Question Paper 2016 East Outside Delhi Set-3

CBSE Class XII Mathematics

General Instruction:

- All question are compulsory.
- Please check that this Question Paper contains 26 Questions.
- Questions 1 to 6 in Section-A are Very Short Answer Type Questions carrying one mark each.
- Questions 7 to 19 in Section-B are Long Answer I Type Questions carrying 4 marks each.
- Questions 20 to 26 in Section-C are Long Answer II Type Questions carrying 6 marks each.
- Please write down the serial number of the Question before attempting it.

Section A

1. Write the coordinates of the point which is the reflection of the point (α, β, λ) in the XZ-plane.

Sol. $(\alpha, -\beta, \gamma)$

2. Find the position vector of the point which divides the join of points with position vector $\bar{a} + 3\bar{b}$ and $\bar{a} - \bar{b}$ internally in the ratio 1:3.

Sol. $\dfrac{1(\bar{a} - \bar{b}) + 3(\bar{a} + 3\bar{b})}{4}$ (i.e., using correct formula)

$= \bar{a} + 2\bar{b}$

3. If $|\bar{a}| = 4$, $|\bar{b}| = 3$ and $\bar{a}.\bar{b} = 6\sqrt{3}$, then find the value of $|\bar{a} \times \bar{b}|$.

Sol. Finding $\cos\theta = \frac{\sqrt{3}}{2}$

$\left|\bar{a} \times \bar{b}\right| = 6$

4. Write the value of $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

Sol. $R_1 \rightarrow R_1 + R_2 + R_3$ **and** $C_1 + C_1 + C_2 + C_3$

Ans. 0

5. If $A = \begin{pmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{pmatrix}$ **and** $B = \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{pmatrix}$ **and BA = (b_{ij}), find b_{21} + b_{32}.**

Sol. b_{21} = -16, b_{23} = -2 [For any one correct value]

b_{21} + b_{23} = -16 + (-2) = -18

6. Write the number of all possible matrices of order 2×3 **with each entry 1 or 2.**

Sol. 2^6 or 64

Section B

7. Find the equation of the tangent line to the curve $y = \sqrt{5x-3} - 5$, **which is parallel to the line 4x – 2y + 5 = 0.**

Sol. $y = \sqrt{5x-3} - 5$

$$\frac{dy}{dx} = \frac{5}{2\sqrt{5x-3}}$$

Slope of line 4x – 2y + 5 = 0 is $\frac{-4}{-2} = 2$.

$$\therefore \qquad \frac{52\sqrt{5x-3}}{} = 2 \quad x\frac{73}{80}$$

Putting $x = \frac{73}{80}$ in eqn. (i), we get $y = \frac{-15}{4}$

Equation of tangent

$$y + \frac{15}{4} = 2\left(x - \frac{73}{80}\right)$$

or 80x – 40y – 223 = 0

8. Solve the differential equation:

(x^2 + 3xy + y^2) dx – x^2dy = 0 given that y = 0, when x = 1.

Sol. (x^2 + 3xy + y^2) dx – x^2 dy = 0

$$\frac{dy}{dx} = \frac{x^2 + 3xy + y^2}{x^2}$$

Let y = vx

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore \qquad v + x\frac{dv}{dx} = 1 + 3v + v^2$$

$$\Rightarrow \qquad x\frac{dv}{dx} = v^2 + 2v + 1$$

$$\Rightarrow \qquad \frac{dv}{(v+1)^2} = \frac{dx}{x}$$

Integrating both sides

$$\Rightarrow \qquad -\frac{1}{v+1} = \log|x| + C$$

$$\Rightarrow \qquad \frac{-x}{x+y} = \log|x| + C$$

When x = 1, y = 0 ⇒ C = -1

$$\Rightarrow \qquad \frac{-x}{x+y} = \log|x| - 1$$

$$\Rightarrow \qquad y = (x+y)\log|x|$$

or $y = \dfrac{x\log|x|}{1-\log|x|}$

9. On her birthday Seema decided to donate some money to children of an orphanage home. If there were 8 children less, every one would have got Rs 10 more. However, if there were 16 children more, every one would have got Rs 10 less. Using matrix method, find the number of children and the amount distributed by Seema. What value are reflected by Seema's decision?

