

An Analytical Framework for Evaluating the Error Characteristics of Approximate Adders

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Abstract—Approximate adders have been considered as a potential alternative for error-tolerant applications to trade off some accuracy for gains in other circuit-based metrics, such as power, area and delay. Existing approximate adder designs have shown substantial advantages in improving many of these operational features. However, the error characteristics of the approximate adders still remain an issue that is not very well understood. A simulation-based method requires both programming efforts and a time-consuming execution for evaluating the effect of errors. This method becomes particularly expensive when dealing with various sizes and types of approximate adders. In this paper, a framework based on analytical models is proposed for evaluating the error characteristics of approximate adders. Error features such as the error rate and the mean error distance are obtained using this framework without developing functional models of the approximate adders for time-consuming simulation. As an example, the estimate of peak signal-to-noise ratios (PSNRs) in image processing is considered to show the potential application of the proposed analysis. This analytical framework provides an efficient method to evaluate various designs of approximate adders for meeting different figures of merit in error-tolerant applications.

Index Terms—Approximate computing, approximate adder, PSNR estimate, mean error distance, image processing

1 INTRODUCTION

APPROXIMATE computing has become a promising technique to reduce the power, area and delay constraints in VLSI design, albeit at the expense of a loss in computational accuracy [1]. This technique is applicable to error-tolerant applications such as multimedia, mining and recognition [2]. Generally, there are two methodologies for reducing accuracy by approximation. The first methodology uses a voltage-over-scaling (VOS) technique for CMOS circuits to save power, while also introducing errors into the circuit [3], [4], [5]. The second methodology is based on redesigning a logic circuit into an approximate version. While the VOS technique is applicable to most circuits for error-tolerant applications, an approximate redesign requires to consider the different functionalities of logic circuits. As one of the simplest, but key components of arithmetic circuits, adders have attracted an extensive interest for redesigning and implementing approximate schemes. Approximate adders have been proposed by using a reduced number of transistors [6], [7] and by truncating the carry propagation chain for a speculation-based operation [8], [9], [10], [11], [12].

An approximate speculative designs achieve a better performance in terms of area, power and delay compared to conventional (exact) adders. New metrics and simulation-based approaches have been proposed to model and

evaluate approximate adders according to specific computational features [2], [13], [14], [15]. Monte Carlo or exhaustive simulation approaches have been employed to acquire data for analysis. This class of approaches are however time-consuming and require building functional models of the approximate designs. To improve efficiency, a mathematical characterization of the arithmetic accuracy of approximate adders is then required for a better understanding of the design prior to a simulation-based evaluation.

In addition to generic metrics (such as the error rate (ER)), application specific measures (ASMs) such as the peak signal-to-noise ratio (PSNR) for image processing are well suited in practice. Without an approach to modeling the relationship between the generic metrics and the ASMs, extensive programming and simulation efforts are required to obtain the ASMs for assessing the impact and the potential of approximate computing in different applications. Therefore, an effective approach to obtain or estimate the ASMs from generic error metrics is needed; however, there are no formal methodologies or analytical approaches for these purposes in the technical literature.

In this paper, an analytical framework is proposed to assess the arithmetic accuracy, i.e. the error rate and mean error distance (MED), of approximate adders. Three types of approximate adders are considered and their error features are compared using the proposed analysis. The revealed error characteristics provide insights into the quality of an appropriate adder for achieving a desired operational accuracy. As an example of ASMs, the PSNR in image processing is considered. A model is presented for estimating the PSNR from the MED obtained from the proposed framework; experimental results show that the estimated PSNRs are very close to the PSNRs obtained by simulation. The utilization of the proposed framework to PSNR estimate provides an analytical approach for assessing and

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Manuscript received 26 Oct. 2013; revised 7 Mar. 2014; accepted 16 Mar. 2014. Date of publication 13 Apr. 2014; date of current version 8 Apr. 2015.

Recommended for acceptance by P. A. Montuschi.

For information on obtaining reprints of this article, please send e-mail to: reprints@ieee.org, and reference the Digital Object Identifier below.

Digital Object Identifier no. 10.1109/TC.2014.2317180

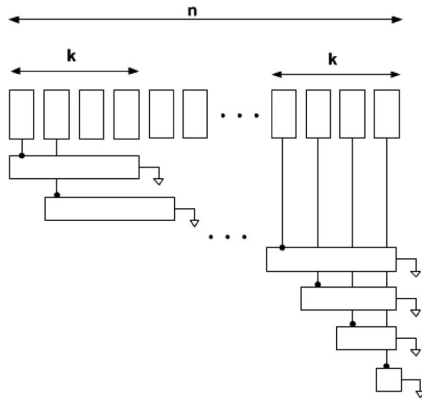


Fig. 1. The almost correct adder. n : the adder size; k : the maximum carry chain length.

designing a feasible approximate image processing system based on approximate adders.

The major contributions of this paper are as follows:

- An analytical framework is developed for modeling and evaluating the error characteristics of three types of approximate adders found in the technical literature.
- A comparative study is performed for various approximate adders using different carry speculation schemes to gain an insight into the qualitative features of a design with respect to several error metrics.
- An analytical approach is developed to model the relationship between the PSNR and the MED obtained from the proposed framework in approximate adder-based image processing. This approach can effectively estimate the PSNR from the approximate adders used in an image precessing application.

The organization of this paper is as follows. Section 2 reviews the approximate designs applicable to the framework proposed in this paper. Section 3 describes the analysis for modeling the error characteristics of the approximate adders. Discussion follows in Section 4. Section 5 presents the comparison of approximate adder designs using the proposed framework; as an application, the PSNR estimate in image processing is investigated. Conclusion is given in Section 6.

2 REVIEW OF EXISTING APPROXIMATE ADDERS

2.1 The Speculative and Almost Correct Adder (ACA)

The so-called almost correct adder [9] is based on the speculative adder design in [8]. The ACA utilizes insufficient information, i.e., k LSBs for predicting the sum of each bit in an n -bit adder ($n > k$). The same illustration (Fig. 1) as in [15] is used for the ACA (and the ESA in the following section). In Fig. 1, 4 bits (i.e., $k = 4$) are used to calculate each bit in the sum of an n -bit adder. The identical vertical rectangular blocks on the top denote the inputs, while the horizontal rectangles under them show the carry propagation paths for each sum bit. This design is based on the observation that the carry propagation chain is usually shorter than n , i.e. in practice, the truncation of the chain up to some length has a very low probability to be erroneous.

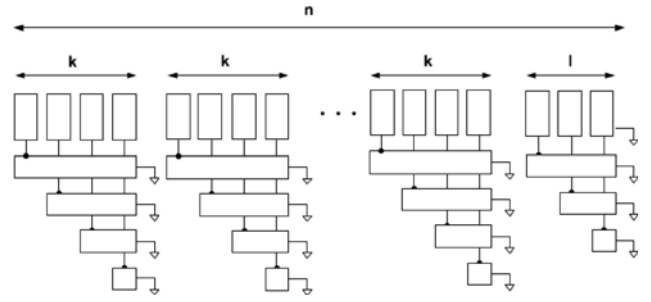


Fig. 2. The equal segmentation adder. n : the adder size; k : the maximum carry chain length; l : size of the first sub-adder ($l \leq k$).

2.2 The Equal Segmentation Adder (ESA)

A dynamic segmentation and error compensation (DSEC) scheme is presented in [5] for an approximate adder design. This approximate adder consists of several sub-adders of different sizes divided from an n -bit adder; each of the sub-adders operates in parallel and has a truncated carry input. For convenience, but with no loss in correctness, sub-adders of equal size are considered in this manuscript. Moreover the error compensation part [5] is neglected because the focus of this manuscript is on analyzing the approximate operation. Thus, a simplified DSEC adder referred to as an equal segmentation adder is analyzed in this paper (Fig. 2).

2.3 The Error-Tolerant Adder Type II (ETAIL), the Speculative Carry Select Adder (SCSA) and the Accuracy-Configurable Approximate Adder

The ETAIL is also based on the truncation of the carry propagation chain and the segmentation of a full-sized adder [12]. Compared to the ESA, the predicted carry input for each segmented k -bit sub-adder (or the sum generator in Fig. 3) is generated by k LSBs. The ETAIL has an improved accuracy compared to the ESA, because it uses more information to predict the carry when the same k is used.

