Constraint → soft

End Sem

28/10/2022

Friday

Covering:- $K \subseteq V \ni$ every edge of G has atleast one end is K

Covering Number - $\beta(G)$

- Cardinality of minimum covering

Edge Covering:-

A $L \subseteq E \ni$ each vertex of G is an end point of some edge is L.

Q. whether always L exist?

A. We cannot always define L.

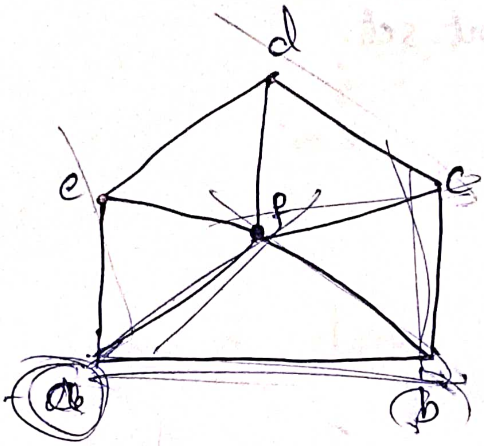
Independent Set:-

A Subset S of V is called independent set of G if no two vertices of S are adjacent in G.

An independent set is maximum if G has no

~~adjacent~~
independent set 's'' with

$|S'| > |S|$



Let $S = \{a\}$ ~~S'~~ $= \{c, d\}$

Not an independent subset $|S'| > |S|$

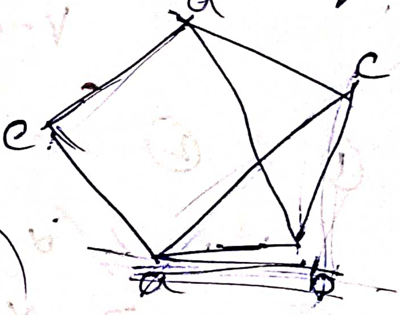
Now

$S = \{a, c\}$ → Independent subset as no S'

maximum independent subset

Clique

A Clique of a Simple graph G is a subset S of V such that G[S] is complete.



$\{a, b, c\}$ is a clique

$\alpha(G)$: Independence number of G

→ Cardinality of maximum independent set of G.

Theorem:- A Set $S \subseteq V$ is an independent set iff $V \setminus S$ is a covering of G.

Proof:- S is independent

⇒ no edges of G has both ends in S

⇒ no edges of G has both ends in S
⇒ no edges of G has both ends in S

⇒ Each edge has atleast one end in V/S

that is $V \setminus S$ is a covering

Corollary:- $\alpha + \beta = V$

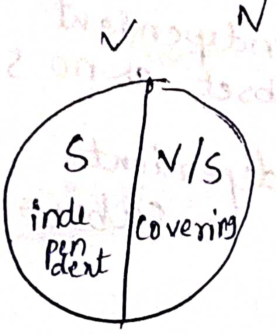
Proof:-

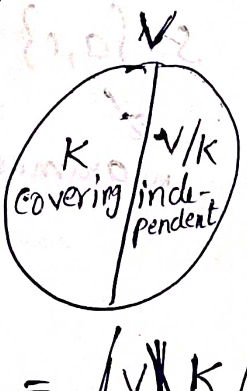
S- Maximum independent set

K- minimum covering

Then V/K is an independent set

V/S is a covering





$$\nu - \beta = |V \setminus K| \leq \alpha \quad — ①$$

$$\nu - \alpha = |V \setminus S| \geq \beta \quad — ②$$

① & ②

$\nu \leq \alpha + \beta$ This is possible only when $\alpha + \beta = \nu$

$\nu \geq \alpha + \beta$

from ① & ② $\Rightarrow$ Theorem i.e $\alpha + \beta = \nu$:

α' - edge independent number) cardinality of maximum matching.

β' - edge ~~in~~ covering number) cardinality of minimum edge covering.

Theorem:- If $\delta > 0$, then $\alpha' + \beta' = \nu$

H/W

01/01/2022
Tuesday

Ramsey's Theorem

Independent set $S \subseteq V$ ← Non adjacent

clique $K \subseteq G$ G^c

Adjacent

* S is a clique of G iff S is independent set of G^c

$\boxed{r(S,t)\ ,\ S,t \geq 1}$ → coloring problem
coloring a graph with 2 colors eg:-
Red, Blue

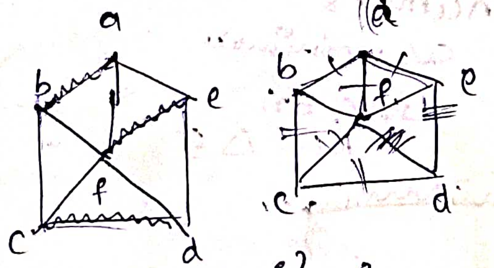
S clique with Blue colour
T clique with Red colour

K_n → Two colour

$G \cup G^c = K_n$

→ G

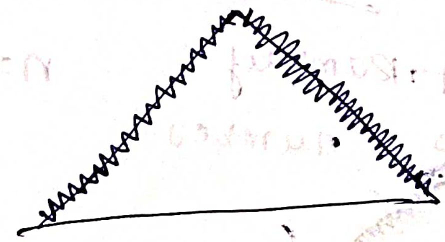
→ G^c



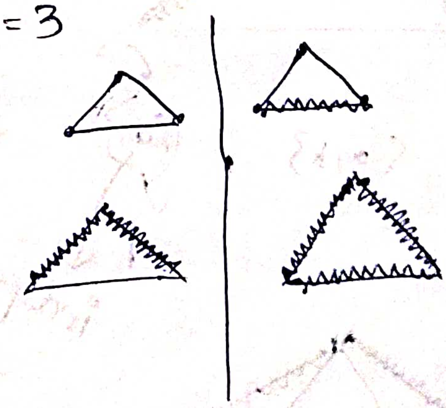
$M = \{ab, cd, ef\} = 3$

$K = \{a, c, e\} = 3$

$M + K = 3 + 3$
$= 6$



for $n = 3$



Ramsay's Theorem:-

Statement :-

The Smallest n such that every 2-coloring of K_n conta a monochromatic clique of order s or t. It is denoted by $r(s,t)$ known as Ramsay number

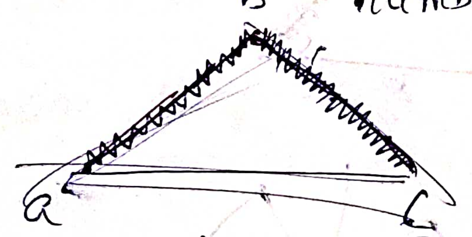
$K(2,1)$ possible?

