

*BioE215 Physics-based Simulation of Biological Structures*  
 Equations for Modeling the Forces Generated by Muscles and Tendons  
 Clay Anderson, May 11, 2007

This purpose of this document is to provide relatively simple, concrete examples of how force generation by muscle and tendon can be modeled. Equations for five lumped-parameter models of increasing complexity are developed below. This first page defines the symbols used in the equations.

**Definitions**

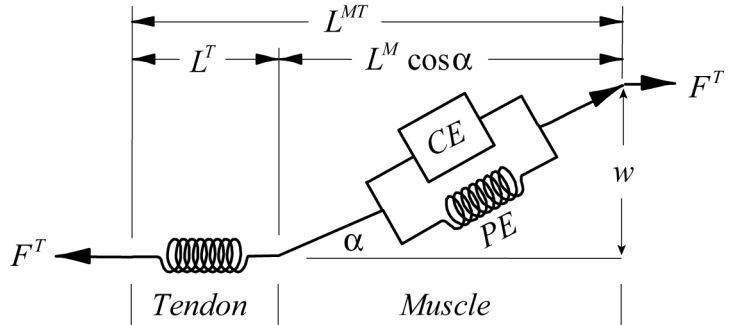
$T$  = Tendon  
 $M$  = Muscle  
 $CE$  = Contractile Element  
 $PE$  = Parallel Elastic Element

$L$  = Length  
 $V$  = Velocity  
 $F$  = Force  
 $A$  = Area

$E$  = Elasticity Modulus (stress / strain)  
 $\sigma$  = Stress (force / area)  
 $\epsilon$  = Strain (% change in length)

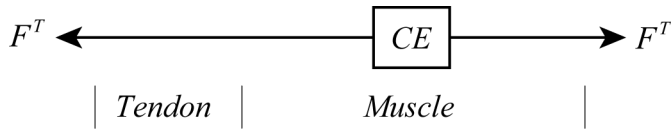
$a$  = Muscle activation. Variable used to quantify the concentration of calcium  $Ca^{++}$  in the lumen of the fibers of a muscle.  
 $a=1.0$  means full activation;  $a=0.0$  means no activation  
 $x$  = Muscle excitation  
 Muscle excitation is a variable used to quantify the net motor signal to a muscle.  
 $x=1.0$  means full excitation;  $x=0.0$  means no excitation  
 $\alpha$  = Pennation angle. Angle between the line of action of a muscle's fibers and the line of action of its tendon.

$L_s^T$  = Tendon slack length. The length at which tendon begins to generate force.  
 $L_o^M$  = Optimal muscle fiber length. Fiber length at which muscle generates its largest force.  
 $v_{max}^M$  = Maximum shortening velocity of muscle, normalized by optimal fiber length ( $L_o^M$ ).  
 $F_o^M$  = Optimal muscle force. Force generated by muscle when its fibers are at  $L_o^M$ .  
 $\alpha_o$  = Optimal pennation angle. Pennation angle when a muscle's fibers are at  $L_o^M$ .  
 $w$  = Width of muscle.  
 $\sigma_o^T$  = Optimal tendon stress. Stress developed by tendon when the force is  $F_o^M$ .