In the so-called speculative carry select addition (SCSA) [10], an n -bit adder is first divided into $\lceil \frac{n}{k} \rceil$ sub-adders (also known as “window adders”); each sub-adder consists of two k -bit adders: adder0 and adder1 (Fig. 4). The only difference between the two k -bit adders is the carry input; the carry of adder0 is “0” while it is “1” for adder1. The output of the i th sub-adder is selected from adder0 and adder1 based on the carry out signal generated by the $(i - 1)$ th sub-adder. The carry out of each sub-adder is

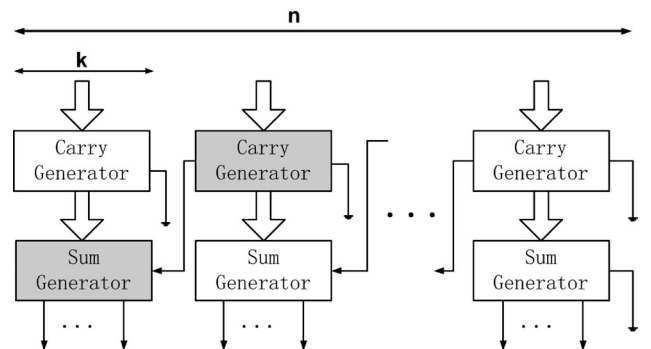


Fig. 3. Block diagram of error-tolerant adder type II. n : the adder size; k : half of the maximum carry chain length.

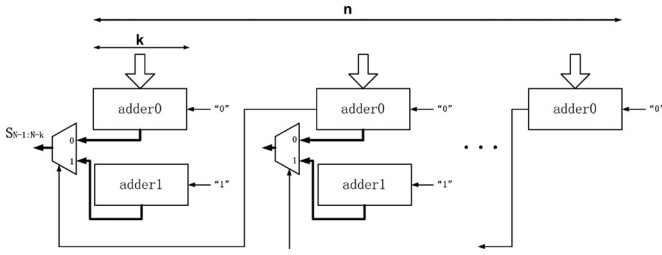


Fig. 4. The speculative carry selection adder. n : the adder size; k : the maximum carry chain length.

generated based on the k -bit in the sub-adder rather than all previous bits. Therefore, the carry selection process is still approximate and faster than a traditional carry selection scheme. Even though the SCSA and the ETAIL have different circuit implementations, they share a similar functionality if their sub-adders have the same length. The SCSA and the ETAIL generate the same carry signal for each sub-adder (or the Sum Generator in the ETAIL) even though by different circuit implementations.

The accuracy-configurable approximate adder proposed in [11] can adjust the accuracy during runtime. For a given accuracy, the approximate configuration of the adder performs a similar function as the ETAIL.

For establishing the error characteristics, adders with the same functionality are considered to be the same type. For example, an ETAIL and a SCSA with the same k and n values generate the same output for the same inputs. Thus, they have the same error characteristics. However, characteristics related to a circuit implementation such as delay and power are not necessarily the same. Some of the approximate adders also have an error correction circuit that permits an additional accurate operation mode; only the approximate operation of each adder is considered in this paper.

3 ERROR ANALYSIS

3.1 Preliminaries

3.1.1 Metrics

The *error distance* (ED) and the *mean error distance* are proposed in [13] to evaluate the arithmetic performance of approximate circuits. For an approximate adder, ED is defined as the absolute value of the difference between the accurate and approximate sums, i.e.,

$$ED = |S' - S|, \quad (1)$$

where S' is the sum of the approximate adder and S is the sum of an accurate adder. MED is defined as the average ED for a given set of input vectors, i.e.,

$$MED = E[ED] = \sum_i ED_i P(ED_i), \quad (2)$$

where $P(ED_i)$ is the probability of ED_i . The *error rate* is defined as the percentage of erroneous outputs among all outputs [16], i.e.,

$$ER = \sum_i P(ED_i), \text{ for any } ED_i \neq 0. \quad (3)$$

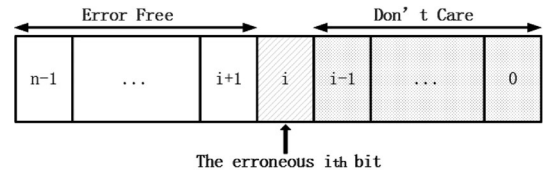


Fig. 5. Sub error set Π_i in the ACA.

The above metrics (ER and MED) are of interest for evaluating the arithmetic performance of approximate adders. In the following section, an analytical method is presented to calculate these metrics for different types of adders.

3.1.2 Notation

The notation used in the error analysis throughout this paper is introduced next. We consider an n -bit approximate adder with inputs A, B and C_0 , and an output S . A_i, B_i, S_i are the corresponding input and output bits at the i th position. C_i is the carry to be added to the i th bit. Let $p_i = P(C_i = 1) = P(A_{i-1}B_{i-1} = 1) + P(A_{i-1} \oplus B_{i-1} = 1, C_{i-1} = 1)$, for uniformly-distributed inputs, $p_i = \frac{1}{4} + \frac{1}{2}p_{i-1}$, which leads to

$$p_i = \frac{1}{2} + \frac{1}{2^i} \left(p_0 - \frac{1}{2} \right), \quad (4)$$

where p_0 is the probability that the initial carry bit is 1. Assume $p_0 = 0$, then

$$p_i = \frac{1}{2} \left(1 - \frac{1}{2^i} \right). \quad (5)$$

Let \bar{X}_i and \tilde{X}_i denote the events that the i th approximate sum bit is the same as or different from the i th exact sum bit, respectively, i.e., $\bar{X}_i = \{S_i = S'_i\}$, $\tilde{X}_i = \{S_i \neq S'_i\}$. An \mathbf{X} vector consisting of \bar{X}_i 's or \tilde{X}_i 's is used to denote a set of outputs of the approximate adder compared to the accurate one. For example, for a 4-bit approximate adder, $\{\bar{X}_3 \bar{X}_2 \tilde{X}_1\}$ denotes a set of outputs in which S_1 is incorrect, both S_2 and S_3 are correct and S_0 could be either correct or incorrect.

3.2 ACA Error Characteristics

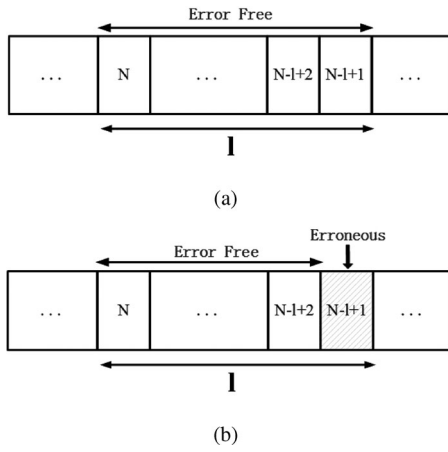
A *universal error set* is said to be formed by all possible error patterns of the ACA. To calculate the MED of an n -bit ACA, this error set is divided into n disjoint subsets and then the MED is calculated for each subset. The universal error set, denoted by Π , is divided into the subsets of $\Pi_i (i = 0, 1, \dots, n-1)$, i.e.,

$$\Pi = \cup \Pi_i, \quad (6)$$

where $\Pi_i = \{\bar{X}_{n-1}, \bar{X}_{n-2}, \dots, \tilde{X}_i\}$. The error patterns in Π_i are those whose i th bit is erroneous, while the more significant bits are correct and the less significant bits (LSBs) are "don't cares", i.e., they can be either correct or erroneous, as shown in Fig. 5. Based on the error set division, the mean error distance of the approximate adder is calculated as:

$$MED = \sum_i E[|e_i|], \quad (7)$$

where $E[|e_i|]$ is the mean error distance of the subset Π_i .

Fig. 6. (a) \bar{P}_l , (b) \tilde{P}_l .

In the subset Π_i , the errors in the i th bit (i.e. $\pm 2^i$) are dominant. Moreover, some of the errors in the lower bits can cancel each other. Therefore, the errors in Π_i have on average a magnitude of approximately 2^i ; so the MED of Π_i is calculated as

$$E[|e_i|] \approx 2^i q_i, \quad (8)$$

where q_i is the probability that the error patterns fall in Π_i . The total mean error distance is then given by

$$MED = \sum_i 2^i q_i. \quad (9)$$

The error rate of the ACA is given by

$$ER = \sum_i q_i. \quad (10)$$

Next the calculation of q_i is presented. Let \bar{P}_l be the probability that l consecutive bits in the approximate sum are correct (Fig. 6a) and \tilde{P}_l be the probability that $l-1$ consecutive bits in the approximate sum are correct, but the next lower bit is erroneous (Fig. 6b), i.e.,

$$\bar{P}_l = P(\bar{X}_N, \bar{X}_{N-1}, \dots, \bar{X}_{N-l+2}, \bar{X}_{N-l+1}), \quad (11)$$

$$\tilde{P}_l = P(\bar{X}_N, \bar{X}_{N-1}, \dots, \bar{X}_{N-l+2}, \tilde{X}_{N-l+1}). \quad (12)$$

Furthermore, let \bar{Q}_l denote the conditional probability that one approximate sum bit is correct given that $l-1$ consecutive lower bits in the approximate sum are correct, and \tilde{Q}_l denote the conditional probability that one approximate sum bit is correct given that $l-2$ consecutive lower bits in the approximate sum are correct but the next lower bit is erroneous, i.e.,

$$\bar{Q}_l = P(\bar{X}_N | \bar{X}_{N-1}, \dots, \bar{X}_{N-l+2}, \bar{X}_{N-l+1}), \quad (13)$$

$$\tilde{Q}_l = P(\bar{X}_N | \bar{X}_{N-1}, \dots, \bar{X}_{N-l+2}, \tilde{X}_{N-l+1}). \quad (14)$$

Note that in these probabilities, only the length of the pattern is important, i.e., the exact index of each pattern, N , is less important.

