Graph Isomorphism

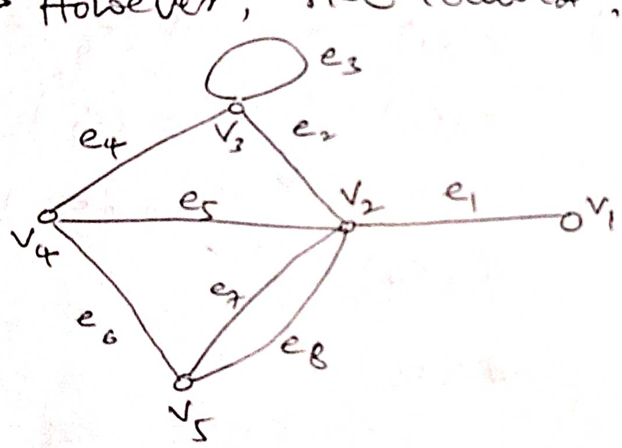
Identical: $G = H$, if $V(G) = V(H)$, $E(G) = E(H)$ and

$\psi_G = \psi_H$.

→ can be represented with identical diagrams.

→ However, not identical, but can have same diagram.

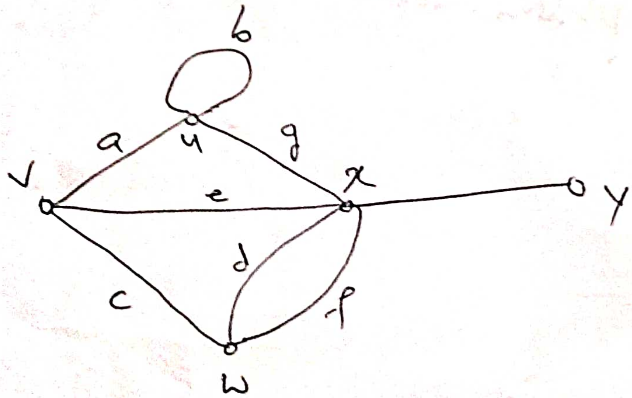
G:



$V = \{v_1, v_2, v_3, v_4, v_5\}$

$E = \{e_1, e_2, e_3, e_4, e_5 \ldots e_8\}$

H:



$V = \{u, v, w, x, y\}$

$E = \{a, b, c, d, e, f, g\}$,

→ G & H are having same diagrams,
But, not identical.

→ However, isomorphic graphs $(G \cong H)$.

<u>G and H are isomorphic</u>, if there are bijections

$\theta : V(G) \to V(H)$ and $\phi : E(G) \to E(H)$ such that

$\psi_G(e) = uv$ iff $\psi_H(\phi(e)) = \theta(u)\theta(v)$.

such pair (θ, ϕ) of mappings is called an isomorphism between G and H.

Ex:

$\theta(v_1) = y$ and $\phi(e_1) = h$ $\phi(e_6) = c$

$\theta(v_2) = x$ $\phi(e_2) = g$ $\phi(e_7) = d$

$\theta(v_3) = u$ $\phi(e_3) = b$ $\phi(e_8) = f$.

$\theta(v_4) = v$ $\phi(e_4) = a$

$\theta(v_5) = w$ $\phi(e_5) = e$

is an isomorphism between the graphs G and H.

Complete graph: upto isomorphism, there is only one complete graph on n vertices.

→ Bipartite graph:
→ complete bipartite graph.

① Let $V = \{1, 2, \cdots n\}$. How many different graphs with vertex set V are there?

Sol: Each graph G with vertex set V is uniquely determined by its edge set E.

Set of all pairs in V is $\binom{V}{2}$

$\Rightarrow E \subseteq \binom{V}{2}$

$\Rightarrow$ There are $2^{\binom{n}{2}}$ graphs with vertex set V.

② How many non-isomorphic graphs with ~~four~~ four vertices are there?

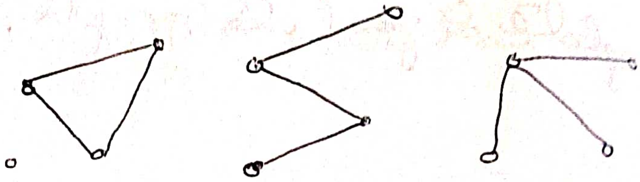
Sol: There are 11 graphs.

Assume $V = \{1, 2, 3, 4\}$.

1) graph with zero edge : only one

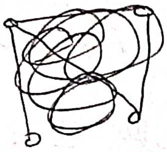
2). graph with one edge ;  only one

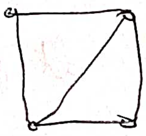
3). graph with two edges ;  two

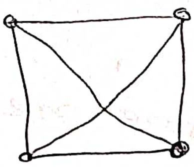
4) graph with three edges ;  two

5) graph with 4 edge ;

 two



6). graph with 5 edges ;  only one

7). graph with 6 edges  only one.

⇒ Isomorphic graphs have same no. of vertices, edges and same degree sequence.

⇒ But converse may not true.

Representation of graph:

→ picture of the graph
→ list of vertices and edges
→ Adjacency matrix: $|V| \times |V|$ with $a_{uv} = 0$ if $uv \notin E$
$= 1$ if $uv \in E$.
→ Adjacency list.

* Walk: sequence of vertices $v_1, v_2, \dots v_m$ such that v_j is adjacent to v_{j+1} for all $j = 1, 2, \dots m-1$.

The length of walk is $m-1$, the no of edges.

* Trail: A walk with all distinct edges.

* Path: A walk with all distinct vertices.

* Connected graph: For every two vertices u and v, we can find a walk $v_1, v_2, \dots v_m$ such that $v_1 = u$ and $v_m = v$.

→ A walk is closed if it has +ve length ~~with~~ and first and last vertex are same

→ A closed trail is called tour.

→ A closed path is a cycle.

→ Euler trail in a graph is a trail in which every edge of the graph appears exactly once.

→ Hamilton cycle is a cycle in which every vertex of a graph appears exactly once in the cycle.

Thm: Let a graph have at least one edge and no vertex of degree one. Then the graph contains a cycle.

Thm: A connected graph contain an Euler tour iff every vertex has even degree.

Thm: A connected graph contain an Euler trail iff the graph has exactly two vertices of odd degree.

* Degree sequence of a graph:

list all degrees of its vertices, in non-decreasing order.

Q: prove that two isomorphic graphs must have the same degree sequence. Is converse true. Justify your ans.

sol: let $G(V, E) \cong H(V', E')$

$\Rightarrow |V(G)| = |V'(H)|$

suppose $V = \{v_1, v_2, \cdots v_n\}$

$V' = \{w_1, w_2, \cdots w_n\}$

$\Rightarrow$ the degree sequences of G and H are

$deg(v_1), deg(v_2) \cdots, deg(v_n)$

$deg(w_1), deg(w_2) \cdots deg(w_n)$

Now, we have to show that both list contains the same numbers (in different order).

let $f: V \to V'$ be the isomorphism of G and H.

ie for $x, y \in V$, $xy \in E$ iff $f(x)f(y) \in E'$.

Now, for every v_i, $y \in N_G(v_i)$ iff $v_i y \in E$,

but $v_i y \in E$ iff $f(v_i) f(y) \in E'$

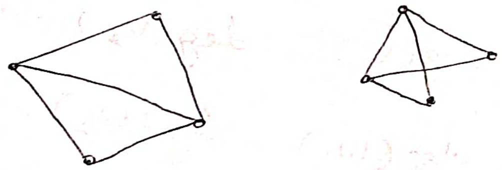
and $f(v_i) f(y) \in E'$ iff $f(y) \in N_H(f(v_i))$

Therefore, for every i,

$$\deg_G(v_i) = |N_G(v_i)| = |N_H(f(v_i))| = \deg_H(f(v_i))$$

that is, both list contains the same sequence.

<u>Converse</u>:



?? ISO

2, 2, 3, 3 2, 2, 3, 3

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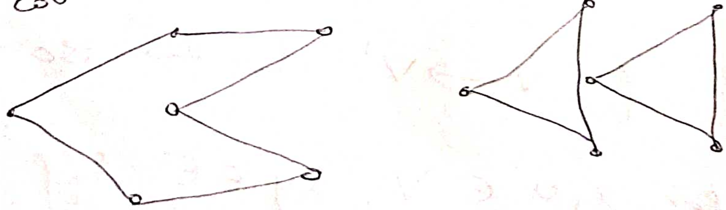
connected but not connected.



2, 2, 2, 2, 2, 2. But not isomorphism.

* A graph is K-regular, if all vertices have degree k.

Q: How many non-isomorphic 3-regular graphs with

6 vertices are there?

7 vertices are there?

