(*) **Hamilton cycle:-**

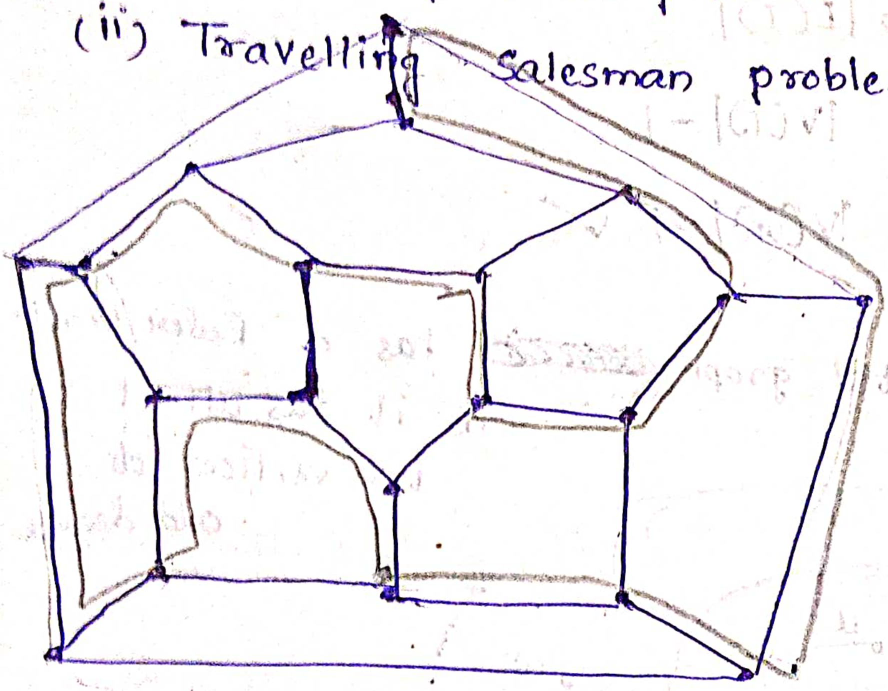
A path that contains every vertex of G.

→ A cycle which contains every vertex of the graph exactly once.

Applications:-

(i) Chinese postman problem.

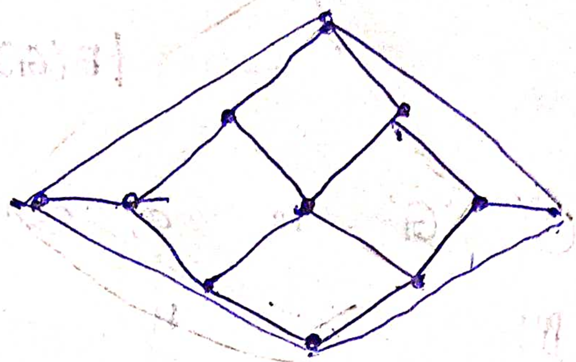
(ii) Travelling salesman problem.



dodecahedron.

⇓

→ Contains hamilton cycle.



Herschal graph.

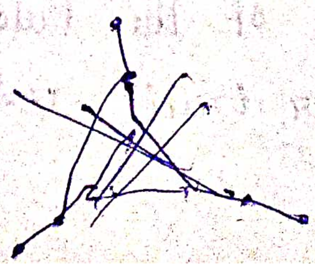
⇓

→ doesnot contains Hamilton cycle.

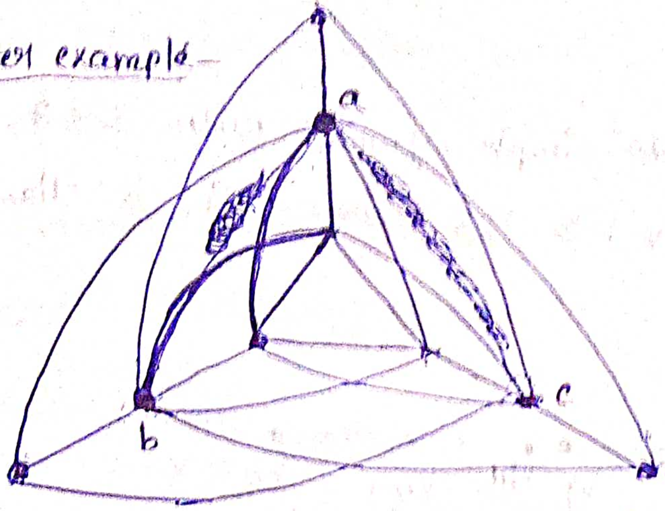
Statement:-

If G is Hamilton graph, then for every non-empty proper subset 'S' of 'V',

$$comp(G-S) \leq |S|.$$



Counter example -



Non-Hamilton.

