

Question Paper 2016 Delhi Set-1
CBSE Class XII Mathematics

General Instruction:

- All question are compulsory.
- Please check that this Question Paper contains 26 Questions.
- Questions 1 to 6 in Section-A are Very Short Answer Type Questions carrying one mark each.
- Questions 7 to 19 in Section-B are Long Answer I Type Questions carrying 4 marks each.
- Questions 20 to 26 in Section-C are Long Answer II Type Questions carrying 6 marks each.
- Please write down the serial number of the Question before attempting it.

Section A

1. Find the maximum value of

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 + \sin \theta & 1 \\ 1 & 1 & 1 + \cos \theta \end{vmatrix}$$

Sol. Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 + \sin \theta & 1 \\ 1 & 1 & 1 + \cos \theta \end{vmatrix}$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 0 & \sin \theta & 0 \\ 0 & 0 & \sin \theta \end{vmatrix}$$

2. If A is a square matrix such that $A^2 = I$, then find the simplified value of

$$(A - I)^3 + (A + I)^3 - 7A$$

Sol. Given: $(A - I)^3 + (A + I)^3 - 7A$

$$= A^3 - I^3 - 3A^2I + 3AI^2 + A^3 + I^3 + 3A^2I + 3AI^2 - 7A$$

$$= 2A^3 + 6AI^2 - 7A$$

$$= 2AA^2 + 6AI^2 - 7A$$

$$= 8A - 7A = A$$

3. Matrix $A = \begin{vmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{vmatrix}$ is given to be symmetric, find value of a and b .

Sol. We have

$$A = \begin{vmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{vmatrix}$$

$$A' = \begin{vmatrix} 0 & 3 & 3a \\ 2b & 1 & 3a \\ -2 & 3 & -1 \end{vmatrix}$$

We know that a matrix is symmetric if $A = A'$.

Thus,

$$A = \begin{vmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 3 & 3a \\ 2b & 1 & 3 \\ -2 & 3 & -1 \end{vmatrix}$$

Now,

$$2b = 3$$

$$\Rightarrow b = \frac{3}{2}$$

Also, $3a = -2$

$$\Rightarrow a = \frac{-2}{3}$$

Therefore, $a = \frac{-2}{3}$ and $b = \frac{3}{2}$

4. Find the position vector of a point which divides the join of points with position vectors $\vec{a} - 2\vec{b}$ and $2\vec{a} + \vec{b}$ externally in the ratio 2:1.

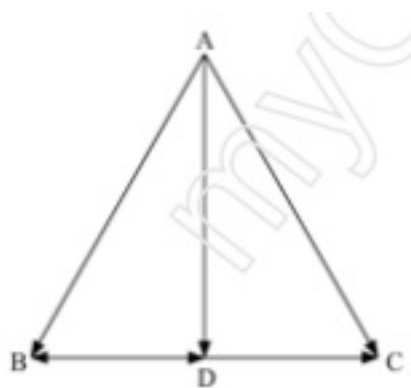
Sol. Let A and B be the points with position vectors $\vec{a} - 2\vec{b}$ and $2\vec{a} + \vec{b}$, respectively.

Also, let R divide AB externally in the ratio 2:1

$$\therefore \text{Position vector of } R = \frac{2 \times (2\vec{a} + \vec{b}) - 1 \times (\vec{a} - 2\vec{b})}{2 - 1} = 3\vec{a} + 4\vec{b}$$

5. The two vectors $\hat{j} + \hat{k}$ and $3\hat{i} - \hat{j} + 4\hat{k}$ represent the two sides AB and AC, respectively of a triangle. Find the length of the median through A.

Sol. In ΔABC ,



Using the triangle law of vector addition, we have

$$\begin{aligned}\overline{BC} &= \overline{AC} - \overline{AB} \\ &= (3\hat{i} - 4 + 4\hat{k}) - (\hat{j} + \hat{k}) \\ &= 3\hat{i} - 2\hat{j} + 3\hat{k}\end{aligned}$$

(Since AD is the median)

In $\triangle ABD$, using the triangle law of vector addition, we have

$$\begin{aligned}\overline{AD} &= \overline{AB} + \overline{BD} \\ &= (\hat{j} + \hat{k}) + \left(\frac{3}{2}\hat{i} - \hat{j} + \frac{3}{2}\hat{k}\right) \\ &= \frac{3}{2}\hat{i} + 0\hat{j} + \frac{5}{2}\hat{k}\end{aligned}$$

$$\therefore AD = \sqrt{\left(\frac{3}{2}\right)^2 + 0^2 + \left(\frac{5}{2}\right)^2} = \frac{1}{2}\sqrt{34}$$

Hence, the length of the median through A is $\frac{1}{2}\sqrt{34}$ units.

6. Find the vector equation of plane which is at a distance of 5 units from the origin and its normal vector is $2\hat{i} - 3\hat{j} + 6\hat{k}$.

Sol. Given:

Normal vector, $\hat{n} = 2\hat{i} - 3\hat{j} + 6\hat{k}$

Perpendicular distance, $d = 5$ units

The vector equation of a plane that is at a distance of 5 units from the origin and has its normal vector $\hat{n} = 2\hat{i} - 3\hat{j} + 6\hat{k}$ is as follows:

$$\vec{r} \cdot \hat{n} = d$$

$$\vec{r} \cdot (2\hat{i} - 3\hat{j} + 6\hat{k}) = 5$$

Section B

7. Prove that: $\tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{8} = \frac{\pi}{4}$

OR

Solve for X:

$$2 \tan^{-1}(\cos x) = \tan^{-1}(2 \operatorname{cosec} x)$$

Sol. LHS : $\left(\tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} \right) + \left(\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{8} \right)$

$$= \tan^{-1} \left(\frac{\frac{1}{5} + \frac{1}{7}}{1 - \frac{1}{5} \times \frac{1}{7}} \right) + \tan^{-1} \left(\frac{\frac{1}{3} + \frac{1}{8}}{1 - \frac{1}{3} \times \frac{1}{8}} \right)$$

$$\left[\because \tan^{-1} A + \tan^{-1} B = \tan^{-1} \left(\frac{A+B}{1-AB} \right) \right]$$

$$= \tan^{-1} \left(\frac{\frac{6}{17} + \frac{11}{23}}{1 - \frac{6}{17} \times \frac{11}{23}} \right)$$

$$= \tan^{-1} 1$$

$$= \frac{\pi}{4}$$

OR

$$2 \tan^{-1}(\cos x) = \tan^{-1}(2 \operatorname{cosec} x)$$

$$\Rightarrow \tan^{-1} \left(\frac{2 \cos x}{1 - \cos^2 x} \right)$$

$$= \tan^{-1}(2 \operatorname{cosec} x)$$

$$\left[\because 2 \tan^{-1} x = \tan^{-1} \left(\frac{2x}{1 - x^2} \right) \right]$$

$$\Rightarrow \frac{2 \cos x}{\sin^2 x} = 2 \operatorname{cosec} x$$

$$\Rightarrow \cos x = \sin x$$

$$\Rightarrow \tan x = 1$$

$$\Rightarrow x = \frac{\pi}{4}$$

8. The monthly incomes of Aryan and Babban are in the ratio 3:4 and their monthly expenditures are in the ratio 5:7. If each saves Rs 15,000 per month, find their monthly incomes using matrix method. This problem reflects which value?

Sol. Let the monthly incomes of Aryan and Babban be $3x$ and $4x$, respectively.