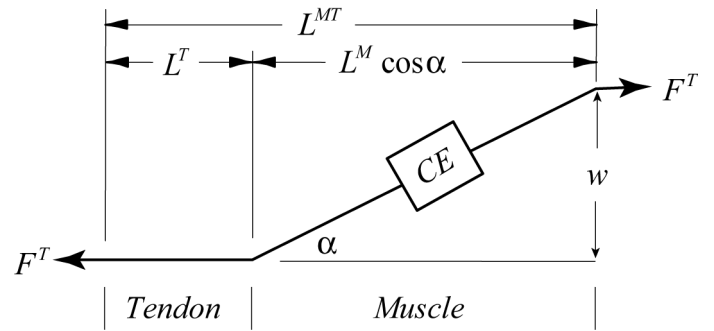
## Overview of Models

### Parallel Fibered

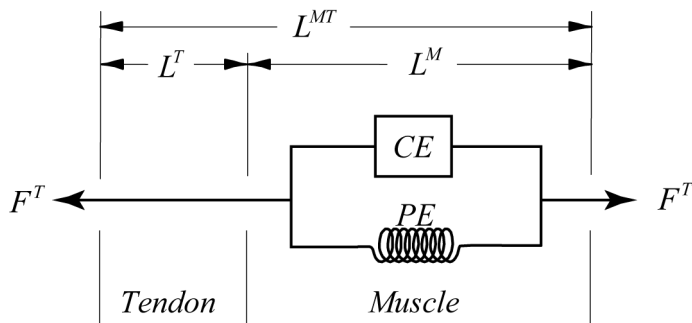


1a) Simple parallel fibered  
(p. 3)

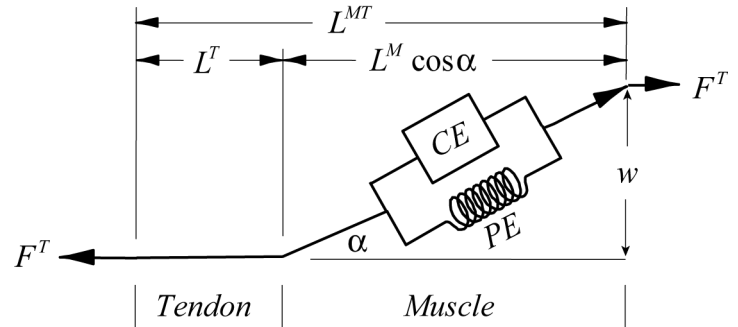
### Pennate



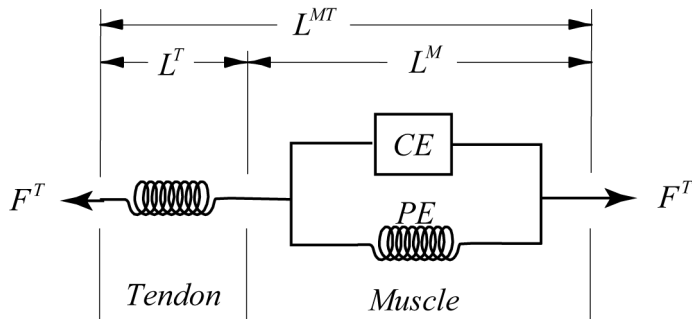
1b) Simple pennate with pennation  
(p. 3)



2a) Force-length-velocity  
(p. 5)



2b) Force-length-velocity with pennation  
(p. 7)



3a) Force-length-velocity with elastic tendon  
(p. 9)

## REFERENCES (p. 13)

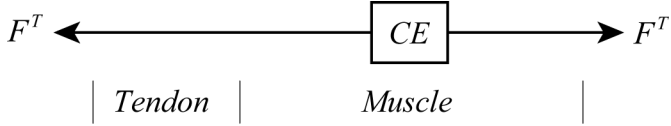
### Model 1a: Simplest

1. Tendon is infinitely stiff.
2. Force production by the muscle has no dependence on its length or velocity.
3. The muscle fibers act in the same direction as the tendon.

parameters:  $F_o^M$

time-varying inputs:  $a$

The simplest possible muscle neglects all but the strength ( $F_o^M$ ) and activation level ( $a$ ) of a muscle.



$$F^T = F^{CE} = a(t) \cdot F_o^M \quad (1)$$

The activation,  $a(t)$ , is like a control knob for turning up and down the force. Its value is usually chosen based on experimental conditions or comes from a control algorithm that is used to drive a musculoskeletal model to track some desired motion.  $F_o^M$ , usually expressed in Newtons, is typically obtained from the literature. Literature values can be scaled to match the strength of an individual, if strength measurements are available for that individual.

