Image Compression using EVD and SVD

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I. Motivation

The main objective of this experiment is to perform *Eigenvalue Decomposition*(EVD) and *Singular Value Decomposition*(SVD) on a greyscale image. Then, reconstruct the image by taking rank-one matrices corresponding to top k eigenvalues/singular values(in descending order of magnitude). We shall plot the frobenius norm between the original and reconstructed images for various values of k, and compare the plots obtained in case of EVD/SVD. And then try one of the methods on a color image too.
Let us assume the given matrix as A, EVD/SVD would decompose A as $U\Sigma V$ where Σ is a diagonal matrix.

Fig. 1. *Original greyscale image*

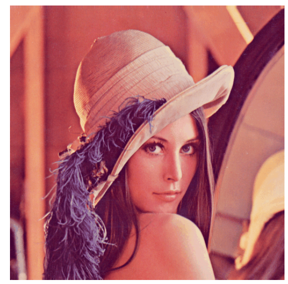

Fig. 2. *Original color image*

II. EVD

Let A be a real square matrix. $\Sigma = diag(\lambda_1, \lambda_2, \lambda_3, ...)$ such that $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq ....$ are eigenvalues of A. And E is a matrix whose columns are eigenvectors corresponding to eigenvalues $\lambda_1, \lambda_2, \lambda_3, ...$ in the exact order.

$$A.E = \Sigma.E$$

Assuming E is invertible, we have;

$$A = E.\Sigma.E^{-1}$$

We would approximate A by taking first k eigenvalues;

$$A = E[:, : k].\Sigma[: k, : k].E^{-1}[: k, :]$$

Note that some of the eigenvalues could be complex conjugates. We shall reconstruct images for only those values of 'k' such that all conjugate pairs are considered and we will get a real-valued image.

III. SVD

Let A be any real $m \times n$ matrix. We can write A as:

$$A = U.\sigma.V^T$$

where:
σ - $m \times n$ diagonal matrix whose principal diagonal consists of positive square root of eigenvalues of $A.A^T$ or $A^T.A$ in descending order.(note that both of them have the same non-zero eigenvalues. Also the eigenvalues are both real and positive given that $A.A^T/A^T.A$ are self-adjoint operators) and positive semi-definite).

U - $m{\times}m$ matrix whose columns are normalised eigenvectors of $A.A^T$ corresponding to eigenvectors in descending order.

V - $n{\times}n$ matrix whose columns are normalised eigenvectors of $A^T.A$ corresponding to eigenvectors in descending order.

We would approximate A by taking first k singular values;

