Master Theorem

- How do you solve a recurrence of the form

$$T(n) = aT\left(\frac{n}{b}\right) + O(n^d)$$

We will use the master theorem.

Summation Lemma

Consider the summation

$$\sum_{k=0}^{n} r^k$$

It behaves differently for different values of r.

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It behaves differently for different values of r.

If $r < 1$ then this sum converges. This means that the sum is bounded above by some constant c. Therefore

$$if \;\; r < 1, \qquad then \sum_{k=0}^{n} r^k < c \;\; for \;\; all \;\; n \;\; so \;\; \sum_{k=0}^{n} r^k \in O(1)$$

Summation Lemma

Consider the summation

$$\sum_{k=0}^{n} r^k$$

It behaves differently for different values of r.

$\boxed{\text{If } r = 1}$ then this sum is just summing 1 over and over n times. Therefore

$$if \;\; r = 1, \qquad then \sum_{k=0}^{n} r^k = \sum_{k=0}^{n} 1 = n + 1 \;\; \epsilon \;\; O(n)$$

Summation Lemma

Consider the summation

$$\sum_{k=0}^{n} r^k$$

It behaves differently for different values of r.

$\boxed{\text{If } r > 1}$ then this sum is exponential with base r.

$$if\ \ r > 1, then\ \sum_{k=0}^{n} r^k < cr^n\ for\ all\ n, \qquad so \sum_{k=0}^{n} r^k \in\ O(r^n) \qquad \left(c > \frac{r}{r-1}\right)$$

Summation Lemma

Consider the summation

$$\sum_{k=0}^{n} r^k$$

It behaves differently for different values of r.

$$\sum_{k=0}^{n} r^k \in \begin{cases} O(1) & if \;\; r < 1 \\ O(n) & if \;\; r = 1 \\ O(r^n) & if \;\; r > 1 \end{cases}$$

Master Theorem

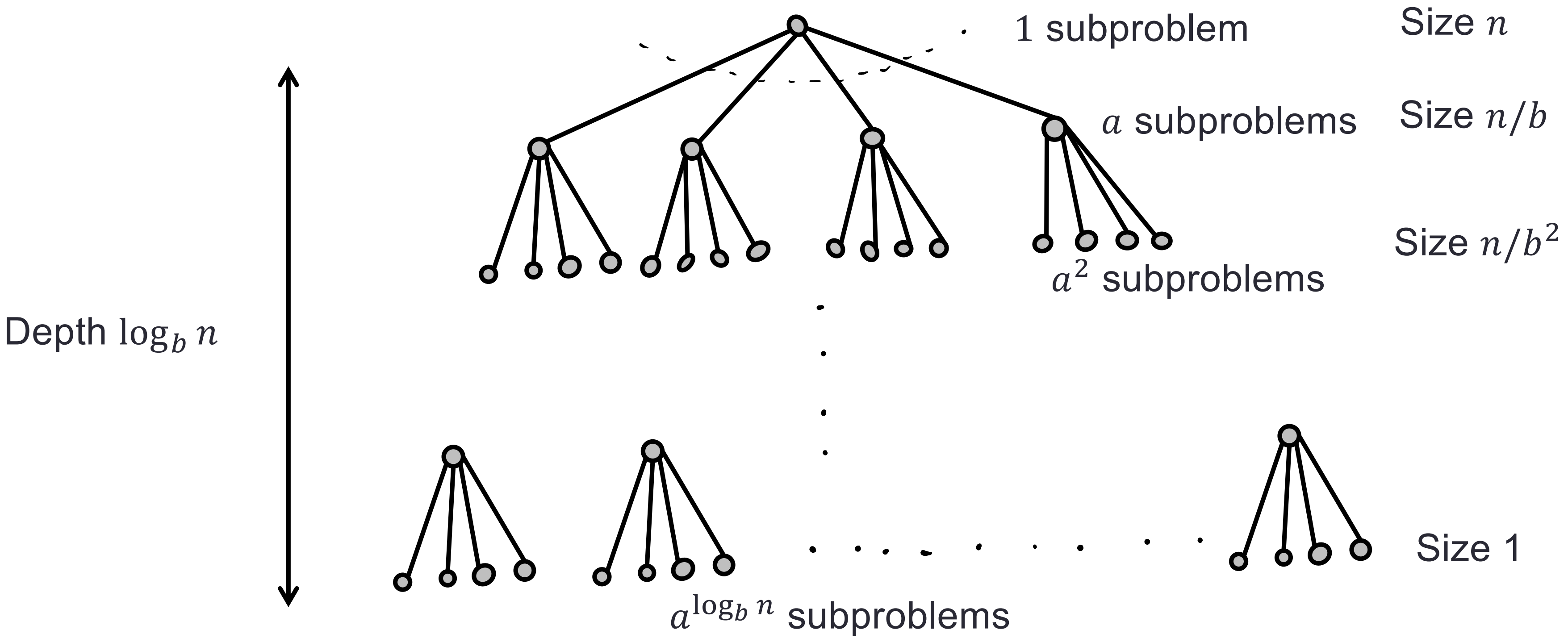
Master Theorem: If $T(n) = aT(n/b) + O(n^d)$ for some constants $a > 0, b > 1, d \geq 0$,

Then

$$T(n) \in \begin{cases} O\left(n^d\right) & if \;\; a < b^d \\ O\left(n^d \log n\right) & if \;\; a = b^d \\ O\left(n^{\log_b a}\right) & if \;\; a > b^d \end{cases}$$

Master Theorem: Solving the recurrence

$$T(n) = aT(n/b) + O(n^d)$$



Master Theorem: Solving the recurrence

After k levels, there are a^k subproblems, each of size n/b^k.

