CHAPTER - 8
Application of Integrals

EE24BTECH11061 - Rohith Sai

Exercise : 8.3 (Miscellaneous)

6) Find the area enclosed between the parabola $y^2 = 4ax$ and the line $y = mx$.

Solution (using the General Method):
The given curves are the parabola $y^2 = 4ax$ and the line $y = mx$. To find the points of intersection, we represent the curves in the matrix form of a conic section.
The parabola $y^2 = 4ax$ can be written in matrix form as:

$$y^\top V y + u^\top y + f = 0 \tag{1}$$

where:

$$V = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ u = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \ f = 0 \tag{2}$$

The line $y = mx$ can be represented as:

$$m^\top y + d = 0 \tag{3}$$

where:

$$m = \begin{pmatrix} 1 \\ -m \end{pmatrix}, \ d = 0 \tag{4}$$

$$\implies m = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{5}$$

Using the matrix form for the intersection points:

$$y_i = h + k_i m \tag{6}$$

where:

$$k_i = \frac{-m^\top (Vh + u) \pm \sqrt{(m^\top (Vh + u))^2 - g(h)\, m^\top V m}}{m^\top V m} \tag{7}$$

Here, h is the center of the conic. Since it's a parabola, we take $h = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Substitute the values:

$$k_i = \frac{-\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}0\\-2\end{pmatrix} \pm \sqrt{\left(\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}0\\-2\end{pmatrix}\right)^2}}{\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}} \tag{8}$$

Simplifying:

$$k_i = \frac{2 \pm \sqrt{4}}{1} \tag{9}$$

Thus:

$$k_1 = 4,\ k_2 = 0 \tag{10}$$

Substitute back to find y_i:

$$y_1 = \begin{pmatrix}0\\0\end{pmatrix} + 4\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}4\\4\end{pmatrix} \tag{11}$$

$$y_2 = \begin{pmatrix}0\\0\end{pmatrix} \tag{12}$$

Therefore, the points of intersection are $(0, 0)$ and $(4, 4)$.
The enclosed area is the integral of the difference between the line and the parabola from $x = 0$ to $x = 4$:

$$A = \int_0^4 \left(x - \sqrt{4x}\right) dx \tag{13}$$

Solution (using the Trapezoidal Rule):

The total area can be approximated by dividing the region into small trapezoidal strips and summing their areas:

$$A = \frac{1}{2}h\left(y\left(x_1\right) + y\left(x_0\right)\right) + \frac{1}{2}h\left(y\left(x_2\right) + y\left(x_1\right)\right) + \cdots + \frac{1}{2}h\left(y\left(x_n\right) + y\left(x_{n-1}\right)\right) \tag{14}$$

Simplifying:

$$A = h\left[\frac{1}{2}\left(y\left(x_0\right) + y\left(x_n\right)\right) + y\left(x_1\right) + \cdots + y\left(x_{n-1}\right)\right] \tag{15}$$

Let $A\left(x_n\right)$ be the area enclosed by the curve from $x = x_0$ to $x = x_n$, where $\left(x_0, x_1, \ldots, x_n\right)$ are equidistant points with step size h:

$$A\left(x_n + h\right) = A\left(x_n\right) + \frac{1}{2}h\left(y\left(x_n + h\right) + y\left(x_n\right)\right) \tag{16}$$

Repeating this process until we reach the desired area and discretizing the steps, let

$A(x_n) = A_n$ and $y(x_n) = y_n$:

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2 y_n' \tag{17}$$

In the given question:

$$y_n = \sqrt{4 - x_n^2} + x_n - 2 \tag{18}$$

and:

$$y_n' = \frac{-x_n}{\sqrt{4 - x_n^2}} + 1 \tag{19}$$

Thus, the general difference equation becomes:

$$A_{n+1} = A_n + h\left(\sqrt{4 - x_n^2} + x_n - 2\right) + \frac{1}{2}h^2\left(\frac{-x_n}{\sqrt{4 - x_n^2}} + 1\right) \tag{20}$$

Final Result:
Theoretical area: $\frac{8}{3} \approx 2.67$ sq. units.
Computed area using the trapezoidal rule: 2.658 sq. units.
The plot showing the area enclosed between the parabola $y^2 = 4x$ and the line $y = x$ is shown below: