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Randomized Algorithm (CS 9072)

Department of Computer Science and Engineering
Assignment - CLRS Exercise 5.2

Question Number 1

In HIRE-ASSISTANT, assuming that the candidates are presented in a random order, what is the probability that you hire exactly one time? What is the probability that you hire exactly n times?

You will hire exactly one time if the best candidate is presented first. There are (n - 1)! orderings with the best candidate first, so, it is with probability

$$\frac{(n-1)!}{n!} = \frac{1}{n}$$

that you only hire once. You will hire exactly n times if the candidates are presented in increasing order. This fixes the ordering to a single one, and so this will occur with probability

$$\frac{1}{n}.$$

Question Number 2

HIRE-ASSISTANT, assuming that the candidates are presented in a random order, what is the probability that you hire exactly twice?

Since the first candidate is always hired, we need to compute the probability that that exactly one additional candidate is hired. Since we view the candidate ranking as reading an random permutation, this is equivalent to the probability that a random permutation is a decreasing sequence followed by an increase, followed by another decreasing sequence. Such a permutation can be thought of as a partition of [n] into 2 parts. One of size k and the other of size n - k, where $1 \leq k \leq n-1$. For each such partition, we obtain a permutation with a single increase by ordering the numbers each partition in decreasing order, then concatenating these sequences. The only thing that can go wrong is if the numbers n through n - k + 1 are in the first partition. Thus there are $\binom{n}{k}$ - 1 permutations which correspond to hiring the second and final person on step k + 1. Summing, we see that the probability you hire exactly twice is

$$\frac{\sum_{k=1}^{n-1} \binom{n}{k} - 1}{n!} = \frac{2^n - 2 - (n-1)}{n!} = \frac{2^n - n - 1}{n}$$

Question Number 3

Use indicator random variables to compute the expected value of the sum of n dice.

Let X_j be the indicator of a dice coming up j. So, the expected value of a single dice roll X is

$$E[X] = \sum_{j=1}^{6} j \; Pr(X_j) = \frac{1}{6} \sum_{j=1}^{6} j$$

So, the sum of n dice has probability

$$E[nX] = nE[X] = \frac{n}{6}\sum_{j=1}^{6} j = \frac{n6(6+1)}{12} = 3.5n$$

Question Number 4

Use indicator random variables to solve the following problem, which is known as the hat-check problem. Each of n customers gives a hat to a hat-check person at a restaurant. The hat-check person gives the hats back to the customers in a random order. What is the expected number of customers who get back their own hat?

Let X be the number of customers who get back their own hat and X_i be the indicator random variable that customer i gets his hat back. The probability that an individual gets his hat back is 1/n. Then we have

$$E[X] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \frac{1}{n} = 1$$

Question Number 5

Let A[1. . . n] be an array of n distinct numbers. If $i < j$ and $A[i] > A[j]$, then the pair (i, j) is called an inversion of A. Suppose that the elements of A form a uniform random permutation of $\langle 1, 2, ..., n\rangle$ Use indicator random variables to compute the expected number of inversions.

Let X_{ij} for $i < j$ be the indicator of $A[i] > A[j]$. Then, we have that the expected number of inversions is

$$E[\sum_{i<j}] = \sum_{i<j} E[X_{ij}] = \sum_{i=1}^{n-1}\sum_{j=1+1}^{n} Pr(A[i] > A[j]) = \frac{1}{2}\sum_{i=1}^{n-1} n - i$$

$$= \frac{n(n-1)}{2} - \frac{n(n-1)}{4} = \frac{n(n-1)}{4}$$