Sol: we know that $\sum_{v} \deg(v) = 2|E|$ —— ①

7- vertices, each vertex have degree 3,

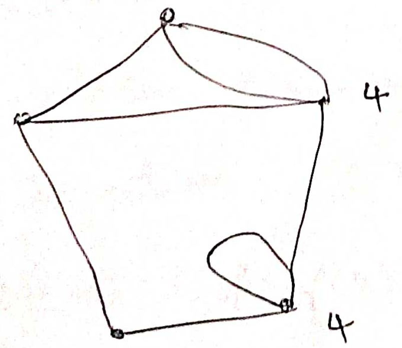
$\Rightarrow \sum_{v} \deg(v) = 7 * 3 = 21$ not even

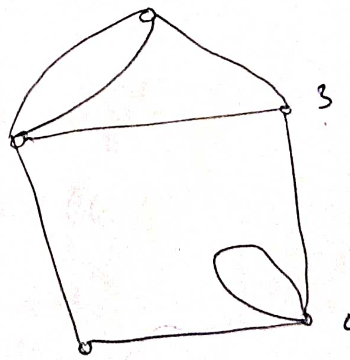
Contradiction to ①.

So, this graphs are zero.

6- vertices:

Q: Show that the following graphs are not isomorphic





Class-III
18/08/2022

Adjacency Matrix :

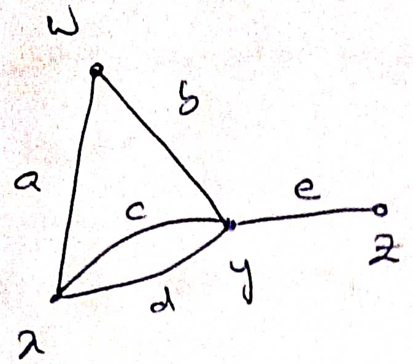
Def: Let G be a loopless graph with vertex set $V(G) = \{v_1, v_2, .. v_n\}$ and edge set $E(G) = \{e_1, e_2, ... e_m\}$. The adjacency matrix of G, written $A(G)$, is the $n \times n$ matrix in which entry a_{ij} is the number of edges in G with endpoints $\{v_i, v_j\}$.

Incident matrix $M(G)$: $n \times m$ matrix in which entry m_{ij} is 1 if v_i is an end point of e_j and otherwise it is 0.

$$A(G) = [a_{ij}]_{n\times n}, \quad a_{ij} = \begin{cases} \text{No. of edges with } \{v_i, v_j\} \\ . \end{cases}$$

$$M(G) = [m_{ij}]_{n\times m}, \quad m_{ij} = \begin{cases} 1, & v_i \text{ end point of } e_j \\ 0, & \text{otherwise.} \end{cases}$$

EX:



$$A(G) = \begin{array}{c|cccc} & w & x & y & z \\ \hline w & 0 & 1 & 1 & 0 \\ x & 1 & 0 & 2 & 0 \\ y & 1 & 2 & 0 & 1 \\ z & 0 & 0 & 1 & 0 \end{array}$$

$$M(G) = \begin{array}{c|ccccc} & a & b & c & d & e \\ \hline w & 1 & 1 & 0 & 0 & 0 \\ x & 1 & 0 & 1 & 1 & 0 \\ y & 0 & 1 & 1 & 1 & 1 \\ z & 0 & 0 & 0 & 0 & 1 \end{array}$$

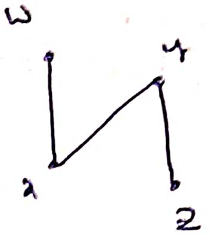
$\deg(w) = 2$
$\deg(x) = 3$
$\deg(y) = 4$
$\deg(z) = 1$.

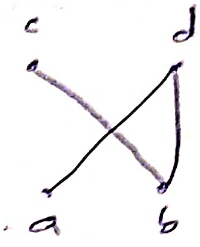
Remark:

1. Every adjacency matrix is symmetric, that is, $a_{ij} = a_{ji}$.
2. Every adjacency matrix of a simple graph has entries either 0 or 1, and with 0's on the diagonal.
3. The degree of v is the sum of the entries in the row for v in either $A(G)$ or $M(G)$.
4. The vertex bijection preserves the adjacency relation.

Def: An isomorphism from a simple graph G to simple graph H is a bijection $f: V(G) \rightarrow V(G)$ such that $uv \in E(G)$ iff $f(u)f(v) \in E(H)$. We denoted $G \cong H$.

EX: Given two graphs





G :   H :

$f : V(G) \rightarrow V(H)$ defined by

$f(w) = a, \ f(x) = d, \ f(y) = b, \ f(z) = c.$

$$\begin{array}{c} \\ w \\ x \\ y \\ z \end{array}\begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \quad \begin{array}{c} \\ w \\ y \\ z \\ x \end{array}\begin{array}{c} \begin{array}{cccc} w & y & z & x \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array} \quad \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Note: Isomorphic relation is an equivalence relation on the set of simple graphs.

→ Reflexive : $G \cong G$

→ Symmetric : $G \cong H \Rightarrow H \cong G$

→ Transitive : $G \cong H$ & $H \cong F \Rightarrow G \cong F$.

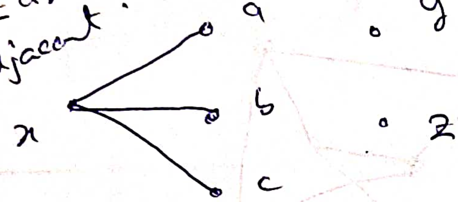
* How many non-isomorphic 3-regular graphs with 6 vertices are there?

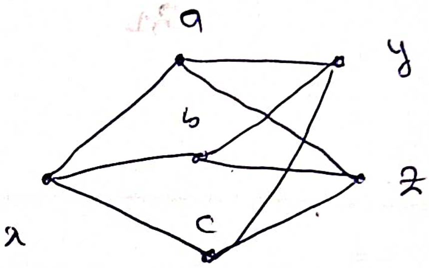
Sol: Let x be any vertex of 3-regular graph and a, b, c be its 3 neighbours.

~~The~~ Suppose y and z are the remaining two vertices.

We can observe that y and z are not adjacent to x.

Case-I: y and z are not adjacent.

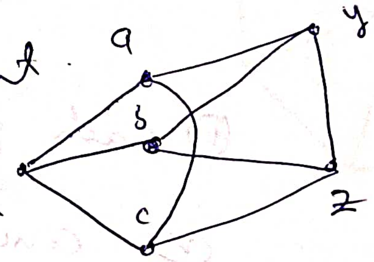




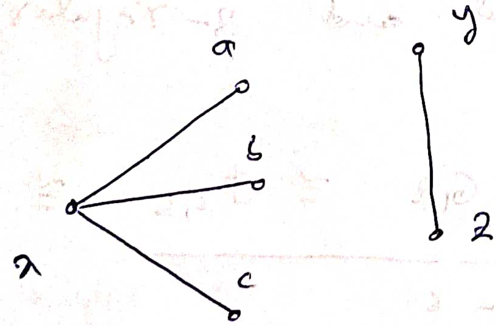
: $G1$

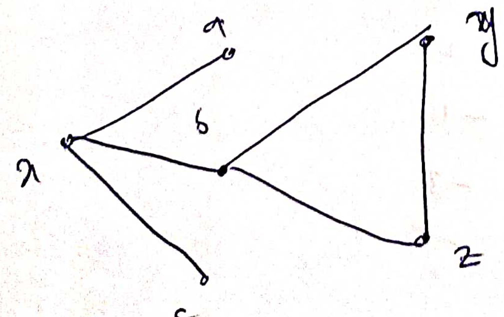
$\cong K_{3,3}$.

Case-II: y and z are adjacent.



: $G2$ contains K_3





$G1 \not\cong G2$ because

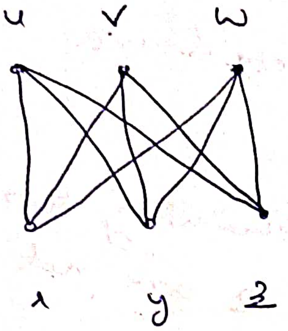
$G1$: is bipartite

$G2$: contain cycle with 3-vertices.

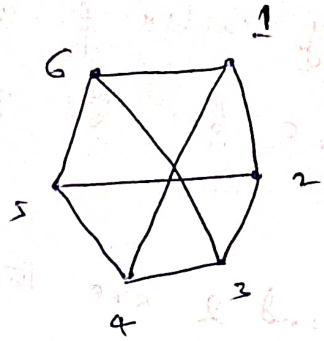
By pigeonhole principle.

y, z common neibour.

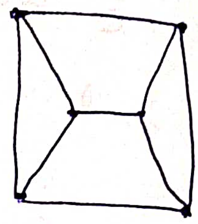
→ Determine which pairs of graphs below are isomorphic. Justify your answer.?



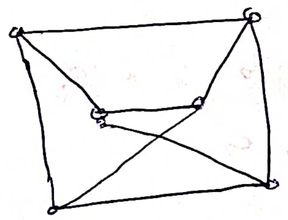
G1



G2.



G3



G4.

① Each graph has 6 vertices and 9 edges.

② connected and 3-regular graphs.

