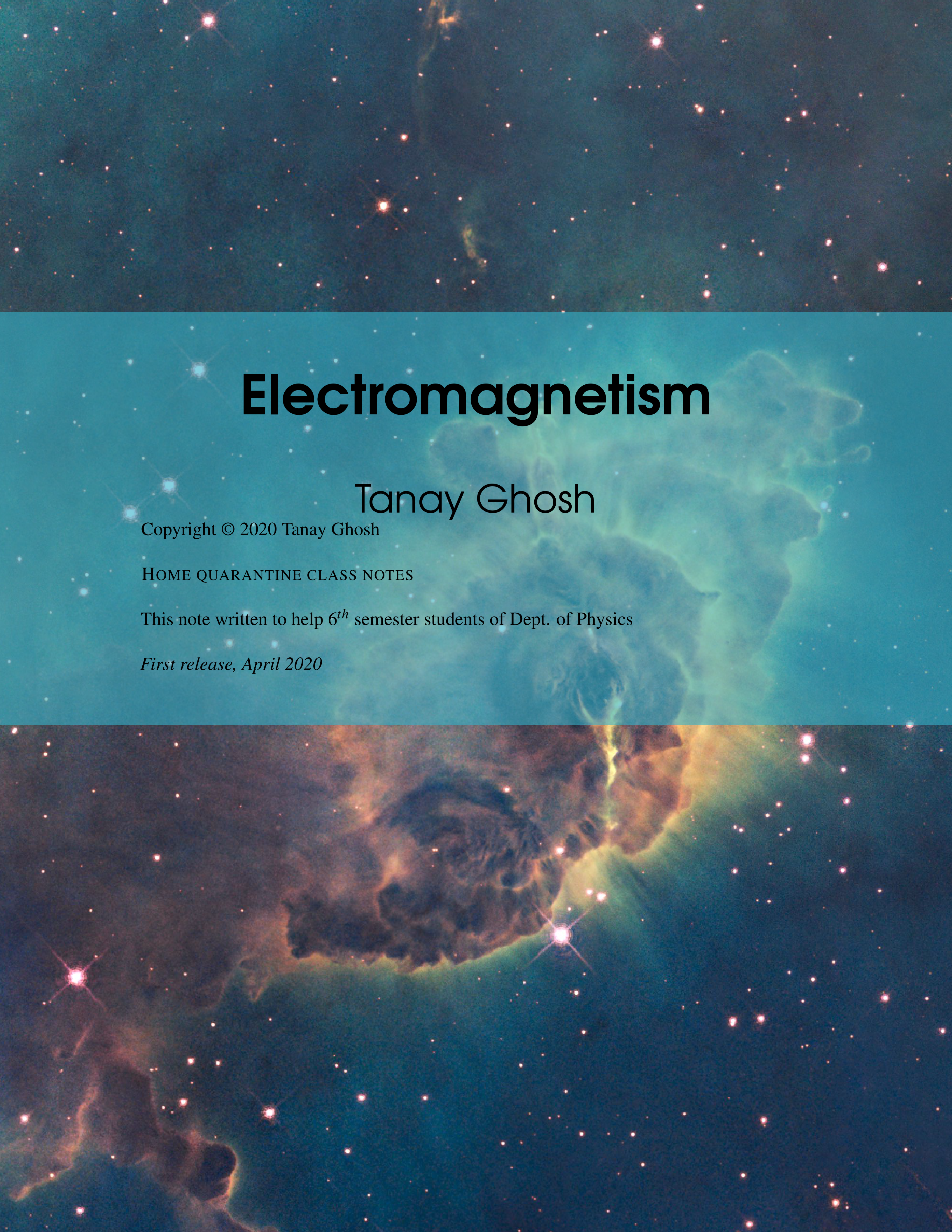

Electromagnetism

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HOME QUARANTINE CLASS NOTES

This note written to help 6^{th} semester students of Dept. of Physics

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1. The Origin of Solar System

1.1 Introduction

We develop the tools to model the magnetic field that is produced by an electric current. We will introduce the Biot-Savart Law, which is analoguous to Coulomb's Law in that it can be used to calculate the magnetic field produced by any current. We will also introduce Ampere's Law, which can be thought of as the analogue to Gauss' Law, allowing us to easily determine the magnetic field when there is a high degree of symmetry.

- Understand how to apply the Biot-Savart Law to determine the magnetic field from an electric current.
- Understand how to apply Ampere's Law.
- Understand how to model the forces that are exerted on each other by two wires carrying current.
- Understand how to model a solenoid and a toroid.

How does an electromagnet work?

- Current is passed through a magnet, increasing its strength.
- Current is passed through a circular coil, creating a magnetic field

1.2 The Biot-Savart Law

The Biot-Savart law allows us to determine the magnetic field at some position in space that is due to an electric current. More precisely, the Biot-Savart law allows us to calculate the infinitesimal magnetic field, $d\vec{B}$, that is produced by a small section of wire, $d\vec{l}$, carrying a current, I, such that $d\vec{l}$ is co-linear with the wire and points in the direction of the electric current:

$$d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{d\vec{l} \times \hat{r}}{r^2}$$

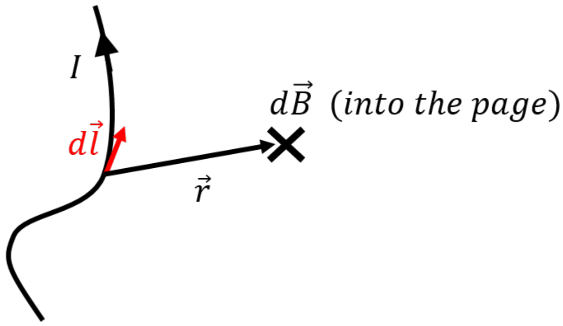

Figure 1.1: The infinitesimal magnetic field, $d\vec{B}$, that is created by an infinitesimal section of wire, $d\vec{l}$, carrying current I. Note that the vector, $\vec{r}$, goes from $d\vec{l}$ to the point where we wish to calculate the field.

where $\vec{r}$ is the vector from the element of wire, $d\vec{l}$, to the point where we would like to determine the magnetic field, as illustrated in Figure 1.2 μ_0 is a constant of proportionality called the "permeability of free space", and has the value $\mu_0 = 4\pi e - 7T \cdot m/A$.

The infinitesimal magnetic field, $d\vec{B}$, that is created by an infinitesimal section of wire, $d\vec{l}$, carrying current I. Note that the vector, $\vec{r}$, goes from $d\vec{l}$ to the point where we wish to calculate the field.

The Biot-Savart Law has some similarities with the Coulomb Law to calculate the electric field, as the magnitude of the magnetic field decreases as the inverse of the square distance between the source and the field. However, it can only be expressed in differential form (i.e. as an infinitesimal) and it requires working in three dimensions because of the cross product. It is usually more convenient to use the Biot-Savart Law in the form:

$$d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{d\vec{l} \times \vec{r}}{r^3}$$

where the unit vector $\hat{r}$ was replaced by $\vec{r}/r$.

The procedure for applying the Biot-Savart Law is as follows:

1. Make a really good diagram, as you will have to include some 3D aspects.
2. Choose an infinitesimal section of wire, $d\vec{l}$.
3. Determine the vector $\vec{r}$.
4. Determine the cross-product, $d\vec{l} \times \vec{r}$, which will point in the direction of the magnetic field from that infinitesimal section of wire.
5. Write out the infinitesimal vector $d\vec{B}$, and determine its components.
6. Think about symmetry! As you sum the $d\vec{B}$, will some components cancel? If yes, you do not need to do those integrals.
7. Determine the total magnetic field, component by component, by summing (integrating) the components of $d\vec{B}$ over the wire.

1.2.1 Magnetic field from a straight current-carrying wire

In this section, we use the Biot-Savart Law to determine the magnetic field a distance, h, from the centre of a finite straight wire of length L, carrying current I, as illustrated in

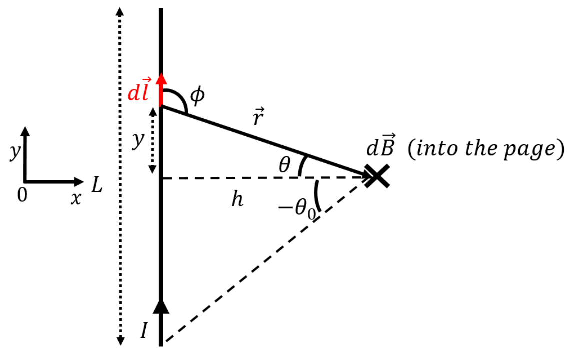

Figure 1.2: Setting up the model to use the Biot-Savart Law to calculate the magnetic field a distance h from the centre of a current-carrying wire of length L.

Figure 1.2.1.

