Question Paper 2016 merit Delhi Set-3

CBSE Class-12 Mathematics

General Instructions:

(i) All questions are compulsory.

(ii) Please check that this Question Paper contains **26** *Questions.*

(iii) Marks for each question are indicated against it.

(iv) Questions **1** *to* **6** *in Section-***A** *are Very Short Answer Type Questions carrying* **one** *mark each.*

(v) Questions 7 *to* **19** *in Section-***B** *are Long Answer* **I** *Type Questions carrying* **4** *marks each.*

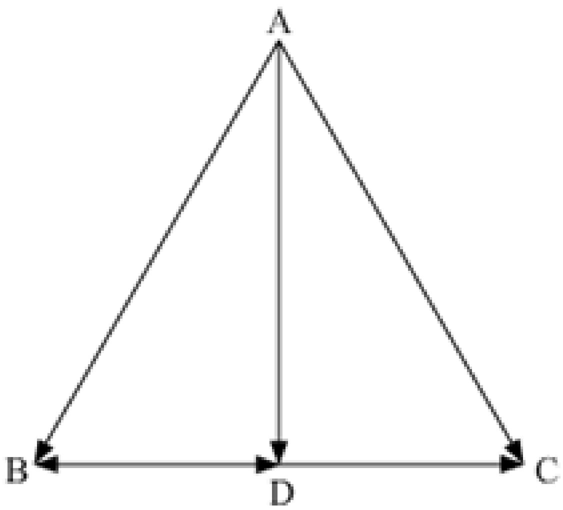
(vi) Questions **20** *to* **26** *in Section-***C** *are Long Answer* **II** *Type Questions carrying* **6** *marks each.*

(vii) Please write down the serial number of the Question before attempting it.

Section A

1 The two vectors $\hat{j}+\hat{k}$ and $3\hat{i}-\hat{j}+4\hat{k}$ represent the two sides AB and AC, respectively of a. Find the length of the median through A.

Ans: In Δ ABC,



Using the triangle law of vector addition, we have

$$\overrightarrow{BC}=\overrightarrow{AC}-\overrightarrow{AB}$$

= $(3\hat{i}-\hat{j}+4\hat{k})-(\hat{j}+\hat{k})$

= $3\hat{i}-2\hat{j}+3\hat{k}$

(Since AD is the median)

In ΔABD, using the triangle law of vector addition, we have

$$\overrightarrow{AD}=\overrightarrow{AB}+\overrightarrow{BD}$$

$$=(\hat{j}+\hat{k})+\left(\frac{3}{2}\hat{i}-\hat{j}+\frac{3}{2}\hat{k}\right)$$

$$=\frac{3}{2}\hat{i}+0\hat{j}+\frac{5}{2}\hat{k}$$

$$\therefore AD=\sqrt{\left(\frac{3}{2}\right)^2+0^2+\left(\frac{5}{2}\right)^2}=\frac{1}{2}\sqrt{34}$$

Hence, the length of the median through A is $\frac{1}{2}\sqrt{34}$ units.

Q2 Find the vector equation of a plane which is at a distance of 5 units from the origin and its normal vector is $2\hat{i}-3\hat{j}+6\hat{k}$

Ans: Given:

Normal vector, $\hat{n}=2\hat{i}-3\hat{j}+6\hat{k}$

Perpendicular distance, d = 5 units

The vector equation of a plane that is at a distance of 5 units from the origin and has its normal vector $\hat{n}=2\hat{i}-3\hat{j}+6\hat{k}$ is as follows:

$$\overrightarrow{r}.\hat{n}=d$$

$$\overrightarrow{r}.\ (2\hat{i}-3\hat{j}+6\hat{k})=5$$

Q.3 Find the maximum value of

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$$

Ans: Let Δ= $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$

Applying $R_2 \rightarrow R_2 - R_1$ *and* $R_3 \rightarrow R_3 - R_1$, , we get

Δ= $\begin{vmatrix} 1 & 1 & 1 \\ 0 & \sin\theta & 0 \\ 0 & 0 & \cos\theta \end{vmatrix}$

= $sin\theta cos\theta$

= $\frac{sin2\theta}{2}$

We know that $-1 \leq sin2\theta \leq 1$.

∴ Maximum value of $\Delta = \frac{1}{2} \times 1 = \frac{1}{2}$

Q4 If A is a square matrix such that A^2 = I, then find the simplified value of $(A – I)^3$ + $(A + I)^3$ – 7A.

Ans: Given = $(A-I)^3 + (A+I)^3 - 7A$ b

$= A^3 - I^3 - 3A^2I + 3AI^2 + A^3 + I^3 + 3A^2I + 3AI^2 - 7A$

$= 2A^3 + 6AI^2 - 7A$

$= 2A.A^2 + 6AI^2 - 7A$

$= 8A - 7A = A$

Q5 M atrix $A = \begin{vmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{vmatrix}$ **is given to be symmetric, find values of a and b.**

Ans: We have

$$A=\begin{vmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{vmatrix}$$

$$A'=\begin{vmatrix} 0 & 3 & 3a \\ 2b & 1 & 3 \\ -2 & 3 & -1 \end{vmatrix}$$

We know that a matrix is symmetric if $A = A'$.

Thus,

$$A=\begin{vmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 3 & 3a \\ 2b & 1 & 3 \\ -2 & 3 & -1 \end{vmatrix}$$

Now, $2b=3 \Rightarrow b=\frac{3}{2}$

Also, $3a=-2 \Rightarrow a=\frac{-2}{3}$

Therefore, $a=\frac{-2}{3}$ *and* $b=\frac{3}{2}$

Q.6 Find the position vector of a point which divides the join of points with position vectors $\vec{a}-2\vec{b}$ and $2\vec{a}+\vec{b}$ externally in the ratio 2 : 1.

Ans: Let A and B be the points with position vectors $\vec{a}-2\vec{b}$ and $2\vec{a}+\vec{b}$ respectively.

Also, let R divideAB externally in the ratio 2 : 1.

$\therefore$ Position vector of $\frac{2\times(2\vec{a}+\vec{b})-1\times(\vec{a}-2\vec{b})}{2-1}=3\vec{a}+4\vec{b}$

Section B

Q7 Find the general solution of the following differential equation :

$$\left(1+y^2\right)+\left(x-e^{\tan^{-1}y}\right)\frac{dy}{dx}=0$$

Ans: Given:

$$\left(1+y^2\right)+\left(x-e^{\tan^{-1}y}\right)\frac{dy}{dx}=0$$

$$Let\, tan^{-1}y=t$$

$$\Rightarrow y=tan\ t$$

$$\Rightarrow \frac{dy}{dx}=sec^2t\frac{dt}{dx}$$

Therefore, the equation becomes

$$\left(1+tan^2t\right)+\left(x-e^t\right)sec^2t\frac{dt}{dx}=0$$

$$\Rightarrow sec^2t+\left(x-e^t\right)\left(sec^2t\right)\frac{dt}{dx}=0$$

$$\Rightarrow 1+\left(x-e^t\right)\frac{dt}{dx}=0$$

$$\Rightarrow \left(x-e^t\right)\frac{dt}{dx}=-1$$

$$\Rightarrow x-e^t=\frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt}+1.x=e^t$$

$$IF\quad e^{\int 1.dt}=e^t$$

$$\therefore e^t.\left(\frac{dx}{dt}+1.x\right)=e^t.e^t$$

$$e\Rightarrow \frac{d}{dt}\left(xe^t\right)=e^{2t}$$

Integrating both the sides, we get

$$xe^t=\int e^{2t}dt$$

$$\Rightarrow xe^t=\frac{1}{2}e^{2t}+C \qquad .....(1)$$

Substituting the value of *t* in (1), we get

$$xe^{\tan^{-1}y}=\frac{1}{2}e^{2\tan^{-1}y}+C1$$

$$\Rightarrow e^{2\tan^{-1}y}=2xe^{\tan^{-1}y}+C$$

It is the required general Solution

Q8 Show that the vectors $\vec{a},\ \vec{b}\ and\ \ \vec{c}$ are coplanar if $\vec{a},\ \vec{b},\vec{b}\ +\ \vec{c}$ and $\vec{c}\ +\ \vec{a}$ are coplanar.