$G1 \cong G2 \cong K_{3,3} \cong G4$

u	1	6
v	3	4
w	5	2
x	2	1
y	4	3
z	6	5

But not G3 isomorphic to G1 / G2 / G3.

because, ~~$\overline{G1}$ consists of two disjoint 3-~~cycles, not connected

but, $\overline{G3}$ contains 6-cycle and connected

Reason: G1 $\cong K_{3,3}$

but G3 contains K_3, so it's vertices cannot be partitioned.

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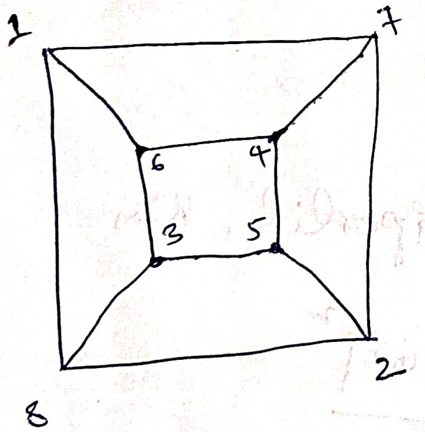
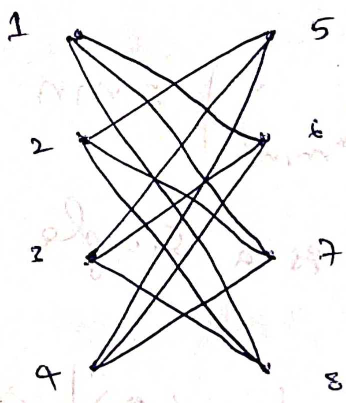
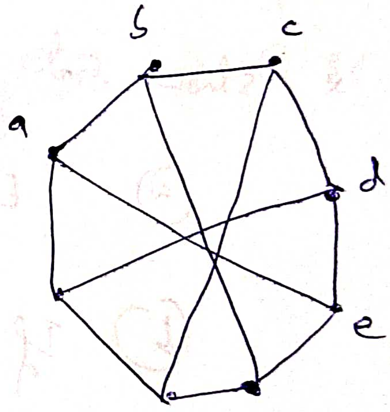
*





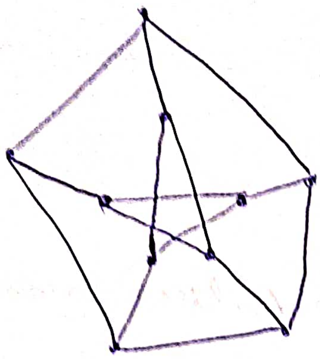


G1 $\cong$ G2 $\ncong$ G3

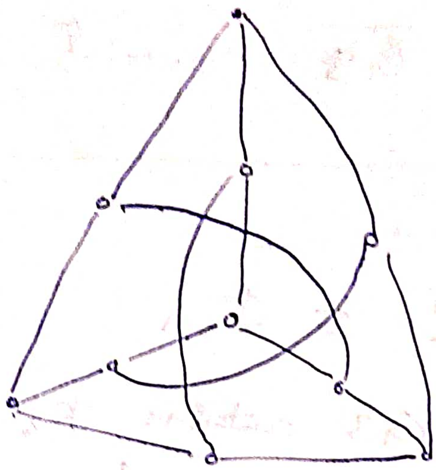
bipartite, so no odd cycle.

5-cycle odd cycle.

* Show that the following graphs are isomorphic.



petersen graph.



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* show that

ⓐ $|E(K_{m,n})| = mn$

ⓑ If G is a simple and bipartite, then

$$|E(G)| \leq \frac{|V(G)|^2}{4}$$

Theorem: A graph is bipartite iff it has no odd cycle.

* Let G be bipartite. Show that the vertices of G can be enumerated so that the adjacency matrix of G has the form

$$\left[\begin{array}{c|c} 0 & A_{12} \\ \hline A_{21} & 0 \end{array}\right]$$

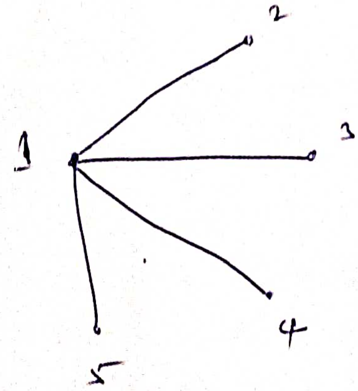
where A_{21} is transpose of A_{12}

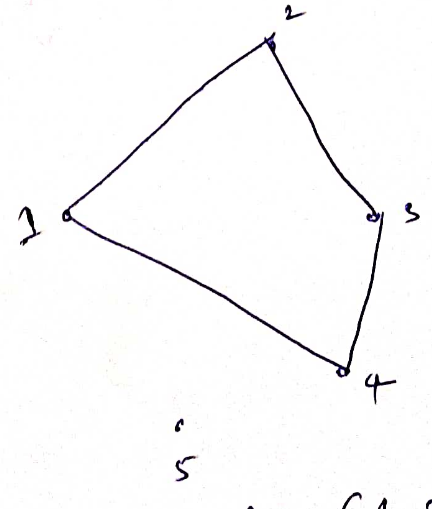
* Show that if G is simple and eigen value of A are distinct, then the automorphism of G is abelian.

State: If G and H are isomorphic, then Adjacency matrix A & B have the same eigenvalues.

converse, not true.

Ex:





eigenvalues: $0, -2,$ But $G1 \not\cong G2$.

Ex: Let M be the incident and A be the adjacency matrix of a graph G.

(a) show that every column sum of M is 2 :

(b) What are the column sum of A ? (degree(V))

* Automorphism of graph is an isomorphism from the graph to itself.

$\Rightarrow$ it form a group, denoted by aut(G).

if f and g are automorphism of A

then fg also.

$f : G \to G$ auto - bijection map.

$g : G \to G$ also - bijection map.

$A = PDP^T$, D - diagonal matrix with distinct entries.

$B = QDQ^T$, P, Q - orthogonal matrices.

($PP^T = I$).

A permutation matrix Π satisfies $\Pi A \Pi^T = B$

iff $\exists$ a diagonal $\pm$ matrix S for

which $\Pi P = QS$.

So, $\Pi \in aut(G)$, and $A \in S$ -set of

* <u>Subgraphs:</u>

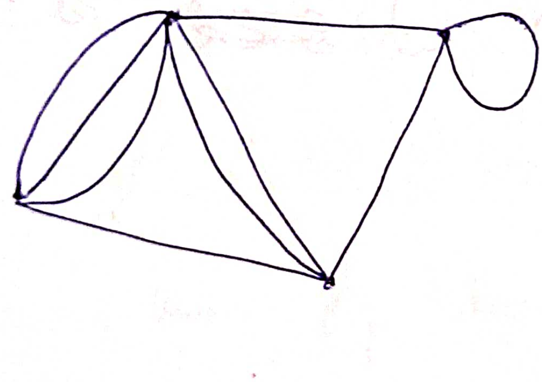
$H \subseteq G$ iff $V(H) \subseteq V(G)$

$E(H) \subseteq E(G)$

ψ_H is the restriction of ψ_G to $E(H)$.

<u>proper subgraph:</u> $H \subset G$

<u>spanning subgraph:</u> subgraph with $V(H) = V(G)$.



spanning - simple graph.

Trees

* **Cycle** : A closed trail whose origin and internal vertices are distinct is a cycle.

→ A cycle of length K is called K-cycle.

Def (Tree): A tree is a connected acyclic graph.

Theorem: In a tree, any two vertices are connected by a unique path.

proof: By contradiction.

Let G be a tree

Assume there are two distinct paths ~~(u,v)~~ P_1 and P_2 with (u,v) in G.

Since $P_1 \neq P_2$, there is an edge $e = xy$ of P_1 is not an edge of P_2.

clearly, graph $(P_1 \cup P_2) - e$ is connected.

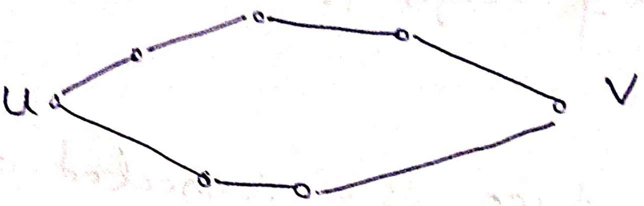
Therefore, it contains (x,y)-path P.

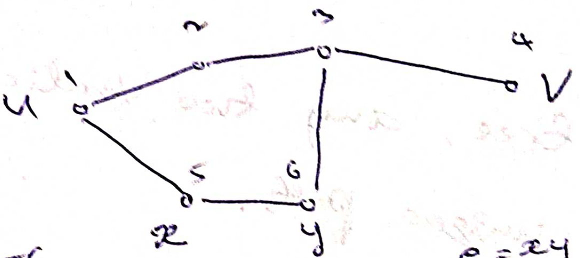
But, then $P + e$ is a cycle in the acyclic graph G,

$\rightarrow \leftarrow$.

