BERNSTEIN-BEZIER BASES FOR TETRAHEDRAL FINITE ELEMENTS

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Local Spaces and Their Basis (Table 2)

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\begin{aligned}
S_n^E &= span\{\frac{n!}{\alpha_i!\alpha_j!}\lambda_i^{\alpha_i}\lambda_j^{\alpha_j}\} \\
&= span\{B_\alpha^n\} \\
\nabla S_n^E &= span\{\frac{n!}{\alpha_i!\alpha_j!}(\alpha_i\lambda_i^{\alpha_i-1}\lambda_j^{\alpha_j}\nabla\lambda_i + \alpha_j\lambda_i^{\alpha_i}\lambda_j^{\alpha_j-1}\nabla\lambda_j)\} \\
&= span\{B_{\alpha-1}^{n-2}\cdot n(n-1)\cdot(\frac{\lambda_j\nabla\lambda_i}{\alpha_j}+\frac{\lambda_i\nabla\lambda_j}{\alpha_i})\} \\
& where\ E=\{x_i,x_j\}\ and\ \alpha\in\mathbb{Z}_+^4:\alpha_k>0\ if\ k\in\{i,j\},\ else\ \alpha_k=0 \\
S_n^F &= span\{\frac{n!}{\alpha_i!\alpha_j!\alpha_k!}\lambda_i^{\alpha_i}\lambda_j^{\alpha_j}\lambda_k^{\alpha_k}\} \\
&= span\{B_\alpha^n\} \\
\nabla S_n^F &= span\{\frac{n!}{\alpha_i!\alpha_j!\alpha_k!}(\alpha_i\lambda_i^{\alpha_i-1}\lambda_j^{\alpha_j}\lambda_k^{\alpha_k}\nabla\lambda_i + \alpha_j\lambda_i^{\alpha_i}\lambda_j^{\alpha_j-1}\lambda_k^{\alpha_k}\nabla\lambda_j + \alpha_k\lambda_i^{\alpha_i}\lambda_j^{\alpha_j}\lambda_k^{\alpha_k-1}\nabla\lambda_k)\} \\
&= span\{B_{\alpha-1}^{n-3}\cdot n(n-1)(n-2)\cdot(\frac{\lambda_j\lambda_k\nabla\lambda_i}{\alpha_j\alpha_k}+\frac{\lambda_i\lambda_k\nabla\lambda_j}{\alpha_i\alpha_k}+\frac{\lambda_i\lambda_j\nabla\lambda_k}{\alpha_i\alpha_j})\} \\
& where\ F=\{x_i,x_j,x_k\}\ and\ \alpha\in\mathbb{Z}_+^4:\alpha_l>0\ if\ l\in\{i,j,k\},\ else\ \alpha_l=0\ and\ \sum_{l=1}^4\alpha_l=n \\
S_n^T &= span\{\frac{n!}{\alpha_1!\alpha_1!\alpha_3!\alpha_4!}\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\lambda_3^{\alpha_3}\lambda_4^{\alpha_4}\} \\
&= span\{B_\alpha^n\} \\
\nabla S_n^T &= span\left\{\begin{aligned}&\frac{n!}{\alpha_1!\alpha_1!\alpha_3!\alpha_4!}(\alpha_1\lambda_1^{\alpha_1-1}\lambda_2^{\alpha_2}\lambda_3^{\alpha_3}\lambda_4^{\alpha_4}\nabla\lambda_1+\alpha_2\lambda_1^{\alpha_1}\lambda_2^{\alpha_2-1}\lambda_3^{\alpha_3}\lambda_4^{\alpha_4}\nabla\lambda_2\\ &\qquad+\alpha_3\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\lambda_3^{\alpha_3-1}\lambda_4^{\alpha_4}\nabla\lambda_3+\alpha_4\lambda_1^{\alpha_1-1}\lambda_2^{\alpha_2}\lambda_3^{\alpha_3}\lambda_4^{\alpha_4-1}\nabla\lambda_4)\end{aligned}\right\} \\