Let the truncated carry propagation length be k ; $\tilde{P}_l (l \leq k)$ is calculated first. Further denote the accurate sum as S and the inaccurate sum as S' . As S'_{N-l+1} is the most significant erroneous bit, i.e., all higher bits in the sum are correct, the input carry, $C'_{N-l+1-k}$ to calculate S'_{N-l+1} , must propagate all the way to S'_{N-l+1} , i.e., $P_i = A_i \oplus B_i = 1$, for $i = N-l-k+2, N-l-k+3, \dots, N-l$. Also, $P_{N-l+1} = 0$, because, otherwise, the wrongly estimated carry would propagate to S'_{N-l+2} . As $P_{N-l+1} = 0$, it is sufficient to show that $S'_{N-l+2}, S'_{N-l+3}, \dots, S'_N$ are correct, because $P_{N-l+1} = 0$ prevents the error from propagating to the higher $k-1$ bits. Therefore,

$$\begin{aligned} \tilde{P}_l &= P(P_{N-l-k+2}, \dots, P_{N-l} = 1, P_{N-l+1} = 0, \\ &C'_{N-l-k+2} \neq C_{N-l-k+2}) = \frac{1}{2^{k+1}}, \quad l \geq 2. \end{aligned} \quad (15)$$

Let $\tilde{P}_1 = P(\tilde{X}_N)$, then

$$\begin{aligned} \tilde{P}_1 &= P(P_{N-k+1}, \dots, P_{N-2} = 1, P_{N-1} = 0, \\ &C'_{N-k+1} \neq C_{N-k+1}) = \frac{1}{2^k}. \end{aligned} \quad (16)$$

Thus,

$$\tilde{P}_l = \begin{cases} \frac{1}{2^{k+1}}, & l \geq 2, \\ \frac{1}{2^k}, & l = 1. \end{cases} \quad (17)$$

Since $\bar{P}_1 = \tilde{P}_2 + \bar{P}_2 = \tilde{P}_2 + \tilde{P}_3 + \bar{P}_3 = \dots = \tilde{P}_2 + \tilde{P}_3 + \dots + \tilde{P}_l + \bar{P}_l$, then

$$\bar{P}_l = \bar{P}_1 - \sum_{i=2}^l \tilde{P}_i, \quad (18)$$

where $\bar{P}_1 = 1 - \tilde{P}_1$. For $l \leq k$,

$$\tilde{Q}_l = \frac{\tilde{P}_l}{\bar{P}_{l-1}}, \quad l \leq k. \quad (19)$$

For $l > k$, since only k bits are used to calculate the current sum, the bits that are less significant than these k bits, have no influence on \tilde{Q}_l . Thus,

$$\tilde{Q}_l = \bar{Q}_k = \frac{\bar{P}_k}{\bar{P}_{k-1}}, \quad l > k. \quad (20)$$

From all of the above, we obtain

$$\tilde{Q}_l = \begin{cases} \frac{\tilde{P}_l}{\bar{P}_{l-1}} = 1, & k \geq l > 2, \\ \frac{\tilde{P}_2}{\bar{P}_1} = \frac{1}{2}, & l = 2, \\ \frac{\bar{P}_k}{\bar{P}_{k-1}} = \frac{2^{k+1} - k - 1}{2^{k+1} - k}, & l > k. \end{cases} \quad (21)$$

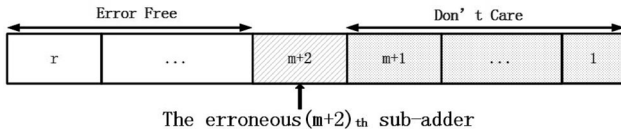


Fig. 7. An example of the error in ESA dominated by the $(m+2)$ th sub-adder.

This leads to

$$\begin{aligned}
 q_i &\approx P(\bar{X}_N, \bar{X}_{N-1}, \dots, \bar{X}_{i+1}, \tilde{X}_i) \\
 &= P(\bar{X}_N | \bar{X}_{N-1}, \dots, \bar{X}_{i+1}, \tilde{X}_i) P(\bar{X}_{N-1}, \dots, \bar{X}_{i+1}, \tilde{X}_i) \\
 &= P(\bar{X}_N | \bar{X}_{N-1}, \dots, \bar{X}_{i+1}, \tilde{X}_i) P(\bar{X}_{N-1} | \bar{X}_{N-2}, \dots, \bar{X}_{i+1}, \tilde{X}_i) \\
 &\quad P(\bar{X}_{N-2}, \dots, \bar{X}_{i+1}, \tilde{X}_i) \\
 &= P(\bar{X}_N | \bar{X}_{N-1}, \dots, \bar{X}_{i+1}, \tilde{X}_i) P(\bar{X}_{N-1} | \bar{X}_{N-2}, \dots, \bar{X}_{i+1}, \tilde{X}_i) \\
 &\quad \dots P(\bar{X}_{i+1} | \tilde{X}_i) P(\tilde{X}_i) \\
 &= \tilde{Q}_{N-i+1} \tilde{Q}_{N-i} \dots \tilde{Q}_2 \tilde{P}_1
 \end{aligned}$$

Hence, (17), (21) and (22) can be used to calculate q_i in the ACA error analysis.

3.3 ESA Error Characteristics

Consider an n -bit ESA divided into r ($r = \lceil \frac{n}{k} \rceil - 1$) sub-adders of equal size k and one sub-adder of size $l = n - kr$. Thus there are $(r+1)$ sub-adders in total. Since the lowest sub-adder (i.e. the first sub-adder) is always error-free, only the higher r sub-adders that can be erroneous are considered. Initially the error in the $(m+2)$ th sub-adder ($m = 0, 1, \dots, r-1$) is considered; this is always 2^{mk+l} , as introduced by the wrong estimate of the input carry to this sub-adder.

When the speculative carry into each sub-adder is truncated to 0, the error rate of the $(m+2)$ th sub-adder is given by

$$\begin{aligned}
 P(\text{error} = -2^{mk+l}) &= P(C'_{mk+l} < C_{mk+l}) \\
 &= p_{mk+l} \approx \frac{1}{2}.
 \end{aligned} \tag{23}$$

The error rate of each sub-adder (except for the first one) is approximately $\frac{1}{2}$. All sub-adders are independent, because there is no connection between them; so, the error rate of the entire adder is

$$ER = 1 - \left(1 - \frac{1}{2}\right)^r = 1 - \left(\frac{1}{2}\right)^r. \tag{24}$$

An approximate method is then introduced to calculate the mean error distance of the ESA. Since the errors in a lower sub-adder are significantly smaller than those in a higher sub-adder, the error magnitude of the approximate adder is dominated by the highest erroneous sub-adder. For example, for 8-bit sub-adders, the error in the third sub-adder is 256 times greater than the error in the second sub-adder. Therefore, if a sub-adder is erroneous, the errors in the lower sub-adders become insignificant and thus, they can be ignored. Hence, for an ESA with $(r+1)$ sub-adders, the error magnitudes can be approximately divided into r levels, i.e. $\{2^l, 2^{k+l}, \dots, 2^{(r-1)k+l}\}$. Fig. 7 shows a case when

the error is dominated by the $(m+2)$ th sub-adder, and the error probability is given by

$$P(\text{error} = -2^{mk+l}) \approx \left(\frac{1}{2}\right)^{r-m-1} \times \frac{1}{2}. \tag{25}$$

The mean error distance is therefore given by

$$\begin{aligned}
 MED &= \sum_{m=0}^{r-1} 2^{mk+l} \left(\frac{1}{2}\right)^{r-m-1} \times \frac{1}{2} \\
 &= \frac{2^{kr+r} - 1}{2^{k+1} - 1} \times 2^{l-r} \approx 2^{n-k-1}.
 \end{aligned} \tag{26}$$

3.4 ETAIL Error Characteristics

Similar to the analysis of the ESA, the ETAIL uses the same partition scheme and its error analysis starts with the evaluation of the error rate in the $(m+2)$ th sub-adder. The $(m+1)$ th sub-adder generates the approximate carry, C'_{mk+l} , to the $(m+2)$ th sub-adder based on the assumption that the input carry to itself, $C'_{(m-1)k+l}$ is 0. Thus, when the exact carry $C_{(m-1)k+l}$ is 1 and propagates through the $(m+1)$ th sub-adder, the carry generated by the $(m+1)$ th sub-adder is erroneous; thus, this results in an error in the $(m+2)$ th sub-adder. Hence, the error rate of the $(m+2)$ th sub-adder is given by

$$\begin{aligned}
 P(\text{error} = -2^{mk+l}) &= P(C'_{mk+l} < C_{mk+l}) \\
 &= \left(\frac{1}{2}\right)^k p_{mk+l} = \left(\frac{1}{2}\right)^{k+1} \left(1 - \frac{1}{2^{mk+l}}\right) \approx \frac{1}{2^{k+1}}.
 \end{aligned} \tag{27}$$