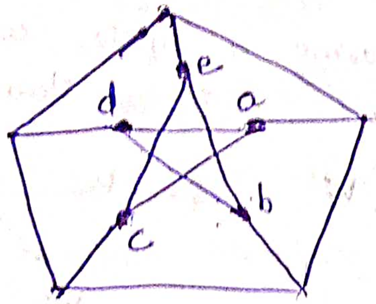
$|V| = 9$.

delete $S = \{a, b, c\}$.

$comp(G - S) = 4$.

$|S| = 3$.

does not satisfies theorem so, non-hamilton



Non-Hamilton.

$S = \{a, b, c, d, e\} \Rightarrow |S| = 5$

$comp(G - S) = 1$

$comp(G-S) \leq |S|$

but not Hamilton cycle.

$\Rightarrow$ Satisfies theorem but non-hamilton.

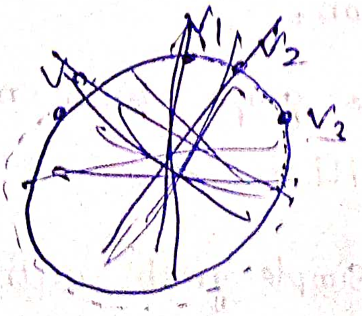
Hamilton graph :-

A cycle that contains every vertex of 'G' is Hamilton graph.

Proof :-

'C' is Hamilton cycle of 'G' $\Rightarrow$ for every $S \subset V$, $S \neq \emptyset$

$comp(C - S) \leq |S|$.

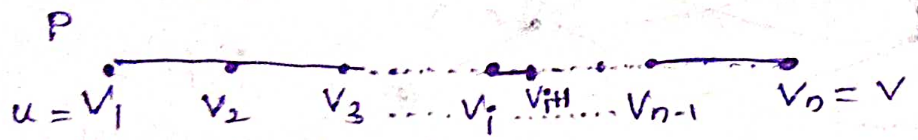


Since, $(C-S)$ is a spanning subgraph of $G - S$ and therefore the theorem.

$comp(G-S) \leq comp(C-S) \leq |S|$.

Statement :-

If 'G' is a simple graph with $n \geq 3$, where $n = |V|$ and $\delta(v) \geq n/2$ then 'G' is Hamilton.



$\Downarrow$

Maximum non-Hamilton path.

$\Rightarrow P + uv \longrightarrow$ forms a cycle which is $\Rightarrow$ Hamilton.

$S = \{v_i, \quad \}$, $T = \{v_i \quad \}$. $u \leftarrow v_{i+1}$

$\swarrow$

v_i

$v_i \notin S \cup T$

$S \cap T = \phi$

$uv_{i+1} \in E$

$v_i v \in E$

$deg(u) + deg(v) = |S| + |T|$

$= |S \cup T| + |S \cap T| < n$ $(\because 2\delta(v) \geq n)$

(0)

This is a contradiction.

* Statement :-

If G is a simple graph with $n = |V| \geq 3$ and $\delta \geq \frac{n}{2}$, then 'G' is Hamilton.

Pf :- Suppose 'G' is a simple graph with $n \geq 3$ and $\delta \geq n/2$, but not Hamilton.

$\exists$ A maximal non-hamilton simple graph with $n \geq 3$ and $\delta \geq n/2$.

Since $n \geq 3$, G can not be complete.

Then, $\exists$ nonadjacent u & v in G.

Since G is non hamiltonian, each hamilton cycle of $G+uv$ must contain edge uv.

$\exists$ hamilton path $v_1, v_2, \ldots, v_n$ in G with $v_1 = u$ and $v_n = v$.

Set $S = \{ v_i / uv_{i+1} \in E \}$.

$T = \{ v_i / v_i v \in E \}$.

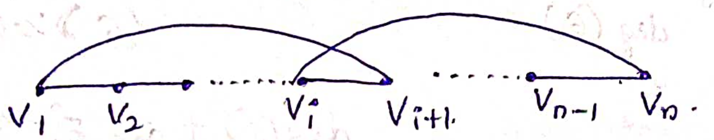
Since $v_n \notin (S \cup T)$, we have

$|S \cup T| < n$ —— (1)

$|S \cap T| = 0$ —— (2)

Suppose $v_i \in S \cap T$, then 'G' would have to contain hamilton cycle

$v_1, v_2, \ldots \ldots v_i, v_n, v_{n-1} \ldots \ldots v_{i+1} v_1$.



Using (1) & (2),

$\deg(u) + \deg(v) = |S| + |T|$

$= |S \cup T| + |S \cap T| < n$ —— (3)

But $\delta \geq n/2$, so (3) is contradicting with assumption.

