Gibbs Phenomenon

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> This is a test document on MathJax support on various platforms, mainly on printed material.

For large N, the graph of the Nth partial sum of the Fourier series of f on $[-L, L]$ overshoots the graph of the function at a jump discontinuity by approx. 9% of the magnitude of the jump.

(a) Nth Partial Sum

We write out the Nth partial sum of the Fourier series of $f(x)$ and expand its coefficients:

$$\begin{aligned}
S_N(x) &= a_0 + \sum_{n=1}^{N}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right] \\
&= \frac{1}{2L}\int_{-L}^{L} f(t)dt \\
&+ \frac{1}{L}\sum_{n=1}^{N}\left[\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt\right]\cos\left(\frac{n\pi x}{L}\right) \\
&+ \frac{1}{L}\sum_{n=1}^{N}\left[\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt\right]\sin\left(\frac{n\pi x}{L}\right) \\
&= \frac{1}{L}\int_{-L}^{L} f(t)\left\{\frac{1}{2} + \sum_{n=1}^{N}\cos\left[\frac{n\pi(t-x)}{L}\right]\right\}dt.
\end{aligned}$$

(b) Trigonometric Identity

It can be shown that

$$\sum_{n=1}^{N}\cos(n\xi) = \frac{\sin\left[(N+\frac{1}{2})\xi\right]}{2\sin\left(\frac{\xi}{2}\right)} - \frac{1}{2}.$$

Here is a trick: multiply it by $\sin\left(\frac{\xi}{2}\right)$ and try to lead to an identity. It can also be simplified by transforming $\cos x = \frac{e^{ix}+e^{-ix}}{2}$ and transforming back to $\sin$ terms. (It would be tricky if you didn't know RHS, though.)

(c) Approximating (a)

For small x, $\sin x \approx x$, and using (b), (a) can be expressed as

$$S_N(x) \approx \frac{1}{2L}\int_{-L}^{L} f(t)\frac{\sin\left[\frac{\pi}{L}(N+\frac{1}{2})(t-x)\right]}{\frac{\pi}{2L}(t-x)}dt.$$

(d) Estimating a Discontinuity

To find out the amount, we use the following function with a jump discontinuity 1 at $x = L$,

$$f(x) = \begin{cases} 0 & \text{if } -L < x \leq x_0, \\ 1 & \text{if } x_0 < x < L. \end{cases}$$

Inserting it into S_N yields

$$\begin{aligned} S_N(x) &\approx \frac{1}{2L}\int_{x_0}^{L} \frac{\sin\left[\frac{\pi}{L}(N+\frac{1}{2})(t-x)\right]}{\frac{\pi}{2L}(t-x)}dt \\ &= \frac{1}{\pi}\int_{(\pi/L)(N+1/2)(x_0-x)}^{(\pi/L)(N+1/2)(L-x)} \frac{\sin s}{s}ds. \end{aligned}$$

(e) Putting $N \to \infty$

The integral $\int_0^z \frac{\sin t}{t}dt$ is called the sine integral, denoted as $\mathrm{Si}(z)$.

Computing S_N when $N \to \infty$ diverges to three cases depending on x, in which we are only interested in the case that $x > x_0$, that is $(\pi/L)(N+1/2)(x_0 - x) \to -\infty$:

$$
\begin{aligned}
\lim_{N\to\infty} S_N(x) &\approx \frac{1}{\pi}\int_{(\pi/L)(N+1/2)(x_0-x)}^{\infty} \frac{\sin s}{s} ds \\
&= \frac{1}{\pi}\left\{\lim_{z\to\infty} \mathrm{Si}(z) - \mathrm{Si}\left[\left(\frac{\pi}{L}\right)\left(N+\frac{1}{2}\right)(x_0-x)\right]\right\},
\end{aligned}
$$

(f) Evaluating the Sine Integral

$\mathrm{Si}(z)$ at infinity can be computed as follows,

$$
\begin{aligned}
\lim_{z\to\infty} \mathrm{Si}(z) &= \int_0^\infty \frac{\sin t}{t} dt \\
&= \int_0^\infty \int_0^\infty e^{-st} \frac{\sin t}{t} ds\, dt \\
&= \int_0^\infty \int_0^\infty e^{-st} \frac{\sin t}{t} dt\, ds \\
&= \int_0^\infty \frac{1}{s^2+1} ds \\
&= \frac{\pi}{2}.
\end{aligned}
$$

$\mathrm{Si}(z)$ is maximized when $\mathrm{Si}'(z^*) = 0$, that is $\frac{\sin z^*}{z^*} = 0$, and the minimal x^* satisfying the condition is $z^* = \pi$.

Thus the overshoot is at most

$$
\frac{1}{\pi}\left[\frac{\pi}{2} - \mathrm{Si}(\pi)\right] = \frac{1}{2} - \frac{\mathrm{Si}(\pi)}{\pi} \approx 0.0895,
$$

which is approximately 9%.

> Written with StackEdit.