CHAPTER - 3: Pair of Linear Equations in Two Variables

EE24BTECH11061 - Rohith Sai

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Exercise 3.3 - Question

Solve the following pair of linear equations using LU decomposition:

$$x + y = 14$$
$$x - y = 4$$

Solution Outline

Steps to Solve:

1. Convert the given equations into matrix form.
2. Apply LU decomposition to the coefficient matrix.
3. Solve for intermediate variables (y_1, y_2) using forward substitution.
4. Solve for unknowns (x, y) using backward substitution.

Step 1: Matrix Form

Matrix Representation:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 14 \\ 4 \end{bmatrix}$$
$$A\vec{x} = \vec{b}$$

Here:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 14 \\ 4 \end{bmatrix}.$$

Objective: Decompose A into L (Lower triangular) and U (Upper triangular) matrices.

Step 2: LU Decomposition

LU Decomposition Steps:

1. Start with A and initialize:

$$L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad U = A$$

2. For each row and column:

$$U_{k,j} = A_{k,j} - \sum_{m=1}^{k-1} L_{k,m} \cdot U_{m,j}, \quad j \geq k,$$

$$L_{i,k} = \frac{1}{U_{k,k}} \left(A_{i,k} - \sum_{m=1}^{k-1} L_{i,m} \cdot U_{m,k} \right), \quad i > k.$$

LU Decomposition Results

Decomposed Matrices:

$$L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}.$$

Verification:

$$A = L \cdot U$$

Step 3: Forward Substitution

Solve for intermediate variables $\vec{y}$:

$$L \cdot \vec{y} = \vec{b}.$$

Substituting:

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 14 \\ 4 \end{bmatrix}.$$

Solution:

$$y_1 = 14,$$
$$y_1 + y_2 = 4 \implies y_2 = -10.$$

Step 4: Backward Substitution

Solve for unknowns $\vec{x}$:

$$U \cdot \vec{x} = \vec{y}.$$

Substituting:

$$\begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 14 \\ -10 \end{bmatrix}.$$

Solution:

$$\begin{aligned} x_2 &= 5, \\ x_1 + x_2 &= 14 \implies x_1 = 9. \end{aligned}$$

Final Solution

The solution is:

$$x = 9, \quad y = 5.$$

Graphical Representation

Graph of the equations:

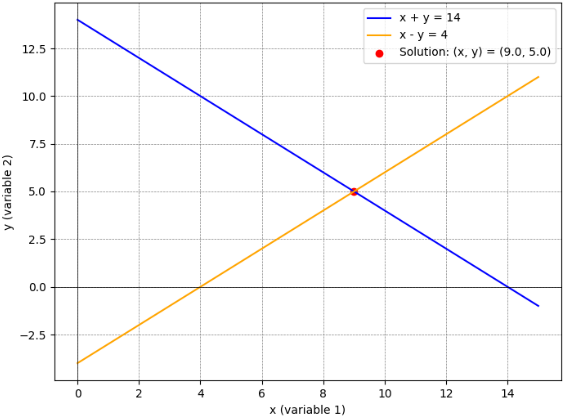

Figure: Graphical Representation of the Solution