$\therefore$ 'G' is Hamilton.

* **Lemma** :- Let G be a simple graph and Let u and v be two non-adjacent vertices in G such that

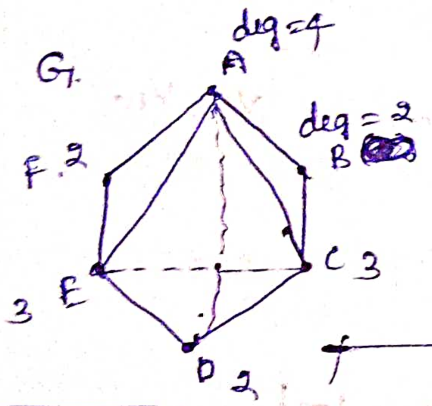
$$d(u) + d(v) \geq n.$$

then 'G' is hamiltonian iff $G+uv$ is Hamiltonian.

Pf:-

* **Closure of 'G' :- ($C(G)$).**

Graph obtained from 'G' by recursively joining pairs of non-adjacent vertices whose degree sum is atleast $n = |V|$ until no such pair remains.



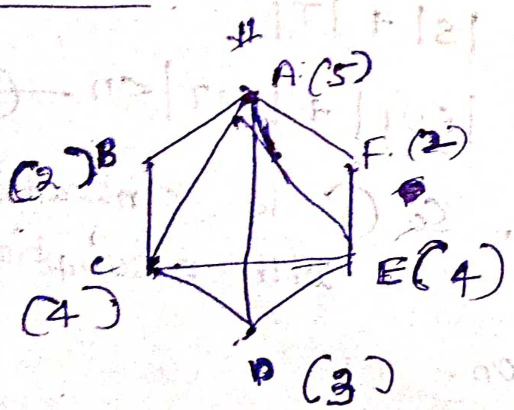
$n = 6$

deg (G)