We start by choosing an infinitesimal element of wire, $d\vec{l}$, a distance y above the centre of the wire, as shown (we choose the origin to be located at the centre of the wire). The vector $d\vec{l}$ is thus given by:

$$d\vec{l} = dl\hat{y}$$

The vector, $\vec{r}$, from $d\vec{l}$ to the point at which we would like to know the magnetic field is given by:

$$\vec{r} = r\cos\theta\hat{x} - r\sin\theta\hat{y}$$
$$r = \sqrt{h^2 + y^2} = \frac{h}{\cos\theta}$$

The cross-product between $d\vec{l}$ and $\vec{r}$ is easily found with the right-hand rule to point into the page (corresponding to the negative z direction). The magnitude of the cross-product is given by:

$$||d\vec{l} \times \vec{r}|| = dlr\sin\phi$$

where $\phi = \pi/2 + \theta$ is the angle between $d\vec{l}$ and $\vec{r}$, so that $\sin\phi = \cos\theta$. The cross-product can thus be written in terms of θ as:

$$d\vec{l} \times \vec{r} = -dlr\cos\theta\hat{z}$$

Note that we can also determine the cross-product algebraically instead of using the right-hand rule and the magnitude:

$$\begin{aligned}
d\vec{l} \times \vec{r} &= (dl\hat{y}) \times (r\cos\theta\hat{x} - r\sin\theta\hat{y}) \\
&= dlr\cos\theta(\hat{y} \times \hat{x}) - rdl\sin\theta(\hat{y} \times \hat{y}) \\
&= -dlr\cos\theta\hat{z}
\end{aligned}$$

The infinitesimal magnetic field element, $d\vec{B}$, is given by:

$$d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{d\vec{l}\times\vec{r}}{r^3} = -\frac{\mu_0 I}{4\pi}\frac{dl\cos\theta}{r^2}\hat{z}$$

Any segment along the wire will result in a magnetic field that is into the page (negative z direction), thus there will be no cancellations due to any symmetries. We can now proceed to perform the integral.

We can use either θ or y to label the wire elements and carry out the integration. We will choose to integrate over θ, requiring us to express dl and r in terms of θ (and constants), as those are the only quantities in $d\vec{B}$ that depend on the position of $d\vec{l}$. In order to express dl in terms of $d\theta$, we first relate θ to y, the position of the wire element:

$$y = h\tan\theta \quad \rightarrow \quad dl = dy = \frac{dy}{d\theta}d\theta = \frac{h}{\cos^2\theta}d\theta$$

and r is given by:

$$r = \frac{h}{\cos\theta} \quad \rightarrow \quad \frac{1}{r^2} = \frac{\cos^2\theta}{h^2}$$

Putting this altogether into $d\vec{B}$:

$$d\vec{B} = -\frac{\mu_0 I}{4\pi}\frac{dl\cos\theta}{r^2}\hat{z} = -\frac{\mu_0 I}{4\pi}\left(\frac{h}{\cos^2\theta}d\theta\right)\left(\frac{\cos^2\theta}{h^2}\right)\cos\theta\hat{z} = -\frac{\mu_0 I}{4\pi h}\cos\theta d\theta\hat{z} = dB_z\hat{z}$$

We define the angle, θ_0, to be the maximum amplitude of the angle θ when integrating over the wire (see Figure 1.2.1), so that we integrate θ from $-\theta_0$ to $+\theta_0$:

$$B_z = \int_{-\theta_0}^{+\theta_0} dB_z = -\frac{\mu_0 I}{4\pi h}\int_{-\theta_0}^{+\theta_0}\cos\theta d\theta = -\frac{\mu_0 I}{4\pi h}(2\sin\theta_0) = -\frac{\mu_0 I}{2\pi h}\sin\theta_0$$

Using the given dimensions:

$$\sin\theta_0 = \frac{L/2}{\sqrt{h^2+\frac{L^2}{4}}}$$

Thus, the magnetic field, $\vec{B}$, a distance, h, from the centre of a wire of length, L, carrying current, I, in the positive y direction is given by:

$$\vec{B} = -\frac{\mu_0 I}{2\pi h}\frac{L/2}{\sqrt{h^2+\frac{L^2}{4}}}\hat{z} \quad \text{(finite wire)}$$

The magnetic field must be rotationally symmetric; that is, if the wire is vertical, the magnetic field at a distance h must look the same regardless of the angle from which we

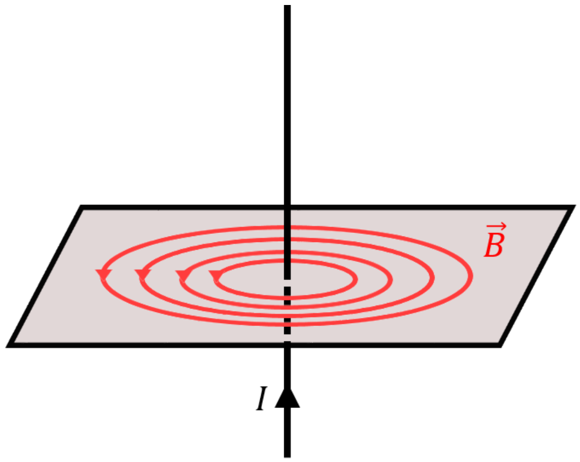

Figure 1.3: The magnetic field from a current-carrying wire forms concentric circles centred on the wire.

view the vertical wire (we should always see the magnetic field going into the page at the point that we use in Figure 1.2.1). Thus, the magnetic field lines must form circles around the wire, as illustrated in Figure 1.2.1. Note that the direction of the magnetic field is given by the right-hand rule for axial vectors; when you align your thumb with the current, your fingers curl in the direction of the magnetic field.

It is of particular interest to investigate the limiting case of an infinitely long wire, in the limit of $L \to \infty$, or equivalently, $\theta_0 \to \frac{\pi}{2}$. The latter is easiest to evaluate, since $\sin\theta_0 \to 1$. The magnitude of the magnetic field, $\vec{B}$, a distance, h, from an infinite wire carrying current, I, is given by:

$$B = \frac{\mu_0 I}{2\pi h} \quad \text{(infinite wire)}$$

One can often make the approximation that the wire is infinite in length, when the distance, h, is small compared to the length, L, of the wire.

1.2.2 Magnetic field from a circular current-carrying wire

In this section, we examine the magnetic field that is created by a circular current-carrying loop of wire. We can determine the shape of the magnetic field, by considering small sections as straight wires, with circular magnetic field lines around them. As we move closer to the centre of the ring, those fields sum together, as illustrated in Figure 1.2.2. Note that the magnetic field from a loop of current is identical to that from a bar magnet (as a bar magnet is, of course, a collection of current loops).

Below, we use the Biot-Savart Law to derive an expression for the magnitude of the magnetic field at a distance, h, from the centre of a ring of radius, R, along its axis of symmetry, when there is a current, I, in the ring. While the mathematics are much easier than the case for the straight wire, the challenge in this case is to visualize the calculation in three dimensions! Figure 1.2.2 shows the loop of current, as well as our choice of coordinate

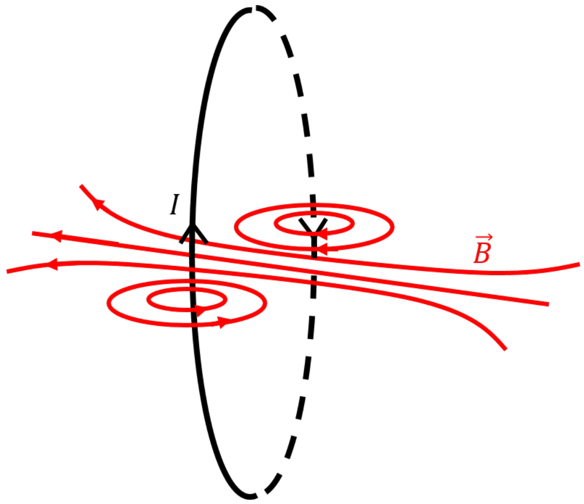

Figure 1.4: The magnetic field from a current-carrying loop of wire can be thought of as the sum of the fields from small straight sections of wire.

system (with the origin at the centre of the ring). In particular, we wish to calculate the magnetic field at a distance, h, along the z axis. The x axis goes into the page.

In order to apply the Biot-Savart Law, we choose an element, $d\vec{l}$, of wire at the top of the ring, as illustrated. At this position, the element, $d\vec{l}$, points in the positive x direction (into the page):

$$d\vec{l} = dl\hat{x}$$

The vector, $\vec{r}$, from the wire element to the point where we wish to determine the magnetic field is given by:

$$\vec{r} = -r\sin\theta\hat{y} + r\cos\theta\hat{z}$$

and the angle θ will be the same for all wire elements along the ring. The cross-product, $d\vec{l}\times\vec{r}$, can be evaluated algebraically:

$$\begin{aligned}
d\vec{l}\times\vec{r} &= (dl\hat{x})\times(-r\sin\theta\hat{y} + r\cos\theta\hat{z}) \\
&= -rdl\sin\theta(\hat{x}\times\hat{y}) + rdl\cos\theta(\hat{x}\times\hat{z}) \\
&= -rdl\sin\theta\hat{z} + rdl\cos\theta(-\hat{y}) \\
&= -rdl\sin\theta\hat{z} - rdl\cos\theta\hat{y}
\end{aligned}$$

so that the element of magnetic field, $d\vec{B}$, corresponding to that choice of $d\vec{l}$, will lie in the $y-z$ plane, as illustrated in Figure 1.2.2. Note that the vector $d\vec{B}$ is perpendicular to the vector $\vec{r}$ (since it is the cross-product of $\vec{r}$ with another vector). The magnetic field element,

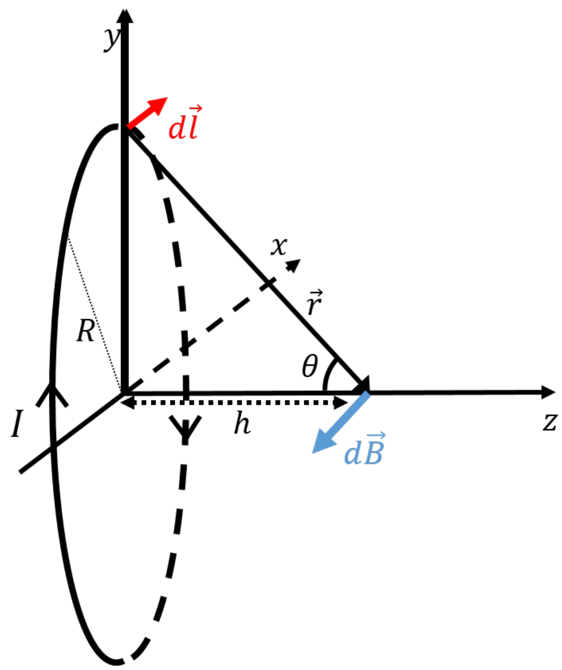

Figure 1.5: Diagram to apply the Biot-Savart Law in order to determine the magnetic field along the symmetry axis of a ring carrying current, I. The x axis goes into the page.