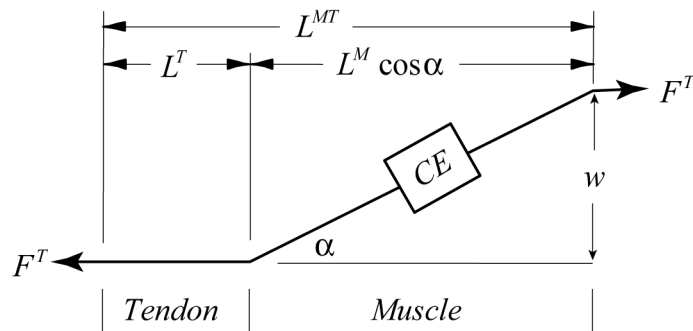
### Model 1b: Simplest Possible with Pennation

1. Tendon is infinitely stiff.
2. Force production by the muscle has no dependence on its length or velocity.
3. The muscle fibers act at a pennation angle,  $\alpha$ , to the line of action of the tendon.

parameters:  $F_o^M, L_o^M, \alpha_o, L_s^T$

time-varying inputs:  $a, L^{MT}$

This model is the same as Model 1a, except that the pennation angle of the muscle fibers,  $\alpha$ , is taken into account.



The force exerted by the tendon on bone is the force in the contractile element multiplied by the cosine of the pennation angle:

$$F^T = F^{CE} \cdot \cos \alpha = a(t) \cdot F_o^M \cdot \cos \alpha, \quad (2)$$

where we have used Eq. (1) to substitute for  $F^{CE}$ .

The pennation angle changes as the length of the muscle changes. To compute how  $\alpha$  changes, it is often assumed that the width of the muscle,  $w$ , remains constant, so that the following is always true

$$w = L^M \sin \alpha. \quad (3)$$

The value of  $w$  can be computed from the optimal fiber length and the optimal pennation angle:

$$w = L_o^M \sin \alpha_o. \quad (4)$$

$L_o^M$  and  $\alpha_o$  are input parameters that are typically obtained from the literature for the particular muscle in question (e.g., Delp et al., 1990; Anderson et al., 1999; Garner et al., 2001, Holzbaur et al., 2005). Together with the knowledge of the total actuator length,  $L^{MT}$ , and the tendon slack length,  $L_s^T$ , the pennation angle (or even better the  $\cos \alpha$ ) can be computed. The total length of the actuator is given by

$$L^{MT} = L_s^T + L^M \cos \alpha. \quad (5)$$

Note that because tendon is assumed to be infinitely stiff, we can use  $L_s^T$  for  $L^T$ . Using  $\sin^2 \alpha + \cos^2 \alpha = 1$  along with Eqs. (3) and (5), you can show that

$$\cos \alpha = \sqrt{\frac{\left(\frac{L^{MT} - L_s^T}{w}\right)^2}{1 + \left(\frac{L^{MT} - L_s^T}{w}\right)^2}} \quad (6)$$

Substituting for  $\cos \alpha$  in Eq. (2), the equation for the force generated by the actuator is thus given by

$$F^T = a(t) \cdot F_o^M \cdot \sqrt{\frac{\left(\frac{L^{MT} - L_s^T}{w}\right)^2}{1 + \left(\frac{L^{MT} - L_s^T}{w}\right)^2}} \quad (7)$$

As long as  $L^{MT} > L_s^T$ , Eq. (7) produces a well defined value. If ever  $L^{MT} \leq L_s^T$  (i.e., the muscle is slack), the force generated by the tendon is zero. If a muscle's fibers are not pennate and the width of the muscle is zero ( $w = 0$ ), Eq. (1) should be used.

The total length of an actuator,  $L^{MT}$ , is determined by the path geometry of the actuator. It can be determined in many cases from the kinematics of the musculoskeletal model (e.g., by computing the distance from the muscle origin to the muscle insertion). The tendon slack length, on the other hand, is more difficult to determine; a length cannot generally be measured. The value is typically chosen by the modeler (you) so that the muscle generates force over the appropriate range of total actuator lengths or joint angles.

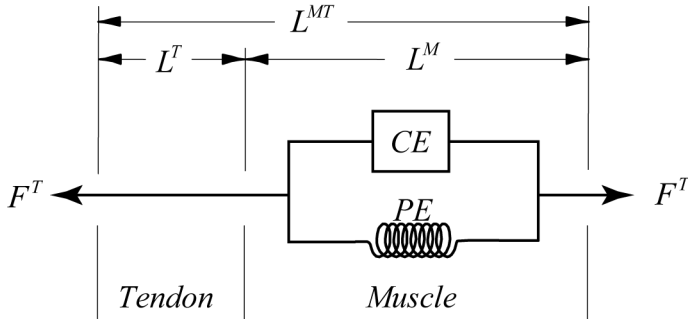
### Model 2a: Force-Length-Velocity Properties of Muscle with Inelastic Tendon

1. Tendon is infinitely stiff.
2. Force production by the muscle depends non-linearly on its length and velocity and is the sum of an active component, the contractile element ( $F^{CE}$ ), and a passive component, the parallel elastic element ( $F^{PE}$ ).
3. The muscle fibers act in the same direction as the tendon.

parameters:  $L_s^T$ ,  $F_o^M$ ,  $L_o^M$ ,  $v_{max}^{\%}$

time-varying inputs:  $a$ ,  $L^{MT}$ ,  $\dot{L}^M$

This model is similar to Model 1a, but now there is parallel elastic element in addition to the contractile element, and force production depends on the length and shortening velocity of the muscle,  $L^M$  and  $\dot{L}^M$  respectively.



The force in the tendon is given by

$$F^T = F^{CE} + F^{PE} \quad (8)$$

The contractile element force will be assumed to have the same form as Eq. (1) except that it is modified by the product of two normalized terms, one expressing the length dependence,  $\hat{F}_L^M(L^M)$ , and the other the velocity dependence,  $\hat{F}_V^M(\dot{L}^M)$ :

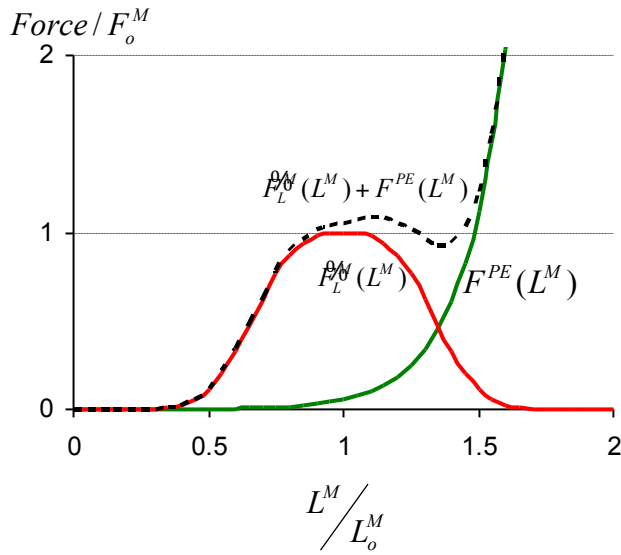
$$F^{CE} = a(t) \cdot F_o^M \cdot \hat{F}_L^M(L^M) \cdot \hat{F}_V^M(\dot{L}^M) \quad (9)$$

$\hat{F}_L^M(L^M)$  can be modeled by a Gaussian-like function (Thelen, 2003), and  $\hat{F}_V^M(\dot{L}^M)$  can be modeled by a sigmoidal:

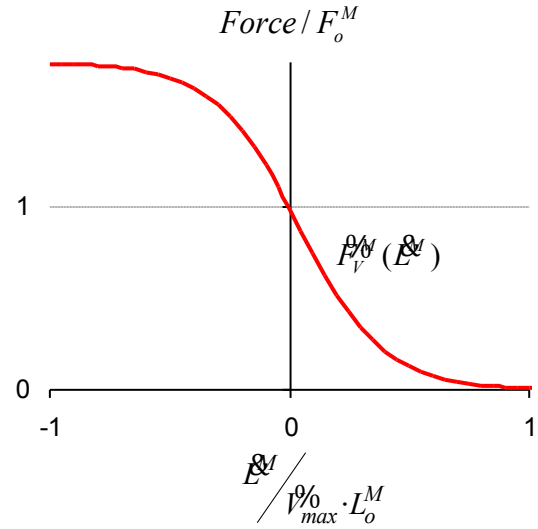
$$\hat{F}_L^M(L^M) = \exp[17.33 \cdot \left| \frac{L^M}{L_o^M} - 1.0 \right|^3] \quad (10)$$

$$\hat{F}_V^M(\dot{L}^M) = 1.8 - \frac{1.8}{1.0 + \exp\left[\frac{0.04 - \frac{\dot{L}^M}{v_{max}^{\%} \cdot L_o^M}}{0.18}\right]} \quad (11)$$