Ans: It is given that $\vec{a},\ \vec{b},\vec{b}\ +\ \vec{c}$ and are coplanar.

Therefore, Scalar triple product=Volume of the parallelopoid=0

$$(\vec{a}+\vec{b}).\left[\left(\vec{b}+\vec{c}\right)\times\left(\vec{c}+\vec{a}\right)\right]=0$$

$$\Rightarrow\left(\vec{a}+\vec{b}\right).\left[\left(\vec{b}\times\vec{c}\right)+\left(\vec{b}\times\vec{a}\right)+0+\left(\vec{c}\times\vec{a}\right)\right]=0$$

$$\Rightarrow\vec{a}.\left(\vec{b}\times\vec{c}\right)+\vec{a}.\left(\vec{b}\times\vec{a}\right)+\vec{a}.\left(\vec{c}\times\vec{a}\right)+\vec{b}.\left(\vec{b}\times\vec{c}\right)+\vec{b}.\left(\vec{b}\times\vec{a}\right)+\vec{b}.\left(\vec{c}\times\vec{a}\right)=0$$

$$\Rightarrow\left[abc\right]+0+0+0+0+\left[bca\right]=0$$

$$\Rightarrow 2\left[abc\right]=0$$

$$\Rightarrow\left[abc\right]=0$$

Therefore, the vectors $\vec{a},\ \vec{b}\ and\ \ \vec{c}$ are coplanar.

Q9 Find the vector and Cartesian equations of the line through the point (1, 2, −4) and perpendicular to the two lines.

$\vec{r}=\left(8\hat{i}-19\hat{j}+10\hat{k}\right)+\lambda\left(3\hat{i}-16\hat{j}+7\hat{k}\right)$ **and**

$\vec{r}=\left(15\hat{i}+29\hat{j}+5\hat{k}\right)+\mu\left(3\hat{i}+8\hat{j}-5\hat{k}\right).$

Ans: The equations of the given lines are

$$\vec{r}=\left(8\hat{i}-19\hat{j}+10\hat{k}\right)+\lambda\left(3\hat{i}-16\hat{j}+7\hat{k}\right)\qquad.....(1)$$

$$\vec{r}=\left(15\hat{i}+29\hat{j}+5\hat{k}\right)+\mu\left(3\hat{i}+8\hat{j}-5\hat{k}\right)\qquad.....(2)$$

Normal parallel to (1) is $\overline{n_1} = 3\hat{i} - 16\hat{j} + 7\hat{k}$.

Normal parallel to (2) is $\overline{n_2} = 3\hat{i} + 8\hat{j} - 5\hat{k}$.

The required line is perpendicular to the given lines. So, the normal $\vec{n}$ parallel to the required line is perpendicular to $\vec{n}_1$ *and* $\vec{n}_2$.

$$\therefore \vec{n} = \vec{n}_1 - \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix}$$

$$= 24\hat{i} + 36\hat{j} + 72\hat{k}$$

Thus, the vector equation of the required line is

$$\vec{r} = \left(\hat{i} + 2\hat{j} - 4\hat{k}\right) + \gamma\left(24\hat{i} + 36\hat{j} + 72\hat{k}\right)$$

$$\Rightarrow \vec{r} = \left(\hat{i} + 2\hat{j} - 4\hat{k}\right) + k\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right) \qquad (Where\, k = 12\gamma)$$

Also, the Cartesian equation of the required line is

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$$

Q10 Three persons A, B and C apply for a job of Manager in a Private Company. Chances of their selection (A, B and C) are in the ratio 1 : 2 :4. The probabilities that A, B and C can introduce changes to improve profits of the company are 0.8, 0.5 and 0.3, respectively. If the change does not take place, find the probability that it is due to the appointment of C.

Ans: Let E_1, E_2 *and* E_3 be the events denoting the selection of A, B and C as managers, respectively.

$P(E_1)$ = Probability of selection of A $= \frac{1}{7}$

$P(E_2)$ = Probability of selection of B $= \frac{2}{7}$

$P(E_3)$ = Probability of selection of C $= \frac{4}{7}$

Let *A* be the event denoting the change not taking place.

$$P\left(\frac{A}{E_1}\right) = \text{Probability that } A \text{ does not introduce change} = 0.2$$

$$P\left(\frac{A}{E_2}\right) = \text{Probability that } B \text{ does not introduce change} = 0.5$$

$$P\left(\frac{A}{E_3}\right) = \text{Probability that } C \text{ does not introduce change} = 0.7$$

∴ Required probability = $P\left(\frac{E_3}{A}\right)$

By Bayes' theorem, we have

$$P\left(\frac{E_3}{A}\right)$$

$$= \frac{P(E_3)P\left(\frac{A}{E_3}\right)}{P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{B^2}\right) + P(E_3)P\left(\frac{A}{E_3}\right)}$$

$$\text{v} = \frac{\frac{4}{7}\times 0.7}{\frac{1}{7}\times 0.2 + \frac{2}{7}\times 0.5 + \frac{4}{7}\times 0.7}$$

$$= \frac{2.8}{0.2+1+2.8} = \frac{2.8}{4} = 0.7$$

OR

A and B throw a pair of dice alternately. A wins the game if he gets a total of 7 and B wins the game if he gets a total of 10. If A starts the game, then find the probability that B wins.

Ans: Total of 7 on the dice can be obtained in the following ways:

(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)

Probability of getting a total of 7 = $\frac{3}{36} = \frac{1}{6}$

Probability of not getting a total of 10 = $1 - \frac{1}{12} = \frac{11}{12}$

Total of 10 on the dice can be obtained in the following ways:

(4, 6), (6, 4), (5, 5)

Probability of getting a total of 10 = $\frac{3}{36}=\frac{1}{12}$

Probability of not getting a total of 10 = $1-\frac{1}{12}=\frac{11}{12}$

Let E and F be the two events, defined as follows:

E = Getting a total of 7 in a single throw of a dice

F = Getting a total of 10 in a single throw of a dice

$$P(E) = \frac{1}{6},\ P(\bar{E})=\frac{5}{6}\ P(F) = \frac{1}{12},\ P(\bar{F})=\frac{11}{12}$$

A wins if he gets a total of 7 in 1st, 3rd or 5th … throws.

Probability of A getting a total of 7 in the 1st throw = $\frac{1}{6}$

A will get the 3rd throw if he fails in the 1st throw and B fails in the 2nd throw.

