Unit I : Single Variable Calculus

Rolle's Theorem:

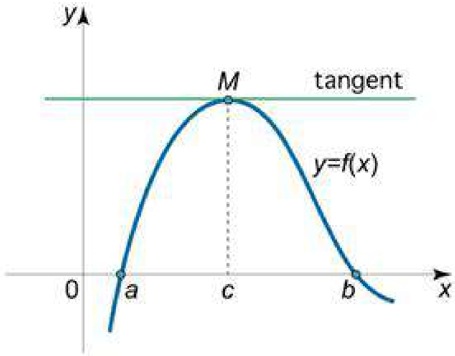
Let f(x) be a function defined in [a, b]

i) function f(x) is continuous on the closed interval [a, b]

ii) differentiable on the open interval (a, b)

iii) f(a) = f(b)

then there exists at least one point $x = c$ in the open interval (a, b) such that $f'(c) = 0$.



Geometric interpretation

There is a point c on the interval $(a,\ b)$ where the tangent to the graph of the function is horizontal.

Lagrange's Mean Value Theorem:

Let f(x) be a function defined in [a, b]

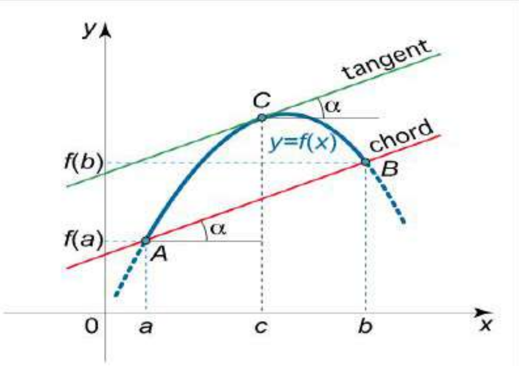
i) Function f(x) is continuous on a closed interval [a , b]

ii) Function f(x) differentiable on the open interval (a, b)

then there is at least one point x = c on this interval (a, b), such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Geometric Interpretation



The chord passing through the points of the graph corresponding to the ends of the segment a and b has the slope equal to $k = \tan\alpha = \frac{f(b) - f(a)}{b - a}$

Then there is a point $x = c$ inside the interval [a , b] where the tangent to the graph is parallel to the chord.

Cauchy's Mean Value Theorem:

Let f(x) be a function defined in [a, b]

i) Function f(x) and $g(x)$ is continuous on a closed interval [a , b]

ii) Function f(x) and $g(x)$ differentiable on the open interval (a, b)

iii) $g'(x) \neq 0$ for all value of x in (a, b)

then there is at least one point x = c on this interval (a, b), such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

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Example 1: Verify Rolle's Mean Value theorem for $\boldsymbol{f(x) = x^2 - 5x + 4}$ *in* $[\mathbf{1}, \mathbf{4}]$

Solution : $f(x) = x^2 - 5x + 4$

$As\ f(x) = x^2 - 5x + 4$ is a polynomial

Every polynomial is continuous and differentiable everywhere

$\therefore f(x) = x^2 - 5x + 4$ is continuous in [1 , 4] and differetiable in (1, 4)

$f(x) = x^2 - 5x + 4$

$f(a) = f(1) = 1^2 - 5(1) + 4 = 1 - 5 + 4 = 0$

$f(b) = f(4) = 4^2 - 5(4) + 4 = 16 - 20 + 4 = 0$

$f(a) = f(b)$

All condition of Rolle's Mean Value theorem satisfied.

then their exist at least one c ∈ (1, 4) such that $f'(c) = 0$

$f(x) = x^2 - 5x + 4$

$f'(x) = 2x - 5$

put $x = c$

$f'(c) = 2c - 5$

$\therefore f'(c) = 0 \quad \Rightarrow 2c - 5 = 0 \qquad \therefore 2c = 5 \qquad \therefore c = \frac{5}{2} = 2.5$

$\therefore c = 2.5 \in (1, 4)$ hence Lagrange's Mean Value theorem verified.

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Example 2: Verify Rolle's Mean Value theorem for

$$f(x) = e^x(sin\ x - cos\ x) \quad in \quad \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$$

Solution : $f(x) = e^x(sin\ x - cos\ x)$

$As\ f(x)$ is a combination of exponational and sine, cosine functions

Exponational and Sine, Cosine function are continuous and differentiable

$\therefore f(x) = e^x(sin\ x - cos\ x)$ is continuous in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$ and differetiable in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

$f(x) = e^x(sin\ x - cos\ x) =$

$$f(a) = f\left(\frac{\pi}{4}\right) = e^{\frac{\pi}{4}}\left(sin\ \frac{\pi}{4} - cos\ \frac{\pi}{4}\right) = e^{\frac{\pi}{4}}\left(\ \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = 0$$

$$f(b) = f\left(\frac{5\pi}{4}\right) = e^{\frac{5\pi}{4}}\left(sin\ \frac{5\pi}{4} - cos\ \frac{5\pi}{4}\right) = e^{\frac{5\pi}{4}}\left(-\frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right)\right) = 0$$

$f(a) = f(b)$

All condition of Rolle's Mean Value theorem satisfied.

then their exist at least one c $\in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ such that $f'(c) = 0$

$f(x) = e^x(sin\ x - cos\ x)$

$f'(x) = e^x(cos\ x + sin\ x) + e^x(sin\ x - cos\ x)$

$f'(x) = 2e^x \sin x$

put $x = c$

$f'(c) = 2e^c \sin c$

$\therefore f'(c) = 0 \quad \Rightarrow 2e^c \sin c\ = 0$

$\therefore \sin c = 0 \qquad \therefore c = \sin^{-1} 0 \qquad \therefore c = n\pi \qquad n = 0,1,2,3 \ldots.$

$c = 0, \pi, 2\pi, 3\pi, 4\pi \ldots$

$\therefore c = \pi\ \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ hence Rolle's Mean Value theorem verified.

Example 3: Verify Lagrange's Mean Value theorem for

$$\boldsymbol{f(x) = (x-1)(x-2)(x-3)} \quad in \ [\mathbf{0\ 3}]$$

Solution : $f(x) = (x-1)(x-2)(x-3)$

$f(x) = x^3 - 6x^2 + 11x - 6$

$As\ f(x) = (x-1)(x-2)(x-3)$ is a polynomial

Every polynomial is continuous and differentiable everywhere

$\therefore f(x) = (x-1)(x-2)(x-3)$ is continuous in $[0,\ 3]$ and differetiable in $(0\ 3)$

All condition of Lagrange's Mean Value theorem satisfied.

then their exist at least one c $\in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

$f(x) = x^3 - 6x^2 + 11x - 6$

$f(a) = f(0) = (0)^3 - 6(0)^2 + 11(0) - 6 = -6$

$f(b) = f(3) = (3)^3 - 6(3)^2 + 11(3) - 6 \ = \ 27 - 54 + 33 - 6 \ = \ 0$

$f'(x) = 3x^2 - 12x + 11$

put $x = c$

$f'(c) = 3c^2 - 12c + 11$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$\therefore 3c^2 - 12c + 11 = \frac{0-(-6)}{3-0} = \frac{6}{3}$

$\therefore 3c^2 - 12c + 11 = 2$

$\therefore 3c^2 - 12c + 11 - 2 = 0$

$\therefore 3c^2 - 12c + 9 = 0$

$\therefore c^2 - 4c + 3 = 0$

$\therefore c^2 - 3c - c + 3 = 0$

$\therefore c(c-3) - 1(c-3) = 0$

$\therefore (c-3)(c-1) = 0 \qquad \therefore c = 3\,,1$

$\therefore c = 1 \in (0,3)$ hence Lagrange's Mean Value theorem verified.

Example 4: Verify Lagrange's Mean Value theorem for $\boldsymbol{f(x) = \log x}$ in $[\mathbf{1},\ \boldsymbol{e}]$

Solution : $f(x) = \log x$

$As\ f(x) = \log x$ is a logarithmic function

Every logarithmic function is continuous and differentiable in its domain

$\therefore f(x) = \log x$ is continuous in [1, e] and differetiable in (1 e)

All condition of Lagrange's Mean Value theorem satisfied.

then their exist at least one c ∈ (1 e) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

$f(x) = \log x$

$f(a) = f(0) = \log 1 = 0$

$f(b) = f(e) = \log e = 1$

$f'(x) = \frac{1}{x}$ put $x = c$ $f'(c) = \frac{1}{c}$

$\frac{1}{c} = \frac{1-0}{e-1}$ $\therefore c = e - 1$

$\therefore c = e - 1 \in (1\ \ e)$ hence Lagrange's Mean Value theorem verified.

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Example 5: Verify Cauchy's Mean value theorem theorem for

$$\boldsymbol{f(x) = x^3}\ \ and\ \ \mathbf{g}(\boldsymbol{x}) = \boldsymbol{x^4}\ \ in\ \ [\mathbf{0}\ \ \mathbf{2}]$$

Solution : $f(x) = x^3$ and $g(x) = x^4$

$As\ f(x) = x^3$ and $g(x) = x^4$ are a polynomials

Every polynomial is continuous and differentiable everywhere

$\therefore f(x) = x^3$ and $g(x) = x^4$ is continuous in [0, 2] and differetiable in (0 2)

As $g(x) = x^4$ $g'(x) = 4x^3 \neq 0$ for x in $(0, 2)$

All condition of Cauchy's Mean Value theorem satisfied.

then their exist at least point $c \in (0,2)$ such that $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$As\ f(x) = x^3$ and $g(x) = x^4$

$f(b) = f(2) = 2^3 = 8$ and $g(b) = g(2) = 2^4 = 16$

$f(a) = f(0) = 0$ and $g(a) = g(0) = 0$

$f'(x) = 3x^2$ and $g'(x) = 4x^3$

Put $x = c$

$f'(c) = 3c^2$ and $g'(c) = 4c^3$

$$\frac{3c^2}{4c^3} = \frac{8-0}{16-0}$$

$$\frac{3}{4c} = \frac{1}{2} \quad \therefore \quad 4c = 6 \quad \therefore \quad c = \frac{6}{4} \quad \therefore \quad c = \frac{3}{2}$$

$\therefore \quad c = \frac{3}{2} \in (0,2)$ hence Cauchy's Mean value theorem verified.

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Expansion of Function

Taylor's Theorem:

Statement : Let $f(a+h)$ be a function of h which can be expanded in powers of h and let the expansion be differentiable term by term any number of times w.r.t. h

then $f(a+h) = f(a) + h\,f'(a) + \frac{h^2}{2!}\,f''(a) + \frac{h^3}{3!}\,f'''(a) + \dots\dots + \frac{h^n}{n!}\,f^n(a) + \dots$

Expansion of f(x + h) in power of 'h'

$$f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2!}\,f''(x) + \frac{h^3}{3!}\,f'''(x) + \dots\dots + \frac{h^n}{n!}\,f^n(x) + \dots\dots$$

Expansion of f(x + h) in power of 'x'

$$f(x+h) = f(h) + x\,f'(h) + \frac{x^2}{2!}f''(h) + \frac{x^3}{3!}f'''(h) + \dots\dots + \frac{x^n}{n!}f^n(h) + \dots\dots$$

Expansion f(x) in power of (x − a) = 0 or about x = a

$$f(x) = f(a) + (x-a)\ f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots + \frac{(x-a)^n}{n!}f^n(a) \dots$$

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Example 1: Using Taylor's theorem express

$$(x-2)^4 - 3(x-2)^3 + 4(x-2)^2 + 5 \quad \text{in powers of } x$$

Solution: Let $f(x+h) = (x-2)^4 - 3(x-2)^3 + 4(x-2)^2 + 5$

Here $x + h = x - 2 \qquad \therefore h = -2$

By using Taylor's theorem, **Expansion of $\mathbf{f(x+h)}$ in power of x**

$$f(x+h) = f(h) + x\,f'(h) + \frac{x^2}{2!}\,f''(h) + \frac{x^3}{3!}\,f'''(h) + \frac{x^4}{4!}\,f^{\prime v}(h) + \ldots\ldots\ldots\ldots$$

$$f(x-2) = f(-2) + x\,f'(-2) + \frac{x^2}{2!}\,f''(-2) + \frac{x^3}{3!}\,f'''(-2) + \frac{x^4}{4!}\,f^{\prime v}(-2) + \ \ldots\ldots \quad (A)$$

$$\therefore f(x) = x^4 - 3x^3 + 4x^2 + 5$$

$f(x) = x^4 - 3x^3 + 4x^2 + 5 \qquad \therefore\ f(h) = f(-2) = 61$

$f'(x) = 4x^3 - 9x^2 + 8x \qquad \therefore\ f'(h) = f'(-2) = -84$

$f''(x) = 12x^2 - 18x + 8 \qquad \therefore\ f''(-2) = f''(-2) = 92$

$f'''(x) = 24x - 18 \qquad \therefore\ f'''(-2) = f'''(-2) = -66$

$f^{\prime v}(x) = 24 \qquad \therefore\ f^{\prime v}(-2) = f^{\prime v}(-2) = 24$

$f^{v}(x) = 0 \qquad \therefore\ f^{v}(-2) = f^{v}(-2) = 0$

Equation (A) becomes

$$f(x-2) = 61 + x\,(-84) + \frac{x^2}{2}\,(92) + \frac{x^3}{6}\,(-66) + \frac{x^4}{24}\,(24)$$

$$f(x-2) = 61 - 84x + 46x^2 - 11x^3 + x^4$$

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Example 2: Using Taylor's theorem express

$$(x+2)^4 + 3(x+2)^3 + (x+2) + 7 \quad \text{in powers of x}$$

Solution: Let $f(x+h) = (x+2)^4 + 3(x+2)^3 + (x+2) + 7$

Here $x + h = x + 2$ $\quad\therefore h = 2$

By using Taylor's theorem, **Expansion of f(x + h) in power of x**

$$f(x+h) = f(h) + x\,f'(h) + \frac{x^2}{2!}\,f''(h) + \frac{x^3}{3!}\,f'''(h) + \frac{x^4}{4!}\,f^{iv}(h) + \dots\dots\dots\dots$$

$$f(x-2) = f(2) + x\,f'(2) + \frac{x^2}{2!}\,f''(2) + \frac{x^3}{3!}\,f'''(2) + \frac{x^4}{4!}\,f^{iv}(2) + \dots\dots \quad (A)$$

$$\therefore f(x) = x^4 + 3x^3 + x + 7$$

$f(x) = x^4 + 3x^3 + x + 7$	$\therefore f(2) = 49$
$f'(x) = 4x^3 + 9x^2 + 1$	$\therefore f'(2) = 69$
$f''(x) = 12x^2 + 18x$	$\therefore f''(2) = 84$
$f'''(x) = 24x + 18$	$\therefore f'''(2) = 66$
$f^{iv}(x) = 24$	$\therefore f^{iv}(2) = 24$
$f^{v}(x) = 0$	$\therefore f^{v}(2) = 0$