The parameters  $L_o^M$  and  $v_{max}^{\%}$  are typically found in the literature (e.g., Delp et al., 1990; Anderson and Pandy, 1999; Garner and Pandy, 2001; Holzbaur et al., 2005).  $L_o^M$  varies from muscle to muscle.  $v_{max}^{\%}$  is about 4.0 optimal fiber lengths per second for slow-twitch muscle and as high as 10.0 optimal fiber lengths per second for fast-twitch muscle (refs??).



**Fig. 1:** Force-length properties of fully activated muscle. Eqs (10) and (12).



**Fig. 2:** Force-velocity properties of fully activated muscle. Eq (11).

The parallel elastic element can be modeled with an exponential as well:

$$F^{PE} = F_o^M \cdot 3 \cdot 10^4 \cdot \exp[6 \cdot (\frac{L^M}{L_o^M} - 3.2)]. \quad (12)$$

The problem now is to find the length and shortening velocity of muscle, given the overall length and shortening velocity of the actuator. The length of the muscle is given by

$$L^M = L^{MT} - L_s^T. \quad (13)$$

Time differentiation of Eq. (13), and noting that  $\dot{L}_s^T = 0$  because tendon is assumed to be infinitely stiff, yields the following equation for the shortening velocity of muscle:

$$\dot{L}^{MT} = \dot{L}^M. \quad (14)$$

Eqs. (10) - (12) only approximate the length and velocity dependencies of muscle. The nice thing about them is that they are simple and can be expressed in computer code easily. For more accurate models, one can use curve fits (polynomials, splines, ...) of experimental data.

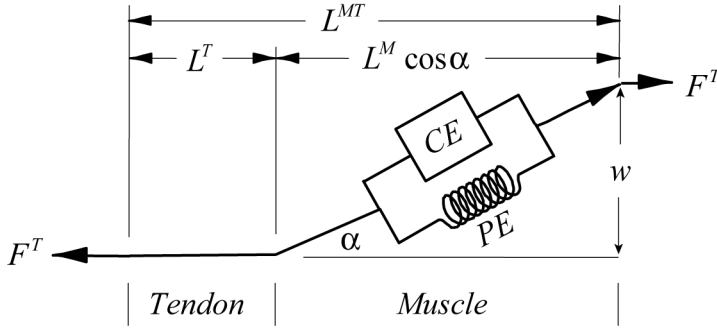
### **Model 2b: Force-Length-Velocity Properties of Muscle with Inelastic Tendon and Pennation**

1. Tendon is infinitely stiff.
2. Force production by the muscle depends non-linearly on its length and velocity and is the sum of an active component, the contractile element force ( $F^{CE}$ ), and a passive component, the parallel elastic element force ( $F^{PE}$ ).
3. The muscle fibers act at a pennation angle,  $\alpha$ , to the line of action of the tendon.

parameters:  $L_s^T, F_o^M, L_o^M, v_{max}^M, \alpha_o$

time-varying inputs:  $a, L^{MT}, \dot{L}^{MT}$

This model is similar to Model 2a, but pennation angle is included.