Remark :-

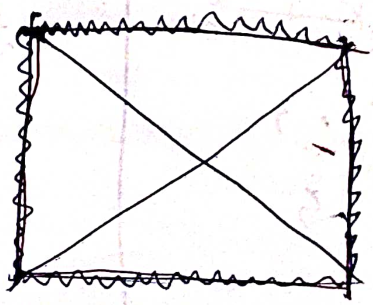
1. For all possible integers s,t $\exists$ $r(s,t)$ Such that if $n \geq r(s,t)$ and edges of K_n are colored with Red or Blue, then there is a "Red k-clique or " Blue l-clique "

$r(3,3) = 6$ ~~~~~ Red
—— Blue

for $n=3$ not Ramsay number

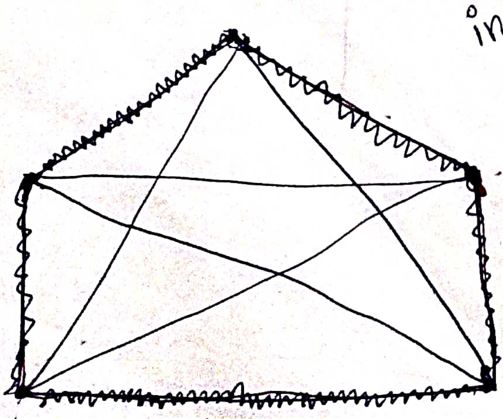


$n=4$ not Ramsay number as minimum Blue (or) Red is 3



$\{a,b\}$ $\{a,\}$ independent

for $n=5$



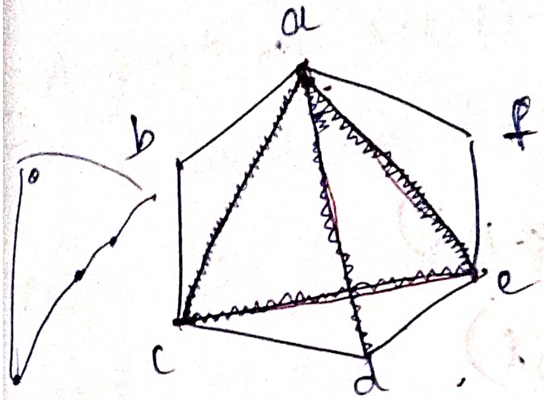
$n=6$

$r(s,1)=1$ $r(3,3)=6$

$r(1,t)=1$

$r(2,t)=t$

$r(s,2)=s$



$r(3,4)=?$

$r(2,t)$

K_2 - Blue

K_t - red

?

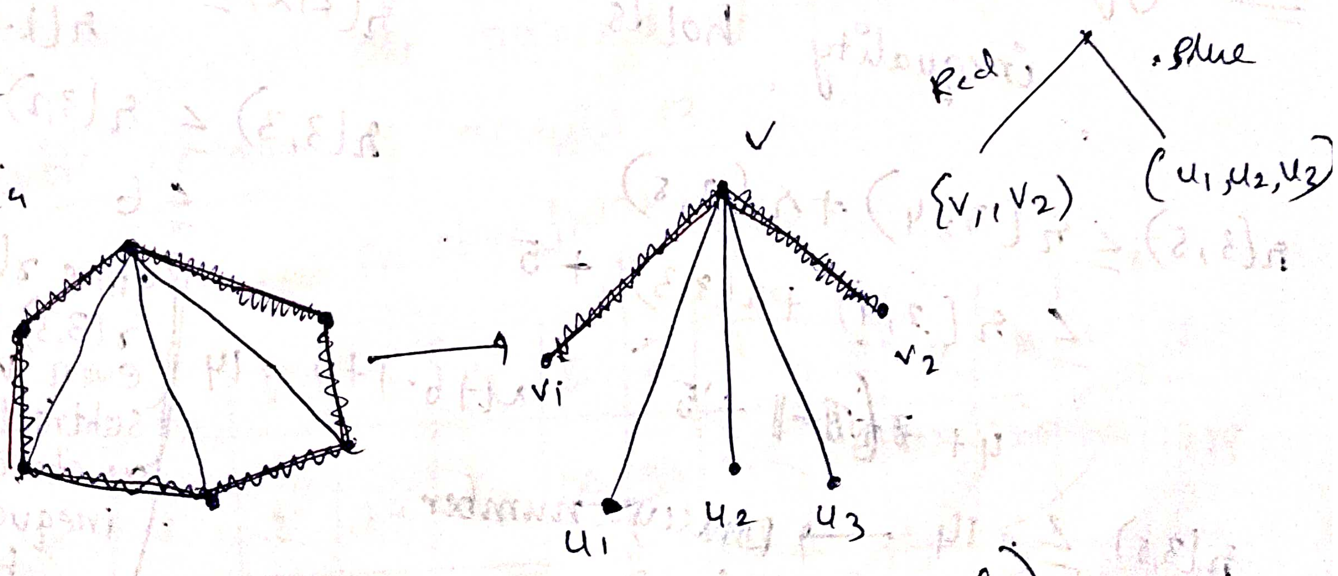
Statement:- $r(s,t) \le r(s,t-1) + r(s-1,t)$