So, during the kth level of recursion, the time complexity is

$$O\left(\left(\frac{n}{b^k}\right)^d\right)a^k = O\left(a^k\left(\frac{n}{b^k}\right)^d\right)$$

$$= O\left(n^d\left(\frac{a}{b^d}\right)^k\right)$$

Master Theorem: Solving the recurrence

After k levels, there are a^k subproblems, each of size n/b^k.

So, during the kth level, the time complexity is $O\left(\left(\frac{n}{b^k}\right)^d\right)a^k = O\left(a^k\left(\frac{n}{b^k}\right)^d\right)$

$$= O\left(n^d\left(\frac{a}{b^d}\right)^k\right)$$

After $\log_b n$ levels, the subproblem size is reduced to 1, which usually is the size of the base case.

So the entire algorithm is a sum of each level.

$$T(n) = O\left(n^d \sum_{k=0}^{\log_b n}\left(\frac{a}{b^d}\right)^k\right)$$

Master Theorem: Proof

$$T(n) = O\left(n^d \sum_{k=0}^{\log_b n} \left(\frac{a}{b^d}\right)^k\right)$$

Case 1: $a < b^d$

Then we have that $\frac{a}{b^d} < 1$ and the series converges to a constant so

$$T(n) = O(n^d)$$

Master Theorem: Proof

$$T(n) = O\left(n^d \sum_{k=0}^{\log_b n} \left(\frac{a}{b^d}\right)^k \right)$$

Case 2: $a = b^d$

Then we have that $\frac{a}{b^d} = 1$ and so each term is equal to 1

$$T(n) = O(n^d \log_b n)$$

Master Theorem: Proof

$$T(n) = O\left(n^d \sum_{k=0}^{\log_b n} \left(\frac{a}{b^d}\right)^k\right)$$

Case 2: $a > b^d$

Then the summation is exponential and grows proportional to its last term $\left(\frac{a}{b^d}\right)^{\log_b n}$ so

$$T(n) = O\left(n^d \left(\frac{a}{b^d}\right)^{\log_b n}\right) = O\left(n^{\log_b a}\right)$$

Master Theorem

Theorem: If $T(n) = aT(n/b) + O(n^d)$ for some constants $a > 0, b > 1, d \geq 0$,

Then

$$T(n) \in \begin{cases} O(n^d) & if \;\; a < b^d \\ O(n^d \log n) & if \;\; a = b^d \\ O(n^{\log_b a}) & if \;\; a > b^d \end{cases}$$

Top-heavy

Steady-state

Bottom-heavy

Master Theorem Applied to Multiply

$$T(n) \epsilon \begin{cases} O(n^d) & if \;\; a < b^d \\ O(n^d \log n) & if \;\; a = b^d \\ O(n^{\log_b a}) & if \;\; a > b^d \end{cases}$$

The recursion for the runtime of Multiply is
T(n) = 4T(n/2) + cn

So we have that a=4, b=2, and d=1. In this case, $a > b^d$ so

$$T(n) \epsilon O\left(n^{\log_2 4}\right) = O(n^2)$$

Not any improvement of grade-school method.

Master Theorem Applied to MultiplyKS

$$T(n) \epsilon \begin{cases} O(n^d) & if \;\; a < b^d \\ O(n^d \log n) & if \;\; a = b^d \\ O(n^{\log_b a}) & if \;\; a > b^d \end{cases}$$

The recursion for the runtime of Multiply is
T(n) = 3T(n/2) + cn

So we have that a=3, b=2, and d=1. In this case, $a > b^d$ so

$$T(n) \epsilon O\left(n^{\log_2 3}\right) = O(n^{1.58})$$

An improvement on grade-school method!!!!!!

Poll: What is the fastest known integer multiplication time?

- $O\left(n^{log3}\right)$
- $O(n\ logn\ (\log(logn)\ )^2)$
- $O(n\ logn\ 2\text{^}\{\log^* n\})$
- $O(n \log n)$
- O(n)

Poll: What is the fastest known integer multiplication time? All have/will be correct

- $O\left(n^{log3}\right)$ Kuratsuba
- $O(n\ logn\ \ \log\log n\ )$ Schonhage-Strassen, 1971
- $O(n\ logn\ 2\text{^}\{c\log^* n\})$ Furer, 2007
- $O(n\log n)$ Harvey and van der Hoeven, 2019
- O(n), you, tomorrow?

Can we do better than $n^{1.58}$?

- Could any multiplication algorithm have a faster asymptotic runtime than $\Theta(n^{1.58})$?
- Any ideas?????

Can we do better than $n^{1.58}$?

- What if instead of splitting the number in half, we split it into thirds.

- x= | x_L | x_M | x_R |
- y= | y_L | y_M | y_R |

Can we do better than $n^{1.58}$?