Equation (A) becomes

$$f(x+2) = 49 + x\,(69) + \frac{x^2}{2}\,(84) + \frac{x^3}{6}\,(66) + \frac{x^4}{24}\,(24) + 0$$

$$f(x+2) = 49 + 69x + 42x^2 + 11x^3 + x^4$$

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Example 3: Using Taylor's theorem express

$$49 + 69x + 42x^2 + 11x^3 + x^4 \quad \text{in powers of } (x+2)$$

Solution: Let $f(x) = 49 + 69x + 42x^2 + 11x^3 + x^4$

Here $x - a = x + 2 \qquad \therefore -a = 2 \quad \therefore a = -2$

By using Taylor's theorem, **Expansion $\mathbf{f(x)}$ in power of $\mathbf{(x - a)}$ or about $\mathbf{x = a}$**

$$f(x) = f(a) + (x-a)\ f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \ldots + \frac{(x-a)^n}{n!} f^n(a) \ldots.$$

$$f(x) = f(-2) + (x+2)f'(-2) + \frac{(x+2)^2}{2} f''(-2) + \frac{(x+2)^3}{6} f'''(-2) + \frac{(x+2)^4}{24} f'^v(-2) \quad \ldots.(A)$$

$$\therefore f(x) = 49 + 69x + 42x^2 + 11x^3 + x^4$$

$f(x) = 49 + 69x + 42x^2 + 11x^3 + x^4 \qquad \therefore f(-2) = 7$

$f'(x) = 69 + 84x + 33x^2 + 4x^3 \qquad \therefore f'(-2) = 1$

$f''(x) = 84 + 66x + 12x^2 \qquad \therefore f''(-2) = 0$

$f'''(x) = 66 + 24x \qquad \therefore f'''(-2) = 18$

$f'^v(x) = 24 \qquad \therefore f'^v(-2) = 24$

$f^v(x) = 0 \qquad \therefore f^v(-2) = 0$

Equation (A) becomes

$$f(x) = 7 + (x+2)1 + \frac{(x+2)^2}{2}(0) + \frac{(x+2)^3}{6} 18 + \frac{(x+2)^4}{24} 24$$

$$f(x) = 7 + (x+2) + 3(x+2)^3 + (x+2)^4$$

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Example 4: Using Taylor's theorem express $x^3 + 7x^2 + x - 6$ in powers of $(x - 3)$

Solution:

Let $f(x) = x^3 + 7x^2 + x - 6$

Here $x - a = x - 3 \qquad \therefore a = 3$

By using Taylor's theorem, **Expansion f(x) in power of (x − a) or about x = a**

$$f(x) = f(a) + (x-a)\ f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \ \dots\ + \frac{(x-a)^n}{n!} f^n(a) \dots.$$

$$f(x) = f(3) + (x-3)f'(3) + \frac{(x-3)^2}{2!}\ f''(h) + \frac{(x-3)^3}{3!}\ f'''(3) + \frac{(x-3)^4}{4!}\ f'^v(3) + \dots (A)$$

$$\therefore f(x) = x^3 + 7x^2 + x - 6$$

$f(x) = x^3 + 7x^2 + x - 6 \qquad \therefore f(3) = 87$

$f'(x) = 3x^2 + 14x + 1 \qquad \therefore f'(3) = 70$

$f''(x) = 6x + 14 \qquad \therefore f''(3) = 32$

$f'''(x) = 6 \qquad \therefore f'''(3) = 6$

$f'^v(x) = 0 \qquad \therefore f'^v(3) = 0$

Equation (A) becomes

$$f(x) = 87 + (x-3)69 + \frac{(x-3)^2}{2}\ 32 + \frac{(x-3)^3}{6}\ 6$$

$$f(x) = 87 + 70(x-3) + 16(x-3)^2 + (x-3)^3$$

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Example 5: Using Taylor's theorem express $(x-1)^4 - 3(x-1)^3 + 4(x-1)^2 + 5$ in powers of x.

Example 6: Using Taylor's theorem express $2(x-2)^3 + 19(x-2)^2 + 53(x-2) + 40$ in powers of x.

Example 7: Using Taylor's theorem express $3x^3 - 2x^2 + x - 6$ in powers of $x - 2$

Example 8: Using Taylor's theorem express $1 + 2x + 3x^2 + 4x^3$ in powers of $x + 1$

Maclaurin's Theorem:

Statement: Let f(x) be a function of x which can be expanded in ascending powers and let the expansion be differentiable term by term any number of times then

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots.. + \frac{x^n}{n!}f^n(0) + \ldots\ldots$$

Note:

1) If $y = f(x)$ then $f(0) = (y)_0$, $f'(0) = (y_1)_0$, $f''(0) = (y_2)_0 \ldots\ldots f^n(0) = (y_n)_0$

   Maclaurin's Theorem stated as

$$y = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \ldots\ldots.. + \frac{x^n}{n!}(y_n)_0 + \ldots\ldots$$

2) The $(n+1)^{th}$ term of expansion $\frac{x^n}{n!}f^n(0)$ is called general term.

3) $\cosh x = \frac{e^x + e^{-x}}{2}$ $\qquad \sinh x = \frac{e^x - e^{-x}}{2}$

4) $\frac{d}{dx}(\cosh x) = \sinh x$

5) $\frac{d}{dx}(\sinh x) = \cosh x$

6) $\int \sinh x\, dx = \cosh x + c$

7) $\int \cosh x\, dx = \sinh x + c$

Example : Expansion of e^x

Solution : $Let\ f(x) = e^x$

by Maclaurin's theorem

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ \dots\dots.. + \frac{x^n}{n!}f^n(0) + \ \dots\dots$$

$f(x) = e^x$ $\therefore f(0) = e^0 = 1$

$f'(x) = e^x$ $\therefore f'(0) = e^0 = 1$

$f''(x) = e^x$ $\therefore f''(0) = e^0 = 1$

$f'''(x) = e^x$ $\therefore f'''(0) = e^0 = 1$

$f'^v(x) = e^x$ $\therefore f'^v(0) = e^0 = 1$

$f^v(x) = e^x$ $\therefore f^v(0) = e^0 = 1$

....

$f^n(x) = e^x$ $\therefore f^n(0) = e^0 = 1$

by Maclaurin's Theorem:

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ \dots\dots.. + \frac{x^n}{n!}f^n(0) + \ \dots\dots$$

$$f(x) = 1 + x\,(1) + \frac{x^2}{2!}1 + \frac{x^3}{3!}1 + \ \dots\dots.. + \frac{x^n}{n!}1 + \ \dots\dots$$

$$f(x) = e^x = 1 + x\ + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\dots\dots.. + \frac{x^n}{n!} + \ \dots\dots$$

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Example : Expansion of e^{-x}

Solution : $Let\ f(x) = e^{-x}$

by Maclaurin's Theorem:

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots.. + \frac{x^n}{n!}f^n(0) + \ldots\ldots$$

$f(x) = e^{-x}$ $\quad\therefore f(0) = e^0 = 1$

$f'(x) = -e^{-x}$ $\quad\therefore f'(0) = -e^0 = -1$

$f''(x) = e^{-x}$ $\quad\therefore f''(0) = e^0 = 1$

$f'''(x) = -e^{-x}$ $\quad\therefore f'''(0) = -e^0 = -1$

$f'^{v}(x) = e^{-x}$ $\quad\therefore f'^{v}(0) = e^0 = 1$

$f^{v}(x) = -e^{-x}$ $\quad\therefore f^{v}(0) = -e^0 = -1$

…………..

$f^n(x) = (-1)^n e^{-x}$ $\quad\therefore f^n(0) = (-1)^n e^0 = (-1)^n$

by Maclaurin's Theorem:

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots.. + \frac{x^n}{n!}f^n(0) + \ldots\ldots$$

$$f(x) = 1 + x\,(-1) + \frac{x^2}{2!}1 + \frac{x^3}{3!}(-1) + \ldots\ldots.. + \frac{x^n}{n!}(-1)^n + \ldots\ldots$$

$$f(x) = e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots\ldots.. + (-1)^n\frac{x^n}{n!} + \ldots\ldots$$

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Example : Expansion of sin x

Solution : $Let\ f(x) = \sin x$

by Maclaurin's Theorem:

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ \ldots\ldots.. + \frac{x^n}{n!}f^n(0) + \ \ldots\ldots$$

$f(x) = \sin x$ $\quad\therefore f(0) = \sin 0 \quad = 0$

$f'(x) = \cos x$ $\quad\therefore f'(0) = \cos 0 \ = 1$

$f''(x) = -\sin x$ $\quad\therefore f''(0) = -\sin 0 \quad = 0$

$f'''(x) = -\cos x$ $\quad\therefore f'''(0) = -\cos 0 \ = -1$

$f'^{v}(x) = \sin x$ $\quad\therefore f'^{v}(0) = \sin x \quad = 0$

$f^{v}(x) = \cos x$ $\quad\therefore f^{v}(0) = \cos 0 \ \ = 1$

by Maclaurin's Theorem:

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ \ldots\ldots.. + \frac{x^n}{n!}f^n(0) + \ \ldots\ldots$$

$$f(x) = 0 + x\,(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1)\ \ \ldots\ldots$$

$$f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} - \ \ldots\ldots..$$

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Example : Expansion of sinh x

Solution : $Let\ f(x) = \sinh\ x$

by Maclaurin's Theorem:

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ \dots\dots..+ \frac{x^n}{n!}f^n(0) + \ \dots....$$

$$f(x) = \sinh\ x \qquad \therefore f(0) = \sin 0 \quad = 0$$

$$f'(x) = \cosh x \qquad \therefore f'(0) = \cos 0 \ = 1$$

$$f''(x) = \sinh\ x \qquad \therefore f''(0) = \sin 0 \quad = 0$$

$$f'''(x) = \cosh x \qquad \therefore f'''(0) = \cos 0 \ = 1$$

$$f'^{v}(x) = \sinh\ x \qquad \therefore f'^{v}(0) = \sin x \quad = 0$$

$$f^{v}(x) = \cosh x \qquad \therefore f^{v}(0) = \cos 0 \ = 1$$

by Maclaurin's Theorem:

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ \dots\dots..+ \frac{x^n}{n!}f^n(0) + \ \dots....$$

$$f(x) = 0 + x\,(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1)\ \dots....$$

$$f(x) = \sin x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!} - \ \dots\dots$$

****

Standard Expansions:

1) $e^x \ = \ 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ \dots\dots\dots\dots$

2) $e^{-x} \ = \ 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \ \dots\dots\dots\dots$

3) $\sin x \ = \ x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ \dots\dots\dots\dots$

4) $\sinh x \ = \ x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \ \dots\dots\dots\dots$

5) $\cos x \ = \ 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ \dots\dots\dots\dots$

6) $\cosh x \ = \ 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \ \dots\dots\dots\dots$

7) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \dots\dots\dots\dots$

8) $\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \dots\dots\dots\dots$

9) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots\dots\dots\dots$

10) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \dots\dots\dots\dots$

11) $(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \dots\dots\dots\dots\dots\dots$

12) $\frac{1}{(1+x)} = (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots\dots\dots\dots\dots$

13) $\frac{1}{(1-x)} = (1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + x^5 + \dots\dots\dots\dots\dots$

14) $\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7} + \dots\dots\dots\dots$

15) $\sinh^{-1} x = x - \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} - \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7} + \dots\dots\dots\dots$

16) $\cos^{-1} x = \frac{\pi}{2} - \left[x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7} + \dots\dots\dots\dots\right]$

17) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\dots\dots\dots$

18) $\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots\dots\dots\dots$

****

Examples: Expand of $e^x \cos x$ in ascending powers of x upto a term in x^4

Solution: Let $f(x) = e^x \cos x$

We know that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\dots\dots\dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots\dots\dots\dots$$

$$f(x) = e^x \cos x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\dots\dots\dots\dots\right)$$

$$e^x \cos x = 1\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\dots\right) + x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\dots\right) + \frac{x^2}{2!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\dots\right) + \frac{x^3}{3!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\dots\right) + \frac{x^4}{4!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\dots\right)$$

$$e^x \cos x = 1\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\dots\right) + x\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\dots\right) + \frac{x^2}{2}\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\dots\right) + \frac{x^3}{6}\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\dots\right) + \frac{x^4}{24}\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\dots\right)$$

$$e^x \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + x - \frac{x^3}{2} + \frac{x^2}{2} - \frac{x^4}{4} + \frac{x^3}{6} + \frac{x^4}{24}$$

$$e^x \cos x = 1 + x - \frac{x^3}{3} - \frac{x^4}{6}$$

OR

Examples: Expand of $e^x \cos x$ in ascending powers of x upto a term in x^4

Solution: $Let\ f(x) = e^x \cos x$

by Maclaurin's Theorem

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0)\dots\dots.. + \frac{x^n}{n!}f^n(0) + \dots\dots$$

$f(x) = e^x \cos x \qquad \therefore f(0) = e^0 \cos 0 = 1$

$f'(x) = e^x \cos x - e^x \sin x \qquad \therefore f'(0) = e^0 \cos 0 - e^0 \sin 0 = 1$

$f''(x) = e^x \cos x - e^x \sin x - e^x \cos x - e^x \sin x = -2e^x \sin x \qquad \therefore f''(0) = 0$

$$f'''(x) = -2e^x \sin x - 2e^x \cos x \qquad \therefore f'''(0) = -2$$

$$f^{iv}(x) = \quad -2e^x \sin x - 2e^x \cos x + 2e^x \sin x - 2e^x \cos x \qquad \therefore f^{iv}(0) = -4$$

by Maclaurin's Theorem:

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \dots\dots$$

$$f(x) = 1 + x\,(1) + \frac{x^2}{2}(0) + \frac{x^3}{6}(-2) + \frac{x^4}{24}(-4)$$

$$f(x) = 1 + x - \frac{x^3}{3} - \frac{x^4}{6}$$

****

Examples : Expand $\sqrt{1+\sin x}$ in ascending powers of x upto a term in x^6

Solution : *Let* $f(x) = \sqrt{1+\sin x}$

$$f(x) = \sqrt{\sin^2\left(\frac{x}{2}\right) + \cos^2\left(\frac{x}{2}\right) + 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}$$

$$f(x) = \sqrt{\left(\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)\right)^2} \qquad \because (a+b)^2 = a^2 + 2ab + b^2$$

$$f(x) = \sqrt{1+\sin x} = \sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)$$

We know that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots\dots\dots\dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots\dots\dots\dots$$

Put $x = \frac{x}{2}$ in above expansion

$$\sin\left(\frac{x}{2}\right) = \frac{x}{2} - \frac{1}{3!}\left(\frac{x}{2}\right)^3 + \frac{1}{5!}\left(\frac{x}{2}\right)^5 - \dots\dots\dots\dots$$

$$\cos\left(\frac{x}{2}\right) = 1 - \frac{1}{2!}\left(\frac{x}{2}\right)^2 + \frac{1}{4!}\left(\frac{x}{2}\right)^4 - \dots\dots\dots\dots$$

$$f(x) = \sqrt{1+\sin x} = \sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)$$

$$f(x) = \frac{x}{2} - \frac{1}{3!}\left(\frac{x}{2}\right)^3 + \frac{1}{5!}\left(\frac{x}{2}\right)^5 - \dots\dots\dots + 1 - \frac{1}{2!}\left(\frac{x}{2}\right)^2 + \frac{1}{4!}\left(\frac{x}{2}\right)^4 - \frac{1}{6!}\left(\frac{x}{2}\right)^6 \dots\dots$$

$$f(x) = \frac{x}{2} - \frac{1}{6} * \frac{x^3}{8} + \frac{1}{120} * \frac{x^5}{32} - \dots\dots\dots + 1 - \frac{1}{2} * \frac{x^2}{4} + \frac{1}{24} * \frac{x^4}{16} - \frac{1}{720} * \frac{x^6}{64} \dots\dots$$

$$f(x) = 1 + \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} - \frac{x^6}{46080}$$

Type: Expansions of functions by using substitution:-

Ex.1 – Expand $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ in ascending powers of x

Solution:

Let $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Put $x = \tan\theta \longrightarrow (\therefore \theta = \tan^{-1} x)$

$$\therefore f(x) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= \sin^{-1}(\sin 2\theta)$$

$$= 2\theta$$

$$= 2(\tan^{-1} x) \qquad (\text{If } x = \tan\theta \text{ then } \theta = \tan^{-1} x)$$

We know that $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\dots\dots\dots$

$$\therefore \sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\dots\dots\dots\right]$$

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Ex.2- Prove that $\sec^{-1}\left[\frac{1}{1-2x^2}\right] = 2\left[x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5}\dots..\right]$

Solution:

Let $f(x) = \sec^{-1}\left[\frac{1}{1-2x^2}\right]$

Put $x = \sin\theta \longrightarrow (\therefore \theta = \sin^{-1} x)$

$$\therefore f(x) = \sec^{-1}\left[\frac{1}{1-2\sin^2\theta}\right]$$

$$= \sec^{-1}\left[\frac{1}{\cos 2\theta}\right]$$

$$= \sec^{-1}[\sec 2\theta]$$

$$= 2\theta \qquad (If\ x = \sin\theta\ then\ \theta = \sin^{-1} x)$$

$$= 2\sin^{-1} x$$

$$\sec^{-1}\left[\frac{1}{1-2x^2}\right] = 2\left[x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5}\dots..\right]$$ Hence Proved.