Suppose their monthly expenditure are $5y$ and $7y$, respectively.

Since each saves Rs 15,000 per month,

Monthly saving of Aryan: $3x - 5y = 15,000$

Monthly saving of Babban: $4x - 7y = 15,000$

The above system of equations can be written in the matrix form as follows:

$$\begin{bmatrix} 3 & -5 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$$

or,

$$AX = B, \text{ where } A = \begin{bmatrix} 3 & -5 \\ 4 & -7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$$

Now,

$$|A| = \begin{vmatrix} 3 & -5 \\ 4 & -7 \end{vmatrix} = -21 - (-20) = -1$$

$$\text{Adj } A = \begin{bmatrix} -7 & -4 \\ 5 & 3 \end{bmatrix}^T = \begin{bmatrix} -7 & 5 \\ -4 & 3 \end{bmatrix}$$

$$\text{So, } A^{-1} = \frac{1}{|A|} \text{adj } A = -1 \begin{bmatrix} -7 & 5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ 4 & -3 \end{bmatrix}$$

$$\therefore X = A^{-1} B$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 105000 - 75000 \\ 60000 - 45000 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$$

$$\Rightarrow x = 30,000 \text{ and } y = 15,000$$

Therefore,

$$\text{Monthly income of Aryan} = 3 \times \text{Rs}30,000 = \text{Rs}90,000$$

$$\text{Monthly income of Babban} = 4 \times \text{Rs}30,000 = \text{Rs}1,20,000$$

From the problem, we are encouraged to understand the power of savings. We should

Save certain part of our monthly income for the future.

9. If $x = a \sin 2t(1 + \cos 2t)$ and $y = b \cos 2t(1 - \cos 2t)$, find the values of $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$ and $t = \frac{\pi}{3}$.

OR

If $y = X^x$, prove that $\frac{d^2y}{dx^2} - \frac{1}{y} \left(\frac{dy}{dx} \right)^2 - \frac{y}{x} = 0$.

Sol. $x = a \sin 2t(1 + \cos 2t)$

$$y = b \cos 2t(1 - \cos 2t)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{b - [-2 \sin 2t + 2 \cos 2t \sin 2t \times 2]}{a[2 \cos 2t + 2 \cos 4t]}$$

$$\frac{dy}{dx} = \frac{b}{a} \left[\frac{-2 \sin 2t + 2 \sin 4t}{2 \cos 2t + 2 \cos 4t} \right]$$

$$\left. \frac{dy}{dx} \right|_{t=\frac{\pi}{4}} = \frac{b}{a} \left[\frac{-2+0}{0-2} \right] = \frac{b}{a}$$

$$\text{And } \left. \frac{dy}{dx} \right|_{t=\frac{\pi}{4}} = \frac{b}{a} \left[\frac{-2\sqrt{3}}{-2} \right] = \frac{\sqrt{3}b}{a}$$

OR

$$y = x^x$$

Applying logarithm,

$$\log y = x \log x$$

$$\frac{1}{y} \frac{dy}{dx} = \log x + x \times \frac{1}{x} = 1 + \log x$$

$$\frac{dy}{dx} = X^x [1 + \log x]$$

$$\frac{d^2y}{dx^2} = \frac{d(x^x)}{dx} (1 + \log x) + x^x \left[\frac{d}{dx} (1 + \log x) \right]$$

$$= x^x (1 + \log x)(1 + \log x) + x^x \left[\frac{1}{x} \right]$$

$$= x^x (1 + \log x)^2 + x^{x-1}$$

$$\frac{d^2y}{dx^2} - \frac{1}{y} \left(\frac{dy}{dx} \right)^2 - \frac{y}{x} = x^x (1 + \log x)^2 x^{x-1} - \frac{1}{x^x} (x^x (1 + \log x)^2) - \frac{x^x}{x}$$

$$= x^x (1 + \log x)^2 + x^{x-1} - x^x (1 + \log x)^2 - x^{x-1}$$

$$= 0$$

Hence proved.

10. Find the values of p and q for which

$$f(x) = \begin{cases} \frac{1 - \sin^3 x}{3 \cos^2 x}, & \text{if } x < \frac{\pi}{2} \\ P & , \text{if } x = \frac{\pi}{2} \\ \frac{q(1 - \sin x)}{(\pi - 2x)^2}, & \text{if } x > \frac{\pi}{2} \end{cases}$$

Is continuous at $x = \frac{\pi}{2}$

Sol. $f(x) = \begin{cases} \frac{1 - \sin^3 x}{3 \cos^2 x}, & \text{if } x < \frac{\pi}{2} \\ P & , \text{if } x = \frac{\pi}{2} \\ \frac{q(1 - \sin x)}{(\pi - 2x)^2}, & \text{if } x > \frac{\pi}{2} \end{cases}$

for continuity,

$$\lim_{x \rightarrow \frac{\pi}{2}^-} f(x) = \lim_{x \rightarrow \frac{\pi}{2}^+} f(x) = f\left(\frac{\pi}{2}\right)$$

$$\lim_{x \rightarrow \frac{\pi}{2}^-} f(x) = \lim_{x \rightarrow \frac{\pi}{2}^-} \left(\frac{1 - \sin^3 x}{3 \cos^2 x} \right) = \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{(1 - \sin x)(1 + \sin^2 x + \sin x)}{3[1 - \sin^2 x]}$$

$$\lim_{x \rightarrow \frac{\pi}{2}^-} f(x) = \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{(1 + \sin^2 x + \sin x)}{3(1 + \sin x)} = \frac{1 + 1 + 1}{3(2)} = \frac{1}{2}$$

$$\text{Let } \frac{\pi}{2} - x = \theta \Rightarrow x = \frac{\pi}{2} - \theta$$

$$\lim_{x \rightarrow \frac{\pi}{2}} = \lim_{\theta \rightarrow 0} \frac{\left[1 - \sin\left(\frac{\pi}{2} - \theta\right)\right]}{(2\theta)^2} = \frac{q}{4} \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta^2}$$

$$= \frac{q}{4} \lim_{\theta \rightarrow 0} \frac{2 \sin^2 \frac{\theta}{2}}{\theta^2} = \frac{q}{4} \lim_{\theta \rightarrow 0} \frac{\sin^2 \frac{\pi}{2}}{4 \times (\theta/2)} = \frac{q}{8}$$