$\deg(A) + \deg(D) \geq n.$

$(6) \geq (6)$

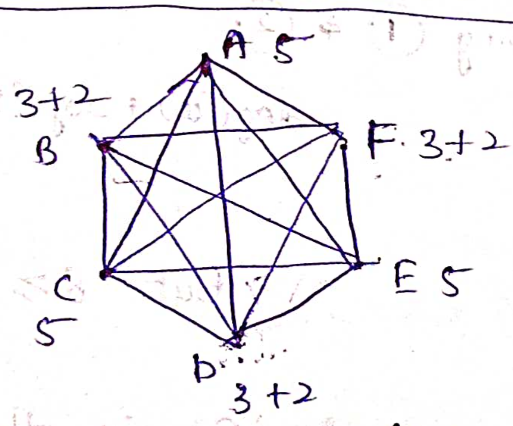
$\deg(C) + \deg(E) = 6 \geq n.$



$\deg(C) + \deg(F) \geq n$

$6 \geq 6$

$\deg(E) + \deg(B) = 6 \geq n$



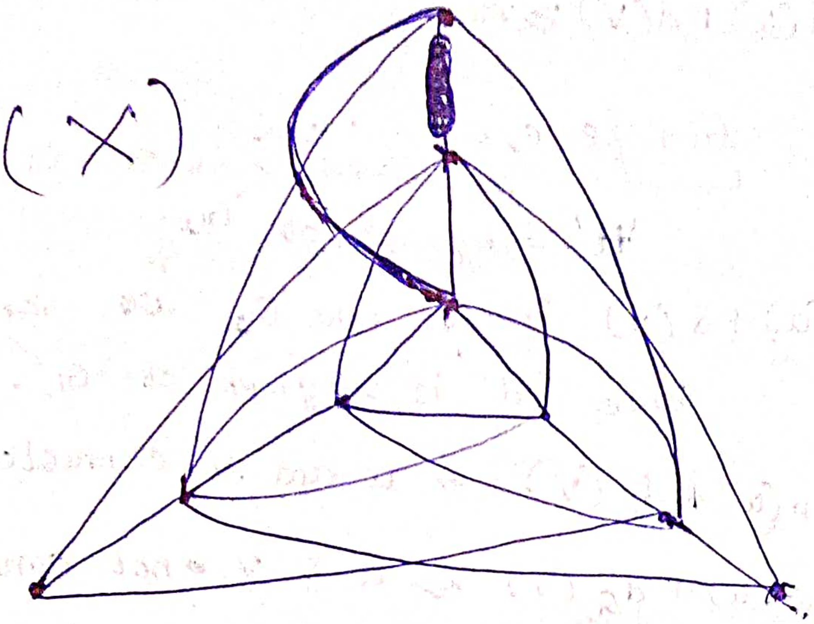
K_6

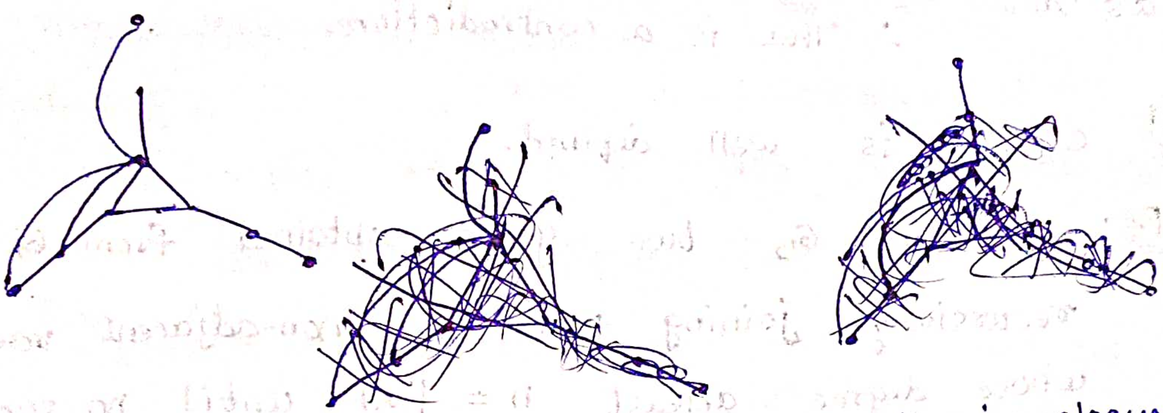
$\deg(B) + \deg(F) = 6 \geq n$

$d(D) + d(F) = 6 \geq n$

$d(B) + d(B) = 6 \geq n,$

(X)





① A simple graph is Hamilton iff it is closure hamilton.

② Let G be a simple graph with $n = |V| \geq 3$. If $c(G)$ is complete, then G is Hamilton.
(can't say if G is hamilton then $c(G)$ is complete).

*** Lemma :-**

$c(G)$ is well defined.

Pf:- G_1 G_2

Edges ⇒ $e_1, e_2, \ldots e_n$ $f_1, f_2, \ldots f_m$ ← edges.

(need to proove both G_1 and G_2 are same)

$G_1 + e_1 + e_2 + \ldots e_n$, $G_2 + f_1 + f_2 + \ldots f_n$ ⇒ same ✓

$e_1, e_2, \ldots e_k, \ldots, e_n$

↑

$uv = e_{k+1} \notin G_2$

$d(u)+d(v) \geq n$.

$$\underbrace{H = G + \{e_1, e_2, \ldots, e_k\}}_{\Downarrow};$$

'H' subgraph of 'G_2'.

$d(u)+d(v)$ in G_1 and G_2 are same.

Since 'H' is subgraph of 'G_2'.

Both sum of degrees are same

$$\begin{cases} d_H(u)+d_H(v) \Rightarrow u \text{ and } v \text{ connected.} \\ d_{G_2}(u)+d_{G_2}(v) \Rightarrow u \text{ \& } v \text{ not connected} \end{cases}$$

$\therefore$ This is a contradiction.

(*) $C(G)$ is well defined.

Pf:- G_1 and G_2 two graphs obtained from G by recursively joining pairs of non-adjacent vertices whose degree atleast $n = |V|$ until no such pair remains.

Suppose.

$e_1, e_2, e_3, \ldots\ldots, e_m$ — (1)

$f_1, f_2, f_3, \ldots\ldots, f_n$ — (2)

Sequence of edges added to 'G' in obtaining G_1 and G_2 respectively.

We have to show that each e_i is edge of G_2 and f_i is edge of G_1.

If possible, let $e_{k+1} = uv$ be the first edge in sequence (1), that is not an edge of G_2.

Set $H = G + \{e_1, e_2, \ldots\ldots e_k\}$

then from G_1,

$$d_H(u) + d_H(v) \geq n.$$

By the ~~[illegible]~~ closure of H, 'H' is a subgraph of G_2.

$$d_{G_2}(u) + d_{G_2}(v) \geq n.$$

$\therefore$ $e_{K+1} = uv \in G_2$

$\Rightarrow$ $G_1 = G_2$ ✓

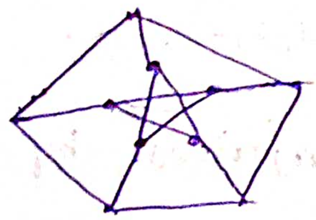
(*)

<u>Statement</u> :- Let 'G' be a simple graph with degree sequence $(d_1, d_2, d_3, \ldots\ldots, d_n)$ where $d_1 \leq d_2 \leq \ldots\ldots \leq d_n$ and $n \geq 3$. Suppose there is no value of $m < n/2$ for which

$$d_m \leq m \quad \& \quad d_{n-m} < n-m.$$

Then 'G' is Hamilton.