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Ex.3 – Expand $\cos^{-1}(4x^3 - 3x)$ in ascending powers of x .

Solution:

Let $f(x) = \cos^{-1}(4x^3 - 3x)$

Put $x = cos\theta$

$$f(x) = \cos^{-1}(4cos^3\theta - 3cos\theta)$$

$$= \cos^{-1}(cos3\theta)$$

$$= 3\,\theta$$

$$= 3\ \cos^{-1}x \qquad (If\ x = cos\theta\ then\ \theta = \cos^{-1}x\ )$$

$$= 3\ \left[\frac{\pi}{2} - \left(x\ + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7} + \dots\dots\dots\dots\right)\right]$$

$$=3\,\frac{\pi}{2} - 3\left[x\ + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7} + \dots\dots\dots\dots\right]$$

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Ex.4- Prove that $\cot^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right) = \frac{\pi}{2} - 3\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\dots\dots\dots\right)$

Ex.5- Prove that $\sin^{-1}(3x - 4x^3) = 3\left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \dots\dots\dots\dots\right)$

Indeterminate Forms

Defination: Let f(x) and g(x) be any two function of x such that $f(a) = 0$ and $g(a) = 0$ then the ratio $\frac{f(x)}{g(x)}$ is said to be the indeterminate form $\frac{0}{0}$ at $x = a$

There are several Indeterminate Forms $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 0^0, ∞^0, 1^∞

True Value (Limit):

The limiting value of an indeterminate form is called its true value.

Type I : Indeterminate Form $\frac{0}{0}$ (L'Hospital Rule)

Let f(x) and g(x) be any two function of x such that $f(a) = 0$ and $g(a) = 0$

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Type II : Indeterminate Form $\frac{\infty}{\infty}$

If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$ then $\lim_{x \to a} \frac{f(x)}{g(x)}$ in $\frac{\infty}{\infty}$ form then reduces to $\frac{0}{0}$ by $\frac{f(x)}{g(x)} = \frac{1/f(x)}{1/g(x)}$ and L'Hospital Rule is applicable.

L'Hospital Rule is applied to the $\frac{\infty}{\infty}$ form Thus $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Type III : Indeterminate Form $0 \times \infty$

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$ then $\lim_{x \to a} f(x) \cdot g(x)$ takes $0 \times \infty$

$f(x) \cdot g(x) = \frac{f(x)}{\frac{1}{g(x)}}$ or $\frac{g(x)}{\frac{1}{f(x)}}$ and the limit reduces to either $\frac{0}{0}$ form or $\frac{\infty}{\infty}$ form and L'Hospital Rule is applicable.

Type IV : Indeterminate Form $\infty - \infty$

If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$ then $\lim_{x \to a} [f(x) - g(x)]$ takes $\infty - \infty$ simplify the expression $f(x) - g(x)$ and the limit reduces to either $\frac{0}{0}$ form or $\frac{\infty}{\infty}$ form and L'Hospital Rule is applicable.

Note:

1) If $\lim\limits_{x\to a} \dfrac{f_1(x)}{f_2(x)} - \dfrac{g_1(x)}{g_2(x)}$ in $\infty - \infty$ form then $\lim\limits_{x\to a} \dfrac{f_1(x)g_2(x) - g_1(x)f_2(x)}{f_2(x)g_2(x)}$ is in $\dfrac{0}{0}$

2) If $f'(x),\ f''(x) \dots\dots f^{n-1}(x)$ and $g'(x),\ g''(x) \dots\dots g^{n-1}(x)$ all are zero,

But $f^n(x)$ and $g^n(x)$ are not both zero then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f^n(x)}{g^n(x)}$

3) Use of L'Hospital Rule : Differentiate numerator and denominator separately and then put $x = a$. If this reduces to indeterminate form then apply the rule again.

4) If logrithmic term is present in $0 * \infty$ form then keep logritmic term in numerator

Type V : Indeterminate Form $0^0, \infty^0, 1^\infty$

1) If $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$ then $\lim\limits_{x\to a} \{f(x)\}^{g(x)}$ takes 0^0

2) If $\lim\limits_{x\to a} f(x) = \infty$ and $\lim\limits_{x\to a} g(x) = 0$ then $\lim\limits_{x\to a} \{f(x)\}^{g(x)}$ takes ∞^0

3) If $\lim\limits_{x\to a} f(x) = 1$ and $\lim\limits_{x\to a} g(x) = \infty$ then $\lim\limits_{x\to a} \{f(x)\}^{g(x)}$ takes 1^0

If the true value of limit is denoted by L

Then $L = \lim\limits_{x\to a} \{f(x)\}^{g(x)}$

Taking Log on both sides

$$\log L = \log \lim_{x\to a} \{f(x)\}^{g(x)}$$

$$\log L = \lim_{x\to a} g(x) \log f(x)$$

limit can be determined by $0 \times \infty$ form, true value is b

$$\log L = b$$

$$L = e^b$$

Note: $e^\infty = \infty \quad e^{-\infty} = 0$

Standard Limits:

1) $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ 3) $\lim\limits_{x\to 0}\frac{\tan x}{x} = 1$ 6) $\lim\limits_{x\to 0}(1+x)^{1/x} = e$

2) $\lim\limits_{x\to 0}\frac{\sin^{-1} x}{x} = 1$ 4) $\lim\limits_{x\to 0}\frac{\tan^{-1} x}{x} = 1$ 7) $\lim\limits_{x\to \infty}\left(1+\frac{1}{x}\right)^{x} = e$

3) $\lim\limits_{x\to 0}\frac{\sinh x}{x} = 1$ 5) $\lim\limits_{x\to 0}\frac{e^x - 1}{x} = 1$ 8) $\lim\limits_{x\to 0}\frac{a^x - 1}{x} = \log a$

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Example 1) Evaluate $\lim\limits_{x\to 0}\frac{xe^x - \log(1+x)}{x^2}$

Solutoin: Let $L = \lim\limits_{x\to 0}\frac{xe^x - \log(1+x)}{x^2}$ .... (1)

$x \to 0$ **put** $x = 0$

$$L = \frac{(0)e^0 - \log(1+0)}{0^2} = \frac{0(1) - \log 1}{0} = \frac{0-0}{0} = \frac{0}{0} \qquad \therefore \frac{0}{0} \text{ form}$$

From (1) $L = \lim\limits_{x\to 0}\frac{xe^x + e^x - \frac{1}{(1+x)}}{2}$ .... (2)

$x \to 0$ **put** $x = 0$

$$L = \frac{(0)e^0 + e^0 - \frac{1}{1+0}}{2(0)} = \frac{0(1) + 1 - \frac{1}{1-0}}{2(0)} = \frac{0}{0} \qquad \therefore \frac{0}{0} \text{ form}$$

From (2) $L = \lim\limits_{x\to 0}\frac{xe^x + e^x + e^x - \frac{-1}{(1+x)^2}}{2}$

$$L = \lim_{x\to 0}\frac{xe^x + e^x + e^x + \frac{1}{(1+x)^2}}{2} \qquad \text{............ (3)}$$

$x \to 0$ **put** $x = 0$

$$L = \frac{(0)e^0 + e^0 + e^0 + \frac{1}{(1+0)^2}}{2}$$

$$L = \frac{0+1+1+1}{2} = \frac{3}{2}$$

"The Only things that will stop you from fulfilling your dreams is you"

Example 2) **Evaluate** $\lim\limits_{x\to\frac{1}{2}} \dfrac{\cos^2(\pi x)}{e^{2x}-2xe}$

Solutoin: Let $L = \lim\limits_{x\to\frac{1}{2}} \dfrac{\cos^2 \pi x}{e^{2x}-2xe}$ .... (1)

$x \to \frac{1}{2}$ **put** $x = \frac{1}{2}$

$$L = \frac{\cos^2 \pi(1/2)}{e^{2(1/2)}-2(1/2)e} = \frac{0}{e-e} = \frac{0}{0}$$

From (1) $L = \lim\limits_{x\to\frac{1}{2}} \dfrac{2\cos\pi x\ (-\sin\pi x)\,\pi}{2e^{2x}-2e}$ $\because \sin 2x = 2\sin x\ \cos x$

$$L = \lim_{x\to\frac{1}{2}} \frac{-\pi\ \sin(2\pi x)}{2e^{2x}-2e} \qquad \ldots\ldots\ldots\ldots (2)$$

$x \to \frac{1}{2}$ **put** $x = \frac{1}{2}$

$$L = \frac{-\pi\ \sin(2\pi/2)}{2e^{2(1/2)}-2e} = \frac{-\pi(0)}{2e-2e} = \frac{0}{0}$$

$$L = \lim_{x\to\frac{1}{2}} \frac{-\pi\ \cos(2\pi x)\ (2\pi)}{4e^{2x}-0} \qquad \ldots\ldots\ldots\ldots (3)$$

$x \to \frac{1}{2}$ **put** $x = \frac{1}{2}$

$$L = \frac{-2\pi^2\ \cos(2\pi/2)}{4e^{2(1/2)}}$$

$$L = \frac{-2\pi^2\ (-1)}{4e}$$

$$L = \frac{2\pi^2}{4e} = \frac{\pi^2}{2e}$$

****

Example 3) Evaluate $\lim\limits_{x\to 0} \dfrac{\log \sin 2x}{\log \sin x}$

Solutoin: Let $L = \lim\limits_{x\to 0} \dfrac{\log \sin 2x}{\log \sin x}$ ……………(1)

$\mathbf{x \to 0 \ put \ x = 0}$

$$L = \frac{\log \sin 2(0)}{\log \sin (0)} = \frac{\log 0}{\log 0} = \frac{\infty}{\infty} \text{ from}$$

From (1) $L = \lim\limits_{x\to 0} \dfrac{\frac{1}{\sin 2x} \cos 2x \ (2)}{\frac{1}{\sin x} \cos x}$ $\quad \because \dfrac{d}{dx} \log x = \dfrac{1}{x}$

$L = \lim\limits_{x\to 0} \dfrac{2 \cot 2x}{\cot x}$ $\quad \because \dfrac{\cos x}{\sin x} = \cot x \quad \dfrac{1}{\cot x} = \tan x$

$L = \lim\limits_{x\to 0} \dfrac{\frac{2}{\tan 2x}}{\frac{1}{\tan x}} = \lim\limits_{x\to 0} \dfrac{2 \tan x}{\tan 2x}$ ………(2)

$\mathbf{x \to 0 \ put \ x = 0}$

$$L = \frac{2 \tan 0}{\tan 20} = \frac{0}{0} \text{ from}$$

$$L = \lim_{x\to 0} \frac{2 \sec^2 x}{\sec^2 2x \ (2)} = \frac{2 \sec^2 0}{2 \sec^2 2(0)} = \frac{1}{1} = 1$$

OR

Example 3) Evaluate $\lim\limits_{x\to 0} \dfrac{\log \sin 2x}{\log \sin x}$

Solutoin: Let $L = \lim\limits_{x\to 0} \dfrac{\log \sin 2x}{\log \sin x}$ ……………(1)

$\mathbf{x \to 0 \ put \ x = 0}$

$$L = \frac{\log \sin 2(0)}{\log \sin (0)} = \frac{\log 0}{\log 0} \quad \frac{\infty}{\infty} \text{ form}$$

$$L = \lim_{x\to 0} \frac{\frac{1}{\sin 2x} \cos 2x \ (2)}{\frac{1}{\sin x} \cos x}$$

$$L = \lim_{x \to 0} \frac{2\cos 2x \ \ \sin x}{\sin 2x \ \cos x}$$

$$L = \lim_{x \to 0} \frac{2\cos 2x \ \ \sin x}{2\sin x \ \cos x \ \cos x} \qquad \because \sin 2x = 2\sin x \ \cos x$$

$$L = \lim_{x \to 0} \frac{\cos 2x}{\cos^2 x} \qquad \because \cos 2x = \cos^2 x - \sin^2 x$$

$$L = \lim_{x \to 0} \frac{\cos^2 x - \sin^2 x}{\cos^2 x}$$

$$L = \lim_{x \to 0} \frac{\cos^2 x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x}$$

$$L = \lim_{x \to 0} 1 - \tan^2 x$$

$\mathbf{x \to 0 \ \ put \ x = 0}$

$$L = 1 - \tan^2 0 \qquad L = 1 - 0 = 1$$

****

Example 4) **Evaluate** $\displaystyle \lim_{x \to 0} \frac{\mathbf{e^x - 1 - x}}{\mathbf{\log(1+x) - x}}$

Solutoin: Let $\displaystyle L = \lim_{x \to 0} \frac{e^x - 1 - x}{\log(1+x) - x}$ ………… (1)

$\mathbf{x \to 0 \ \ put \ x = 0}$

$$L = \frac{e^0 - 1 - 0}{\log(1+0) - 0} = \frac{1-1}{\log(1)} \qquad \frac{0}{0} \text{ form}$$

From (1) $\displaystyle L = \lim_{x \to 0} \frac{e^x - 0 - 1}{\frac{1}{1+x} - 1}$ ………… (2)

$$L = \frac{e^0 - 0 - 1}{\frac{1}{1+0} - 1} \qquad \frac{0}{0} \text{ form}$$

From (2) $\displaystyle L = \lim_{x \to 0} \frac{e^x}{\frac{-1}{(1+x)^2}}$ ………… (3)

$\mathbf{x \to 0 \ \ put \ x = 0}$

$$L = \frac{e^0}{\frac{-1}{(1+0)^2}} = -1$$

Example 5) Find a and b if $\lim\limits_{x \to 0} \dfrac{x(1 + a\cos x) - b\sin x}{x^3} = 1$

Solutoin: Let $L = \lim\limits_{x \to 0} \dfrac{x(1 + a\cos x) - b\sin x}{x^3}$ ....(1)

$x \to 0$ put $x = 0$

$$L = \frac{(0)(1 + a\cos 0) - b\sin 0}{0^3} = \frac{0}{0} \text{ form}$$

From (1) $L = \lim\limits_{x \to 0} \dfrac{x(0 - a\sin x) + (1 + a\cos x) - b\cos x}{3x^2}$

$L = \lim\limits_{x \to 0} \dfrac{-a x\sin x + 1 + a\cos x - b\cos x}{3x^2}$ ....(2)