Now,

$$\lim_{x \rightarrow \frac{\pi}{2}} f(x) = \lim_{x \rightarrow \frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \frac{1}{2} = p = \frac{q}{8}$$

$$\Rightarrow p = \frac{1}{2} \text{ and } q = \frac{8}{2} = 4$$

11. Show that the equation of normal at any point t on the curve $x = 3 \cos t - \cos^3 t$ and $y = 3 \sin t - \sin^3 t$ is

$$4(y \cos^3 t - \sin^3 t) = 3 \sin 4t.$$

Sol. Given:

$$\begin{aligned} x &= 3 \cos t - \cos^3 t \\ y &= 3 \sin t - \sin^3 t \end{aligned} \quad \text{Slope of the tangent, } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3 \cot t - 3 \sin^2 t \cos t}{-3 \sin t + 3 \cos^2 t \sin t}$$

$$= \frac{3 \cos t - [\cos^2 t]}{-3 \sin t [\sin^2 t]}$$

$$\frac{dy}{dx} = \frac{-\cos^3 t}{\sin^3 t}$$

$$\therefore \text{Slope of the normal} = \frac{\sin^3 t}{\cos^3 t}$$

The equation of the normal is given by

$$\frac{y - (3 \sin t - \sin^3 t)}{x - (3 \cos t - \cos^3 t)} = \frac{\sin^3 t}{\cos^3 t}$$

$$\Rightarrow y \cos^3 t - 3 \sin t \cos^3 t + \sin^3 t \cos^3 t$$

$$= x \sin^3 t - 3 \cos t \sin^3 t + \sin^3 t$$

$$\Rightarrow y \cos^3 t - x \sin^3 t = 3(\sin t \cos t - \cos t \sin^3 t)$$

$$\Rightarrow y \cos^3 t - x \sin^3 t = 3 \sin t \cos t (\cos^2 t - \sin^2 t)$$

$$\Rightarrow y \cos^3 t - x \sin^3 t = \frac{3}{2} \sin 2t \cos 2t = \frac{3}{4} \sin 4t$$

$$\Rightarrow y \cos^3 t - x \sin^3 t = 3(\sin t \cos t - \cos t \sin^3 t)$$

$$\Rightarrow 4(y \cos^3 t - x \sin^3 t) = 3 \sin 4t$$

Hence proved.

12. Find $\int \frac{(3 \sin \theta - 2) \cos \theta}{5 - \cos^2 \theta - 4 \sin \theta} d\theta$.

OR

Evaluate $\int_0^\pi e^{2x} \cdot \sin \left(\frac{\pi}{4} + x \right) dx$

Sol. $I = \int \frac{(3 \sin \theta - 2) \cos \theta}{5 - \cos^2 \theta - 4 \sin \theta} d\theta$

$$\Rightarrow I = \int \frac{(3 \sin \theta - 2) \cos \theta}{5 - (1 - \sin^2 \theta) - 4 \sin \theta} d\theta$$

$$\Rightarrow I = \int \frac{(3 \sin \theta - 2) \cos \theta}{\sin^2 \theta - 4 \sin \theta + 4}$$

No, let $\sin \theta = t$

$$\Rightarrow \cos \theta d\theta = dt$$

$$\therefore I = \int \frac{(3t - 2) dt}{t^2 - 4t + 4}$$

$$\Rightarrow 3t - 2 = A \frac{d}{dx} (t^2 - 4t + 4) + B$$

$$\Rightarrow 3t - 2 = A(2t - 4) + B$$

$$\Rightarrow 3t - 2 = (2A) + B - 4A$$

Comparing the coefficients of the like powers of t, we get

$$2A = 3 \Rightarrow A = \frac{3}{2} \text{ and } B - 4A = -2$$

$$\Rightarrow B - 4 \times \frac{3}{2} = -2$$

$$\Rightarrow B = -2 + 6 = 4$$

Substituting the value of A and B, we get

$$3t - 2 = \frac{3}{2} (2t - 4) + 4$$

$$\therefore I = \int \frac{(3t - 2) dt}{t^2 - 4t + 4}$$

$$\begin{aligned}
 &= \int \left(\frac{\frac{3}{2}(2t-4) + 4}{t^2 - 4t + 4} \right) dt \\
 &= \frac{3}{2} \int \left(\frac{2t-4}{t^2 - 4t + 4} \right) dt + 4 \int \frac{dt}{t^2 - 4t + 4} \\
 &= \frac{3}{2} I_1 + 4 I_2 \dots (1)
 \end{aligned}$$

Here,

$$I_1 = \int \frac{(2t-4)dt}{t^2 - 4t + 4} \text{ and } I_2 = \int \frac{dt}{t^2 - 4t + 4}$$

Now

$$I_1 = \int \frac{(2t-4)dt}{t^2 - 4t + 4}$$

$$\text{Let } t^2 - 4t + 4 = p$$

$$\Rightarrow (2t-4)dt = dp$$

$$I_1 = \int \frac{(2t-4)dt}{t^2 - 4t + 4}$$

$$= \int \frac{dp}{p} = \log |P| + C_1$$

$$= \log |t^2 - 4t + 4| + C_1 \dots (2)$$

and

$$I_2 = \int \frac{dt}{t^2 - 4t + 4}$$

$$\begin{aligned}
 &= \int \frac{dt}{(t-2)^2} \\
 &= \int (t-2)^{-2} dt \\
 &= \frac{(t-2)^{-2+1}}{-2+1} + C_2 \\
 &= \frac{-1}{t-2} + C_2 \dots (3)
 \end{aligned}$$

From (1), (2) and (3), we get

$$\begin{aligned}
 I &= \frac{3}{2} \log |t^2 - 4t + 4| + 4 \times \frac{-1}{t-2} + C_1 + C_2 \\
 &= 3 \log |\sin^2 \theta - 4 \sin \theta + 4| + \frac{4}{2-t} + C \quad (\text{where } C = C_1 + C_2) \\
 &= \frac{3}{2} \times 2 \log |(\sin \theta - 2)| + \frac{4}{2 - \sin \theta} + C \\
 &= 3 \log |2 - \sin \theta| + \frac{4}{2 - \sin \theta} + C
 \end{aligned}$$

OR

$$\text{Let } I = \int_0^\pi e^{2x} \sin\left(\frac{\pi}{4} + x\right) dx$$

Integrating by parts, we get

$$\begin{aligned}
 \Rightarrow I &= \frac{1}{2} \left[e^{2x} \sin\left(\frac{\pi}{4} + x\right) \right]_0^\pi - \frac{1}{2} \left\{ \left[\frac{1}{2} e^{2x} \cos\left(\frac{\pi}{4} + x\right) \right]_0^\pi + \frac{1}{2} \int_0^\pi e^{2x} \sin\left(\frac{\pi}{4} + x\right) dx \right\} \\
 \Rightarrow I &= \frac{1}{2} \left[e^{2x} \sin\left(\frac{\pi}{4} + x\right) \right]_0^\pi - \frac{1}{4} \left[e^{2x} \cos\left(\frac{\pi}{4} + x\right) \right]_0^\pi - \frac{1}{4} I
 \end{aligned}$$

$$\Rightarrow \frac{5}{4}I = \frac{1}{2} \left[e^{2x} \sin \left(\pi + \frac{\pi}{4} \right) - \sin \left(\frac{\pi}{4} \right) \right] - \frac{1}{4} \left[e^{2x} \cos \left(\pi + \frac{\pi}{4} \right) - \cos \left(\frac{\pi}{4} \right) \right]$$

$$\Rightarrow \frac{5}{4}I = \frac{1}{2} \left[-e^{2x} \times \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right] - \frac{1}{4} \left[e^{2x} \times \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right]$$

$$\Rightarrow I = -\frac{1}{5\sqrt{2}}(e^{2x} + 1)$$

13. Find $\int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx$.

Sol. $I = \int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx$

Let $x^{\frac{3}{2}} = t$

$$\Rightarrow \frac{3}{2} x^{\frac{1}{2}} dx = dt$$

$$x^{\frac{1}{2}} dx = \frac{2}{3} dt$$

Putting the values in I, we get

$$I = \int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx$$

$$= \frac{2}{3} \int \frac{1}{\sqrt{a^3 - t^2}} dt$$

Using the following formula of integration, we get

$$\int \frac{dx}{\sqrt{a^3 - x^3}} = \sin^{-1} \left(\frac{x}{a} \right)$$

$$\therefore \frac{2}{3} \int \frac{1}{\sqrt{a^3 - t}} dt = \frac{2}{3} \sin^{-1} \left(\frac{t}{a^{\frac{3}{2}}} \right) + C$$