Theorem:-

$$r(s,t) \le r(s-1,t) + r(s,t-1)$$

Proof:-

- idea



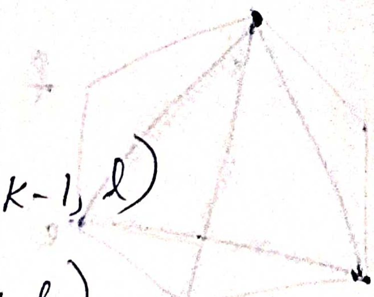
Proof by induction:-

$$N = r(k, l-1) + r(k-1, l)$$

03 | 11 | 2022

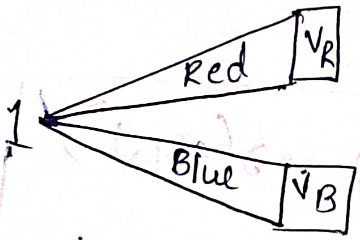
Thursday

State ment :-



$$r(k,l) \leq r(k,l-1) + r(k-1,l)$$

$$N = r(k,l-1) + r(k-1,l)$$



$\{(1,x)\ x \in V$ redcoloured$\}$

$\{(1,x)\ x \in V$ blue coloured

$$\boxed{N-1 = |V_R| + |V_B|}$$

Note:- If both $r(k,l-1)$ & $r(k-1,l)$ are even then strict inequality holds

$r(k,l) \leq r(k,l-1) + r(k-1,l$

$r(3,3) \leq r(3,2) +$

≤ 6

$r(3,5) \leq r(3,4) + r(2,5)$

$\leq r(2,4) + r(3,3) + 5$

$4 + 6 - 1 + 5$

$4 + 6 - 1 + 5 = 14$

since $r(2,4)$ $r(3,3)$ are even we subtract 1 as strict inequality hold

$r(3,5) \leq 14 \longrightarrow$ Ramsey number

~~$r(3,3) \leq r(3,2) + r(2,3)$~~

$r(4,5) \leq r(4,4) + r(3,5)$

$\leq r(4,3) + r(3,4) + r(3,5)$

$\leq [r(4,2) + r(3,3)] + [r(2,4) + r(3,3)] + [r(3,4) + r(2,5)]$

even even even even

$\leq 4+6-1 + 4+6-1 + r(3,4) + 5$

$\leq 23 + r(3,3) + r(2,4) \Rightarrow \leq 23 + 6 + 4 - 1$

≤ 32 | Ramsay number (32)

K-l Ramsey Graph:-

A graph on $r(k;l)-1$ vertices that contains neither a k-clique nor l-independent set

Note:- By definition of $r(k,l)$, the (k,l) Ramsey graph exists for $k\geq 2$, $l \geq 2$

) Can you draw (3,3)-Ramsey graph?

(3,5) → Ramsey graph

Number of vertices 13

Finite field $(\mathbb{Z}_{13}, +_{13}, \times_{13})$ → where we can apply addition and multiplication freely.

$\mathbb{Z}_{13} = \{0,1,2,\dots 12\}$

↳ set of all ~~integers~~ remainders when we divide all integers with '13' $\mathbb{Z}_{13}$

* Two vertices are adjacent if their difference is a cubic residue of modulo 13

cubic residue → cubic ~~root~~ in class fields

eg:-

x	1	2	3	4	5	6	7	8	9	10	11	12
x^3	1	8	27	64	125	216	343	512	729	1000	1331	1728
Modulo 13 $x^3 \% 13$	1	8	1	12	8	8	5	5	1	12	5	12

Cubic residue of modulo 13 = $\{1, 5, 8, 12\}$

↳ set of elements $\exists x$ $\exists$ If $a = x^3$

$(4,4) \to$ Ramsey graph $\le r(4,3) + r(3,4)$

$[r(4,2) + r(3,3)] + [r(3,3) + r(2,3)]$

$= [4+6-1] + [6+3] \le 10-1+9 = \le 18$

Statement:-

$$r(k,l) \le \binom{k+l-2}{k-1}$$

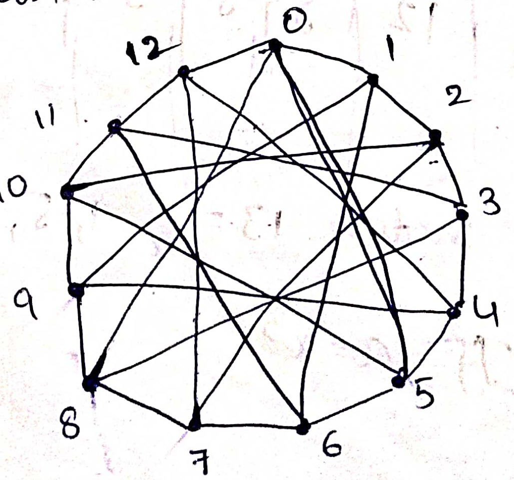
Proof:- By induction

from $(1,l) = r(k,1) = 1$

$r(2,l) = l$

$r(k,2) = k$

Theorem holds when $k+l \le 5$



statement

① $r(k,k) \geq 2^{k/2}$

② If $m = \min\{k, l\}$ then $r(k,l) \geq 2^{m/2}$

04 | 11 | 2022
Friday

Schur's Theorem:-

eg:- $\{1, 2, \ldots, 13\}$

partions $\{1, 4, 10, 13\}$
$\{2, 3, 11, 12\}$
$\{5, 6, 7, 8, 9\}$

Mutually exclusive

$x+y=z$
No partition that satisfies this condition.

Let $\{S_1, S_2, \ldots S_n\}$ be any partition of the Set of integers $\{1, 2, \ldots, r_n\}$ where r_n Ramsay number. Then for some i, S_i contains three integers x, y, z satisfy the $x+y=z$

Proof:- K_{r_n} with vertices $\{1, 2, \ldots r_n\}$.

→ color the edges of K_{r_n} in colors $1, 2, \ldots n$
by the rule:
The edge uv is assigned color j iff $|u-v| \in S_j$

By Ramsey's theorem

→ there exits a monochromatic triangle, that is,

→ there are 3 vertices say a, b, c such that ab, bc, ca colored with same color, say i.

→ $x = a-b$, $y = b-c$, $z = c-a$ (or) $(a-c)$ → absolute value

$x+y=z$

Statement:-

- Every k-chromatic graph has atleast k-vertices of degree $k-1$.

Statement:- for any positive integer k, there exists a k-chromatic graph containing no triangle.

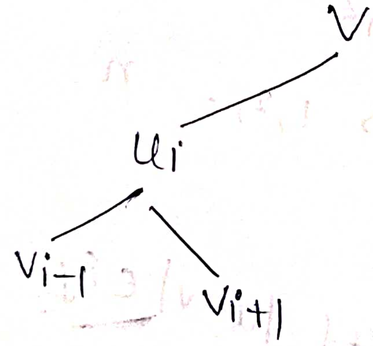
Idea:

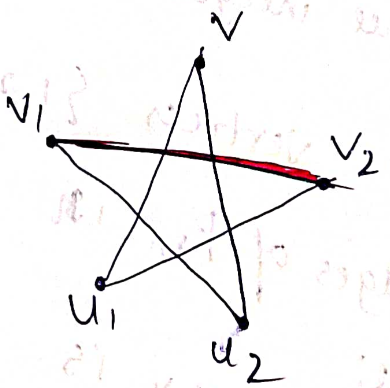
Proof by Induction.