$d\vec{B}$, is given by:

$$\begin{aligned} d\vec{B} &= \frac{\mu_0 I}{4\pi}\frac{d\vec{l}\times\vec{r}}{r^3} = \frac{\mu_0 I}{4\pi r^3}(-rdl\sin\theta\hat{z} - rdl\cos\theta\hat{y}) \\ &= \frac{\mu_0 I}{4\pi r^2}(-dl\sin\theta\hat{z} - dl\cos\theta\hat{y}) = dB_z\hat{z} + dB_y\hat{y} \end{aligned}$$

As the wire element, $d\vec{l}$, moves around the circle, the tip of the resulting magnetic field vector element traces a circle centred on the z axis, as illustrated in Figure 1.2.2. Note that, in general, $d\vec{B}$ will also have an x component (the x component was only 0 before because we chose $d\vec{l}$ to be at the top of the ring). When we sum together the magnetic field elements, the x and y components will cancel, so that we are left with the z component. The total magnetic field will be in the negative z direction, as anticipated from Figure 1.2.2. Summing together the z components of the infinitesimal magnetic fields:

$$\begin{aligned} dB_z &= -\frac{\mu_0 I}{4\pi r^2}dl\sin\theta \\ B_z &= \int dB_z = -\int \frac{\mu_0 I}{4\pi r^2}dl\sin\theta \end{aligned}$$

Note that in this case, both r and θ are constant for all of the $d\vec{l}$, allowing us to take them out of the integral. The integral is then just a sum of the dl elements, which must add up to

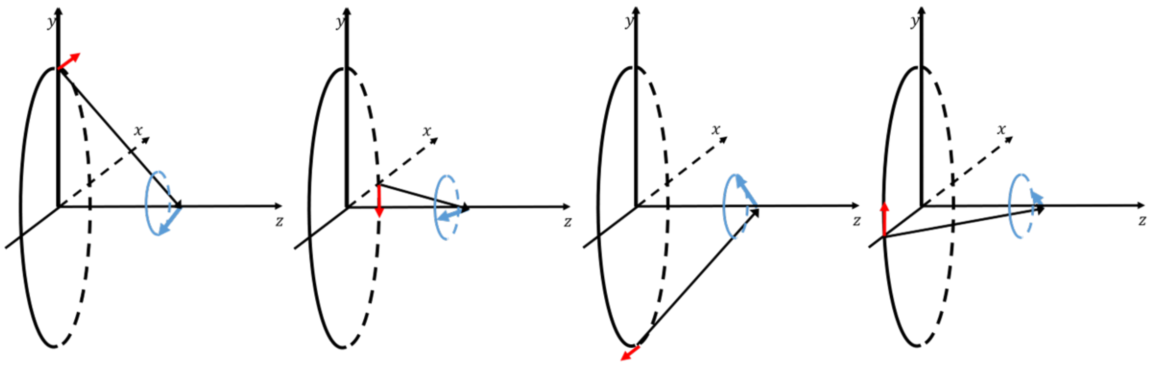

Figure 1.6: As the wire element, $d\vec{l}$, moves along the ring, the tip of corresponding magnetic field element vector, $d\vec{B}$, describes a circle centred on the z axis. Thus, only the (negative) z component of $d\vec{B}$ will survive when these are all added together.

the circumference of the ring:

$$B_z = \int dB_z = -\frac{\mu_0 I}{4\pi r^2} \sin\theta \int_0^{2\pi R} dl = -\frac{\mu_0 I}{4\pi r^2} \sin\theta (2\pi R) = -\frac{\mu_0 I}{2r^2} R \sin\theta$$

In terms of the variables that we are given:

$$\begin{aligned} r &= \sqrt{R^2 + h^2} \\ \sin\theta &= \frac{R}{r} = \frac{R}{\sqrt{R^2+h^2}} \\ \therefore \quad \vec{B} &= -\frac{\mu_0 I}{2} \frac{R^2}{(R^2+h^2)^{\frac{3}{2}}} \hat{z} \quad \text{(field from a loop of current)} \end{aligned}$$

In this case, the math was relatively straightforward (no substitutions to evaluate the integral), however it is challenging to visualise the problem in three dimensions.

A coil is made of N loops of current-carrying wire packed closely together. What is the magnetic field at the centre of the coil?

$\frac{\mu_0 I}{2R}$

$\frac{N\mu_0 I}{2R}$ (correct)

$\frac{N\mu_0 I}{2R^2}$

$\frac{\mu_0 I}{R}$

1.3 Force between two current-carrying wires

Consider two infinite parallel straight wires, a distance h apart, carrying upwards currents, I_1 and I_2, respectively, as illustrated in Figure 1.3. The first wire will create a magnetic field,

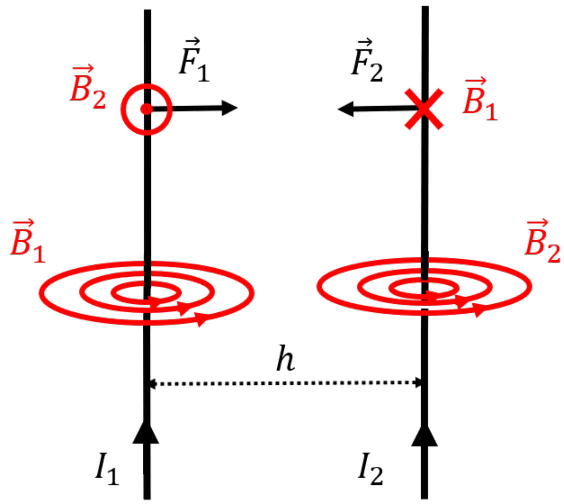

Figure 1.7: Two parallel current-carrying wires will exert an attractive force on each other, if their currents are in the same direction.

$\vec{B}_1$, in the shape of circles concentric with the wire. At the position of the second wire, the magnetic field B_1 is into the page, and has a magnitude:

$$B_1 = \frac{\mu_0 I_1}{2\pi h}$$

Since the second wire carries a current, I_2, upwards, it will experience a magnetic force, $\vec{F}_2$, from the magnetic field, B_1, that is towards the left (as illustrated in Figure 1.3 and determined from the right-hand rule). The magnetic force, $\vec{F}_2$, exerted on a section of length, l, on the second wire has a magnitude given by:

$$F_2 = I_2||\vec{l}\times\vec{B}_1|| = I_2 l B_1 = \frac{\mu_0 I_2 I_1 l}{2\pi h}$$

where we used the fact that the angle between $\vec{l}$ and $\vec{B}$ is 90. We expect, from Newton's Third Law, that an equal and opposite force should be exerted on the first wire. Indeed, the second wire will create a magnetic field, $\vec{B}_2$, that is out of the page at the location of the first wire, with magnitude:

$$B_2 = \frac{\mu_0 I_2}{2\pi h}$$

This leads to a magnetic force, $\vec{F}_1$, exerted on the first wire, that points to the right (from the right-hand rule). On a section of length, l, of the first wire, the magnetic force from the magnetic field, $\vec{B}_2$, has magnitude:

$$F_1 = I_1||\vec{l}\times\vec{B}_2|| = I_1 l B_2 = \frac{\mu_0 I_1 I_2}{2\pi h}$$

which does indeed have the same magnitude as the force exerted on the second wire. Thus, when two parallel wires carry current in the same direction, they exert equal and opposite attractive forces on each other. Two parallel wires carry current in opposite directions. What force do they exert on each other?

There will be no force, since the currents cancel.

There will be an attractive force between the wires.

There will be a repulsive force between the wires. (correct)

The attractive force between two wires used to be the basis for defining the Ampere, the S.I. (base) unit for electric current. Before 2019, the Ampere's was defined to be "that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed one metre apart in vacuum, would produce between these conductors a force equal to $2e-17N$ per metre of length". Recently, the definition was updated to be based on defining the Coulomb in such a way that the elementary charge has a numerical value of $e = 1.602176634e-19C$, and the Ampere corresponds to one Coulomb per second. The force between two wires is a good system to understand how any physical quantity cannot depend on our choice of the right-hand to define cross-products. As mentioned in the previous chapter, any physical quantity, such as the direction of the force exerted on a wire, will always depend on two successive uses of the right hand. In this system, we first used the right-hand rule for axial vectors to determine the direction of the magnetic field from one of the wires. We then used the right-hand rule to determine the direction of the cross-product to determine the direction of the force on the other wire. You can verify that you get the same answer if you, instead, use your left-hand to define the direction of the magnetic field (which will be in the opposite direction), and then again for the cross-product. This also highlights that the magnetic field (and the electric field) is just a mathematical tool that we use to, ultimately, describe the motion of charges or compass needles.