Again, putting the value of t, we get

$$\frac{2}{3} \int \frac{1}{\sqrt{a^3 - t^2}} dt = \frac{2}{3} \sin^{-1} \left(\frac{t}{a^{\frac{3}{2}}} \right) + C$$

$$= \frac{2}{3} \sin^{-1} \left(\frac{x^{\frac{3}{2}}}{a^{\frac{3}{2}}} \right) + C$$

Here, C is constant of integration

14. Evaluate $\int_{-1}^2 |x^3 - x| dx$.

Sol. Let: $\int_{-1}^2 |x^3 - x| dx$

$$f(x) = x^3 - x$$

$$f(x) = x^3 - x = x(x-1)(x+1)$$

The signs of f(x) for the different values are shown in the figure given below:



$$f(x) > 0 \text{ for all } x \in (-1, 0) \cup (1, 2)$$

$f(x) < 0$ for all $x \in (0, 1)$

Therefore,

$$|x^3 - x| = \begin{cases} x^3 - x, & x \in (-1, 0) \cup (1, 2) \\ -(x^3 - x), & x \in (0, 1) \end{cases}$$

$$\therefore I = \int_{-1}^2 |x^3 - x| dx$$

$$= \int_{-1}^0 (x^3 - x) dx - \int_0^1 (x^3 - x) dx + \int_1^2 (x^3 - x) dx$$

$$= \int_{-1}^0 |x^3 - x| dx + \int_0^1 |x^3 - x| dx + \int_1^2 |x^3 - x| dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2} \right]_{-1}^0 + \left[\frac{x^4}{4} - \frac{x^2}{2} \right]_0^1 + \left[\frac{x^4}{4} - \frac{x^2}{2} \right]_1^2$$

$$= -\left(\frac{1}{4} - \frac{1}{2}\right) - \left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{16}{4} - \frac{4}{4}\right) - \left(\frac{1}{4} - \frac{1}{2}\right)$$

$$= \frac{3}{4} + (4 - 2) = \frac{11}{4}$$

15. Find the particular solution of the differential equation

$$(1 - y^2)(1 + \log x) dx + 2xy dy = 0$$

Given that $y = 0$ when $x = 1$.

Sol. Given

$$(1 - y^2)(1 + \log x) dx + 2xy dy = 0$$

$$\Rightarrow (1 - y^2)(1 + \log x) dx = -2xy dy$$

$$\Rightarrow \left(\frac{1 - \log x}{2x} \right) dx = - \left(\frac{y}{1 - y^2} \right) dy \dots\dots (1)$$

Let: $1 + \log x = t$

$$\text{and } (1 - y^2) = p \Rightarrow \frac{1}{x} dx = dt \text{ and } -2y dy = dp$$

Therefore, (1) becomes

$$\int \frac{t}{2} dt = \int \frac{1}{2} dp$$

$$\Rightarrow \frac{t^2}{4} = \frac{\log p}{2} + C \dots\dots (2)$$

Substituting the values of t and p in (2), we get

$$\frac{(1 + \log x^2)}{4} = \frac{\log(1 - y^2)}{2} + C \dots\dots (3)$$

$$\text{At } x = 1 \text{ and } y = 0, (3) \text{ becomes } C = \frac{1}{4}$$

Substituting the value of C in (3), we get

$$\frac{(1 + \log x^2)}{4} = \frac{\log(1 - y^2)}{2} + \frac{1}{4}$$

$$\Rightarrow (1 + \log x^2) = 2 \log(1 - y^2) + 1$$

OR

$$(\log x)^2 + \log x^2 = \log(1 - y^2)^2$$

It is the required particular solution.

16. Find the general solution of the following differential equation:

$$(1 + y^2) + (x - e^{\tan^{-1}y}) \frac{dy}{dx} = 0$$

Sol. Given:

$$(1 + y^2) + (x - e^{\tan^{-1}y}) \frac{dy}{dx} = 0$$

Let $\tan^{-1}y = t$

$$\Rightarrow y = \tan t$$

$$\Rightarrow \frac{dy}{dx} = \sec^2 t \frac{dt}{dx}$$

Therefore, the equation becomes

$$(1 + \tan^2 t) + (x - e^t) \sec^2 t \frac{dt}{dx} = 0$$

$$\Rightarrow \sec^2 t + (x - e^t) (\sec^2 t) \frac{dt}{dx} = 0$$

$$\Rightarrow 1(x - e^t) \frac{dt}{dx} = 0$$

$$\Rightarrow (x - e^t) \frac{dt}{dx} = -1$$

$$\Rightarrow x - e^t = \frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt} + 1x = e^t$$

$$\text{IF } \int 1 \cdot dt = e^t$$

$$\therefore e^t \left(\frac{dx}{dt} + 1x \right) = e^t \cdot e^t$$

$$e \Rightarrow \frac{d}{dt}(xe^t) = e^{2t} \text{ Integrating both the sides, we get}$$

$$xe^t = \int e^{2t} dt$$

Substituting the value of t in (1), we get

$$\Rightarrow xe^t = \frac{1}{2} e^{2t} + C \dots\dots (1)$$

$$xe^{\tan^{-1}y} = \frac{1}{2} e^{2\tan^{-1}y} + C1$$

$$\Rightarrow e^{2\tan^{-1}y} = 2xe^{\tan^{-1}y} + C \text{ It is the required general solution.}$$

17. Show that the vector \vec{a}, \vec{b} and \vec{c} are coplanar if $\vec{a}, \vec{b}, \vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ are coplanar.

Sol. It is given that $\vec{a}, \vec{b}, \vec{b} + \vec{c}$ and are coplanar.

Therefore, Scalar triple product=Volume of the parallelopoid = 0

$$(\vec{a} + \vec{b}) \cdot [(\vec{b} + \vec{c}) \times (\vec{c} + \vec{a})] = 0$$

$$\Rightarrow (\vec{a} + \vec{b}) \cdot [(\vec{b} \times \vec{c}) + (\vec{b} \times \vec{a}) + 0 + (\vec{c} \times \vec{a})] = 0$$

$$\Rightarrow \vec{a} \cdot (\vec{b} \times \vec{c}) + \vec{a} \cdot (\vec{b} \times \vec{a}) + \vec{a} \cdot (\vec{c} \times \vec{a}) + \vec{b} \cdot (\vec{b} \times \vec{c}) + \vec{b} \cdot (\vec{b} \times \vec{a}) + \vec{b} \cdot (\vec{c} \times \vec{a}) = 0$$

$$\Rightarrow [abc] + 0 + 0 + 0 + 0 + [bca] = 0$$

$$\Rightarrow 2[abc] = 0$$

$$\Rightarrow [abc] = 0$$

Therefore, the vector \vec{a}, \vec{b} and \vec{c} are coplanar.

18. Find the vector and Cartesian equations of the line through the point (1, 2, -4) perpendicular to the two lines.

$$\vec{r} = (8\hat{i} - 19\hat{j} + 10\hat{k}) + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k}) \text{ and}$$

$$\vec{r} = (15\hat{i} + 29\hat{j} + 5\hat{k}) + \mu(3\hat{i} - 8\hat{j} - 5\hat{k}).$$

Sol. The equations of the given lines are

$$\vec{r} = (8\hat{i} - 19\hat{j} + 10\hat{k}) + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k}) \dots (1)$$

$$\vec{r} = (15\hat{i} + 29\hat{j} + 5\hat{k}) + \mu(3\hat{i} - 8\hat{j} - 5\hat{k}) \dots (2)$$

Normal parallel to (1) is $\vec{n}_1 = 3\hat{i} - 16\hat{j} + 7\hat{k}$.