Sol. Let the number of children be x and the amount distributed by Seema for one student be Rs y.

So, $(x-8)(y+10) = xy$

$$\Rightarrow \qquad 5x - 4y = 40 \quad \text{....(i)}$$

and (x + 16)(y – 10) = xy

$\Rightarrow 5x - 8y = -80$ ..... (ii)

Here $A = \begin{matrix} 5 & -4 \\ 5 & -8 \end{matrix};\ X = \begin{matrix} x \\ y \end{matrix};\ B = \begin{matrix} 40 \\ -80 \end{matrix}$

$$AX = B \Rightarrow X = A^{-1}B$$

$$A^{-1} = -\frac{1}{20}\begin{matrix}-8 & 4 \\ -5 & 5\end{matrix}$$

$$\Rightarrow \quad \begin{matrix}x \\ y\end{matrix} = \begin{matrix}32 \\ 30\end{matrix}$$

$$\Rightarrow \quad x = 32,\ y = 30$$

No. of students = 32

Amount given to each student = Rs 30.

Value reflected: To help needy people.

10. Show that the lines $\frac{x-1}{3} = \frac{y-1}{-1} = \frac{z+1}{0}$ and $\frac{x-4}{2} = \frac{y}{0} = \frac{z+1}{3}$ intersect. Find their point of intersection.

Sol. $\frac{x-1}{3} = \frac{y-1}{-1} = \frac{z+1}{0} = \lambda\ (let)$ ...(i)

$$\Rightarrow \quad x = 3\lambda + 1,\ y = -\lambda + 1,\ z = -1$$

$\frac{x-4}{2} = \frac{y}{0} = \frac{z+1}{3} = \mu$ .....(ii)

$$\Rightarrow \quad x = 2\mu + 4,\ y = 0,\ z = 3\mu - 1$$

If the lines intersect, then they have a common point for some value of λ and μ.

So $3\lambda + 1 = 2\mu + 4$ ....(iii)

$-\lambda + 1 = 0 \Rightarrow \lambda = 1$

$3\mu - 1 = -1 \Rightarrow \mu = 0$

Since $\lambda = 1$ & $\mu = 0$ satisfy equation (iii) so the given lines intersect and the point of intersection is (4, 0, -1).

11. Show that the function f given by:

$$f(x)=\begin{cases}\dfrac{e^{1/x}-1}{e^{1/x}+1} & ,if\ x\neq 0\\ -1 & ,if\ x=0\end{cases}$$

Is discontinuous at x = 0.

Sol. $f(x)=\dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}\quad x\neq 0$

-1 = x = 0

$$LHL:\lim_{x\to 0^-}\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$$

$$=\lim_{h\to 0}\frac{e^{-\frac{1}{h}}-1}{e^{-\frac{1}{h}}+1}=\frac{0-1}{0+1}=-1$$

RHL: $\lim_{h\to 0}\dfrac{e^{\frac{1}{h}}-1}{e^{\frac{1}{h}}+1}=\lim_{h\to 0}\dfrac{1-e^{-\frac{1}{h}}}{1+e^{-\frac{1}{h}}}=1$

LHL ≠ RHL

∴ f(x) is discontinuous at x = 0

12. Find: $\displaystyle\int\frac{2x+1}{(x^2+1)(x^2+4)}dx$

Sol. Let $I=\displaystyle\int\frac{2x+1}{(x^2+1)(x^2+4)}dx$

Let $\dfrac{2x+1}{(x^2+1)(x^2+4)}dx = \dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{x^2+4}$

Getting $A=\dfrac{2}{3}, B=\dfrac{1}{3}, C=\dfrac{-2}{3}, D=\dfrac{-1}{3}$

$$\therefore \qquad I=\frac{2}{3}\int\frac{x}{x^2+1}dx+\frac{1}{3}\int\frac{1}{x^2+1}dx+\frac{-2}{3}\int\frac{xdx}{x^2+4}+\frac{-1}{3}\int\frac{dx}{x^2+4}$$

$$=\frac{1}{3}\log\left|x^2+1\right|+\frac{1}{3}\tan^{-1}x-\frac{1}{3}\log\left|x^2+4\right|-\frac{1}{6}\tan^{-1}\frac{x}{2}+C$$

13. If $x=e^{\cos 2t}$ **and** $y=e^{\sin 2t}$**, prove that** $\dfrac{dy}{dx}=-\dfrac{y\log x}{x\log y}$.

OR

Verify Mean value theorem for the function f(x) = 2 sinx + sin2x on [0, π].