For this problem, we can determine whether the force is attractive or repulsive by finding the directions of the forces on wire 1 ($\vec{F}_1$) and wire 2 ($\vec{F}_2$). Figure 1.3 shows how to find the directions of $\vec{F}_1$ and $\vec{F}_2$ step by step using the right hand rules. The process is as follows:

(a) Use the axial right hand rule to find the magnetic field at wire 2 due to the current in wire 1 ($\vec{B}_1$).
(b) Use the right hand rule to find the direction of the force on wire 2 due to its current and the magnetic field $\vec{B}_1$.
(c) Use the axial right hand rule to find the magnetic field at wire 1 due to the current in wire 2 ($\vec{B}_2$).
(d) Use the right hand rule to find the direction of the force on wire 1 due to its current and the magnetic field $\vec{B}_2$.

Both $\vec{F}_1$ and $\vec{F}_2$ point towards the centre, so the force is attractive. See what happens when you use your left hand!

When current is flowing in a straight cable, how to you expect the charges to be distributed radially through the cross-section of the cable?

Uniformly in radius (current density does not depend on r).

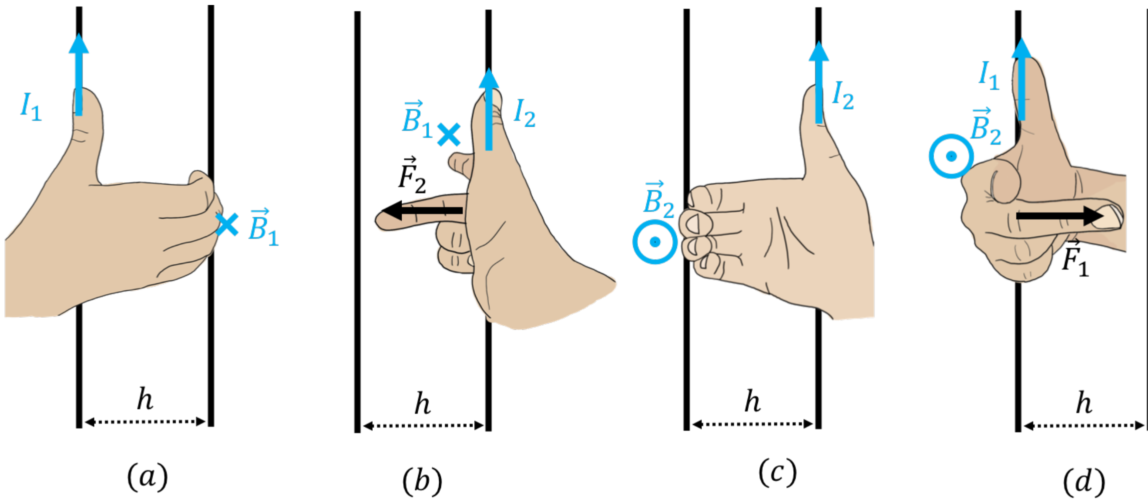

Figure 1.8: (a) Axial right hand rule to find $\vec{B}_1$. (b) Right hand rule to find $\vec{F}_2$. (c) Axial right hand rule to find $\vec{B}_2$. (d) Right hand rule to find $\vec{F}_1$.

There will be an excess of positive charges on the outside of the cable. (correct)

There will be an excess of negative charges on the outside of the cable.

1.4 Ampere's Law

Ampere's Law is similar to Gauss' Law, as it allows us to (analytically) determine the magnetic field that is produced by an electric current in configurations that have a high degree of symmetry. Ampere's Law states:

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I^{enc}$$

where the integral on the left is a "path integral", similar to how we calculate the work done by a force over a particular path. The circle sign on the integral means that this is an integral over a "closed" path; a path where the starting and ending points are the same. I^{enc} is the net current that crosses the surface that is defined by the closed path, often called the "current enclosed" by the path. This is different from Gauss' Law, where the integral is over a closed surface (not a closed path, as it is here). In the context of Gauss' Law, we refer to "calculating the **flux** of the electric field **through** a closed surface"; in the context of Ampere's Law, we refer to "calculating the **circulation** of the magnetic field **along** a closed path".

We apply Ampere's Law in much the same way as we apply Gauss' Law:

1. Make a good diagram, identify symmetries.
2. Choose a closed path over which to calculate the circulation of the magnetic field (see below for how to choose the path). The path is often called an "Amperian loop" (think "Gaussian surface").

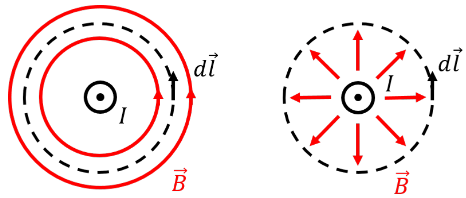

Figure 1.9: By symmetry, the magnetic field from a current-carrying infinite wire (illustrated with current coming out of the page), must either form concentric circles (left panel), or be in the radial direction (right panel). We know that the former (circles, left panel) is the correct choice. The dotted lines show "Amperian loops" that one can use to calculate the integral in Ampere's Law.

3. Evaluate the circulation integral.
4. Determine how much current is "enclosed" by the Amperian loop.
5. Apply Ampere's Law.

Similarly to Gauss' Law, we need to **choose the path** (instead of the surface) over which we will evaluate the integral. The integral will be easy to evaluate if:

1. **The angle between $\vec{B}$ and $d\vec{l}$ is constant along the path**, so that:

$$\oint \vec{B}\cdot d\vec{l} = \oint Bdl\cos\theta = \cos\theta \oint Bdl$$

   where θ is the angle between $\vec{B}$ and $d\vec{l}$.

2. **The magnitude of $\vec{B}$ is constant along the path**, so that:

$$\cos\theta \oint Bdl = B\cos\theta \oint dl$$

Choosing a path that meets these two conditions is only possible if there is a high degree of symmetry. Consider an infinitely long straight wire, carrying current, I, out of the page, as illustrated in Figure 1.4. The magnetic field from the wire must look the same regardless of the angle from which we view the wire ("azimuthal symmetry"). Thus, the magnetic field must either form concentric circles around the wire (which we know is the case from the Biot-Savart Law) or it must be in the radial direction (pointing towards or away from the wire). These two possibilities are illustrated in Figure 1.4, and we will pretend, for now, that we do not know which is correct. In order to apply Ampere's Law, we choose an Amperian loop (instead of a "Gaussian surface"). In the case of an infinite current-carrying wire, a circle that is concentric with the wire will meet the properties above, regardless of the two possible configurations of the magnetic field: with a circular Amperian loop, the angle between the magnetic field and the element $d\vec{l}$ is constant along the entire loop, and the magnitude of the magnetic field is constant along the loop. Our choice of loop is illustrated in Figure 1.4, where we have illustrated the magnetic field for the case where it forms concentric circles. The circulation of the magnetic field along a circular path of

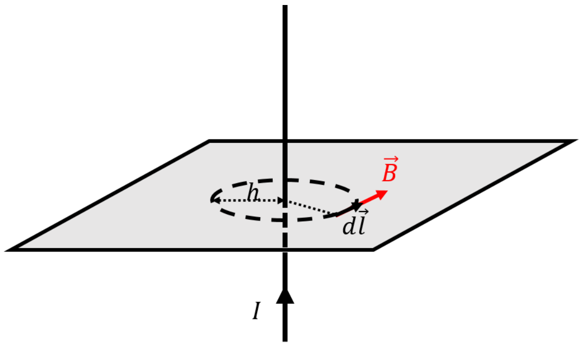

Figure 1.10: An Amperian loop that is a circle of radius, h, will allow us to determine the magnetic field at a distance, h, from an infinitely-long current-carrying wire.

radius, h, is given by:

$$\oint \vec{B} \cdot d\vec{l} = \oint Bdl\cos\theta = \cos\theta \oint Bdl = B\cos\theta \oint dl = B\cos\theta(2\pi h)$$

where $\cos\theta$ is 1 if the field forms circles (correct) or 0 if the field is radial (incorrect). We can now evaluate the current that is enclosed by the Amperian loop. The current that is enclosed is given by the net current that traverses the surface defined by the Amperian loop (in this case, a circle of radius h). Since the loop encloses the entire wire, the enclosed current is simply I. Applying Ampere's Law:

$$\begin{aligned}\oint \vec{B} \cdot d\vec{l} &= \mu_0 I^{enc}\\ B\cos\theta(2\pi h) &= \mu_0 I\end{aligned}$$

At this point, it is clear that $\cos\theta$ cannot be zero, since the right-hand side of the equation is not zero. We can thus conclude that the magnetic field must indeed make concentric circles, as we had previously determined. The magnitude of the magnetic field is given by:

$$B = \frac{\mu_0 I}{2\pi h}$$

as we found previously with the Biot-Savart Law. Again, in analogy with Gauss' Law, one needs to apply some knowledge of symmetry and argue in which direction the magnetic field should point, in order to use Ampere's Law effectively. Ampere's law proves that the magnetic field at the centre of a current-carrying loop is zero because there is no enclosed current:

True.

False. (correct) Let's compare Ampere's law to Gauss's law. Recall that Gauss's law is given by:

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q^{enc}}{\epsilon_0}.$$

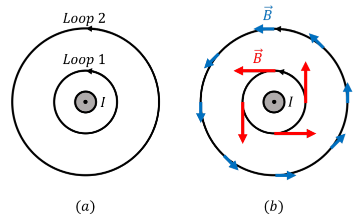

Figure 1.11: Left: 2 Amperian loops around a current carrying wire. Right: Magnetic field vectors around each loop.

We want to construct a Gaussian surface such that we can write this as

$$EA = \frac{Q^{enc}}{\varepsilon_0},$$

where A is the surface area that is perpendicular to the field lines and E is the (constant) magnitude of the electric field over the surface. The flux is given by Q^{enc}/ε_0. To find the electric field at any point on the surface, we use

$$E = \frac{Q^{enc}/\varepsilon_0}{A}.$$

In other words, the electric field is the flux per unit area. Ampere's law is very similar, except that instead of flux, we have the new concept of "circulation" and instead of a surface integral, we have a line integral. When we have a high degree of symmetry in Ampere's law, we are often able to write:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I^{enc} \rightarrow Bl = \mu_0 I^{enc}$$

where l is the length of the Amperian loop that is parallel to the field, B is the magnitude of the magnetic field, and $\mu_0 I^{enc}$ gives us the circulation. To find the magnetic field, we use

$$B = \frac{\mu_0 I^{enc}}{l},$$

so the magnetic field is like the circulation per unit length. In the Figure1.4, I have shown two Amperian loops with different radii, for the example of the straight wire with a current coming out of the page. Ampere's law tells us that the circulation will be the same along each path. However, because the inner loop is smaller, the circulation per unit length (the magnetic field) will be larger at each point. This is represented in the right panel, where the length of the vectors is proportional to the magnetic field strength.

A long solid uniform cable of radius, R, carries current, I, with a current density that is uniform through the cross-section of the cable. Determine the strength of the magnetic field as a function of r, the distance from the centre of the cable, inside *and* outside of the cable. In this case, we need to determine the magnetic field both inside and outside of the cable. Figure **??** shows two circular Amperian loops that we can use to apply Ampere's Law to

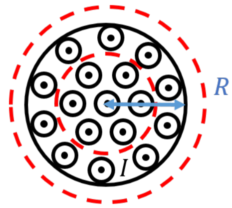

Figure 1.12: Two circular Amperian loops to determine the magnitude of the magnetic field inside and outside of a current-carrying cable of radius, R (with uniform current coming out of the page).

determine the magnetic field inside and outside of the cable. By symmetry, and following the discussion in this chapter, we know that the magnetic field must form concentric circles, both inside and outside of the cable. Outside the cable, we proceed in the same fashion as above, choosing an Amperian loop of radius, $r > R$, such that the circulation is given by:

$$\oint \vec{B} \cdot d\vec{l} = B2\pi r$$

The entire cable is enclosed by the loop, so that the enclosed current is, I. Thus, Ampere's Law gives:

$$\begin{aligned} \oint \vec{B} \cdot d\vec{l} &= \mu_0 I^{enc} \\ B(2\pi r) &= \mu_0 I \\ \therefore B &= \frac{\mu_0 I}{2\pi r} \quad (r \geq R) \end{aligned}$$

Inside of the cable, the circulation integral around a circular path of radius, $r < R$, is the same:

$$\oint \vec{B} \cdot d\vec{l} = B2\pi r$$

However, in this case, the smaller Amperian loop does not enclose all of the current flowing through the cable. We are told that the current density, j, is uniform in the cable. We can thus determine the current per unit area (i.e. the current density) that flows through the whole cable, and use that to determine how much current flows through the surface with area πr^2 that is defined by the Amperian loop:

$$\begin{aligned} j &= \frac{I}{A} = \frac{I}{\pi R^2} \\ \therefore I^{enc} &= j(\pi r^2) = \frac{I}{\pi R^2}(\pi r^2) = I\frac{r^2}{R^2} \end{aligned}$$

Finally, we can apply Ampere's Law to determine the magnitude of the magnetic field inside the cable:

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I^{enc}$$
$$B(2\pi r) = \mu_0 I \frac{r^2}{R^2}$$
$$\therefore B = \frac{\mu_0 I}{2\pi R^2} r$$

and we find that the magnetic field is zero at the centre of the cable ($r = 0$), and increases linearly up to the edge of the cable ($r = R$).

Discussion: In this example, we used Ampere's Law to model the strength of the magnetic field inside and outside of a current-carrying cable. In order to apply Ampere's Law inside the cable, we took into account that only a fraction of the current is enclosed by the Amperian loop. This problem is analogous to applying Gauss' Law to determine the electric field inside and outside of a uniformly charged sphere.

1.4.1 Interpretation of Ampere's Law and vector calculus

In this section, we discuss Ampere's Law in the context of vector calculus and provide a different perspective, mostly for informational purposes. The integral that appears in Ampere's Law is called the "circulation" of the vector field, $\vec{B}$:

$$\oint \vec{B}\cdot d\vec{l}$$

The circulation, as its name implies, is a measure of "how much rotation there is in the field". To visualize this, imagine that the vector field is a velocity field for points in a fluid. Regions of the fluid where there are little whirlpools (so called "eddies"), correspond to regions of the field with non-zero circulation (the sign of the integral tells us the direction of rotation, using the right-hand rule for axial vectors). Examples of field with and without circulation are shown in Figure 1.4.1. You will recognize that static electric charges create electric fields with no circulation (right panel), whereas static currents create magnetic fields with circulation. Ampere's Law is thus a statement that an electric current will result in a field with a magnitude proportional to the current, that has some degree of rotation to it. The direction of rotation of that field corresponds to the right-hand rule for axial vectors as applied to the current (your thumb points in the direction of the current so that your fingers curl in the direction of the rotation of the associated field).

Circulation, as defined by the integral over a closed loop, is not a local property of the field, since it depends on what the field is doing as a whole over the path of the loop. Just as one can obtain a "local" version of Gauss' Law, one can also obtain a local version of Ampere's Law using techniques from advanced vector calculus (that are beyond the scope of this textbook).

Stokes' theorem allows one to convert the circulation integral (a path integral on a closed

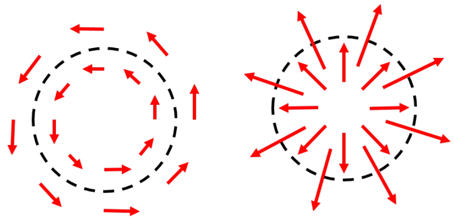

Figure 1.13: Examples of field with (left panel) and without (right panel) circulation, as evaluated along the closed loop shown with the dashed line.

loop) into a integral over the (open) surface that is defined by the loop:

$$\oint_C \vec{B}\cdot d\vec{l} = \int_S (\nabla\times\vec{B})\cdot d\vec{A}$$

where the subscript C indicates that the integral is over a one-dimensional path, whereas the subscript S indicates that the integral is over a two-dimensional surface. The term, $\nabla\times\vec{B}$, is called the "curl" of the magnetic field and is a local measure of the amount of rotation in the field. Applying Stokes' theorem to Ampere's Law yield:

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I^{enc}$$
$$\int_S (\nabla\times\vec{B})\cdot d\vec{A} = \mu_0 I^{enc}$$

Note that we can also write the current, I^{enc}, that is enclosed by the loop as the integral of the current density, $\vec{j}$, over the surface defined by the loop:

$$I^{enc} = \int_S \vec{j}\cdot d\vec{A}$$

Thus, we can write Ampere's Law with integrals over the same surface on either side of the equation, implying that the integrands must be the same:

$$\int_S (\nabla\times\vec{B})\cdot d\vec{A} = \mu_0 \int_S \vec{j}\cdot d\vec{A}$$
$$\therefore\quad \nabla\times\vec{B} = \mu_0\vec{j}$$

This last equation now relates a local property (current density) to the magnetic field at that point, and is the usual form in which Ampere's Law is presented (the so-called "differential form", rather than the "integral form").

The curl of the magnetic field, $\nabla\times\vec{B}$, is a vector that is given by the following:

$$\nabla\times\vec{B} = (B_z y - B_y z)\,\hat{x} + (B_x z - B_z x)\,\hat{y} + (B_y x - B_x y)\,\hat{z}$$

and the name "curl" is chosen because this is a measure of the amount of rotation (curl) in the field. In differential form, Ampere's Law can read as: "a current density will create a (magnetic) field that has non-zero curl".

Since Ampere's Law in differential form is a vector equation (both sides are vectors), it really corresponds to three equations in Cartesian coordinates, one per component. For example, the x component of the equation is a "partial differential equation" for the y and z components of the magnetic field:

$$(B_z y - B_y z) = \mu_0 j_x$$

that is in general difficult to solve without a computer (and all three equations are required, as these are "coupled", since a given component of the magnetic field appears in two of three equations).