- What if instead of splitting the number in half, we split it into thirds.

- $x = 2^{2n/3}x_L + 2^{n/3}x_M + x_R$
- $y = 2^{2n/3}y_L + 2^{n/3}y_M + y_R$

Multiplying trinomials

- $(ax^2 + bx + c)(dx^2 + ex + f)$

Multiplying trinomials

- $(ax^2 + bx + c)(dx^2 + ex + f)$
  $= adx^4 + (ae + bd)x^3 + (af + be + cd)x^2 + (bf + ce)x$
  $+ cf$

9 multiplications means 9 recursive calls.

Each multiplication is 1/3 the size of the original.

Multiplying trinomials

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$$T(n) = 9T\left(\frac{n}{3}\right) + O(n)$$

Multiplying trinomials

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$$T(n) = 9T\left(\frac{n}{3}\right) + O(n)$$

$$T(n) \epsilon \begin{cases} O(n^d) & if \;\; a < b^d \\ O(n^d \log n) & if \;\; a = b^d \\ O(n^{\log_b a}) & if \;\; a > b^d \end{cases}$$

a=9
b=3
d=1

$9 > 3^1$

$T(n) = O(n^{\log_3 9})$

$T(n) = O(n^2)$

Multiplying trinomials

- $(ax^2 + bx + c)(dx^2 + ex + f)$
  $= adx^4 + (ae + bd)x^3 + (af + be + cd)x^2 + (bf + ce)x + cf$
- There is a way to reduce from 9 multiplications down to just 5!!!
- Then the recursion becomes
- $T(n) = 5T(n/3)$ + O(n)
- So by the master theorem

Multiplying trinomials

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- There is a way to reduce from 9 multiplications down to just 5!!!

- Then the recursion becomes
- $T(n) = 5T(n/3)$ + O(n)

- So by the master theorem T(n)=O($n^{\log_3 5}$) $= O(n^{1.43})$

Dividing into k subproblems

- What happens if we divide into k subproblems each of size n/k.

- $(a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots a_1x + a_0)(b_{k-1}x^{k-1} + b_{k-2}x^{k-2} + \cdots b_1x + b_0)$

- How many terms are there? (multiplications.)

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- $(a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots a_1x + a_0)(b_{k-1}x^{k-1} + b_{k-2}x^{k-2} + \cdots b_1x + b_0)$
- How many terms are there? (multiplications.)
- There are k^2 multiplications. The recursion is

$$T(n) = k^2T\left(\frac{n}{k}\right) + O(n) \dots \dots \dots a = k^2, b = k, d = 1$$

$$T(n) = O(n^{\log_k k^2}) = O(n^2)$$

Cook-Toom algorithm

- In fact, if you split up your number into k equally sized parts, then you can combine them with 2k-1 multiplications instead of the k^2 individual multiplications.

- This means that you can get an algorithm that runs in

- $T(n) = (2k-1)T(n/k)$ + O(n)

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- This means that you can get an algorithm that runs in

- $T(n) = (2k-1)T(n/k)$ + O(n)

- $T(n) = O\left(n^{\frac{\log(2k-1)}{\log k}}\right)$ time!!!!

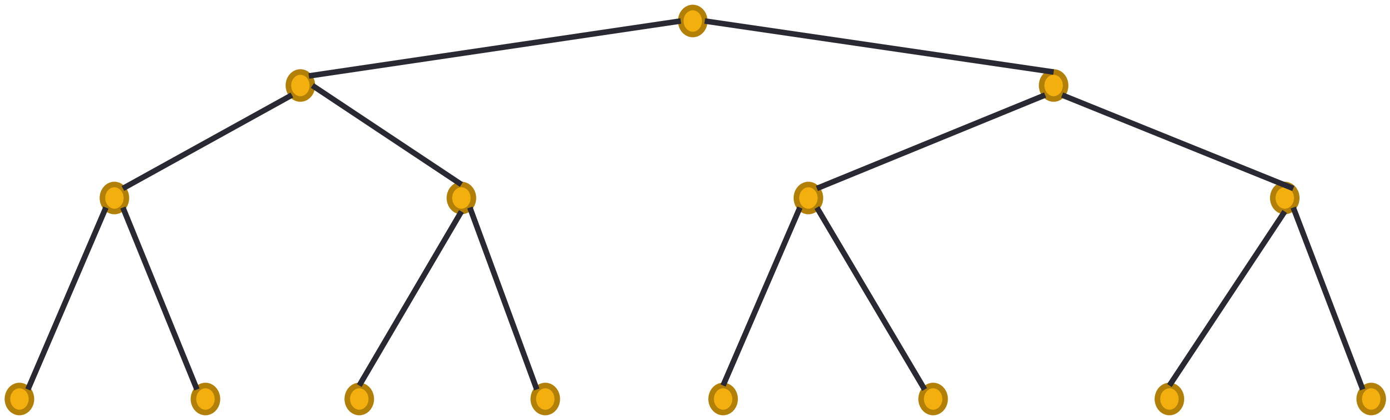
Cook-Toom algorithm

$T(n) = (2k - 1)T(n/k)$ + O(n)

- $T(n) = O\left(n^{\frac{\log 2k-1}{\log k}}\right)$ time.
- So we can have a near-linear time algorithm if we take k to be sufficiently large. The O(n) term in the recursion takes a lot of time the bigger k gets. So is it worth it to make k very large?