<u>Ex</u> :-



$n = 10$ ✓ $\Rightarrow m < 5$

$m/2 = 5$ $\quad m = 1, 2, 3, 4$

is hamilton?

$\Downarrow$

can not decide a Hamilton cycle.

$d_1 \leq 1$, $\quad d_2 \leq 2$, $\quad d_3 \leq 3$, $\quad d_4 \leq 4$,

$3 \leq 1$ $\quad 3 \leq 2$ $\quad 3 \leq 3$

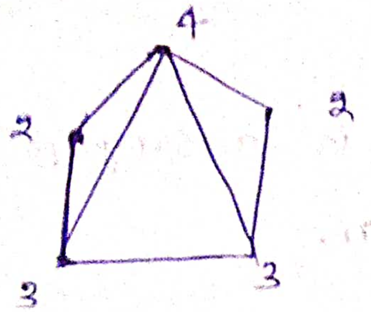
✗ $\quad$ ✗ $\quad$ ✓

$d_{10-1} < 10-1$, $d_{10-2} < 10-2$, ……………, $d_{10-4} < 10-4$

$\Rightarrow$ $d_1\ d_2\ d_3\ d_4\ d_5\ d_6\ d_7\ d_8\ d_9\ d_{10}$

$3, 3, 3, 3, 3, 3, 3, 3, 3, 3.$

$d_i \leq d_j$, $i \leq j$ ✓



$d_1 d_2 d_3 d_4 d_5$
2, 2, 3, 3, 4

$n = 5$

$m = 2 < n/2$

$m = 1, 2.$

$d_1 \le 1$, $d_2 \le 2$,
× ✓

$d_{5-1} < 5-1.$

$d_{5-2} < 5-2.$

⇒ ~~cast~~ can't say Hamilton ✓

(*) Degree majorized:-

A sequence of degrees $(p_1, p_2, \dots, p_n)$ is said to be majorized by $(q_1, q_2, \dots, q_n)$ if

$$p_i \le q_i, \text{ for } i = 1, 2, \dots, n.$$

Ex:- (2,2,2,2,2) is majorized by (2,2,2,3,3).

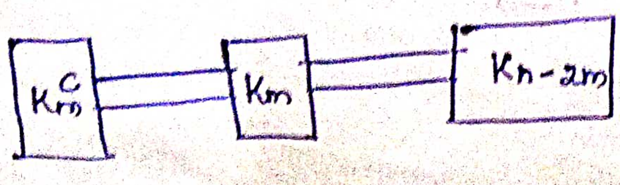
~~(*) Join G and H~~

⇒ Join G and H, denoted by $G \vee H$. where G and H are disjoint graphs

$G \vee H \leftarrow G + H$ joint each vertex of G. to each vertex H.

Now, for $1 \le m < n/2$, $C_{m,n}$ denotes

$$K_m \vee (K_m^c + K_{n-2m})$$

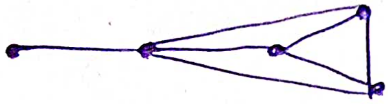


* Theorem (Chvatal 1972) :-

If 'G' is a non-hamiltonian simple graph with $n \geq 3$ then G is degree majorized by some $C_{m,n}$.

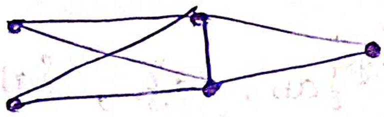
$$C_{1,5} = K_1 \vee (K_1^c + K_{5-2(1)}).$$

$$= K_1 \vee (K_1^c + K_3).$$



$$C_{2,5} = K_2 \vee (K_2^c + K_{5-2(2)}).$$

$$= K_2 \vee (K_2^c + K_1).$$



$$C_{3,5} = K_3 \vee (K_3^c + \underset{(\times)}{K_{-1}})$$

Here $m \nless n/2$

$3 < 5/2$

$\Rightarrow 3 < 2.5 (\times)$

(*) Statement :-

If 'G' is a simple graph with $n = |V| \geq 3$ and $|E| > \binom{n-1}{2} + 1$, then 'G' is hamilton.

