Question Paper 2016 Central Outside Delhi Set-3

CBSE Class XII Mathematics

General Instruction:

- All question are compulsory.
- Please check that this Question Paper contains 26 Questions.
- Questions 1 to 6 in Section-A are Very Short Answer Type Questions carrying one mark each.
- Questions 7 to 19 in Section-B are Long Answer I Type Questions carrying 4 marks each.
- Questions 20 to 26 in Section-C are Long Answer II Type Questions carrying 6 marks each.
- Please write down the serial number of the Question before attempting it.

Section A

1. Write the position vector of the point which divides the join of points with position vectors $3\bar{a}-2\bar{b}$ and $2\bar{a}+3\bar{b}$ in the ratio $2:1$.

Sol. $\dfrac{2(2\bar{a}+3\bar{b})+1(3\bar{a}-2\bar{b})}{2+1}$

$=\dfrac{7}{3}\hat{a}+\dfrac{4}{3}\hat{b}$ **(or enternal division may also be considered)**

2. Write the number of vectors of unit length perpendicular to both vectors $\bar{a}=2\hat{i}+\hat{j}+2\hat{k}$ and $\bar{b}=\hat{j}+\hat{k}$.

Sol. 2

3. Find the vector equation of the plane with intercepts 3, -4 and 2 on x, y and zaxis respectively.

Sol. $\frac{x}{3}+\frac{y}{-4}+\frac{z}{2}=1$

$\Rightarrow \vec{r}\cdot(4\hat{i}-3\hat{j}+6\hat{k})=12$ or $\vec{r}\cdot\left(\frac{\hat{i}}{3}-\frac{\hat{j}}{4}+\frac{\hat{k}}{2}\right)=1$

4. If $x \in N$ and $\begin{vmatrix} x+3 & -2 \\ -3x & 2x \end{vmatrix} = 8$, then find the value of x.

Sol. $(x+3)2x-(-2)(-3x)=8$

$x=2$

5. Use elementary column operation $C_2 \to C_2 + 2C_1$ in the following matrix equation:

$$\begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

Sol. $\begin{pmatrix} 2 & 5 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$

6. Write the number of all possible matrices of order 2×2 with each entry 1, 2 or 3.

Sol. $\left.\begin{array}{r} \text{No. of possible matrices} = 3^4 \\ or\ \ 81 \end{array}\right\}$

Section B

7. Evaluate: $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} dx$

OR

Evaluate: $\int_0^{\frac{3}{2}} |x\cos \pi x|\, dx$

Sol. Let $I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx,$ Also $I = \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx$

$$= \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x} dx$$

Adding to get, $2I = \int_0^{\pi/2} \frac{1}{\sin x + \cos x} dx = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{1}{\cos (x - \pi/4)} dx$

$$\Rightarrow 2I = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \sec(x - \pi/4) dx = \frac{1}{\sqrt{2}} \log \left| \sec\left(x - \frac{\pi}{4}\right) + \tan\left(x - \frac{\pi}{4}\right) \right|_0^{\pi/2}$$

$$\Rightarrow 2I = \frac{1}{\sqrt{2}} \left\{ \log\left|\sqrt{2}+1\right| - \log\left|\sqrt{2}-1\right| \right\}$$

$$\Rightarrow I = \frac{1}{2\sqrt{2}} \left\{ \log\left|\sqrt{2}+1\right| - \log\left|\sqrt{2}-1\right| \text{ or } \frac{1}{2\sqrt{2}} \log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right| \right.$$

OR

$$\int_0^{3/2} |x \cos \pi x| dx = \int_0^{1/2} x \cos \pi x\, dx - \int_{1/2}^{3/2} x \cos \pi x\, dx$$

$$= \left\{ \frac{x \sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2} \right\}_0^{1/2} - \left\{ \frac{x \sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2} \right\}_{1/2}^{3/2}$$

$$= \frac{1}{2\pi} - \frac{1}{\pi^2} - \left(-\frac{3}{2\pi} - \frac{1}{2\pi} \right) = \frac{5}{2\pi} - \frac{1}{\pi^2}$$

8. In a game, a man wins Rs 5 for getting a number greater than 4 and loses Rs 1 otherwise, when a fair die is thrown. The man decided to throw a die thrice but to quit as and when he gets a number greater than 4. Find the expected value of the amount he wins/loses.