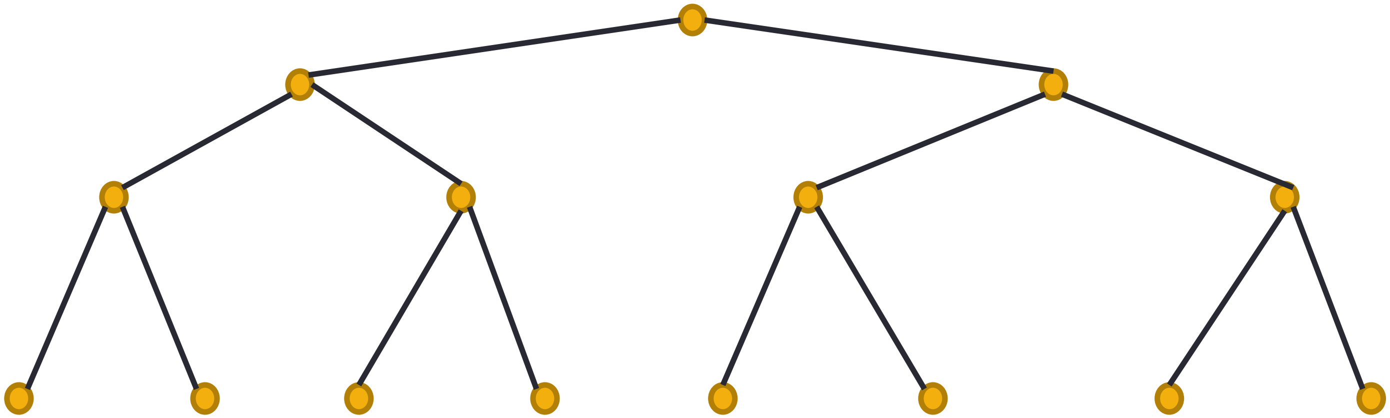
Divide and Conquer Trees

- Let's say we have a full and balanced binary tree (all parents have two children and all leaves are on the bottom level.)



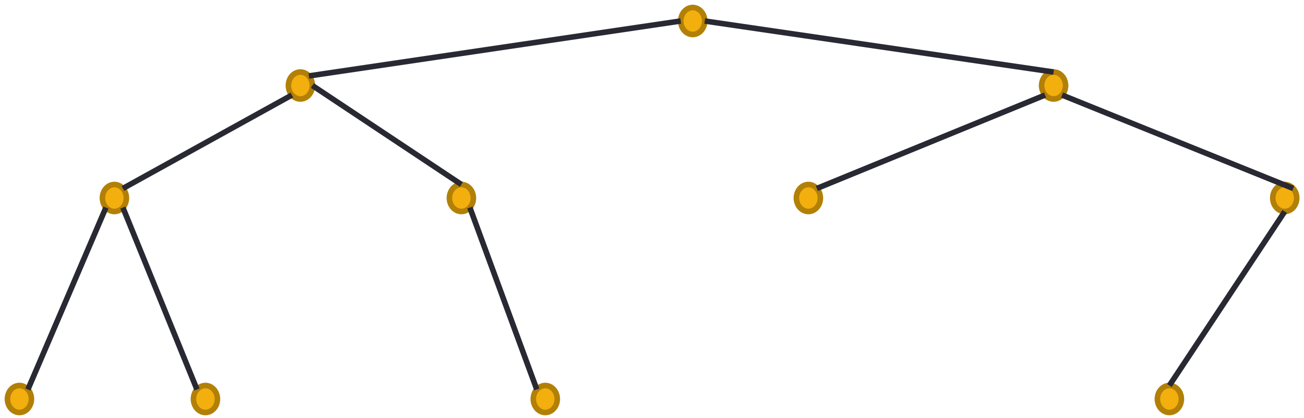
Divide and Conquer Trees

- Notice that each child's subtree is half of the problem so we get a nice divide and conquer structure.



Divide and Conquer Trees

- If the tree is uneven, we can still use the same strategy but we need to take a bit of care when calculating runtime.