Normal parallel to (2) is $\vec{n}_2 = 3\hat{i} - 8\hat{j} - 5\hat{k}$.

The required line is perpendicular to the given lines. So, the normal \vec{n} parallel to the required line is perpendicular to \vec{n}_1 and \vec{n}_2 .

$$\therefore \vec{n} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix}$$

$$= 24\hat{i} + 36\hat{j} + 72\hat{k}$$

Thus, the vector equation of the required line is

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \gamma(24\hat{i} + 36\hat{j} + 72\hat{k})$$

$$\Rightarrow \vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + k(2\hat{i} + 3\hat{j} + 6\hat{k}) \text{ (Where } k = 12\gamma)$$

Also, the Cartesian equation of the required line is

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$$

19. Three persons A, B and C apply for a job of manager in a Private Company. Chances of their selection (A, B and C) are in the ratio 1:2:4. The probabilities that A, B and C can introduce changes to improve profits of the company are 0.8, 0.5 and 0.3, respectively. If the change does not take place, find the probability that it is due to the appointment of C.

OR

A and B throw a pair of dice alternately. A wins the game if he gets a total of 7 and B wins the game if he gets a total of 10. If A starts the game, then find the probability that B wins.

Sol. Let E_1 , E_2 and E_3 be the events denoting the selection of A, B and C as managers, respectively.

$$P(E_1) = \text{Probability of selection of } A = \frac{1}{7}$$

$$P(E_2) = \text{Probability of selection of } B = \frac{2}{7}$$

$$P(E_3) = \text{Probability of selection of } C = \frac{4}{7}$$

$$P\left(\frac{A}{E_1}\right) = \text{Probability that A does not introduce change} = 0.2$$

$$P\left(\frac{A}{E_2}\right) = \text{Probability that B does not introduce change} = 0.5$$

$$P\left(\frac{A}{E_3}\right) = \text{Probability that C does not introduce change} = 0.7$$

$$\therefore \text{required probability} = P\left(\frac{E_3}{A}\right)$$

By Bayes' theorem, we have

$$\begin{aligned} P\left(\frac{E_3}{A}\right) &= \frac{P(E_3)P\left(\frac{A}{E_3}\right)}{P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right) + P(E_3)P\left(\frac{A}{E_3}\right)} \\ &= \frac{\frac{4}{7} \times 0.7}{\frac{1}{7} \times 0.2 + \frac{2}{7} \times 0.5 + \frac{4}{7} \times 0.7} \\ &= \frac{2.8}{0.2 + 1 + 2.8} = \frac{2.8}{4} = 0.7 \end{aligned}$$

OR

Total of 7 on the dice can be obtained in the following ways:

(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)

$$\text{Probability of getting a total of 7} = \frac{3}{36} = \frac{1}{6}$$

$$\text{Probability of not getting a total of 10} = 1 - \frac{1}{12} = \frac{11}{12}$$

Total of 10 on the dice can be obtained in the following ways:

(4, 6), (6, 4), (5, 5)

$$\text{Probability of getting a total of 10} = \frac{3}{36} = \frac{1}{12}$$

Probability of not getting a total of 10 = $1 - \frac{1}{12} = \frac{11}{12}$

Let E and F be the two events, defined as follows:

E = Getting a total of 7 in a single throw of a dice

F = Getting a total of 10 in single throw of a dice

$$P(E) = \frac{1}{6}, P(\bar{E}) = \frac{5}{6}, P(F) = \frac{1}{12}, P(\bar{F}) = \frac{11}{12}$$

A wins if he gets a total of 7 in 1st, 3rd or 5th ... throws.

Probability of A getting a total of 7 in the 1st throw = $\frac{1}{6}$

A will get the 3rd throw if he fails in the 1st throw and B fails in the 2nd throw.

Probability of A getting total 7 in the 3rd throw = $P(\bar{E})P(\bar{F})P(E) = \frac{5}{6} \times \frac{11}{12} \times \frac{1}{6}$

Similarly, probability of getting a total of 7 in the 5th throw =

$$P(\bar{E})P(\bar{F})P(\bar{E})P(\bar{F})P(E) = \frac{5}{6} \times \frac{11}{12} \times \frac{5}{6} \times \frac{11}{12} \times \frac{1}{6} \text{ and so on}$$

Probability of winning of A = $\frac{1}{6} + \left(\frac{5}{6} \times \frac{11}{12} \times \frac{1}{6} \right) + \left(\frac{5}{6} \times \frac{11}{12} \times \frac{5}{6} \times \frac{11}{12} \times \frac{1}{6} \right) + \dots$

$$= \frac{\frac{1}{6}}{1 - \frac{5}{6} \times \frac{11}{12}} = \frac{12}{17}$$

\therefore Probability of winning of B = $1 - \text{probability of winning of A} = 1 - \frac{12}{17} = \frac{5}{17}$

Section C

20. Let $f: N \rightarrow N$ be a function defined as $f(x) = 9x^2 + 6x - 5$. Show that $f: N \rightarrow S$, where S is the range of f , is invertible. Find the inverse of f and hence find $f^{-1}(4)$ and $f^{-1}(163)$.

Sol. Given: $f(x) = 9x^2 + 6x - 5$

$$\text{Let } y = 9x^2 + 6x - 5$$

$$\Rightarrow y = (3x + 1)^2 - 1 - 5 = (3x + 1)^2 - 6$$

$$\Rightarrow (3x + 1)^2 = y + 6$$

$$\Rightarrow 3x + 1 = \sqrt{y + 6}$$

$$\Rightarrow x = \frac{\sqrt{y + 6} - 1}{3} \text{ as } x \in N$$

$$\Rightarrow \sqrt{y + 6} - 1 > 0$$

$$\Rightarrow y + 6 > 1$$

$$\Rightarrow y > -5 \text{ and } y \in N$$

So, the function is invertible if the range of the function $f(x)$ is $\{1, 2, 3, \dots\}$.

Therefore, the inverse of the function $f(x)$ is $f^{-1}(y)$, i.e. x .

Now,

$$f^{-1}(y) = \frac{\sqrt{y + 6} - 1}{3}$$

$$f^{-1}(43) = \frac{\sqrt{43 + 6} - 1}{3} = 2$$

$$f^{-1}(163) = \frac{\sqrt{163 + 6} - 1}{3} = 4$$

21. Prove that $\begin{vmatrix} yz - x^2 & zx - y^2 & xy - z^2 \\ zx - y^2 & xy - z^2 & yz - x^2 \\ xy - z^2 & yz - x^2 & zx - y^2 \end{vmatrix}$ is divisible by $(x + y + z)$ and hence find the quotient.