$x \to 0$ put $x = 0$

$$L = \frac{0(0 - a\sin 0) + (1 + a\cos 0) - b\cos 0}{3(0)^2} = \frac{(1 + a) - b}{0} = \infty$$

but limit is finite i.e. 1 $\quad 1 \neq \dfrac{(1 + a) - b}{0} = \infty$

$\therefore$ For Finite limit it should be $\dfrac{0}{0}$ from $\dfrac{(1 + a) - b}{0} = \dfrac{0}{0}$ from

$\therefore (1 + a) - b = 0 \qquad \therefore a - b = -1$ ....(A)

From (2) $L = \lim\limits_{x \to 0} \dfrac{-a x\cos x - a\sin x + 0 - a\sin x + b\sin x}{6x}$ ....(3)

$x \to 0$ put $x = 0$

$$L = \frac{-a(0)\cos(0) - a\sin(0) - a\sin(0) + b\sin(0)}{6(0)} = \frac{0}{0}$$

From (3) $L = \lim\limits_{x \to 0} \dfrac{a x\sin x - a\cos x - a\cos x - a\cos x + b\cos x}{6}$ ....(4)

$x \to 0$ put $x = 0$

$$L = \frac{a(0)\sin(0) - a\cos(0) - a\cos(0) - a\cos(0) + b\cos(0)}{6}$$

$$L = \frac{0 - a - a - a + b}{6}$$

$L = \dfrac{-3a + b}{6}$ but limit value $L = 1$ is given but limit is finite 1 $\therefore 1 = \dfrac{-3a + b}{6}$

$\therefore -3a + b = 6 \ldots\ldots\ldots (B)$

$\therefore\ a - b = -1 \ \ldots\ldots\ldots (A)$

Solving (A) and (B) $a - b = -1 \quad -3a + b = 6$

Add (A) and (B) $\quad -2a = 5 \quad \therefore \quad a = -\dfrac{5}{2}$

Put $a = -\dfrac{5}{2}$ in (A) $\quad \therefore \quad -\dfrac{5}{2} - b = 1 \quad \therefore b = -\dfrac{3}{2}$

$$\therefore a = -\frac{5}{2} \qquad b = -\frac{3}{2}$$

****

Example 6) Find a and b if $\lim\limits_{x\to 0} [x^{-3}\sin x + ax^{-2} + b] = 0$

Solutoin: Let $\quad L = \lim\limits_{x\to 0} [x^{-3}\sin x + ax^{-2} + b]$

$$L = \lim_{x\to 0}\left[\frac{\sin x}{x^3} + \frac{a}{x^2} + b\right]$$

$$L = \lim_{x\to 0}\left[\frac{\sin x + ax + bx^3}{x^3}\right] \ldots\ldots\ldots (1)$$

x → 0 put x = 0

$$L = \frac{\sin(0) + a(0) + b(0)^3}{(0)^3} = \frac{0}{0} \text{ form}$$

From (1) $\quad L = \lim\limits_{x\to 0}\left[\dfrac{\cos x + a + 3x^2 b}{3x^2}\right] \ldots\ldots\ldots (2)$

$$L = \frac{\cos(0) + a + 3x(0)^2 b}{3(0)^2} = \frac{1+a}{0} = \infty$$

but limit is finite $L = 0$

$\therefore$ For Finite limit it should be $\dfrac{0}{0} \quad \therefore \dfrac{1+a}{0} = \dfrac{0}{0}$

$\therefore\ 1 + a = 0 \quad \therefore a = -1$

From (2) $\quad L = \lim_{x \to 0} \left[\frac{-\sin x + 6xb}{6x}\right] \qquad \ldots\ldots\ldots(3)$

$x \to 0$ put $x = 0$

$$L = \frac{-\sin(0) + 6(0)b}{6(0)} \qquad \frac{0}{0} \text{ form}$$

$x \to 0$ put $x = 0$

From (3) $\quad L = \lim_{x \to 0} \left[\frac{-\cos x + 6b}{6}\right] \qquad \ldots\ldots\ldots(4)$

$$L = \frac{-\cos(0) + 6b}{6}$$

$$L = \frac{-1 + 6b}{6}$$

but limit is finite 0 $\quad \therefore \frac{-1 + 6b}{6} = 0$

$\therefore -1 + 6b = 0 \qquad b = \frac{1}{6}$

$\therefore a = -1 \qquad b = \frac{1}{6}$

****

Example 7) Find a and b if $\lim_{x \to 0} \frac{a\cos x - a + bx^2}{x^4} = \frac{1}{12}$

Solutoin: Let $\quad L = \lim_{x \to 0} \frac{a\cos x - a + bx^2}{x^4} \qquad \ldots\ldots\ldots(1)$

$x \to 0$ put $x = 0$

$$L = \frac{a\cos(0) - a + b(0)^2}{(0)^4} = \frac{a \ - a}{0} \qquad \frac{0}{0} \text{ form}$$

From (1) $\quad L = \lim_{x \to 0} \frac{-a\sin x + 2xb}{4x^3} \qquad \ldots\ldots\ldots(2)$

$x \to 0$ put $x = 0$

$$L = \frac{-a\sin(0) + 2(0)b}{4(0)^3} \qquad \frac{0}{0} \text{ form}$$

From (2) $\quad L = \lim_{x \to 0} \left[\frac{-a\cos x + 2b}{12x^2}\right] \qquad \ldots\ldots\ldots(3)$

$x \to 0$ put $x = 0$

$$L = \frac{-a\cos(0) + 2b}{12(0)^2} = \frac{-a + 2b}{0} \neq \frac{1}{12}$$

but limit is finite $\frac{1}{12}$

$\therefore$ For Finite limit it should be $\frac{0}{0}$

$\therefore \quad -a + 2b = 0 \quad \ldots\ldots\ldots (A)$

From (3) $\quad L = \lim_{x \to 0} \left[\frac{a \sin x}{24x}\right] \quad \ldots\ldots\ldots (3)$

$$L = \frac{a}{24} \lim_{x \to 0} \left[\frac{\sin x}{x}\right]$$

$$\frac{1}{12} = \frac{a}{24}(1)$$

$$a = 2$$

$\therefore \quad -a + 2b = 0 \quad \therefore -2 + 2b = 0 \qquad b = 1$

$a = 2$ and $b = 1$

****

Example 8) Find a and b if $\lim_{x \to 0} \frac{a \sin^2 x + b \log \cos x}{x^4} = -\frac{1}{2}$

Example 9) If $\lim_{x \to 0} \frac{\sin 2x + p \sin x}{x^3}$ is finite then find the value of p

and hence find the value of limit

****

Example 10) Evaluate $\lim\limits_{x\to 0} \sin x \log x$

Solution: Let $L = \lim\limits_{x\to 0} \sin x \log x \quad \ldots\ldots\ldots\ldots (1)$

$$x \to 0 \text{ put } x = 0$$

$$L = \sin 0 \log 0 \qquad 0 \times \infty \text{ form}$$

$$L = \lim_{x\to 0} \frac{\log x}{\frac{1}{\sin x}} = \lim_{x\to 0} \frac{\log x}{\operatorname{cosec} x} \ldots\ldots\ldots\ldots (2)$$

$$x \to 0 \text{ put } x = 0$$

$$L = \frac{\log 0}{\operatorname{cosec} 0} \qquad \frac{\infty}{\infty} \text{ form}$$

From (2) $$L = \lim_{x\to 0} \frac{1/x}{-\operatorname{cosec} x \cot x}$$

$$L = \lim_{x\to 0} \frac{\sin x \tan x}{-x}$$

$$L = \lim_{x\to 0} \frac{\sin x \tan x}{-x} \qquad \frac{0}{0} \text{ form} \quad \ldots\ldots\ldots\ldots (3)$$

From (3) $$L = \lim_{x\to 0} \frac{\sin x \sec^2 x + \tan x \cos x}{-1} \quad \ldots\ldots\ldots\ldots (4)$$

$$x \to 0 \text{ put } x = 0$$

$$L = \frac{\sin(0) \sec^2(0) + \tan(0)\cos(0)}{-1}$$

$$L = 0$$

****

Example 11) Evaluate $\lim_{x \to \pi/2} (1 - \sin x)\tan x$

Solution: Let $L = \lim_{x \to \pi/2} (1 - \sin x) \tan x \quad \ldots\ldots\ldots\ldots (1)$

$x \to \pi/2$ **put** $x = \pi/2$

$$L = (1 - \sin \pi/2)\tan \pi/2 \qquad 0 \times \infty \text{ form}$$

From (1) $$L = \lim_{x \to \pi/2} \frac{(1 - \sin x)}{\frac{1}{\tan x}} = \lim_{x \to \pi/2} \frac{(1 - \sin x)}{\cot x} \quad \ldots\ldots\ldots\ldots (2)$$

$x \to \pi/2$ **put** $x = \pi/2$

$$L = \frac{(1 - \sin \pi/2)}{\cot \pi/2} \qquad \frac{0}{0} \text{ form}$$

From (2) $$L = \lim_{x \to \pi/2} \frac{-\cos x}{-\operatorname{cosec}^2 x}$$

$x \to \pi/2$ **put** $x = \pi/2$

$$L = \frac{-\cos \pi/2}{-\operatorname{cosec}^2 \pi/2}$$

$$L = \frac{0}{1^2}$$

$$L = 0$$

****

Example 12) Evaluate $\lim_{x \to 0} \left[\frac{\pi}{4x} - \frac{\pi}{2x(e^{\pi x} + 1)}\right]$

Solution : Let $L = \lim_{x \to 0} \left[\frac{\pi}{4x} - \frac{\pi}{2x(e^{\pi x} + 1)}\right] \quad \ldots\ldots\ldots\ldots (1)$

$x \to 0$ **put** $x = 0$

$$L = \frac{\pi}{4(0)} - \frac{\pi}{2(0)(e^{\pi 0} + 1)} \qquad \infty - \infty \text{ form}$$

From (1) $$L = \lim_{x \to 0} \left[\frac{2x\pi(e^{\pi x} + 1) - 4x\pi}{8x^2(e^{\pi x} + 1)}\right]$$

$$L = \lim_{x \to 0} \left[\frac{\pi(e^{\pi x} + 1) - 2\pi}{4x(e^{\pi x} + 1)}\right]$$

$$L = \lim_{x \to 0} \left[\frac{\pi e^{\pi x} + \pi - 2\pi}{4x(e^{\pi x} + 1)}\right]$$

$$L = \lim_{x \to 0} \left[\frac{\pi e^{\pi x} - \pi}{4x(e^{\pi x} + 1)}\right] \qquad \ldots\ldots\ldots..(2)$$

$x \rightarrow 0$ put $x = 0$

$$L = \frac{\pi e^{0x} - \pi}{4(0)(e^{\pi 0} + 1)} = \frac{\pi - \pi}{0} \qquad \frac{0}{0} \text{ form}$$

From (2)

$$L = \lim_{x \to 0} \left[\frac{\pi^2 e^{\pi x}}{4x(\pi e^{\pi x} + 0) + 4(\pi e^{\pi x} + 1)}\right]$$

$$L = \frac{\pi^2 e^{\pi 0}}{4(0)(\pi e^0 + 0) + 4(e^0 + 1)}$$

$$L = \frac{\pi^2}{4(1 + 1)}$$

$$L = \frac{\pi^2}{8}$$

****

Example 13) Evaluate $\lim_{x \to 0} \left[\frac{1}{x} - \frac{1}{x^2}\log(1 + x)\right]$

Solution:

$$\text{Let } L = \lim_{x \to 0} \left[\frac{1}{x} - \frac{1}{x^2}\log(1 + x)\right] \qquad \ldots\ldots\ldots\ldots(1)$$

$x \rightarrow 0$ put $x = 0$

$$L = \frac{1}{0} - \frac{1}{0}\log(1 + 0) \qquad \infty - \frac{0}{0} \text{ form}$$

From (1)

$$L = \lim_{x \to 0} \left[\frac{x}{x^2} - \frac{1}{x^2}\log(1 + x)\right]$$

$$L = \lim_{x \to 0} \left[\frac{x * x^2 - x^2\log(1 + x)}{x^2 x^2}\right]$$

$$L = \lim_{x \to 0} \left[\frac{x^2(x - \log(1 + x))}{x^2 x^2}\right]$$

$$L = \lim_{x \to 0} \left[\frac{x - \log(1 + x)}{x^2}\right] \qquad \frac{0}{0} \text{ form} \qquad \ldots\ldots\ldots\ldots(1)$$

From (1) $L = \lim_{x \to 0} \left[\frac{1 - \frac{1}{1+x}}{2x}\right]$ ……… (2)

$x \to 0$ put $x = 0$

$$L = \frac{1 - \frac{1}{1+0}}{2(0)} \qquad \frac{0}{0} \text{ form}$$

From (2) $L = \lim_{x \to 0} \left[\frac{0 - \frac{-1}{(1+x)^2}}{2}\right]$

$x \to 0$ put $x = 0$

$$L = \frac{\frac{1}{(1+0)^2}}{2}$$

$$L = \frac{1}{2}$$

****

Example 14) Evaluate $\lim_{x \to 1} \left[\frac{1}{x-1} - \frac{2}{x^2-1}\right]$

Solution : Let $L = \lim_{x \to 1} \left[\frac{1}{x-1} - \frac{2}{x^2-1}\right]$ ………… (1)

$x \to 1$ put $x = 1$

$$L = \frac{1}{1-1} - \frac{2}{1^2-1} = \frac{1}{0} - \frac{2}{0} \qquad \infty - \infty \text{ form}$$

From (1) $L = \lim_{x \to 0} \left[\frac{(x^2-1) - 2(x-1)}{(x-1)(x^2-1)}\right]$

$$L = \lim_{x \to 0} \left[\frac{(x-1)(x+1) - 2(x-1)}{(x-1)(x^2-1)}\right]$$

$$L = \lim_{x \to 0} \left[\frac{(x+1) - 2}{(x^2-1)}\right]$$

$$L = \lim_{x \to 0} \left[\frac{x-1}{(x^2-1)}\right]$$

$$L = \lim_{x\to 0}\left[\frac{x-1}{(x-1)(x+1)}\right]$$

$$L = \lim_{x\to 0}\left[\frac{1}{(x+1)}\right]$$

$\mathbf{x \to 1 \ \ put\ x = 1}$

$$L = \frac{1}{1+1}$$

$$L = \frac{1}{2}$$

****

Example 15) **Evaluate** $\lim\limits_{x\to 0}\{\sin x\}^{\tan x}$

Solution: Let $\quad L = \lim\limits_{x\to 0}\{\sin x\}^{\tan x}$ ………… (1)

$\mathbf{x \to 0 \ \ put\ \ x = 0}$

$L = \{\sin 0\}^{\tan 0}$ $\quad$ $\mathbf{0^0}$ **form**

Taking log on both sides

$$\log L = \log \lim_{x\to 0}\{\sin x\}^{\tan x}$$

$$\log L = \lim_{x\to 0} \log\{\sin x\}^{\tan x} \qquad \because \ \log a^n = n * \log a$$

$$\log L = \lim_{x\to 0} \tan x \log \sin x \qquad \text{………… (2)}$$

$\mathbf{x \to 0 \ \ put\ x = 0}$

$$\log L = \tan 0 \log \sin 0 = \tan 0 \log 0 \qquad \mathbf{0 \times \infty\ form}$$

From (2) $\quad \log L = \lim\limits_{x\to 0}\dfrac{\log \sin x}{1/\tan x}$

$$\log L = \lim_{x\to 0}\frac{\log \sin x}{\cot x} \qquad \therefore \ \frac{1}{\tan x} = \cot x \qquad \text{………… (3)}$$