Probability of A getting a total of 7 in the 3rd throw

$$= P(\bar{E})P(\bar{F})P(E)=\frac{5}{6}\times\frac{11}{12}\times\frac{1}{6}$$

Similarly, probability of getting a total of 7 in the 5th throw

$$= P(\bar{E})P(\bar{F})P(\bar{E})P(\bar{F})P(E)$$

$=\frac{5}{6}\times\frac{11}{12}\times\frac{5}{6}\times\frac{11}{12}\times\frac{1}{6}$ and so on

Probability of winning of

$$A = \frac{1}{6}+\left(\frac{5}{6}\times\frac{11}{12}\times\frac{1}{6}\right)+\left(\frac{5}{6}\times\frac{11}{12}\times\frac{5}{6}\times\frac{11}{12}\times\frac{1}{6}\right)+\ldots$$

$$=\frac{\frac{1}{6}}{1-\frac{5}{6}\times\frac{11}{12}}=\frac{12}{17}$$

∴ Probability of winning of B = 1 − Probability of winning of A = $1-\frac{12}{17}=\frac{5}{17}$

Q11 Prove that: $tan^{-1}\frac{1}{5}+tan^{-7}\frac{1}{7}+tan^{-3}\frac{1}{3}+tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

Ans: LHS:

$$\left(tan^{-1}\frac{1}{5}+tan^{-1}\frac{1}{7}\right)+\left(tan^{-1}\frac{1}{3}+tan^{-1}\frac{1}{8}\right)$$

$$=tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right)+tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right)$$

$$\left[\because tan^{-1}A+tan^{-1}B=tan^{-1}\left(\frac{A+B}{1-AB}\right)\right]$$

$$=tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right)$$

$$=tan^{-1}1$$

$$=\frac{\pi}{4}$$

OR

Solve for *x*: $2\,tan^{-1}(cos\,x)=tan^{-1}(2cosec\,x)$

Ans:

$$2tan^{-1}(cosx)=tan^{-1}(2cosecx)$$

$$\Rightarrow tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right)$$

$$=tan^{-1}(2cosecx)$$

$$\left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$\Rightarrow \frac{2\cos}{\sin^2 x} = 2\operatorname{cosec} x$$

$$\Rightarrow \cos x = \sin x$$

$$\Rightarrow \tan x = 1$$

$$\Rightarrow x = \frac{\pi}{4}$$

Q12 The monthly incomes of Aryan and Babban are in the ratio 3 : 4 and their monthly expenditures are in the ratio 5 : 7. If each saves Rs 15,000 per month, find their monthly incomes using matrix method. This problem reflects which value?

Ans: Let the monthly incomes of Aryan and Babban be $3x$ and $4x$, respectively.

Suppose their monthly expenditures are $5y$ and $7y$, respectively.

Since each saves Rs 15,000 per month,

Monthly saving of Aryan: $3x-5y = 15{,}000$

Monthly saving of Babban: $4x-7y = 15{,}000$

The above system of equations can be written in the matrix form as follows:

$$\begin{bmatrix} 3 & -5 \\ 4 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$$

or,

AX = B, where

$$A = \begin{bmatrix} 3 & -5 \\ 4 & -7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$$

Now,

$$|A| = \begin{bmatrix} 3 & -5 \\ 4 & -7 \end{bmatrix} = -21-(-20) = -1$$

$$Adj\ A = \begin{bmatrix} -7 & -4 \\ 5 & 3 \end{bmatrix}^T = \begin{bmatrix} -7 & 5 \\ -4 & 3 \end{bmatrix}$$

So, $A^{-1} = \dfrac{1}{|A|} adj\ A = -1\begin{bmatrix} -7 & 5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ 4 & -3 \end{bmatrix}$

$\therefore X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 105000 - 75000 \\ 60000 - 45000 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$$

$$\Rightarrow x = 30,000 \text{ and } y = 15,000$$

Therefore,

Monthly income of Aryan = $3 \times Rs\,30,000 = Rs\,90,000$

Monthly income of Babban = $4 \times Rs\,30,000 = Rs\,1,20,000$

From this problem, we are encouraged to understand the power of savings. We should save certain part of our monthly income for the future.

Q13 If x = a sin 2t (1 + cos 2t) and y = b cos 2t (1 – cos 2t), find the values of $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$ and $t = \frac{\pi}{3}$.

Ans: x = a sin 2t (1 + cos 2t)

y = b cos 2t (1 – cos 2t)

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{b - [-2\sin 2t + 2\cos 2t \sin 2t \times 2]}{a[2\cos 2t + 2\cos 4t]}$$

$$\frac{dy}{dx} = \frac{b}{a}\left[\frac{-2\sin 2t + 2\sin 4t}{2\cos 2t + 2\cos 4t}\right]$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = \frac{b}{a}\left[\frac{-2+0}{0-2}\right] = \frac{b}{a}$$

and $\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{3}} = \frac{b}{a}\left[\frac{-2\sqrt{3}}{-2}\right] = \frac{\sqrt{3}b}{a}$

OR

If $y = X^x$, *prove that* $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$.

Ans: $y = x^x$

Applying logarithm,

log y = x log x

$$\frac{1}{y}\frac{dy}{dx} = \log x + x \times \frac{1}{x} = 1 + \log x$$

$$\frac{dy}{dx} = X^x\left[1 + \log x\right]$$

$$\frac{d^2y}{dx^2} = \frac{d(x^x)}{dx}(1 + \log x) + x^x\left[\frac{d}{dx}(1 + \log x)\right]$$

$$= x^x(1 + \log x)(1 + \log x) + x^x\left[\frac{1}{x}\right]$$

$$= x^x(1 + \log x)^2 + x^{x-1}$$

$$\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = x^x(1 + \log x)^2 + x^{x-1} - \frac{1}{x^2}\left(x^x(1 + \log x)^2\right) - \frac{x^x}{x}$$

$$= x^x(1 + \log x)^2 + x^{x-1} - x^x(1 + \log x)^2 - x^{x-1}$$

=0

Hence proved.

Q14 Find the values of p and q for which

$$f(x) = \begin{cases} \frac{1 - \sin^3 x}{3\cos^2 x}, & if\ x < \frac{x}{2} \\ P, & if\ x = \frac{\pi}{2} \\ \frac{q(1 - \sin x)}{(\pi - 2x)^2}, & if\ x > \frac{\pi}{2} \end{cases}$$

is continuous at $x = \frac{\pi}{2}$

Ans: $f(x)=\begin{cases}\dfrac{1-\sin^3 x}{3\cos^2 x}, & if\ x<\dfrac{x}{2}\\ P, & if\ x=\dfrac{\pi}{2}\\ \dfrac{q(1-\sin x)}{(\pi-2x)^2}, & if\ x>\dfrac{\pi}{2}\end{cases}$

for continuity,

$$\lim_{x\to\frac{\pi}{2}} f(x)=\lim_{x\to\frac{\pi+}{2}} f(x)=f\left(\frac{\pi}{2}\right)$$

$$\lim_{x\to\frac{\pi}{2}} f(x)=\lim_{x\to\frac{\pi-}{2}}\left(\frac{1-\sin^3 x}{3\cos^2 x}\right)$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{(1-\sin x)(1+\sin^2 x+\sin x)}{3\left[1-\sin^2 x\right]}$$

$$\lim_{x\to\frac{\pi}{2}} f(x)=\lim_{x\to\frac{\pi}{2}}\frac{(1+\sin^2 x+\sin x)}{3(1+\sin x)}=\frac{1+1+1}{3(2)}=\frac{1}{2}$$

$$Let\ \frac{\pi}{2}-x=\theta\Rightarrow x=\frac{\pi}{2}-\theta$$

$$\lim_{x\to\frac{\pi+}{2}}=\lim_{\theta\to0}\frac{\left[1-\sin(\frac{\pi}{2}-\theta)\right]}{(2\theta)^2}=\frac{q}{4}\lim_{\theta\to0}\frac{1-\cos\theta}{\theta^2}$$

$$=\frac{q}{4}\lim_{\theta\to0}\frac{2\sin^2\frac{\theta}{2}}{\theta^2}=\frac{q}{2}\lim_{\theta\to0}\frac{\sin^2\frac{\theta}{2}}{4\times(\theta/2)}=\frac{q}{8}$$