P_1 : $u\, v_1 v_2 v_3 v_4 v_5\, v$

P_2 : $u\, u_1 u_2 u_3 u_4\, v$





P_2 : u 1 2 3 4 v

P_1 : u x y 5 v

$e = xy \in P_1$

$\notin P_2$

Note: $P : x \cdots y$

$P + e$: 1 2 3 5 6.

(x, y)-path P.

The converse of this theorem holds for graphs without loops.

Theorem: If G is a tree, then ~~$|V|$~~ $|E| = |V| - 1$.

proof: By induction on ~~$|E|$~~ $|V| = n$.

when $n = 1$, $G \cong K_1$ and $|E| = 0 = n-1$.

suppose theorem true for all trees on less than n vertices

let G be a tree on $n \geq 2$ vertices.

Let $uv \in E$, then $G - uv$ contains no (u,v)-path,

since uv is the unique in (u,v)-path in G.

Thus, $G - uv$ is disconnected. ~~and so~~

suppose G_1 and G_2 ~~of~~ are two components of $G - uv$.

and both are acyclic, and then trees.

since each have vertices less than n,

$$|E(G_1)| = |V(G_1)| - 1$$

$$|E(G_2)| = |V(G_2)| - 1$$

$$\Rightarrow |E(G)| = |E(G_1)| + |E(G_2)| + 1$$

$$= |V(G_1)| - 1 + |V(G_2)| - 1 + 1$$

$$= |V(G)| - 1.$$

Cor: Every nontrivial tree has atleast two vertices of degree one.

proof: Let G be a non-trivial tree.

Then $\deg(v) \geq 1$ for all $v \in V$.

We know that, $\sum_{v \in V} \deg(v) = 2|E|$

$$= 2(|V|-1)$$

$$= 2|V|-2.$$

It now follows that $\deg(v) = 1$ for atleast two vertices v.

Def: A cutpoint in a connected graph G is a vertex whose removal disconnects the graph.

Def: Forest: A graph with no cycles.

Tree: connected forest.

* If G is a forest, then $\text{comp}(G) = |V(G)| - |E(G)|$.

* Leaf is a vertex of degree 1.

* Let T be a tree with $|V(T)| \geq 2$. Then T has ≥ 2 leaf vertices. Further,

  if T has exactly 2 leaf vertices, then T is a path.

* If T is a tree and v is a leaf of T, then $T-v$ is a tree.

<u>proof</u>: We know that $T-v$ has no cycle.

If $u, w \in V(T-v)$, then

$\exists$ a path P from u to w in T

and this path cannot contain v,

so P is also a path in $(T-v)$.

Thus, $T-v$ is connected.

<u>Spanning tree</u>: If $T \subseteq G$ is a tree and $V(T) = V(G)$,

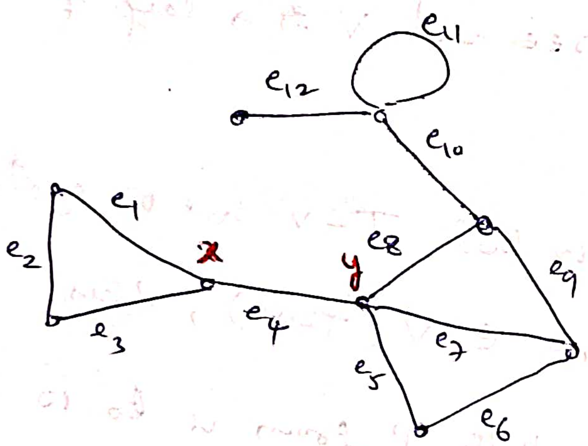
we call T is a spanning tree of G.

* Def: CUT EDGES and BONDS.

A cut edge is an edge e such that, (removal of it (disconnect the graph)) increase the components in the graph.

Theorem: An edge e of G is a cut edge of G iff e contained in no cycle of G. (proof) ??

EX:



cut edges ; e_4, e_{10}, e_{12}.

Theorem: A connected graph is a tree iff every edge is a cut edge.

proof: G is a tree and e is edge in G.

Since G is acyclic, e is contained in no cycle of G.

Therefore, e is a cut edge (from above theorem).

Theorem: An edge e of G is a cut edge iff e is contained in no cycle of G.

proof: $G = (E, V)$ is a graph.

Let $e \in E$ is a cut edge., prove e is contained in no cycle of G.

then, $\omega(G-e) > \omega(G)$ # component of G, denoted $\omega(G)$.

$\exists$ $u, v \in V$ that connected in G but, not in $G-e$.

$\therefore$ there is some (u,v)-path P in G, that traverse through e.

suppose, x and y are ends of e, and that x preceds y on P.

In $G-e$, u is connected to x by section of P and v is connected to y by section of P.

If e is in a cycle C, then x and y would be connected in $G-e$ by path $C-e$.

$\Rightarrow$ u and v are connected in $G-e$

$\rightarrow\leftarrow$.

Converse, Suppose $e = xy \in E$ is not a cut edge of G.

$\Rightarrow w(G-e) = w(G)$.

Since there is an (x,y)-path in G,

x and y are in the same component of G

$\Rightarrow$ x and y are in the same component of $G-e$, and hence, there is an (x,y)-path P in $G-e$.

$\Rightarrow$ e is ~~not~~ in the cycle $P+e$ of G.

$P: \; x v_1 v_2 \cdots v_k y$

$e = xy$

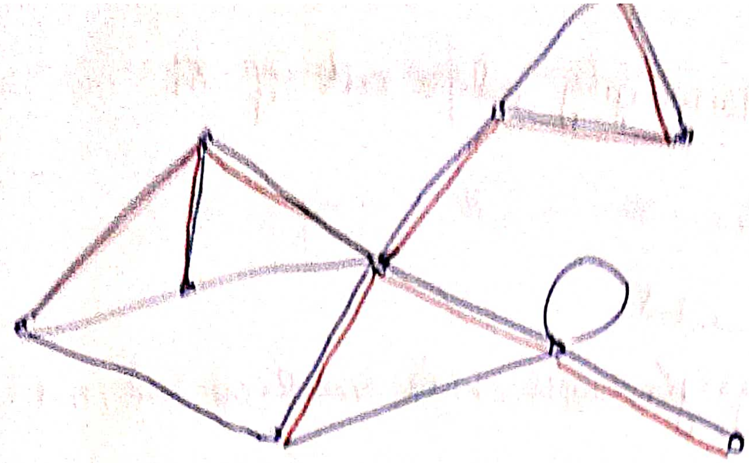
$P+e; \quad x v_1 v_2 \cdots v_k y x$ cycle.

Cor: Every connected graph contains a spanning tree.

Cor: If G is connected, then $|E| \geq |V| - 1$.

In spanning tree T, $|E| = |V| - 1$.

$|E(G)| \geq |E(T)| = |V(T)| - 1 = |V(G)| - 1$.



Spanning tree in a connected graph.

Th: Let T be a spanning tree of a connected graph G and let e be an edge of G not in T. Then T+e contains a unique cycle.

proof: Since T is acyclic graph,

each cycle of T+e contains e.

We know that, C is cycle of T+e iff

C−e is a path in T connecting two ends of e.

Also we know ~~we~~ from theorem, T has a unique path,

therefore, T+e contains a unique cycle.

Bond : A minimal nonempty edge cut of G

State: prove G is a connected graph with n vertices and n-1 edges, then G is a tree.

Sol: proof by induction on n.

→ For $n=1$, the only graph with one vertex and 0 edges is K_1 and it is tree.

Inductive assumption: Suppose that every connected graph with n-1 vertices and n-2 edges is a tree.

→ Let G be a connected graph with n vertices and n-1 edges.

Claim: G has a vertex v of degree 1.

we see that

$$2(n-1) = 2|E(G)| = \sum_v \deg(v).$$

If every vertex of G had degree atleast 2, then RHS of above equation would be atleast 2n, and that is not possible.

Since G is connected, every vertex has degree at least 1.

therefore, there must be a vertex v of degree 1.

Let u be the only neighbor of v in G.

<u>Claim:</u> $G-v$ is connected.

let x and y be two vertices in $G-v$.

Since G is connected, there is a path from x to y in G. (because $\deg(v)$ is 1, v is not part of (x,y)-path).

$\Rightarrow$ (x,y)-path in G is also a path in $G-v$.

$\Rightarrow$ $G-v$ connected.

Now, $G-v$ has $n-1$ vertices, $n-2$ edges, and connected.

By induction it must be a tree.

$\Rightarrow$ Hence, $G-v$ does not contain a cycle.

By adding vu back, it cannot create a cycle.

$\Rightarrow$ G is a tree, because G is acyclic.

-* prove by induction that every tree is a bipartite graph.

proof: Let T be a tree

$\Rightarrow$ T contains a vertex of degree 1, say v.

$\Rightarrow$ T−V is also a tree.

Now assume induction on vertices ~~of~~ $|V| = n$.

1) suppose $n = 1, 2$
then graphs are K_1 and K_2.
both are bipartite.

2) suppose any tree with $n-1$ ~~edges~~ and less ~~of~~ vertices
is bipartite graph.

3) Then we have to prove a tree with n vertices
is bipartite.

