CHAPTER - 4
Quadratic Equations

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Exercise : 4.2

4.1) Find the roots of the following equation $2x^2 - x + \frac{1}{8} = 0$
Solution:
First, we simplify the given equation:

$$2x^2 - x + \frac{1}{8} = 0 \tag{1}$$

$$\implies 16x^2 - 8x + 1 = 0 \tag{2}$$

Companion Matrix

For a quadratic equation of the form:

$$ax^2 + bx + c = 0, \tag{3}$$

the corresponding companion matrix is given by:

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}. \tag{4}$$

Substitute the coefficients $a = 16$, $b = -8$, and $c = 1$ into the companion matrix formula:

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{1}{16} & \frac{8}{16} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{16} & \frac{1}{2} \end{pmatrix}. \tag{5}$$

QR Algorithm

The QR algorithm iteratively decomposes the matrix A_n into an orthogonal matrix Q_n and an upper triangular matrix R_n, and updates the matrix as:

$$A_{n+1} = R_n Q_n. \tag{6}$$

This process continues until A_n converges to an upper triangular matrix, where the diagonal elements are the eigenvalues of A.

Steps of the Algorithm

a) Initialize the companion matrix:

$$A_0 = \begin{pmatrix} 0 & 1 \\ -\frac{1}{16} & \frac{1}{2} \end{pmatrix}. \tag{7}$$

b) Perform QR decomposition of A_n:

$$A_n = Q_n R_n, \tag{8}$$

where Q_n is orthogonal and R_n is upper triangular.

c) Update the matrix:

$$A_{n+1} = R_n Q_n. \tag{9}$$

d) Repeat the above steps until A_n converges to an upper triangular matrix. The diagonal elements of this matrix are the eigenvalues, which correspond to the roots of the quadratic equation.

Roots of the Quadratic Equation

The eigenvalues of the companion matrix A are the roots of the quadratic equation. Applying the QR algorithm numerically to A, we find:

$$\lambda_1 = 0.25, \quad \lambda_2 = 0.25. \tag{10}$$

Conclusion

The QR decomposition method applied to the companion matrix of $2x^2 - x + \frac{1}{8} = 0$ finds the roots of the equation. Both roots are real and equal:

$$x = 0.25. \tag{11}$$

This demonstrates the utility of the QR algorithm in computing eigenvalues, which are the roots of polynomial equations.

Newton's Method

Newton's Method is given by the update formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{12}$$

where:

$$f(x) = 16x^2 - 8x + 1 \quad \text{and} \quad f'(x) = 32x - 8 \tag{13}$$

The update equation becomes:

$$x_{n+1} = x_n - \frac{16x_n^2 - 8x_n + 1}{32x_n - 8} \tag{14}$$

Using an initial guess $x_0 = 0.5$, we observe that x_n converges at the 18th iteration to:

$$x = 0.25 \tag{15}$$

Secant Method

Alternatively, we can use the Secant Method, which avoids the derivative:

$$x_{n+1} = x_n + f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \tag{16}$$

Taking initial guesses $x_0 = 0.5$ and $x_1 = 0.4$, we observe that x_n converges at the 25th iteration to:

$$x = 0.25 \tag{17}$$

The graph below shows the equation and the root of the equation