$\mathbf{x \to 0 \ \ put\ x = 0}$

$$\log L = \frac{\log \sin 0}{\cot 0} \qquad \frac{\infty}{\infty}\ \textbf{form}$$

From (3) $\quad \log L = \lim\limits_{x\to 0}\dfrac{\left(\frac{1}{\sin x}\right)\ \frac{d}{dx}\sin x}{-\mathrm{cosec}^2 x} = \dfrac{\frac{\cos x}{\sin x}}{-\mathrm{cosce}^2 x}$

$$\log L = \lim_{x\to 0} \frac{\cot x}{-\mathrm{cosec}^2 x} \qquad \frac{\infty}{\infty} \textbf{ form} \qquad \ldots\ldots\ldots.(4)$$

From (4) $\quad \log L = \lim_{x\to 0} \dfrac{-\mathrm{cosec}^2 x}{-2\mathrm{cosec}^1 x\ \mathrm{cosec}\, x \cot x} \qquad \because\ x^n = nx^{n-1}\dfrac{d}{dx}(x)$

$$\log L = \lim_{x\to 0} \frac{1}{2\cot x} = \lim_{x\to 0} \frac{1}{2}\tan x$$

$\mathbf{x \to 0\ \ put\ \ x = 0}$

$$\log L = \frac{1}{2}\tan 0$$

$$\log L = 0$$

$$e^{\log L} = e^0 \quad L = e^0 = 1$$

****

Example 16) Evaluate $\lim\limits_{x\to 0}\{\cot x\}^{\sin x}$

Solution: Let $\quad L = \lim\limits_{x\to 0}\{\cot x\}^{\sin x}$

$\mathbf{x \to 0\ \ put\ x = 0}$

$$L = \lim_{x\to 0}\{\cot 0\}^{\sin 0} \qquad \infty^0 \textbf{ form} \qquad \ldots\ldots\ldots(1)$$

Taking log on both sides

$$\log L = \log \lim_{x\to 0}\{\cot x\}^{\sin x}$$

$$\log L = \lim_{x\to 0} \log \{\cot x\}^{\sin x}$$

$$\log L = \lim_{x\to 0} \sin x \log \cot x \qquad \ldots\ldots\ldots(2)$$

$\mathbf{x \to 0\ \ put\ x = 0}$

$$\log L = \sin 0 \log \cot 0 \qquad \mathbf{0 \times \infty \text{ form}}$$

From (2) $\quad \log L = \lim\limits_{x\to 0} \dfrac{\log \cot x}{1/\sin x}$

$$\log L = \lim_{x\to 0} \frac{\log \cot x}{\mathrm{cosec}\, x} \qquad \because\ \frac{1}{\sin x} = \mathit{cosec}\ x \qquad \ldots\ldots\ldots(3)$$

$\mathbf{x \to 0\ \ put\ x = 0}$

$$\log L = \frac{\log \cot 0}{\mathrm{cosec}\, 0} \qquad \frac{\infty}{\infty} \textbf{ form}$$

From (3) $\log L = \lim_{x\to 0} \dfrac{-\operatorname{cosec}^2 x / \cot x}{-\operatorname{cosec} x \cot x}$

$\log L = \lim_{x\to 0} \dfrac{\operatorname{cosec} x}{\cot^2 x}$ $\quad \dfrac{\infty}{\infty}$ **form** ……….(**4**)

From (4) $\log L = \lim_{x\to 0} \dfrac{\tan^2 x}{\sin x}$ taking reciprocal $\dfrac{0}{0}$ **form**

$\log L = \lim_{x\to 0} \dfrac{2\tan x \sec^2 x}{\cos x}$ $\quad \because \dfrac{d}{dx}(x^n) = nx^{n-1}$

x → 0 put x = 0

$\log L = \dfrac{2\tan 0 \sec^2 0}{\cos 0} = \dfrac{0}{1}$

$\log L = 0$

$L = e^0 = 1$

****

Example 17) Evaluate $\lim_{x\to 0}\left[\dfrac{a^x + b^x}{2}\right]^{1/x}$

Solution: Let $L = \lim_{x\to 0}\left[\dfrac{a^x + b^x}{2}\right]^{1/x}$

x → 0 put x = 0

$L = \left[\dfrac{a^0 + b^0}{2}\right]^{1/0} = \left[\dfrac{1+1}{2}\right]^{\infty}$ $\quad 1^\infty$ **form** ………….(1)

Taking log on both sides

$\log L = \log \lim_{x\to 0}\left[\dfrac{a^x + b^x}{2}\right]^{1/x}$

$\log L = \lim_{x\to 0} \log\left[\dfrac{a^x + b^x}{2}\right]^{1/x}$

$\log L = \lim_{x\to 0} \dfrac{\log\left[\dfrac{a^x + b^x}{2}\right]}{x}$ $\quad \because \log a^n = n \log a$ ……(2)

x → 0 put x = 0

$\log L = \log\left[\dfrac{a^0 + b^0}{2}\right]/0$ $\quad \dfrac{0}{0}$ **form**

From (2) $\log L = \lim_{x\to 0}\left[\dfrac{1}{\dfrac{a^x + b^x}{2}} \dfrac{(a^x \log a + b^x \log b)}{2}\right]/1$

$$\log L = \lim_{x \to 0} \frac{2}{a^x + b^x} \frac{(a^x \log a + b^x \log b)}{2}$$

$$\log L = \lim_{x \to 0} \frac{a^x \log a + b^x \log b}{a^x + b^x}$$

$\mathbf{x \to 0 \ \ put \ \ x = 0}$

$$\log L = \frac{a^0 \log a + b^0 \log b}{a^0 + b^0}$$

$$\log L = \frac{\log a + \log b}{1 + 1}$$

$$\log L = \frac{\log(a\,b)}{2} \qquad \because \ \mathbf{\log a + \log b = \log ab}$$

$$\log L = \frac{1}{2} \log ab$$

$$\log L = \log(ab)^{\frac{1}{2}}$$

$$L = (ab)^{\frac{1}{2}}$$

****

Fourier Series

Periodic Functions:

For every real $f(x)$ and there exists some positive number T such that $F(x+nT) = F(x)$ is callled Periodic Function.

T is called permitive period or fundamental period of $f(x)$

Example: The fundamental period of $sin\,x$, $cos\,x$, $sec\,x$, $cosec\,x$ is 2π and $tan\,x, cot\,x$ is π

Even Function: Function $f(x)$ is defined in $-l < x < l$ is said to be even if

$f(x) = f(-x)$ Example: $cos\,x$, x^2

Odd Function: Function $f(x)$ is defined in $-l < x < l$ is said to be odd if

$f(x) = -f(-x)$ Example: $sin\,x$, x^3, $tan\,x$

Note:

1) If $f(x)$ is even, the values of y for $-x$ and x are same, therefore graph of $y = f(x)$ is symmetric about $x-$ axis. 2) If $f(x)$ is odd, the values of y for $-x$ and x differ by sign only therefore graph of $y = f(x)$ is symmetric about origin(opposite quadrants).

3) If $f(x)$ is Even function of x, $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

4) If $f(x)$ is Odd function of x, $\int_{-a}^{a} f(x)dx = 0$

5) Any function $f(x)$ can be expressed as sum of even and odd functions

$$f(x) = \left[\frac{f(x)+f(-x)}{2}\right] + \left[\frac{f(x)-f(-x)}{2}\right]$$

6)

Sr. No	$f(x)$	$g(x)$	$f(x) \pm g(x)$	$f(x) \genfrac{}{}{0pt}{}{\times}{\div} g(x)$
1)	Even	Even	Even	Even
2)	Odd	Odd	Odd	Even
3)	Even	Odd	Neither Odd nor Even	Odd
4)	Odd	Even	Neither Odd nor Even	Odd

"The Only things that will stop you from fulfilling your dreams is you"

Formula:

1) $\int uv\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\int v\,dx\right]dx$

2) $\int \sin x \sin nx\ dx \ = \frac{1}{2}\int[\cos(1-n)x - \cos(1+n)\,x]\,dx$

3) $\int \cos x \cos nx\,dx \ = \frac{1}{2}\int[\cos(1+n)x + \cos(1-n)\,x]\,dx$

4) $\int \sin x \cos nx\,dx \ = \frac{1}{2}\int[\sin(1+n)x + \sin(1-n)\,x]\,dx$

5) $\int \cos x \sin nx\,dx \ = \frac{1}{2}\int[\sin(1+n)x + \sin(1-n)\,x]\,dx$

6) $\cos n\pi = (-1)^n$ $\qquad\qquad$ $\cos 2n\pi = 1$

7) $\sin n\pi = 0$ $\qquad\qquad$ $\sin 2n\pi = 0$

8) $\int e^{ax}\sin bx\ dx \ = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$

9) $\int e^{ax}\cos bx\ dx \ = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$

10) $\int uvdx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + u''''v_5 - u'''''v_6 + \ldots\ldots\ldots$

where dashes($'\ ''\ '''\ ''''\ \ldots$)indicate derivitives and suffixes (1,2,3..)indicates integrals

Dirchlet's Condition:

Let $f(x)$ be function defined in $C < x < C + 2L$ such that

i) $f(x)$is defined and single valued in the given interval also $\int_C^{C+2L} f(x)dx$ exits

ii) $f(x)$ may have finite number of finite discontinuities in the interval.

iii) $f(x)$ may have finite number of maxima or minima in the given interval.

Fourier Series :

Let $f(x)$ be periodic function of period $2L$ defined in the interval $C < x < C + 2L$ and satisfies Dirchlet's Conditions then $f(x)$ can be expressed as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)$$

where a_0 , a_n , b_n are called Fourier constants or Fourier coeficients and given by

$$a_0 = \frac{1}{L}\int_C^{C+2L} f(x)dx$$

$$a_n = \frac{1}{L}\int_C^{C+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_C^{C+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Fourier Series For different interval

Fourier Series in the interval $(0, 2L)$	Fourier Series in the interval $(0, 2\pi)$
$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$
$a_0 = \frac{1}{L}\int_0^{2L} f(x)dx$	$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$
$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$	$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$
$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$	$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$

Example: *Find the Fourier series of the function* $f(x) = e^{-x};\ 0 \le x \le 2\pi$ *and* $f(x + 2\pi) = f(x)$

Solution: The Fourier series of $f(x)$ in $0 \le x \le 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$$

$$\mathbf{a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx} \qquad \mathbf{a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx} \qquad \mathbf{b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx}$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} e^{-x}dx$$

$$= \frac{1}{\pi}(-e^{-x})_0^{2\pi} = \frac{1}{\pi}[-e^{-2\pi} - (-e^{0})] = \frac{1}{\pi}[-e^{-2\pi} + 1] = \frac{1}{\pi}[1 - e^{-2\pi}]$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\, dx \qquad \because \int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$$

$a = -1$ **and** $b = n$

$$a_n = \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-\cos nx + n\sin nx)\right]_0^{2\pi}$$

"The Only things that will stop you from fulfilling your dreams is you"

$$a_n = \frac{1}{\pi}\left\{\left[\frac{e^{-2\pi}}{1+n^2}(-\cos 2n\pi + n\sin 2n\pi)\right] - \left[\frac{e^0}{1+n^2}(-\cos 0 + n\sin 0)\right]\right\}$$

$$a_n = \frac{1}{\pi}\left\{\left[\frac{e^{-2\pi}}{1+n^2}(-1+0)\right] - \left[\frac{1}{1+n^2}(-1+0)\right]\right\} \qquad \because \cos 2n\pi = 1 \quad \sin 2n\pi = 0$$

$$a_n = \frac{1}{\pi}\left\{\left[\frac{-e^{-2\pi}}{1+n^2}\right] - \left[\frac{-1}{1+n^2}\right]\right\} = \frac{-e^{-2\pi}+1}{\pi(1+n^2)} = \frac{1-e^{-2\pi}}{\pi(1+n^2)}$$

$$\boldsymbol{b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\, dx \qquad \because \int e^{ax}\sin bx\; dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$$

$a = -1$ and $b = n$

$$b_n = \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-\sin nx - n\cos nx)\right]_0^{2\pi}$$

$$b_n = \frac{1}{\pi}\left\{\left[\frac{e^{-2\pi}}{1+n^2}(-\sin 2n\pi - n\cos 2n\pi)\right] - \left[\frac{e^0}{1+n^2}(-\sin 0 - n\cos 0)\right]\right\}$$

$$b_n = \frac{1}{\pi}\left\{\left[\frac{e^{-2\pi}}{1+n^2}(0-n)\right] - \left[\frac{1}{1+n^2}(0-n)\right]\right\} \qquad \because \cos 2n\pi = 1 \quad \sin 2n\pi = 0$$

$$b_n = \frac{1}{\pi}\left\{\left[\frac{-ne^{-2\pi}}{1+n^2}\right] - \left[\frac{-n}{1+n^2}\right]\right\} = \frac{-ne^{-2\pi}+n}{\pi(1+n^2)} = \frac{n(1-e^{-2\pi})}{\pi(1+n^2)}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos nx + b_n\sin nx]$$

$$e^{-x} = \frac{1-e^{-2\pi}}{2\pi} + \sum_{n=1}^{\infty}\left[\frac{1-e^{-2\pi}}{\pi(1+n^2)}\cos nx + \frac{n(1-e^{-2\pi})}{\pi(1+n^2)}\sin nx\right]$$

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Example: Find the Fourier series of the functions $f(x) = x^2;\ \ 0 \le x \le 2\pi$ and $f(x + 2\pi) = f(x)$

Solution: The Fourier series of $f(x)$ in $0 \le x \le 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

$$\boldsymbol{a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx \qquad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx}$$

$$\boldsymbol{a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2 dx}$$

$$= \frac{1}{\pi}\left(\frac{x^3}{3}\right)_0^{2\pi} \quad = \frac{1}{\pi}\left[\frac{(2\pi)^3}{3} - 0\right) \quad = \frac{1}{\pi}\left[\frac{8\pi^3}{3}\right] \quad = \frac{8\pi^2}{3}$$

$$\boldsymbol{a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\, dx$$

$$a_n = \frac{1}{\pi}\left\{x^2 \frac{\sin nx}{n} - (2x)\left(\frac{-\cos nx}{n*n}\right) + (2)\left(\frac{-\sin nx}{n*n*n}\right)\right\}_0^{2\pi}$$

$$a_n = \frac{1}{\pi}\left\{(2\pi)^2 \frac{\sin 2n\pi}{n} + (2*2\pi)\left(\frac{\cos 2n\pi}{n^2}\right) + (2)\left(\frac{-\sin 2n\pi}{n^3}\right)\right\}$$

$$-\left\{(0)^2 \frac{\sin 0}{n} + (2*0)\left(\frac{\cos 0}{n^2}\right) + (2)\left(\frac{-\sin 0}{n^3}\right)\right\}$$

$$a_n = \frac{1}{\pi}\left\{0 + \frac{4\pi}{n^2} + 0\right\} - \{0 - 0 + 0\} \quad \because \boldsymbol{\cos 2n\pi = 1 \quad \sin 2n\pi = 0}$$

$$a_n = \frac{4}{n^2}$$

$$\boldsymbol{b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\, dx$$

$$b_n = \frac{1}{\pi}\left\{x^2\left(\frac{-\cos nx}{n}\right) - (2x)\left(\frac{-\sin nx}{n*n}\right) + (2)\left(\frac{\cos nx}{n*n*n}\right) - 0\right\}_0^{2\pi}$$

$$b_n = \frac{1}{\pi}\left\{(2\pi)^2\left(\frac{-\cos 2\pi n}{n}\right) + (2*2\pi)\left(\frac{\sin 2n\pi}{n^2}\right) + (2)\left(\frac{\cos 2n\pi}{n^3}\right)\right\}$$

$$-\left\{(0)^2\frac{\cos 0}{n}+(2*0)\left(\frac{\sin 0}{n^2}\right)+(2)\left(\frac{\cos 0}{n^3}\right)\right\}$$

$$b_n=\frac{1}{\pi}\left\{-\frac{4\pi^2}{n}\quad+0+\frac{2}{n^3}\right\}-\left\{0+0+\frac{2}{n^3}\right\}\qquad \because \cos 2n\pi=1\quad \sin 2n\pi=0$$

$$b_n=\frac{1}{\pi}\left\{-\frac{4\pi^2}{n}\quad+0+\frac{2}{n^3}-\frac{2}{n^3}\right\}$$

$$b_n=-\frac{4\pi}{n}$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}[a_n\cos nx+b_n\sin nx]$$

$$x^2=\frac{8\pi^2}{3}\quad+\sum_{n=1}^{\infty}\left[\frac{4}{n^2}\cos nx-\frac{4\pi}{n}\sin nx\right]$$