In the ETAIL, the first and second sub-adders are always error-free, so there are totally $(r-1)$ sub-adders that can be erroneous. Hence, the ER of the ETAIL is given by

$$ER \approx 1 - \left(1 - \frac{1}{2^{k+1}}\right)^{r-1}. \tag{28}$$

Similarly as for the ESA, the error in the ETAIL is approximately divided among the $(r-1)$ levels. Hence

$$P(\text{error} = -2^{mk+l}) = \left(1 - \frac{1}{2^{k+1}}\right)^{r-m-1} \times \frac{1}{2^{k+1}}, \tag{29}$$

and

$$\begin{aligned}
 MED &= \sum_{m=1}^{r-1} 2^{mk+l} \left(1 - \frac{1}{2^{k+1}}\right)^{r-m-1} \times \frac{1}{2^{k+1}} \\
 &= \frac{2^{kr-k+l} - 2^l \left(1 - \frac{1}{2^{k+1}}\right)^{r-1}}{2^{k+1} - 2 + 2^{-k}}.
 \end{aligned}$$

Equation (30) can be simplified by ignoring $-2 + 2^{-k}$ in the denominator and $2^l \left(1 - \frac{1}{2^{k+1}}\right)^{r-1}$ in the numerator, hence

$$MED \approx 2^{n-2k-1}. \tag{31}$$

3.5 Error Characteristics under Different Carry Estimation Methods

All approximate adders considered so far use a fixed carry (with a value of 0) estimate such that the carry propagation chain is truncated. For example, for $k = 10$ in the ACA, an assumed input carry $C'_1 = 0$ is used to calculate S'_{10} . This assumption may lead to under-estimated results and the average error is non-zero. A straightforward solution to avoid the non-zero average error encountered in a method using a fixed carry is to use a random carry. If the input operands are uniformly distributed, one of the less significant input bits can be used as the random carry signal [14]. For example, $C'_1 = A_0$ or $C'_1 = B_0$ can be used to estimate C_1 in the previous ACA example. Hereafter, these two methods are referred to as the *fixed carry estimate* and *1-LSB carry estimate*, respectively.

In the previous section, the error characteristics of approximate adders with a fixed carry estimate have been discussed. Next, the error characteristics under the 1-LSB carry estimate case are considered.

3.5.1 ACA

A detailed solution under the 1-LSB carry estimate case is not provided, because it can be obtained in a manner similar to the fixed carry case. When 1-LSB is used to estimate the carry, (17) and (21) are replaced by (32) and (33), respectively, while (22) remains the same.

$$\begin{aligned} \tilde{P}_l &= P(q_{N-l-k+2}, \dots, q_{N-l} = 1, q_{N-l+1} = 0, \\ C'_{N-l-k+2} \neq C_{N-l-k+2}) &= \frac{1}{3} \frac{1}{2^k} \left(1 + \frac{1}{2^{2l-1}} \right), \quad l \leq k \end{aligned} \quad (32)$$

$$\tilde{Q}_l = \begin{cases} \frac{\tilde{P}_l}{\tilde{P}_{l-1}} = \frac{2^{2l-1} + 1}{4(2^{2l-3} + 1)}, & k \geq l \geq 2, \\ \frac{\tilde{P}_k}{\tilde{P}_{k-1}} = \frac{19 \times 2^{3k-1} - (6k+4)2^{2k-2} - 1}{49 \times 2^{3k-3} - (6k-2)2^{2k-4} - 1}, & l > k. \end{cases} \quad (33)$$

After q_i is calculated by (22), (9) and (10) can still be used to obtain the MED and ER in the 1-LSB carry estimate case.

3.5.2 ESA

The error rate of each sub-adder in the 1-LSB carry estimate case is half of the error rate in the fixed carry case (as discussed later in this manuscript). Thus, the procedure in the previous section can be utilized by changing the error rate of each sub-adder at the beginning. If 1-LSB is used to estimate the truncated carry, (23) is changed to:

$$\begin{aligned} P(\text{error} = 2^{mk+l}) &= P(C'_{mk+l} > C_{mk+l}) \\ &= P(A_{mk+l-1} = 1, B_{mk+l-1} = C_{mk+l-1} = 0) \\ &= \frac{1}{4} (1 - p_{mk+l}) \approx \frac{1}{8}, \end{aligned} \quad (34)$$

and

$$P(\text{error} = -2^{mk+l}) = P(C'_{mk+l} < C_{mk+l}) \approx \frac{1}{8}. \quad (35)$$

The mean error distance is then given by

$$\begin{aligned} MED &= \sum_{m=0}^{r-1} 2^{mk+l} \left(\frac{3}{4} \right)^{r-m-1} \times \frac{1}{4} \\ &= \frac{2^{kr} - \left(\frac{3}{4} \right)^r}{2^{k+2} - 3} \times 2^l \approx 2^{n-k-2}. \end{aligned} \quad (36)$$

The ER is:

$$ER = 1 - \left(1 - \frac{1}{4} \right)^r = 1 - \left(\frac{3}{4} \right)^r. \quad (37)$$

3.5.3 ETAIL

If 1-LSB is used to estimate the truncated carry, the analysis of the ETAIL is very similar to the ESA, so the ER and MED for the ETAIL are given as follows:

$$ER \approx 1 - \left(1 - \frac{1}{2^{k+2}} \right)^{r-1}, \quad (38)$$

and

$$\begin{aligned} MED &= \sum_{m=1}^{r-1} 2^{mk+l} \left(1 - \frac{1}{2^{k+2}} \right)^{r-m-1} \times \frac{1}{2^{k+2}} \\ &\approx 2^{n-2k-2}. \end{aligned} \quad (39)$$

3.6 Monte Carlo Simulation

3.6.1 Error Analysis for Monte Carlo Simulation

Assume the accurate MED is μ and $\hat{\mu}_m$ is an estimate for MED obtained by averaging EDs from m iterations of Monte Carlo simulation. This can be modeled as a Monte Carlo integration approach [17]. The variance of $\hat{\mu}_m$ is

$$\text{var}(\hat{\mu}_m) = \frac{v}{m}, \quad (40)$$

where v is the variance of EDs given by

$$v = \sum_i (ED_i)^2 P(ED_i) - \mu^2. \quad (41)$$

For large m , by the Law of Large Numbers,

$$\hat{\mu}_m \sim N\left(\mu, \frac{v}{m}\right). \quad (42)$$

For a given confidence level, a parameter z_c can be determined to show the corresponding confidence interval. Therefore the error of $\hat{\mu}_m$ becomes

$$e = \frac{z_c}{\mu} \sqrt{\frac{v}{m}}. \quad (43)$$

For a confidence level of 95 percent, $z_c = 1.96$. Therefore, for $m = 1,000,000$, the error is

$$e = \frac{0.00196\sqrt{v}}{\mu} = 0.00196CV, \quad (44)$$

TABLE 1
CV (i.e., \sqrt{v}/μ) Values for ACA

	fixed carry estimate	1-LSB carry estimate
k=6	6.8	7.7
k=7	9.7	11.0
k=8	13.6	15.5
k=9	19.6	21.9
k=10	27.0	31.4
k=11	39.0	43.7
k=12	53.7	63.7

where $CV = \sqrt{v}/\mu$, the coefficient of variance.

Equation (44) will be used in the following to analyze the error for Monte Carlo simulations of three approximate designs with a confidence level of 95 percent.

Considering (25), (26) and (41), the variance of the EDs of ESA for a fixed carry estimate is given by

$$v = \sum_{m=0}^{r-1} (2^{mk+l})^2 \left(\frac{1}{2}\right)^{r-m-1} \times \frac{1}{2} - (2^{n-k-1})^2 \quad (45)$$

$$\approx 2^{2n-2k-2}.$$

By taking into consideration (26) and (45), the percentage error is 0.196 percent by (44), i.e., we are 95 percent confident that a simulated MED is within 0.196 percent of the true MED. Similarly, the variance of EDs of ESA for the 1-LSB carry estimate is

$$v \approx 3 \times 4^{n-k-2}. \quad (46)$$

The corresponding error is 0.34 percent.