Let v be a vertex in T with degree 1.
and u be only its neighbour.

$\Rightarrow$ T−V is a tree with vertices $n-1$.

$\Rightarrow$ T−V is a bipartite.

$\Rightarrow$ suppose X, Y are bipartitions of T

If $u \in X$, then $X, Y \cup \{v\}$ is a bipartition of T.

Otherwise, the addition of v in X with $u \in X$ forms a cycle, $\rightarrow\leftarrow$.

(7) Let T be a tree with $|V(T)| \geq 2$. Then T has ≥ 2 leaf vertices.

Further, if T has exactly 2 leaf vertices, then T is a path.

Proof: If T has exactly 2 leaf vertices, then every other ~~vector~~ vertex of T has degree 2.

$\Rightarrow T$ is a path.

Theorem: Let G be a connected graph with $|V(G)| > 1$. If H is a subgraph of G chosen according to one of the following conditions, then H is a spanning tree.

(i) $H \subseteq G$ is minimal so that H is connected and $V(H) = V(G)$.

(ii) $H \subseteq G$ is maximal so that H has no cycles.

proof: (i). If H has a cycle C and $e \in E(C)$,

then $H-e$ is connected.

which contradicts the minimality of H.

Thus, H has no cycles, and it is a spanning tree

(ii). From the maximality of H, we have $V(H) = V(G)$

Suppose, X is the vertex set of a component of H,

and $X \neq V(G)$.

from the connectivity of G,

$\exists$ an edge e of G with one end point

in X and one end point in $V(G) \setminus X$.

Now, e is a cut edge of $H+e$.

So, $H+e$ has no cycles

which contradicting the maximality of H.

Thus, H has only one component, and

it is a spanning tree.

Forest: A forest is a graph with no cycles.

Theorem: If G is a forest, then $comp(G) = |V(G)| - |E(G)|$

proof: proof by induction on $|E(G)|$

Base step

1) if $|E(G)| = 0$,

then every component is isolated vertext.

So, $comp(G) = |V| - |E| = |V|$.

Inductive step.

2) Assume $|E(G)| > 0$ and a edge $e \in E(G)$.

Then we have for $G-e$,

$$comp(G) = comp(G-e) \cancel{+1} -1.$$

$$= |V(G-e)| - (|E(G-e)| + 1)$$

$$= |V(G)| - |E(G)|$$

Hence proved.

30/08/2021

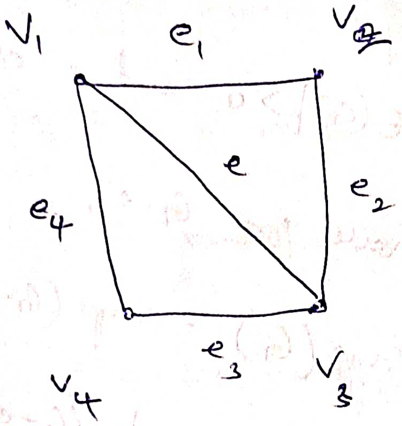
Quiz-01

CAYLEY'S FORMULA :

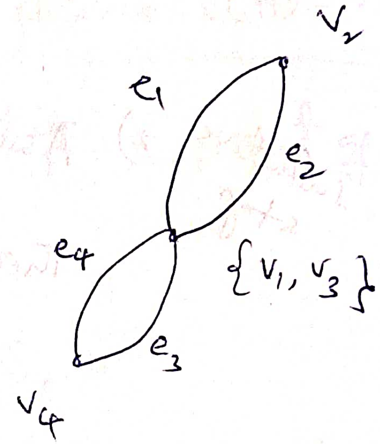
Contracted edge : edge e of G is contracted, if it is deleted and its ends are identified,

resultant graph denoted by $G.e$.

EX:



G



$G.e$

contraction of an edge e.

If e is a link of G, then

$$v(G.e) = v(G) - 1$$

$$\varepsilon(G.e) = \varepsilon(G) - 1.$$

$$\omega(G.e) = \omega(G).$$

→ We denote no. of spanning trees of G by $\tau(G)$.
→ An edge with distinct ends is a link.

Theorem: If e is a link of G, then $\tau(G) = \tau(G-e) + \tau(G \cdot e)$

proof: Suppose G is a graph and e an edge of G.

Then spanning tree of G, that does not contain 'e' is also a spanning tree of G - e.

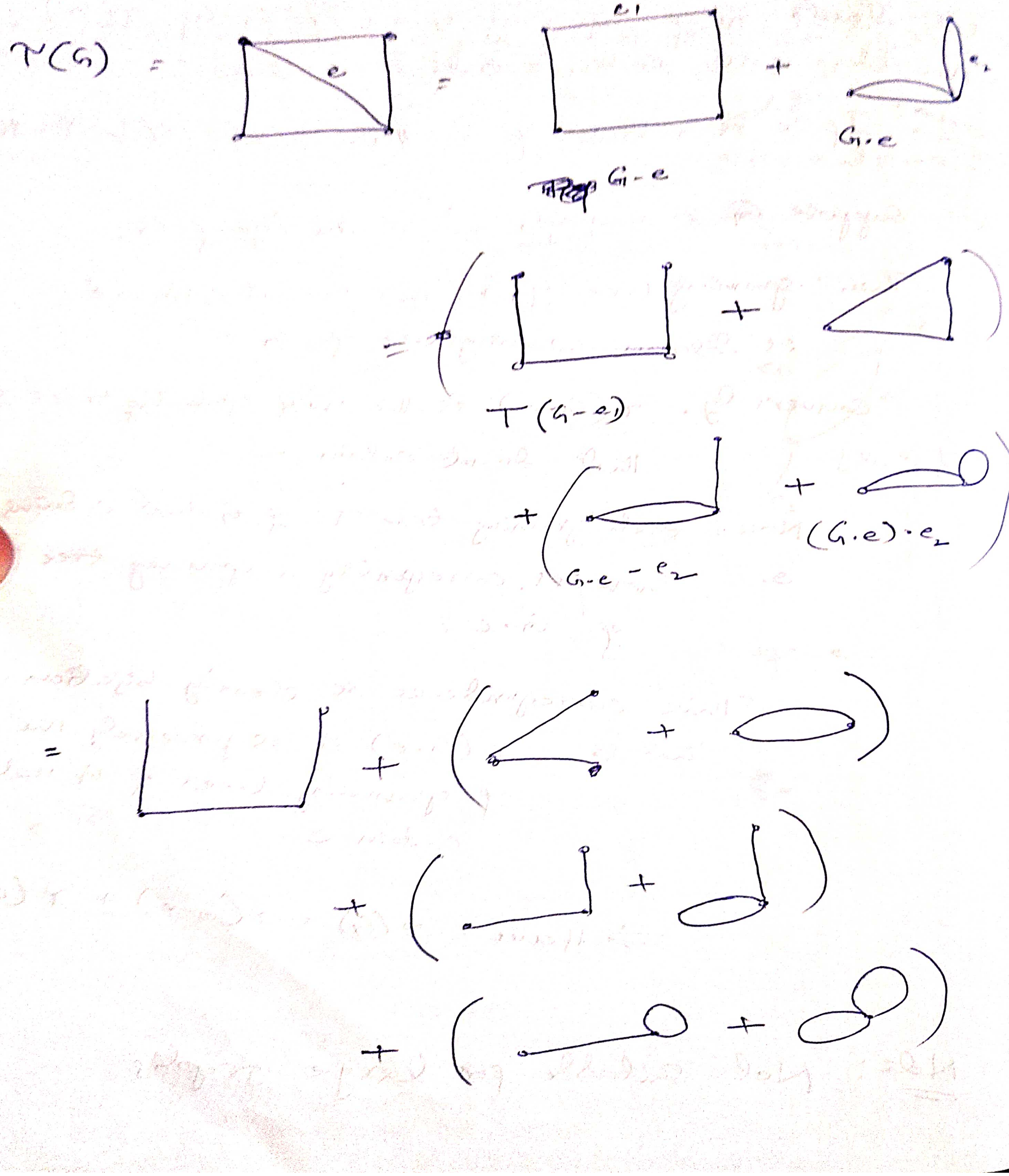
Conversely, $\tau(G-e)$ is the no. of spanning trees of G that do not contain e.