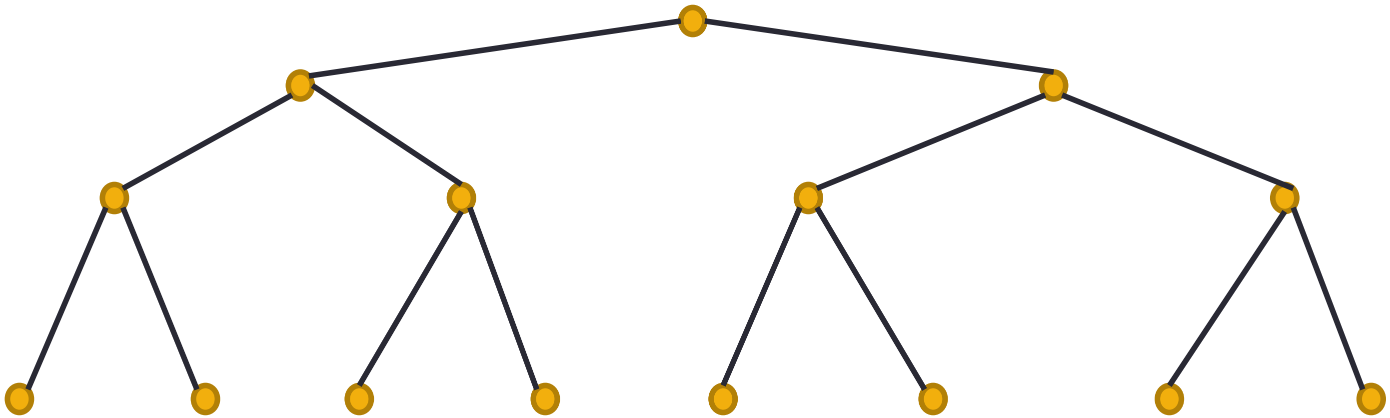


Least common ancestor

- Given a binary tree with n vertices, we wish to compute $LCA(x, y)$ for each pair of vertices x, y.
- $LCA(x, y)$ is the least common ancestor of x and y. Or in other words, the "youngest" common ancestor of x and y.
- For example, the LCA of me and my brother is our parent. The LCA of me and my uncle is my grandparent (his parent.) A vertex can be its own ancestor so the LCA of me and my father is my father.

Least common ancestor

- What pairs of vertices will have the root r as their least common ancestor?



Least common ancestor

- What pairs of vertices will have the root r as their least common ancestor?
- For each vertex v, set $lca(v, r) = r$.
- For each pair of vertices u, v such that u is in the left subtree and v is in the right subtree, set $lca(u, v) = r$.
- Now what? Are we done?
- Recurse on the left and right subtrees!!!!!

Pseudocode

```
Def LCA(r):
    Lsubtree = explore(r.lc)
    Rsubtree = explore(r.rc)
    for all vertices u in Lsubtree:
        lca(u, r) = r
    for all vertices v in Rsubtree:
        lca(r, v) = r
    for all vertices u in Lsubtree:
        for all vertices v in Rsubtree:
            lca(u, v) = r
    LCA(r.lc)
    LCA(r.rc)
```

Pseudocode (runtime)

Def **LCA**(r):

Lsubtree = **explore**(r.lc)

Rsubtree = **explore**(r.rc)

for all vertices u in Lsubtree:

$lca(u, r) = r$

for all vertices v in Rsubtree:

$lca(r, v) = r$

for all vertices u in Lsubtree:

for all vertices v in Rsubtree:

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LCA(r.lc)

LCA(r.rc)

If the binary tree is balanced, then each recursive call is of size $\frac{n-1}{2}$ or roughly half.

How long does the non-recursive part take?

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  LCA(r.lc)

  LCA(r.rc)

If the binary tree is balanced, then each recursive call is of size $\frac{n-1}{2}$ or roughly half.

How long does the non-recursive part take?

$$T(n) = 2T\left(\frac{n-1}{2}\right) + O(\mathrm{n}^2)$$

Using the master theorem with a=2, b=2, d=2,

$$T(n) = O(n^2)$$

Pseudocode (runtime uneven)

Def **LCA**(r):

  Lsubtree = **explore**(r.lc)

  Rsubtree = **explore**(r.rc)

  for all vertices u in Lsubtree:

    $lca(u, r) = r$

  for all vertices v in Rsubtree:

    $lca(r, v) = r$

  for all vertices u in Lsubtree:

    for all vertices v in Rsubtree:

      $lca(u, v) = r$

  LCA(r.lc)

  LCA(r.rc)

If the binary tree is uneven then the runtime recurrence is

$$T(n) = T(L) + T(R) + O(LR)$$

Where L is the size of the left subrtree and R is the size of the right subtree.

What do you think the total runtime will be? Take a guess and we can check it!!!

Uneven DC runtime

- $T(n) = T(L) + T(\mathrm{R}) + \mathrm{O}(\mathrm{LR})$
- We guess that it would take $O(n^2)$. So let's try to prove this using induction.
- Claim: $T(n) \leq cn^2$ for all $n \geq 1$ and for some constant c that is bigger than $T(1)$ and bigger than the coefficient in the $O(LR)$ term.

Uneven DC runtime

- Base case. $T(1) < c(1^2)$. True by choice of c.
- Suppose that for some $n > 1$, $T(k) < ck^2$ for all k such that $1 \leq k < n$.
- Then

$$T(n) < T(L) + T(R) + cLR \leq cL^2 + cR^2 + cLR$$
$$< cL^2 + cR^2 + 2cLR = c(L+R)^2 = c(n-1)^2 < cn^2$$

Make Heap

- Problem: Given a list of n elements, form a heap containing all elements.

Divide and conquer strategy

- Assume $n = 2^k - 1$. (Add blank elements if needed)
- Divide the list into two lists of size $\frac{n-1}{2}$ and a left-over element
- Make heaps with both (in sub-trees of root)
- Put left-over element at root.
- “Trickle down” top element to reinstate heap property

Time analysis

- To solve one problem, we solve two problems of half the size, and then spend constant time per depth of the tree.

