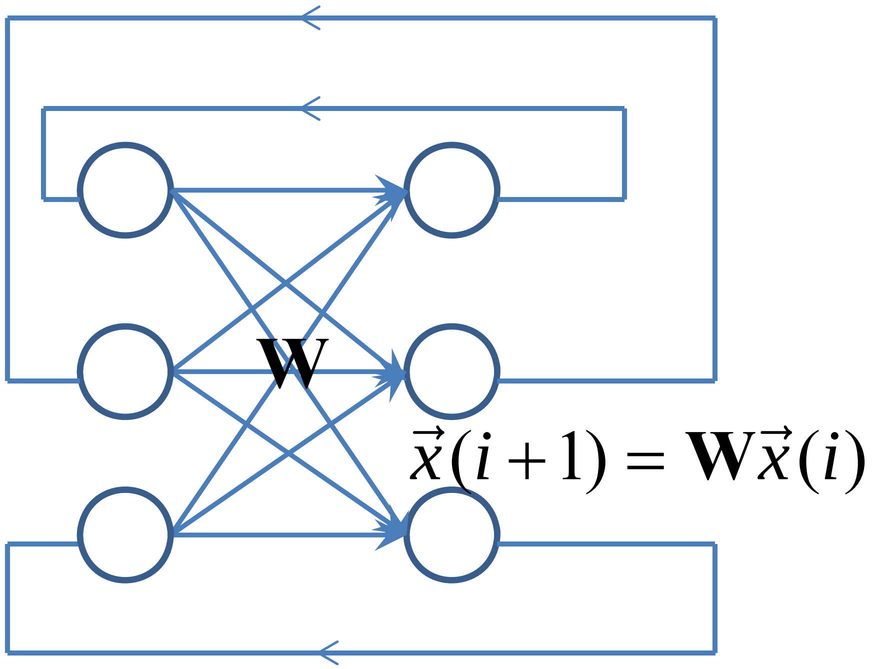


without feedback
(recall is a feedforward step)

Linear associative memory networks

- Simple single layer or recurrent networks





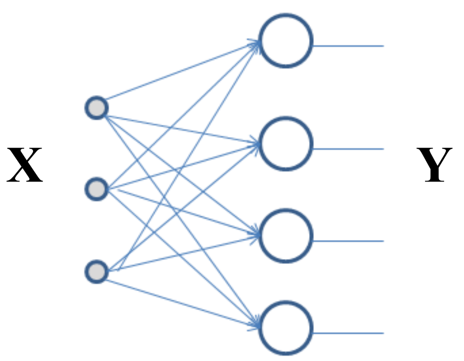
without feedback
(recall is a feedforward step)

autoassociative recurrent network, with feedback
(recall is an iterative process)

Associative learning in a single layer network

- Bipolar coding {-1, 1} with sign transform:

$$\mathrm{sgn}(x) = \begin{cases} 1 \;, x \geq 0 \\ -1 \;, x < 0 \end{cases}$$



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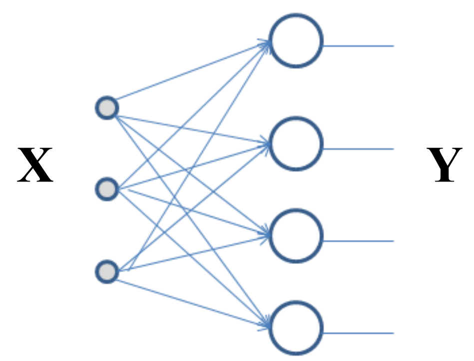
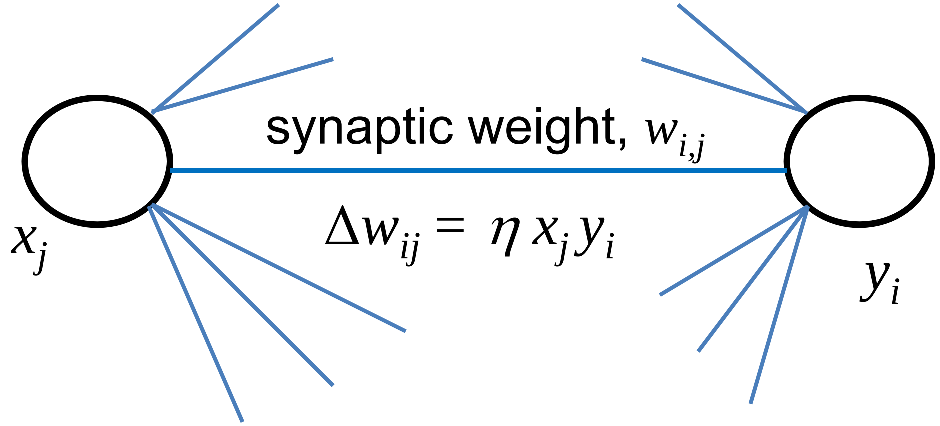


- Hebbian learning (correlation learning, outer product)

$$\mathbf{W} = \mathbf{W}^1 + \mathbf{W}^2 + \ldots + \mathbf{W}^m$$

$$\mathbf{W}^k = [w_{ij}] = [x_j^k \; y_i^k] \quad (\textit{outer product})$$



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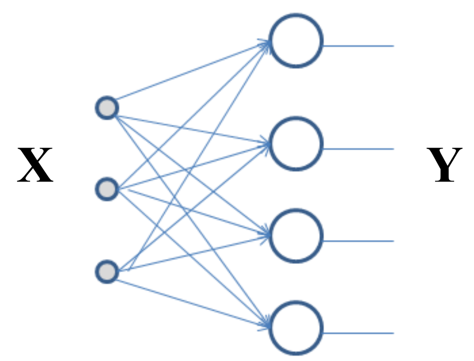

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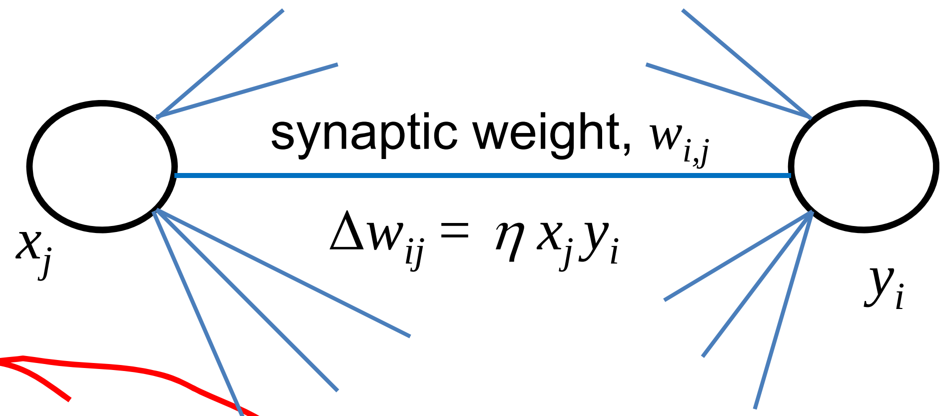

$$\vec{y}_p(\vec{x}_p^{\mathrm{T}} \vec{x}_p) + \sum_{p \neq k}^{m} \vec{y}_k (\vec{x}_p^{\mathrm{T}} \vec{x}_k)$$

crosstalk

Hebbian learning for associative memory

- Autoassociative case

$$\mathbf{W} = \mathbf{X}\mathbf{X}^{\mathrm{T}}$$

$$\mathrm{sgn}(\mathbf{W}\vec{x}) = \vec{x}, \quad \mathrm{sgn}(\mathbf{W}\mathbf{X}) = \mathbf{X}$$

Essentially, $\vec{x}$ are the eigenvectors of nonlinear sgn operation so the idea is to find $\mathbf{W}$ for which sgn($\mathbf{W}\mathbf{X}$) has these patterns as eigenvectors,
<u>but we do not want $\mathbf{W} = \mathbf{I}$ as a trivial solution of $\mathrm{sgn}(\mathbf{W}\mathbf{X}) = \mathbf{X}$</u>

$$\text{for } \mathbf{W} = \mathbf{X}\mathbf{X}^{\mathrm{T}}, \ \mathrm{sgn}(\mathbf{W}\mathbf{X}) = \mathrm{sgn}(\mathbf{X}\underbrace{\mathbf{X}^{\mathrm{T}}\mathbf{X}}) = \mathrm{sgn}(\mathbf{X}) = \mathbf{X}$$

For orthogonal $\mathbf{X}$ (or nearly),
$\mathbf{X}^{\mathrm{T}}\mathbf{X}$ is a scaled identity $\mathbf{I}$ matrix

Hebbian learning for associative memory

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Essentially, $\vec{x}$ are the eigenvectors of nonlinear sgn operation so the idea is to find $\mathbf{W}$ for which sgn($\mathbf{WX}$) has these patterns as eigenvectors,

$$\mathbf{W} = \mathbf{X}\mathbf{X}^{\mathrm{T}}$$

From a geometrical perspective:

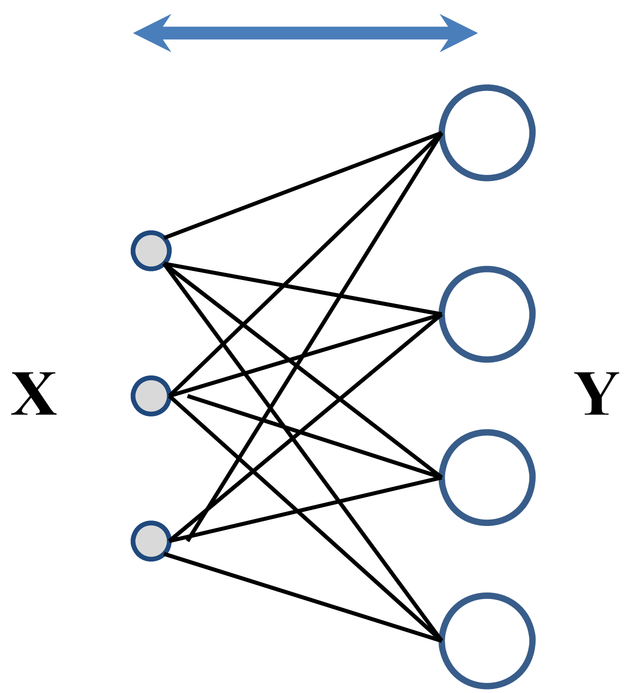
$\mathbf{W}$ describes *non-orthogonal* projection on the subspace spanned by $\vec{X}$.