1.5 Summary

Magnetic fields are created by moving charges. The Biot-Savart Law allows us to determine the infinitesimal magnetic field, $d\vec{B}$, that is produced by the current, I, flowing in an infinitesimal section of wire, $d\vec{l}$:

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l} \times \hat{r}}{r^2}$$

where μ_0 is a constant called the permeability of free space. The vector $\vec{r}$ points from the wire element, $d\vec{l}$, to the point at which we want to determine the magnetic field. In order to determine the magnetic field from a finite wire, one must sum (integrate) the contributions that come from each section of wire. It is often easier to work with the Biot-Savart law written without the unit vector, $\hat{r}$:

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l} \times \vec{r}}{r^3}$$

The magnetic field at a distance, h, from an infinitely long wire carrying current, I, is given by:

$$B = \frac{\mu_0 I}{2\pi h}$$

The magnetic field from a straight current-carrying wire forms concentric circles centred around the wire. The direction of the magnetic field is given by the right-hand rule for axial vectors; with the thumb pointing in the direction of current, the fingers curl in the direction of the magnetic field.

The magnitude of the magnetic field, a distance, h, from the centre of a circular loop of wire with radius, R, carrying current, I, along the axis of symmetry of the loop is given by:

$$B = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + h^2)^{\frac{3}{2}}}$$

The direction of the magnetic field can also be found using the right-hand rule for axial currents. In this case, if your fingers curl in the direction of the current loop, your thumb points in the same direction as the magnetic field at the centre of the loop.

Two parallel wires carrying currents, I_1 and I_2, separated by a distance, h, will exert equal and opposite forces on each other with a magnitude:

$$F = \frac{\mu_0 I_1 I_2}{2\pi h}$$

The force is attractive if the two currents flow in the same direction and repulsive otherwise.

Ampere's Law is the magnetism analogue to Gauss' Law. Just like Gauss' Law, it requires a high degree of symmetry to be applied analytically, although it is always valid. Ampere's Law relates the circulation of the magnetic field around a closed path to the current enclosed by that path:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I^{enc}$$

In order to apply Ampere's Law, we must first choose an Amperian loop over which to compute the closed path integral (instead of choosing a Gaussian surface to calculate the flux of the electric field on a closed surface). The circulation integral will be straightforward to evaluate if:

1. **The angle between $\vec{B}$ and $d\vec{l}$ is constant along the path**, so that:

$$\oint \vec{B} \cdot d\vec{l} = \oint Bdl \cos\theta = \cos\theta \oint Bdl$$

   where θ is the angle between $\vec{B}$ and $d\vec{l}$.
2. **The magnitude of $\vec{B}$ is constant along the path**, so that:

$$\cos\theta \oint Bdl = B\cos\theta \oint dl$$

The current enclosed, I^{enc}, corresponds to the net current that crosses the surface that is defined by the Amperian loop (a closed path always defines a surface).

Ampere's Law is straightforward to use in situations with a high degree of symmetry, such as infinitely long wires carrying current.

Solenoids are formed by combining many loops of current together, in order to form a strong and uniform magnetic field. The magnetic field inside of a solenoid has a magnitude of:

$$B = \mu_0 n I$$

where, I, is the current in the solenoid, and n, is the number of loops per unit length in the solenoid. The magnetic field just outside of a solenoid is zero, and generally, the magnetic field is negligible outside of a solenoid.

A toroid is formed by bending a solenoid into a circle to form a torus. The magnetic field lines inside of a toroid form concentric circles. The magnetic field decreases with radius inside of a toroid and is identically zero everywhere outside a toroid.

Biot-Savart law:

$$d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{d\vec{l}\times\hat{r}}{r^2} = \frac{\mu_0 I}{4\pi}\frac{d\vec{l}\times\vec{r}}{r^3}$$

Magnetic field from a finite wire:

$$B = \frac{\mu_0 I}{2\pi h}$$

Magnetic field from an infinitely long wire:

$$B = \frac{\mu_0 I}{2\pi h}$$

Magnetic field from a circular loop of current:

$$B = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+h^2)^{\frac{3}{2}}}$$

Force between two wires:

$$F = \frac{\mu_0 I_1 I_2}{2\pi h}$$

Ampere's law:

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I^{enc}$$

How does one make electricity with a hydroelectric dam?

By running water through a coil to induce a current.

By using water to rotate a coil inside of a fixed magnetic field. (correct)

By using water to charge a metallic surface by friction, and then maintaining that potential difference.

1.6 Faraday's Law

In the previous chapter, we described how an electric current produces a magnetic field. In this chapter, we describe how an electric current can be produced (or rather, "induced") by a magnetic field. The most important aspect of electromagnetic induction is that it always involves quantities that change with time. In past chapters, we have only dealt with static electric and magnetic fields, static charges (for electric fields), and static currents (for magnetic fields).

Faraday's law connects the flux of a **time-varying** magnetic field to an induced voltage (rather than a current). For historical reasons, the induced voltage is also called an induced "electromotive force" (emf), even if it is a voltage and not a force. Faraday's law is as follows:

$$\Delta V = -\frac{d\Phi_B}{dt}$$

where ΔV is the induced voltage, and Φ_B is the flux of the magnetic field through an open surface, defined in the same way as the flux of the electric field (Section **??**):

$$\Phi_B = \int_S \vec{B} \cdot d\vec{A}$$

If the magnetic field has a constant magnitude over the surface, S, and always makes the same angle with the surface, then the flux can be written as:

$$\Phi_B = \vec{B} \cdot \vec{A}$$

where the magnitude of the vector $\vec{A}$ is equal to the area of the surface, and the vector $\vec{A}$ is normal to the surface.

The surface, S, is defined by a closed path. The induced voltage can be thought of as an ideal battery placed in the closed path that defines the surface (right-hand panel of Figure **??**). The minus sign in Faraday's Law indicates the direction of the current associated with the induced voltage. It is important to note that an induced voltage only exists if the flux of the magnetic field changes (since the induced voltage is given by the time-derivative of the flux). Remember, induction is all about time-varying fields! This is better illustrated with an example.

Consider a loop of wire that is immersed in a uniform magnetic field, $\vec{B}$, that is perpendicular to the plane of the loop, as illustrated in Figure **??**. As time goes by, the magnetic field increases in strength, as shown in going from the left panel to the right panel. The flux of the magnetic field through the loop increases in magnitude, and a voltage is thus induced across the wire (illustrated by the ideal battery on the loop in the right panel), leading to an induced current, I. When calculating the flux of the magnetic field, we have to choose the surface element vector, $d\vec{A}$, to be perpendicular to the surface over which we calculate the flux. There are two choices (upwards or downwards, referring to Figure **??**); we **chose** to define $d\vec{A}$ to point upwards. Thus, the magnetic flux is positive in both panels, and increases with time. The derivative, $d\vec{B}/dt$, is positive and the right-hand side of Faraday's equation

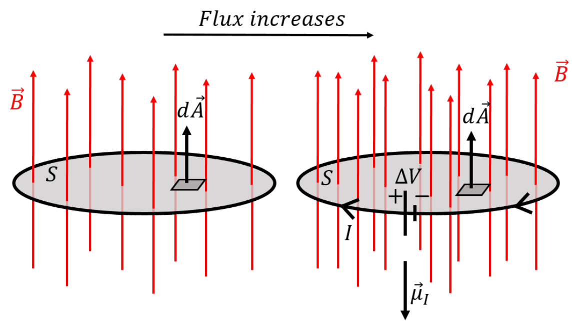

Figure 1.14: As the magnetic field increases, so does the flux through the loop that is shown. The changing flux results in an induced voltage, which produces an induced current that has a magnetic moment, $\vec{\mu}_I$. The induced current produces a magnetic field in a direction to oppose the changing flux.

is negative because of the negative sign in front. Had we chosen to define $d\vec{A}$ to point downwards, the right-hand side of Faraday's law would be negative.

We can describe the direction of the induced current, I, in terms of its magnetic dipole moment $\vec{\mu}_I$, also shown in Figure **??**. The overall sign on the right-hand side of Faraday's law is determined by our (arbitrary) choice of the direction $d\vec{A}$. With this choice, we found that the right-hand side of Faraday's law is negative:

$$\Delta V = -\frac{d\Phi_B}{dt} = \text{a negative number}$$

The overall sign of ΔV indicates whether the magnetic moment of the induced current is parallel (ΔV positive) or anti-parallel (ΔV negative) to $d\vec{A}$. This allows us to determine the direction of the induced current, and thus the direction of the ideal battery that represents the induced voltage. In general, when possible, it is common to choose the direction of $d\vec{A}$ to be parallel to the magnetic field vector, so that the flux is positive (although this does not guarantee that its derivative is positive).

1.6.1 Lenz's law

The minus sign in Faraday's law is sometimes called "Lenz's law", and ultimately comes from the conservation of energy. In Figure **??** above, we found that as the magnetic flux increases through the loop, a current is induced. That **induced current will also produce a magnetic field** (in the direction of its magnetic dipole moment vector, $\vec{\mu}_I$).

Lenz's law states that the "induced current will always be such that the magnetic field that it produces counteracts the changing magnetic field that induced the current". In Figure **??**, the magnetic field points in the upwards direction, and increases in magnitude with time. The induced current produces a magnetic field that points downwards to counteract the changing magnetic field, and preserve a constant flux through the loop. If this were not the case, the induced current would be in the opposite direction, contributing to the increasing magnetic flux through the loop, inducing more current, producing more flux, inducing more

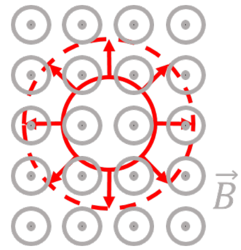

Figure 1.15: A loop whose radius increases with time.

current, etc. Clearly, this would lead to an infinite current and solve the world's energy crisis. Unfortunately, conservation of energy (expressed here as Lenz's law) prevents this from happening.