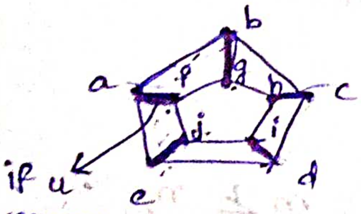
Moreover, the only non-Hamilton simple graph with 'n' vertices and $\binom{n-1}{2} + 1$ edges are

$C_{1,n}$ and for $n=5$, $C_{2,5}$

Pf :-

(*) Matchings :-

Def :- A subset 'M' of E(G) is called a matching in G if its elements are links and no two are adjacent in 'G'.



if u remove these links

⇓

2 cycles are formed

Matchings :- {ab, cd}, {ab, ed}.
~~{bc, ed}~~, {bc, ae}.
{cd, ae}, {fg, ih}.
{fg, ij}, {gh, ij}.
{gh, jf}, {ih, fj}.

~~{ab, cd, fg}~~

$M_1 = \{af\}$. ⇒ 'a' and 'f' are M_1 saturated vertices.
'i' and 'j' vertices are M_1 unsaturated.
⇓ remaining

$M_2 = \{af, ji\}$.

$M_3 = \{af, ji, ed\}$.

$M_4 = \{af, ji, ed, ch\}$.

$M_5 = \{af, ji, ed, ch, gb\} = M_4 \cup \{gb\}$

⇓

Here, all vertices are M_5 saturated. (⇒ Perfect matching)

Def:- A matching 'M' saturated a vertex v, then v is called M-saturated.

Note:- The vertex incident to the edge of a matching M are called saturated by 'M'.

~~Eg:-~~

Perfect Matching:- If every vertex of 'G' is M-saturated, then we call 'M' is perfect matching in 'G'.

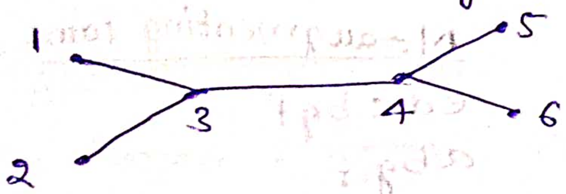
Maximum matching:-

'M' is maximum matching if 'G' has no matching M' with $|M'| > |M|$.

~~Note:-~~

~~$\Rightarrow$ Maximum matching $\Rightarrow$ Perfect matching.~~

~~Perfect matching $\not\Rightarrow$ Max. matching.~~



Max. matching:-

$\{13, 46\}$

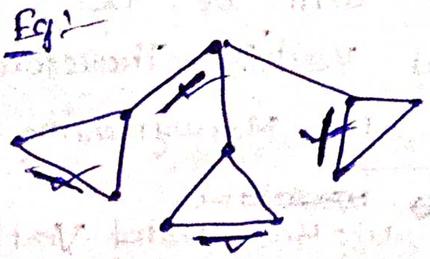
Perfect matching:-

Not exists.

*

$\Rightarrow$ $\boxed{\begin{array}{l}\text{Perfect matching} \Rightarrow \text{Max. matching.} \\ \text{Max. matching} \not\Rightarrow \text{Perfect matching}\end{array}}$

Eg:-

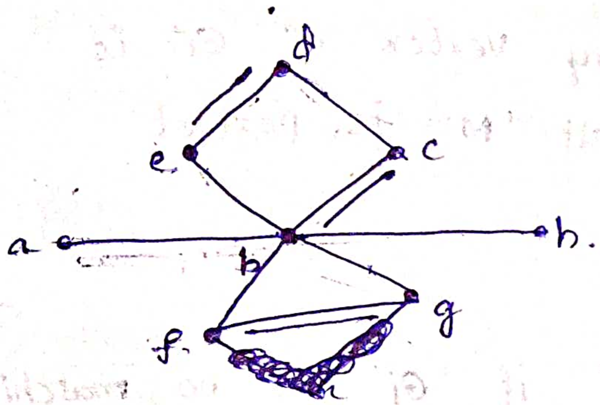


$\Rightarrow$ no perfect matching.

$\Rightarrow$

* Def:- (M-alternating Path)

Let 'M' be a matching in G. A 'M'-alternating path in G is a path whose edges are alternatively in E\M, M
(E-M)
(set diff.)



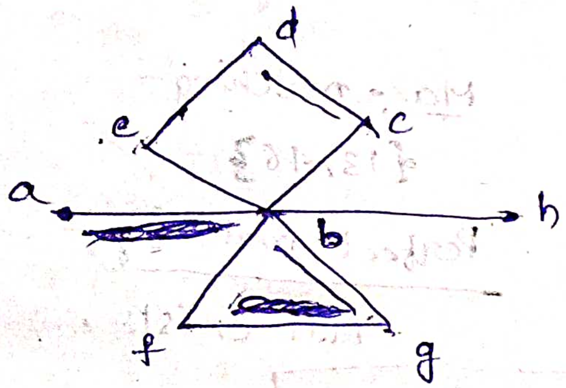
abcde ⇒ M-alternating path.

edcbfg ⇒ " "

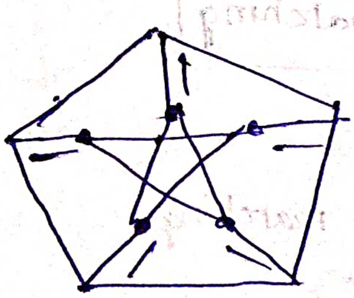
** ⇒ single edge is also a ~~[...]~~ perfect matching. (•——•)
Trivial path, (M)alternating path.