- T(n) =   T(   ) + O(   )

Time analysis

- To solve one problem, we solve two problems of half the size, and then spend constant time per depth of the tree.

- $T(n) = 2\,T(\ n/2\ ) + O(\log n\ )$
- Doesn't fit master theorem.

Time analysis: sandwiching

- To solve one problem, we solve two problems of half the size, and then spend constant time per depth of the tree.

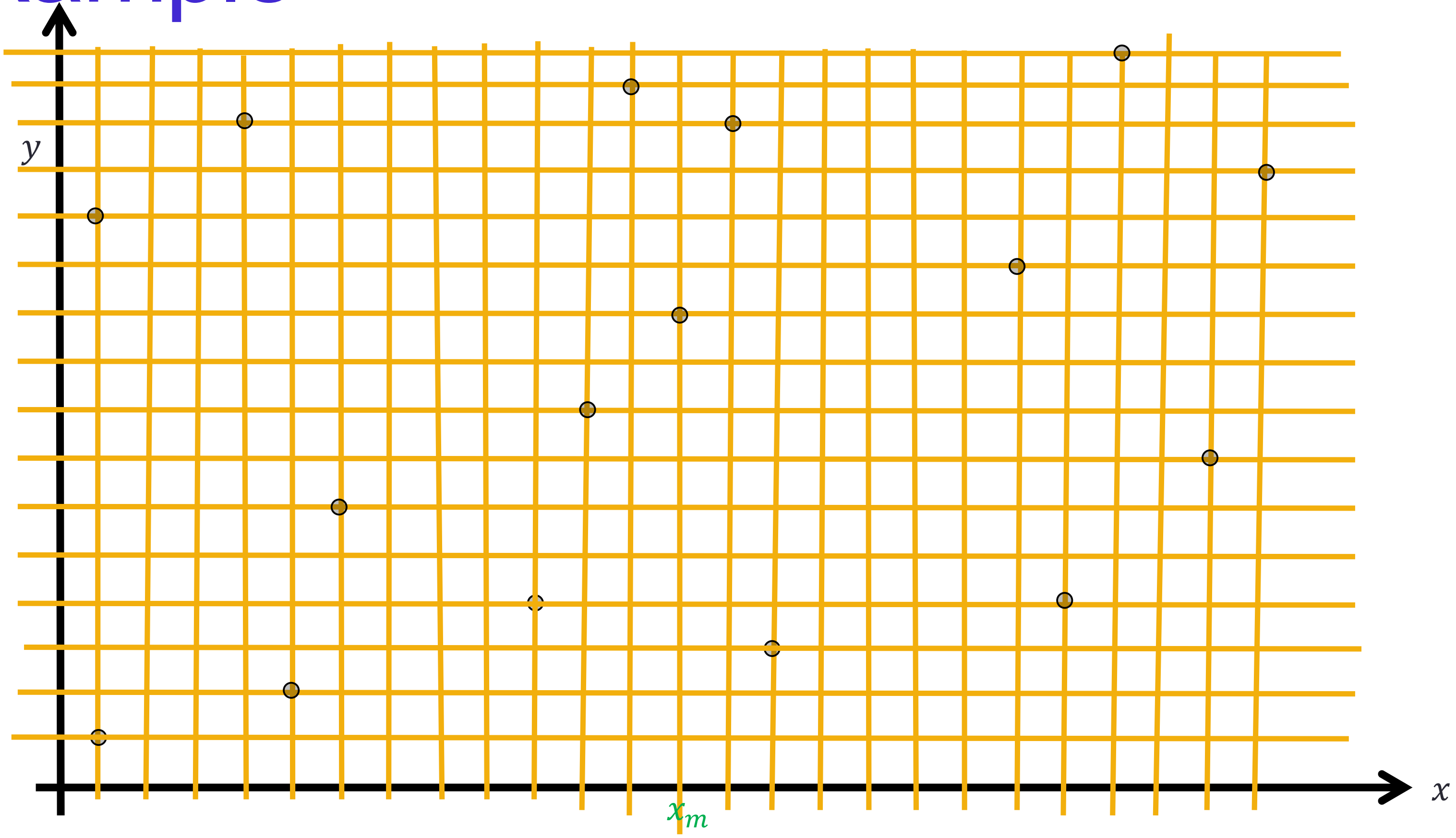
- T(n) = 2 T(n/2) + O(log n)
- Define L(n) =2 T(n/2) + O(1), H(n) = 2T(n/2) +$O\left(n^{\left\{\frac{1}{2}\right\}}\right)$
- L(n) < T(n) < H(n)
- Apply Master Theorem: Both L(n) and H(n) are O(n),
- So T(n) is O(n)