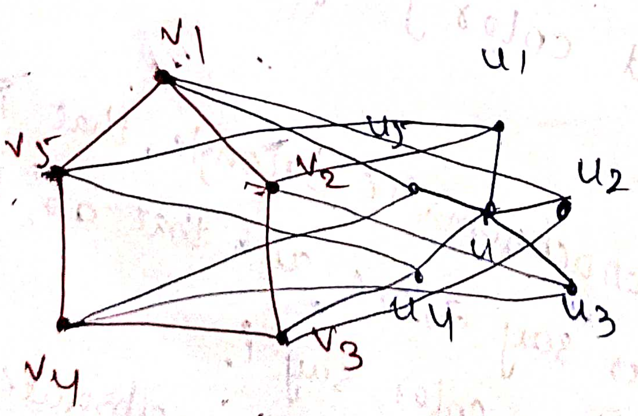
$G(2) \longrightarrow$ 2 color graph (trivial)

$v_1, v_2, \ldots v_n$ - G_k

$u_1, u_2, \ldots u_n, v$ - G_{k+1}







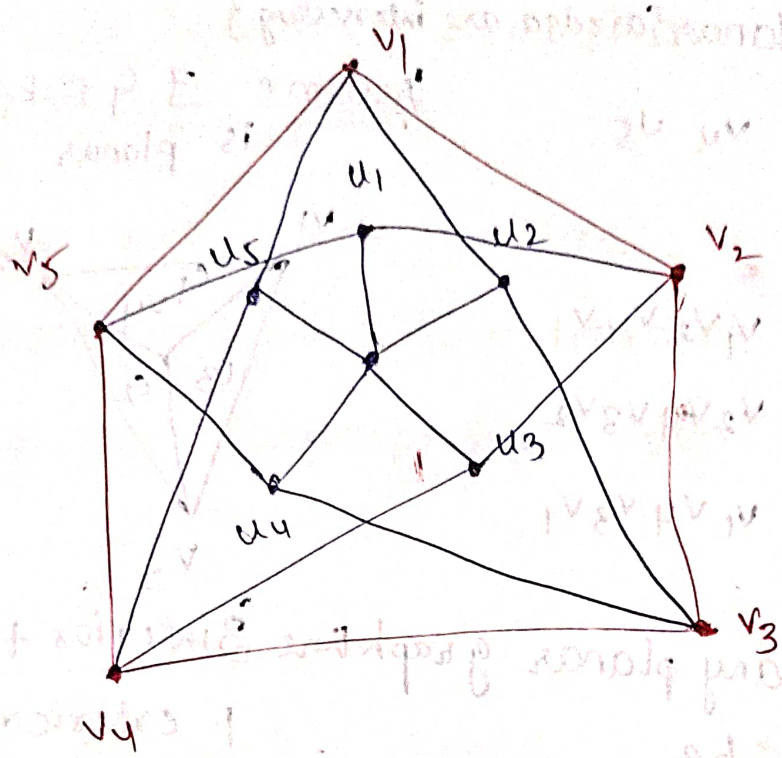
G_3

v_1 adjacent vertices are v_2, v_5

(or) so u_1 join to v_2, v_5

v_1 v_2 v_3 v_4 v_5

u_1 u_2 u_3 u_4 u_5



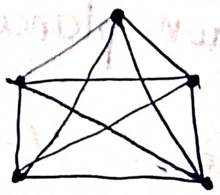
15/11/2022 / Tuesday.

Planar graph:-

A graph is said to be planar if it can be drawn in a plane so that it's edges intersect only at their ends.

* Drawn graph called planar graph $\tilde{G}$ of G and $\tilde{G} \cong G$

K_5 is non-planar

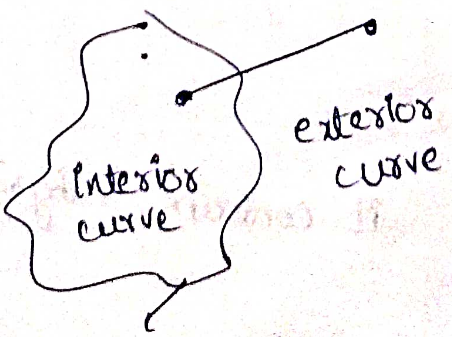


$\rightarrow$ 7 faces.

K_5

Jordan Curve

It is a continuous nonself intersecting curve whose origin and terminus coincide

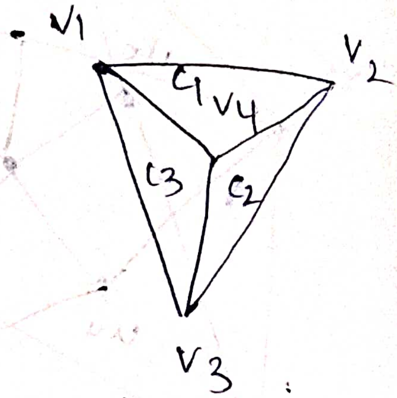


$\rightarrow$ Any curve line joining interior point and exterior point must intersect at a point in C-Curve

Proof for k_5 non planar {as edges are intersecting}

Assume $\exists G \subseteq k_5$ wh is planar

$V_1 \quad V_2 \quad V_3 \quad V_4 \quad V_5$



Now

interior c_1 $V_1V_2V_4V_1$

interior c_2 $V_2V_4V_3V_2$

interior c_3 $V_1V_4V_3V_1$

for any planar graph there 3 interior + 1 exterior.

will be

1 exterior and 3 interior

Now we can take V_5 anywhere within interior c_1, interior c_2, interior c_3, ~~interior c_3~~, exterior.

→ If we place V_5 exterior then

edge V_4 to V_5 is interior to exterior

so non planar

→ If we place V_5 in either of interior c_1 c_2 or c_3 then one of the point will be external in each plane so, this is also non planar.

so k_5 is non planar.

*** A graph is non planar iff it contains subgraph turned to k_{33} or k_5

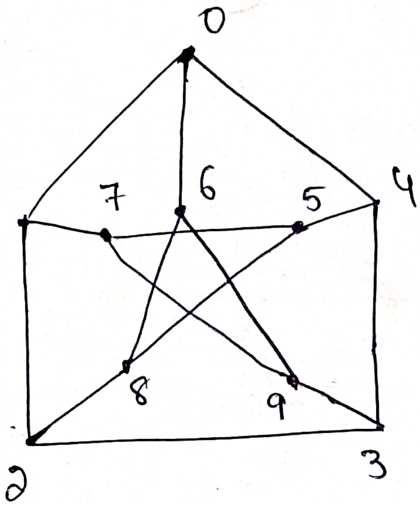
$K_{3,3} \longrightarrow$ bipartite $\longrightarrow$ nonplanar

$K_5 \longrightarrow$ non planar

① Remove edges and vertices

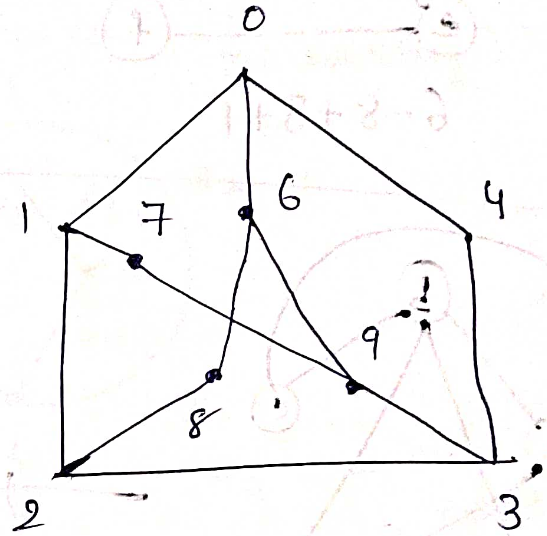
② Collapse degree two vertices into Single edge

③ Apply an isomorphisim into $K_{3,3}$ or K_5

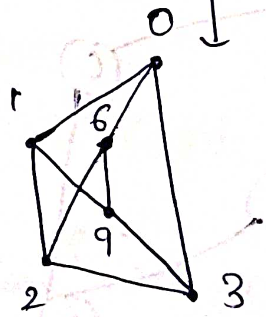


has $K_{3,3}$

Remove 5 $\longrightarrow$



Collage 4,7,8



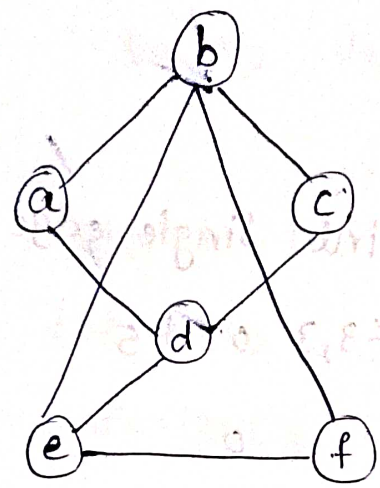
$K_{3,3}$ $\begin{pmatrix} 0 & 2 & 6 \\ 1 & 9 & 3 \end{pmatrix}$

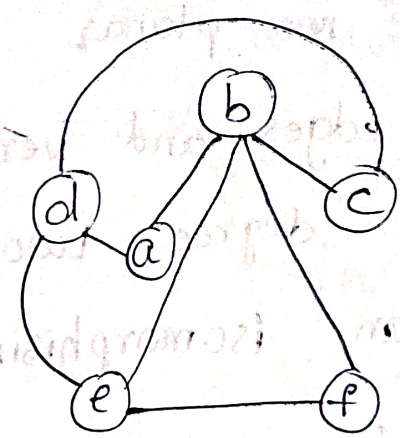
<u>Euler's formula</u>:-

let $G(V,E)$ be a connected planar graph and f be a set of faces of planar drawing of G then (H.W)

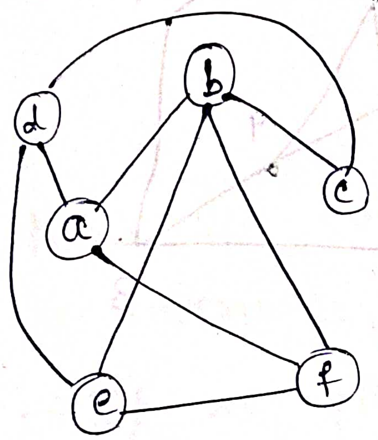
$$|V| - |E| + |F| = 2$$

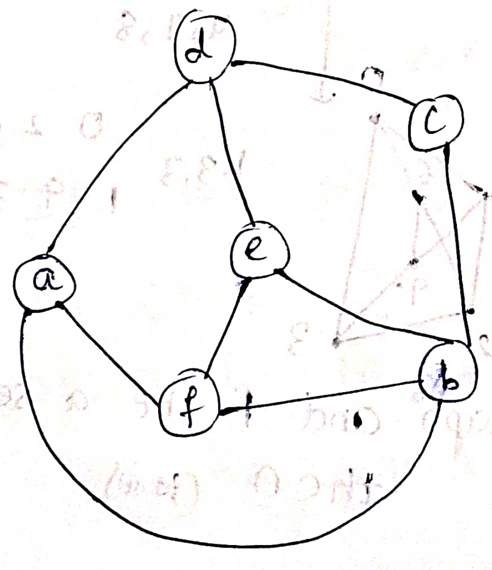
(Necessary condition but not sufficient)





6-8+3+1





17/11/2022
Thursday

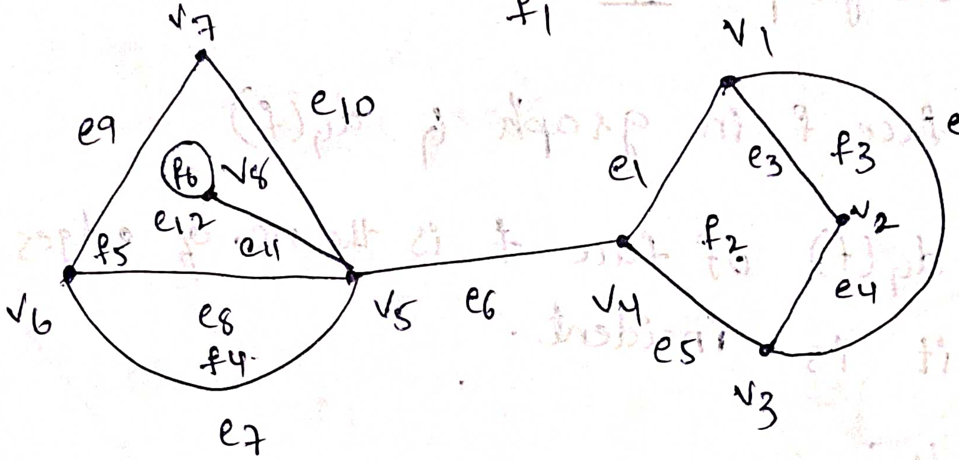
Dual of a planar graph

Let G be a planar graph

→ $F(G)$ - set of faces

→ $\phi(G)$ - No. of faces

* Each planar has exactly one unbounded face — {exterior face} (no boundary)