Now, each spanning tree T of G that contains e, there is corresponding a spanning tree $T \cdot e$ of $G \cdot e$;

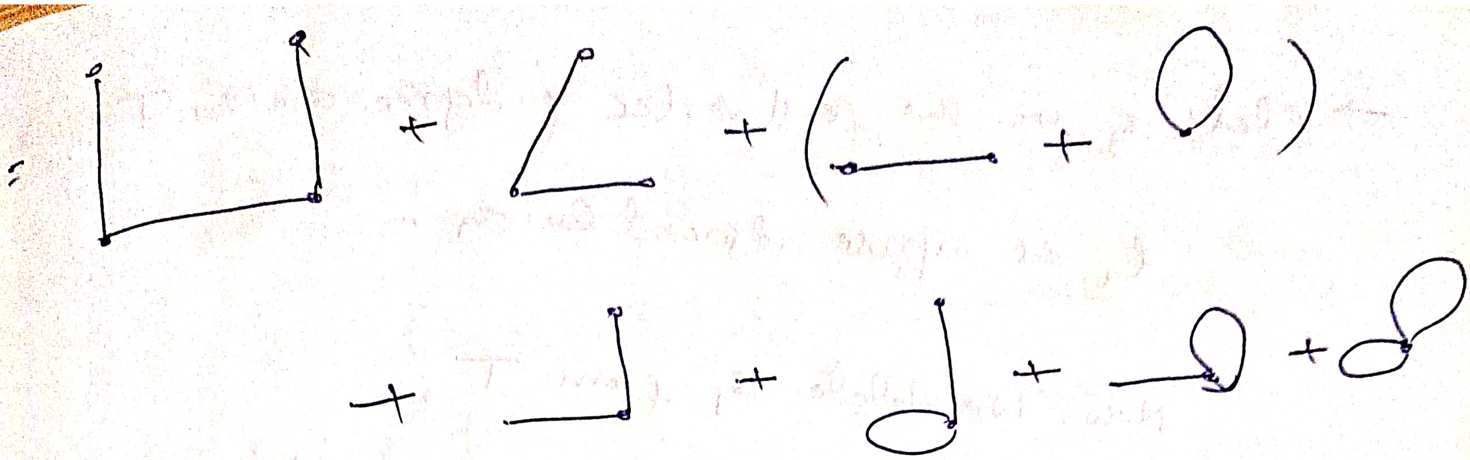
This correspondence is clearly bijection, that is, $\tau(G \cdot e)$ is is precisely the number of spanning trees of G that contain e.

$\Rightarrow$ Hence, $\tau(G) = \tau(G-e) + \tau(G \cdot e)$.

Note: Not suitable for large graphs



τ(G) =
e
e1
+
G·e
G - e
=
+
T(G - e1)
+
G·e - e2
(G·e)·e2
=
+
+
+



Recursive calculation of $\tau(G)$.

Theorem: $\tau(K_n) = n^{n-2}$.

proof: $V(K_n) = N = \{1, 2, \dots n\}$

We denote n^{n-2} is no. of sequences of length $n-2$ that can be formed from N.

Now, it is sufficient to establish 1-1 correspondence between $\tau(K_n)$ and set of n^{n-2} sequences.

With each spanning tree T of K_n, we associate a unique sequence $(t_1, t_2, \dots t_{n-2})$ as follows:

→ Let s_1 be the first vertex of degree one in T

t_1 is suppose adjacent to s_1.

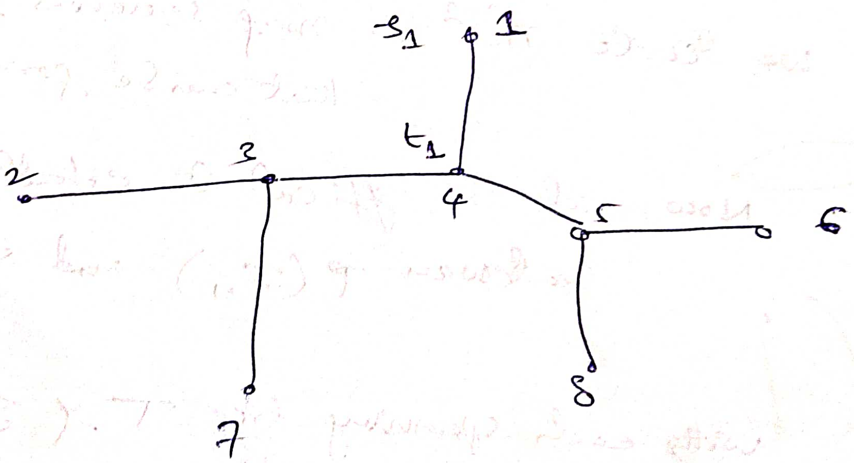
Now, we delete s_1 from T,

then denote by s_2 the first vertex of degree one in $T - s_1$.

Then, next take vertex t_2 adjacent to s_2.

repeat, this until t_{n-2} has been defined.

and a tree with two vertices remains.



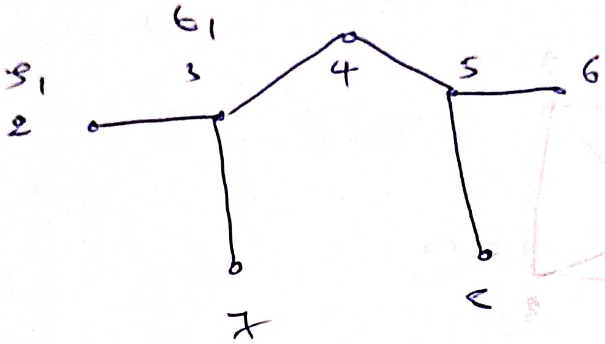
Tree - spanning tree T of K_n.

s_1 – Vertex : 1 2 6 7 3 8

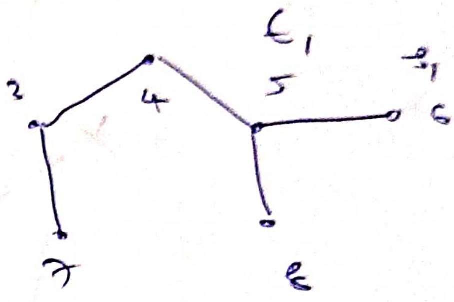
t_1 – ~~degree~~ adjacent : 4 3 5 3 4 5 Sequence $(t_1, t_2, \ldots t_{n-2})$.

(4, 3, 5, 3, 4, 5) – sequence for T.

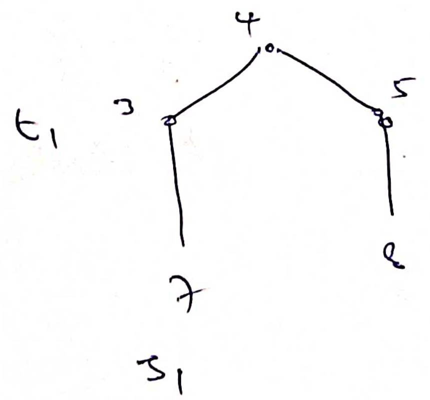
T = (1, 2, 6, 7, 8, 3, 4, 5) – degree sequence.



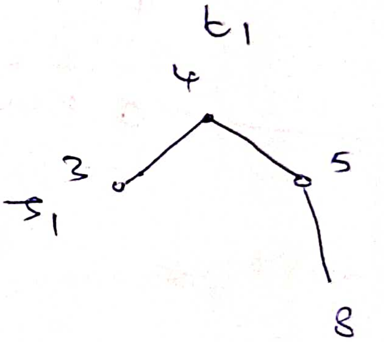
(2, 6, 7, 8, 3, 4, 5)



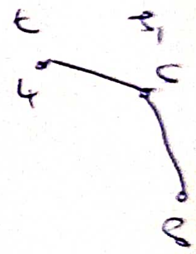
(6, 7, 8, 3, 4, 5)



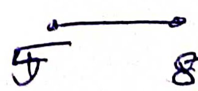
(7, 8, 3, 4, 5)



(3, 8, 4, 5)



(4, 8, 5).

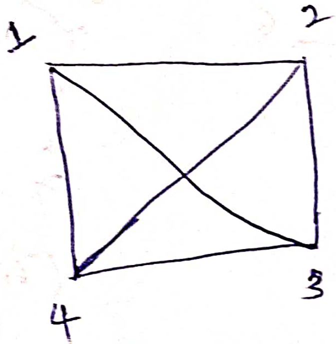


∴ Different spanning trees of K_n determine different sequences.

* There are 6 nonisomorphic spanning trees for K_6,

but $6^4 = 1296$ distinct spanning trees of K_6.

*



o End o

* Show that if e is an of K_n, then

$$\tau(K_n - e) = (n-2)n^{n-3}.$$

Sol: We know that

① $\tau(K_n) = n^{n-2}$.

② $\tau(G) = \tau(G-e) + \tau(G \cdot e)$

$$\nu(G \cdot e) = \nu(G) - 1$$

$$\varepsilon(G \cdot e) = \varepsilon(G) - 1$$

$$\omega(G \cdot e) = \omega(G)$$

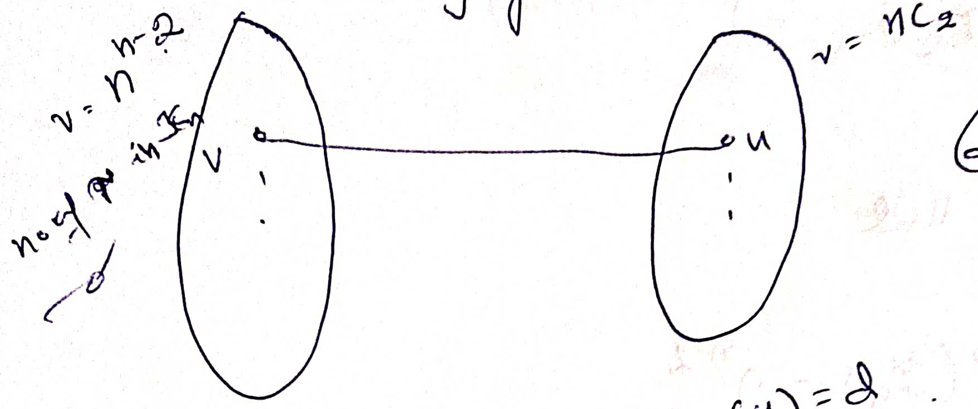
about K_n, → n^{n-2} - spanning trees

→ nC_2 - edges.