minimum distance

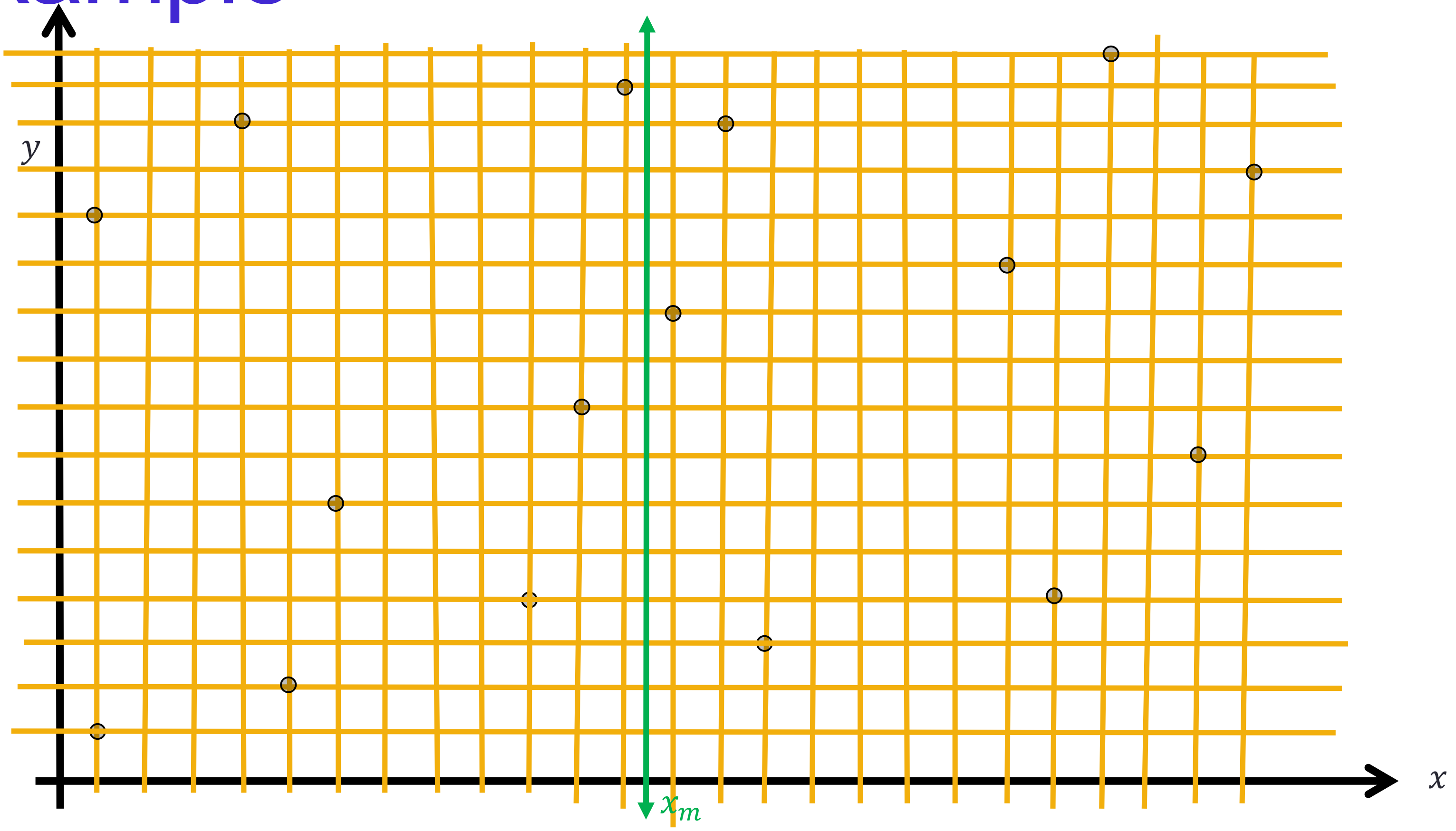
- Given a list of coordinates, $[(x_1, y_1), \ldots, (x_n, y_n)]$, find the distance between the closest pair.

- Brute force solution?
- min = 0
- for i from 1 to n-1:
  - for j from i+1 to n:
    - if min > distance$((x_i, y_i), (x_j, y_j))$
- return min