OR

A bag contains 4 balls. Two balls are drawn at random (without replacement) and are found to be white. What is the probability that all balls in the bag are white?

Sol. Let X = Amount he wins then $x = Rs\,5, 4, 3, -3$

P = Probability of getting a no. $> 4 = \frac{1}{3}, q = 1 - p = \frac{2}{3}$

X:	5	4	3	-3
P(x)	$\frac{1}{3}$	$\frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$	$\left(\frac{2}{3}\right)^2 \cdot \frac{1}{3} = \frac{4}{27}$	$\left(\frac{2}{3}\right)^3 = \frac{8}{27}$

Expected amount he wins $= \sum XP(X) = \frac{5}{3} + \frac{8}{9} + \frac{12}{27} - \frac{24}{27}$

$= Rs\frac{19}{9} \text{ or } Rs\,2\frac{1}{9}$

OR

E_1 = Event that all balls are white,

E_2 = Event that 3 balls are white and 1 balls is non white

E_3 = Event that 2 balls are white and 2 balls are non-white

A = Event that 2 balls without replacement are white

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

$$P(A/E_1) = 1,\ P(A/E_2) = \frac{3}{4}\cdot\frac{2}{3} = \frac{1}{2},\ P(A/E_3) = \frac{2}{4}\cdot\frac{1}{3} = \frac{1}{6}$$

$$P(E_1/A) = \frac{1.\,1/3}{1.\,1/3 + 1/3\,.\,1/2 + 1/3\,.\,1/6} = \frac{3}{5}$$

9. Find: $\int \frac{x^2}{x^4 + x^2 - 2}\,dx$

Sol. Let $x^2 = t$ $\therefore \frac{x^2}{x^4 + x^2 - 2} = \frac{x^2}{(x^2 - 1)(x^2 + 2)} = \frac{t}{(t-1)(t+2)} = \frac{A}{t-1} + \frac{B}{t+2}$

Solving for A and B to get, $A = \frac{1}{3},\ B = \frac{2}{3}$

$$\int \frac{x^2}{x^4 + x^2 - 2}\,dx = \frac{1}{3}\int \frac{1}{x^2 - 1}\,dx + \frac{2}{3}\int \frac{1}{x^2 + 2}\,dx = \frac{1}{6}\log\left|\frac{x-1}{x+1}\right| + \frac{\sqrt{2}}{3}\tan^{-1}\frac{x}{\sqrt{2}} + C$$

10. If $x = a\sin 2t(1 + \cos 2t)$ **and** $y = b\cos 2t(1 - \cos 2t)$, **find** $\frac{dy}{dx}$ $at\ t = \frac{\pi}{4}$.

Sol. $\frac{dx}{dt} = 2a\cos t\ 2t\,(1 + \cos 2t) - 2a\sin 2t \cdot \sin 2t$

$$\frac{dy}{dt} = -2b\sin 2t\,(1 - \cos 2t) + 2b\cos 2t \cdot \sin 2t$$

$$\left.\frac{dy}{dx}\right]_{t=\frac{\pi}{4}} = \left.\frac{2b\cos 2t \cdot \sin 2t - 2b\sin 2t\,(1 - \cos 2t)}{2a\cos 2t\,(1 + \cos 2t) - 2a\sin 2t \cdot \sin 2t}\right]_{t=\frac{\pi}{4}} = \frac{b}{a}$$

11. Find the coordinates of the point where the line through the points A(3, 4, 1) and

B(5, 1, 6) crosses the XZ plane. Also find the angle which this line makes with the XZ plane.