e_{11} → cut edge

$V = \{v_1, v_2, \ldots v_8\}$

$E = \{e_1, e_2, \ldots e_{12}\}$

$F = \{f_1, f_2, \ldots f_6\}$

Note:-

① G is planar graph and f is a face

② $b(f)$ → boundary of a face f

$b(f_2) = v_1 e_1 v_4 e_5 v_3 e_4 v_2 e_3 v_1$

$b(f_5) =$ ~~$v_6 e_9 v_7 e_{10} v_5 e_8 v_6$~~ (or) $v_7 e_{10} v_5 e_{11} v_8 e_{12} v_8 e_{11} v_5 e_8 v_6 e_9 v_7$

(or)

(or) $v_7 e_{10} v_5 e_{11} v_8 e_{12} v_8 e_{11} v_5 e_8 v_6 e_9 v_7$

(or)

$v_6 e_9 v_7 e_{10} v_5 e_{11} v_8 e_{12} v_8 e_{11} v_5 e_8 v_6$

3) f is said to be incident with vertices & edges in it's boundary.

④ If e is a cut edge in a planar graph just one face is incident with e. Otherwise there are two faces incident with e. So in the above graph e_{11} is cutedge.

⑤ Degree of face f in graph G $d_G(f)$

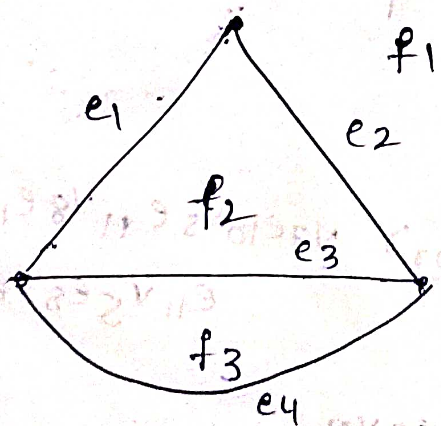
The degree, $d_G(f)$ of face f is the no. of edges with which it is incident

$d(f_2) = 4$

$d(f_5) = 6$ { No. of edges in $b(f_5)$

Note:- No. of edges in $b(f)$

Dual graph of a planar graph G



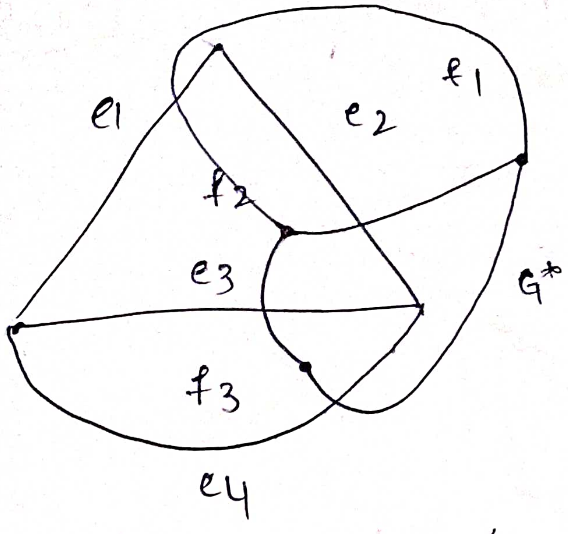
G

{planar graph}

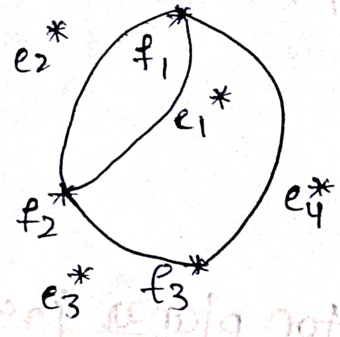
① for each face f in G, there is a vertex f^* is G^*

② For each edge in G, there is an edge e^* in G^*

③ Two vertices f^* and g^* in G^* are joined by an edge e^* in G^* iff their corresponding face f and g in G are seperated by an edge e in G.