Now,

$$\lim_{x\to\frac{\pi}{2}} f(x)=\lim_{x\to\frac{\pi+}{2}} f(x)=f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow\frac{1}{2}=p=\frac{q}{8}$$

$$\Rightarrow P=\frac{1}{2}\ and\ q=\frac{8}{2}=4$$

Q15. Show that the equation of normal at any point t on the curve $x=\ 3\cos t\ -\cos^3 t$ and $y\ =\ 3\sin t\ -\sin^3 t$ is $4\left(y\cos^3 t\ -\ \sin^3 t\right)\ =\ 3\sin 4t.$

Ans: Given: $\begin{array}{l} x=3\cos t-\cos^3 t \\ y=3\sin t-\sin^3 t \end{array}$

Slope of the tangent,

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dx}}=\frac{3\cos t-3\sin^2\cos t}{-3\sin t+3\cos^2 t\sin t}$$

$$=\frac{3\cos t-\left[\cos^2 t\right]}{-3\sin t\left[\sin^2 t\right]}$$

$$\frac{dy}{dx}=\frac{-\cos^{3t}}{\sin^3 t}$$

$\therefore$ Slope of the normal= $\frac{\sin^3 t}{\cos^3 t}$

The equation of the normal is given by

$$\frac{y-(3\sin t-\sin^3 t)}{x-(3\cos t-\cos^3 t)}=\frac{\sin^3 t}{\cos^3 t}$$

$$\Rightarrow y\cos^3 t-3\sin t\cos^3 t+\sin^3 t\cos^3 t$$

$$=x\sin^3 t-3\cos t\sin^3 t+\sin^3 t\cos^3 t$$

$$\Rightarrow y\cos^3 t-x\sin^3 t=3(\sin t\cos t-\cos t\sin^3 t)$$

$$\Rightarrow y\cos^3 t-x\sin^3 t=3\sin t\cos t(\cos^2 t-\sin^3 t)$$

$$\Rightarrow y\cos^3 t-x\sin^3 t=\frac{3}{2}\sin 2t\cos 2t=\frac{3}{4}\sin 4t$$

$$\Rightarrow y\cos^3 t-x\sin^3 t=3(\sin t\cos t-\cos t\sin^3 t)$$

$$\Rightarrow 4(y\cos^3 t-x\sin^3 t)=3\sin 4t$$

Hence proved.

Q16. Find $\int \frac{(3\sin\theta-2)\cos\theta}{5-\cos^2\theta-4\sin\theta}\theta.$

Ans: $I=\int \frac{(3\sin\theta-2)\cos\theta}{5-\cos^2\theta-4\sin\theta}d\theta$

$\Rightarrow I=\int \frac{(3\sin\theta-2)\cos\theta}{5-(1-\sin^2\theta)-4\sin\theta}d\theta$

$\Rightarrow I=\int \frac{(3\sin\theta-2)\cos\theta}{\sin^2\theta-4\sin\theta+4}d\theta$

$No, let\ \sin\theta=t.$

$\Rightarrow \cos\theta\, d\theta=dt$

$\therefore I=\int \frac{(3t-2)dt}{t^2-4t+4}$

$\Rightarrow 3t-2=A\frac{d}{dx}\left(t^2-4t+4\right)+B$

$\Rightarrow 3t-2=A\left(2t-4\right)+B$

$\Rightarrow 3t-2=\left(2A\right)t+B-4A$

Comparing the coefficients of the like powers of *t*, we get

$2A=3\Rightarrow A=\frac{3}{2}$

And B−4 A=−2

$\Rightarrow B-4\times\frac{3}{2}=-2$

$\Rightarrow B=-2+6=4$

Substituting the values of A and B, we get

$3t-2=\frac{3}{2}(2t-4)+4$

$\therefore I=\int \frac{(3t-2)dt}{t^2-4t+4}$

$$= \int \left(\frac{\frac{3}{2}(2t-4)+4}{t^2-4t+4} \right) dt$$

$$= \frac{3}{2} \int \left(\frac{2t-4}{t^2-4t+4} \right) dt + 4 \int \frac{dt}{t^2-4t+4}$$

$$= \frac{3}{2} I_1 + 4 I_2 \qquad ......(1)$$

Here,

$$I_1 = \int \frac{(2t-4)dt}{t^2-4t+4} \quad and \ I_2 = \int \frac{dt}{t^2-4t+4}$$

Now

$$I_1 = \int \frac{(2t-4)dt}{t^2-4t+4}$$

$$Let\ t^2 - 4t + 4 = p$$

$$\Rightarrow (2t-4)dt = dp$$

$$I_1 = \int \frac{(2t-4)dt}{t^2-4t+4}$$

$$= \int \frac{dp}{p} = \log |p| + C_1$$

$$= \log \left| t^2 - 4t + 4 \right| + C_1 \qquad ....(2)$$

And $I_2 = \int \frac{dt}{t^2-4t+4}$

$$= \int \frac{dt}{(t-2)^2}$$

$$= \int (t-2)^{-2} dt$$

$$= \frac{(t-2)^{-2+1}}{-2+1} + C_2$$

$$= \frac{-1}{t-2} + C_2 \qquad ....(3)$$

From (1), (2) and (3), we get

$$I=\frac{3}{2}log|t^2-4t+4|+4\times\frac{-1}{t-2}+C_1+C_2$$

$$=32\,log|\,sin^2\theta-4\,sin\theta+4\,|+\frac{4}{2-t}+C \qquad (where\ C=C_1+C_2)$$

$$=\frac{3}{2}\log\left|(\sin\theta-2)^2\right|+\frac{4}{2-\sin\theta}+C$$

$$=\frac{3}{2}\times 2\log\left|(\sin\theta-2)\right|+\frac{4}{2-\sin\theta}+C$$

$$=3\log\left|(2-\sin\theta\right|+\frac{4}{2-\sin\theta}+C$$

OR

Evaluate $\int_0^{\pi} e^{2x}.\quad \sin\left(\frac{\pi}{4}+x\right)dx$

Ans: $Let\ I=\int_0^{\pi} e^{2x}\sin(\frac{\pi}{4}+x)dx$

Integrating by parts, we get

$$\Rightarrow I=\frac{1}{2}\left[e^{2x}\,sin\left(\frac{\pi}{4}+x\right)_0^{\pi}\right]-\frac{1}{2}\left\{\left[\frac{1}{2}e^{2x}\cos\left(\frac{\pi}{4}+x\right)_0^{\pi}\right]+\frac{1}{2}\int_0^{\pi}e^{2x}\sin\left(\frac{\pi}{4}+x\right)dx\right\}$$

$$\Rightarrow I=\frac{1}{2}\left[e^{2x}\,sin\left(\frac{\pi}{4}+x\right)_0^{\pi}\right]-\frac{1}{4}\left[e^{2x}\cos\left(\frac{\pi}{4}+x\right)_0^{\pi}\right]-\frac{1}{4}I$$