→ Tree contains $n-1$ edges.

* T E

Tree   biparlile graph.   edges



Graph G.

$deg(v) = n-1$ $deg(u) = d$.

→ establish a relation as follows:

* Draw an edge between two vertices whenever a tree contains an edge.

→ Tree have n-1 edges, so degree of every tree vertex in the above graph is (n-1).

→ By symmetry, every edge in T_n belongs to the same no of trees, say d, which is also degree of any edge-vertex in the graph.

Hence, the no. of edges in G,

$$\boxed{\text{edges(Tree)} = \text{edges (edges)}}$$

$$n^{n-2}(n-1) = d\, {}^nC_2$$

Means, no. of vertices × ~~no. of~~ the degree of vertices.

$$d = \frac{n^{n-2}(n-1)}{\frac{n(n-1)}{2}} = 2n^{n-3}$$

The no. of tree that contain a given edge is d.

So, the no. of tree that do not contain edge,

is $$n^{n-2} - d = n^{n-2} - 2n^{n-3}$$

$$= n^{n-3}(n-2).$$

Other sol:

$\tau(K_n) = n^{n-2}$.

x - no. of spanning tree that ~~does not contain~~ contains e

We have to find $n^{n-2} - x$.

Let m be the no. of pairs (T, e)
where T spanning tree of K_n
and e is edge of T.

$$m = n^{n-2}(n-1) \quad -① $$

$$m = \binom{n}{2} x \quad -② $$

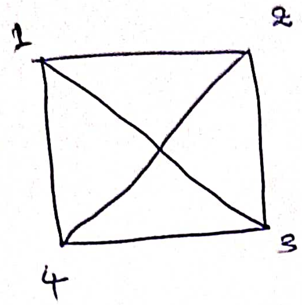
$$x = \frac{n^{n-2}(n-1)}{nC_2} = 2n^{n-3}.$$

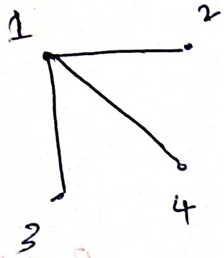
$\Rightarrow$ no. of spanning tree that donot contain e

are $n^{n-2} - 2n^{n-3} = (n-2)n^{n-3}$.

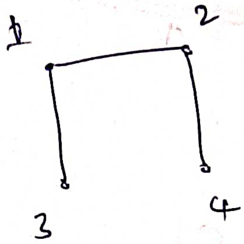
* Draw all sixteen spanning trees of K_4

$\tau(K_4) = 4^{4-2} = 4^2 = 16$ ⇒ n^{n-2} # sequences of length $(n-2)$.

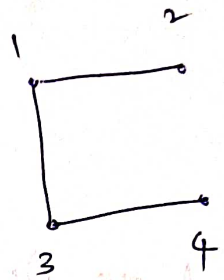




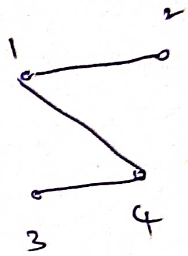
(1, 1)



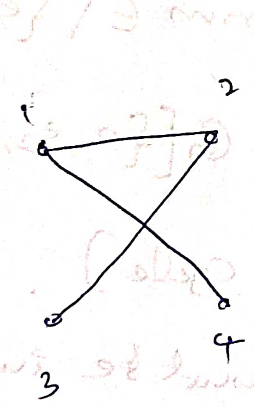
(1, 2)



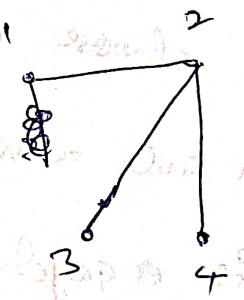
(1, 3



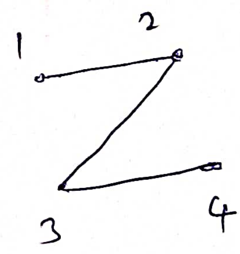
(1, 4)



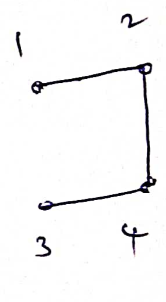
(2, 1)



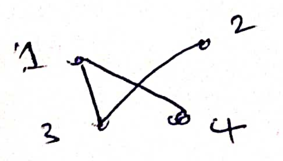
(2, 2)



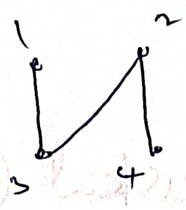
(2, 3



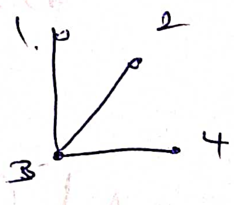
(2, 4)



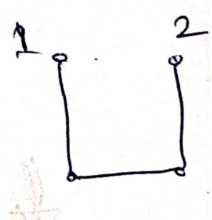
(3, 1)



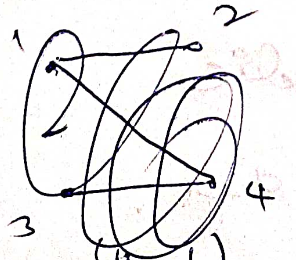
(3, 2)



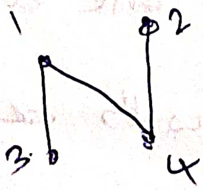
(3, 3



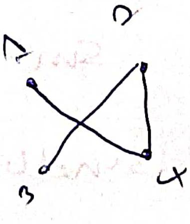
(3, 4)



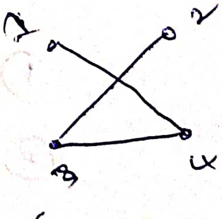
~~(4, 1)~~



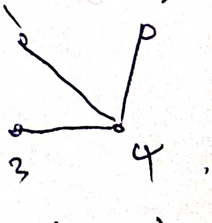
(4, 1)



(4, 2)

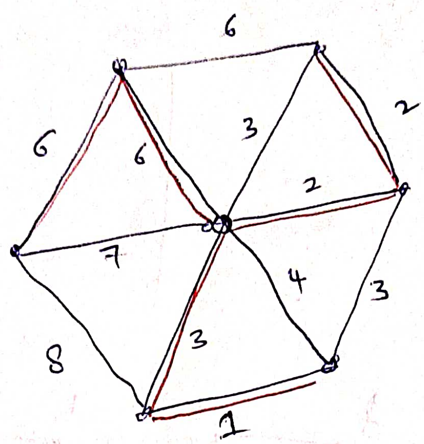


(4, 3)



(4, 4)

Minimum spanning tree :



Algo: 1) choose link e_1.

2) If edges $e_1, e_2, \cdots e_i$ have been chosen, then choose e_{i+1} from $E \setminus \{e_1, e_2, \cdots e_i\}$ in such a way that $G[\{e_1, e_2, \cdots, e_{i+1}\}]$ is ~~a~~ acyclic (no cycle).

3). stop when step 2 cannot be implemented.

Extended by Kruskal (1956).

1) small one choose

2) small with acyclic

3) stop if ~~cycl~~ 2 stops.

Theorem: Any spanning tree $T^* = G[\{e_1, e_2, \ldots e_{v-1}\}]$ constructed by Kruskal's algo is an optimal tree.

Unit-3: Hamilton cycles & Euler Tours.

Euler Trail : Traverse every edge of G.

Tour : A closed walk that traverses each edge of G at least once.

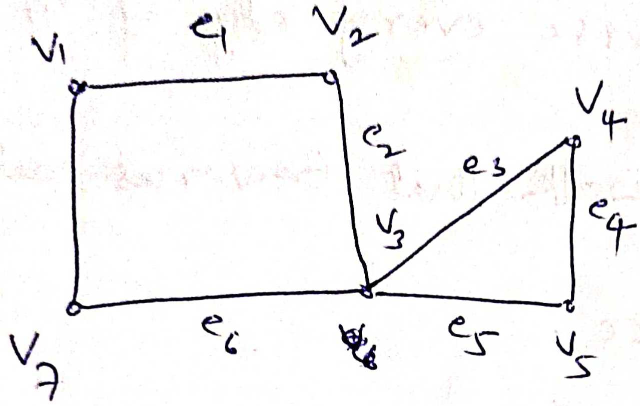
Euler Tour : A Tour which traverses each edge exactly once.

Eulerian graph ; it contains an Euler tour.

