Rahul Bhadani
Course Code: CSCI 5352
NetID: rahul.bhadani
Instructor: Prof. Jeana Ginning

Q1) a)

Definition of convolution 1D:

$$y[n,m] = g[n,m] * h[n,m]$$
$$= \sum_{k=-\infty}^{\infty} g[k] \sum_{l=-\infty}^{\infty} h[l] \times [n-k, m-l]$$

Convolution is commutative:

$$g[n,m] * h[n,m] = h[n,m] * g[n,m]$$
$$\therefore \sum_{k=-\infty}^{\infty} g[k] \sum_{l=-\infty}^{\infty} h[l] \times [n-k, m-l]$$
$$= \sum_{l=-\infty}^{\infty} h[l] \sum_{k=-\infty}^{\infty} g[k] \times [n-k, m-l]$$

Eq. 1 separable to $(n \times n)$ and $(1 \times m)$,

$$h[n,m] = h_1[n] \times h_2[m]$$

$$\therefore g[n,m] * h[n,m] = \sum_{k=-\infty}^{\infty} g[k] \sum_{l=-\infty}^{\infty} h[l] \times [n-k, m-l]$$
$$= \sum_{k=-\infty}^{\infty} g[k] \sum_{l=-\infty}^{\infty} h_1[l] \times h_2[l] \times [n-k, m-l]$$
$$= \sum_{k=-\infty}^{\infty} g[k] \left(\sum_{l=-\infty}^{\infty} h_1[l] \times [n-k] \right) \times h_2[m-l]$$

Definition of convolution 1D:

$$y[n] = g[n] * h_1[n]$$
$$= \sum_{k=-\infty}^{\infty} g[k] \times h_1[n-k]$$

The separable 2D convolution is performed twice of 1D convolution in horizontal and vertical dimension.

$$g[n,m] = (g[n] * h_1[n]) * h_2[m]$$
$$= (\sum_{k=-\infty}^{\infty} g[k] \times h_1[n-k]) * h_2[m]$$
$$= \sum_{l=-\infty}^{\infty} h_2[l] \times (\sum_{k=-\infty}^{\infty} g[k] \times h_1[n-k]) \times [m-l]$$

Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$
Input Kernel

(1) Using convolution 2D

$$g[n,m] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} * \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$
$$= 1 \times 1 + 2 \times 2 + 3 \times 1 + 4 \times 0 + 5 \times 0 + 6 \times 0 + 7 \times -1 + 8 \times -2 + 9 \times -1$$
$$= 1 + 4 + 3 + 0 + 0 + 0 - 7 - 16 - 9$$
$$= 4 - 32 = -28$$
$$= \mathbf{20} \ //$$

(2) Using separable convolution

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} * \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} * \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$
Separable kernel

Horizontal convolution 1D

$$g[n,m] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} * \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \times 1 + 2 \times 0 + 3 \times -1 & 1 \times 2 + 2 \times 0 + 3 \times -2 & 1 \times 3 + 2 \times 0 + 3 \times -1 \\ 4 \times 1 + 5 \times 0 + 6 \times -1 & 4 \times 2 + 5 \times 0 + 6 \times -2 & 4 \times 3 + 5 \times 0 + 6 \times -1 \\ 7 \times 1 + 8 \times 0 + 9 \times -1 & 7 \times 2 + 8 \times 0 + 9 \times -2 & 7 \times 3 + 8 \times 0 + 9 \times -1 \end{bmatrix}$$
$$= \begin{bmatrix} -2 & -4 & -2 \\ -2 & -4 & -2 \\ -2 & -4 & -2 \end{bmatrix}$$

Vertical convolution 1D

$$g[n,m] = \begin{bmatrix} -2 & -4 & -2 \\ -2 & -4 & -2 \\ -2 & -4 & -2 \end{bmatrix} * \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} -2 \times 1 + -4 \times 2 + -2 \times 1 & -2 \times 1 + -4 \times 2 + -2 \times 1 & -2 \times 1 + -4 \times 2 + -2 \times 1 \\ -2 \times 1 + -4 \times 2 + -2 \times 1 & -2 \times 1 + -4 \times 2 + -2 \times 1 & -2 \times 1 + -4 \times 2 + -2 \times 1 \\ -2 \times 1 + -4 \times 2 + -2 \times 1 & -2 \times 1 + -4 \times 2 + -2 \times 1 & -2 \times 1 + -4 \times 2 + -2 \times 1 \end{bmatrix}$$
$$= \begin{bmatrix} -8 & -16 & -8 \\ -8 & -16 & -8 \\ -8 & -16 & -8 \end{bmatrix}$$

Q1) b) Given: Kernel image and (1×3) + (3×1) kernel.

Horizontal convolution and vertical convolution are commutative.