Sol. Equation of line through A(3, 4, 1) and B(5, 1, 6)

$$\frac{x-3}{2}=\frac{y-4}{-3}=\frac{z-1}{5}=k(say)$$

General point on the line:

$$x=2k+3,\ y=-3k+4,\ z=5k+1$$

Line crosses xz plane i.e. y = 0 if -3k + 4 = 0

$$\therefore \qquad k=\frac{4}{3}$$

Co-ordinate of required point $\left(\frac{17}{3}, 0, \frac{23}{3}\right)$

Angle, which line makes with xz plane:

$$\sin\theta=\left|\frac{2(0)+(-3)(1)+5(0)}{\sqrt{4+9+25}\sqrt{1}}\right|=\frac{3}{\sqrt{38}}\Rightarrow\theta\sin^{-1}\left(\frac{3}{\sqrt{38}}\right)$$

12. Find: $\int(3x+1)\sqrt{4-3x-2x^2}\,dx$

Sol. $\int(3x+1)\sqrt{4-3x-2x^2}\,dx=-\frac{3}{4}\int(-4x-3)\sqrt{4-3x-2x^2}\,dx-\frac{5}{4}$
$\int\sqrt{4-3x-2x^2}\,dx$

$$=-\frac{1}{2}(4-3x-2x^2)^{3/2}-\frac{5}{4}\sqrt{2}\int\sqrt{\left(\frac{\sqrt{41}}{4}\right)^2-\left(x+\frac{3}{4}\right)^2}\,dx$$

$$= -\frac{1}{2}(4-3-2x^2)^{3/2} - \frac{5}{4}\sqrt{2}\left\{\frac{4x+3}{8}\sqrt{\frac{41}{16}-\left(x+\frac{3}{4}\right)^2} + \frac{41}{32}\cdot\sin^{-1}\left(\frac{4x+3}{\sqrt{41}}\right)\right\} + C$$

$$= -\frac{1}{2}(4-3x-2x^2)^{3/2} - \frac{5}{4}\left\{\frac{4x+3}{8}\sqrt{4-3x-2x^2} + \frac{41\sqrt{2}}{32}\cdot\sin^{-1}\left(\frac{4x+3}{\sqrt{41}}\right)\right\} + C$$

13. The equation of tangent at (2, 3) on the curve $y^2 = ax^3 + b$ is $y = 4x - 5$. Find the value of a and b.

Sol. $y^2 = ax^3 + b \Rightarrow 2y\frac{dy}{dx} = 3ax^2 \therefore \frac{dy}{dx} = \frac{3a}{2}\frac{x^2}{y}$

Slope of tangent at $(2, 3) = \left.\frac{dy}{dx}\right]_{(2,3)} = \frac{3a}{2}\cdot\frac{4}{3} = 2a$

Comparing with slope of tangent y = 4x – 5, we get, 2a = 4 $\therefore \boxed{a = 2}$

Also (2, 3) lies on the curve ∴ 9 = 8a + b, put a = 2, we get b = -7

14. A trust invested some money in two type of bonds. The first bond pays 10% interest and second bond pays 12% interest. The trust received Rs 2,800 as interest. However, if trust had interchanged money in bonds, they would have got Rs 100 less as interest. Using matrix method, find the amount invested by the trust. Interest received on this amount will be given to Helpage India as donation. Which value is reflected in this question?

Sol. let Rs x be invested in first bond

and Rs y be invested in second bond

then the system of equation is:

$$\left.\begin{array}{l}\frac{10x}{100}+\frac{12y}{100}=2800\\ \frac{12x}{100}+\frac{10y}{100}=2700\end{array}\right\}\Rightarrow\left.\begin{array}{l}5x+6y=140000\\ 6x+5y=135000\end{array}\right\}$$

Let $A=\begin{bmatrix}5 & 6\\ 6 & 5\end{bmatrix}; X=\begin{bmatrix}x\\ y\end{bmatrix}; B=\begin{bmatrix}140000\\ 135000\end{bmatrix}$

$\therefore A\cdot X=B$

$|A|=11; A^{-1}=\frac{1}{-11}\begin{bmatrix}5 & -6\\ -6 & 5\end{bmatrix}$

$\therefore$ Solution is X = A^{-1}B $\Rightarrow\begin{bmatrix}x\\ y\end{bmatrix}=\frac{1}{-11}\begin{bmatrix}5 & -6\\ -6 & 5\end{bmatrix}\begin{bmatrix}140000\\ 135000\end{bmatrix}=\begin{bmatrix}10000\\ 15000\end{bmatrix}$

$\therefore$ x = 10000, y = 15000, $\therefore$ Amount invested = Rs 25000

Value caring elders

15. Solve the differential equation:

$$y+x\frac{dy}{dx}=x-y\frac{dy}{dx}$$

Sol. The differential equation can be re-written as:

$\frac{dy}{dx}=\frac{x-y}{x+y}$, put y = vx, $\frac{dy}{dx}=v+x\frac{dv}{dx}$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{1-v}{1+v}\quad\Rightarrow\frac{1+v}{1-2v-v}dv=\frac{1}{x}dx$$

integrating we get

$$\Rightarrow\frac{1}{2}\int\frac{2V+2}{V^2+2V-1}dv=-\int\frac{1}{x}dx=\frac{1}{2}\log|V^2+2V-1|=-\log x+\log C$$

∴ Solution of the differential equation is:

$$\frac{1}{2}\log\left|\frac{y^2}{x^2}+\frac{2y}{x}-1\right| = \log C - \log x \ or, \ y^2+2xy-x^2 = C^2$$

16. The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}-5\hat{k}$ **and** $2\hat{i}+2\hat{j}+3\hat{k}$. **Find the two unit vectors parallel to its diagonals. Using the diagonal vectors, find the area of the parallelogram.**

Sol. let $\overline{d}$ **&** $\overline{d_2}$ be the two diagonal vectors:

$$\therefore \quad \overline{d_1} = 4\hat{i}-2\hat{j}-2\hat{k}, \ \overline{d_2} = -6\hat{j}-8\hat{k}$$

or $\overline{d_2} = 6\hat{j}+8\hat{k}$

Unit vectors parallel to the diagonals are:

$$\hat{d}_1 = \frac{2}{\sqrt{6}}\hat{i}-\frac{1}{\sqrt{6}}\hat{j}-\frac{1}{\sqrt{6}}\hat{k}$$

$$\hat{d}_2 = -\frac{3}{5}j-\frac{4}{5}\hat{k} \quad \left(or\ \hat{d}_2 = \frac{3}{5}\hat{j}+\frac{4}{5}\hat{k}\right)$$

$$\overline{d_1}\times\overline{d_2} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & -2 & -2\\ 0 & -6 & -8\end{vmatrix} = 4\hat{i}+32\hat{j}-24\hat{k}$$

Area of parallelogram $= \frac{1}{2}\left|\overline{d_1}\times\overline{d_2}\right| = \sqrt{404} \ or \ 2\sqrt{101} \ sq.units$

17. Solve the equation for $x : \sin^{-1}x+\sin^{-1}(1-x) = \cos^{-1}x$

OR

If $\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b}=\alpha$, **prove that** $\frac{x^2}{a^2}-2\frac{xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha$

Sol. $\sin^{-1} x + \sin^{-1}(1-x) = \cos^{-1} x \Rightarrow \sin^{-1}(1-x)=\frac{\pi}{2}-2\sin^{-1}x$

$$\Rightarrow 1-x=\sin\left(\frac{\pi}{2}-2\sin^{-1}x\right)\Rightarrow 1-x=\cos(2\sin^{-1}x)\Rightarrow 1-x=1-2\sin^2(\sin^{-1}x)$$

$\Rightarrow 1-x=1-2x^2$

Solving we get, $x = 0$ or $x=\frac{1}{2}$

OR

From the equation: $\cos^{-1}\frac{x}{a}=\alpha-\cos^{-1}\frac{y}{b}$

$$\frac{x}{a}=\cos\left(\alpha-\cos^{-1}\frac{y}{b}\right)\Rightarrow\frac{x}{a}=\cos\alpha\cdot\cos\left(\cos^{-1}\frac{y}{b}\right)+\sin\alpha\cdot\sin\left(\cos^{-1}\frac{y}{b}\right)$$

$$\Rightarrow\frac{x}{a}=\frac{y\cdot\cos\alpha}{b}+\sin\alpha\sqrt{1-\frac{y^2}{b^2}}\Rightarrow\frac{x}{a}-\frac{y}{b}\cos\alpha=\sin\alpha\sqrt{1-\frac{y^2}{b^2}}$$

Squaring both sides,

$$\Rightarrow\left(\frac{x}{a}-y\frac{\cos\alpha}{b}\right)^2=\left(\sin\alpha\sqrt{1-\frac{y^2}{b^2}}\right)^2$$

$$\Rightarrow\frac{x^2}{a^2}-\frac{2xy}{ab}\cdot\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha.$$

18. Differentiate $x^{\sin x}+(\sin x)^{\cos x}$ **with respect to x.**

OR

If $y = 2\cos(\log x) + 3\sin(\log x)$, **prove that** $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$.