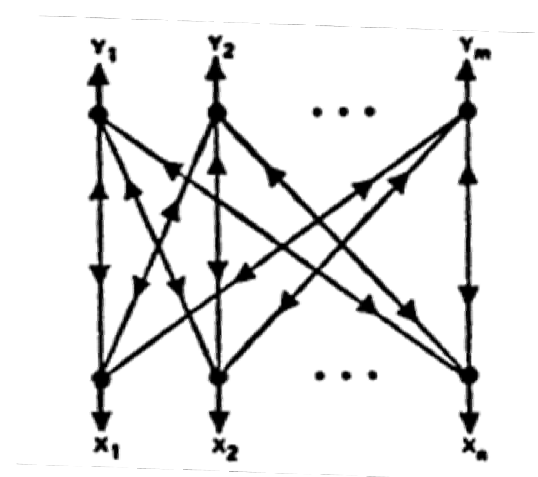
Bidirectional associative memory (resonance)

Builds on the concept of memory networks with feedback (recursive)

- bipolar {-1, 1} coding

- sign activation function





B. Kosko, 1988

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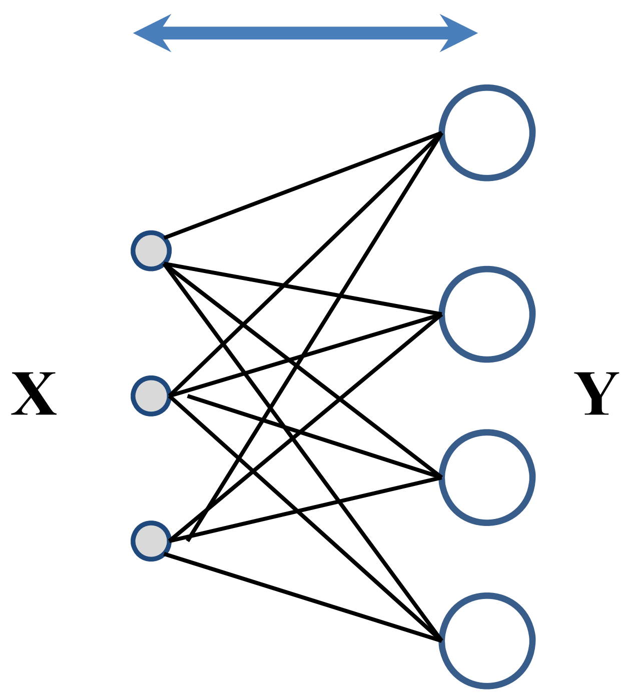

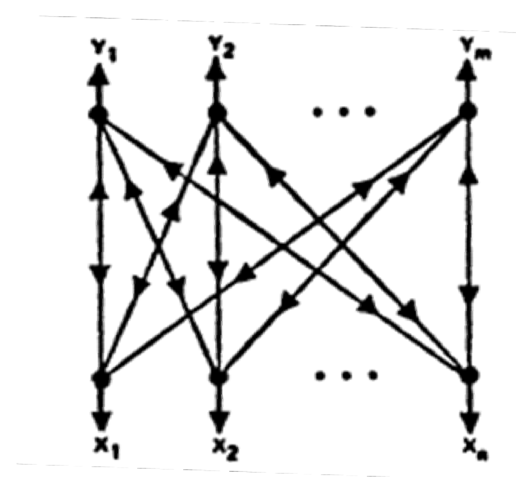

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Bidirectionality (feedback) imposes extra challenges

- synchronous vs asynchronous update
- different properties depending on updating mode

Bidirectional associative memory (resonance)

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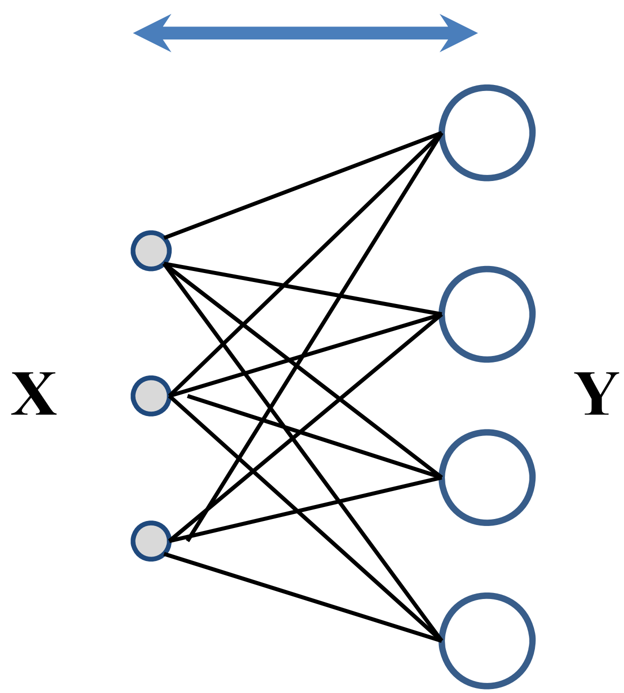
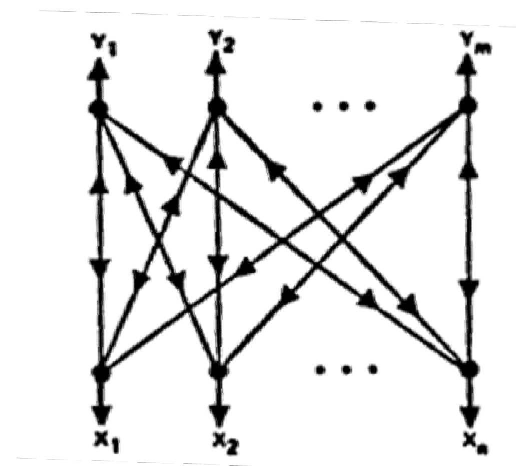
$$\boxed{\begin{aligned} \vec{y}(t) &= \mathrm{sgn}(\mathbf{W}\vec{x}(t)) \\ \vec{x}(t+1) &= \mathrm{sgn}(\mathbf{W}\vec{y}(t)) \end{aligned}}$$



Does it converge?

What are stable points?



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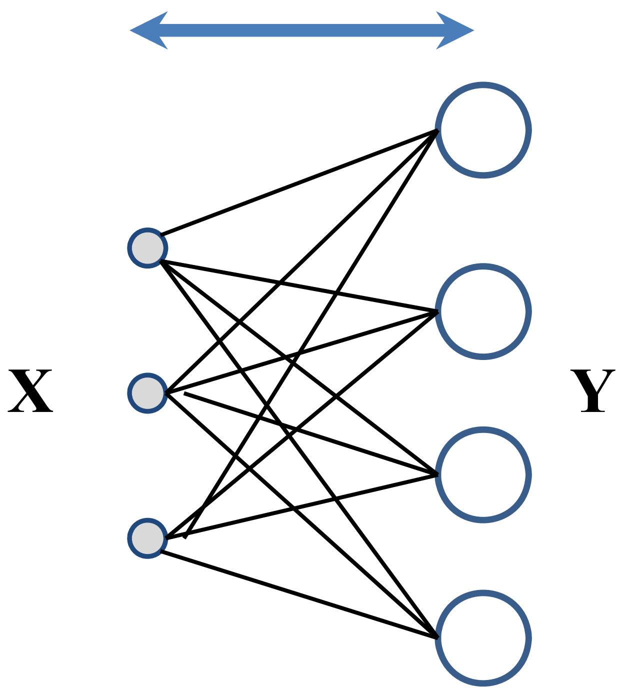
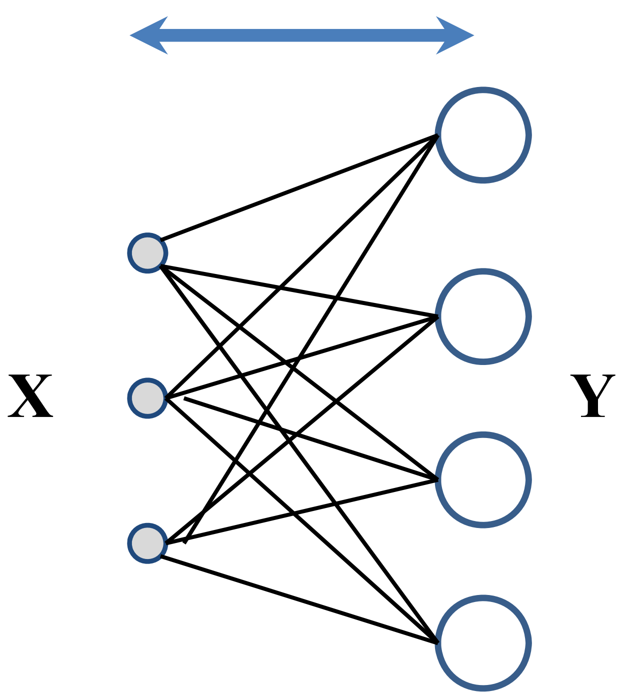
Concept of energy in BAM



If $(\vec{x}, \vec{y})$ is a stable point, then nearby points like $(\vec{x}_0, \vec{y}_0)$ should converge.