(*) Def:-

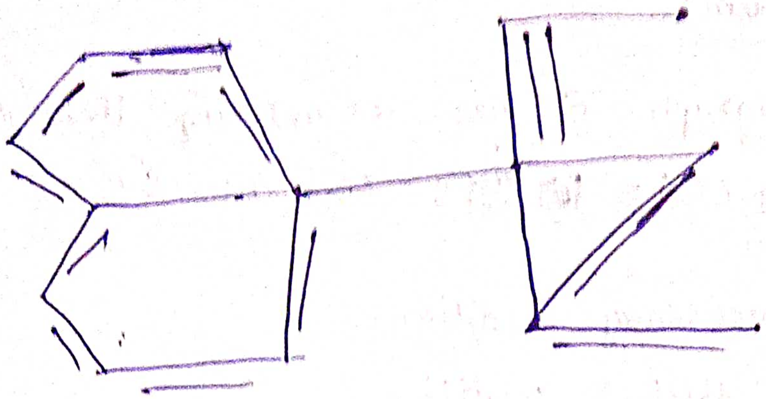
M-augmenting path is a M-alternating path whose origin and terminals are M-unsaturated.



M-augmenting paths
edcbgf
abgf.
hbgf.



⇒ Perfect matching exists. So, there will be no unsaturated vertices. Therefore, there is no M-augmenting path. (~~[...]~~ M-unsaturated vertices does not exists).



M, M'

$$M \Delta M' \begin{cases} (M - M') \\ (M' - M) \end{cases}$$

$(M - M') \cup (M' - M)$.

⊛ HALL'S MATCHING CONDITION :- (1935)

→ Filling jobs.

Appli → Jobs.

Model prblm :-

Personal assignment prblms.

Necessary condition :-

X, Y - bipartitions of bipartite graph.

A matching that saturates 'X'.

(i) If a matching 'M' that saturates 'X' then for every $S \subseteq X$, there must be atleast $|S|$ vertices that have neighbour in 'S'.

$N(S)$ - set of vertices having a neighbour in 'S'.

$|N(S)| \geq |S|$.

* <u>Hall's Theorem</u> :-

An X,Y- bigraph G has a matching that saturates 'X' iff $|N(S)| \geq |S|$ for all $S \subseteq X$.

<u>Pf</u> :- (i) Necessary condition.

(ii) Sufficient condition.

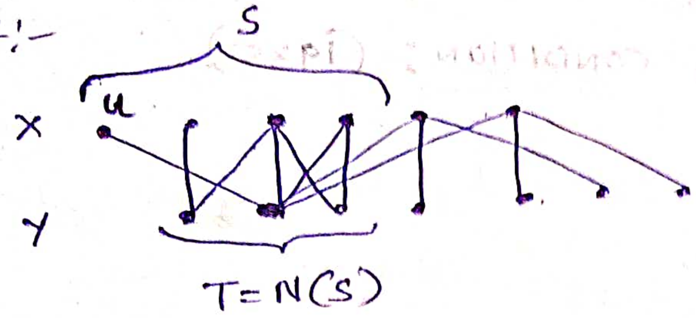
Proof by <u>contrapositive</u>.

$A \Rightarrow B$

$\sim B \Rightarrow \sim A$

* If 'M' is maximum matching and does not saturate 'X', then we obtain a set $S \subseteq X \ni |N(S)| < |S|$

<u>Pf</u>:-



$u \in X$ is M-saturated

Let $u \in X$ be a vertex unsaturated by 'M'.

Define,

S - all vertices reachable from 'u' by M-alternating paths in 'G' that are in 'X'.

T - all vertices reachable from 'u' by M-alternating paths in 'G' that are in 'Y'.

Now, we can see $u \in S$,

Claim-1 :- 'M' matches 'T' with $S-\{u\}$.

Pf :- The M-alternating paths from 'u'

" → reach 'Y' along with edges not in 'M'.

→ return to 'X' along with edges in 'M'.

Hence, every vertex of "$S-\{u\}$" is reached by an edge in M from a vertex in 'T'.

Sub cliam :- There is no augmented path.

Pf :- ??

→ Since, there is no M-augmented path every vertex of 'T' is saturated.