For ETAIL in the fixed carry estimate case, based on (29), (31) and (41), the variance of EDs is

$$v = \sum_{m=1}^{r-1} (2^{mk+l})^2 \left(1 - \frac{1}{2^{k+1}}\right)^{r-m-1} \times \frac{1}{2^{k+1}} - (2^{n-2k-1})^2 \quad (47)$$

$$\approx 2^{2n-3k-1}.$$

The corresponding error for a confidence level of 95 percent and 1,000,000 iterations of simulation is

$$e = 0.00196\sqrt{2^{k+1}}. \quad (48)$$

In the simulation, the smallest and largest k for ETAIL is 4 and 10, thus the error is in the range of 1.11-8.87 percent. For ETAIL in the 1-LSB carry estimate case, the variance can be obtained in a similar way as in the fixed carry estimate case:

$$v \approx 2^{2n-3k-2}. \quad (49)$$

The corresponding error is

$$e = 0.00196\sqrt{2^{k+2}}. \quad (50)$$

The error is in the range of 1.57-12.54 percent for k between 4 and 10.

Since it is difficult to get the theoretical variance for ACA, simulated variances are used. The CVs (i.e., \sqrt{v}/μ) for different k and different carry estimate methods are presented in Table 1. According to Table 1 and (44), the errors of ACA are in the ranges of 1.3-10.5 percent and 1.5-12.5 percent for the fixed and 1-LSB carry estimates, as shown in Table 2.

TABLE 2
Errors of Monte Carlo Simulation for ACA

	fixed carry estimate	1-LSB carry estimate
k=6	1.3%	1.5%
k=7	1.9%	2.2%
k=8	2.7%	3.0%
k=9	3.8%	4.3%
k=10	5.3%	6.1%
k=11	7.7%	8.6%
k=12	10.5%	12.5%

3.6.2 Simulation Results

Figs. 8 and 9 shows the simulation results and the analytical results for the ACA, ESA and ETAIL. The functional models of both accurate and approximate adders are implemented in Matlab. 1,000,000 random input combinations are used to find the MED and ER values.

The simulation and the analytical MEDs are well matched especially for ESA for which the theoretical and simulated curves overlap; there are mainly two sources of discrepancy. The first source is due to the simulation method, i.e., Monte Carlo simulation is not exhaustive. The second source is caused by the approximation used in the analytical framework. As shown in the figures, the MED drops exponentially as k is increased: the MED drops approximately to half of its previous value when k is increased by 1 for all three approximate adders. The difference between MED values with different k values for the same adder is very large; therefore, the small discrepancy between the analytical and simulated results is rather negligible, i.e., the discrepancy will not result in an incorrect assessment of k .

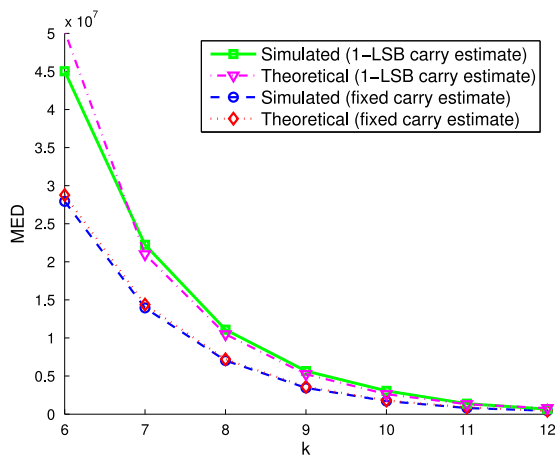
The simulation results for the error rate are shown in Fig. 9. For the ACA, ER decreases to half of its previous value when the value of k is increased by 1. For any k larger than 9, the ER drops below 2 percent for both 1-LSB and fixed carry estimates. Due to the design, the ER of the ESA is nearly constant for a value of k in a range between 11 and 15; it starts to decrease significantly when k is 16. In general, the ESA has a very high error rate, even though it has a very small MED. The ETAIL can significantly reduce ER. When $k = 6$, for example, the ER of the ETAIL with 1-LSB carry estimate is below 2 percent.

4 DISCUSSION

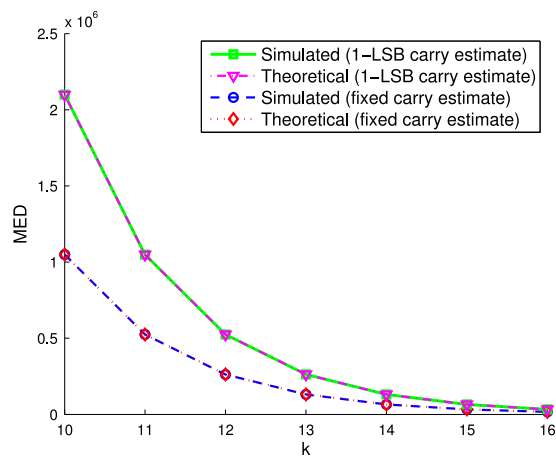
In this section, different features as related to carry estimate are analyzed for the various approximate adders.

4.1 Generalizing Carry Estimation Methods

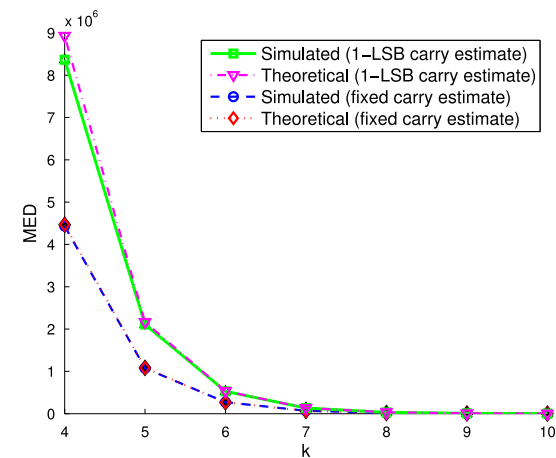
The 1-LSB carry estimate generates rather symmetrical and centralized errors, while a fixed carry tends to generate more biased errors. Another significant advantage of 1-LSB carry estimate is that it reduces the MED. As shown in the simulation results (Fig. 8), the MED of 1-LSB carry estimate is approximately half of the MED of a fixed carry for all three types of approximate adder. Consider the carry estimate accuracy (EA), i.e., the probability that the estimated carry is equal to the exact carry. If the carry is fixed to 0, then $EA = P(C_i = 0) = 1 - p_i \approx \frac{1}{2}$. For the 1-LSB carry estimate, $EA = P(C_i = A_{i-1}) = P(C_i = 1|A_{i-1} = 1)P(A_{i-1} = 1) + P(C_i = 0|A_{i-1} = 0)P(A_{i-1} = 0) = \frac{3}{4}$. It is also intuitively



(a)



(b)

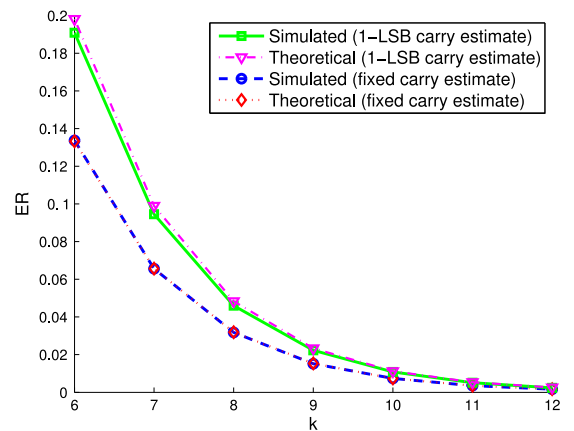


(c)

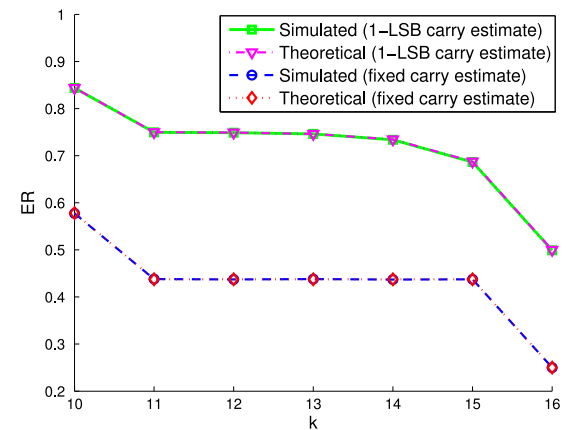
Fig. 8. Simulated and theoretical MED for a 32-bit (a) ACA, (b) ESA, and (c) ETAIL.

true that the 1-LSB carry estimate method has a higher accuracy than a fixed carry estimate, because it uses more information (when $A_{i-1} = 1$, C_i is more likely to be 1). It is then evident that the use of the 1-LSB to estimate carry reduces the ER as well as the MED.

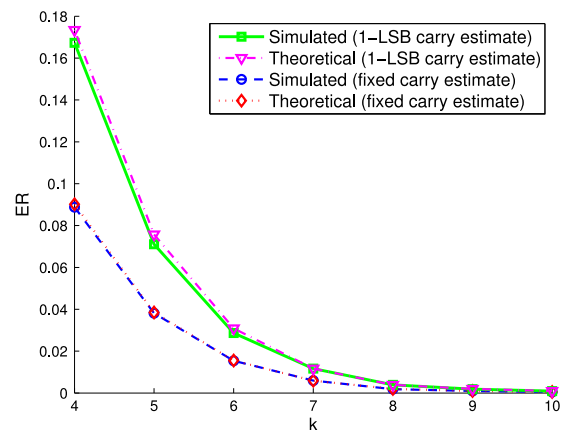
The approximation in the estimate of the carry signals is a significant issue for an approximate adder design.