Concept of energy in BAM



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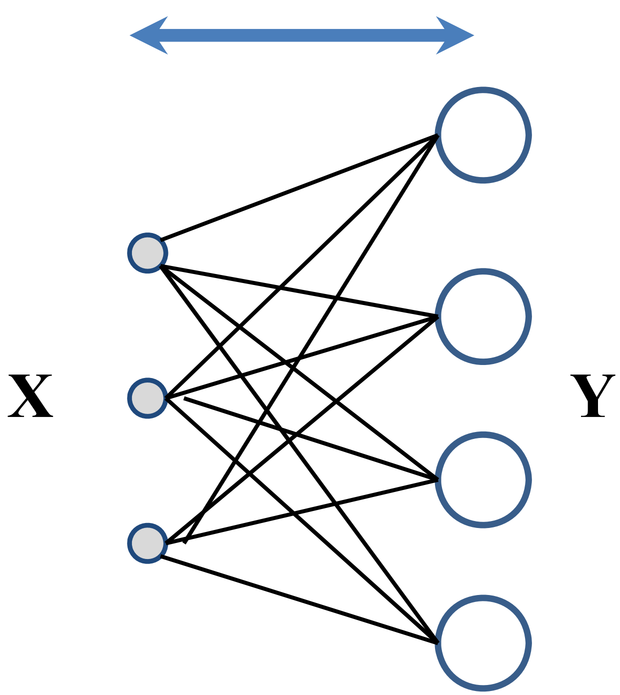
$$\vec{y}_0 = \mathbf{W}\vec{x}_0 \text{ , next } \quad \vec{e} = \mathbf{W}^{\mathrm{T}}\vec{y}_0$$

How far is $\vec{e}$ from $\vec{x}_0$?

Concept of energy in BAM



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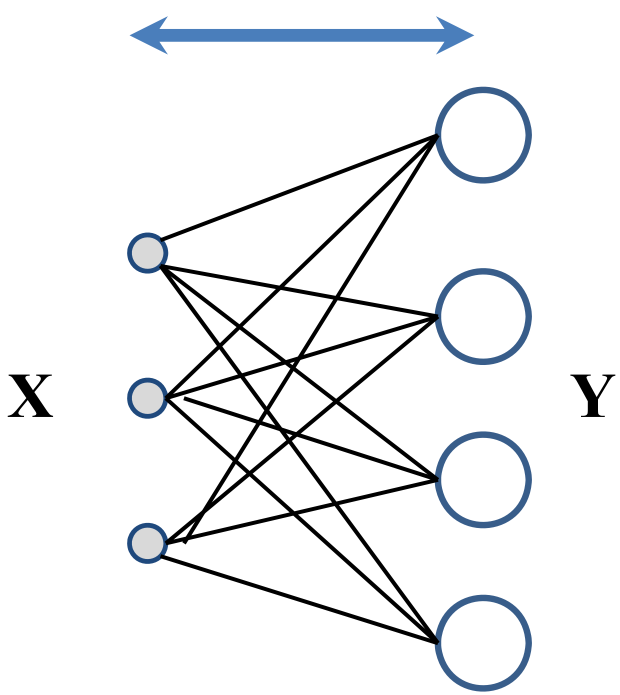
$$E = -\vec{x}_0^{\ \mathrm{T}}\vec{e} = -\vec{x}_0^{\ \mathrm{T}}\mathbf{W}^{\mathrm{T}}\vec{y}_0 = -\vec{y}_0^{\ \mathrm{T}}\mathbf{W}\vec{x}_0$$

Concept of energy in BAM



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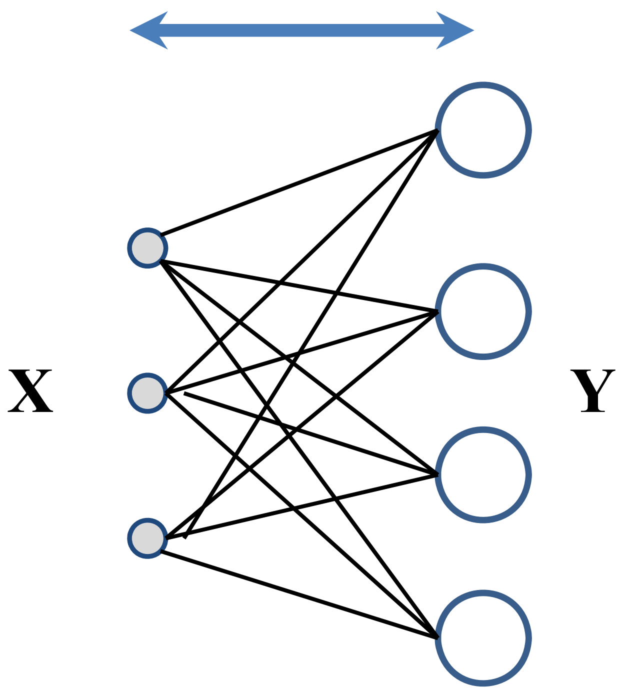
For the autoassociative BAM with **W**, energy in the state $\vec{x}$:

$$E(\vec{x},\vec{x}) = -\frac{1}{2}\vec{x}^{\mathrm{T}}\mathbf{W}\vec{x}$$

$$E(\vec{x}) = -\frac{1}{2}\sum_{i,j=1}^{n} w_{i,j}x_i x_j$$

Concept of energy in BAM



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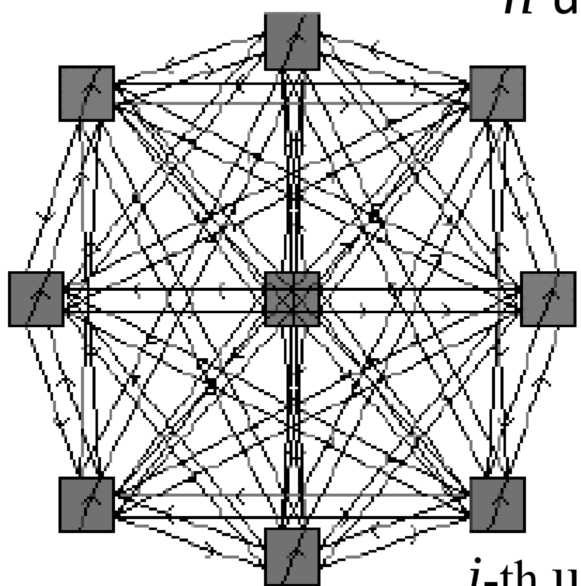
For the autoassociative BAM with $\mathbf{W}$, energy in the state $\vec{x}$:

$$E(\vec{x}, \vec{x}) = -\frac{1}{2}\vec{x}^{\mathrm{T}}\mathbf{W}\vec{x} + \vec{x}^{\mathrm{T}}\vec{\theta}$$

If bias is added

$$E(\vec{x}) = -\frac{1}{2}\sum_{i,j=1}^{n} w_{i,j}x_i x_j + \sum_{i=1}^{n}\theta_i x_i$$

Hopfield network



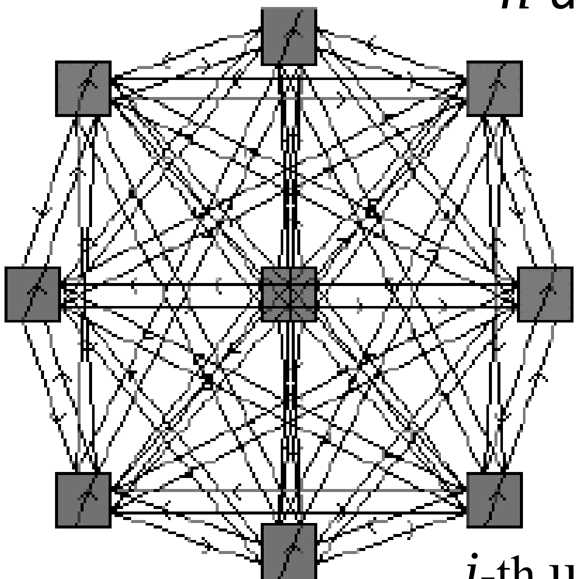
$$\forall_i \, w_{i,i} = 0$$ no self-connections

$$\vec{x}\,' = \mathrm{sgn}(\mathbf{W}\vec{x} + \vec{\theta})$$

$$E(state = \vec{x}) = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{i,j} x_i x_j + \sum_{i=1}^{n} \theta_i x_i$$

Hopfield network

n units



j-th unit

$$\forall_i \, w_{i,i} = 0$$ no self-connections

$$\vec{x}' = \mathrm{sgn}(\mathbf{W}\vec{x} + \vec{\theta})$$

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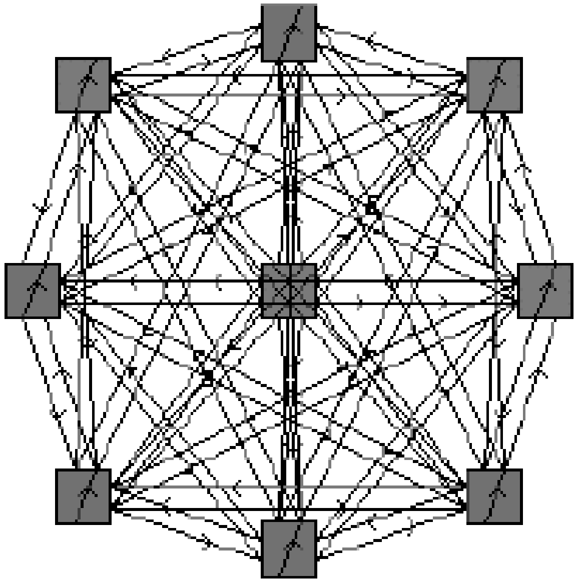
Iterative recall with asynchronous update

1) Apply input probe $\boldsymbol{\xi}_{\boldsymbol{p}} = [\xi_{1,p}, \xi_{2,p}, \ldots, \xi_{n,p}]$, i.e. $x_j(0) = \xi_{j,p}$