The separable kernel can be decomposed into 1 column and 1 row kernel. Hence, separable kernel is the eq. with kernel.

Total no. of multiplications eq. 1D convolution $= 3 \times 3 = 9$

The no. of multiplications eq. 2D convolution $= 3 \times 3 \times 3 \times 3 = 81$

$\therefore$ Total operations $= 2 \times (3 \times 3) \times 3 \times 3 = 162$

Therefore eq. separable kernel is

Total operations eq. 2D convolution $= 81$ (2D convolution)

$$= (3 \times 3)^2 \left(\frac{(3 \times 3) + (3 \times 3)}{(3 \times 3)^2} \right) = 3 \times (3 \times 3) \times \left(\frac{(3 \times 3) + (3 \times 3)}{(3 \times 3)} \right)$$
$$= (3 \times 3) \times (3 \times 3) \times \left(\frac{(3 \times 3) + (3 \times 3)}{(3 \times 3)} \right)$$
$$= (3 \times 3) \times (3 \times 3) \times \left(\frac{(3 \times 3) + (3 \times 3)}{(3 \times 3)} \right)$$
$$= (3 \times 3) \times (3 \times 3) \times \left(\frac{(3 \times 3) + (3 \times 3)}{(3 \times 3)} \right) \ //$$

Q2) a) A gaussian function is eq. the form

$$g(x) = ae^{-\frac{(x-b)^2}{2c^2}}, \text{ where } b \in \mathbb{R}, c \in \mathbb{R}^+, a \in \mathbb{R}$$

$\therefore$ it is even and $\sigma^2 = c^2$ variance.

The convolution eq. two gaussian is a gaussian.

Let mean, variance, transition $\frac{1}{2}$ eq. a gaussian g is a gaussian. Hence, the product gg eq. gaussian is sub gaussian. g is a gaussian.

Suppose $g(x)$ is a separable gaussian in two variables eq. the form

$$g(x,y) = ae^{-\frac{(x-b)^2}{2c^2}} \times ae^{-\frac{(y-b)^2}{2c^2}} \times a$$

Let $\frac{1}{2}\sigma = \frac{1}{2}c$

$$\therefore g(x,y) = ae^{-\frac{(x-b)^2}{2\sigma^2}} \times ae^{-\frac{(y-b)^2}{2\sigma^2}}$$
$$\therefore g(x,y) = a^2 e^{-\frac{(x-b)^2}{2\sigma^2} - \frac{(y-b)^2}{2\sigma^2}}$$
$$= a^2 e^{-\frac{(x-b)^2 + (y-b)^2}{2\sigma^2}}$$
$$\therefore g(x,y) = a^2 e^{-\frac{(x-b)^2 + (y-b)^2}{2\sigma^2}}$$

Now, let $\sigma = \frac{1}{2}c$

$$\therefore g(x,y) = a^2 e^{-\frac{(x-b)^2 + (y-b)^2}{2\left(\frac{1}{2}c\right)^2}}$$
$$= a^2 e^{-\frac{(x-b)^2 + (y-b)^2}{\frac{1}{2}c^2}}$$
$$= a^2 e^{-\frac{2((x-b)^2 + (y-b)^2)}{c^2}}$$

As output was from a to a^2, we want a to be equal to the value of a^2.

$\therefore \frac{1}{2} \times g(x) = \frac{1}{2} \times e^{-\frac{(x-b)^2}{2c^2}}$ which is a gaussian, as the output yields a constant c.

Q2) b) Downscaling reduction refers to techniques for reducing the number of input variables in training data. The higher the number of features, the harder it gets to work on it. Also, some features may be redundant or correlated to each other such as underdetermined.

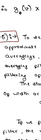
When working with multiple images, dimensionality could be an issue and therefore we would be applying huge amounts of storage. To solve this issue, dimensionality reduction methods such as PCA can be used to compress the images and preserve the important data or features as much as possible. Therefore, we can apply such image preservation to non-important application as dimensionality reduction is to invert to best memory and improve speed of execution for image processing. Dimensionality reduction methods can also be used as dimensionality image transformation such as vectors and images.

The disadvantages of using dimensionality reduction methods for image processing techniques could be lead to important features which is the trade-off for dimension reduction.

Q3) a) Original binary displacement

Objects: $\Omega = \mathbb{C}^2$

map: $\Omega \to \mathbb{C}^2$



i) A displacement is a transformation eq. points.