(a)



(b)



(c)

Fig. 9. Simulated and theoretical ER for a 32-bit (a) ACA, (b) ESA, and (c) ETAIL.

Intuitively, the utilization of more less significant bits results in a better performance to predict the carry signal. Hereinafter, “ k -LSB carry estimate” refers to using the k bits in both A and B that are less significant than the current index to estimate the current carry. The fixed carry approach fails to use the LSBs in the estimate process, while the 1-LSB carry estimate method uses only 1 LSB. For the ETAIL and SCSA, if the Sum Generator is considered as a

TABLE 3
Truth Table of C_{i+1} Given $A_i B_i$

$A_i B_i$	C_{i+1}
00	0
01	U
10	U
11	1

sub-adder, then the Carry Generator (Fig. 3) is the circuitry for the k -LSB carry estimate. Thus, the ETAIL uses more LSBs for the carry estimate than the ESA and this increases the accuracy. Nevertheless, the use of more LSBs incurs a larger area overhead and longer delay; a tradeoff must be made between the number of LSBs for carry prediction and circuit performance.

Table 3 shows the truth table of C_{i+1} given 1-LSB, i.e., A_i and B_i , where “U” means “unknown”. More LSBs must be used to determine the unknown values. Without the information provided by the additional LSBs, the unknown values have approximately a probability of 0.5 to be either ‘1’ or ‘0’. Therefore, the best EA using 1-LSB information is $\frac{2+2/2}{4} = \frac{3}{4}$. The 1-LSB estimated carry, i.e., $C'_{i+1} = A_i$, uses a logic function based on 1-LSB information for achieving an EA of $\frac{3}{4}$. Consider the case in which k LSBs are used to estimate the carry C'_m . If the propagates in these k positions are all 1's, i.e., $P_i = A_i \oplus B_i = 1, i = m-1, m-2, \dots, m-k$, C'_{m-k} is required to determine C'_m . Therefore, without the information of C'_{m-k} , there are 2^k unknown entries in the truth table of C'_m based on previous k LSBs. In the k -LSB carry estimate method, these entries are arbitrarily assigned with certain values (1 or 0), which on average can successfully estimate half (i.e., 2^{k-1}) of the unknown entries. There are totally 2^{2k} input combinations, with 2^k unknown entries and $2^{2k} - 2^k$ known entries in the truth table. Thus, the highest EA based on k LSBs is $\frac{2^{2k} - 2^{k-1}}{2^{2k}} = 1 - 2^{-k-1}$. However, the utilization of more LSBs for carry prediction increases both the delay and the circuit complexity (i.e., the number of transistors). To reduce complexity, additional “approximate” carry estimates based on k LSBs can be derived by slightly changing some entries in the truth table. Two examples are provided by using two LSBs, as given in (51) and (52), with EA being 0.8125 and 0.875, respectively. Note that (52) achieves the highest EA of the 2-LSB carry estimate while (51) has a lower EA with a simpler logic implementation compared to (52):

$$C_{i+1} = A_i B_i + A_i A_{i-1}, \quad (51)$$

$$C_{i+1} = A_i B_i + A_{i-1} (A_i \oplus B_i). \quad (52)$$

4.2 Comparison among Different Approximate Adders

The ACA has an area complexity of $O(n \log \log n) \approx O(n)$, and a delay complexity of $O(\log k)$ [9], where n is the size of the adder and k is the maximum carry chain length. The ESA has the same complexity as the ACA if a parallel adder structure such as carry look-ahead (CLA) is used to implement each sub-adder in the ESA. However for the same k , the ACA has a more

complicated structure than the ESA and thus, it has a larger area overhead and probably a longer delay. As an example consider $k = 10$ and the 1-LSB carry estimate case. The simulated MEDs of the ACA and the ESA are 1.54×10^6 and 1.05×10^6 respectively. In terms of MED, the ESA is better than the ACA. If area and delay are taken into consideration, the ESA is certainly a better scheme, because it has a smaller MED as well as smaller area and delay. However, the ACA has a smaller ER: the ER of the ACA is 0.0074 while the ER of the ESA is 0.5775, which is 78 times of the ER of the ACA.

In the ESA, $\lceil \frac{n}{k} \rceil$ sub-adders have to be implemented. The area and delay complexities are $\lceil \frac{n}{k} \rceil A(k)$ and $\tau(k)$, respectively, where $A(k)$ and $\tau(k)$ are the area and delay complexities of a k -bit adder. In the ETAIL, the area complexity is $(2\lceil \frac{n}{k} \rceil - 2)A(k)$ because every sub-adder is duplicated except for the first and last sub-adders. The delay complexity of the ETAIL is $2\tau(k)$ because the critical path contains two k -bit sub-adders in series. The SCSA has the same error characteristics as the ETAIL, but it has a different circuit implementation. The delay of the SCSA is $\tau(k)$ with the delay of a multiplexer. Compared to ETAIL, the last k -bit sub-adder still needs duplication and $(\lceil \frac{n}{k} \rceil - 1)$ k -bit multiplexers are needed (they are not used in the ETAIL). Therefore, the SCSA is approximately two times faster than the ETAIL at the cost of an increased area overhead.

For comparison purposes, choose $k = 10$ for the ESA and $k = 5$ for the ETAIL with a 1-LSB carry estimate, because these two implementations have relatively the same delay (even though the ETAIL requires more area). The MED of the ESA is 1.05×10^6 , while the MED of the ETAIL is 1.07×10^6 ; the ERs are 0.5775 and 0.0381 for the ESA and the ETAIL, respectively. Therefore for the compared schemes, the ESA and the ETAIL have relatively similar MED, but the ETAIL has a significantly smaller ER. Hence, the ETAIL tends to generate large error magnitudes, because it has a similar MED as the ESA, but a very small ER.

Consider $k = 10$ for the ACA, the ESA and the SCSA, and $k = 5$ for the ETAIL as further examples. These three adders have similar critical path delays. The ESA has the least area overhead, but the largest ER. The ACA has the smallest ER, but it has the largest MED. According to [11], the ACA occupies 36 percent more area but it incurs in a 30 percent smaller delay compared to ETAILM (a modified version of the ETAIL), with both adders having the same carry propagation length. The ETAIL has a significantly reduced MED compared to the ACA and an acceptable ER of 3.81 percent. The SCSA is faster than the ETAIL at the cost of an increase in area (due to the additional multiplexers). If the design objective is an extremely fast approximate adder, then the SCSA is the best choice among these four approximate adders, because it achieves a similar MED to other approximate adders with a smaller k (i.e., a shorter critical path delay). The SCSA ($k = 10$) has approximately the same delay as the ETAIL ($k = 5$). However, the SCSA with $k = 10$ has an extremely small MED and ER: the ER is only 0.05 percent, while the MED is 10^{-3} of the ETAIL ($k = 5$), as shown in Table 4.

TABLE 4
Comparison between the Adders

	ACA	ESA	ETAIL	SCSA
k	10	10	5	10
Delay	$O(\log k)$	$O(\log k)$	$O(\log 2k)$	$O(\log k)$
Area	$O(n \log(\log(n)))$	$O(n)$	$O(n)$	$O(n)$
MED ($\times 10^3$)	1540	1050	1074	1.074
ER (%)	0.74	57.8	3.81	0.05

5 APPLICATION: IMAGE QUALITY EVALUATION USING PSNRs

5.1 Peak Signal-to-Noise Ratio

PSNR is widely used in many DSP applications (such as image processing) as an important figure of merit. In image processing, if \mathbf{I} is the noise-free image and \mathbf{K} is the noisy image, the PSNR is defined as [18]:

$$PSNR = 20 \log(MAX_I / \sqrt{MSE_K}), \quad (53)$$

where MAX_I is the maximum possible pixel value of image \mathbf{I} and MSE is the mean squared error defined as

$$MSE_K = \frac{1}{mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} [\mathbf{I}(i, j) - \mathbf{K}(i, j)]^2. \quad (54)$$

When approximate circuits are used for image processing, \mathbf{I} can be the resulting image using an exact computation while \mathbf{K} is the image obtained by approximate computing. For a good agreement with a precisely processed image, the PSNR of a noisy image should be very large—usually larger than a threshold P , i.e.,

$$PSNR > P,$$

or

$$\sqrt{MSE_K} < C, \quad (55)$$

where $C = 10^{\log(MAX_I) - \frac{P}{20}}$.