2) Iterate *asynchronous* update until convergence (until the state $\boldsymbol{x}$ remains unchanged)

$$x_j(t+1) = \mathrm{sgn}\left(\sum_{i=1}^{n} w_{j,i}\, x_i(t)\right)$$ j=1,..,n is randomly selected one at a time

Hopfield network



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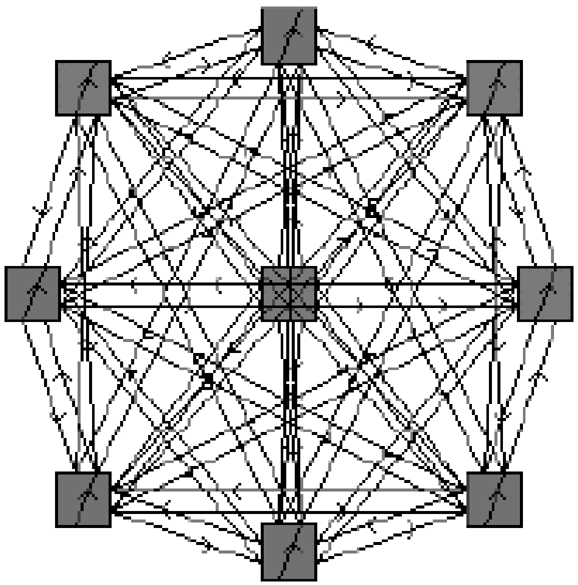
Update occurs only when the state changes, so.....

$$\Delta E_{x_j \to x_j^*} = -\frac{1}{2}\left(\sum_{i}^{n} w_{i,j} x_i x_j^* - \sum_{i}^{n} w_{i,j} x_i x_j\right) = -\frac{1}{2}\left(x_j^* - x_j\right)\sum_{i}^{n} w_{i,j} x_i \le 0$$

Hopfield network



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W should be symmetric with diag=0 for convergence

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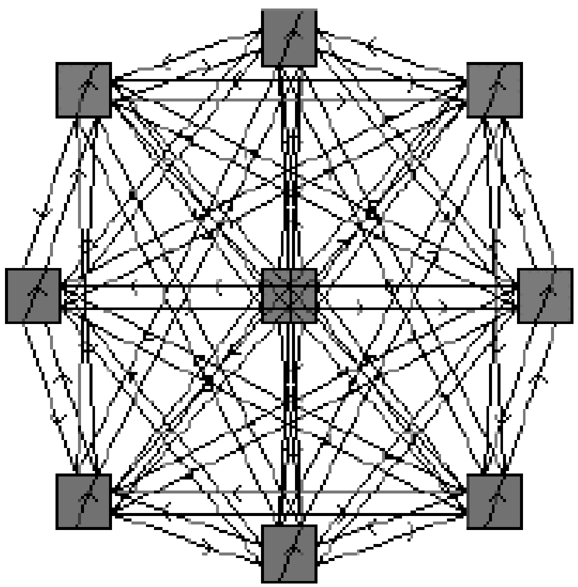
towards lower energy – convergence!

Hopfield network



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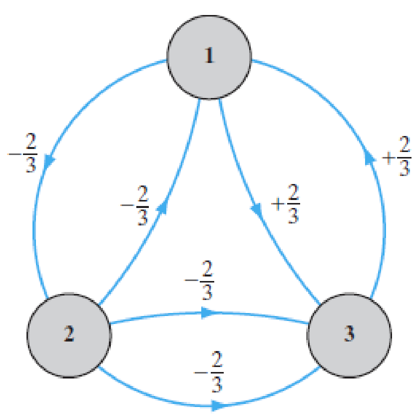
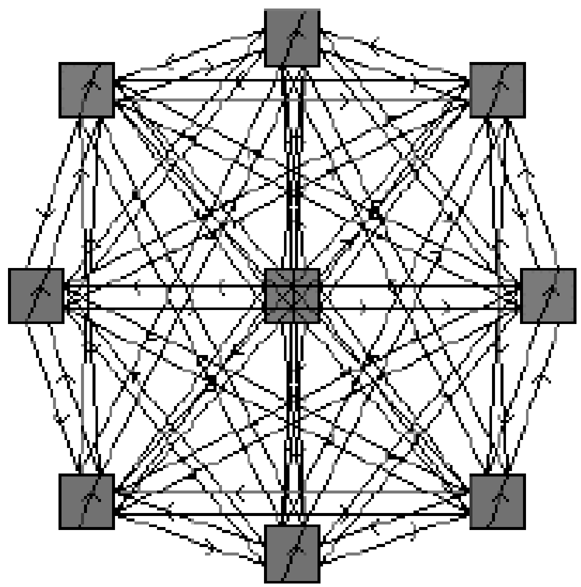


How many states are candidates for fixed states?

Hopfield network



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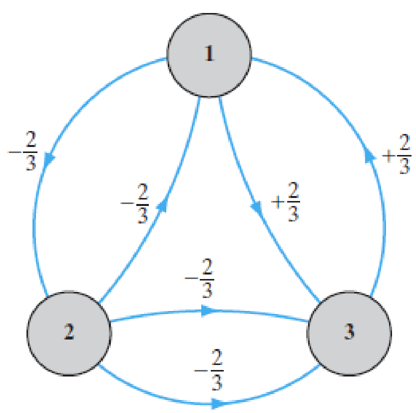
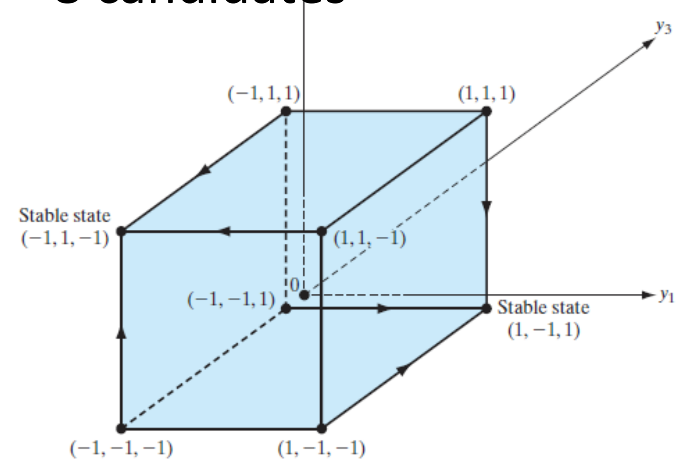
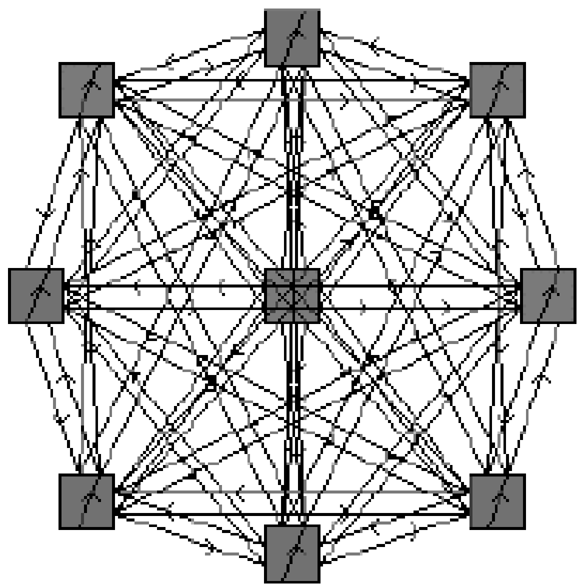


8 candidates



Hopfield network



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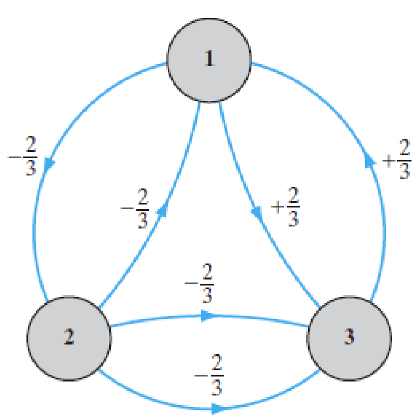
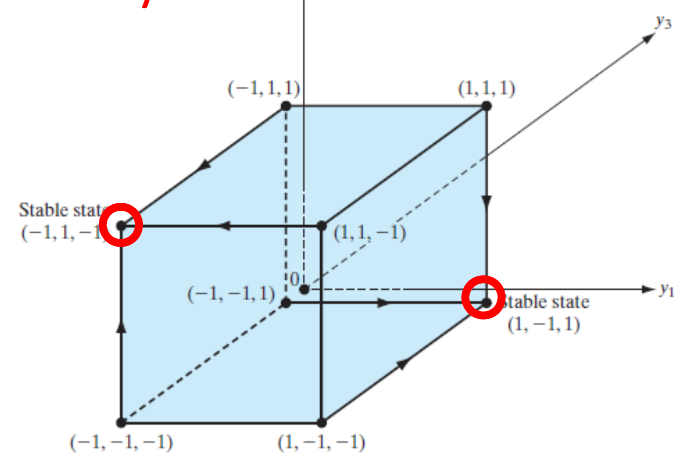
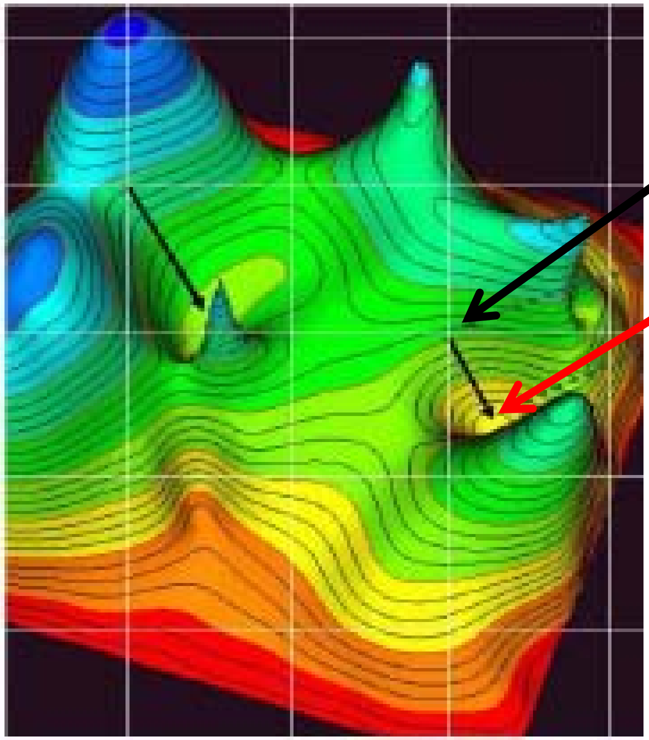


Only **2** out of 8 turn out to be stable!