Sol. $\dfrac{dx}{dt}=e^{\cos 2t}\left(-2\sin 2t\right)$ or $-2x\sin 2t$

$\dfrac{dy}{dt}=e^{\sin 2t}2\cos 2t$ or $2y\cos 2t$

$\dfrac{dy}{dx}=\dfrac{-e^{\sin 2t}2\cos 2t}{e^{\cos 2t}2\sin 2t}$ or $-\dfrac{y}{x}\dfrac{\cos 2t}{\sin 2t}$

$=\dfrac{-y\log x}{x\log y}$

OR

f(x) = 2 sin x + sin 2x on [0, π]

f(x) is continuous in [0, π]

f(x) is differentiable in (0, π)

∴ Mean value theorem is applicable

f(0) = 0, f(π) = 0

f'(x) = 2 cos x + 2 cos 2x

f'(c) = 2 cos c + 2cos 2c

$$f'(c) = \frac{f(\pi) - f(0)}{\pi - 0} = 0$$

∴ 2 cos c + 2 cos 2c = 0

$$\Rightarrow \quad \cos c + 2\cos 2c - 1 = 0$$

$$\Rightarrow \quad (2\cos c - 1)(\cos c + 1) = 0$$

$$\Rightarrow \quad \cos c = \frac{1}{2}$$

$$\Rightarrow \quad c = \frac{\pi}{3} \in (0, \pi)$$

Hence mean value theorem is verified.

14. Solve for $x: \tan^{-1}\left(\frac{2-x}{2+x}\right) = \frac{1}{2}\tan^{-1}\frac{x}{2}, x > 0.$

OR

Prove that $2\sin^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{17}{31}\right) = \frac{\pi}{4}.$

Sol. $\tan^{-1}\left(\frac{2-x}{2+x}\right) = \frac{1}{2}\tan^{-1}\frac{x}{2}$

$$\Rightarrow 2\tan^{-1}\left(\frac{2-x}{2+x}\right)=\tan^{-1}\frac{x}{2}$$

$$\Rightarrow \tan^{-1}\frac{2\left(\frac{2-x}{2+x}\right)}{1-\left(\frac{2-x}{2+x}\right)^2}=\tan^{-1}\frac{x}{2}$$

$$\Rightarrow \tan^{-1}\frac{4-x^2}{4x}=\tan^{-1}\frac{x}{2}$$

$$\Rightarrow \frac{4-x^2}{4x}=\frac{x}{2}$$

$$\Rightarrow x=\frac{2}{\sqrt{3}}\ (\because x>0)$$

OR

$$2\sin^{-1}\left(\frac{3}{5}\right)-\tan^{-1}\left(\frac{17}{31}\right)$$

$$=2\sin^{-1}\left(\frac{3}{4}\right)-\tan^{-1}\left(\frac{17}{31}\right)$$

$$=\tan^{-1}\frac{2\times\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2}-\tan^{-1}\left(\frac{17}{31}\right)$$

$$=\tan^{-1}\frac{24}{7}-\tan^{-1}\frac{17}{31}$$

$$= \tan^{-1}\left(\frac{\frac{24}{7}-\frac{17}{31}}{1+\frac{24}{7}\times\frac{17}{31}}\right)$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

15. Evaluate: $\int_1^5 \{|x-1|+|x-2|+|x-3|\}\, dx$

OR

Evaluate: $\int_0^{\pi} \frac{x\sin x}{1+3\cos^2 x}\, dx$

Sol. $\int_1^5 \{|x-1|+|x-2|+|x-3|\}\, dx$

$$= \int_1^5 (x-1)\, dx + \int_1^2 (2-x) dx + \int_2^5 (x-2) dx + \int_1^3 (3-x)\, dx + \int_1^5 (x-3)\, dx$$

$$= \left[\frac{x^2}{2}-x\right]_1^5 + \left[2x-\frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2}-2x\right]_2^5 + \left[3x-\frac{x^2}{2}\right]_1^3 + \left[\frac{x^2}{3}-3x\right]_3^5$$