Sol. let y = u + v, u = $x^{\sin x}$, v = $(\sin x)^{\cos x}$

$$\log u = \sin x.\log x \Rightarrow \frac{du}{dx} = x^{\sin x}.\left\{\cos x.\log x + \frac{\sin x}{x}\right\}$$

$$\log v = \cos x.\log(\sin x) \Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x}\cdot\{\cos x\cdot\cot x - \sin x\cdot\log(\sin x)\}$$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} = x^{\sin x}\cdot\left\{\cos x\cdot\log x = \frac{\sin x}{x}\right\} + (\sin x)^{\cos x}$$

$$(\cos x\cdot\cot x - \sin x\cdot\log(\sin x))$$

OR

$$\frac{dy}{dx} = \frac{-2\sin(\log x)}{x} + \frac{3\cos(\log x)}{x}$$

$\Rightarrow \quad x\frac{dy}{dx} = -2\sin(\log x) + 3\cos(\log x)$, differentiate w.r.t 'x'

$$\Rightarrow \quad x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{-2\cos(\log x)}{x} - \frac{3\sin(\log x)}{x}$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y \Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

19. Form the differential equation of the family of circles in the second quadrant and touching the coordinate axes.

Sol. Let radius of any of the circle touching co-ordinate axes in the second quadrant be "a" then centre is (-a, a)

∴ Equation of the family of circles is:

$(x + a)^2 + (y – a)^2 = a^2$, $a \in R$

$\Rightarrow x^2 + y^2 + 2ax - 2ay + a^2 = 0$

Differentiate w.r.t. "x", 2x + 2yy' + 2a – 2ay' = 0 $\Rightarrow a = \dfrac{x+yy'}{y'-1}$

∴ The differential equation is:

$$\left(x+\frac{x+yy'}{y'-1}\right)^2+\left(y-\frac{x+yy'}{y'-1}\right)^2=\left(\frac{x+yy'}{y'-1}\right)^2$$

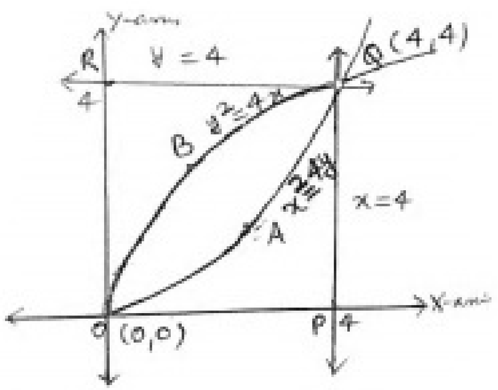
$$\Rightarrow\left(\frac{xy'+yy'}{y'-1}\right)^2+\left(\frac{x+y}{y'-1}\right)^2=\left(\frac{x+yy'}{y'-1}\right)^2$$

Section C

20. prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of square bounded by x = 0, x = 4, y = 4 and y = 0 into three equal parts.

Sol. Point of intersection of $y^2 = 4x$ and $x^2 = 4y$ are (0, 0) and (4, 4);

Correct graph



are $(OAQBO) = \int_0^4 \left(2\sqrt{4} - \frac{x^2}{4}\right) dx$

$$= \left\{ \frac{4}{3} x^{3/2} - \frac{x^3}{12} \right\} \Bigg]_0^4$$

$$= \frac{32}{3} - \frac{16}{3} = \frac{16}{3}$$

are (OPQAO) $= \int_0^4 \frac{x^2}{4} dx = \frac{1}{12} x^3 \Big]_0^4 = \frac{16}{3}$

are (OBQRO) $= \int_0^4 \frac{y^2}{4} dy = \frac{1}{12} y^3 \Big]_0^4 = \frac{16}{3}$

Hence the areas of the three regions are equal.

21. Show that the binary operation * on defined as a*b = a + b+ ab for all $a, b \in A$ is commutative and associative on A. Also find the identity element of * in A and prove that every element of A is invertible.