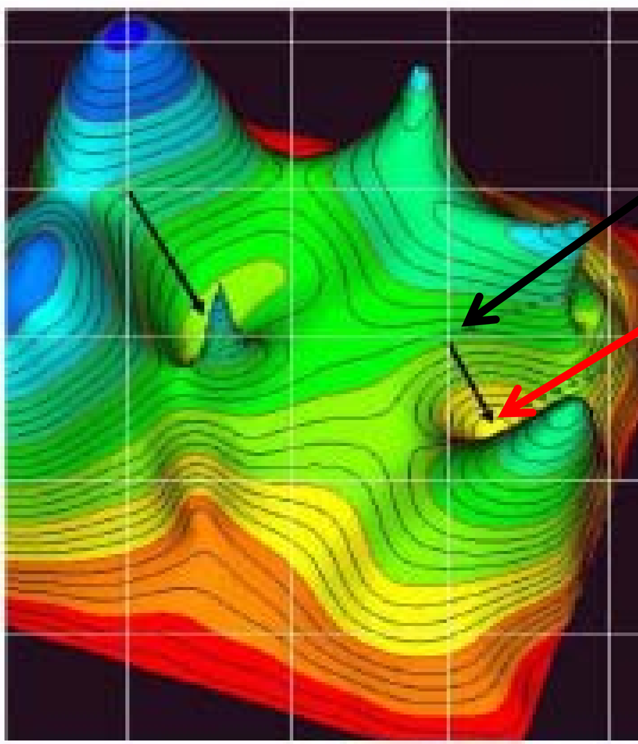
Attractor dynamics



Memory cue
(within the basin of attractor)

Memory state
(local energy minimum,
stable point, attractor)

Attractor dynamics



Memory cue
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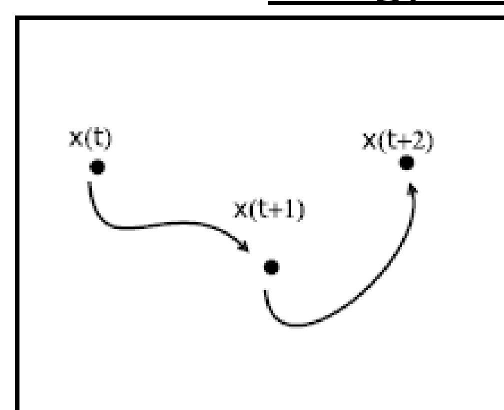
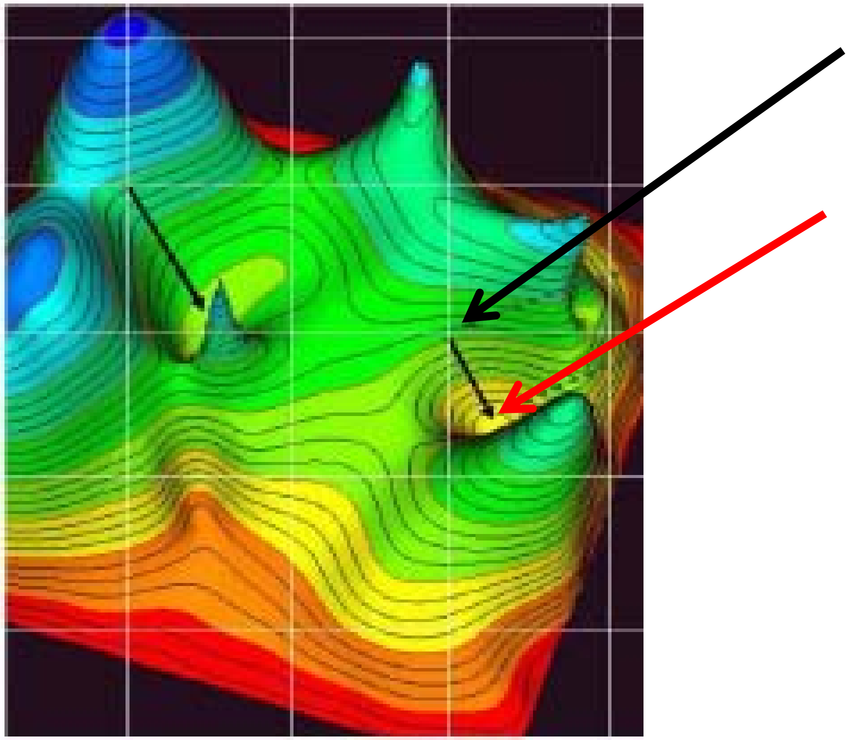
Memory state
(local energy minimum,
stable point, **fixed-point attractor**)

Dynamics travelling in the energy landscape
and attracted to the energy minimum



In *discrete* Hopfield network,
the energy landscape is discrete!

Attractor dynamics



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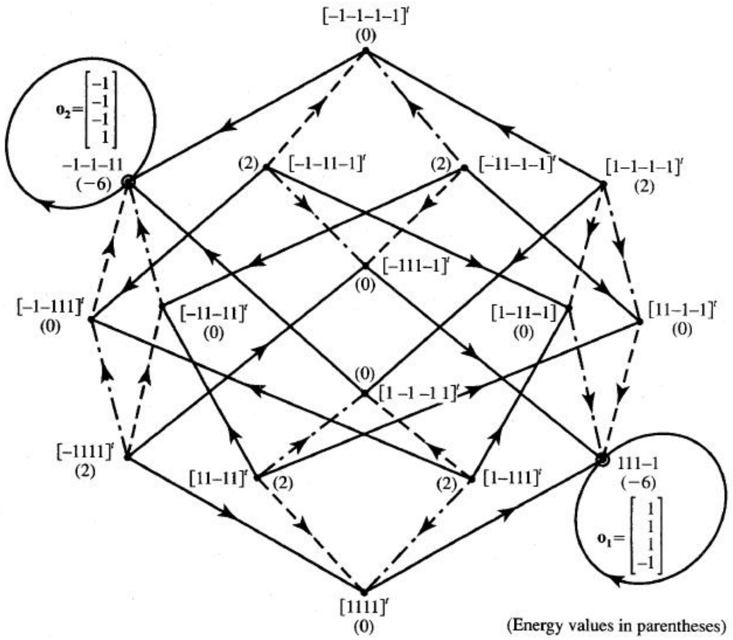

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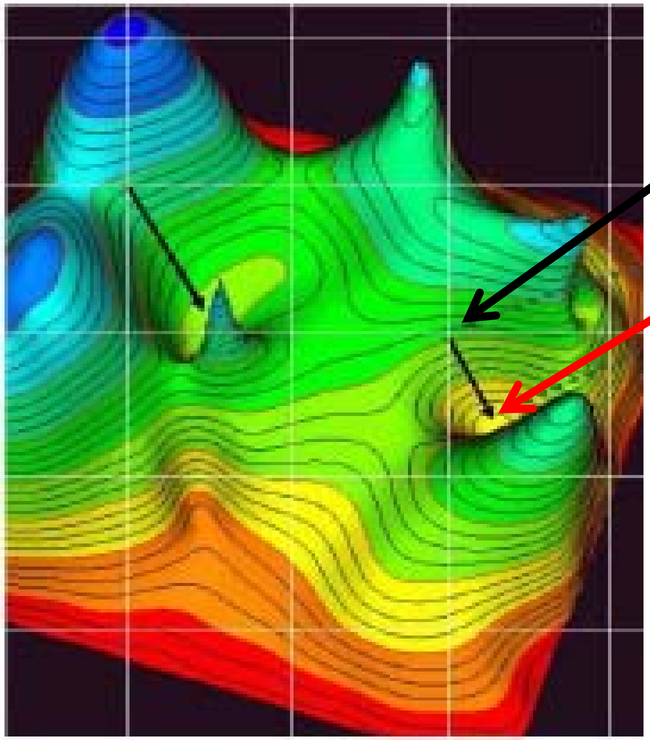

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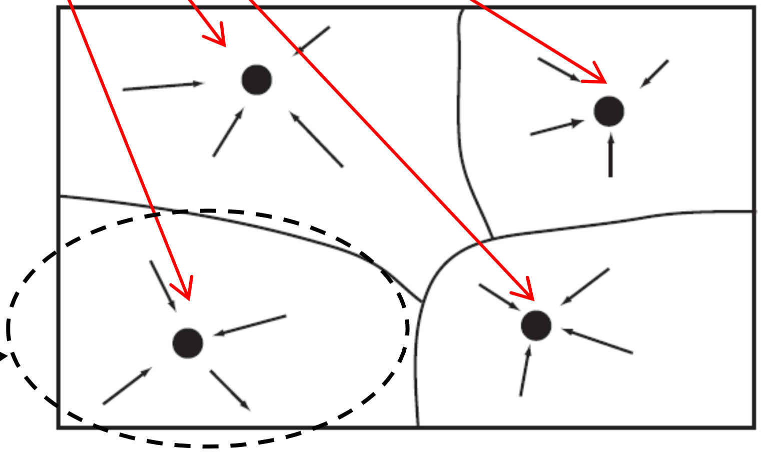


Around each fixed point (attractor), there is a ***basin of attraction***

How do we learn memories for storage?