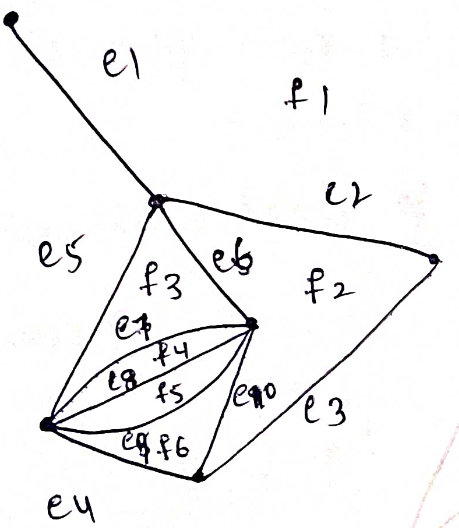
G

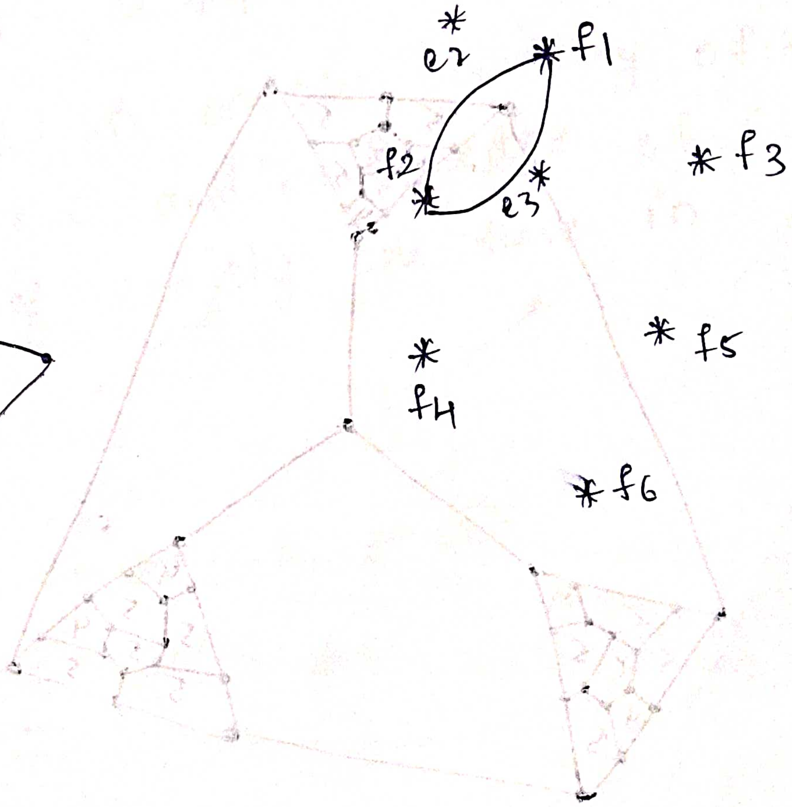


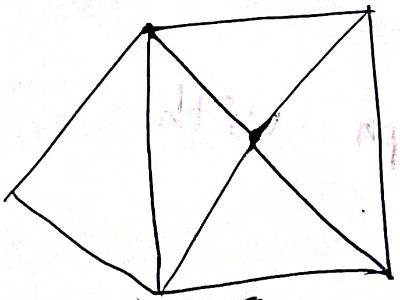
G*

Dual graph

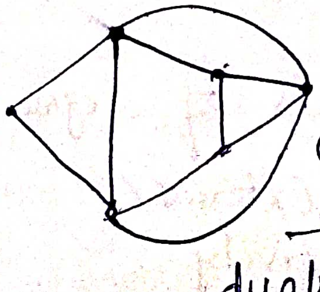
H.W







G_1



G_2

$G_1 \cong G_2$

$\boxed{22 \quad G_1^* \cong G_2^*}$ → check

→ Even two graph are isomorphic there duals need no to be isomorphic

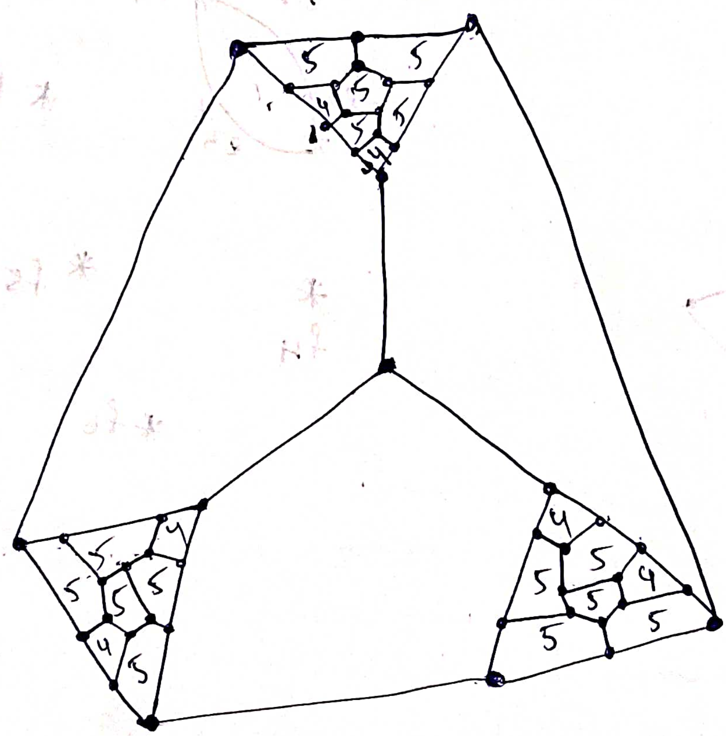
→ But $(G^*)^*$ ~~are~~ a isomorphic

22 | 11 | 2022
Tuesday

Non-hamilton planar graph

Grinberg's Theorem:-

(* 3-regular, 3-connected non hamilton planar graph

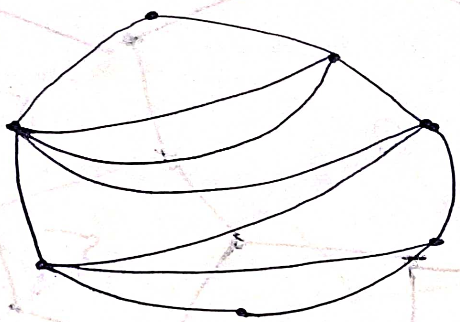


let G be a loopless planar graph with hamilton ~~graph~~ cycle C. Then

$$\sum_{i=1}^{v} (i-2)(\phi_i' - \phi_i'') = 0$$

where ϕ_i' and ϕ_i'' are number of faces of degree i contained in Interior C and Exterior C.

Proof:- E' subset of $E(G) \setminus E(C)$ contained in C and $\varepsilon' = |E'|$



Then C contains exactly $(\varepsilon' + 1)$ faces

$$\sum_{i=1}^{v} \phi_i' = \varepsilon' + 1 \quad \text{— (1)}$$

Now each edge in E' is on the boundary of two faces in Interior C and each edge of C is on the boundary of exactly one face in Interior C