$$A = U[:, : k].\sigma[: k, : k].V^T[: k, :]$$

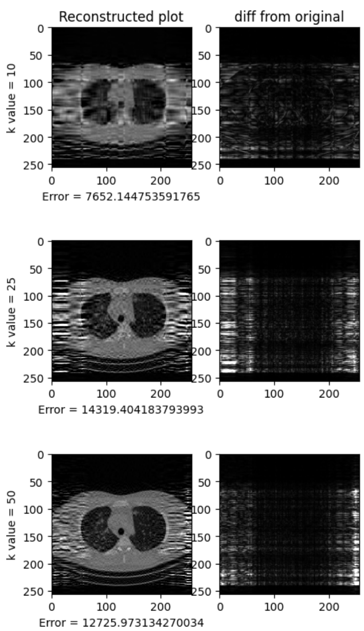

Fig. 3. $EVD\ Compressed\ image\ for\ k = 10, 25, 50$

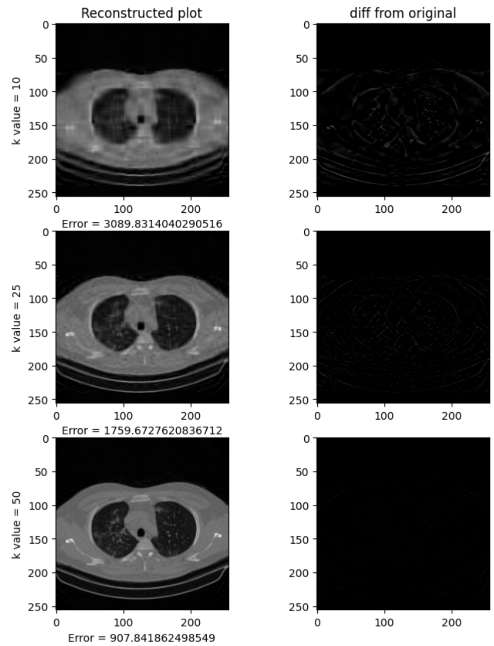

Fig. 4. $SVD\ Compressed\ image\ for\ k = 10, 25, 50$

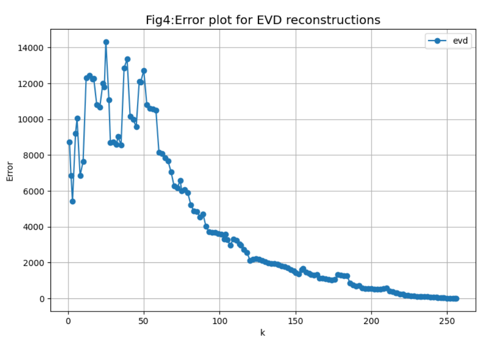

Fig. 5. $Error\ plot\ for\ EVD\ reconstruction$

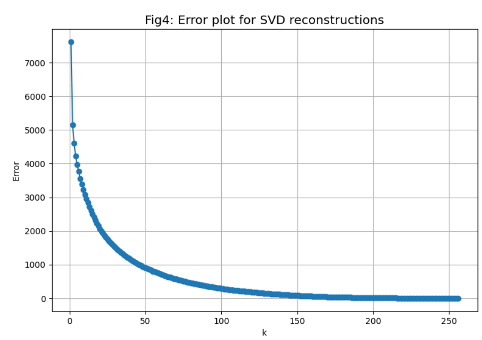

Fig. 6. $Error\ plot\ for\ SVD\ reconstruction$

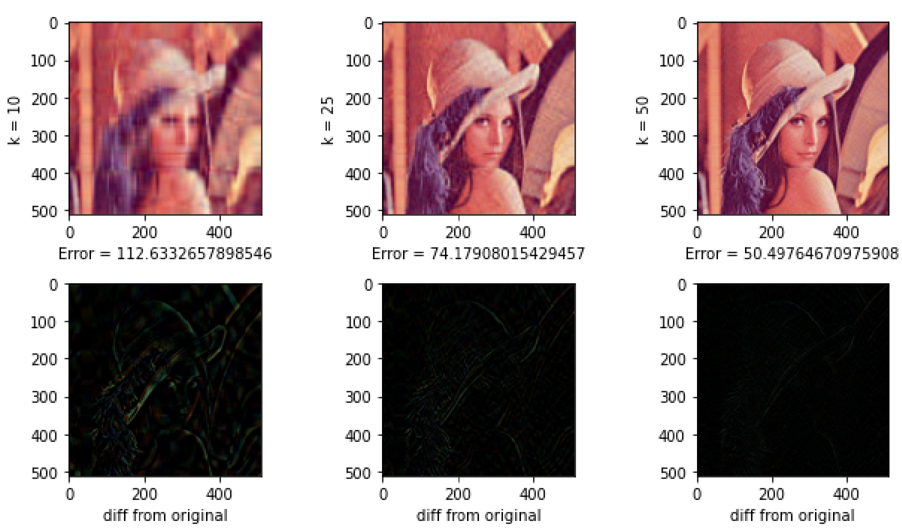

Fig. 7. $Color\ img\ reconstruction\ by\ SVD$

IV. **Experiments AND Inferences**

In the previous page you can have a look at the experiment results, where we have compressed images with EVD/SVD and also the error plot.

For the color image I've considered an RGB image. It's compression(done by svd) is similar to the previous case, just that we'll now have 3 matrices(corresponding to R,G,B colors) to decompose.

- The SVD reconstructions are better than the EVD ones.It is also evident from the error plot and norm values.
- Unlike the SVD case, the EVD error plot is neither smooth nor monotonically decreasing.

In the case of SVD we can write:

$$A = u_1.\sigma_1.v_1^T + u_2.\sigma_2.v_2^T + u_3.\sigma_3.v_3^T + ...$$

where $u_1, u_2...$ and $v_1, v_2, ...$ are columns of eigenvector matrices U and V
Note that each component of the sum is a unit-rank matrix and that too the row-space is span of v_i^T, which are the eigenvectors of $A^T.A$. Note that eigenvectors of self-adjoint matrices are orthogonal(in this case orthonormal). So, therefore each unit-rank matrix's rowspace and even columnspace(which are eigenvectors of $A.A^T$) is orthogonal.

Each time we increment 'k' in SVD , we are adding a rank-one orthogonal matrix which sort of adds an independent feature to the reconstructed image, bringing it closer to the actual image in high-dimensional space(of the images) and thereby reducing the Frobenius norm.
In case of EVD, such a thing cannot be said, in fact in EVD too, with increment in 'k' we are adding rank-one matrices, but these may not independently represent a feature that would take it closer to the actual image(in high-dimensional space). That is evident from the error plot where the norm locally moves up and down,i.e. instantaneously adding one such rank-one matrices reduces the norm for several values of k.

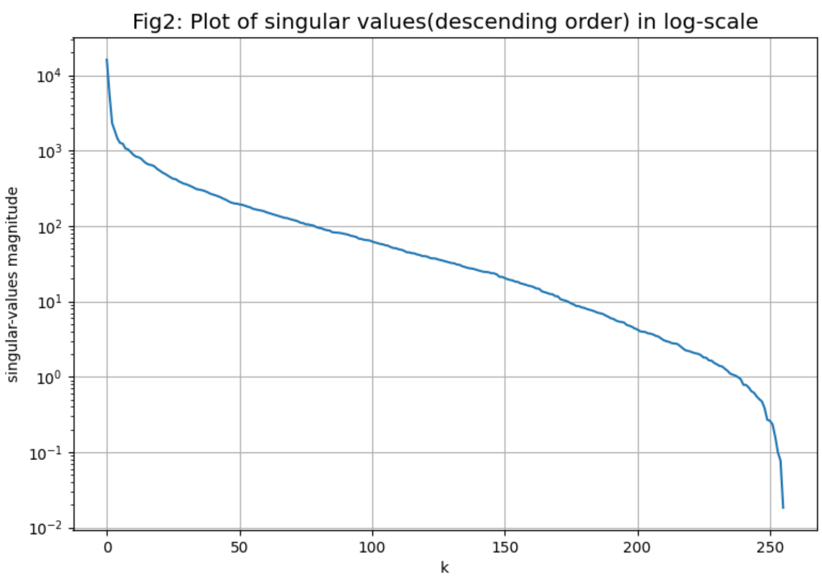

Fig. 8. Using the above plot we could choose values of k to compress our image