OR

Using elementary transformations, find the inverse of the matrix $A = \begin{pmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ and

use it to solve the following system of linear equations:

$$8x + 7y + 3z = 19$$

$$2x + y + z = 5$$

$$x + 2y + 2z = 7$$

Sol. $\begin{vmatrix} yz - x^2 & zx - y^2 & xy - z^2 \\ zx - y^2 & xy - z^2 & yz - x^2 \\ xy - z^2 & yz - x^2 & zx - y^2 \end{vmatrix}$

Applying $C_1 \rightarrow C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} xy + yz + zx - x^2 - y^2 - z^2 & zx - y^2 & xy - z^2 \\ xy + yz + zx - x^2 - y^2 - z^2 & xy - z^2 & yz - x^2 \\ xy + yz + zx - x^2 - y^2 - z^2 & yz - x^2 & zx - y^2 \end{vmatrix}$$

$$\Rightarrow \Delta = (xy + yz + zx - x^2 - y^2 - z^2) \begin{vmatrix} 1 & zx - y^2 & xy - z^2 \\ 1 & xy - z^2 & yz - x^2 \\ 1 & yz - x^2 & zx - y^2 \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we get

$$\Delta = (xy + yz + zx - x^2 - y^2 - z^2) \begin{vmatrix} 1 & zx - y^2 & xy - z^2 \\ 0 & (x + y + z)(y - z) & (x + y + z)(z - x) \\ 0 & (x + y + z)(y - x) & (x + y + z)(z - y) \end{vmatrix}$$

$$\Rightarrow \Delta = (x + y + z)^2 (xy + yz + zx - x^2 - y^2 - z^2) \\ [\{(y - z)(z - y) - (z - x)(y - x)\} - 0 + 0]$$

$$\Rightarrow \Delta = (x + y + z)^2 (xy + yz + zx - x^2 - y^2 - z^2)^2$$

So, Δ is divisible by $(x + y + z)$.

The quotient when Δ is divisible by $(x + y + z)$ is $(x + y + z)(xy + yz + zx - x^2 - y^2 - z^2)$.

OR

$$A = IA$$

i.e.

$$\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \leftrightarrow R_3$, we get

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 8 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

Applying $R_2 \rightarrow R_2 - 2R_1$, we get

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -3 \\ 8 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & 0 \end{bmatrix} A$$

Applying $R_3 \rightarrow R_3 - 8R_1$, we get

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -3 \\ 0 & -12 & -13 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & -8 \end{bmatrix} A$$

Applying $R_3 \rightarrow \frac{R_2}{-3}$, we get

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & -12 & -13 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{3} & \frac{2}{3} \\ 1 & 0 & -8 \end{bmatrix} A$$

Applying $R_1 \rightarrow R_1 - 2R_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -12 & -13 \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} \\ 1 & 0 & -8 \end{bmatrix} A$$

Applying $R_3 \rightarrow R_3 + 12R_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ 1 & -\frac{13}{3} & \frac{2}{3} \\ -1 & 4 & 0 \end{bmatrix} A$$

Thus, we have

$$A^{-1} = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ 1 & -\frac{13}{3} & \frac{2}{3} \\ -1 & 4 & 0 \end{bmatrix}$$

The given system of equation is

$$8x + 4y + 3z = 19$$

$$2x + y + z = 5$$

$$x + 2y + 2z = 7$$

This system of equations can be written as $AX = B$, where

$$A = \begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 19 \\ 5 \\ 7 \end{bmatrix}$$

$$\therefore X = A^{-1} B$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ 1 & -\frac{13}{3} & \frac{2}{3} \\ -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 19 \\ 5 \\ 7 \end{bmatrix}$$

$$\therefore x = 1, y = 2 \text{ and } z = 1$$

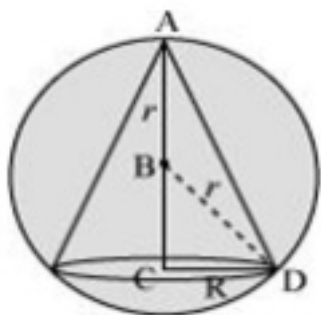
22. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$. Also find maximum volume in terms of volume of the sphere.

OR

Find the intervals in which $f(x) = \sin 3x - \cos 3x$, $0 < x < \pi$, is strictly increasing or strictly decreasing.

Sol. A sphere of fixed radius (r) is given.

Let R and h be the radius and the height of the cone, respectively.



The volume (V) of the cone is given by,

$$V = \frac{1}{3} \pi R^2 h$$

Now, from the right triangle BCD , we have:

$$BC = \sqrt{r^2 - R^2}$$

$$\therefore h = r + \sqrt{r^2 - R^2}$$

$$\therefore v = \frac{1}{3} \pi R^2 (r + \sqrt{r^2 - R^2}) = \frac{1}{3} \pi R^2 r + \frac{1}{3} \pi R^2 \sqrt{r^2 - R^2}$$

$$\therefore \frac{dV}{dR} = \frac{2}{3} \pi R r + \frac{2}{3} \pi R \sqrt{r^2 - R^2} + \frac{\pi R^2}{3} \cdot \frac{(-2R)}{2\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3} \pi R r + \frac{2}{3} \pi R \sqrt{r^2 - R^2} - \frac{\pi R^3}{3\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3} \pi R r + \frac{2\pi R(r^2 - R^2) - \pi R^3}{3\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3} \pi R r + \frac{2\pi R r^2 - 3\pi R^3}{3\sqrt{r^2 - R^2}}$$

Now, $\frac{dV}{dR} = 0$

$$\Rightarrow \frac{2\pi r R}{3} = \frac{3\pi R^3 - 2\pi R r^2}{3\sqrt{r^2 - R^2}}$$

$$\Rightarrow 2r\sqrt{r^2 - R^2} = 3R^2 - 2r^2$$

$$\Rightarrow 4r^2(r^2 - R^2) = (3R^2 - 2r^2)^2$$

$$\Rightarrow 4r^4 - 4r^2 R^2 = 9R^4 + 4r^4 - 12R^2 r^2$$

$$\Rightarrow 9R^4 - 8r^2 R^2 = 0$$

$$\Rightarrow 9R^2 = 8r^2$$

$$\Rightarrow R^2 = \frac{8r^2}{9}$$

Now,

$$= \frac{2\pi r}{3} + \frac{3\sqrt{r^2 - R^2} (2\pi r^2 - 9\pi R^2) + (2\pi Rr^2 - 3\pi R^3)(3R)}{9(r^2 - R^2)} \frac{1}{2\sqrt{r^2 - R^2}}$$

$$= \frac{2\pi r}{3} + \frac{3\sqrt{r^2 - R^2} (2\pi r^2 - 9\pi R^2) + (2\pi Rr^2 - 3\pi R^3)(3R)}{9(r^2 - R^2)} \frac{1}{2\sqrt{r^2 - R^2}}$$

Now, when $R^2 = \frac{8r^2}{9}$, it can be shown that $\frac{d^2V}{dR^2} < 0$.

\therefore The volume is the maximum when $R^2 = \frac{8r^2}{9}$.

When $R^2 = \frac{8r^2}{9}$, height of the cone $= r + \sqrt{r^2 - \frac{8r^2}{9}} = r + \sqrt{\frac{r^2}{9}} = r + \frac{r}{3} = \frac{4r}{3}$.