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Example: Find the Fourier expansion of the periodic function $f(x)=\cos ax \quad (0,2\pi)$

a is not an integer

Solution: $f(x)=\cos ax \quad (0,2\pi) \quad period=2\pi$

The Fourier series of $f(x)$ is given by $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}[a_n\cos nx+b_n\sin nx]$ ……. (1)

$$a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)dx \qquad a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx \qquad b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx\,dx$$

$$a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)dx$$

$$=\frac{1}{\pi}\int_0^{2\pi}\cos ax\,dx$$

$$=\frac{1}{\pi}\left[\frac{\sin ax}{a}\right]_0^{2\pi}$$

$$=\frac{1}{\pi}\left[\frac{\sin 2\pi a}{a}-\frac{\sin 0}{a}\right]$$

$$=\frac{\sin 2\pi a}{a\pi}-\frac{0}{a}=\frac{\sin 2\pi a}{a\pi}$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}\cos ax\cos nx\,dx \qquad \because \cos A\cos B=\frac{1}{2}[\cos(A+B)+\cos(A-B)]$$

$A = ax \quad \text{and} \quad B = nx$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi}[\cos(ax+nx)+\cos(ax-nx)]dx$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi}[\cos(a+n)x+\cos(a-n)x]dx$$

$$a_n = \frac{1}{2\pi}\left[\frac{\sin(a+n)x}{(a+n)}+\frac{\sin(a-n)x}{(a-n)}\right]_0^{2\pi}$$

$$a_n = \frac{1}{2\pi}\left\{\frac{\sin(a+n)2\pi}{(a+n)}+\frac{\sin(a-n)2\pi}{(a-n)}-\frac{\sin(a+n)0}{(a+n)}-\frac{\sin(a-n)0}{(a-n)}\right\}$$

$$a_n = \frac{1}{2\pi}\left\{\frac{\sin 2a\pi}{(a+n)}+\frac{\sin 2a\pi}{(a-n)}\right\} \qquad \because \sin(a+n)2\pi = \sin 2a\pi, \sin(a-n)2\pi = \sin 2a\pi$$

$$a_n = \frac{\sin 2a\pi}{2\pi}\left\{\frac{1}{(a+n)}+\frac{1}{(a-n)}\right\}$$

$$a_n = \frac{\sin 2a\pi}{2\pi}\left\{\frac{(a-n)+(a+n)}{(a+n)(a-n)}\right\} \qquad \because (a^2-b^2) = (a+b)(a-b)$$

$$a_n = \frac{\sin 2a\pi}{2\pi}\left\{\frac{2a}{a^2-n^2}\right\} = \frac{a\sin 2a\pi}{\pi(a^2-n^2)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi}\sin nx\,\cos ax\,dx \qquad \because \sin A\cos B = \frac{1}{2}[\sin(A+B)+\sin(A-B)]$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi}[\sin(nx+ax)+\sin(nx-ax)]dx$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi}[\sin(n+a)x+\sin(n-a)x]dx$$

$$b_n = \frac{1}{2\pi}\left[\frac{-\cos(n+a)x}{(n+a)}+\frac{-\cos(n-a)x}{(n-a)}\right]_0^{2\pi}$$

$$b_n = \frac{1}{2\pi}\left\{\left[\frac{-\cos(n+a)2\pi}{(n+a)}+\frac{-\cos(n-a)2\pi}{(n-a)}\right]-\left[\frac{-\cos(n+a)0}{(n+a)}-\frac{\cos(n-a)0}{(n-a)}\right]\right\}$$

$$\because \cos(n+a)2\pi = \cos 2a\pi, \qquad \cos(n-a)2\pi = \cos 2a\pi$$

$$b_n = \frac{1}{\pi}\left\{\frac{-\cos 2a\pi}{(n+a)}+\frac{-\cos 2a\pi}{(n-a)}+\frac{1}{(n+a)}+\frac{1}{(n-a)}\right\}$$

$$b_n = \frac{1}{\pi}\left\{\frac{1-\cos 2a\pi}{(n+a)} + \frac{1-\cos 2a\pi}{(n-a)}\right\}$$

$$b_n = \frac{1-\cos 2a\pi}{2\pi}\left\{\frac{1}{(n+a)} + \frac{1}{(n-a)}\right\}$$

$$b_n = \frac{1-\cos 2a\pi}{2\pi}\left\{\frac{(n-a)+(n+a)}{(n+a)(n-a)}\right\}$$

$$b_n = \frac{1-\cos 2a\pi}{2\pi}\left\{\frac{2n}{n^2-a^2}\right\} = \frac{n(1-\cos 2a\pi)}{\pi(n^2-a^2)}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos nx + b_n \sin nx]$$

$$f(x) = \frac{1}{2}\frac{\sin 2\pi a}{a\pi} + \sum_{n=1}^{\infty}\left[\frac{a\sin 2a\pi}{\pi(a^2-n^2)}\cos nx + \frac{n(1-\cos 2a\pi)}{\pi(n^2-a^2)}\sin nx\right]$$

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Example: Find the Fourier expansion of the periodic function

$$f(x) = \begin{cases} -\pi & 0 < x < \pi \\ x-\pi & \pi < x < 2\pi \end{cases}$$

State the value of the series at $x = \pi$ $\quad i.e. f(\pi)$

Solution: $f(x) = \begin{cases} -\pi & 0 < x < \pi \\ x-\pi & \pi < x < 2\pi \end{cases}$ here $(0, 2\pi)$

The Fourier series of $f(x)$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos nx + b_n\sin nx]$

$\boldsymbol{a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx}$ $\quad$ $\boldsymbol{a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx}$ $\quad$ $\boldsymbol{b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx}$

$\boldsymbol{a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx}$

$$a_0 = \frac{1}{\pi}\left\{\int_0^{\pi} f(x)dx + \int_{\pi}^{2\pi} f(x)dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi}(-\pi)dx + \int_{\pi}^{2\pi}(x-\pi)dx\right\}$$

$$= \frac{1}{\pi}\left\{(-\pi)\int_0^{\pi}dx + \int_{\pi}^{2\pi}(x-\pi)dx\right\}$$

$$= \frac{1}{\pi}\left\{[-\pi x]_0^{\pi} + \left[\frac{(x-\pi)^2}{2}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{[(-\pi*\pi)-(0)] + \left[\frac{(2\pi-\pi)^2}{2} - \frac{(\pi-\pi)^2}{2}\right]\right\}$$

$$= \frac{1}{\pi}\left\{-\pi^2 + \frac{\pi^2}{2} - 0\right\} = \frac{\pi^2}{\pi}\left\{-1 + \frac{1}{2}\right\}$$

$$= \frac{\pi^2}{\pi}\left\{-\frac{1}{2}\right\}$$

$$= -\frac{\pi}{2}$$

$$\boldsymbol{a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx}$$

$$a_n = \frac{1}{\pi}\left\{\int_0^{\pi}(-\pi)\cos nx\, dx + \int_{\pi}^{2\pi}(x-\pi)\cos nx\, dx\right\}$$

$$a_n = \frac{1}{\pi}\left\{\left[\frac{-\pi\sin nx}{n}\right]_0^{\pi} + \left[\left((x-\pi)\frac{\sin nx}{n}\right) - \left((1-0)\frac{-\cos nx}{n^2}\right) + (0)\frac{-\sin nx}{n^3}\right]_{\pi}^{2\pi}\right\}$$

$$a_n = \frac{1}{\pi}\left\{\begin{array}{c}\frac{-\pi\sin n\pi}{n} - \frac{-\pi\sin 0}{n} + \left[\left((2\pi-\pi)\frac{\sin 2\pi n}{n}\right) - \left(\frac{-\cos 2\pi n}{n^2}\right)\right] \\ -\left[\left((\pi-\pi)\frac{\sin 0}{n}\right) - \left(\frac{-(-1)^n}{n^2}\right)\right]\end{array}\right\}$$

$$a_n = \frac{1}{\pi}\left\{0 - 0 + \left[\left((\pi)0\right) + \frac{1}{n^2}\right] - \left[(0) + \left(\frac{(-1)^n}{n^2}\right)\right]\right\} \qquad \because \cos n\pi = (-1)^n \quad \sin n\pi = 0$$

$$a_n = \frac{1}{\pi}\left\{\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right\}$$

$$a_n = \frac{1}{\pi}\left\{\frac{1-(-1)^n}{n^2}\right\}$$

$$\boldsymbol{b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx}$$

$$b_n = \frac{1}{\pi}\left\{\int_0^{\pi}(-\pi)\sin nx\, dx + \int_{\pi}^{2\pi}(x-\pi)\sin nx\, dx\right\}$$

$$b_n = \frac{1}{\pi}\left\{\left[\frac{(-\pi)(-\cos nx)}{n}\right]_0^{\pi} + \left[\left((x-\pi)\left(\frac{-\cos nx}{n}\right)\right) - \left((1-0)\left(\frac{-\sin nx}{n^2}\right)\right)\right]_{\pi}^{2\pi}\right\}$$

$$b_n = \frac{1}{\pi}\left\{\begin{array}{c}\frac{\pi\cos n\pi}{n} - \frac{\pi\cos 0}{n} + (2\pi-\pi)\left(\frac{-\cos 2n\pi}{n}\right) + \left(\frac{-\sin 2n\pi}{n^2}\right) \\ -(\pi-\pi)\left(\frac{-\cos n\pi}{n}\right) - \left(\frac{-\sin n\pi}{n^2}\right)\end{array}\right\}$$

$$b_n = \frac{1}{\pi}\left\{\frac{\pi(-1)^n}{n} - \frac{\pi}{n} + (\pi)\left(\frac{-1}{n}\right) + 0 - 0 + 0\right\} \qquad b_n = \frac{(-1)^n - 2}{n}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$$

$$f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{\pi}\left\{\frac{1-(-1)^n}{n^2}\right\}\cos nx + \frac{(-1)^n-2}{n}\sin nx\right] \ldots\ldots\ldots\ldots\ldots(1)$$

$Find \quad f(\pi)$: $\quad f(x) = \begin{cases} -\pi & 0 < x < \pi \\ x - \pi & \pi < x < 2\pi \end{cases}$ $\quad$ here $\quad (0, 2\pi)$

$$f(\pi^-) = \log_{x\to\pi^-} -\pi \;= -\pi \;\; and \quad f(\pi^+) = \log_{x\to\pi^+} x - \pi \;= \pi - \pi = 0$$

As $f(x)$ is discontinuous at $x = \pi$,

$$f(\pi) = \frac{f(\pi^-) + f(\pi^+)}{2} = \frac{-\pi + 0}{2} = \frac{-\pi}{2}$$

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Example: Find the Fourier expansion of the function $: f(x) = 2x - x^2; \;\; 0 \le x \le 3$

Solution: $\quad f(\mathrm{x}) = 2\mathrm{x} - \mathrm{x}^2; \qquad 0 \le \mathrm{x} \le 3$

Here $0 \le \mathrm{x} \le 3$ i.e. $0 \le \mathrm{x} \le 2\mathrm{L}$

$2\mathrm{L} = 3 \quad \Rightarrow \quad \mathrm{L} = \frac{3}{2}$

The Fourier series of $f(x)$ in $0 \le \mathrm{x} \le 2\mathrm{L}$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$\boldsymbol{a_0 = \frac{1}{L}\int_0^{2L} f(x)dx \qquad a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx}$$

$$\boldsymbol{a_0 = \frac{1}{L}\int_0^{2L} f(x)dx}$$

$$= \frac{1}{3/2}\int_0^3 (2x - x^2)dx$$

$$= \frac{2}{3}\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^3 \quad = \frac{2}{3}\left[x^2 - \frac{x^3}{3}\right]_0^3$$

$$= \frac{2}{3}\left\{\left[3^2 - \frac{3^3}{3}\right] - \left[0^2 - \frac{0^3}{3}\right]\right\} \quad = \frac{2}{3}[9 - 9]$$

$$a_0 = 0$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$a_n = \frac{1}{3/2}\int_0^{3} (2x - x^2)\cos\left(\frac{n\pi x}{3/2}\right)dx$$

$$= \frac{2}{3}\int_0^{3} (2x - x^2)\cos\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left\{\frac{(2x-x^2)\sin\left(\frac{2n\pi x}{3}\right)}{\frac{2n\pi}{3}} - \frac{(2-2x)\left[-\cos\left(\frac{2n\pi x}{3}\right)\right]}{\left(\frac{2n\pi}{3}\right)^2} + \frac{(0-2)\left[-\sin\left(\frac{2n\pi x}{3}\right)\right]}{\left(\frac{2n\pi}{3}\right)^3} - (0)\right\}_0^3$$

$$= \frac{2}{3}\left\{\frac{(2x-x^2)\sin\left(\frac{2n\pi x}{3}\right)}{\frac{2n\pi}{3}} + \frac{(2-2x)\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2} + \frac{2\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^3}\right\}_0^3$$

$$= \frac{2}{3}\left\{\frac{(2(3)-3^2)\sin\left(\frac{2n\pi(3)}{3}\right)}{\frac{2n\pi}{3}} + \frac{(2-2(3))\cos\left(\frac{2n\pi(3)}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2} + \frac{2\sin\left(\frac{2n\pi\,(3)}{3}\right)}{\left(\frac{2n\pi}{3}\right)^3}\right.$$
$$\left. - \frac{(2(0)-0^2)\sin(0)}{\frac{2n\pi}{3}} - \frac{(2-2(0))\cos(0)}{\left(\frac{2n\pi}{3}\right)^2} - \frac{2\sin(0)}{\left(\frac{2n\pi}{3}\right)^3}\right\}$$

$$= \frac{2}{3}\left\{\frac{(-3)\sin(2n\pi)}{\frac{2n\pi}{3}} + \frac{(-4)\cos(2n\pi)}{\left(\frac{2n\pi}{3}\right)^2} + \frac{2\sin(2n\pi)}{\left(\frac{2n\pi}{3}\right)^3} - \frac{(0)\sin(0)}{\frac{2n\pi}{3}} - \frac{(2)\cos(0)}{\left(\frac{2n\pi}{3}\right)^2} - \frac{2\sin(0)}{\left(\frac{2n\pi}{3}\right)^3}\right\}$$