$$\Rightarrow\frac{5}{4}I=\frac{1}{2}\left[e^{2x}\,sin\left(\pi+\frac{\pi}{4}\right)-\sin\left(\frac{\pi}{4}\right)\right]-\frac{1}{4}\left[e^{2\pi}\cos\left(\pi+\frac{\pi}{4}\right)-\cos\left(\frac{\pi}{4}\right)\right]$$

$$\Rightarrow\frac{5}{4}I=\frac{1}{2}\left[-e^{2x}\times\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right]-\frac{1}{4}\left[e^{2\pi}\times\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right]$$

$$\Rightarrow I=-\frac{1}{5\sqrt{2}}(e^{2\pi}+1)$$

Q17. Find $\int\frac{\sqrt{x}}{\sqrt{a^3-x^3}}dx.$

Ans: $I=\int\frac{\sqrt{x}}{\sqrt{a^3-x^3}}dx$

$$Let:\ x^{\frac{3}{2}} = t$$

$$\Rightarrow \frac{3}{2} x^{\frac{1}{2}} dx = dt$$

$$x^{\frac{1}{2}} dx = \frac{2}{3} dt$$

Putting the values in *I*, we get

$$I = \int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx$$

$$= \frac{2}{3} \int \frac{1}{\sqrt{a^3 - t^2}} dt$$

Using the following formula of integration, we get

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$$

$$\therefore \frac{2}{3} \int \frac{1}{\sqrt{a^3 - t}} dt = \frac{2}{3} sin^{-1}\left(\frac{t}{a^{\frac{3}{2}}}\right) + C$$

Again, putting the value of *t*, we get

$$\frac{2}{3} \int \frac{1}{\sqrt{a^3 - t^2}} dt = \frac{2}{3} sin^{-1}\left(\frac{t}{a^{\frac{3}{2}}}\right) + C$$

$$= \frac{2}{3} sin^{-1}\left(\frac{x^{\frac{3}{2}}}{a^{\frac{3}{2}}}\right) + C$$

Here, *C* is constant of integration.

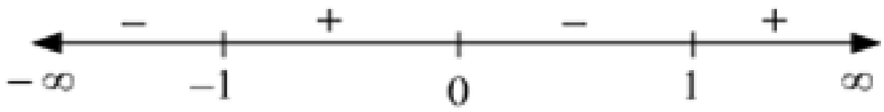
Q18 Evaluate ${}^{2}\int_{-1} |x^3 - x| dx.$

Ans: Let: ${}^{2}\int_{-1} |x^3 - x| dx.$

$$f(x) = x^3 - x$$

$$f(x) = x^3 - x = x(x-1)(x+1)$$

The signs of *f*(*x*) for the different values are shown in the figure given below:



$f(x) > 0 \; for\; all\; x \in (-1,0) \cup (1,2)$

$f(x) < 0 \; for\; all\; x \in (0,1)$

Therefore,

$|x^3 - x| = \{x^3 - x,\; x \in (-1,0) \cup (1,2) - (x^3 - x),\; x \in (0,1)$

$\therefore I = \int_{-1}^{2} x^3 - x|\, dx$

$= \int_{-1}^{0} (x^3 - x)dx - \int_{0}^{1} (x^3 - x)dx + \int_{1}^{2} (x^3 - x)dx$

$= \int_{-1}^{0} |x^3 - x|\, dx + \int_{0}^{1} |x^3 - x|\, dx + \int_{1}^{2} |x^3 - x|\, dx$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{10}^{0} + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{0}^{1} + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{1}^{2}$$

$$= -\left(\frac{1}{4} - \frac{1}{2}\right) - \left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{16}{4} - \frac{4}{4}\right) - \left(\frac{1}{4} - \frac{1}{2}\right)$$

$$= \frac{3}{4} + (4 - 2) = \frac{11}{4}$$

Q19 Find the particular solution of the differential equation $\left(1 - y^2\right)\left(1 + logx\right)dx + 2xydy = 0$ given that $y = 0$ when $x = 1$.

Ans: Given:

$\left(1 - y^2\right)\left(1 + logx\right)dx + 2xydy = 0$

$\Rightarrow \left(1 - y^2\right)\left(1 + logx\right)dx = -2xydy$

$$\Rightarrow \left(\frac{1 - \log x}{2x}\right)dx = -\left(\frac{y}{1 - y^2}\right)dy \qquad .....(1)$$

Let: $1 + logx = t$

and $\left(1 - y^2\right) = p \Rightarrow \frac{1}{x}dx = dt\, and\, -2ydy = dp$

Therefore, (1) becomes

$$\int \frac{t}{2}dt = \int \frac{1}{2p}dp$$

$$\Rightarrow \frac{t^2}{4} = \frac{\log p}{2} + C \quad ....(2)$$

Substituting the values of t and p in (2), we get

$$\frac{(1+\log x^2)}{4} = \frac{\log(1-y^2)}{2} + C \qquad .....(3)$$

At x=1 and y=0, (3) becomes $C = \frac{1}{4}$

Substituting the value of C in (3), we get

$$\frac{(1+\log x^2)}{4} = \frac{\log(1-y^2)}{2} + \frac{1}{4}$$

$$\Rightarrow \left(1+logx^2\right) = 2log\left(1-y^2\right)+1$$

Or

$$(logx)^2 + logx^2 = log\left(1-y^2\right)^2$$

It is the required particular solution.

SECTION C

Q20 Find the coordinate of the point P where the line through A(3, –4, –5) and B(2, –3, 1) crosses the plane passing through three points L(2, 2, 1), M(3, 0, 1) and N(4, –1, 0). Also, find the ratio in which P divides the line segment AB.

Ans: The equation of the plane passing through three given points can be given by

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ x-3 & y-0 & z-1 \\ x-4 & y+1 & z-0 \end{vmatrix} = 0$$

Performing elementary row operations $R2 \to R1 - R2 \ and \ R3 \to R1 - R3,$ we get

$$\Rightarrow \begin{vmatrix} x-2 & y-2 & z-1 \\ 3-2 & 0-2 & 0 \\ 4-2 & -1-2 & -1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-2 & y-2 & z-1 \\ 1 & -2 & 0 \\ 2 & -3 & -1 \end{vmatrix} = 0$$

Solving the above determinant, we get

$$\Rightarrow (x-2)(2-0)-(y-2)(-1-0)+(z-1)(-3+4)=0$$

$$\Rightarrow (2x-4)+(y-2)+(z-1)=0$$

$$\Rightarrow 2x+y+z-7=0$$

Therefore, the equation of the plane is $2x+y+z-7=0$.

Now, the equation of the line passing through two given points is

$$\frac{x-3}{2-3}=\frac{y+4}{-3+4}=\frac{z+5}{1+5}=\lambda$$

$$\Rightarrow \frac{x-3}{-1}=\frac{y+4}{1}=\frac{z+5}{6}=\lambda$$

$$\Rightarrow x=(-\lambda+3), y=(\lambda-4), z=(6\lambda-5)$$

At the point of intersection, these points satisfy the equation of the plane

$2x+y+z-7=0$.

Putting the values of x, y and z in the equation of the plane, we get the value of λ .

$$2(-\lambda+3)+(\lambda-4)+(6\lambda-5)-7=0$$

$$\Rightarrow -2\lambda+6+\lambda-4+6\lambda-5-7=0$$

$$\Rightarrow 5\lambda=10$$

$$\Rightarrow \lambda=2$$

Thus, the point of intersection is P(1, −2, 7).