= 17

OR

Let $I = \int_0^{\pi} \frac{x\sin x}{1+3\cos^2 x} dx$ .... (i)

$$I = \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+3\cos^2(\pi-x)} dx$$

$$= \int_0^{\pi} \frac{\pi \sin x}{1+3\cos^2 x}dx - \int_0^{\pi} \frac{x \sin x}{1+3\cos^2 x}dx$$

Adding (i) & (ii), we have

$$2I = \int_0^{\pi} \frac{\pi \sin x}{1+3\cos^2 x}dx$$

Put cos x = t

-sin x dx = dt, when x = 0 ⇒ t = 1, for x = π ⇒ t = -1

$$2I = -\pi \int_1^{-1} \frac{dx}{1+3t^2}$$

$$= \frac{\pi}{3} \int_{-1}^{1} \frac{dt}{\left(\frac{1}{\sqrt{3}}\right)^2 + (t)^2}$$

$$= \frac{\pi}{3} \times \sqrt{3} \left[\tan^{-1}(\sqrt{3t} \right]_{-1}^{1}$$

$$= \frac{\sqrt{3}\pi}{3} [\tan^{-1}\sqrt{3} - (-\tan^{-1}\sqrt{3})]$$

$$I = \frac{\sqrt{3}\pi}{3} \cdot \frac{\pi}{3} = \frac{\sqrt{3}\pi^2}{9}$$

16. Solve the differential equation:

$$x\frac{dy}{dx} + y - x + xy \cot x = 0;\ x \neq 0.$$

Sol. $x\frac{dy}{dx}+y-x+xy\cot x=0$

$$\Rightarrow \quad \frac{dy}{dx}+\left(\frac{1}{x}+\cot x\right)y=1$$

$$I\cdot F=e^{\int\left(\frac{1}{x}+\cot x\right)dx}=e^{\log(x\sin x)}$$

= x sin x

$$\therefore\ y\times x\sin x=x\sin x\,dx$$

$$\Rightarrow \quad xy\sin x=-x\cos x+\sin x+C$$

17. A committee of 4 student is selected at random from a group consisting of 7 boys and 4 girls. Find the probability that there are exactly 2 boys in the committee, given that at least one girl must be there in the committee.

OR

A random variable X has the following probability distribution:

X	0	1	2	3	4	5	6
P(X)	C	2C	2C	3C	C^2	$2C^2$	$7C^2+C$

Find the value of C and also calculate mean of the distribution.

Sol. Let A = exactly 2 boys in the committee

B = at least one girl must be there in the committee.

$$P(B)=\frac{{}^4C_1\times{}^7C_3+{}^4C_2\times{}^7C_2+{}^4C_3\times{}^7C_1+{}^4C_4}{{}^{11}C_4}$$

$$=\frac{59}{66}$$

$$P(A \cap B) = \frac{^4C_2 \times ^7C_2}{^{11}C_4} = \frac{21}{55}$$

$$P(A / B) = \frac{P(A \cap B)}{P(B)} = \frac{21/55}{59/66} = \frac{126}{295}$$

OR

C + 2C + 2C + 3C + C^2 + $2C^2$ + $7C^2$ + C = 1

⇒ $10C^2$ + 9C = 1

⇒ $10C^2$ + 9C – 1 = 0

$\therefore \qquad C = \frac{1}{10}$

Mean = 0 × C + 1 × 2C + 2 × + 2C + 3 × 3C + 4×C^2 + 5 × $2C^2$ + 6($7C^2$ + C)

= $56C^2$ + 21C

$$= 56 \times \frac{1}{100} + 21 \times \frac{1}{10}$$

= 0.56 + 2.1 = 2.66

18. Find the angle between the vectors $\bar{a} + \bar{b}$ **and he** $\bar{a} - \bar{b}$ **if** $\bar{a} = 2\hat{i} - \hat{j} + 3\hat{k}$ **and** $\bar{b} = 3\hat{i} + \hat{j} - 2\hat{k}$, **and hence find a vector perpendicular to both** $\bar{a} + \bar{b}$ **and** $\bar{a} - \bar{b}$.