Thus, an M-alternating path reaching $y \in T$ extends via 'M' to a vertex of 'S'.

Hence, these edges 'M' yields a bijection from 'T' to $S-\{u\}$.

$\therefore |T| = |S-\{u\}|$

We know that, $T = N(S)$.

$\Rightarrow |N(S)| = |S-\{u\}| < |S|$. ✓

Cor :-

Let 'G' be a K-regular bipartite graph for $K>0$, the 'G' has a perbect matching.

Pf :- H/W

~~**Ore's theorem :-**~~

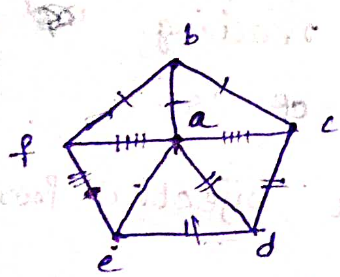
~~If 'G' is a simple graph with $n = |V| \geq 3$ and $\deg(u) + \deg(v) \geq n$ when $u, v \in V$ and $uv \notin E$, then 'G' is Hamiltonian.~~

~~Pf :-~~

Covering :- $G = (V, E)$.

A covering of a graph 'G' is a subset 'K' of 'V', such that every edge of 'G' has atleast one end in 'K'.

Note :- A covering 'K' is minimum covering, if 'G' has no covering K' with $|K'| < |K|$.



$K = \{b, d, f, a\}$.

Here, 'K' is minimum covering.

$K' = \{b, d, f, a, c\} \Rightarrow$ not a minimum covering.

Matching :-

$\{bf, ed, ca\}$.

Remark :-

1. If 'K' is a covering of 'G' and 'M' is a matching of 'G'. Then 'K' contains atleast one end of each edge of 'M'.

   $\therefore |M| \leq |K|$.

2. M^* - Maximum matching.

   $\hat{K}$ - Minimum covering. then,

   $|M^*| \leq |\hat{K}|$.

(3) If 'G' is a bipartite, we do have $|M^*| = |\hat{K}|$.

__Lemma:-__

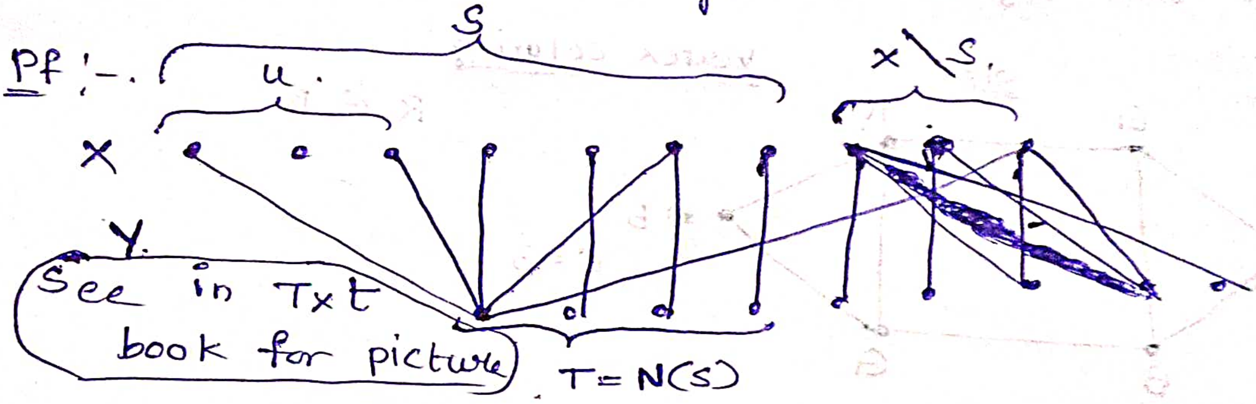
Let 'M' be a matching and 'K' be a covering such that $|M| = |K|$. Then 'M' is a maximum matching and 'K' is minimum covering.

__Pf:-__

(*) __Theorem:-__

In a bipartite graph 'G', the no. of edges in a maximum matching is equal to the no. of vertices in a minimum covering.

__Pf:-__



'G' is bipartite graph with (X, Y).

M^* – Maximum matching.

u – Set of M^* unsaturated vertices in 'X'.

Z – Set of all vertices connected by M^*-alternating paths to vertices of 'u'.

Then set $S = Z \cap X$,

$T = Z \cap Y$

Now, we can see that every vertex in 'T' is M^* saturated and $N(S) = T$.

Now, define $\hat{K} = (X \setminus S) \cup T$

Then, every edge of 'G' must have atleast one end in $\hat{K}$. $\Rightarrow$ (This claim is true)

$\therefore$ $\hat{K}$ is a minimum covering from relation, $|M^*| = |\hat{K}|$.