Now, let P divide the line AB in the ratio $m : n$.

By the section formula, we have

$$1=\frac{2m+3m}{m+n}$$

$$\Rightarrow m+2n=0$$

$$\Rightarrow m=-2n$$

$$\Rightarrow mn=\frac{-2}{1}$$

Hence, P externally divides the line segment AB in the ratio 2 : 1.

Q.21 An urn contains 3 white and 6 red balls. Four balls are drawn one by one with replacement from the urn. Find the probability distribution of the number of red balls drawn. Also find mean and variance of the distribution.

Ans: Let X denote the total number of red balls when four balls are drawn one by one with replacement.

P (getting a red ball in one draw) = 23

P (getting a white ball in one draw) = 13

x	0	1	2	3	4
P(x)	$\left(\frac{1}{3}\right)^4$	$\frac{2}{3}\left(\frac{1}{3}\right)^3 . {}^4C_1$	$\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^3 . {}^4C_2$	$\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) . {}^4C_3$	$\left(\frac{2}{3}\right)^4$
	$\frac{1}{81}$	$\frac{8}{81}$	$\frac{24}{81}$	$\frac{32}{81}$	$\frac{16}{81}$

Using the formula for mean, we have

$$\bar{X} = \sum P_i X_i$$

$$Mean\left(\bar{X}\right) = \left(0 \times \frac{1}{81}\right) + 1\left(\frac{8}{81}\right) + 2\left(\frac{24}{81}\right) + 3\left(\frac{32}{81}\right) + 4\left(\frac{16}{81}\right)$$

$$= \frac{1}{81}(8 + 48 + 96 + 64)$$

$$= \frac{216}{81} = \frac{8}{3}$$

Using the formula for variance, we have

$$Var(X) = \sum P_i X_i^2 - \left(\sum P_i X_i\right)^2$$

$$Var(X) = \left\{\left(0 \times \frac{1}{81}\right) + 1\left(\frac{8}{81}\right) + 4\left(\frac{24}{81}\right) + 9\left(\frac{32}{81}\right) + 16\left(\frac{16}{81}\right)\right\} - \left(\frac{8}{3}\right)^2$$

$$= \frac{64}{81} - \frac{64}{9} = \frac{8}{9}$$

Hence, the mean of the distribution is $\frac{8}{3}$ and the variance of the distribution is $\frac{8}{9}$.

Q22 A manufacturer produces two products A and B. Both the products are processed on two different machines. The available capacity of first machine is 12 hours and that of second machine is 9 hours per day. Each unit of product A requires 3 hours on both machines and each unit of product B requires 2 hours on first machine and 1 hour on second machine. Each unit of product A is sold at Rs 7 profit and B at a profit of Rs 4. Find the production level per day for maximum profit graphically.

Ans: Let the numbers of units of products A and B to be produced be *x* and *y*, respectively.

Product	**Machine**	
	I (h)	II (h)
A	3	3
B	2	1

Total profit: $Z = 7x + 4y$

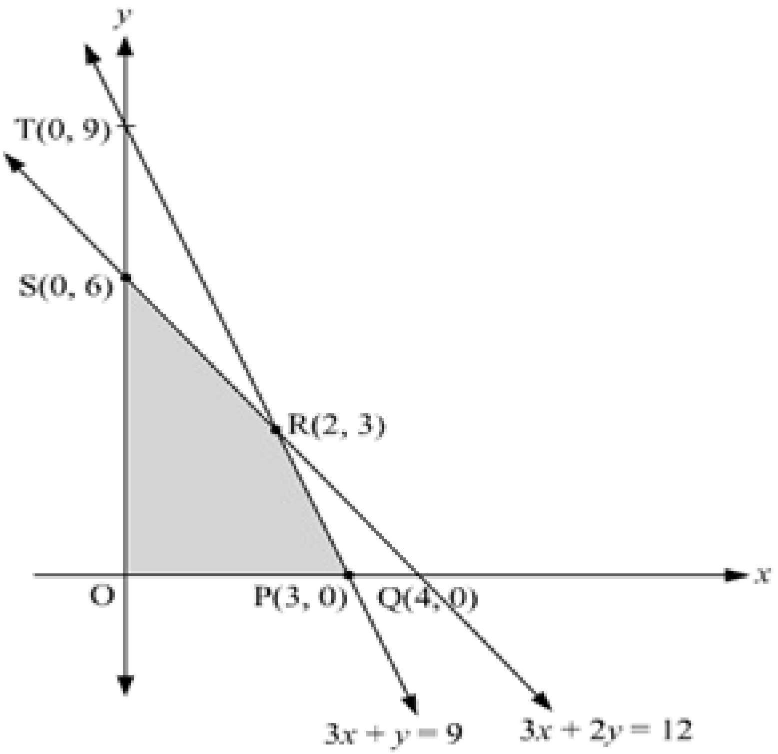
We have to maximise Z = 7*x* + 4*y*, which is subject to constraints.

$3x + 2y \le 12$ (Constraint on machine I)

$3x + y \le 9$ (Constraint on machine II)

$\Rightarrow x \ge 0 \; and \; y \ge 0$

The given information can be graphically expressed as follows:



Values of Z = 7*x* + 4*y* at the corner points are as follows

Corner Point	Z = 7x + 4y
(0, 6)	24
(2, 3)	**26**
(3, 0)	21

Therefore, the manufacturer has to produce 2 units of product A and 3 units of product B for the maximum profit of Rs 26.

Q23 Let $f : N \to N$ be a function defined as $f(x) = 9x^2 + 6x - 5$. Show that $f : N \to S$, where S is the range of f, is invertible. Find the inverse of f and hence find $f^{-1}(43)\, and\, f^{-1}(163)$.

Ans: $Given: f(x) = 9x^2 + 6x - 5$

$Let\ y = 9x^2 + 6x - 5$

$\Rightarrow y = (3x+1)^2 - 1 - 5 = (3x+1)^2 - 6$

$\Rightarrow (3x+1)^2 = y + 6$

$\Rightarrow 3x + 1 = \sqrt{y+6}$

$\Rightarrow x = \dfrac{\sqrt{y+6} - 1}{3}\ as\ x \in N$

$\Rightarrow \sqrt{y+6} - 1 > 0$

$\Rightarrow y + 6 > 1$

$\Rightarrow y > -5\ and\ y \in N$

So, the function is invertible if the range of the function *f*(*x*) is {1, 2, 3, …}.