For (54), define an error matrix as $\mathbf{E} = \mathbf{K} - \mathbf{I}$; then the MSE and MED of image \mathbf{K} are

$$MSE_K = E(\mathbf{E} \circ \mathbf{E}), \quad (56)$$

$$MED_K = E(|\mathbf{E}|). \quad (57)$$

In (56) and (57), $E(\mathbf{X})$ denotes the average value of all the elements in matrix \mathbf{X} , the “ \circ ” operation obtains the element-wise product of two matrices, which is also known as the *Hadamard product* [19], and $|\mathbf{X}|$ obtains the absolute value of each element in \mathbf{X} .

Let d and μ denote the ED and MED of the approximate adder and σ^2 represent the mean squared error distance (MSED):

$$\sigma^2 = MSED = E[d^2]. \quad (58)$$

The MED and MSED of the approximate adders are obtained by the assumption that the inputs are uniformly

distributed. In the analysis in this manuscript, the pixel values of an image are assumed to be sufficiently random, i.e., if an image is processed by an approximate addition for each pixel, the mean and mean squared value of the error matrix are considered to be the same as the MED and the MSED of the approximate adder. Hereinafter, μ and σ^2 denote the MED and MSED of the approximate adder, as well as the mean and mean squared values of the error matrix if a resulting image is obtained by one approximate addition of each pixel. If the image \mathbf{K} is processed by a approximate addition operations, the mean squared error, MSE_K , is given by

$$MSE_K = E \left[\left(\sum_{i=1}^a \mathbf{E}_i \right) \circ \left(\sum_{i=1}^a \mathbf{E}_i \right) \right], \quad (59)$$

where \mathbf{E}_i is the error matrix of the i th addition. Assume each \mathbf{E}_i is independent, then (59) becomes:

$$\begin{aligned} MSE_K &= E \left[\sum_{i=1}^a \mathbf{E}_i \circ \mathbf{E}_i + 2 \sum_{1 \leq i < j \leq a} \mathbf{E}_i \circ \mathbf{E}_j \right] \\ &\leq a\sigma^2 + 2 \sum_{1 \leq i < j \leq a} E(|\mathbf{E}_i|)E(|\mathbf{E}_j|) \\ &\approx a\sigma^2 + a(a-1)\mu^2. \end{aligned} \quad (60)$$

For the ESA and the ETAIL, assume the relationship between \sqrt{MSE} (i.e. σ) and MED (i.e. μ) is given by $\sigma = f(\mu)$ (the function $f(\cdot)$ will be derived next), (60) can be converted to

$$\sqrt{MSE_K} \approx \sqrt{af(\mu)^2 + a(a-1)\mu^2} \leq C. \quad (61)$$

Based on (53) and (61), the PSNR can be estimated by

$$PSNR \approx 20 \log \left(MAX_I / \sqrt{af(\mu)^2 + a(a-1)\mu^2} \right). \quad (62)$$

The solution of (61) gives the maximum value of MED for deriving the parameter k in the ESA or the ETAIL using the corresponding equations (i.e., (26) and (30)) given in Section 3. Hence, this analytical framework can be used to select the proper approximate adder type and parameter (i.e., carry propagation length k) for image processing applications, instead of building functional models of the approximate adder and running time-consuming simulations with different parameters.

5.2 Relationship between μ and σ

5.2.1 ESA

The relationship between μ and σ is analyzed for the fixed carry case. Similar to the MED calculation in (26), the MSE of the ESA is calculated as:

$$\sigma^2 = \sum_{m=0}^{r-1} (2^{mk+l})^2 \left(\frac{1}{2} \right)^{r-m-1} \times \frac{1}{2} \approx 2^{2n-2k-1}. \quad (63)$$

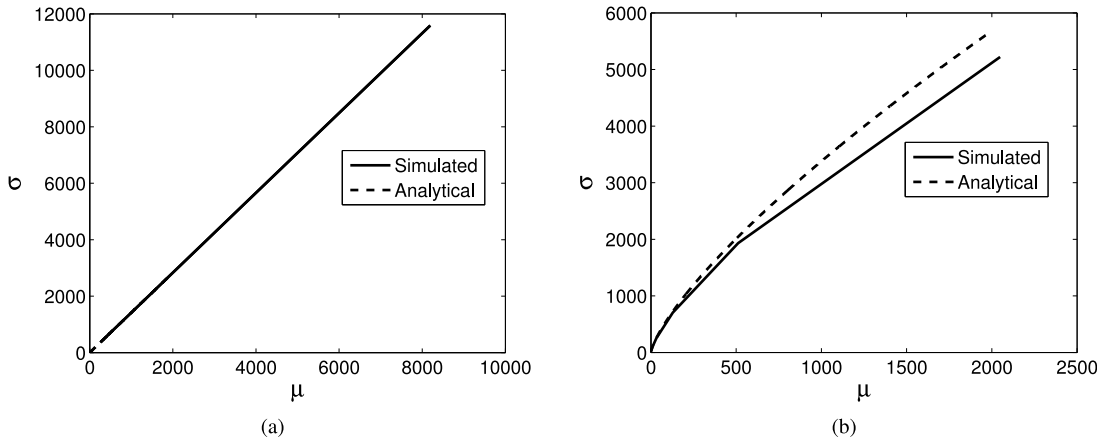


Fig. 10. Simulated and analytical $\sigma - \mu$ relationships for (a) ESA, and (b) ETAIL.

Based on (26) and (63), the relationship between μ and σ for the ESA is given by:

$$\sigma = \sqrt{2}\mu. \quad (64)$$

5.2.2 ETAIL

Similar to the analysis for the ESA, the MED and MSE for the ETAIL are found as $\mu \approx 2^{n-2k-1}$, $\sigma^2 \approx 2^{2n-3k-1}$. The relationship between μ and σ is:

$$\sigma = (2^{n+1}\mu^3)^{\frac{1}{4}}. \quad (65)$$

The simulated relationship between μ and σ and the analytical functions in (64) and (65) are plotted as in Fig. 10 for $n = 16$. For the ESA, the simulated curve and the analytical results match very closely, as shown by the nearly perfect overlapping curves. For the ETAIL, there is a slight discrepancy in the simulated and analytical plots. This discrepancy is due to the approximation when calculating MED and MSE as outlined previously.

5.3 Estimate of the PSNR for Image Processing

Three image processing algorithms are evaluated next: image sharpening, point detection and arithmetic mean filter.

If \mathbf{I} is the original image and \mathbf{S} is the processed image, the sharpening algorithm [20] is performed as

$$\mathbf{S}(x, y) = 2\mathbf{I}(x, y) - \frac{1}{273} \sum_{i=-2}^2 \sum_{j=-2}^2 \mathbf{G}(j+3, j+3) \mathbf{I}(x-i)(y-j), \quad (66)$$

where \mathbf{G} is a matrix given by:

$$\mathbf{G} = \begin{bmatrix} 1 & 4 & 7 & 4 & 1 \\ 4 & 16 & 26 & 16 & 4 \\ 7 & 26 & 41 & 26 & 7 \\ 4 & 16 & 26 & 16 & 4 \\ 1 & 4 & 7 & 4 & 1 \end{bmatrix}. \quad (67)$$

The point detector operation [21] is given by

$$\begin{aligned} \mathbf{S}(x, y) = & 8\mathbf{I}(x, y) - \mathbf{I}(x-1, y-1) - \mathbf{I}(x-1, y) \\ & - \mathbf{I}(x-1, y+1) - \mathbf{I}(x, y-1) - \mathbf{I}(x, y+1) \\ & - \mathbf{I}(x+1, y-1) - \mathbf{I}(x+1, y) - \mathbf{I}(x+1, y+1). \end{aligned} \quad (68)$$

A 3×3 arithmetic mean filter [21] is also implemented as

$$\mathbf{S}(x, y) = \frac{1}{9} \sum_{i=-1}^1 \sum_{j=-1}^1 \mathbf{I}(x+i, y+j). \quad (69)$$

For all three algorithms, the exact additions are replaced by approximate additions using the ESA and ETAIL. All other operations (i.e., multiplication, division and subtraction) are performed accurately. The mean squared error is estimated using (61), in which the number of approximate additions, a , is 25, 8 and 9, respectively for image sharpening, point detection and arithmetic mean filter applications. The PSNR is then given by (53).