Transformation eq. of points indicates an action (g_1) on vectors. For a rigid body displacement, the following properties behave points i.e.

$$\|(g(p) - g(q))\| = \|(p-q)\|$$

g_1 acts on vectors

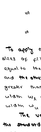
i. Vector $(p-q)$ can be written as

$$g(p) - g(q) \text{ and as } g_1(p-q)$$

$$\therefore \|g_1(p-q)\| = \|(p-q)\|$$

but as $(p-q) \in \mathbb{C}^2$

$$\therefore \|g_1(p-q)\| = \|(p-q)\| \quad \text{— (1)}$$



Since,

$$\|g_1(p-q)\| = \|(p-q)\|$$
$$\|g_1(p)\| = \|(p)\|$$

ii) A rigid body transformation preserves angles as well as distances.

The angle between vectors is the time at the first coordinate is 0 and so on as the angles are preserved. The cross product $g_1(p) \times g_1(q)$ keeps angles and hence the transformation are preserved.

$$\therefore g_1(p) \times g_1(q) = g_1(p \times q) \quad \text{— (2)}$$

Q3) b) To determine the equations by obtaining approximate Gaussian filtering through multiple averagings, we need to determine the average averaging filter required to approximate Gaussian filtering by our desired standard deviation.

The standard deviation eq. averaging filter eq. width w is given by:

$$\sigma_{avg} = \sqrt{\frac{w^2 - 1}{12}} \quad \text{— (1)}$$

We can perform this local width eq. the averaging filter eq. eq. (1).

$$\Rightarrow \sigma_{avg}^2 = \frac{w^2 - 1}{12}$$
$$\Rightarrow 12\sigma_{avg}^2 = (w^2 - 1)$$
$$\Rightarrow \frac{12\sigma_{avg}^2}{w^2 - 1} = 1$$
$$\Rightarrow w^2 - \frac{12\sigma_{avg}^2}{w^2 - 1} = 0$$
$$\Rightarrow w_{local} = \sqrt{\frac{12\sigma_{avg}^2}{w^2 - 1}} \quad \text{— (3)}$$

To apply the averaging filter, we can use back-back eq. filter. The given filter width w_{local} will be equal to the first and larger less than ideal. Let the w_{local} equal to the first and larger integer. Hence, we will apply filter eq. width w_{local} eq. times. We can possibly have with a filter eq. width w_{local} for $(w+1)$ times.

The variance eq. filter's sum, moreover, the desired deviation eq. the equivalent filter, will be:

$$\sigma^2 = \frac{w_{local}^2 + (w_{local}+1)^2}{12} \quad \text{— (4)}$$

From eq. (3), we can solve for w:

$$\sigma^2 = \frac{w_{local}^2 + (w_{local}+1)^2}{12}$$
$$\Rightarrow 12\sigma^2 = w_{local}^2 + (w_{local}+1)^2 = w_{local}^2 + (w_{local}+1)^2$$
$$\Rightarrow 12\sigma^2 = (w_{local}^2 + w_{local}^2 + 2w_{local} + 1)$$
$$\Rightarrow 12\sigma^2 = (2w_{local}^2 + 2w_{local} + 1) = 2(w_{local}^2 + w_{local}) + 1$$
$$\Rightarrow m = \frac{12\sigma^2 - w_{local}^2 - w_{local} - 1}{2} \quad \text{— (5)}$$

In summary, the approximate Gaussian filtering, given σ and w:

→ Calculate local and least w_{local} and w_{local}

→ Calculate m

→ Apply mean filtering eq. width w_{local} eq. m times

→ Apply mean filtering eq. width w_{local} eq. $(w_{local}+1)$ times

To validate this equation, we can take an example, let $\sigma_{avg} = 0.5$ and $w = 5$,

$$w_{local} = \sqrt{\frac{5^2 - 1}{12}} = 1.5811$$
$$m = \frac{12 \times 0.5^2 - 1^2 - 1}{2} = 1.5811$$

The standard deviation eq. the equivalent filter is not that close equal to the $\sigma = 0.5$. Hence, let us try for $\sigma = 0.1$, $\sigma = 0.5$, $\sigma = 1$, $\sigma = 2$, $\sigma = 3$, $\sigma = 4$, $\sigma = 5$.

$\sigma = 0.1$: $w_{local} = 1.5811$, $m = 0.5811$

$\sigma = 0.5$: $w_{local} = 1.5811$, $m = 0.5811$

$\sigma = 1$: $w_{local} = 1.5811$, $m = 0.5811$

$\sigma = 2$: $w_{local} = 1.5811$, $m = 0.5811$

$\sigma = 3$: $w_{local} = 1.5811$, $m = 0.5811$

$\sigma = 4$: $w_{local} = 1.5811$, $m = 0.5811$

$\sigma = 5$: $w_{local} = 1.5811$, $m = 0.5811$

The standard deviation eq. the equivalent filter is not that close equal to the $\sigma = 0.5$ and $w = 5$. Hence, we can see that $\sigma = 0.5$ and $w = 5$ is the best fit.

$\sigma = 0.5$, $w = 5$: $w_{local} = 1.5811$, $m = 0.5811$