$$= \frac{2}{3}\left\{0 + \frac{(-4)}{\left(\frac{2n\pi}{3}\right)^2} + 0 - 0 - \frac{2}{\left(\frac{2n\pi}{3}\right)^2} - 0\right\} \qquad \sin(2n\pi) = 0, \quad \cos 2n\pi = 1$$

$$= \frac{2}{3}\left\{\frac{-4}{\left(\frac{2n\pi}{3}\right)^2} - \frac{2}{\left(\frac{2n\pi}{3}\right)^2}\right\} = \frac{2}{3}\left\{\frac{-6}{\left(\frac{2n\pi}{3}\right)^2}\right\}$$

$$= \frac{-12}{3\left(\frac{4n^2\pi^2}{9}\right)} = \frac{-9}{n^2\pi^2}$$

$$\mathbf{b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx}$$

$$b_n = \frac{1}{3/2}\int_0^3 (2x - x^2)\sin\left(\frac{n\pi x}{3/2}\right)dx$$

$$= \frac{2}{3}\int_0^3 (2x - x^2)\sin\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left\{\frac{(2x - x^2)\left[-\cos\left(\frac{2n\pi x}{3}\right)\right]}{\frac{2n\pi}{3}} - \frac{(2 - 2x)\left[-\sin\left(\frac{2n\pi x}{3}\right)\right]}{\left(\frac{2n\pi}{3}\right)^2} + \frac{(0 - 2)\left[\cos\left(\frac{2n\pi x}{3}\right)\right]}{\left(\frac{2n\pi}{3}\right)^3}\right\}_0^3$$

$$= \frac{2}{3}\left\{\frac{-(2x - x^2)\cos\left(\frac{2n\pi x}{3}\right)}{\frac{2n\pi}{3}} + \frac{(2 - 2x)\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2} - \frac{2\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^3}\right\}_0^3$$

$$= \frac{2}{3}\left\{\frac{-(2(3) - 3^2)\cos\left(\frac{2n\pi(3)}{3}\right)}{\frac{2n\pi}{3}} + \frac{(2 - 2(3))\sin\left(\frac{2n\pi(3)}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2} - \frac{2\cos\left(\frac{2n\pi\,(3)}{3}\right)}{\left(\frac{2n\pi}{3}\right)^3}\right.$$
$$\left. + \frac{(2(0) - 0^2)\cos(0)}{\frac{2n\pi}{3}} - \frac{(2 - 2(0))\sin(0)}{\left(\frac{2n\pi}{3}\right)^2} + \frac{2\cos(0)}{\left(\frac{2n\pi}{3}\right)^3}\right\}$$

$$= \frac{2}{3}\left\{\frac{-(-3)\cos(2n\pi)}{\frac{2n\pi}{3}} + \frac{(-4)\sin(2n\pi)}{\left(\frac{2n\pi}{3}\right)^2} - \frac{2\cos(2n\pi)}{\left(\frac{2n\pi}{3}\right)^3} + \frac{(0)\cos(0)}{\frac{2n\pi}{3}} - \frac{(2)\sin(0)}{\left(\frac{2n\pi}{3}\right)^2}\right.$$
$$\left. + \frac{2\cos(0)}{\left(\frac{2n\pi}{3}\right)^3}\right\}$$

$$= \frac{2}{3}\left\{\frac{3}{\frac{2n\pi}{3}} + 0 - \frac{2}{\left(\frac{2n\pi}{3}\right)^3} + 0 - 0 + \frac{2}{\left(\frac{2n\pi}{3}\right)^3}\right\}$$

$$= \frac{2}{3}\left\{\frac{3}{\frac{2n\pi}{3}} - \frac{2}{\left(\frac{2n\pi}{3}\right)^3} + \frac{2}{\left(\frac{2n\pi}{3}\right)^3}\right\} = \frac{2}{3}\frac{3}{\frac{2n\pi}{3}} = \frac{3}{n\pi}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$2x - x^2 = \frac{0}{2} + \sum_{n=1}^{\infty}\left[\frac{-9}{n^2\pi^2}\cos\left(\frac{n\pi x}{3/2}\right) + \frac{3}{n\pi}\sin\left(\frac{n\pi x}{3/2}\right)\right]$$

$$2x - x^2 = \sum_{n=1}^{\infty}\left[\frac{-9}{n^2\pi^2}\cos\left(\frac{2n\pi x}{3}\right) + \frac{3}{n\pi}\sin\left(\frac{2n\pi x}{3}\right)\right]$$

****

Example: Find the Fourier expansion of the function :

$$f(x) = \begin{cases} \pi x & 0 \le x \le 1 \\ \pi(2-x) & 1 \le x \le 2 \end{cases}$$

Solution: $f(x) = \begin{cases} \pi x & 0 \le x \le 1 \\ 2\pi - x\pi & 1 \le x \le 2 \end{cases}$

Here $0 \le x \le 2$ i.e. $0 \le x \le 2L$

$2L = 2 \Rightarrow L = 1$

The Fourier series of $f(x)$ in $(0, 2L)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$\boldsymbol{a_0 = \frac{1}{L}\int_0^{2L} f(x)dx}$$

$$= \frac{1}{1}\left\{\int_0^1 \pi x dx + \int_1^2 (2\pi - x\pi)\, dx\right\}$$

$$= \left[\frac{\pi x^2}{2}\right]_0^1 + \left[2\pi x - \frac{\pi x^2}{2}\right]_1^2$$

$$= \left\{\left[\frac{\pi}{2} - 0\right] + \left[2\pi(2) - \frac{\pi(2)^2}{2} - 2\pi(1) + \frac{\pi(1)^2}{2}\right]\right\}$$

$$= \left\{\frac{\pi}{2} + 4\pi - 2\pi - 2\pi + \frac{\pi}{2}\right\}$$

$$= \left\{\frac{\pi}{2} + \frac{\pi}{2}\right\}$$

$$\boldsymbol{a_0} = \pi$$

$$\mathbf{a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx}$$

$$a_n = \int_0^1 \pi x\cos(n\pi x)\,dx + \int_1^2 (2\pi - \pi x)\cos(n\pi x)\,dx$$

$$= \left\{\frac{\pi x\sin(n\pi x)}{n\pi} - \frac{-\pi\cos(n\pi x)}{(n\pi)^2}\right\}_0^1 + \left\{\frac{(2\pi - \pi x)\sin(n\pi x)}{n\pi} - \frac{(0-\pi)(-\cos n\pi x)}{(n\pi)^2}\right\}_1^2$$

$$= \left\{\frac{\pi x\sin(n\pi x)}{n\pi} + \frac{\pi\cos(n\pi x)}{(n\pi)^2}\right\}_0^1 + \left\{\frac{(2\pi - \pi x)\sin(n\pi x)}{n\pi} - \frac{\pi\cos n\pi x}{(n\pi)^2}\right\}_1^2$$

$$= \left\{\frac{\pi\sin(n\pi)}{n\pi} + \frac{\pi\cos(n\pi)}{(n\pi)^2} - \frac{\pi(0)\sin(0)}{n\pi} - \frac{\pi\cos(0)}{(n\pi)^2}\right\}$$
$$+ \left\{\frac{(2\pi - 2\pi)\sin(2n\pi)}{n\pi} - \frac{\pi\cos 2n\pi}{(n\pi)^2} - \frac{(2\pi - \pi)\sin(n\pi)}{n\pi} + \frac{\pi\cos n\pi}{(n\pi)^2}\right\}$$

$$= \left\{0 + \frac{\pi\cos(n\pi)}{(n\pi)^2} - 0 - \frac{\pi\cos(0)}{(n\pi)^2}\right\} + \left\{0 - \frac{\pi\cos 2n\pi}{(n\pi)^2} - 0 + \frac{\pi\cos n\pi}{(n\pi)^2}\right\}$$

$$= \frac{\pi(-1)^n}{(n\pi)^2} - \frac{\pi}{(n\pi)^2} - \frac{\pi}{(n\pi)^2} + \frac{\pi(-1)^n}{(n\pi)^2} \qquad \cos(n\pi) = (-1)^n \quad \cos 2n\pi = 1$$

$$= \frac{2\pi(-1)^n}{(n\pi)^2} - \frac{2\pi}{(n\pi)^2} \qquad = \frac{2\pi(-1)^n - 2\pi}{(n\pi)^2}$$

$$a_n = \frac{2(-1)^n - 2}{\pi n^2}$$

$$\mathbf{b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx}$$

$$b_n = \int_0^1 \pi x\sin(n\pi x)\,dx + \int_1^2 (2\pi - \pi x)\sin(n\pi x)\,dx$$

$$= \left\{\frac{\pi x(-\cos(n\pi x))}{n\pi} - \frac{-\pi\sin(n\pi x)}{(n\pi)^2}\right\}_0^1 + \left\{\frac{(2\pi - \pi x)(-\cos(n\pi x))}{n\pi} - \frac{(-\pi)(-\sin n\pi x)}{(n\pi)^2}\right\}_1^2$$

$$= \left\{\frac{-\pi x\cos(n\pi x)}{n\pi} + \frac{\pi\sin(n\pi x)}{(n\pi)^2}\right\}_0^1 + \left\{\frac{-(2\pi - \pi x)\cos(n\pi x)}{n\pi} - \frac{\pi\sin n\pi x}{(n\pi)^2}\right\}_1^2$$

$$= \left\{\frac{-\pi\cos(n\pi)}{n\pi} + \frac{\pi\sin(n\pi)}{(n\pi)^2} + \frac{\pi(0)\cos(0\,)}{n\pi} - \frac{\pi\sin(0)}{(n\pi)^2}\right\}$$

$$+ \left\{\frac{-(2\pi-2\pi)\cos(2n\pi\,)}{n\pi} - \frac{\pi\sin 2n\pi}{(n\pi)^2} + \frac{(2\pi-\pi)\cos(n\pi\,)}{n\pi} + \frac{\pi\sin\, n\pi}{(n\pi)^2}\right\}$$

$$= \left\{\frac{-\pi(-1)^n}{n\pi} + 0 + 0 - 0\right\} + \left\{0 + \frac{\pi(-1)^n}{n\pi} + 0 + 0\right\}$$

$$= \frac{-\pi(-1)^n}{n\pi} + \frac{\pi(-1)^n}{(n\pi)^2} \qquad = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

$$f(x) = \begin{cases}\pi x & 0 \le x \le 1\\ \pi(2-x) & 1 \le x \le 2\end{cases} = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n - 2}{\pi n^2}\cos(\,n\pi x) + (0)\sin(\,n\pi x)\right]$$

$$f(x) = \begin{cases}\pi x & 0 \le x \le 1\\ \pi(2-x) & 1 \le x \le 2\end{cases} = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2(-1)^n - 2}{\pi n^2}\cos(\,n\pi x)$$

****

Fourier Series in the interval $(-\pi, \pi)$ *or* $(-L, L)$

Fourier Series in the interval $(-\mathrm{L}, \mathrm{L})$

function neither even nor odd

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$a_0 = \frac{1}{L}\int_{-\mathrm{L}}^{\mathrm{L}} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-\mathrm{L}}^{\mathrm{L}} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-\mathrm{L}}^{\mathrm{L}} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Fourier Series in the interval $(-\pi, \pi)$

function neither even nor odd

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

Fourier Series in the interval $(-\mathrm{L}, \mathrm{L})$

f(x) is EVEN Function

OR Half Range Cosine Series $(0, \mathrm{L})$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

Where

$$a_0 = \frac{2}{L}\int_0^{\mathrm{L}} f(x)dx$$

$$a_n = \frac{2}{L}\int_0^{\mathrm{L}} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = 0$$

Fourier Series in the interval $(-\pi, \pi)$

f(x) is EVEN Function

OR Half Range Cosine Series $(0, \pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$b_n = 0$$

Fourier Series in the interval $(-\mathrm{L}, \mathrm{L})$

f(x) is ODD Function

OR Half Range Sine Series $(0, \mathrm{L})$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$a_0 = 0 \qquad a_n = 0$$

$$b_n = \frac{2}{L}\int_0^{\mathrm{L}} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Fourier Series in the interval $(-\pi, \pi)$

f(x) is ODD Function

OR Half Range Sine Series $(0, \pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = 0 \qquad a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

"The Only things that will stop you from fulfilling your dreams is you"

Remark :

Whenever the function is defined in the interval $(-\pi,\pi)$ or $(-L,L)$ we have to check if the function is **even** or **odd** or **neither even nor odd**

****

Example: Find the Fourier series for $f(x)=\pi^2-x^2$ in $(-\pi,\pi)$and hence deduce that

i) $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}\dots\dots=\frac{\pi^2}{12}$ $\quad$ $ii)$ $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}\dots\dots=\frac{\pi^2}{6}$ $\quad$ $iii)$ $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{5^2}\dots\dots=\frac{\pi^2}{8}$

Solution: $f(x)=\pi^2-x^2 \quad in\ (-\pi,\pi)$

$f(x)=\pi^2-x^2\dots\dots(1) \qquad f(-x)=\pi^2-(-x)^2=\pi^2-x^2\ \dots\dots(2)$

$f(x)=f(-x) \quad \therefore f(x)$ is even function

The Fourier series of $f(x)$ is given by $\quad f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$

$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx \qquad a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx \qquad b_n=0$

$\boldsymbol{a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx}$

$$a_0=\frac{2}{\pi}\int_0^{\pi}(\pi^2-x^2)dx$$

$$=\frac{2}{\pi}\left[\pi^2x-\frac{x^3}{3}\right]_0^{\pi} \qquad =\frac{2}{\pi}\left[\pi^2\pi-\frac{\pi^3}{3}-0+\frac{0^3}{3}\right]$$

$$=\frac{2}{\pi}\left[\frac{3\pi^3-\pi^3}{3}\right]$$

$$=\frac{2}{\pi}\left[\frac{2\pi^3}{3}\right]$$

$$=\frac{4\pi^2}{3}$$

$\boldsymbol{a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx}$

$$\mathrm{a_n}=\frac{2}{\pi}\int_0^{\pi}(\pi^2-x^2)\cos nx\,dx$$

$$a_n = \frac{2}{\pi}\left[\frac{(\pi^2 - x^2)\sin nx}{n} - \frac{(0-2x)(-\cos nx)}{n^2} + \frac{(-2)(-\sin nx)}{n^3}\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left\{\left[\frac{(\pi^2 - \pi^2)\sin n\pi}{n} - \frac{2\pi(\cos n\pi)}{n^2} + \frac{2(\sin n\pi)}{n^3}\right] - [0 - 0 + 0]\right\}$$

$$a_n = \frac{2}{\pi}\left\{\left[0 - \frac{2\pi(-1)^n}{n^2} + 0\right] - [0 - 0 + 0]\right\}$$

$$a_n = -\frac{4(-1)^n}{n^2}$$

$$\mathbf{b_n = 0}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\pi^2 - x^2 = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} -\frac{4(-1)^n}{n^2}\cos(nx) \quad \dots\dots\dots (1)$$

Put $x = 0$ in (1)

$$\pi^2 = \frac{2\pi^2}{3} - \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos(0)$$

$$\pi^2 = \frac{\pi^2}{3} - \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}$$

$$\pi^2 - \frac{\pi^2}{3} = -\left\{\frac{4(-1)^1}{1^2} + \frac{4(-1)^2}{2^2} + \frac{4(-1)^3}{3^2} + \frac{4(-1)^4}{4^2} + \frac{4(-1)^5}{5^2} + \dots\dots\dots\dots\right\}$$

$$\frac{\pi^2}{3} = -\left\{-\frac{4}{1^2} + \frac{4}{2^2} - \frac{4}{3^2} + \frac{4}{4^2} - \frac{4}{5^2} + \dots\dots\dots\dots\right\}$$

$$\frac{\pi^2}{3} = \frac{4}{1^2} - \frac{4}{2^2} + \frac{4}{3^2} - \frac{4}{4^2} + \frac{4}{5^2} \dots\dots\dots\dots$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \dots\dots\dots\dots \qquad \dots\dots\dots\dots\dots (A)$$