Sol. $\bar{a} + \bar{b} = 5\hat{i} + \hat{k}$

$\bar{a} - \bar{b} = \hat{i} - 2\hat{j} + 5\hat{k}$

Getting $\cos\theta = 0$

$\Rightarrow \qquad \theta = \frac{\pi}{2}$

a vector perpendicular to both $\bar{a}+\bar{b}$ & $\bar{a}-\bar{b}$ is $(\bar{a}+\bar{b})\times(\bar{a}-\bar{b}) = 2\hat{i} - 26\hat{j} - 10\hat{k}$

19. Find: $\int (3x+5)\sqrt{5+4x-2x^2}\,dx$

Sol. $(3x+5)\sqrt{5+4x-2x^2}\,dx$

Let 3x + 5 = A(4 – 4x) + B

$$\Rightarrow \qquad A = -\frac{3}{4},\ B = 8$$

$$I = -\frac{3}{4}(4-4x)\sqrt{5+4x-2x^2}\,dx + 8\sqrt{5+4x-2x^2}\,dx$$

$$= -\frac{3}{4}I_1 + 8I_2 (let)$$

For I_1, put $5 + 4x - 2x^2 = t$

$$\Rightarrow \qquad (4-4x)dx = dt$$

$$-\frac{3}{4}I_1 = -\frac{3}{4}\sqrt{t}\ dt = -\frac{3}{4}\times\frac{2}{3}t^{3/2}$$

$$= -\frac{1}{2}(5+4x-2x^2)^{3/2}$$

$$8I_2 = 8\sqrt{2}\sqrt{\frac{7}{2}-(x-1)^2}\,dx$$

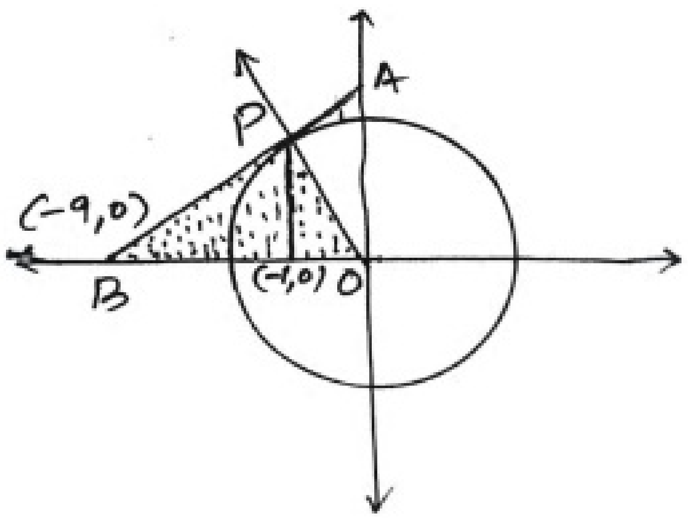
$$= 8\sqrt{2}\left[\frac{x-1}{2}\sqrt{\frac{5}{2}+2x-x^2} + \frac{7}{4}\sin^{-1}\frac{\sqrt{2}(x-)}{\sqrt{7}}\right]$$

$$I = -\frac{1}{2}(5+4x-2x^2)^{3/2} + 4\sqrt{2}(x-1)\sqrt{\frac{5}{2}+2x-x^2} + 14\sqrt{2}\sin^{-1}\frac{\sqrt{2}(x-)}{\sqrt{7}} + C$$

Section C

20. Using integration, find the area of the triangle formed by negative x-axis and tangent and normal to the circle $x^2 + y^2 + = 9$ at $\left(-1, 2\sqrt{2}\right)$.