&= span\left\{\begin{aligned}&B_{\alpha-1}^{n-4}\cdot n(n-1)(n-2)(n-3)\cdot(\frac{\lambda_2\lambda_3\lambda_4\nabla\lambda_1}{\alpha_2\alpha_3\alpha_4}+\frac{\lambda_1\lambda_3\lambda_4\nabla\lambda_2}{\alpha_1\alpha_3\alpha_4}\\ &\qquad+\frac{\lambda_1\lambda_2\lambda_4\nabla\lambda_3}{\alpha_1\alpha_2\alpha_4}+\frac{\lambda_1\lambda_2\lambda_3\nabla\lambda_4}{\alpha_1\alpha_2\alpha_3})\end{aligned}\right\} \\
& where\ T=conv(x_1,x_2,x_3,x_4)\ and\ \alpha\in\mathbb{Z}_+^4:\alpha_i>0\ \forall i\in\{1,2,3,4\}\ and\ \sum_{l=1}^4\alpha_l=n
\end{aligned}
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\begin{aligned}
E_n^F &= span\{\Phi_\alpha^{FT,n}\}\\
&= span\{(n+1)\cdot B_\alpha^n \cdot (\alpha_i(\lambda_j\nabla\lambda_k - \lambda_k\nabla\lambda_j) + \alpha_j(\lambda_i\nabla\lambda_k - \lambda_k\nabla\lambda_i) + \alpha_k(\lambda_i\nabla\lambda_j - \lambda_j\nabla\lambda_i))\}\\
\nabla\times E_n^F &= span\{\nabla\times\Phi_\alpha^{FT,n}\}\\
&\quad where\ F = conv(x_i,x_j,x_k)\ and\ \alpha\in\mathbb{Z}_+^3 : \alpha_l = 0\ \forall l\notin\{i,j,k\}\ and\ \sum_{l=1}^4 \alpha_l = n\\
E_n^T &= span \oplus_{l=1}^2 \{\Psi_{l,\alpha}^{T,n}\}\oplus\{\Psi_{3,\alpha}^{T,n} : \alpha_3 = 1\}\\
&= span\left\{\begin{array}{l}\{(n+1)B_{\alpha-e_1}^n\nabla\lambda_1 - \dfrac{\alpha_1}{n+1}\nabla B_\alpha^{n+1}\}\\ \oplus\{(n+1)B_{\alpha-e_2}^n\nabla\lambda_2 - \dfrac{\alpha_2}{n+1}\nabla B_\alpha^{n+1}\}\\ \oplus\{(n+1)B_{\alpha-e_3}^n\nabla\lambda_3 - \dfrac{1}{n+1}\nabla B_\alpha^{n+1} : \alpha_3 = 1\}\end{array}\right\}\\
\nabla\times E_n^T &= span\oplus_{l=1}^2\{\nabla\times\Psi_{l,\alpha}^{T,n}\}\oplus\{\nabla\times\Psi_{3,\alpha}^{T,n} : \alpha_3 = 1\}\\
&\quad where\ F = \{x_i,x_j,x_k\}\ and\ \alpha\in\mathbb{Z}_+^4 : \alpha_l > 0\ if\ l\in\{i,j,k\},\ else\ \alpha_l = 0\ and\ \sum_{l=1}^4\alpha_l = n+1
\end{aligned}
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\begin{aligned}
V_n^T &= span\{\Upsilon_\alpha^n\}\\
&= span\{(n+1)\cdot B_\alpha^n\sum_{l=1}^4(-1)^l\alpha_l\chi_l\}, where\\
\chi_l &= \lambda_i\nabla\lambda_j\times\nabla\lambda_k - \lambda_j\nabla\lambda_i\times\nabla\lambda_k + \lambda_k\nabla\lambda_i\times\nabla\lambda_j\\
\nabla\cdot V_n^T &= span\{\nabla\cdot\Upsilon_\alpha^n\}\\
&\quad where\ \alpha\in\mathbb{Z}_+^4 : \sum_{l=1}^4\alpha_l = n
\end{aligned}
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