Example



Example



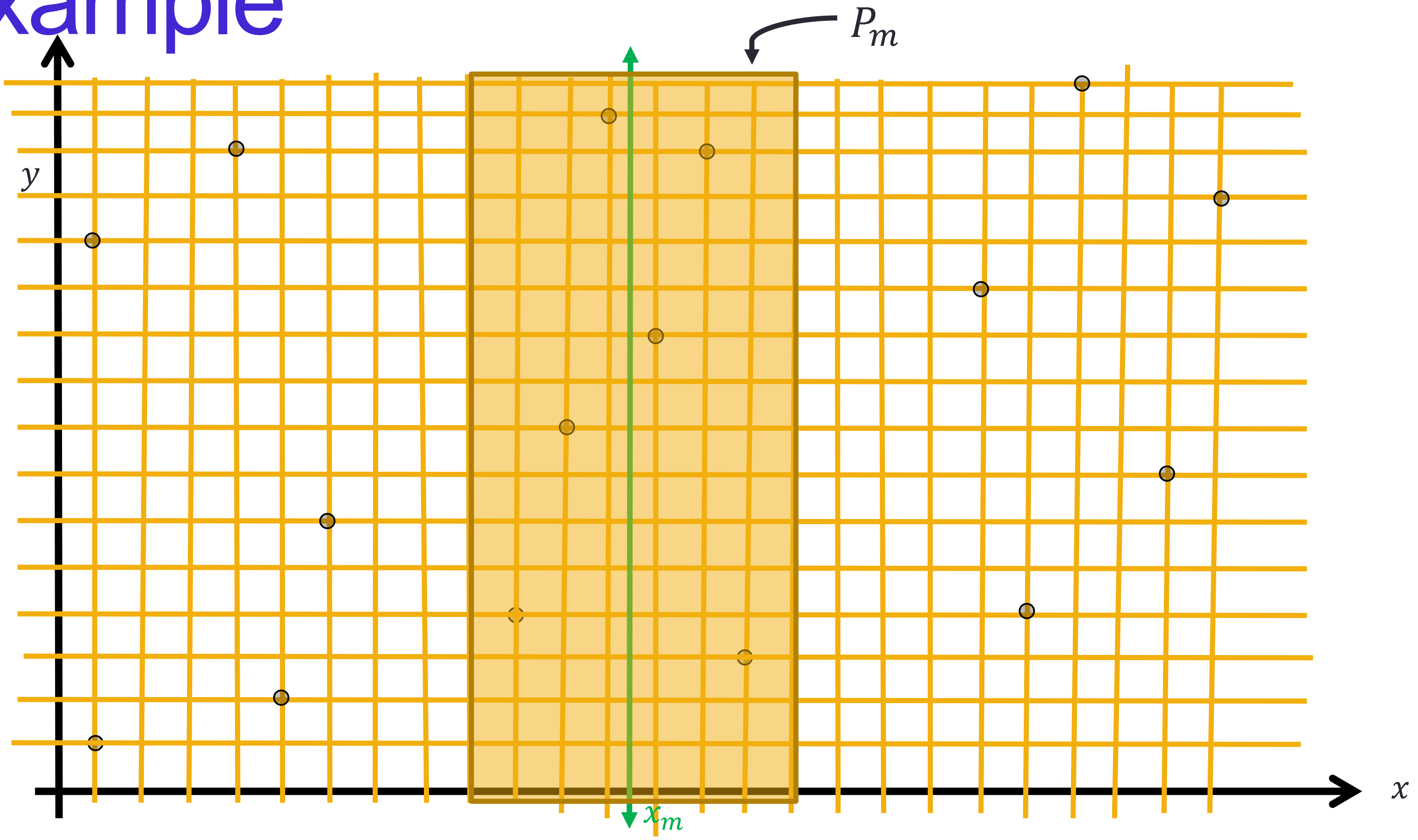
Divide and conquer

- Partition the points by x, according to whether they are to the left or right of the median
- Recursively find the minimum distance points on the two sides.
- Need to compare to the smallest “cross distance” between a point on the left and a point on the right
- Only need to look at “close” points

Combine

- How will we use this information to find the distance of the closest pair in the whole set?
- We must consider if there is a closest pair where one point is in the left half and one is in the right half.
- How do we do this?
- Let $d = \min(d_L, d_R)$ and compare only the points (x_i, y_i) such that $x_m - d \leq x_i$ and $x_i \leq x_m + d$.

Example



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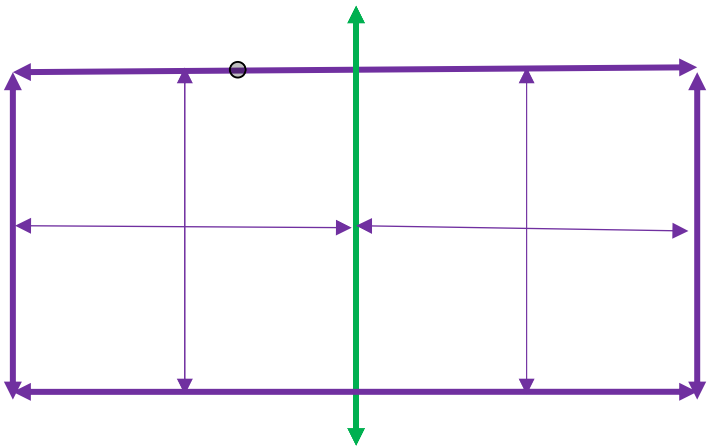
- Worst case, how many points could this be?

Combine step

- Given a point $(x, y) \in P_m$, let's look in a $2d\times d$ rectangle with that point at its upper boundary:



- There could not be more than 8 points total because if we divide the rectangle into 8 $\frac{d}{2}\times\frac{d}{2}$ squares then there can never be more than one point per square.
- Why???

Combine step

- So instead of comparing (x, y) with every other point in P_m we only have to compare it with at most a constant **c** points lower than it (smaller y)
- To gain quick access to these points, let's sort the points in P_m by y values.
- The points above must be in the **c** points before our current point in this sorted list

- Now, if there are k vertices in P_m we have to sort the vertices in $O(k\log k)$ time and make at most ck comparisons in $O(k)$ time for a total combine step of $O(k \log k)$.

- But we said in the worst case, there are n vertices in P_m and so worst case, the combine step takes $O(n \log n)$ time.

Time analysis

- But we said in the worst case, there are n vertices in P_m and so worst case, the combine step takes $O(n \log n)$ time.

- Runtime recursion:

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n \log n)$$

This is T(n) = O(n (log n)^2)

Pre-processing : Sort by both x and y, keep pointers between sorted lists Maintain sorting in recursive calls reduces to T(n) =2 T(n/2) +O(n), so T(n) is O(n log n)