Sol. Equation of circle $x^2 + y^2 = 9$



Diff. w.r.t x, we have

$$\frac{dy}{dx} = -\frac{x}{y}$$

Slope of tangent at $\left(-1, 2\sqrt{2}\right)$

$$m_T = \left(-\frac{x}{y}\right)_{(-1,2\sqrt{2})} = \frac{1}{2\sqrt{2}}$$

eqn. of tangent

$$y - 2\sqrt{2} = \frac{1}{2\sqrt{2}}(x+1)$$

$$\Rightarrow \qquad x - 2\sqrt{2}y + 9 = 0$$

It cuts x-axis at (-9, 0)

eqn. of normal

$$y - 2\sqrt{2} = -2\sqrt{2}(x+1)$$

$$\Rightarrow \qquad 2\sqrt{2}x + y = 0$$

Area of $\triangle$OPB

$$A = \int_{-9}^{-1} \frac{x+9}{2\sqrt{2}}\, dx + \int_{-1}^{0} -2\sqrt{2}x\, dx$$

$$= \frac{1}{2\sqrt{2}}\left[\frac{x^2}{2} + 9x\right]_{-9}^{-1} - 2\sqrt{2}\left[\frac{x^2}{2}\right]_{-1}^{0}$$

$$= 9\sqrt{2}\ sq.unit$$

21. A company manufactures two types of cardigans: type A and type B. It costs Rs 360 to make a type A cardigan and Rs 120 to make a type B cardigan. The company can make at most 300 cardigans and spend at most Rs 72, 000 a day. The number of cardigans of type B cannot exceed the number of cardigans of type A by more than 200. The company makes a profit of Rs 100 for each cardigan of type A and Rs 50 for every cardigan of type B. Formulate this problem as a linear programming problem to maximise the profit to the company. Solve is graphically and find maximum profit.

Sol. Let no. of cardigans of type A be x and that of type B by y.

Then, max. Z = 100 x + 50y

Subject to constant,

$x + y \le 300$ .....(i)

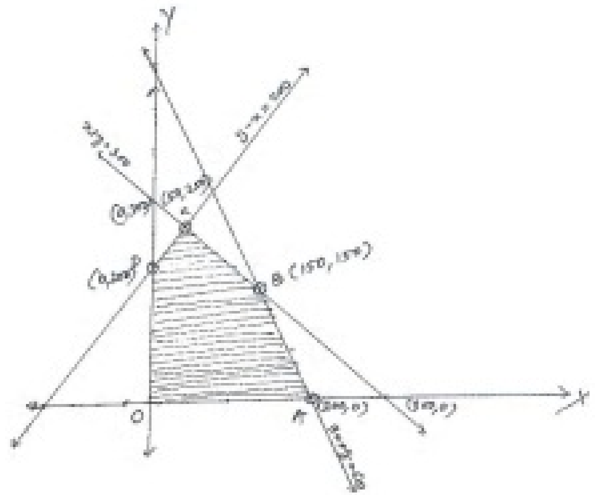
$360x + 120y \le 72{,}000$

$\Rightarrow 3x + y \le 600$ .....(2)

$y - x \le 200$ .....(3)

x, y ≥ 0

correct Figure



Corner points A(200, 0), B(150, 150), C(50, 250), D(0, 200), O(0, 0)

Corner points	Z = 100x + 50y
O(0, 0)	0
A(200, 0)	0
B(150, 150)	22,500 ←maximum
C(50, 250)	17,500
D(0, 200)	10,000

Hence no. of cardigans of type A = 150 and of type B = 150.

and max. profit is Rs 22,500.

22. Find the coordinates of the foot of perpendicular and perpendicular distance from the point P(4, 3, 2) to the plane x + 2y + 3z = 2. Also find the image of P in the plane.

Sol. Eqn. of plane x + 2y + 3z = 2

Eqn. of PR is $\frac{x-4}{1} = \frac{y-3}{2} = \frac{z-3}{3} = \lambda(let)$

⇒ x = λ + 4, y = 2λ + 3, z = 3λ + 2

Let the co-ordinate of R be (λ + 4, 2λ + 3, 3λ + 2)

R also lies on the plane

So, λ + 4 + 2(2λ + 3) + 3(3λ + 2) = 2

⇒ λ = -1

So point R is (3, 1, -1) i.e., foot of perpendicular

Let $Q(\alpha, \beta, \gamma)$ be the image of P

$$\therefore \quad \frac{4+\alpha}{2} = 3, \frac{3+\beta}{2} = 1, \frac{2+\gamma}{2} = -1$$

$$\Rightarrow \quad \alpha = 2, \beta = -1, \gamma = -4$$

So image point Q is (2, -1, -4)

Perpendicular distance $PR = \sqrt{14}$

23. Solve for x: $\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$, using properties of determinants.

OR

Using elementary row operations find the inverse of matrix $A = \begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$ and

hence solve the following system of equations 3x – 3y + 4z = 21, 2x – 3y + 4z = 20, - y + z = 5.