Therefore, the inverse of the function *f*(*x*) is $f^{-1}(y),\ i.e. x.$

Now,

$$f^{-1}(y) = \frac{\sqrt{y+6} - 1}{3}$$

$$f^{-1}(43) = \frac{\sqrt{43+6} - 1}{3} = 2$$

$$f^{-1}(163) = \frac{\sqrt{163+6}-1}{3} = 4$$

Q24 Prove that $\begin{vmatrix} yz-x^2 & zx-y^2 & xy-z^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix}$ **is divisible by (x + y + z) and hence find the quotient.**

Ans: $\begin{vmatrix} yz-x^2 & zx-y^2 & xy-z^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} xy+yz+zx-x^2-y^2-z^2 & zx-y^2 & xy-z^2 \\ xy+yz+zx-x^2-y^2-z^2 & xy-z^2 & xy-x^2 \\ xy+yz+zx-x^2-y^2-z^2 & yz-x^2 & zx-y^2 \end{vmatrix}$$

$$\Rightarrow \Delta = \left(xy+yz+zx-x^2-y^2-z^2\right)\begin{vmatrix} 1 & zx-y^2 & xy-z^2 \\ 1 & xy-z^2 & yz-x^2 \\ 1 & yz-x^2 & zx-y^2 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1 \ and \ R_3 \to R_3 - R_1$, we get

$$\Delta = \left(xy+yz+zx-x^2-y^2-z^2\right)\begin{vmatrix} 1 & zx-y^2 & xy-z^2 \\ 0 & (x+y+z)(y-z) & (x+y+z)(z-x) \\ 0 & (x+y+z)(y-x) & (x+y+z)(z-y) \end{vmatrix}$$

$$\Rightarrow \Delta = (x+y+z)^2\left(xy+yz+zx-x^2-y^2-z^2\right)\left[\{(y-z)(z-y)-(z-x)(y-x)\}-0+0\right]$$

$$\Rightarrow \Delta = (x+y+z)^2\left(xy+yz+zx-x^2-y^2-z^2\right)^2$$

So, Δ is divisible by (x + y + z).

The quotient when Δ is divisible by

$(x+y+z)$ is $(x+y+z)\left(xy+yz+zx-x^2-y^2-z^2\right)$.

OR

Using elementary transformations, find the inverse of the matrix $A=\begin{pmatrix}8&4&3\\2&1&1\\1&2&2\end{pmatrix}$ **and use it to solve the following system of linear equations:**

$8x+4y+3z=19$

$2x+y+z=5$

$x+2y+2z=7$

Ans: A = IA

i.e.

$$\begin{bmatrix}8&4&3\\2&1&1\\1&2&2\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}A$$

Applying $R_1 \leftrightarrow R_3,$ *we get*

$$\begin{bmatrix}1&2&2\\2&1&1\\8&4&3\end{bmatrix}=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}A$$

Applying $R_2 \to R_2-2R_1,$ *we get*

$$\begin{bmatrix}1&2&2\\0&-3&-3\\8&4&3\end{bmatrix}=\begin{bmatrix}0&0&1\\0&1&-2\\1&0&0\end{bmatrix}A$$

Applying $R_3 \to R_3-8R_1,$ *we get*

$$\begin{bmatrix}1&2&2\\0&-3&1\\0&-12&-13\end{bmatrix}=\begin{bmatrix}0&0&1\\0&1&-2\\1&0&-8\end{bmatrix}A$$

Applying $R_3 \to \dfrac{R_2}{-3},$ *we get*

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & -12 & -13 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{3} & \frac{2}{3} \\ 1 & 0 & -8 \end{bmatrix} A$$

Applying $R_1 \to R_1 - 2R_2$, $we\ get$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -12 & -13 \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} \\ 1 & 0 & -8 \end{bmatrix} A$$

Applying $R_3 \to R_3 + 12R_2$, $we\ get$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} \\ 1 & -4 & 0 \end{bmatrix} A$$

Applying $R_3 \to -R_3$ and $R_2 \to R_2 - R_3$, $we\ get$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ 1 & -\frac{13}{3} & \frac{2}{3} \\ -1 & 4 & 0 \end{bmatrix} A$$

Thus, we have

$$A^{-1} = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ 1 & -\frac{13}{3} & \frac{2}{3} \\ -1 & 4 & 0 \end{bmatrix}$$

The given system of equations is

$$8x+4y+3z=19$$

$$2x+y+z=5$$

$$x+2y+2z=7$$

This system of equations can be written as AX = B, where

A= $\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 19 \\ 5 \\ 7 \end{bmatrix}$

$$\therefore \ X = A^{-1}B$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ 1 & -\frac{13}{3} & \frac{2}{3} \\ -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 19 \\ 5 \\ 7 \end{bmatrix}$$

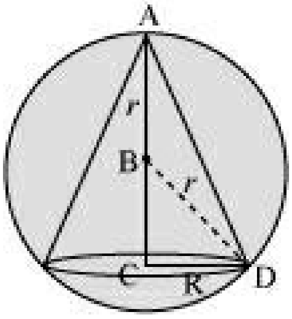
$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0+\frac{10}{3}-\frac{7}{3} \\ 19-\frac{65}{3}+\frac{14}{3} \\ -19+20+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\therefore \ x=1, \ y=2 \ \text{ and } z=1$$

Q.25 Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius $r \ is \ \frac{4r}{3}$ **. Also find maximum volume in terms of volume of the sphere.**

Ans: A sphere of fixed radius (*r*) is given.

Let *R* and *h* be the radius and the height of the cone, respectively.



The volume (V) of the cone is given by,

$$V = \frac{1}{3}\pi R^2 h$$

Now, from the right triangle BCD, we have:

$$\text{BC} = \sqrt{r^2 - R^2}$$

$$\therefore h = r + \sqrt{r^2 - R^2}$$

$$\therefore V = \frac{1}{3}\pi R^2\left(r + \sqrt{r^2 - R^2}\right) = \frac{1}{3}\pi R^2 r + \frac{1}{3}\pi R^2\sqrt{r^2 - R^2}$$

$$\therefore \frac{dV}{dR} = \frac{2}{3}\pi R r + \frac{2}{3}\pi R\sqrt{r^2 - R^2} + \frac{\pi R^2}{3}\cdot\frac{(-2R)}{2\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3}\pi R r + \frac{2}{3}\pi R\sqrt{r^2 - R^2} - \frac{\pi R^3}{3\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3}\pi R r + \frac{2\pi R\left(r^2 - R^2\right) - \pi R^3}{3\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3}\pi R r + \frac{2\pi R r^2 - 3\pi R^3}{3\sqrt{r^2 - R^2}}$$

$$\text{Now}, \frac{dV}{dR^2} = 0$$

$$\Rightarrow \frac{2\pi r R}{3} = \frac{3\pi R^3 - 2\pi R r^2}{3\sqrt{r^2 - R^2}}$$

$$\Rightarrow 2r\sqrt{r^2 - R^2} = 3R^2 - 2r^2$$

$$\Rightarrow 4r^2\left(r^2 - R^2\right) = \left(3R^2 - 2r^2\right)^2$$

$$\Rightarrow 4r^4 - 4r^2R^2 = 9R^4 + 4r^4 - 12R^2r^2$$

$$\Rightarrow 9R^4 - 8r^2R^2 = 0$$

$$\Rightarrow 9R^2 = 8r^2$$

$$\Rightarrow R^2 = \frac{8r^2}{9}$$

Now, $\dfrac{d^2V}{dR^2}=\dfrac{2\pi r}{3}+\dfrac{3\sqrt{r^2-R^2}\left(2\pi r^2-9\pi R^2\right)-\left(2\pi Rr^2-3\pi R^3\right)\left(-6R\right)\dfrac{1}{2\sqrt{r^2-R^2}}}{9\left(r^2-R^2\right)}$

$=\dfrac{2\pi r}{3}+\dfrac{3\sqrt{r^2-R^2}\left(2\pi r^2-9\pi R^2\right)+\left(2\pi Rr^2-3\pi R^3\right)\left(3R\right)\dfrac{1}{2\sqrt{r^2-R^2}}}{9\left(r^2-R^2\right)}$

Now, when $R^2=\dfrac{8r^2}{9}$, it can be shown that $\dfrac{d^2V}{dR^2}<0$.

∴ The volume is the maximum when $R^2=\dfrac{8r^2}{9}$.