Hopfield network as a content addressable memory

A set of memory patterns $\{\boldsymbol{\xi_1}, \boldsymbol{\xi_2}, \ldots \boldsymbol{\xi_M}\}$ to be learnt.

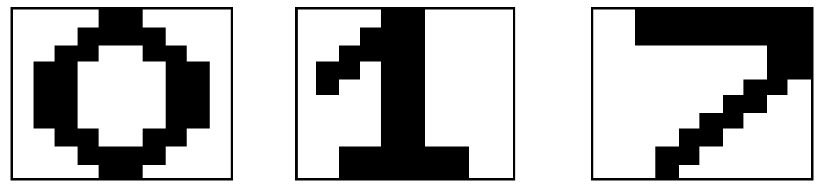
$$\boldsymbol{\xi_k} = [\xi_{k,1}, \xi_{k,2}, \ldots \xi_{k,n}], \quad k=1,.., M$$

Outer product rule (Hebbian-like learning) is used to compute **W**:

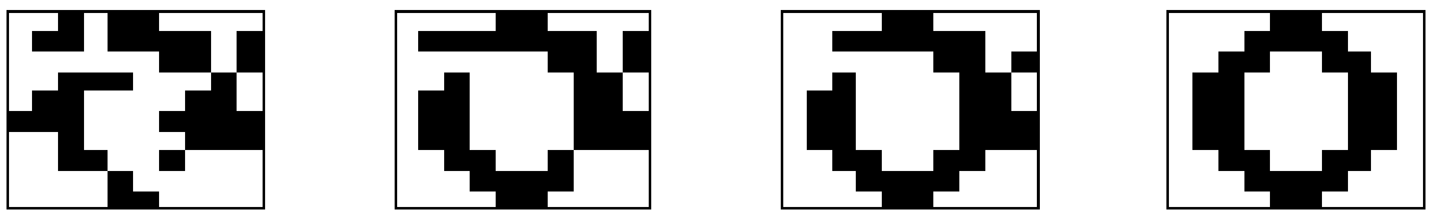
$$w_{j,i} = \begin{cases} \dfrac{1}{n}\displaystyle\sum_{k=1}^{M} \xi_{k,j} \cdot \xi_{k,i}, & j \neq i \\ 0, & j = i \end{cases}$$

Pattern storage and recall example

- The following patterns ξ^1, ξ^2, ξ^3 were stored in the weight matrix W:

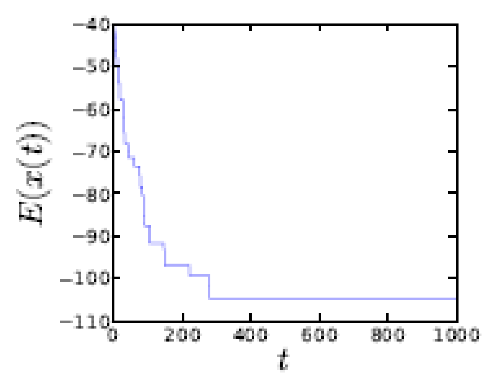


- Four snapshots of the state evolution $x(t)$:



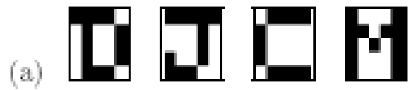
$t = 0$ $t = 50$ $t = 100$ $t = 300$

- Evolution of the energy $E(x(t))$:



adapted from L. Busing (TU Graz)

Pattern storage and recall example



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0	4	.	4	0	-2	-2	-2	-2	-2	-2	0	-4	0	-2	0	0	-2	0	-2	-2	4	4	2	-2
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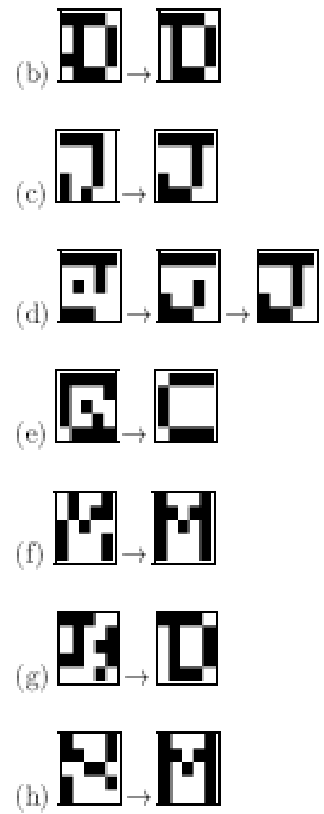
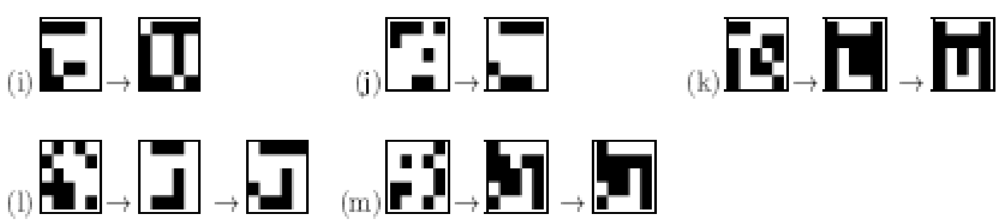
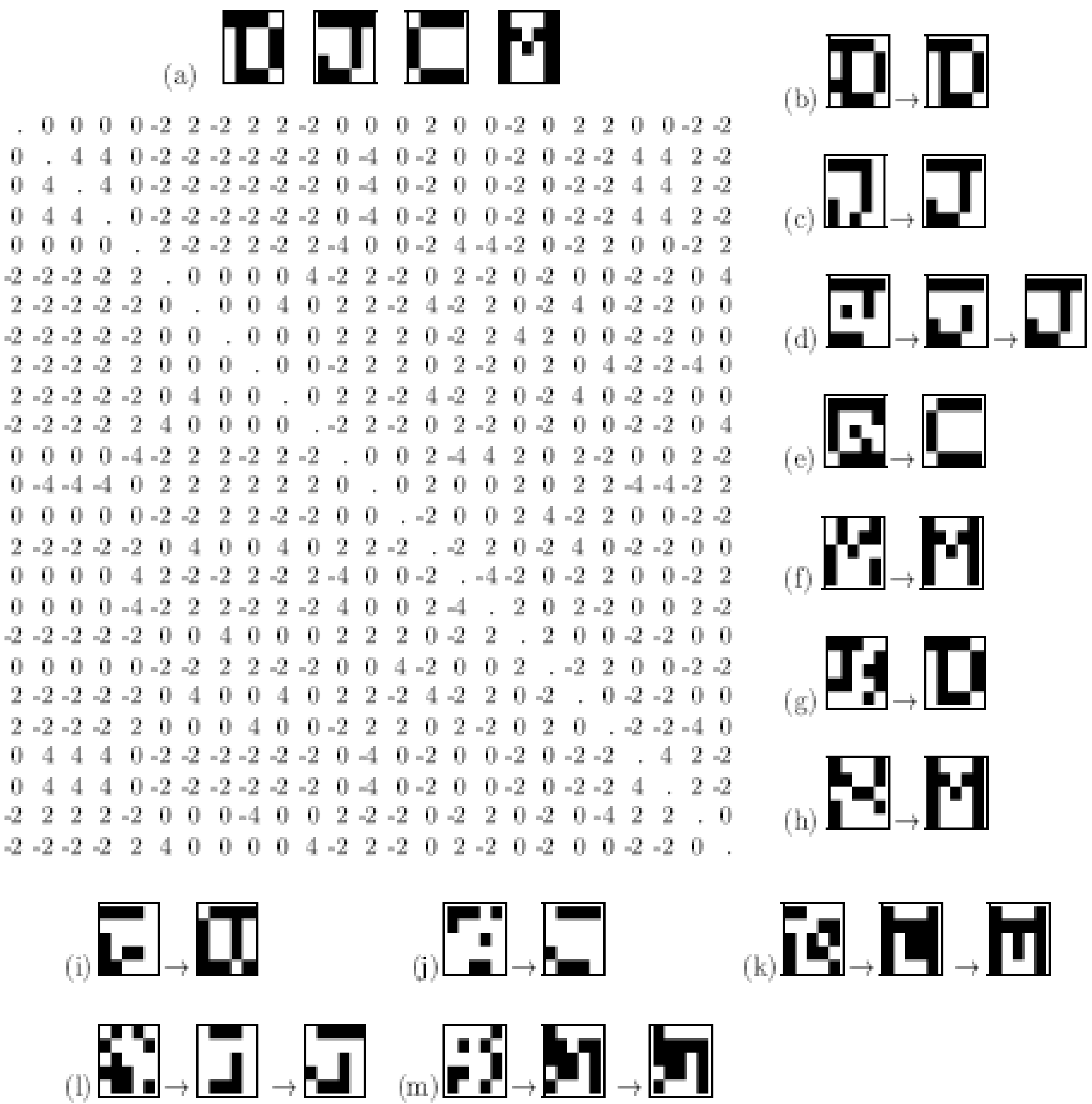




adapted from McKay

Pattern storage and recall example



Common problems

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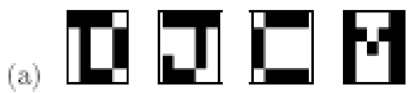