Sol. $\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a+x & a-x & a+x \end{vmatrix} = 0$

Applying $C_1 \to C_1 + C_2 + C_3$

$$\Rightarrow \quad \begin{vmatrix} 3a-x & a-x & a-x \\ 3a-x & a+x & a-x \\ 3a-x & a-x & a+x \end{vmatrix} = 0$$

$$\Rightarrow \quad (3a-x)\begin{vmatrix} 1 & a-x & a-x \\ 1 & a+x & a-x \\ 1 & a-x & a+x \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$

$$\Rightarrow \quad (3a-x)\begin{vmatrix} 1 & a-x & a-x \\ 0 & 2x & 0 \\ 0 & 0 & 2x \end{vmatrix} = 0$$

$$\Rightarrow \quad (3a-x)(4x^2) = 0$$

$$\Rightarrow \quad x = 0 \ or \ 3a$$

OR

$A = I \cdot A$

$$\Rightarrow \quad \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 - R_2$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \rightarrow R_2 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \rightarrow R_2 - 4R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_3 \rightarrow R_3 + R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} A \quad \left[2\frac{1}{2} \textit{ for correct operation to get } A^{-1} \right]$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

The matrix form of given equations

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 21 \\ 20 \\ 5 \end{bmatrix}$$

$\Rightarrow$ AX = B, where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 21 \\ 20 \\ 5 \end{bmatrix}$

$$\Rightarrow \quad X = A^{-1}B$$

$$\Rightarrow \quad \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}\begin{bmatrix} 21 \\ 20 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

$\therefore$ x = 1, y = 2, z = 3

24. A, B and C throw a pair of dice in that order alternately till one of them gets a total of 9 and wins the game. Find their respective probabilities of winning, if A starts first.

Sol. P (winning) $= \frac{1}{9}$

P (not winning) $= \frac{8}{9}$

P(A winning) $= P(A) + P(\overline{A}\overline{B}\overline{C}A) + P(\overline{A}\overline{B}\overline{C}\,\overline{A}\overline{B}\overline{C}A)$ +....

$$= \frac{1}{9} + \left(\frac{8}{9}\right)^3 \frac{1}{9} + \left(\frac{8}{9}\right)^6 \frac{1}{9} + \ldots..$$

$$= \frac{\frac{1}{9}}{1 - \frac{512}{729}} = \frac{81}{217}$$

P(B winning $= P(\overline{A}B) + P(\overline{A}\overline{B}\overline{C}\,\overline{A}B) + p(\overline{A}\overline{B}\overline{C}\,\overline{A}\overline{B}\overline{C}\,\overline{A}B) + \ldots$

$$= \frac{8}{9} \times \frac{1}{9} + \left(\frac{8}{9}\right)^4 \times \frac{1}{9} + \left(\frac{8}{9}\right)^7 \times \frac{1}{9} + \ldots..$$

$$= \frac{\frac{8}{9} \times \frac{1}{9}}{1 - \frac{512}{729}} = \frac{72}{217}$$

P(C winning) = 1 – [P(A winning) + P(B winning)]

$$= 1 - \left[\frac{81}{217} + \frac{72}{217}\right]$$

$$= 1 - \frac{153}{217} = \frac{64}{217}$$

25. Show that the relation R defined by (a, b) R (c, d) $\Rightarrow a + d = b + c$ on the $A \times A$ where A = {1, 2, 3, ……, 10} is an equivalence relation. Hence write the equivalence class [(3, 4)]:a, b, c, $d \in A$

Sol. (a, b) R(c, d) ⇒a + d = b + c

∵ a + b = b + a

$\Rightarrow \quad (a, b)\ R(a, b)\ \forall\ (a, b) \in A \times A$

$\Rightarrow$ R is reflexive

(a, b) R(c, d) ⇒ a + d = b + c

⇒ b + c = a + d

⇒ c + b = d + a

⇒ (c, d) R(a, b)

⇒ R is symmetric

For $(a, b), (c, d)\ \&\ (e, f) \in A \times A$

(a, b) R(c, d) ⇒ a + d = b + c

(c, d) R(e, f) ⇒ c + f = d + e

adding (1) & (2), we get

a + d + c + f = b + c + d + e

⇒ a + f = b + e

⇒ (a, b) R(e, f)

∴ R is transitive.