When $R^2=\dfrac{8r^2}{9}$, height of the cone $=r+\sqrt{r^2-\dfrac{8r^2}{9}}=r+\sqrt{\dfrac{r^2}{9}}=r+\dfrac{r}{3}=\dfrac{4r}{3}$.

Hence, it can be seen that the altitude of a right circular cone of maximum volume that can be inscribed in a sphere of radius *r* is $\dfrac{4r}{3}$.

Let volume of the sphere be $V_s=\dfrac{4}{3}\pi r^3$.

$r=3\sqrt{\dfrac{3V_s}{4\pi}}$

∴ Volume of cone, $V=\dfrac{1}{3}\pi R^2h$

$\Rightarrow R=\dfrac{2\sqrt{2}}{3}r$

$V=\dfrac{1}{3}\pi\left(\dfrac{2\sqrt{2}}{3}r\right)^2\times\dfrac{4r}{3}$

$\Rightarrow V=\dfrac{1}{3}\pi\dfrac{8r^3}{9}\times\dfrac{4r}{3}$

$V=\dfrac{32\pi r^3}{81}=\dfrac{32}{81}\pi\left[\dfrac{3V_s}{4\pi}\right]$

∴ Volume of cone in terms of sphere = $\dfrac{8V_s}{27}$

OR

Find the intervals in which $f(x) = \sin 3x - \cos 3x,\ 0 < x < \pi,$ is strictly increasing or strictly decreasing.

Ans: Consider the function

$f(x) = \sin 3x - \cos 3x.$

$f'(x) = 3\cos 3x + 3\sin 3x$

$= 3(\sin 3x + \cos 3x)$

$$= 3\sqrt{2}\left\{\sin 3x \cos\left(\frac{\pi}{4}\right) + \cos 3x \sin\left(\frac{\pi}{4}\right)\right\}$$

$$= 3\sqrt{2}\left\{\sin\left(3x + \frac{\pi}{4}\right)\right\}$$

For the increasing interval $f'(x) > 0.$

$$3\sqrt{2}\left\{\sin\left(3x + \frac{\pi}{4}\right)\right\} > 0$$

$$\sin\left(3x + \frac{\pi}{4}\right) > 0$$

$$\Rightarrow 0 < 3x + \frac{\pi}{4} < \pi$$

$$\Rightarrow 0 < 3x + \frac{3\pi}{4}$$

$$\Rightarrow 0 < x < \frac{\pi}{4}$$

Also,

$$\sin\left(3x + \frac{\pi}{4}\right) > 0$$

when, $2\pi < 3x + \frac{\pi}{4} < 3\pi$

$$\Rightarrow \frac{7\pi}{4} < 3x < \frac{11\pi}{44}$$

$$\Rightarrow \frac{7\pi}{12} < x < \frac{11\pi}{12}$$

Therefore, intervals in which function is strictly increasing in

$$0 < x < \frac{\pi}{4} \text{ and } \frac{7\pi}{12} < x < \frac{11\pi}{12}.$$

Similarly, for the decreasing interval $f'(x) < 0.$

$$3\sqrt{2}\left\{\sin\left(3x+\frac{\pi}{4}\right)\right\} < 0$$

$$\sin\left(3x+\frac{\pi}{4}\right) < 0$$

$$\Rightarrow \pi < 3x+\frac{\pi}{4} < 2\pi$$

$$\Rightarrow \frac{3\pi}{4} < 3x < \frac{7\pi}{4}$$

$$\Rightarrow \frac{\pi}{4} < x < \frac{7\pi}{12}$$

Also,

$$\sin\left(3x+\frac{\pi}{4}\right) < 0$$

$$When\, 3\pi < 3x+\frac{\pi}{4} < 4\pi,$$

$$\Rightarrow \frac{11\pi}{4} < 3x < \frac{15\pi}{4}$$

$$\Rightarrow \frac{11\pi}{12} < x < \frac{15\pi}{12}$$

The function is strictly decreasing in $\frac{\pi}{4} < x < \frac{7\pi}{12} \text{ and } \frac{11\pi}{12} < x < \pi.$

Q26 Using integration find the area of the region

$$\{(x,y) : x^2+y^2 \leqslant 2ax, y^2 \geqslant ax, x, y \geqslant 0\}.$$

Ans: Given: $x^2+y^2 \le 2ax, y^2 \ge ax, x, y \ge 0$

$$\Rightarrow x^2 + y \;\; - 2ax \le 0,\ y^2 \ge ax, x, y \ge 0$$

$$\Rightarrow x^2 + y^2 - 2ax + a^2 - a^2 \le 0,\ y^2 \ge ax, x, y \ge 0$$

$$\Rightarrow (x-a)^2 + y^2 \le a^2,\ y^2 \ge ax, x, y \ge 0$$

To find the points of intersection of the circle $\left[(x-a)^2 + y^2 = a^2\right]$ and the parabola $\left[y^2 = ax\right]$, we will substitute $y^2 = ax\ in (x-a)^2 + y^2 = a^2$.

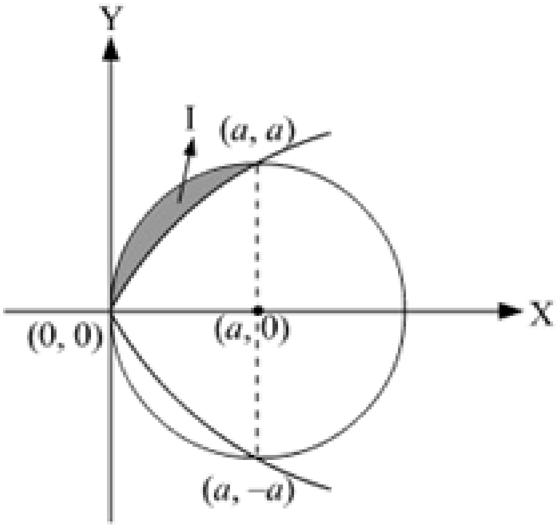
$$(x-a)^2 + ax = a^2$$

$$\Rightarrow x^2 + a - 2ax + ax = a^2$$

$$\Rightarrow x(x-a) = 0$$

$$\Rightarrow x = 0,\ a$$

Therefore, the points of intersection are (0, 0), (a, a) and (a, −a).



Now,

Area of the shaded region= I

Area of I from x=0 to x=a

$$= \left[\int_0^a \left(\sqrt{a^2 - (x-a)^2}\right) dx - \int_0^a \sqrt{ax}\, dx\right]$$ Let $x-a=t$ for the first part of the integral

$$= \int_0^a \left(\sqrt{a^2 - (x-a)^2}\right) dx.$$

$$\Rightarrow dx = dt$$

$$\therefore A_I = \int_{-a}^0 \sqrt{a^2 - t^2}\, dt - 2\frac{\sqrt{a}}{3} \left| x^{\frac{3}{2}} \right|_0^a$$

$$= \left| \frac{t}{2}\sqrt{a^2 - t^2} + \frac{1}{2}a^2 \sin^{-1}\frac{t}{a} \right|_{-a}^{0} - \frac{2a^2}{3}$$

$$= 0 - \left(-\frac{\pi a^2}{4} \right) - \frac{2a^2}{3}$$

$$A_1 = \left(\frac{\pi}{4} - \frac{2}{3} \right) a^2$$

$\therefore$ Area of the shaded region $= \left(\frac{\pi}{4} - \frac{2}{3} \right) a^2$ square units