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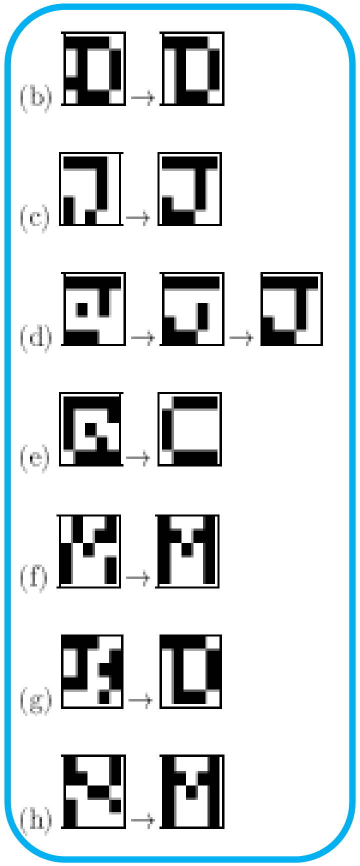

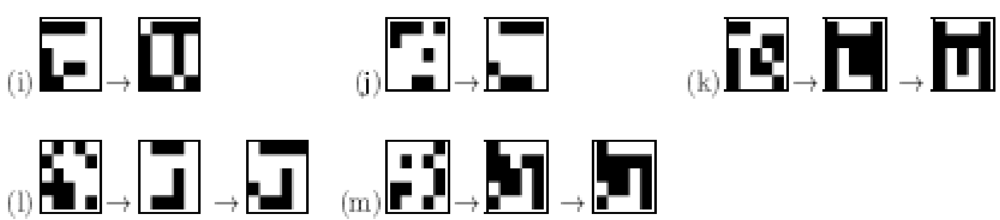

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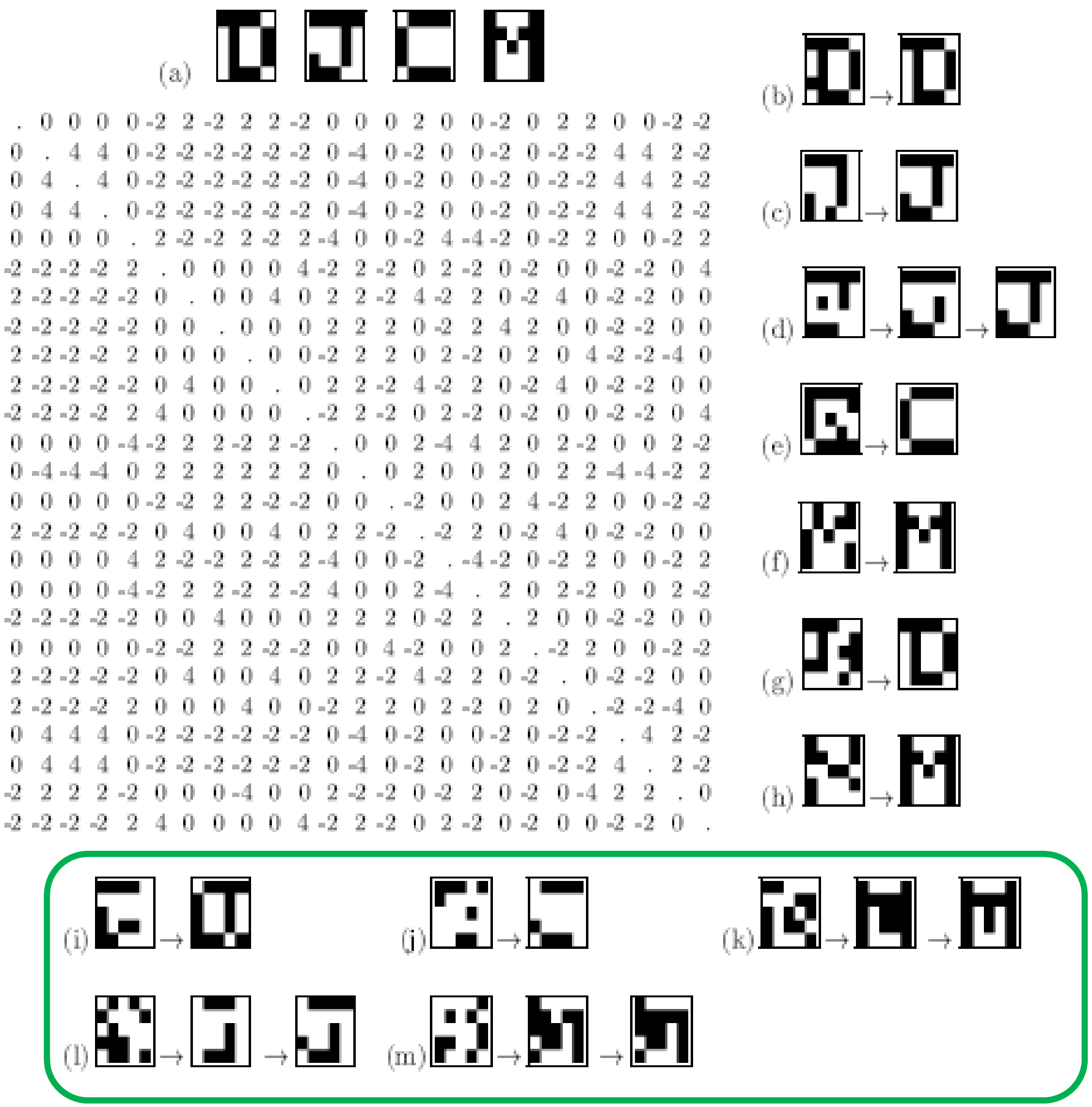
adapted from McKay

Pattern storage and recall example



Common problems

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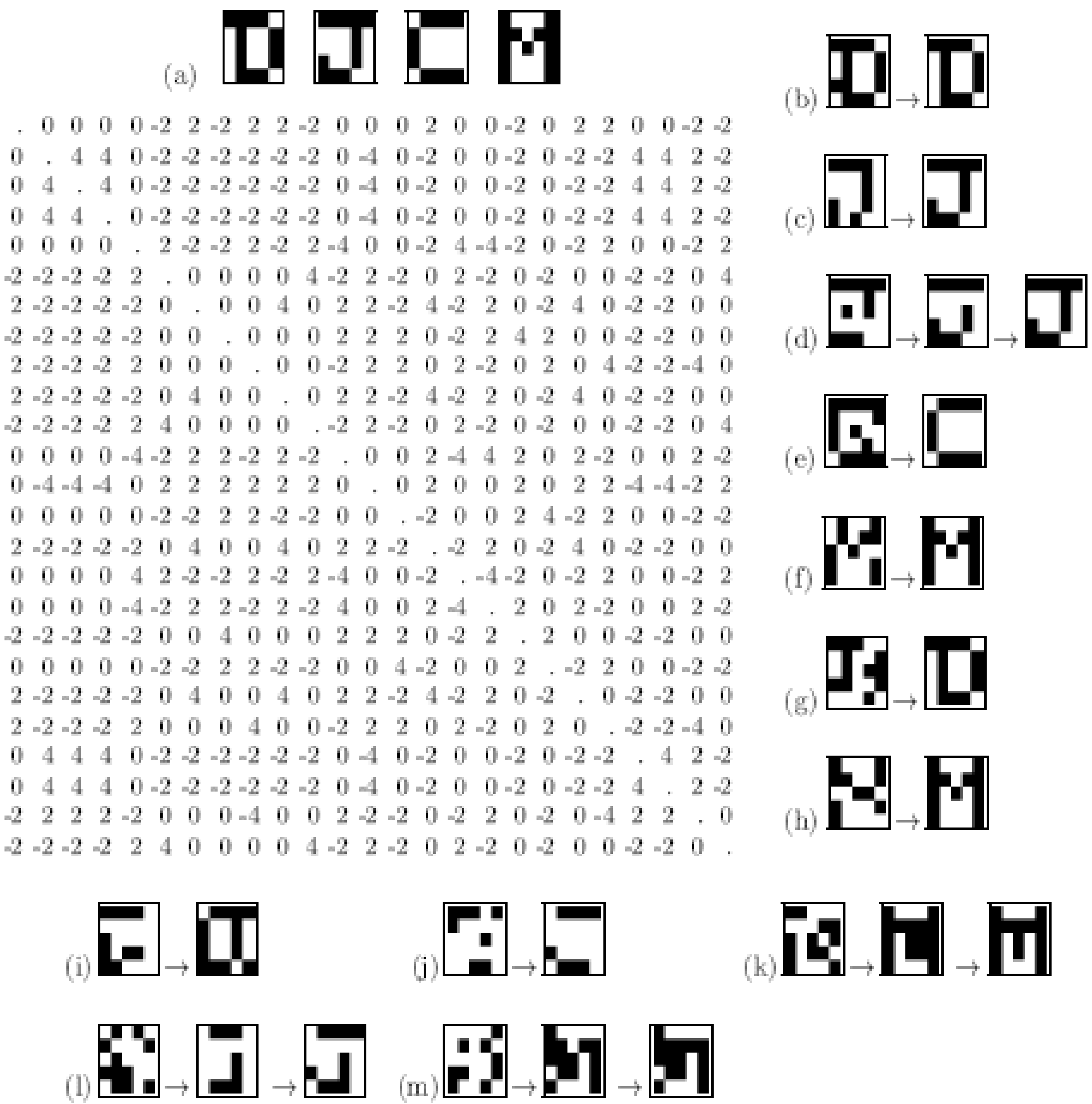
Spurious states often arise out of degenerate eigenvectors.

