CHAPTER - 9
Differential Equations

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Exercise : 9.7 (Miscellaneous)

10) Solve the differential equation $ye^{\frac{x}{y}}\,dx = \left(xe^{\frac{x}{y}} + y^2\right)dy$
Solution (using the Method of Finite Differences):
The given differential equation can be rearranged as:

$$ye^{\frac{x}{y}}\,dx = \left(xe^{\frac{x}{y}} + y^2\right)dy \tag{1}$$

Let's express it in the form of $\frac{dx}{dy}$:

$$\frac{dx}{dy} = \frac{x}{y} + ye^{-\frac{x}{y}} \tag{2}$$

We will solve this using Euler's method, which approximates the solution iteratively. Using the finite difference approximation:

$$x_{n+1} = x_n + h\cdot\left(\frac{x_n}{y_n} + y_n e^{-\frac{x_n}{y_n}}\right) \tag{3}$$

Let y_0 be the initial y-value, x_0 be the initial x-value. Let the step size h be 0.001. The first few iterations are:

$$\begin{aligned}
y_1 &= y_0 + h, & x_1 &= x_0 + h\cdot\left(\frac{x_0}{y_0} + y_0 e^{-\frac{x_0}{y_0}}\right)\\
y_2 &= y_1 + h, & x_2 &= x_1 + h\cdot\left(\frac{x_1}{y_1} + y_1 e^{-\frac{x_1}{y_1}}\right)\\
&\vdots & &\vdots\\
y_n &= y_{n-1} + h, & x_n &= x_{n-1} + h\cdot\left(\frac{x_{n-1}}{y_{n-1}} + y_{n-1} e^{-\frac{x_{n-1}}{y_{n-1}}}\right)
\end{aligned}$$

Solution (using the General Method):
The given differential equation is:

$$ye^{\frac{x}{y}}\,dx = \left(xe^{\frac{x}{y}} + y^2\right)dy \tag{4}$$

Dividing both sides by $ye^{\frac{x}{y}}$, we get:

$$dx = \frac{x + y^2 e^{-\frac{x}{y}}}{y}\,dy \tag{5}$$

Rearranging to express $\frac{dx}{dy}$:

$$\frac{dx}{dy} = \frac{x}{y} + ye^{-\frac{x}{y}} \tag{6}$$

Using the substitution $v = \frac{x}{y}$, we have:

$$x = vy \quad \text{and} \quad \frac{dx}{dy} = v + y\frac{dv}{dy} \tag{7}$$

Substituting these into the equation and simplifying:

$$v + y\frac{dv}{dy} = v + ye^{-v} \tag{8}$$

Dividing by y, we get:

$$\frac{dv}{dy} = e^{-v} \tag{9}$$

Integrating both sides:

$$\int e^{v}\, dv = \int 1\, dy \tag{10}$$

$$\implies e^{v} = y + C \tag{11}$$

Substituting back $v = \frac{x}{y}$:

$$e^{\frac{x}{y}} = y + C \tag{12}$$

Taking the logarithm of both sides:

$$\frac{x}{y} = \ln(y + C) \tag{13}$$

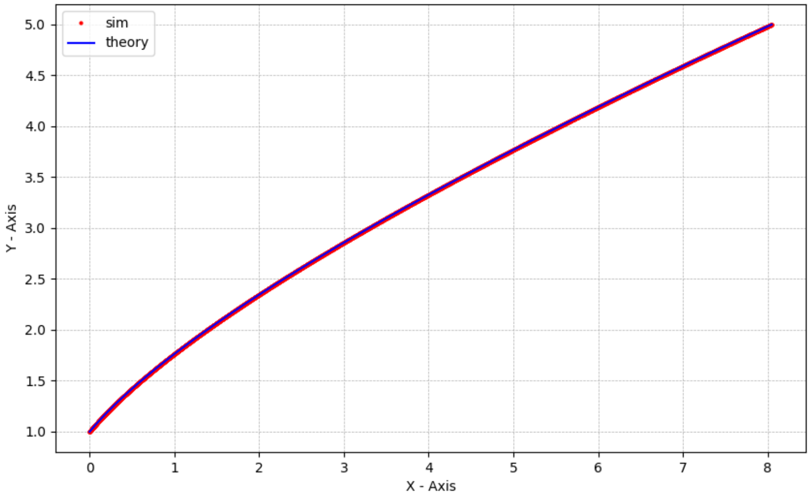
Multiplying by y:

$$x = y\ln(y + C) \tag{14}$$

Let's assume $C = 0$ and $y = 1$. On substituting these values in equation (14), we get $x = 0$.
Therefore, the equation is

$$x = y\ln(y)$$

Therefore, the curve generated using both the above mentioned methods for the given differential equation (1) is shown below:


sim
theory
5.0
4.5
4.0
3.5
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Y - Axis
0
1
2
3
4
5
6
7
8
X - Axis