Sol. Commutative: For any element a, $b \in A$

a * b = a + b + ab = b + a + ba = b * a. Hence * is commutative

Associative: For any three element $a, b, c \in A$

a * (b * c) = a * (b + c + bc) = a + b + c + bc + ab + ac + abc

(a * b) * c = (a + b + ab) * c = a + b + ab + c + ac + bc + abc

∴ a * (b * c) = (a * b) * c, Hence * is Associative.

Identity element: let e ∈ A be the identity element them a * e = e * a = a

⇒ a + e + ae = e + a + ea = a ⇒ e (1 + a) = 0, as a ≠ – 1

e = 0 is the identity element

Invertible: let a, b ∈ A so that 'b' is inverse of a

∴ a * b = b * a = e

⇒ a + b + ab = b + a + ba = 0

As $a \neq -1, b = \frac{-a}{1+a} \in A$ Hence every element of A invertible

22. Using properties of determinants, show that ΔABC is isosceles if:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A+\cos A & \cos^2 B+\cos B & \cos^2 C+\cos C \end{vmatrix}=0$$

OR

A shopkeeper has 3 varieties of pens 'A', 'B' and 'C'. Meenu purchased 1 pen of each variety for a total of Rs 21. Jeevan purchased 4 pens of 'A' variety, 3 pens of 'B' variety, 2 pens of 'B' variety and 3 pens of 'C' variety for Rs 70. Using matrix method, find cost of each variety of pen.

Sol. $$\begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A+\cos A & \cos^2 B+\cos B & \cos^2 C+\cos C \end{vmatrix}=0$$

$$\Leftrightarrow \begin{vmatrix} 1 & 0 & 0 \\ 1+\cos A & \cos B-\cos A & \cos C-\cos A \\ \cos^2 A+\cos A & (\cos B-\cos A)(\cos B+\cos A+1) & (\cos C-\cos A)(\cos C+\cos A+1) \end{vmatrix}=0$$

Taking (cos B – cos A), (cos C – cosA) common form C^2 & C^3

$$\Leftrightarrow (\cos B-\cos A)(\cos C-\cos A)\begin{vmatrix} 1 & 0 & 0 \\ 1+\cos A & 1 & 1 \\ \cos^2 A+\cos A & \cos B+\cos A+1 & \cos C+\cos A+1 \end{vmatrix}=0$$

Expand along R_1

$\Leftrightarrow$ (cos B – cos A) (cos C – cos A) (cos C – cos B) = 0

$\Leftrightarrow$ cos A = cos B $\Leftrightarrow$ A = B $\Leftrightarrow$ ΔABC is an isosceles triangle

Or or

Cos B = cos C B = C

Or or

Cos C = cos A C = A

23. There are two types of fertilisers 'A' and 'B'. 'A' consists of 12% nitrogen and 5% phosphoric acid whereas 'B' consists of 4% nitrogen and 5% phosphoric acid. After testing the soil conditions, farmer finds that he needs at least 12 kg of nitrogen and 12 kg of phosphoric acid for his crops. If 'A' costs Rs 10 per kg and 'B' cost Rs 8 per kg, then graphically determine how much of each type of fertilizer should be used so that nutrient requirements are met at a minimum cost.

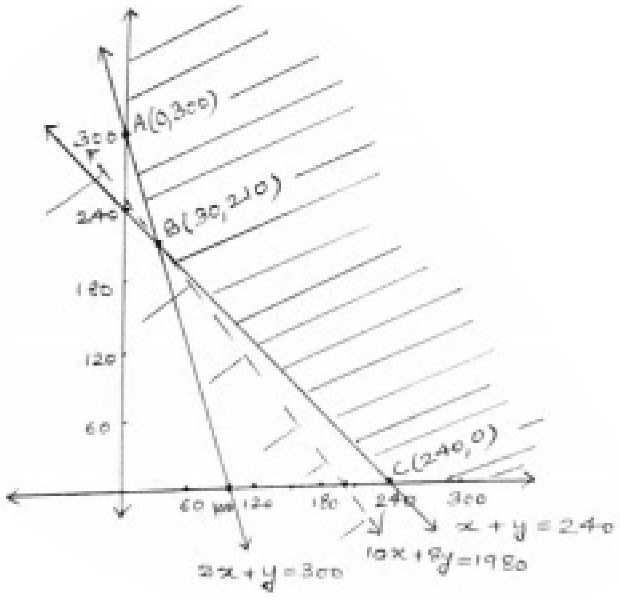
Sol. Let x kg of fertilizer A be used

and y kg of fertilizer B be used

then the linear programming problem is:

Minimize cost: z = 10x + 8y

Subject to
$$\left.\begin{array}{l} \frac{12x}{100}+\frac{4y}{100}\geq 12 \Rightarrow 3x+y\geq 300 \\ \frac{5x}{100}+\frac{5y}{100}\geq 12 \Rightarrow x+y\geq 240 \\ x, y\geq 0 \end{array}\right\}$$



Correct Graph

Value of corners of the unbounded region ABC:

Corner	Value of Z
$A(0,300)$	Rs 2400
$B(30,210)$	Rs 1980 (minimum)
$C(240,0)$	Rs 2400

The region of 10x + 8y < 1980 or 5x + 4y < 990 has no point in common to the feasible region. Hence, minimum cost = Rs 1980 at x = 30 and y = 210

24. Find the position vector of the foot of perpendicular and the perpendicular distance from the point P with position vector $2\hat{i}+3\hat{j}+4\hat{k}$ to the plane $\bar{r}\cdot\left(2\hat{i}+\hat{j}+3\hat{k}\right)-26=0$. Also find image of P in the plane.

Sol. Line through 'P' and perpendicular to plane is:

General point on line is: $\bar{r}=(2+2\lambda)\hat{i}+(3+\lambda)\hat{j}+(4+3\lambda)\hat{k}$

For some $\lambda \in R$ is $\bar{r}$ is the foot of perpendicular say Q, from P to the plane, since it lies on plane

$$\therefore \quad [(2+2\lambda)\hat{i}+(3+\lambda)\hat{j}+(4+3\lambda)\hat{k}]\cdot(2\hat{i}+\hat{j}+3\hat{k})-26=0$$

$$\Rightarrow \quad 4+4\lambda+3+\lambda+12+9\lambda-26=0 \Rightarrow \lambda=\frac{1}{2}$$

$\therefore$ Foot of perpendicular is $Q\left(3\hat{i}+\frac{7}{2}\hat{j}+\frac{11}{2}\hat{k}\right)$

let $P(a\hat{i}+b\hat{j}+c\hat{k})$ be the image of P in the plane then Q is mid point of PP'

$$\therefore \quad Q\left(\frac{a+2}{2}\hat{i}+\frac{b+3}{2}\hat{j}+\frac{c+4}{2}\hat{k}\right)=Q\left(3\hat{i}+\frac{7}{2}\hat{j}+\frac{11}{2}\hat{k}\right)$$

$$\Rightarrow \quad \frac{a+2}{2}=3,\ \frac{b+3}{2}=\frac{7}{2},\ \frac{c+4}{2}=\frac{11}{2} \Rightarrow a=4, b=4, c=7 \therefore P'(4\hat{i}+4\hat{j}+7\hat{k})$$

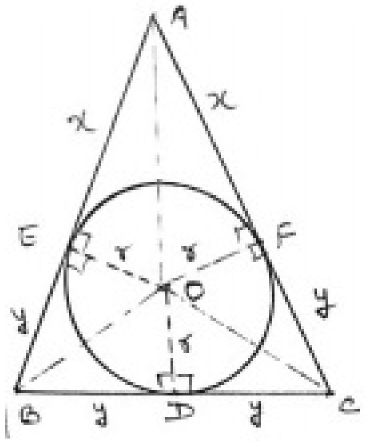
Perpendicular distance of P from plane = PQ $=\sqrt{(2-3)^2+\left(3-\frac{7}{2}\right)^2+\left(4-\frac{11}{2}\right)^2}=\sqrt{\frac{7}{2}}$

25. Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is $6\sqrt{3}\,r$.

OR

If the sum of lengths of hypotenuse and a side of right angled triangle is given, show that area of triangle is maximum, when the angle between them is $\frac{\pi}{3}$.