adapted from McKay

Pattern storage and recall example



Common problems

1. Corruption of individual bits.
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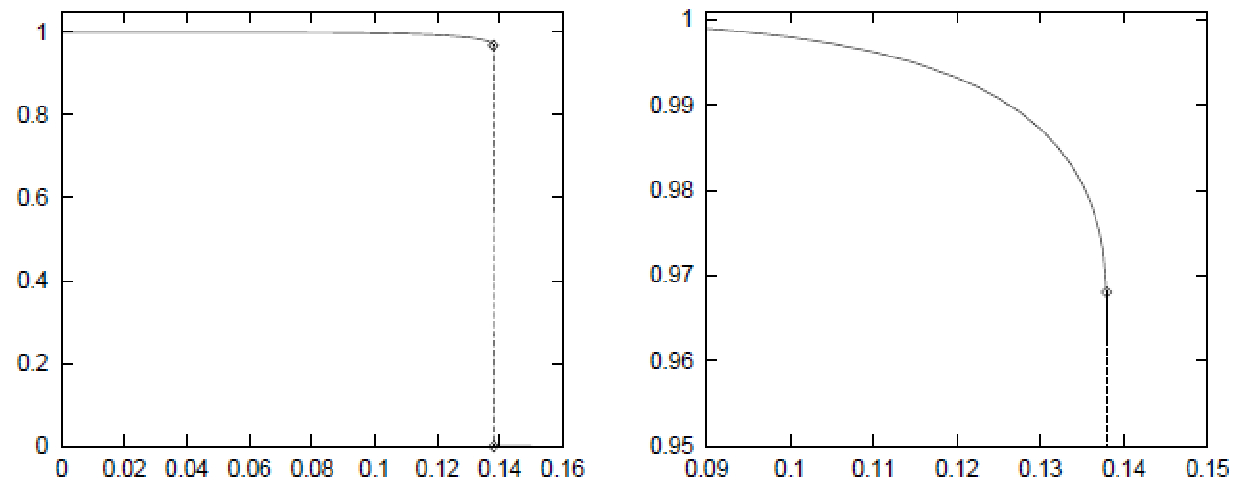
Generally, Hopfield network is robust to noise, data corruption and “brain damage” (zeroed subset of weights).

adapted from McKay

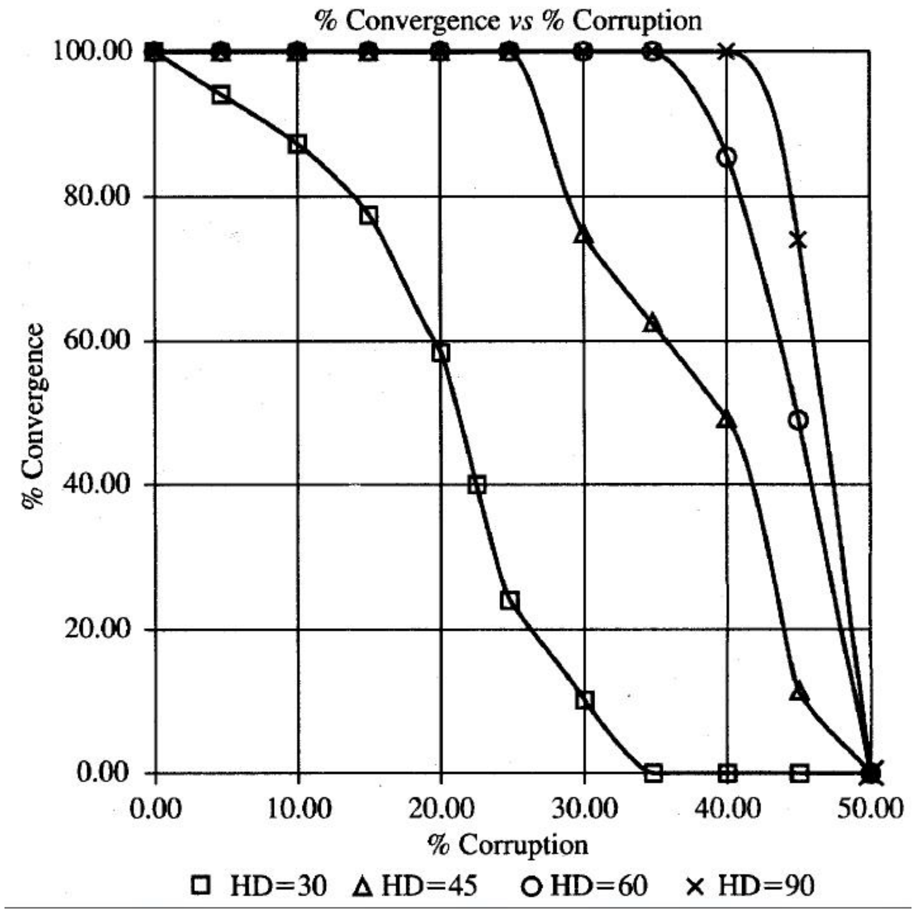
Memory capacity

- Cross-talk between memory patterns is key to limited capacity
- Memory capacity is usually tested on independent random patterns
  - Hopfield network can store roughly *M*<=0.138 *n* of such random patterns (sharp discontinuity)
  - for large *M*/*n*, unstable bits may unfold into an avalanche effect
  - for sparse patterns in the order of n*log(n)
- To guarantee stability of all patterns with high probability, we must ensure

$$M \leq \frac{n}{4 \ln n}$$



Catastrophic forgetting effect



Convergence rate is defined based on the convergence criterion, often expressed as the upper bound on *Hamming distance*.

Network properties are not robust for synchronous updates.

Also, problems for continuous networks.

$$a_i = \sum_j w_{ij} x_j \qquad x_i = \tanh(a_i).$$

Better behaviour for continuous continuous –time Hopfield network

$$a_i(t) = \sum_j w_{ij} x_j(t). \qquad \frac{\mathrm{d}}{\mathrm{d}t} x_i(t) = -\frac{1}{\tau}(x_i(t) - f(a_i)),$$

Hopfield networks for optimisation problems

- Hopfield network's dynamics minimises an energy function
- Some optimisation problems could be mapped to the quadratic energy function (particularly constrain satisfaction problems(CSPs))

Hopfield networks for optimisation problems

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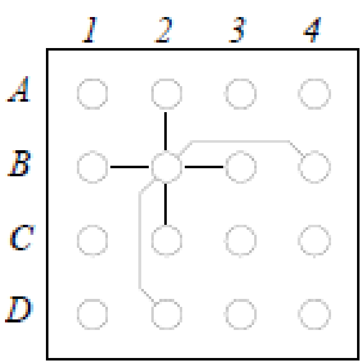
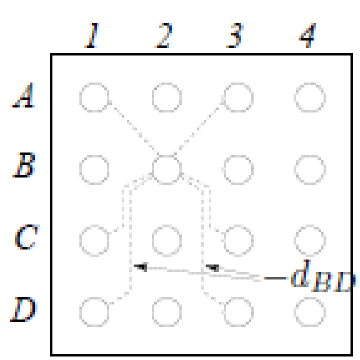
- Travelling salesman problem (TSP) as a classic CSP problem



validity

distances



$$E = \underbrace{\frac{1}{2}\sum_{i,j,k}^{n} d_{ij} x_{ik} x_{j,k+1}}_{\text{sum of distances}} + \underbrace{\frac{\gamma}{2}\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{n} x_{ij} - 1\right)^2 + \sum_{i=1}^{n}\left(\sum_{j=1}^{n} x_{ij} - 1\right)^2\right)}_{\text{validity: single 1s in each column and row}}$$

Hopfield networks for optimisation problems

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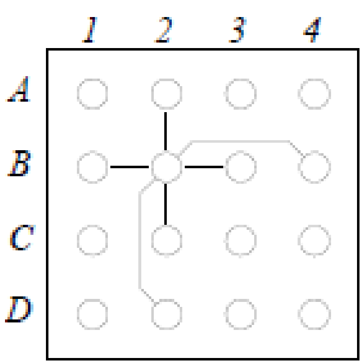

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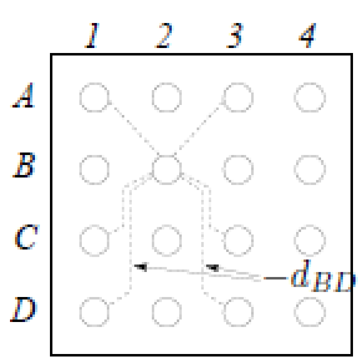

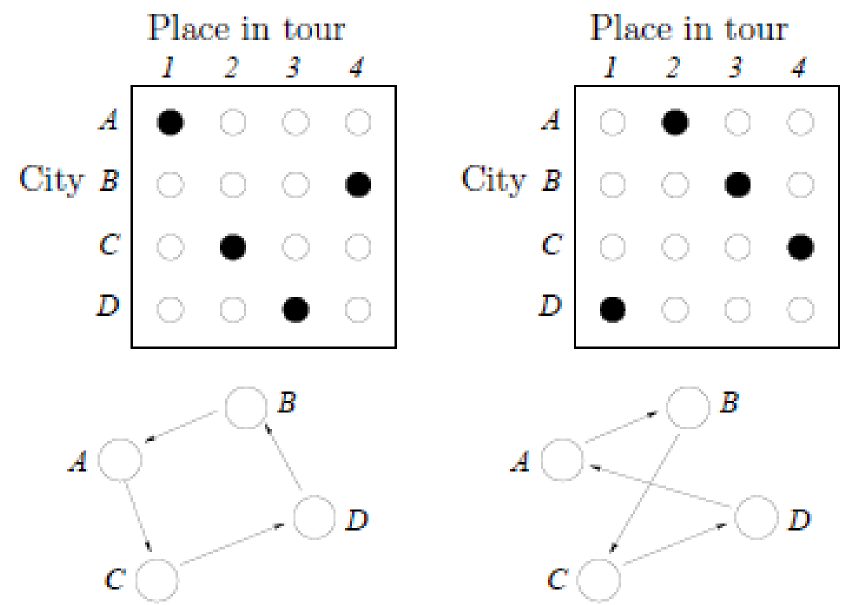

Hopfield networks

In summary

- Hopfield network is a nice model for memory with biological features including Hebbian learning
- It is a very simple, stable and mathematically tractable model
- It has limited capacity and assumes near orthogonal patterns
- It does not allow for storing time series
- The attractor dynamics is limited to fixed points