You can use Lenz's law to determine the direction of induced currents. In general:

- If the magnitude of the magnetic **flux is increasing** in the loop, then the induced current produces a magnetic field that is in the **opposite direction** from the original magnetic field.
- If the magnitude of the magnetic **flux is decreasing** in the loop, then the induced current produces a magnetic field that is in the **same direction** as the original magnetic field.

The negative sign in Faraday's law is not arbitrary (as we saw above, it gives the correct direction for the magnetic moment of the induced current, given our arbitrary choice of direction for $d\vec{A}$). In practice, one can often use Lenz's law to determine the direction of the induced current (so that it counteracts the changing flux), and Faraday's law to determine the magnitude of the induced voltage.

A loop of wire is immersed in a constant and uniform magnetic field out of the page, perpendicular to the plane of the loop, as shown in Figure 1.6.1. If the radius of the loop increases with time, in which direction will be the current induced in the loop?

Since the magnetic field is constant, there is no induced current.

Clockwise. (correct)

Counter-clockwise.

■ **Example 1.1** A uniform time-varying magnetic field is given by:

$$\vec{B}(t) = B_0(1+at)\hat{z}$$

where B_0 and a are positive constants. A coil, made of N circular loops of radius, r, lies in the $x-y$ plane. If the coil has a total resistance, R, what is the magnitude and direction of the current induced in the coil? The coil is made of N loops of wire. Each loop of wire can be treated independently, and each will have its own induced voltage across it. Since each loop is the same, they will all have the same induced voltage, and the total voltage induced across the coil, ΔV, will be given by:

$$\Delta V = -N\frac{d\Phi_B}{dt}$$

where Φ_B is the flux through any one of the loops. That is, each loop is similar to an ideal battery, and the coil is similar to placing all of these batteries in series, so that the voltages from each battery sum together.

The coil lies the $x-y$ plane, perpendicular to the increasing magnetic field, similar to the situation depicted in Figure **??**. Since the magnetic field is uniform over the surface of the coil, we do not need an integral to determine the flux. We define the area vector, $\vec{A}$, to be in the positive z direction (parallel to the magnetic field):

$$\vec{A} = A\hat{z} = \pi r^2 \hat{z}$$

The flux through one circular loop of radius, r, is given by:

$$\Phi_B(t) = \vec{B} \cdot \vec{A} = (B_0(1+at)\hat{z}) \cdot (\pi r^2 \hat{z}) = B_0(1+at)(\pi r^2)$$

We can apply Faraday's law to determine the induced voltage:

$$\begin{aligned}\Delta V &= -N\frac{d\Phi_B}{dt} = -N\frac{d}{dt}B_0(1+at)(\pi r^2)\\ &= -NB_0 a\pi r^2\end{aligned}$$

Since the induced voltage is negative, the magnetic moment of the induced current points in the negative z direction (opposite to our choice of direction for $\vec{A}$). This is consistent with Lenz's law, since the magnetic field increases in the positive z direction, the induced current will produce a magnetic field in the negative z direction to counteract the changing flux. The magnitude of the induced current is given by Ohm's Law:

$$I = \frac{\Delta V}{R} = \frac{NB_0 a\pi r^2}{R}$$

Discussion: In this example, we determined the induced voltage and current in a coil made of N identical loops. We argued that one can sum the induced voltages from the N loops, as these can be thought of as ideal batteries in series. We found that the direction of the induced current as obtained from Faraday's law was consistent with the expectation from Lenz's law. ■

1.7 Maxwell's equations and electromagnetic waves

This section is meant to be informative, as the material is beyond the scope of this textbook. Nonetheless, it is worth summarizing what we have learned about electricity and magnetism, as Maxwell did. We can summarize the main laws from electromagnetism as follows:

$$\oint \vec{E}\cdot d\vec{A} = \frac{Q}{\varepsilon_0} \qquad \text{(Gauss' Law)}$$
$$\oint \vec{B}\cdot d\vec{A} = 0 \qquad \text{(No magnetic monopoles)}$$
$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I^{enc} \qquad \text{(Ampere's Law)}$$
$$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{A} \qquad \text{(Faraday's law)}$$

where we wrote the magnetic flux in Faraday's law using the integral explicitly. As you recall, Gauss' Law is equivalent to Coulomb's Law, relating the electric field to electric charges that produce the electric field. Although we did not explicitly use the second equation, it is the equivalent to Gauss' Law for the magnetic field. The flux of the magnetic field out of a closed surface must always be zero, since there are no magnetic monopoles, so that magnetic field lines never end.

When we covered Ampere's Law, we only considered a static current as the source of the magnetic field. However, if there is an electric field present that is created by charges that are moving, then those can also contribute a current to Ampere's Law:

$$\oint \vec{E}\cdot d\vec{A} = \frac{Q}{\varepsilon_0} \quad \text{(Gauss' Law)}$$
$$\therefore Q = \varepsilon_0 \oint \vec{E}\cdot d\vec{A}$$
$$\therefore I = \frac{dQ}{dt} = \varepsilon_0 \frac{d}{dt}\oint \vec{E}\cdot d\vec{A}$$

so that Ampere's Law, in its most general form, is written:

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \left(I^{enc} + \varepsilon_0 \frac{d}{dt}\oint \vec{E}\cdot d\vec{A} \right) \quad \text{(Ampere's Law)}$$

Writing out the four equations again:

$$\oint \vec{E}\cdot d\vec{A} = \frac{Q}{\varepsilon_0} \qquad \text{(Gauss' Law)}$$
$$\oint \vec{B}\cdot d\vec{A} = 0 \qquad \text{(No magnetic monopoles)}$$
$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \left(I^{enc} + \varepsilon_0 \frac{d}{dt}\oint \vec{E}\cdot d\vec{A} \right) \qquad \text{(Ampere's Law)}$$
$$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{A} \qquad \text{(Faraday's law)}$$

These four equations are known as Maxwell's equations, and form our most complete theory of classical electromagnetism. It is quite interesting to note the similarities and relations between the electric and magnetic field. Maxwell's equations contain equations for the circulation and the total flux out of a closed surface for both fields. Ampere's Law implies that a changing electric field will produce a magnetic field. Faraday's law implies that a changing magnetic field produces an electric field. If a point charge oscillates up and down, it will produce a changing electric field, which will produce a changing magnetic field, which will induce a changing magnetic field, etc. This is precisely what an electromagnetic wave is! The light that we see, the Wi-Fi signals for our phones, and the highly penetrating radiation from nuclear reactors are all examples of electromagnetic waves (of different wavelengths).

In fact, as Maxwell did, we can obtain the wave equation from Maxwell's equations. We sketch out the derivation here, but it is definitely beyond the scope of this textbook. However, you're so close to seeing one of the most exciting revelations of physics that it would be a shame to skip it!

We first write out Maxwell's equations in differential form, as we have already shown for Gauss' Law and Ampere's Law

$$\begin{aligned}
\nabla\cdot\vec{E} &= \frac{\rho}{\varepsilon_0} && \text{(Gauss' Law)}\\
\nabla\cdot\vec{B} &= 0 && \text{(No magnetic monopoles)}\\
\nabla\times\vec{B} &= \mu_0\left(\vec{j}+\varepsilon_0\vec{E}t\right) && \text{(Ampere's Law)}\\
\nabla\times\vec{E} &= -\vec{B}t && \text{(Faraday's law)}
\end{aligned}$$

If we consider a vacuum region in space, with no charges and no currents, these equations reduce to:

$$\begin{aligned}
\nabla\cdot\vec{E} &= 0 & \nabla\cdot\vec{B} &= 0\\
\nabla\times\vec{B} &= \mu_0\varepsilon_0\vec{E}t & \nabla\times\vec{E} &= -\vec{B}t
\end{aligned}$$

We will make use of the following identity from vector calculus:

$$\nabla\times(\nabla\times\vec{E}) = \nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E}$$

where:

$$\begin{aligned}
\nabla^2\vec{E} &= \frac{\partial^2\vec{E}}{\partial x^2}+\frac{\partial^2\vec{E}}{\partial y^2}+\frac{\partial^2\vec{E}}{\partial z^2}\\
&= \left(\frac{\partial^2 E_x}{\partial x^2}+\frac{\partial^2 E_x}{\partial y^2}+\frac{\partial^2 E_x}{\partial z^2}\right)\hat{x}+\left(\frac{\partial^2 E_y}{\partial x^2}+\frac{\partial^2 E_y}{\partial y^2}+\frac{\partial^2 E_y}{\partial z^2}\right)\hat{y}\\
&+\left(\frac{\partial^2 E_z}{\partial x^2}+\frac{\partial^2 E_z}{\partial y^2}+\frac{\partial^2 E_z}{\partial z^2}\right)\hat{z}
\end{aligned}$$

is called the "vector Laplacian".