Put $x = \pi$ in (1)

$$0 = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} -\frac{4(-1)^n}{n^2}\cos(n\pi)$$

$$-\frac{2\pi^2}{3} = \sum_{n=1}^{\infty} -\frac{4(-1)^n}{n^2}(-1)^n$$

$$\frac{2\pi^2}{3} = \sum_{n=1}^{\infty} \frac{4(-1)^{2n}}{n^2}$$

$$\frac{2\pi^2}{3} = \frac{4(-1)^2}{1^2} + \frac{4(-1)^4}{2^2} + \frac{4(-1)^{12}}{3^2} + \frac{4(-1)^{16}}{4^2} + \frac{4(-1)^{20}}{5^2} + \dots\dots\dots\dots$$

$$\frac{2\pi^2}{(3)(4)} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots\dots\dots\dots$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots\dots\dots\dots \qquad \dots\dots\dots\dots\dots(B)$$

Adding (A)and (B)

$$\frac{\pi^2}{12} + \frac{\pi^2}{6} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \dots\dots \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots\dots\dots\dots$$

$$\frac{18\pi^2}{72} = \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \dots\dots\dots\dots$$

$$\frac{\pi^2}{4} = \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \dots\dots\dots\dots$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots\dots\dots(C)$$

****

Example: Find the Fourier series for $f(x) = x^2 \quad in\ (-\pi, \pi)$ and hence deduce that

i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \dots\dots = \frac{\pi^2}{12}$ $\quad ii)\ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \dots\dots = \frac{\pi^2}{6}$ $\quad iii)\ \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{5^2} \dots\dots = \frac{\pi^2}{8}$

Solution: $f(x) = x^2 \quad in\ (-\pi, \pi)$

$f(x) = x^2 \dots\dots (1) \qquad f(-x) = (-x)^2 = x^2 \dots\dots (2)$

$f(x) = f(-x) \quad \therefore f(x)$ is even function

The Fourier series of $f(x)$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx \qquad b_n = 0$

$\boldsymbol{a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx}$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx \quad = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} \quad = \frac{2}{\pi}\left[\frac{\pi^3}{3} - \frac{0^3}{3}\right] \quad = \frac{2\pi^2}{3}$$

$\boldsymbol{a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx}$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx$$

$$a_n = \frac{2}{\pi}\left[\frac{x^2 \sin nx}{n} - \frac{2x(-\cos nx)}{n^2} + \frac{2(-\sin nx)}{n^3}\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left\{\left[\frac{\pi^2 \sin n\pi}{n} - \frac{2\pi(-\cos n\pi)}{n^2} + \frac{2(-\sin n\pi)}{n^3}\right] - [0 - 0 + 0]\right\}$$

$$a_n = \frac{2}{\pi}\left\{\left[0 + \frac{2\pi(-1)^n}{n^2} + 0\right] - [0 - 0 + 0]\right\}$$

$$a_n = \frac{4(-1)^n}{n^2}$$

$\boldsymbol{b_n = 0}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos(nx) \quad \dots\dots\dots (1)$$

Put $x = 0$ in (1)

$$0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(0)$$

$$0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}$$

$$-\frac{\pi^2}{3} = \frac{4(-1)^1}{1^2} + \frac{4(-1)^2}{2^2} + \frac{4(-1)^3}{3^2} + \frac{4(-1)^4}{4^2} + \frac{4(-1)^5}{5^2} + \dots\dots\dots\dots$$

$$-\frac{\pi^2}{3} = \frac{-4}{1^2} + \frac{4}{2^2} - \frac{4}{3^2} + \frac{4}{4^2} - \frac{4}{5^2} + \dots\dots\dots\dots$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \dots\dots\dots\dots \qquad \dots\dots\dots\dots\dots (A)$$

Put $x = \pi$ in (1)

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(n\pi)$$

$$\pi^2 - \frac{\pi^2}{3} = \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} (-1)^n$$

$$\frac{2\pi^2}{3} = \sum_{n=1}^{\infty} \frac{4(-1)^{2n}}{n^2}$$

$$\frac{2\pi^2}{3} = \frac{4(-1)^2}{1^2} + \frac{4(-1)^4}{2^2} + \frac{4(-1)^{12}}{3^2} + \frac{4(-1)^{16}}{4^2} + \frac{4(-1)^{20}}{5^2} + \dots\dots\dots\dots$$

$$\frac{2\pi^2}{(3)(4)} = \frac{4}{1^2} + \frac{4}{2^2} + \frac{4}{3^2} + \frac{4}{4^2} + \frac{4}{5^2} + \dots\dots\dots\dots$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots\dots\dots\dots \qquad \dots\dots\dots\dots\dots (B)$$

Adding (A)and (B)

$$\frac{\pi^2}{12} + \frac{\pi^2}{6} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \dots\dots \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots\dots\dots\dots$$

$$\frac{18\pi^2}{72} = \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \quad \dots\dots\dots\dots$$

$$\frac{\pi^2}{4} = \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \quad \dots\dots\dots\dots$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \quad \dots\dots\dots\dots (C)$$

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Example: Find the Fourier series for $f(x) = x \quad in\ (-\pi, \pi)$and hence deduce that

i) $1 - \frac{1}{3} + \frac{1}{5} \quad \dots\dots = \frac{\pi}{4}$

Solution: $f(x) = x \quad in\ (-\pi, \pi)$

$f(x) = x \dots\dots (1)$ $\qquad\qquad$ $f(-x) = -x \quad \dots\dots (2)$

$f(x) = -f(-x)$ $\qquad\qquad$ $\therefore f(x)$ is odd function

The Fourier series of $f(x)$ is given by $\quad f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$a_0 = 0 \qquad\qquad a_n = \ 0 \qquad\qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx$

$$\boldsymbol{b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx}$$

$$\mathrm{b_n} = \frac{2}{\pi}\int_0^{\pi} x \sin nx\, dx$$

$$\mathrm{b_n} = \frac{2}{\pi}\left[\frac{x(-\cos nx)}{n} - \frac{1(-\sin nx)}{n^2}\right]_0^{\pi}$$

$$\mathrm{b_n} = \frac{2}{\pi}\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$\mathrm{b_n} = \frac{2}{\pi}\left\{\left[\frac{-\pi\cos n\pi}{n} + \frac{\sin n\pi}{n^2}\right] - \left[\frac{-0\cos 0}{n} + \frac{\sin 0}{n^2}\right]\right\}$$

$$\mathrm{b_n} = \frac{2}{\pi}\left\{\left[\frac{-\pi(-1)^n}{n} + 0\right] - [0 - 0\,]\right\}$$

$$\mathrm{b_n} = \frac{-2(-1)^n}{n}$$

The Fourier series of $f(x)$ is given by $\quad f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$x = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n} \sin(nx) \quad \dots\dots\dots (1)$$

$Put\ x = \frac{\pi}{2}\ in\ (1)$

$$\frac{\pi}{2} = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n} \sin\left(\frac{n\pi}{2}\right)$$

$$\frac{\pi}{2} = -2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\left(\frac{n\pi}{2}\right)$$

$$\frac{\pi}{-4} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sin\left(\frac{n\pi}{2}\right)$$

$$\frac{\pi}{-4} = \frac{(-1)^1}{1}\sin\left(\frac{\pi}{2}\right) + \frac{(-1)^2}{2}\sin(\pi) + \frac{(-1)^3}{3}\sin\left(\frac{3\pi}{2}\right) + \frac{(-1)^4}{4}\sin(4\pi) + \frac{(-1)^5}{5}\sin\left(\frac{5\pi}{2}\right) + \dots\dots\dots$$

$$\frac{\pi}{-4} = (-1)\,(1) + \frac{1}{2}(0) - \frac{1}{3}(-1) + \frac{1}{4}(0) - \frac{1}{5}(1) \dots\dots\dots\dots\dots\dots$$

$$\frac{\pi}{-4} = -1 + \frac{1}{3} - \frac{1}{5} \dots\dots\dots\dots\dots\dots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} \dots\dots\dots\dots\dots\dots$$

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Example: Find the Fourier series for $f(x) = \begin{cases} \pi + x\ ;\ -\pi \le x \le 0 \\ \pi - x\ ;\ 0 \le x \le \pi \end{cases}$

$and\ f(x + 2\pi) = f(x)$

Solution: $Interval\ is\ (-\pi\ \pi)$ $\therefore$ Check even or odd

Given $f(x) = \begin{cases} \pi + x\ ;\ -\pi \le x \le 0 \\ \pi - x\ ;\ \quad 0 \le x \le \pi \end{cases}$

put $x = -x$

$f(-x) = \begin{cases} \pi - x\ ;\ \quad -\pi \le -x \le 0 \\ \pi - (-x)\ ;\ \quad 0 \le -x \le \pi \end{cases}$

$$f(-x) = \begin{cases} \pi - x \ ; & \pi \geq x \geq 0 \\ \pi + x \ ; & 0 \geq x \geq -\pi \end{cases}$$

$f(x) = f(-x)$ $\quad\therefore$ function is even

The Fourier series of $f(x)$ is given by $\quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx \qquad\qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx \qquad\qquad b_n = 0$

$$\boldsymbol{a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} (\pi - x)dx$$

$$a_0 = \frac{2}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi}$$

$$a_0 = \frac{2}{\pi}\left\{\left[\pi\pi - \frac{\pi^2}{2}\right] - \left[0 - \frac{0}{2}\right]\right\}$$

$$a_0 = \frac{2}{\pi}\frac{\pi^2}{2} = \pi$$

$$\boldsymbol{a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos nx\, dx$$

$$a_n = \frac{2}{\pi}\left[\frac{(\pi - x)\sin nx}{n} - \frac{(0-1)(-\cos nx)}{n^2}\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left[\frac{(\pi - x)\sin nx}{n} - \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left\{\frac{(\pi - \pi)\sin n\pi}{n} - \frac{\cos n\pi}{n^2} - \frac{(\pi - 0)\sin 0}{n} + \frac{\cos 0}{n^2}\right\}$$

$$a_n = \frac{2}{\pi}\left\{0 - \frac{(-1)^n}{n^2} - 0 + \frac{1}{n^2}\right\}$$

$$a_n = \frac{2}{\pi}\left[\frac{1 - (-1)^n}{n^2}\right]$$

The Fourier series of $f(x)$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi}\left[\frac{1-(-1)^n}{n^2}\right]\cos nx$$

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Example: Find the half range cosine series for $f(x) = x^2$ in $0 < x < \pi$

Solution: The Fourier half range cosine for $f(x)$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$ $\qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$ $\qquad b_n = 0$

$\boldsymbol{a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx}$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^3}{3} - \frac{0^3}{3}\right] = \frac{2\pi^2}{3}$$

$\boldsymbol{a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx}$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx$$

$$a_n = \frac{2}{\pi}\left[\frac{x^2 \sin nx}{n} - \frac{2x(-\cos nx)}{n^2} + \frac{2(-\sin nx)}{n^3}\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left\{\left[\frac{\pi^2 \sin n\pi}{n} - \frac{2\pi(-\cos n\pi)}{n^2} + \frac{2(-\sin n\pi)}{n^3}\right] - [0-0+0]\right\}$$

$$a_n = \frac{2}{\pi}\left\{\left[0 + \frac{2\pi(-1)^n}{n^2} + 0\right] - [0-0+0]\right\}$$

$$a_n = \frac{4(-1)^n}{n^2}$$

$\boldsymbol{b_n = 0}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$f(x) = x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos(nx)$$

****

Example: Find the half range sine series for $f(x) = x \quad in \quad 0 < x < \pi$

Solution: The half range sine series for $f(x)$ is given by $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$a_0 = 0 \qquad a_n = 0 \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$

$$\boldsymbol{b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx$$

$$b_n = \frac{2}{\pi}\left[\frac{x(-\cos nx)}{n} - \frac{1(-\sin nx)}{n^2}\right]_0^{\pi}$$

$$b_n = \frac{2}{\pi}\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$b_n = \frac{2}{\pi}\left\{\left[\frac{-\pi\cos n\pi}{n} + \frac{\sin n\pi}{n^2}\right] - \left[\frac{-0\cos 0}{n} + \frac{\sin 0}{n^2}\right]\right\}$$

$$b_n = \frac{2}{\pi}\left\{\left[\frac{-\pi(-1)^n}{n} + 0\right] - [0 - 0]\right\}$$

$$b_n = \frac{-2(-1)^n}{n}$$

The Fourier series of $f(x)$ is given by $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$\boldsymbol{x = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n} \sin(nx)}$$

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Example: Find the half range cosine series for $f(x) = \pi x - x^2 \ in \ 0 < x < \pi$

Solution: The Fourier half range cosine for $f(x)$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x) dx \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx \qquad b_n = 0$

$$\boldsymbol{a_0 = \frac{2}{\pi}\int_0^{\pi} f(x) dx}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \pi x - x^2 dx$$

$$= \frac{2}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi\pi^2}{2} - \frac{\pi^3}{3} - 0 + 0\right] = \frac{2}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3}\right]$$

$$= \frac{\pi^2}{3}$$

$$\boldsymbol{a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx}$$

$$\mathrm{a_n} = \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\cos nx\, dx$$

$$\mathrm{a_n} = \frac{2}{\pi}\left[\frac{(\pi x - x^2)\sin nx}{n} - \frac{(\pi - 2x)(-\cos nx)}{n^2} + \frac{(-2)(-\sin nx)}{n^3}\right]_0^{\pi}$$

$$\mathrm{a_n} = \frac{2}{\pi}\left\{\left[\frac{(\pi^2 - \pi^2)\sin n\pi}{n} - \frac{(\pi - 2\pi)(-\cos n\pi)}{n^2} + \frac{(-2)(-\sin n\pi)}{n^3}\right] - \left[\frac{(0)\sin 0}{n} - \frac{(\pi - 0)(-\cos 0)}{n^2} + \frac{(-2)(-\sin 0)}{n^3}\right]\right\}$$

$$\mathrm{a_n} = \frac{2}{\pi}\left\{\left[0 - \frac{\pi(-1)^n}{n^2} + 0\right] - \left[0 + \frac{\pi}{n^2} + 0\right]\right\}$$

$$\mathrm{a_n} = \frac{2}{\pi}\left[-\frac{\pi(-1)^n}{n^2} - \frac{\pi}{n^2}\right]$$

$$\mathrm{a_n} = \frac{-2[(-1)^n + 1]}{n^2}$$

$$\mathbf{b_n = 0}$$

$$\mathrm{f(x)} = \frac{\mathrm{a_0}}{2} + \sum_{\mathrm{n=1}}^{\infty} \mathrm{a_n \cos nx}$$

$$\pi x - x^2 = \frac{1}{2}\frac{\pi^2}{3} + \sum_{\mathrm{n=1}}^{\infty}\frac{-2[(-1)^n + 1]}{n^2}\cos(\mathrm{nx}) = \frac{\pi^2}{6} + \sum_{\mathrm{n=1}}^{\infty}\frac{-2[(-1)^n + 1]}{n^2}\cos(\mathrm{nx})$$

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