In the sharpening algorithm, 16-bit approximate adders are used, because the maximum possible sum is 255×273 ,

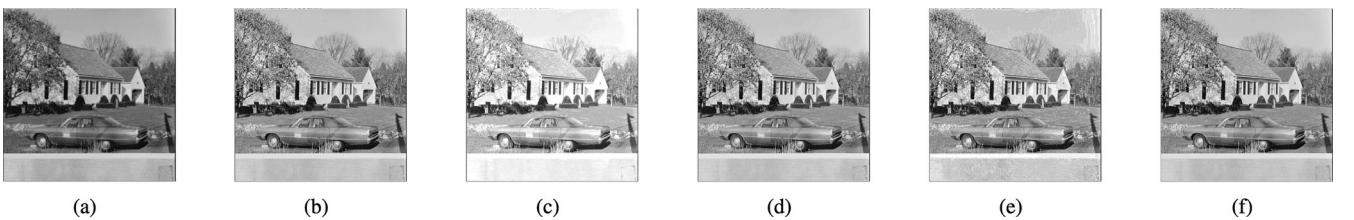


Fig. 11. Image sharpening: (a) original image, (b) using exact adders, (c) using the ESA with $k=6$ and PSNR = 18.1 dB, (d) using the ESA with $k = 10$ and PSNR = 41.4 dB, (e) using the ETAIL with $k = 4$ and PSNR = 27.4 dB and (f) using the ETAIL with $k = 7$ and PSNR = 56.8 dB.

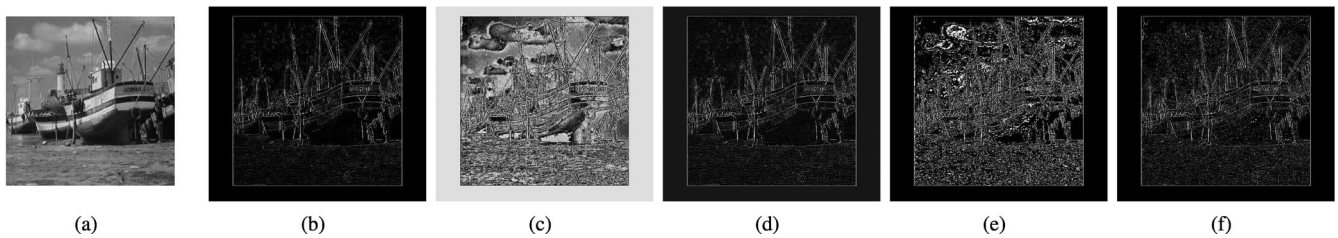


Fig. 12. Point detection: (a) original image, (b) using exact adders, (c) using the ESA with $k = 7$ and $\text{PSNR} = 4.4$ dB, (d) using the ESA with $k = 10$ and $\text{PSNR} = 24.6$ dB, (e) using the ETAll with $k = 4$ and $\text{PSNR} = 10.5$ dB and (f) using the ETAll with $k = 5$ and $\text{PSNR} = 19.9$ dB.



Fig. 13. Arithmetic mean filter: (a) original image with noise, (b) using exact adders, (c) using the ESA with $k = 5$ and $\text{PSNR} = 10.5$ dB, (d) using the ESA with $k = 8$ and $\text{PSNR} = 27.2$ dB, (e) using the ETAll with $k = 3$ and $\text{PSNR} = 16.9$ dB and (f) using the ETAll with $k = 5$ and $\text{PSNR} = 36.3$ dB.

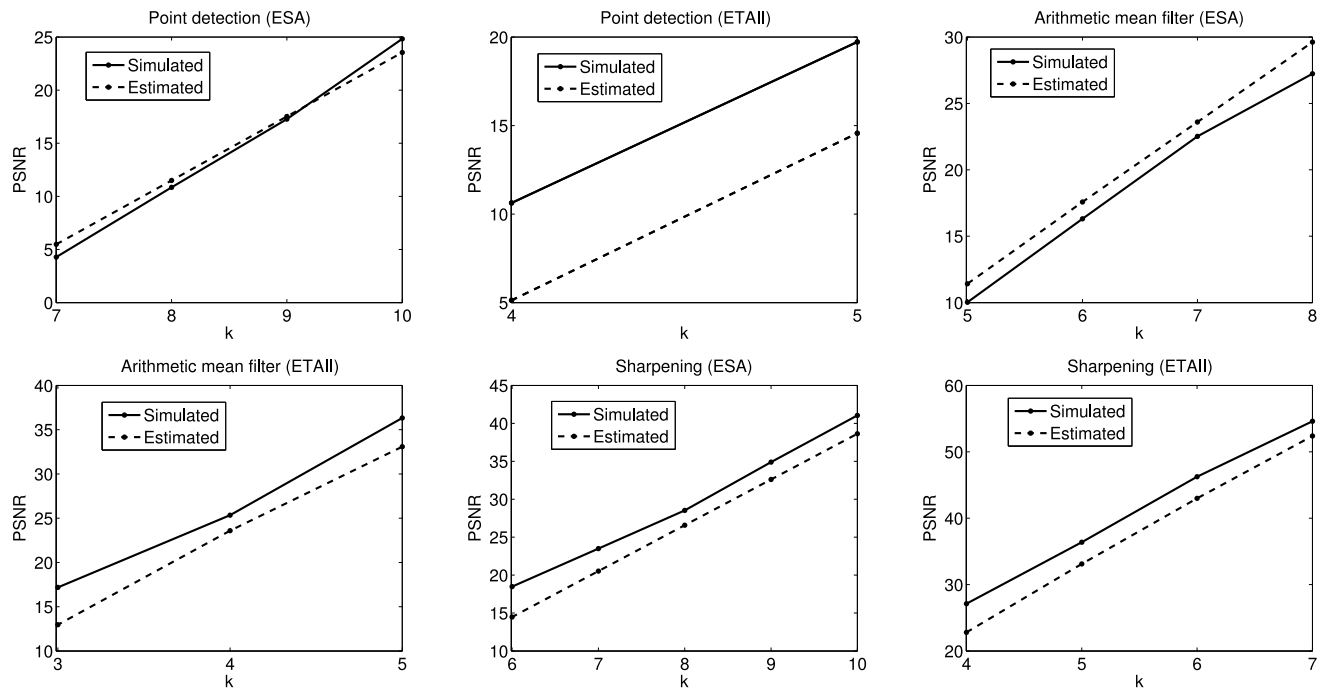


Fig. 14. Simulated and estimated PSNR (dB).

i.e. approximately $2^{16} - 1$. Twelve-bit approximate adders are used for the point detection and arithmetic filter computations. The analytical $\sigma - \mu$ plots shown in Fig. 10 are used for the PSNR.

Six images are selected for the three algorithms and the corresponding PSNR values are obtained; three of them are shown in Figs. 11, 12 and 13. The images are selected such that they are quite “typical” images to be processed, i.e., they show features commonly found in multimedia applications. For the same algorithm and approximate adder, the six images have PSNR values that are very close; this indicates that the PSNR is not strongly correlated to an image, hence the estimated PSNR can be readily applied to them. For further analyzing this feature, define the *relative*

discrepancy (RD) as the maximum difference between the PSNR values of the six images divided by their mean PSNR value. The RD values are shown in Table 5; the largest RD is 13.2 percent, however in most cases the RD is below 5 percent, i.e., at an acceptable level for this type of applications.

The simulated and estimated PSNR values are shown in Fig. 14. The best match between simulated and estimated PSNR values is achieved by the ESA-based point detection application, while the worst occurs for the ETAll-based point detection. In the worst case, the maximum PSNR is about 20 dB, that is generally considered to be low. For most cases, the estimate tends to be less accurate when the PSNR values are too small (< 20 dB). The interesting cases are the ones whose PSNR values are larger than 20 dB,

TABLE 5
Relative Discrepancy

Point detection									
	ESA				ETAI				
k	7	8	9	10	4	5			
RD (%)	13.2	4.8	2.1	1.4	10.5	5.2			
Arithmetic mean filter									
	ESA				ETAI				
k	5	6	7	8	3	4	5		
RD (%)	9.9	5.0	2.7	1.7	5.0	2.0	0.8		
Sharpening									
	ESA				ETAI				
k	6	7	8	9	10	4	5	6	7
RD (%)	5.7	5.0	3.4	1.7	0.9	2.1	1.7	2.6	4.1

otherwise the approximate results are not acceptable. For these cases, the estimate is shown to be accurate in this PSNR range. Moreover, the difference between estimated and simulated results is within a 3 dB margin; so for a given image processing application and approximate adder, the error in the estimate appears to be almost constant. Hence, the simulation results show that with the proposed methodology, the MED is a good indicator for the PSNR. The estimate procedure proposed in this paper can be used to evaluate the performance of approximate adders in image processing applications.

6 CONCLUSION

In this paper, an analytical framework has been proposed for characterizing approximate adder designs. This framework consists of models for the evaluation of three different types of approximate adders targeting several error metrics. Time-consuming simulation can then be avoided by using the proposed analytical models. Design criteria with respect to error characteristics in the operations of these approximate adders have been provided based on the analysis. As an example of the application of the framework, the PSNR in image processing has been evaluated using the proposed framework. The estimated PSNR can then be utilized for selecting the proper scheme of an approximate adder. Extensive simulation results show that there is a good agreement between the analytical outcomes of the proposed framework and the simulation results for three different computational algorithms commonly used in image processing. The PSNR estimate method proposed in this paper shows that there is a close relationship between MED and PSNR, while ER is less important. This may provide insights in the design of approximate arithmetic circuits for error-tolerant applications.

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