The force in the tendon is given by

$$F^T = (F^{CE} + F^{PE}) \cdot \cos \alpha, \quad (15)$$

where  $F^{CE}$  and  $F^{PE}$  are computed the same as above for Model 2a (Eqs. (9)-(12)). Also, as developed previously,  $\cos \alpha$  can be computed using Eq. (6). The computations for the length and shortening velocity of the muscle are different, however. The following relation governs the length of muscle:

$$L^{MT} = L_s^T + L^M \cos \alpha. \quad (16)$$

So, with some algebra, Eq. (16) yields

$$L^M = \frac{L^{MT} - L_s^T}{\cos \alpha}. \quad (17)$$

Derivation of the shortening velocity,  $\dot{L}^M$ , is more involved because the pennation angle,  $\alpha$ , varies as a function of time. Using the product rule and the chain rule for differentiation on Eq. (16) yields

$$\dot{L}^{MT} \cos \alpha - L^M \dot{\alpha} \sin \alpha = \dot{L}^M \cos \alpha. \quad (18)$$

Using the assumption that the muscle width remains constant, time differentiation of Eq. (3) yields

$$\dot{L}^M \sin \alpha + L^M \dot{\alpha} \cos \alpha = 0. \quad (19)$$

Solving Eq. (19) for  $\dot{L}$ , and substituting into Eq. (18) leads to

$$\dot{L}^M \cos \alpha - L^M \sin \alpha \frac{-\dot{L}^M \sin \alpha}{L^M \cos \alpha} = \dot{L}^{MT}. \quad (20)$$

Simplifying Eq. (20) then leads to

$$\dot{L}^M (\cos^2 \alpha + \sin^2 \alpha) = \dot{L}^{MT} \cos \alpha, \quad (21)$$

which, using  $\cos^2 \alpha + \sin^2 \alpha = 1$ , conveniently leads to

$$\dot{L}^M = \dot{L}^{MT} \cos \alpha. \quad (22)$$

Thus, Eqs. (17) and (22) can be used to compute the length and shortening velocity of muscle, and these can be used in Eq. (15) to calculate the force in the tendon.

It is interesting to note for a pennate muscle that, while the force in the tendon is less than the force in the muscle because  $|\cos \alpha| \leq 1$ ,

$$F^T = F^M \cos \alpha, \quad (23)$$

the shortening velocity along the tendon is greater than the shortening velocity of the muscle:

$$\dot{L}^{MT} = \frac{\dot{L}^M}{\cos \alpha}. \quad (24)$$

In this model, since tendon is assumed to be inelastic, it cannot store energy. This means that any work done by the actuator must be the same as that done by the muscle. Multiplying Eqs. (23) and (24) together verifies that this is the case:

$$\underbrace{F^T \dot{L}^{MT}}_{\text{actuator power}} = F^M \cos \alpha \cdot \frac{\dot{L}^M}{\cos \alpha} = \underbrace{F^M \dot{L}^M}_{\text{muscle power}} !! \quad (25)$$

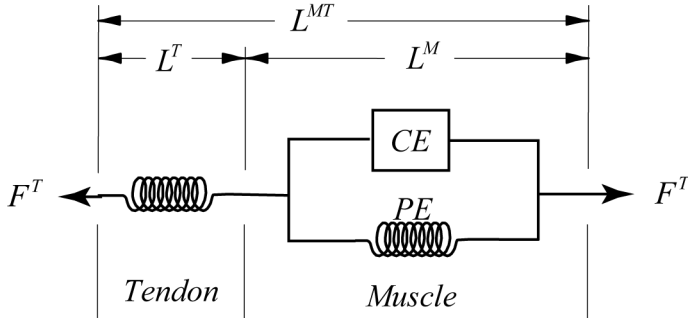


### Model 3a: Force-Length-Velocity Properties of Muscle with Elastic Tendon (1<sup>st</sup>-Order ODE)

1. Tendon is elastic with a linear stress-strain relationship.
2. Force production by the muscle depends non-linearly on its length and velocity and is the sum of an active component, the contractile element force ( $F^{CE}$ ), and a passive component, the parallel elastic element force ( $F^{PE}$ ).
3. The muscle fibers act in the same direction as the tendon.

