Shannon’s Theory

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Entropy (1948):

→ A mathematical measure of information/uncertainty.

X - (finite) r.v.

EX: X toss a coin

$Pr[H] = Pr[T] = 1/2$

1 0

Entropy one bit.

n-independent tosses of a coin

⇒ Entropy ?? n-bit

1 2 3 4 .... n

0/1 0/1 0/1 .... 0/1

4-bit number.

⟹ 0/1 0/1 0/1 0/1

Ex:

$X = x_1, x_2, x_3$

$Pr[X=x] \quad 1/2 \quad 1/4 \quad 1/4$

x_1 as 0

x_2 as 10

x_3 as 11

$$\boxed{\frac{1}{2} \times 1 + \frac{1}{4} \times 2 + \frac{1}{4} \times 2 = \frac{3}{2}}$$

Entropy: Suppose X is a discrete r.v.

$$\boxed{H(X) = -\sum_{x \in X} pr[x] \log_2 pr[x]}$$

Remark: If $y=0$, $\log_2 y$ is undefined

$$\boxed{\lim_{y \to 0} y \log_2 y = 0}$$

?

→ If $|X| = n$, $pr[x] = 1/n \ \forall x \in X$

Then $H(X) = \log_2 n$.

① $H(X) \geq 0 \ \forall$ r.v. X

② $H(X) = 0$ iff $pr[x_0] = 1$ for $x_0 \in X$

$pr[x] = 0$ for $x \neq x_0$.

Ex:

$\mathcal{P} = \{a, b\}$

$\mathcal{K} = \{k_1, k_2, k_3\}$

$\mathcal{C} = \{1, 2, 3, 4\}$

$pr[a] = 1/4$, $pr[b] = 3/4$

$pr[k_1] = 1/2$, $pr[k_2] = pr[k_3] = 1/4$

	a	b
k_1	1	2
k_2	2	3
k_3	3	4

Q: $H(\mathcal{P})$?

$H(\mathcal{K})$?

$H(\mathcal{C})$?

Sol:

$$H(X) = -\sum_{x} pr[x] \log_2 pr[x]$$

$$H(P) = -\left(pr[a] \log_2 pr[a] + pr[b] \log_2 pr[b] \right)$$

$$= -\left(\frac{1}{4} \log_2 \frac{1}{4} + \frac{3}{4} \log_2 \frac{3}{4} \right)$$

$$= 0.81$$

$$H(K) = -\left(pr[k_1] \log_2 pr[k_1] + pr[k_2] \log_2 pr[k_2] + pr(k_3) \log_2 pr[k_1] \right)$$

$$= -\left(\frac{1}{2} \log_2 \frac{1}{2} + \frac{1}{4} \log_2 \frac{1}{4} + \frac{1}{4} \log_2 1/4 \right)$$

$$= \frac{3}{2} = 1.5$$

$$H(C) = -\big(pr[1] \log_2 pr[1] + pr[2] \log_2 pr[2] + pr[3] \log_2 pr[3] + pr[4] \log_2 pr[4] \big)$$

$$= -\left(\frac{1}{8} \log \frac{1}{8} + \frac{7}{16} \log \frac{7}{16} + \frac{1}{4} \log \frac{1}{4} + \frac{3}{16} \log \frac{3}{16} \right)$$

$$= 1.85$$

$$pr[Y=y] = \sum_{\{K :\, y \in C(K)\}} pr[K]\, pr[x = D_K(y)]$$

$$pr[1] = pr[k_1]\, pr[a] = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$$

$$pr[2] = pr[k_1]\, pr[b] + pr[k_2]\, pr[a] = \frac{1}{2} \times \frac{3}{4} + \frac{1}{4} \times \frac{1}{4} = \frac{7}{16}$$

Jensen's Inequality:

Suppose f is a continuous strictly concave function on the interval I,

$$\sum_{i=1}^{n} a_i = 1 \quad \text{and} \quad a_i > 0, \quad 1 \le i \le n.$$

Then

$$\sum_{i=1}^{n} a_i f(x_i) \le f\left(\sum a_i x_i\right)$$

where $x_i \in I$, $1 \le i \le n$

The equality occurs iff $x_1 = x_2 \cdots = x_n$.

Note: $\log_2 x$ is always strictly concave on the $(0, \infty)$

Theorem: Suppose X is a r.v. having prob. distribution, which takes on the values $p_1, p_2, \dots p_n$, where $p_i > 0$, $1 \le i \le n$. Then $H(X) \le \log_2 n$, with equality iff $p_i = 1/n$, $1 \le i \le n$.

proof:

$$H(X) = -\sum_{i=1}^{n} p_i \log_2 p_i$$

$$= \sum_{i=1}^{n} p_i \log_2 1/p_i$$

$$\le \log_2 \left(\sum_{i=1}^{n} p_i \times \frac{1}{p_i} \right)$$ Jensen's inequality

$$= \log_2 n$$

$$\Longrightarrow H(X) \le \log_2 n.$$

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P

	a	b	c
k_1	1	2	3
k_2	2	3	4
k_3	3	4	1

K

$P = \{a, b, c\}$

$K = \{k_1, k_2, k_3\}$

$C = \{1, 2, 3, 4\}$

Q3). What is $H(C)$?

Q1) $Pr[a] = 1/2$, $Pr[b] = 1/3$, $Pr[c] = 1/6$

$H(P)$?

Q2) $Pr[k_1] = Pr[k_2] = Pr[k_3] = 1/3$

$H(K)$?