Theorem : A non-empty connected graph is eulerian iff it has no vertices of degree odd.

Proof :

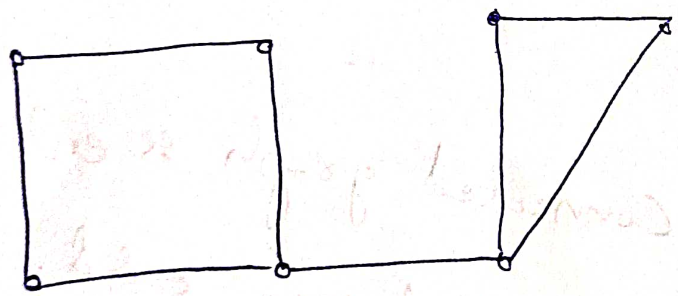
Example: Consider the following graph



Eulerian graph.

$V_1 V_2 V_3 V_4 V_5 V_3 V_7 V_1$

$e_1 e_2 e_3 e_4 e_5 e_6$



Non-Eulerian graph.

Euler Theorem: A connected graph G is an Euler graph iff all the vertices of G are of even degree.

proof: ~~Necessary~~:

Assume that $G(V, E)$ is Euler graph.

Then G contain an Euler tour C.

Now, each time a vertex v occurs as an internal vertex of C, two of the edges incident with v are counted for:

$C: u v_1 v_2 \cdots v_i \cdots v_n u$,

Since Euler tour contains every edge of G, then $\deg(v)$ is even for all $v \neq u$.

Similarly, since C starts and ends with u, $\deg(u)$ is also even.

Hence, G has no vertices of odd degree.

Conversely, Suppose G is a non-eulerian graph connected graph with atleast one edge and no vertices of odd degree.

Choose such a graph G with as few edges as possible.

Since each vertex of G has degree atleast two G contains a closed trail.

Let C be a closed trail of maximum possible length in G.

By assumption C is not Euler tour of G, and so $G - E(C)$ has some component G' with $\varepsilon(G') > 0$.

Since C is itself Elerian, it has no vertices of odd degree.

$\Rightarrow$ G' also has no odd degree vertices.

$\Rightarrow$ since $\varepsilon(G') < \varepsilon(G)$, proof: Let $G(V, E)$ - Euler graph

C - Euler tour of G.

Converse: Assume

G - Non-eulerian graph ~~connected~~.

- ~~with~~ Connected
- Atleast one edge
- No vertices of odd degree.

prove contradicts.

choose G with ~~as~~ few edges as possible.

since $deg(v) \geq 2$, G contains closed trail, ~~C~~.

Now, say C - is a closed trail of maximum length in G.

By assumption, C is not Euler tour of G.

$\Rightarrow$ $G - E(C)$ has some component G' with $\varepsilon(G') > 0$ $\Rightarrow$ $G - E(C)$ is an even degree graph.

because C itself is even ~~degree~~ Euler path.

~~Next, let C_1 be a the~~

Let C_1 be one of the maximum component of $G - E(C)$.

Since $d(v)$ is even for $v \in E(G-E(C))$ $\Rightarrow \exists C_1$ euler tour of $G-E(C)$.

As C_1 has less no. of edges than G, it is Eulerian.

Because G is connected, $\exists$ v in $V(C) \cap V(C_1)$

Without loss of generality, we assume that v is a origin & terminus of both C and C_1.

But, CC' is a closed trail of G with

$\varepsilon(CC') > \varepsilon(C)$,

contradicting the choice of C.

Cor: A connected graph has Euler trail iff it has at most two vertices of odd degree.

proof: from above theorem,

except end vertex, each internal vertex have the even degree.

Converse: G - nontrivial connected graph with atmost two vertices of odd degree.

If G has no such vertices, then G has closed Euler trail.

otherwise, G has exactly two vertices ~~of~~ u and v of odd degree.

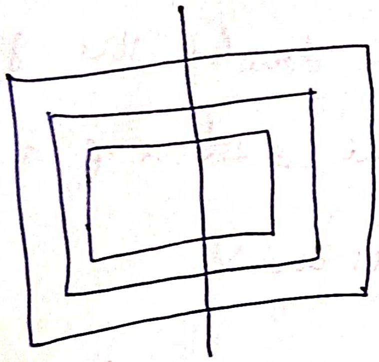
In this case, let $G+e$ denote the graph obtained from G by the addition of a new edge e joining u and v.

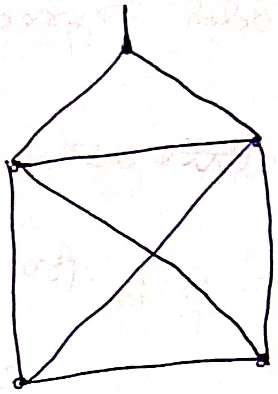
Then each vertex in $G+e$ has even degree and so, $G+e$ has an Euler tour.

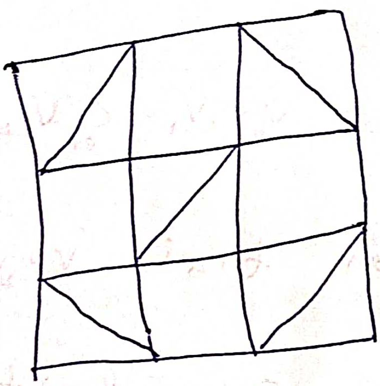
$C = v_0 e_1 v_1 \cdots e_{i+1} v_{i+1}$, where $e_1 = e$.

Then trail $v_1 e_2 v_2 \cdots e_{i+1} v_{i+1}$ is an Euler trail of G.

Ex: Which of the following figures can be drawn without lifting one's pen from the paper or covering a line more than once.







Ex: If possible, draw an eulerian graph G with v even and ε odd; otherwise, explain why there is no such graph.

Theorem: A connected graph G is Eulerian graph iff its edge set can be decomposed into cycles.

Theorem: A graph is Eulerian iff it has an odd number of cycle decompositions.

Theorem: A connected graph G is Eulerian iff its edge set can be decomposed into cycles.

proof: Let G(V, E) - connected graph

G - decomposed into cycles.

Then we have to prove that G is Eulerian. $\boxed{\text{its } \cancel{\text{edge}} \text{ set can be decomposed into cycles}}$.

If K of these cycles incident at a particular vertex V,

then $deg(V) = 2K$.

$\Rightarrow$ degree of every vertex of G is even

$\Rightarrow$ Hence, G is Eulerian.

Converse: Let G ~~be~~ Eulerian.

Then we have to show G can be decomposed into cycles.

we use induction on the number of edges.

Since $\deg(v) \geq 2$ for every $v \in V$,

G has a cycle C.

Then $G - E(C)$ is possibly a disconnected graph

$\Rightarrow$ ~~each~~ of each of the component, say

$C_1, C_2, \cdots C_k$ is an even degree graph.

and hence Eulerian graph.

$\Rightarrow$ By the induction hypothesis, each C_i is

a disjoint union of cycles.

$\Rightarrow$ The cycles together with C provide

a partition of $E(G)$ into cycles.

Theorem: If W is a walk from vertex v, then W contains an odd number of u-v paths. (H/W).

Theorem: A connected graph is Eulerian iff each of its edges lies on an odd number of cycles.

proof: Necessary:

Suppose G - connected Eulerian graph

$e = uv$ be any edge of G.

Then, $G-e$ is a u-v walk, say W.

$\Rightarrow$ $W = G-e$ contains an odd number of u-v paths.

$\Rightarrow$ Each path P in W together with e gives a cycle in G containing e.

$\Rightarrow$ Hence, there are odd no. of cycles in G containing e.

Sufficient:

Suppose G is a connected and its edges lies on odd no. of cycles.

We have to prove G is Eulerian.

Let v be any vertex in G and $E_v = \{e_1, e_2, \ldots e_d\}$ be the set of edges of G incident on v.

then $|E_v| = deg(v) = d$.

For each i, $1 \le i \le d$, let k_i be the no. of cycles of G containing e_i.

By the assumption, each k_i is odd.

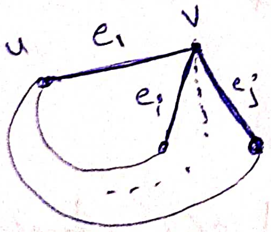
Let $c(v)$ be the no. of cycles of G containing v.

Then, $c(v) = \frac{1}{2} \sum_{i=1}^{d} k_i$

$$\Rightarrow \quad 2c(v) = \sum_{i=1}^{d} k_i$$

$\Rightarrow$ since $2c(v)$ even and each k_i is odd ~~even~~

then d is even.

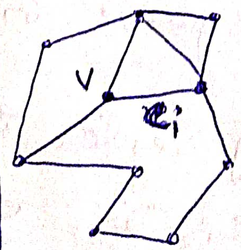


$\Rightarrow$ degree of every vertex is even

$\Rightarrow$ G is Eulerian.

→ for each edge pair e_1, e_i → we get count same cycle.

→ that is, one cycle counted twice which contains v.



$c(v) =$

$deg(v)$

Not Eulerian