Hence R is an equivalence relation.

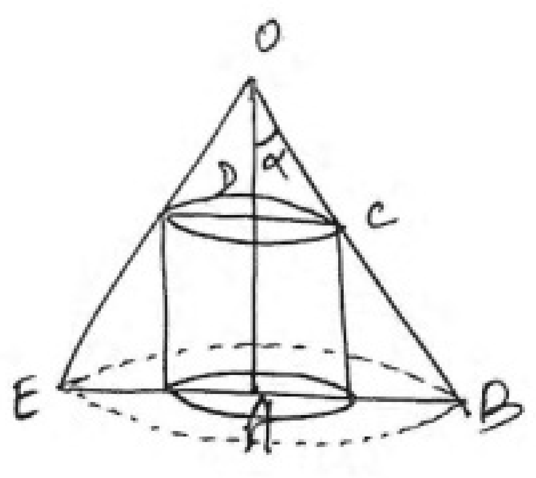
Now [3, 4] = {(1, 2, (2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 8), (8, 9), (9, 10)}

26. Show that height of the cylinder of greatest volume which can be inscribed in a right circular once of height h and semi-vertical angle α is one-third that of the once and the greatest volume of the cylinder is $\frac{4}{27}\pi h^3 \tan^2 \alpha$.

OR

Find the intervals in which the function $f(x) = \frac{4\sin x}{2+\cos x} - x; 0 \le 2\pi$ is strictly increasing or strictly decreasing.

Sol. Let CD = R, AD = x



$$\Rightarrow \qquad OD = h - x$$

$\because \quad ODC \sim \Delta OAB$

$\Rightarrow \quad \frac{h-x}{h} = \frac{R}{AB} \Rightarrow \frac{h-x}{h} = \frac{R}{h\tan\alpha}$

$\Rightarrow \quad R = (h-x)\tan\alpha$

$V = \pi R^2 x$

$= \pi(h-x)^2 \tan^2\alpha \cdot x$

$= \pi \tan^2\alpha (h-x)^2 x$

$\frac{dV}{dx} = \pi\tan^2\alpha(h^2 - 4hx + 3x^2)$

$\frac{dV}{dx} = 0 \Rightarrow h^2 - 4hx + 3x^2 = 0$

$\Rightarrow (h-x)(h-3x) = 0$

$\Rightarrow$ x = h (not possible) or $x = \frac{h}{3}$

$\frac{d^2V}{dx^2} = \pi\tan^2\alpha(-4+6x)$

$\left(\frac{d^2V}{dx^2}\right)_{x=h/3} = \pi\tan^2\alpha(-2h) < 0$

$\Rightarrow$ V is maximum for $x = \frac{h}{3}$

So $V_{max} = \pi\tan^2\alpha(h-x)^2 x$

$= \pi\tan^2\alpha\left(h - \frac{h}{3}\right)^2 \frac{h}{3}$

$$= \frac{4\pi h^3}{27}\tan^2\alpha$$

OR

$$y = \frac{4\sin x}{2+\cos x} - x,\ x \in [0,\ 2\pi]$$

$$\frac{dy}{dx} = \frac{(2+\cos x)4\cos x - 4\sin x(-\sin x)}{(2+\cos x)^2} - 1$$

$$\frac{dy}{dx} = \frac{\cos x(4-\cos x)}{(2+\cos x)^2}$$

f(x) is strictly increasing for f'(x) > 0

i.e., cos x > 0 $\Rightarrow$ $x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right]$

and f(x) is strictly decreasing for f'(x) < 0

i.e., cos x < 0 $\Rightarrow$ $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$