Hence, it can be seen that the altitude of a right of a right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

Let volume of the sphere be $V_s = \frac{4}{3} \pi r^3$.

$$r = 3 \sqrt{\frac{3V_s}{4\pi}}$$

\therefore Volume of cone, $V = \frac{1}{3} \pi R^2 h$

$$\Rightarrow R = \frac{2\sqrt{2}}{3} r$$

$$v = \frac{1}{3} \pi \left(\frac{2\sqrt{2}}{3} r \right)^2 \times \frac{4r}{3}$$

$$\Rightarrow V = \frac{1}{3} \pi \frac{8r^3}{9} \times \frac{4r}{3}$$

$$V = \frac{32\pi r^3}{81} = \frac{32}{81} \pi \left[\frac{3V_s}{4\pi} \right]$$

$$\therefore \text{Volume of cone in terms of sphere} = \frac{8V_s}{27}$$

OR

Consider the function $f(x) = \sin 3x - \cos 3x$.

$$f(x) = 3\cos 3x + 3\sin 3x$$

$$= 3(\sin 3x + \cos 3x)$$

$$= 3\sqrt{2} \left\{ \sin 3x \cos \left(\frac{\pi}{4} \right) + \cos 3x \sin \left(\frac{\pi}{4} \right) \right\}$$

$$= 3\sqrt{2} \left\{ \sin \left(3x + \frac{\pi}{4} \right) \right\}$$

For the increasing interval $f'(x) > 0$.

$$3\sqrt{2} \left\{ \sin \left(3x + \frac{\pi}{4} \right) \right\} > 0$$

$$\sin \left(3x + \frac{\pi}{4} \right) > 0$$

$$\Rightarrow 0 < 3x + \frac{\pi}{4} < \pi$$

$$\Rightarrow 0 < 3x + \frac{3\pi}{4}$$

$$\Rightarrow 0 < x < \frac{\pi}{4}$$

Also,

$$\sin\left(3x + \frac{\pi}{4}\right) > 0$$

$$\text{When, } 2\pi < 3x + \frac{\pi}{4} < 3\pi$$

$$\Rightarrow \frac{7\pi}{4} < 3x < \frac{11\pi}{4}$$

$$\Rightarrow \frac{7\pi}{12} < x < \frac{11\pi}{12}$$

Therefore, intervals in which function is strictly increasing in

$$0 < x < \frac{\pi}{4} \text{ and } \frac{7\pi}{12} < x < \frac{11\pi}{12}.$$

Similarly, for the decreasing intervals $f'(x) < 0$.

$$3\sqrt{2} \left\{ 3x + \frac{\pi}{4} \right\} < 0$$

$$\sin\left(3x + \frac{\pi}{4}\right) < 0$$

$$\Rightarrow \pi < 3x + \frac{\pi}{4} < 2\pi$$

$$\Rightarrow \frac{3\pi}{4} < 3x < \frac{7\pi}{4}$$

$$\Rightarrow \frac{\pi}{4} < x < \frac{7\pi}{12}$$

Also,

$$\sin\left(3x + \frac{\pi}{4}\right) < 0$$

$$\text{When } 3\pi < 3x + \frac{\pi}{4} < 4\pi.$$

$$\Rightarrow \frac{11\pi}{4} < 3x < \frac{15\pi}{4}$$

$$\Rightarrow \frac{11\pi}{12} < x < \frac{15\pi}{12}$$

The function is strictly decreasing in $\frac{\pi}{4} < x < \frac{7\pi}{12}$ and $\frac{11\pi}{12} < x < \pi$.

23. Using integration find the area of the region

$$\{(x, y) : x^2 + y^2 \leq 2ax, y^2 \geq ax, x, y \geq 0\}.$$

Sol. Given: $x^2 + y^2 \leq 2ax, y^2 \geq ax, x, y \geq 0$.

$$\Rightarrow x^2 + y - 2ax \leq 0, y^2 \geq ax, x, y \geq 0$$

$$\Rightarrow x^2 + y^2 - 2ax + a^2 - a^2 \leq 0, y^2 \geq ax, x, y, \geq 0$$

$$\Rightarrow (x - a)^2 + y^2 \leq a^2, y^2 \geq ax, x, y \geq 0$$

Total find the points of intersection of the circle $\left[(x - a)^2 + y^2 = a^2\right]$

and the parabola $\left[y^2 = ax\right]$, we will substitute $y^2 = ax$ in $(x - a)^2 + y^2 = a^2$.

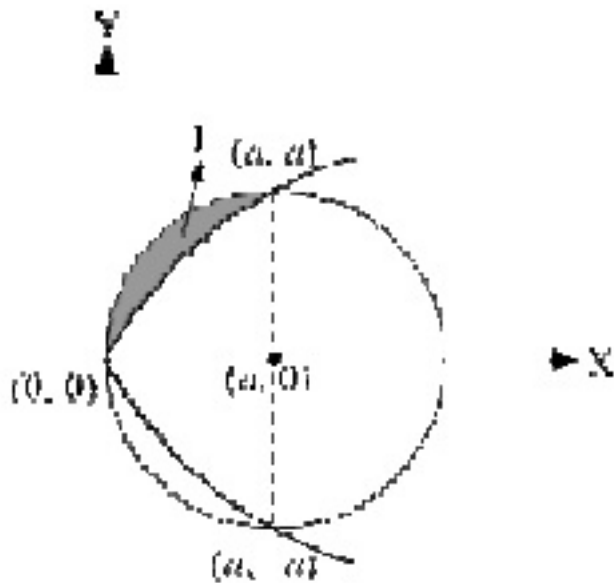
$$(x - a)^2 + ax = a^2$$

$$\Rightarrow x^2 + a - 2ax + ax = a^2$$

$$\Rightarrow x(x - a) = 0$$

$$\Rightarrow x = 0, a$$

Therefore, the points of intersection are (0, 0), (a, a) and (a, -a).



Now,

Area of the shaded region = I

Area of I from $x = 0$ to $x = a$

$$= \left[\int_0^a \left(\sqrt{a^2 - (x-a)^2} \right) dx - \int_0^a \sqrt{ax} dx \right]$$

Let $x - a = t$ for the first part of the integral

$$= \int_0^a \left(\sqrt{a^2 - (x-a)^2} \right) dx.$$

$$\Rightarrow dx = dt$$

$$\therefore A_I = \int_{-a}^0 \sqrt{a^2 - t^2} dt - 2 \frac{\sqrt{a}}{3} \Big| x^{\frac{3}{2}} \Big|_0^a$$

$$= \left[\frac{t}{a} \sqrt{a^2 - t^2} + \frac{1}{2} a^2 \sin^{-1} \frac{t}{a} \right]_0^{-a} - \frac{2a^2}{3}$$

$$= 0 - \left(-\frac{\pi a^2}{4} \right) - \frac{2a^2}{3}$$

$$A_f = \left(\frac{\pi}{4} - \frac{2}{3} \right) a^2$$

$$\therefore \text{Area of the shaded region} = \left(\frac{\pi}{4} - \frac{2}{3} \right) a^2 \text{ square units}$$

24. Find the coordinate of the point P where the line through A (3, -4, -5) and B(2, -3, 1) crosses the plane passing through three points L(2, 2, 1), M(3, 0, 1) and N(4, -1, 0). Also, find the ratio in which P divides the line segment AB.

Sol. The equation of the plane passing through three given points can be given by

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ x-3 & y-0 & z-1 \\ x-4 & y+1 & z-0 \end{vmatrix} = 0$$

Performing elementary row operation $R_2 \rightarrow R_1 - R_2$ and $R_3 \rightarrow R_1 - R_3$, we get

$$\Rightarrow \begin{vmatrix} x-2 & y-2 & z-1 \\ 3-2 & 0-2 & 0 \\ 4-2 & -1-2 & -1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-2 & y-2 & z-1 \\ 1 & -2 & 0 \\ 2 & -3 & -1 \end{vmatrix} = 0$$

Solving the above determinant, we have

$$\Rightarrow (x - 2)(2 - 0) - (y - 2)(-1 - 0) + (z - 1)(-3 + 4) = 0$$

$$\Rightarrow (2x - 4) + (y - 2) + (z - 1) = 0$$

$$\Rightarrow 2x + y + z - 7 = 0$$

Therefore, the equation of the plane is $2x + y + z - 7 = 0$.