Consider taking the curl ($\nabla\times$) of the equation that has the curl of the electric field (Faraday's law):

$$\begin{aligned}
\nabla\times\left(\nabla\times\vec{E}=-\vec{B}t\right)\\
\nabla(\nabla\cdot\vec{E})-\nabla^2\vec{E}&=-\nabla\times\vec{B}t\\
-\nabla^2\vec{E}&=-t\nabla\times\vec{B}\\
-\nabla^2\vec{E}&=-t\mu_0\varepsilon_0 Et\\
-\nabla^2\vec{E}&=-\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}
\end{aligned}$$

where, in the third line, we made use of Gauss' Law ($\nabla\cdot\vec{E}=0$), and, in the fourth line, Ampere's Law ($\nabla\times\vec{B}=\mu_0\varepsilon_0 Et$). The last equation that we obtained is a vector equation (the vector Laplacian has three components, as does the time-derivative of $\vec{E}$ on the right-hand side). Consider the x component of this equation:

$$\frac{\partial^2 E_x}{\partial x^2}+\frac{\partial^2 E_x}{\partial y^2}+\frac{\partial^2 E_x}{\partial z^2}=\mu_0\varepsilon_0\frac{\partial^2 E_x}{\partial t^2}$$

If we define the quantity:

$$c=\frac{1}{\sqrt{\varepsilon_0\mu_0}}$$

then, the x component of the equation can be written as:

$$\frac{\partial^2 E_x}{\partial x^2}+\frac{\partial^2 E_x}{\partial y^2}+\frac{\partial^2 E_x}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2}$$

which is exactly the wave equation for the component, E_x, of the electric field, propagating with speed c, the speed of light! Thus, the speed of light is directly related to the constants ε_0 and μ_0. You can write out similar equations for the y and z components of the electric field, and find the similar equations for the magnetic field if you start by taking the curl of Ampere's Law instead of Faraday's law.

We have just shown that electric and magnetic fields can behave as waves, which we now understand to be the waves that are responsible for light, radio waves, gamma rays, infra-red radiation, etc. All of these are types of electromagnetic waves with different frequencies. Although we did not demonstrate this, the electromagnetic waves that propagate are such that the magnetic and electric field vectors are always perpendicular to each other. Electromagnetic waves also carry energy. Thus, a charge that is oscillating (say on a spring) and creating an electromagnetic wave must necessarily be losing energy (or work must be done to keep the charge oscillating with the same amplitude). Finally, it is worth noting that, according to Quantum Mechanics, light (and the other frequencies of radiation), are really carried by particles called "photons". Those particles are strange, since their propagation is described by a wave equation.

1.8 Summary

Faraday's law connects a **changing** magnetic flux to an induced voltage:

$$\Delta V = -\frac{d\Phi_B}{dt}$$

The magnetic flux, Φ_B, is calculated as the flux of the magnetic field through an open surface, S:

$$\Phi_B = \int_S \vec{B} \cdot d\vec{A}$$

The induced voltage, ΔV, is the potential difference that is induced along the closed path (a "loop") that bounds the surface, S. If a charge, q, were to move around that closed path, it would gain (or lose) energy, $q\Delta V$. Note that the potential difference that is induced corresponds to a non-conservative electric force, as a charge can gain/lose energy by moving along a closed path. The induced voltage is often called an induced electromotive force (emf), even if it is a voltage.

The minus sign in Faraday's law is sometime referred to as "Lenz's law", since it indicates in which direction the induced voltage will be. It is easiest to think of the closed path as a physical wire (e.g. a loop of wire) through which a current will be induced as a result of the induced voltage. The minus sign is easiest to interpret in terms of the relative direction between the area vector used to define the flux, and the magnetic dipole moment vector, $\vec{\mu}$, associated with the induced current (which points in the same direction as the magnetic field that is produced by the induced current).

When calculating the flux of the magnetic field, the surface element vector $d\vec{A}$, must be perpendicular to the surface through which the flux is calculated, which leads to two possible choices. Once a choice is made, and Faraday's law has been applied, the sign of ΔV will indicate if the magnetic dipole moment of the induced current points in the same direction as $d\vec{A}$ (positive ΔV) or in the opposite direction (negative ΔV).

If N loops of wire are combined together into a coil, the voltages across each loop sum together, so that the voltage induced across the coil is given by:

$$\Delta V = -N\frac{d\Phi_B}{dt}$$

Lenz's law is a statement about conservation of energy. Indeed, the induced current must create a magnetic field that **opposes** the change in flux, otherwise, the induced current would grow indefinitely. Lenz's law can be summarized as follows:

- If the magnitude of the magnetic **flux is increasing** in the loop, then the induced current produces a magnetic field that is in the **opposite direction** from the original magnetic field.
- If the magnitude of the magnetic **flux is decreasing** in the loop, then the induced current produces a magnetic field that is in the **same direction** as the original magnetic field.

A voltage is induced along a closed path any time that the flux of the magnetic field through the corresponding surface changes. The flux can change either because the magnetic field is changing, or because the loop is changing (in size or orientation relative to the magnetic field). In the latter case (changing loop), one speaks of a "motional emf". A generator creates a motional emf by rotating a coil (with N loops, each with area, A), inside a fixed uniform magnetic field, $\vec{B}$. The voltage produced by a generator is given by:

$$\Delta V = NAB\omega \sin(\omega t)$$

where ω is the angular speed of the coil. A generator thus produces alternating voltage/current. The current that is induced in the coil of the generator will dissipate energy as it flows through a resistance, R. Thus, one must do work in order to keep the generator spinning. The current induced in the coil of the generator will also result in a magnetic moment, and a "counter torque" will be exerted on the coil. One must thus exert a torque in order to keep the generator spinning (and the work done by exerting that torque is converted into the electrical energy dissipated in the resistor). The counter torque on the generator is always in the same direction, and has a magnitude:

$$\tau = \frac{NA^2B^2\omega \sin^2(\omega t)}{R}$$

When an electric motor is used, a "back emf" is induced in the coil of the motor. The back emf is such that it resists the direction of current (Lenz's law), or else the motor would spin infinitely fast. As the motor spins faster, the back emf grows, until it reaches an equilibrium. Motors thus draw a large current when they first start up, since at low speed, they have no back emf.

Since a changing magnetic flux induces a voltage, an electric field is also induced. We can replace the voltage in Faraday's law with the circulation of the electric field to write a more general version of Faraday's law:

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

The induced electric field forms closed field lines, and is different than the electric field that is produced by static charges, since the latter will have field lines that start and end on charges. The force associated with the induced electric field is not conservative.

When a metallic object passes through a region of magnetic field, the induced electric field will induce current loops in the material called eddy currents. The magnetic field will also exert a force on these eddy currents to oppose the motion that is creating the currents (Lenz's law); as the eddy currents dissipate electrical energy in the material, the metallic object must lose kinetic energy unless a force is acting on it. Magnetic brakes make use of this principle.

Transformers are used to convert an alternating voltage, ΔV_p, into a different alternating voltage, ΔV_s. A "primary" coil, with N_p windings, creates a changing magnetic flux that is guided (e.g. by an iron core) to a "secondary" coil, with N_s windings. The voltage induced in the secondary coil is given by:

$$\Delta V_s = \frac{N_p}{N_s} \Delta V_p$$

Maxwell's four equations form our best classical theory of electromagnetism. Those equations imply that a changing magnetic field produces an electric field (Faraday's law), while a changing electric field can produce a magnetic field (Ampere's Law). By combining Maxwell's equation (with some heavy vector calculus), one can show that this leads to the formation of electromagnetic waves, that propagate with a speed, c, given by:

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

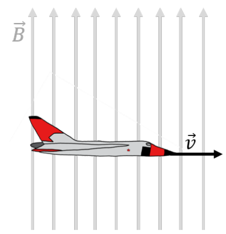

Figure 1.16: The Avro Arrow moving through a magnetic field.

1.8.1 example

In the 1950s, the Royal Canadian Air Force developed a jet airplane called the Avro Arrow. This jet reached a speed of Mach 1.9 ($652ms^{-1}$), and was considered one of the most advanced airplanes that existed at the time. Suppose that the Avro Arrow is travelling at a velocity of $v = 652ms^{-1}$ above the South Pole through Earth's vertical magnetic field, $B = 5.2e-5T$, as shown in Figure 1.8.1. If the Avro Arrow had a wingspan of $l = 15m$, determine the induced voltage across its wings. This is identical to the motional emf that is generated by a bar moving in a magnetic field. As the airplane moves as illustrated (towards the left, in an upwards magnetic field), the electrons in the wing of the airplane will be pushed into the page. Eventually, the electric field from the electrons will prevent further electrons from accumulating at that side of the wing, and there will be a constant (Hall) voltage, ΔV, across the wing tips. This will happen when the magnetic and electric force are equal and opposite:

$$qvB = qE = q\frac{\Delta V}{L}$$

where L is the wingspan of the airplane. The induced potential is thus given by:

$$\Delta V = BLv = (5.2e-5T)(15m)(652ms^{-1}) = 0.51V$$

1.8.2 example

A generator is made of N circular loops of radius $R = 0.3m$, rotating at a frequency of $f = 60Hz$ in a uniform magnetic field, $B = 0.1T$. How many coils must the generator have in order for it to produce an alternating voltage with a maximum amplitude of $\Delta V = 110V$. The voltage produced by a generator is given by:

$$\Delta V = NAB\omega \sin(\omega t)$$

and the angular frequency is given by $\omega = 2\pi f$. The number of required coils is thus:

$$N = \frac{\Delta V}{AB\omega} = \frac{\Delta V}{\pi R^2 B 2\pi f} = \frac{(110V)}{2\pi^2(0.3m)^2(0.1T)(60Hz)} = 10.3$$

Thus, one requires 10 loops in the coil to generate the desired voltage.

Wish you all the best, Tanay Ghosh