parameters:  $L_s^T$ ,  $F_o^M$ ,  $L_o^M$ ,  $v_{max}^M$ ,  $F^T(t=0)$

time-varying inputs:  $a$ ,  $L^{MT}$ ,  $\dot{L}^{MT}$



With elastic tendon, it is generally not possible to write a closed-form expression for tendon force. The reason is that it is not possible to determine the shortening velocity of muscle uniquely given the shortening velocity of the actuator. Both the shortening velocity of tendon and muscle contribute to the shortening velocity of the actuator:

$$L^{MT} = L^T + L^M \quad (26)$$

$$\dot{L}^{MT} = \dot{L}^T + \dot{L}^M \quad (27)$$

Computing the force output of the actuator requires a first-order (or higher) ordinary differential equation that is usually nonlinear. In this model, tendon force is chosen as the dependent variable of the differential equation. Other choices for the dependent variable, such as the length of the muscle, are also possible (Shutte et al., 1993). The equations for modeling force will therefore have the form

$$\dot{F}^T = f(a, L^{MT}, \dot{L}^{MT}). \quad (28)$$

The time history of force can then be obtained by integrating Eq. (28) given an initial value for the force:

$$F^T = F^T(t=0) + \int_{t=0}^t \dot{F}^T dt. \quad (29)$$

The basic relationships for this model are the same as for Model 2a, except now there is an additional equation for the force developed by tendon that depends linearly on the length of the tendon:

$$F^T = k^T \cdot (L^T - L_s^T). \quad (30)$$

The stress-strain properties of tendon are approximated reasonably well by the following linear relationship:

$$\sigma = E^T \cdot \varepsilon = E^T \cdot \frac{L^T - L_s^T}{L_s^T}; \quad E^T \approx 1.2 \text{ GPa}. \quad (31)$$

The stress in tendon, across different muscles, has generally been estimated to be 32 MPa when the muscle is generating its optimal force (Zajac, 1989). We therefore have the following relationships:

$$F^T = A^T \cdot E^T \cdot \frac{L^T - L_s^T}{L_s^T} \quad (32)$$

$$F_o^M = A^T \cdot \sigma_o^T; \quad \sigma_o^T \approx 32 \text{ MPa}, \quad (33)$$

where  $A^T$  is the cross-sectional area of tendon. Solving Eq. (33) for  $A^T$ , and substituting into Eq. (32), yields

$$F^T = \frac{E^T}{\sigma_o^T} \cdot F_o^M \cdot \frac{L^T - L_s^T}{L_s^T} = \frac{1.2 \text{ GPa}}{32 \text{ MPa}} \cdot \frac{F_o^M}{L_s^T} \cdot (L^T - L_s^T) = \frac{37.5 \cdot F_o^M}{L_s^T} \cdot (L^T - L_s^T). \quad (34)$$

Thus, the stiffness for a tendon can be approximated knowing the slack length of the tendon and the optimal force of the muscle:

$$k^T = \frac{37.5 \cdot F_o^M}{L_s^T}. \quad (35)$$

Now, back to the development of first-order differential equation for actuator force... The shortening velocity of tendon is related to the time derivative of tendon force:

$$\dot{F}^T = k^T \dot{L}^T. \quad (36)$$

Using Eq. (27), Eq. (36) can be written in terms of the shortening velocity of the actuator and muscle:

$$\dot{F}^T = k^T \cdot (\dot{L}^{MT} - \dot{L}^M) \quad (37)$$

$\dot{L}^{MT}$  can be computed from the path geometry of the muscle. To complete development of the differential equation, the trick now is to use the force-length-velocity properties of muscle to solve for  $\dot{L}^M$ . This is just the inverse of the force velocity curve expressed by Eq. (11) and shown in Fig. 2, and it will depend upon the muscle length and the force developed by the contractile element:

$$\dot{L}^M = \dot{L}_V^{M-1}(F^{CE}, L^M). \quad (38)$$

How to compute  $L^M$ ,  $F^{CE}$ , and the inverse of the force-velocity curve is described next. Given  $F^T$ , the muscle length can be computed from Eqs. (26) and (30):

$$L^M = L^{MT} - L_s^T - \frac{F^T}{k^T} \quad (39)$$

From Eq. (8), the force in the contractile element is

$$F^{CE} = F^T - F^{PE}. \quad (40)$$

Since  $L^M$  is known from Eq. (39),  $F^{PE}$  can be computed using Eq. (12). And, since  $F^T$  is known at the current time,  $F^{CE}$  can be computed using Eq. (40).