Sol. Let ΔABC be isosceles with inscribed circle of radius ‘r’ touching sides AB, AC and BC at E, F and D respectively.
let AE = AF = x, BE = BD = y, CF = CD = y then area (ΔABC) = ar(ΔAOB) + ar(ΔAOC) + ar(ΔBOC)



$$\Rightarrow \frac{1}{2}.2y\left(r+\sqrt{r^2+x^2}\right)=\frac{1}{2}\{2yr+2(x+y)r\} \Rightarrow x=\frac{2r^2y}{y^2-r^2}$$

Then,

P(Perimeter of ΔABC) = 2x + 4y$= \dfrac{4r^2y}{y^2-r^2}+4y$

$$\frac{dP}{dy}=\frac{-4r^2(r^2+y^2)}{(y^2-r^2)^2}+4 \text{ and } \frac{dP}{dy}=0 \Rightarrow y=\sqrt{3}r$$

$$\left.\frac{d^2P}{dy^2}\right]_{y=\sqrt{3}r}=\frac{4r^2y(2y^2+6r^2)}{(y^2-r^2)^3}=\frac{6\sqrt{3}}{r}>0$$

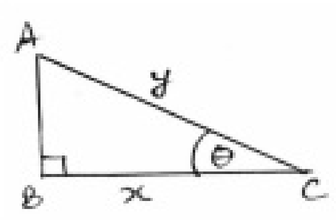
∴ Perimeter is least iff $y=\sqrt{3}r$ and least perimeter is

$$P=4y+\frac{4r^2y}{y^2-r^2}=4\sqrt{3}r+\frac{4r^2\sqrt{3}r}{2r^2}=6\sqrt{3}r$$

OR

let ABC be the right triangle with $\angle B = 90°$

$\angle ACB = \theta$, AC = y, BC = x, x + y = k (constant)



A (Area of triangle) $= \frac{1}{2}.BC.AB = \frac{1}{2}.x\sqrt{y^2 - x^2}$

let $z = A^2 = \frac{1}{4}x^2(y^2 - x^2) = \frac{1}{4}x^2\{(k-x)^2 - x^2\} = \frac{1}{4}(x^2k^2 - 2kx^3)$

$\frac{dz}{dx} = \frac{1}{4}(2xk^2 - 6kx)^2$ and $\frac{dz}{dx} = 0 \Rightarrow x = \frac{k}{3}, y = k - x = \frac{2k}{3}$

$\left.\frac{d^2z}{dx^2}\right]_{x=\frac{k}{3}} = \frac{1}{4}(2k^2 - 12kx)\Big]_{x=\frac{k}{3}} = -\frac{k^2}{2} < 0$

$\therefore$ z and area of ΔABC is max at $x = \frac{k}{3}$

and, $\cos\theta = \frac{x}{y} = \frac{k}{3}.\frac{3}{2k} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

26. Five bad oranges are accidently mixed with 20 good ones. If four oranges are drawn one by one successively with replacement, then find the probability distribution of number of bad oranges drawn. Hence find the mean and variance of the distribution.

Sol. Let X = number of bad oranges out of 4 drawn = 0, 1, 2, 3, 4

P = Probability of a bad orange $= \frac{1}{5}, q = 1 - p = \frac{4}{5}$

$\therefore$ probability distribution is:

X:	0	1	2	3	4

P(X):	${}^4C_0\left(\frac{4}{5}\right)^4=\frac{256}{625}$	${}^4C_1\frac{1}{5}\left(\frac{4}{5}\right)^3$	${}^4C_2\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)$	${}^4C_3\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)$	${}^4C_4\left(\frac{1}{5}\right)^4$
		$=\frac{256}{625}$	$=\frac{96}{625}$	$=\frac{16}{625}$	$=\frac{1}{625}$

$$Mean(\mu)=\sum X.P(X)=0\times\frac{256}{625}+1\times\frac{256}{625}+2\times\frac{96}{625}+3\times\frac{16}{625}+4\times\frac{1}{625}=\frac{4}{5}$$

$$Variance(\sigma^2)=\sum x^2.P(x)-\left[\sum x.P(x)\right]^2$$

$$=0\times\frac{256}{625}+\frac{1\times256}{625}+\frac{4\times96}{625}+\frac{9\times16}{625}+\frac{16}{625}-\left(\frac{4}{5}\right)^2=\frac{16}{25}$$