Now, the equation of the line passing through two given points is

$$\frac{x-3}{2-3} = \frac{y+4}{-3+4} = \frac{z+5}{1+5} = \lambda$$

$$\Rightarrow \frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} = \lambda$$

$$\Rightarrow x = (-\lambda + 3), y = (\lambda - 4), z = (6\lambda - 5)$$

At the point of intersection, these points satisfy the equation of the plane

$$2x + y + z - 7 = 0$$

Putting the values of x, y and z in the equation of the plane, we get the value of λ .

$$2(-\lambda + 3) + (\lambda - 4) + (6\lambda - 5) - 7 = 0$$

$$\Rightarrow -2\lambda + 6 + \lambda - 4 + 6\lambda - 5 - 7 = 0$$

$$\Rightarrow 5\lambda = 10$$

$$\Rightarrow \lambda = 2$$

Thus, the point of intersection is P(1, -2, 7).

Now, let P divide the line AB in the ratio m:n.

By the section formula, we have

$$1 = \frac{2m + 3n}{m + n}$$

$$\Rightarrow m + 2n = 0$$

$$\Rightarrow m = -2n$$

$$\Rightarrow mn = \frac{-2}{1}$$

Hence, P externally divides the line segment AB in the ratio 2:1.

25. An urn contains 3 white and 6 red balls. Four balls are drawn one by one with replacement from the urn. Find the probability distribution of the number of red balls drawn. Also find mean and variance of the distribution.

Sol. Let X denote the total number of red balls when four balls are drawn one by one with replacement.

$$P(\text{getting a red ball in one draw}) = \frac{2}{3}$$

$$P(\text{getting a white ball in one draw}) = \frac{1}{3}$$

X	0	1	2	3	4
P(X)	$\left(\frac{1}{3}\right)^4$	$\frac{2}{3}\left(\frac{1}{3}\right)^3 \cdot {}^4C_1$	$\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) \cdot {}^4C_2$	$\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 \cdot {}^4C_3$	$\left(\frac{2}{3}\right)^4$
	$\frac{1}{81}$	$\frac{8}{81}$	$\frac{24}{81}$	$\frac{32}{81}$	$\frac{16}{81}$

Using the formula for mean, we have

$$\bar{X} = \sum P_i X_i$$

$$\text{Mean}(\bar{X}) = \left(0 \times \frac{1}{81}\right) + 1\left(\frac{8}{81}\right) + 2\left(\frac{24}{81}\right) + 3\left(\frac{32}{81}\right) + 4\left(\frac{16}{81}\right)$$

$$= \frac{1}{81}(8 + 48 + 96 + 64)$$

$$= \frac{216}{81} = \frac{8}{3}$$

Using the formula for variance, we have

$$Var(X) = \sum P_i X_i^2 - \left(\sum P_i X_i \right)^2$$

$$Var(X) = \left\{ \left(0 \times \frac{1}{81} \right) + 1 \left(\frac{8}{81} \right) + 4 \left(\frac{24}{81} \right) + 9 \left(\frac{32}{81} \right) + 16 \left(\frac{16}{81} \right) \right\} - \left(\frac{8}{3} \right)^2$$

$$= \frac{64}{81} - \frac{64}{9} = \frac{8}{9}$$

Hence, the mean of the distribution is $\frac{8}{3}$ and the variance of the distribution is $\frac{8}{9}$.

26. A manufacturer produces two products A and B. Both the products are processed on two different machines. The available capacity of first machine is 12 hours and that of second machine is 9 hours per day. Each unit of product A requires 3 hours on both machines and each unit of product B requires 2 hours on first machine and 1 hour on second machine. Each unit of product A is sold at Rs 7 profit and b at a profit of Rs 4. Find the production level per day for maximum profit graphically.

Sol. Let the number of units of production is be x and y, respectively.

Product	Machine	
	I(h)	II (h)
A	3	3
B	2	1

Total profit: $Z = 7x + 4y$

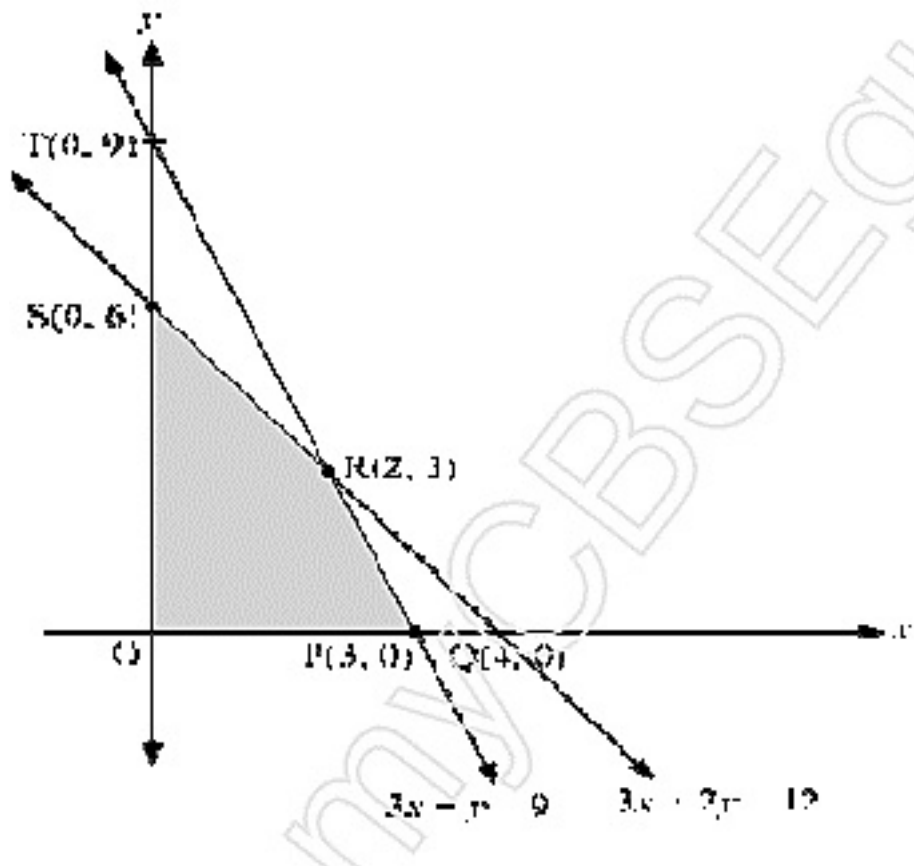
We have to maximise $Z = 7x + 4y$, which is subject to constraints.

$3x + 2y \leq 12$ (Constraint on machine I)

$3x + y \leq 9$ (Constraint on machine II)

$\Rightarrow x \geq 0$ and $y \geq 0$

The given information can be graphically expressed as follows:



Values of $Z = 7x + 4y$ at the corner points are as follows:

Corner Point	$Z = 7x + 4y$
(0, 6)	24
(2, 3)	26
(3, 0)	21

Therefore, the manufacturer has to produce 2 units of product A and 3 units of product B for the maximum profit of Rs 26.