From Eq. (9), the force in the contractile element is also given by

$$F^{CE} = a(t) \cdot F_o^M \cdot F_L^M(L^M) \cdot F_V^M(\dot{L}^M). \quad (41)$$

Isolating  $F_V^M(\dot{L}^M)$  in Eq. (41), we find

$$F_V^M(\dot{L}^M) = \frac{F^{CE}}{a(t) \cdot F_o^M \cdot F_L^M(L^M)} \quad (42)$$

Now, the inverse of Eq. (42) can be used to solve for  $\dot{L}^M$ . That is, given the force-velocity property of muscle, we would like to know the shortening velocity of muscle given its force and length. Defining  $F_0^{CE}$  as the ratio of the actual contractile force to the isometric contractile force,

$$F_0^{CE} = \frac{F^{CE}}{a(t) \cdot F_o^M \cdot F_L^M(L^M)}, \quad (43)$$

and taking the inverse of the normalized force-velocity curve,  $F_V^{M^{-1}}$  (i.e., the inverse of Eq. (11)), the muscle shortening velocity becomes

$$\dot{L}^M = F_V^{M^{-1}}(F_V^M(\dot{L}^M)) = F_V^{M^{-1}}(F_0^{CE}). \quad (44)$$

Note that  $F_L^M(L^M)$  in Eq. (43) is evaluated using Eq. (10).

The inverse expressed in Eq. (44) doesn't need to be an analytic inverse of Eq. (11). It can simply be a curve fit of the inverse. Simply turn the plot in Fig. 2 on its side, making  $\dot{L}^M$  a dependent variable, and do another curve fit. A sum of exponentials, for example, can be used for the curve fit:

$$\dot{L}^M = V_{max}^0 \cdot L_o^M \cdot \left\{ 0.5 + \exp[-3 \cdot (F_0^{CE} - 0.02)] + 0.1 \cdot \exp[12 \cdot (F_0^{CE} - 1.5)] \right\} \quad (45)$$

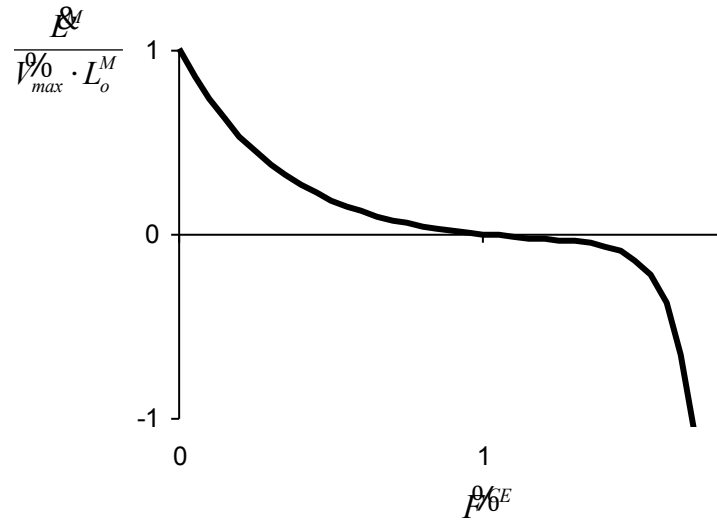


Fig. 4: Inverse of the force-velocity property of muscle. Eq. (45).

So, finally, from Eqs. (37) and (45), we get an expression for the time derivative of force in the tendon:

$$\dot{F}^T = k^T \cdot \left( \dot{L}^T - V_{max}^0 \cdot L_o^M \cdot \left\{ 0.5 + \exp[-3 \cdot (\frac{F^E}{F_0^E} - 0.02)] + 0.1 \cdot \exp[12 \cdot (\frac{F^E}{F_0^E} - 1.5)] \right\} \right) \quad (46)$$

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\* McMahon (1984) and Zajac (1989) are excellent sources for basic muscle mechanics and modeling.