$$\sum_{i=1}^{v} i\,\phi_i' = 2\varepsilon' + v \quad \text{— (2)}$$

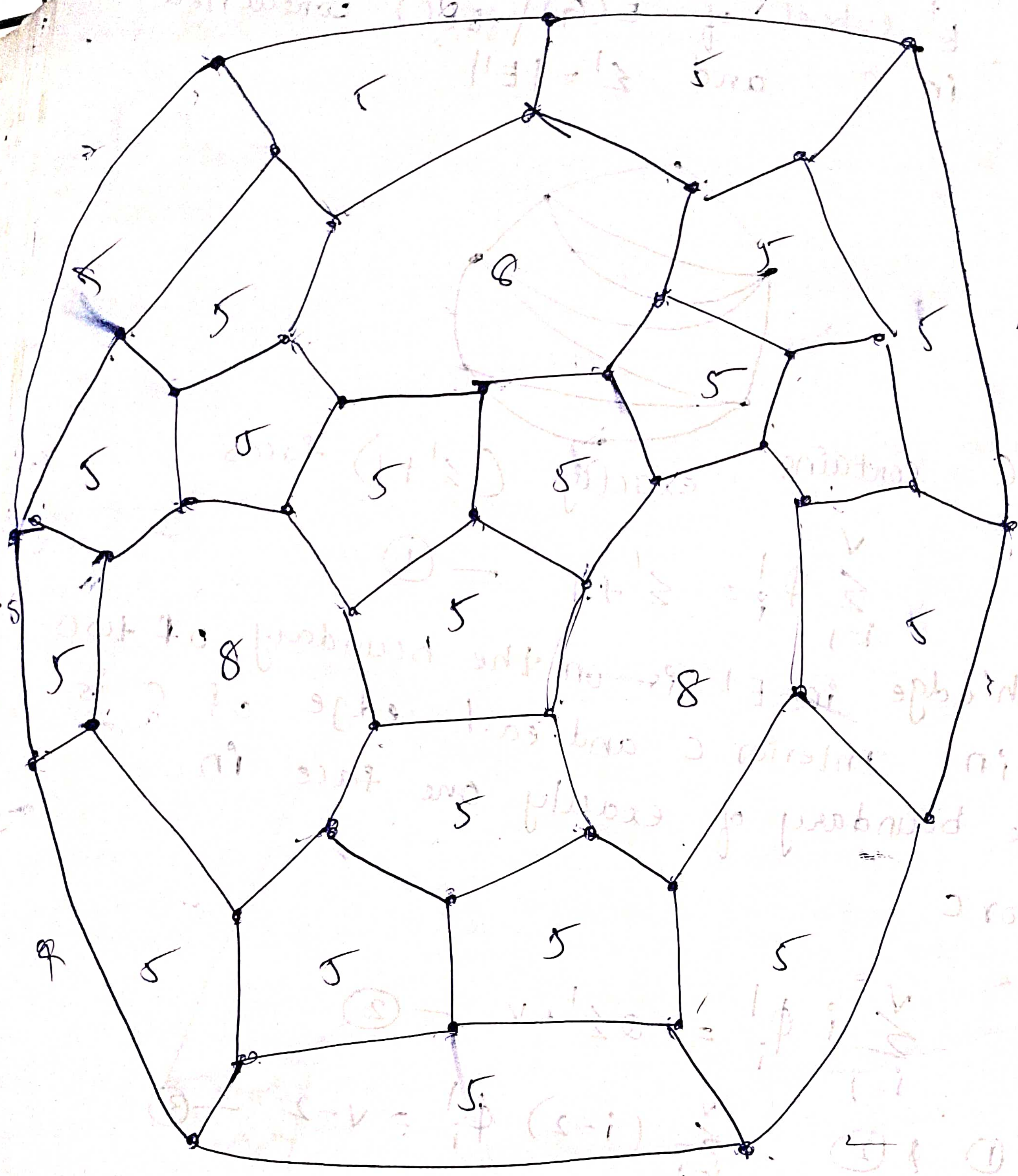
from (1) & (2)

$$\sum_{i=1}^{v} (i-2)\,\phi_i' = v - 2 \quad \text{— (3)}$$

Similarly

$$\sum_{i=1}^{v} (i-2)\,\phi_i'' = v - 2 \quad \text{— (4)}$$

from (3) & (4) the result



Faces of degree

5, 8, 9 ⟶ No. of faces outside

$(5-2)(\phi'_5 - \phi''_5) + (8-2)(\phi'_8 - \phi''_8) + (9-2)(\phi'_9 - \phi''_9)$

6 divisible by %3

3 divisible by %3

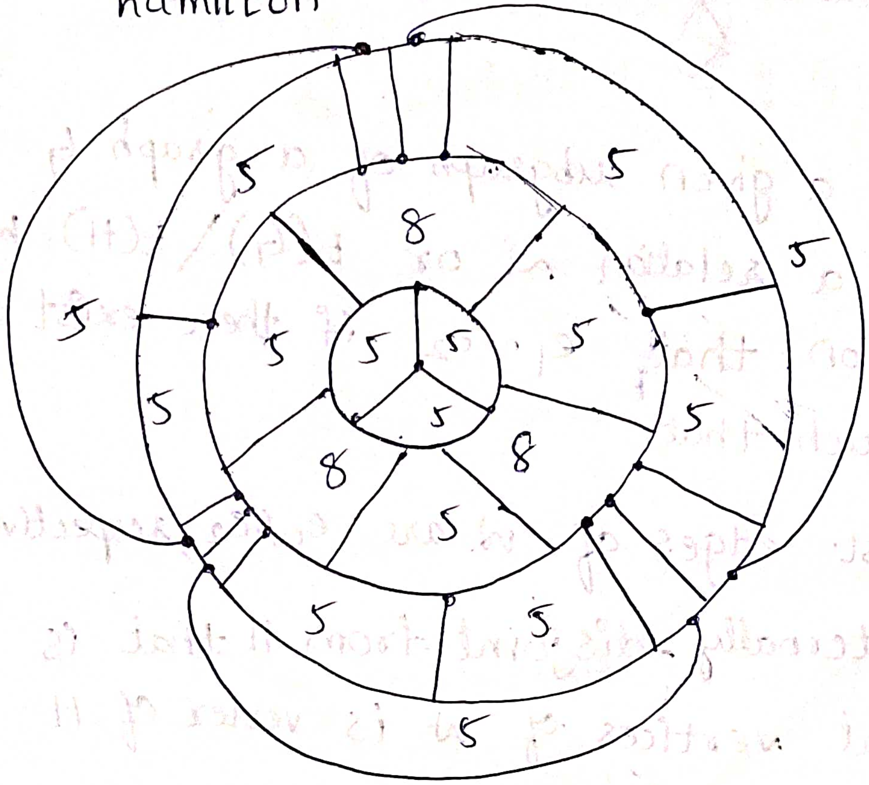
7 ≠ 0

Not divisible by %3

Remaining 2 phases %3 but except one so non hamitonian

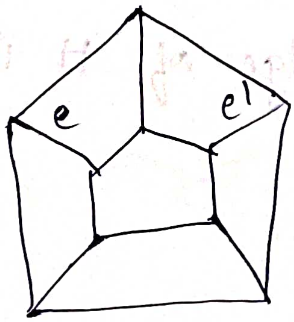
Note:-

Corollary:- Only one face having degree is not 2 mod 3 and others 2 mod 3. Then G is not hamilton



HW

→ Show that no Hamilton cycle contains both the edges e and e' in the following graph.



24/11/2022
Thursday

Planarity Algorithm:-

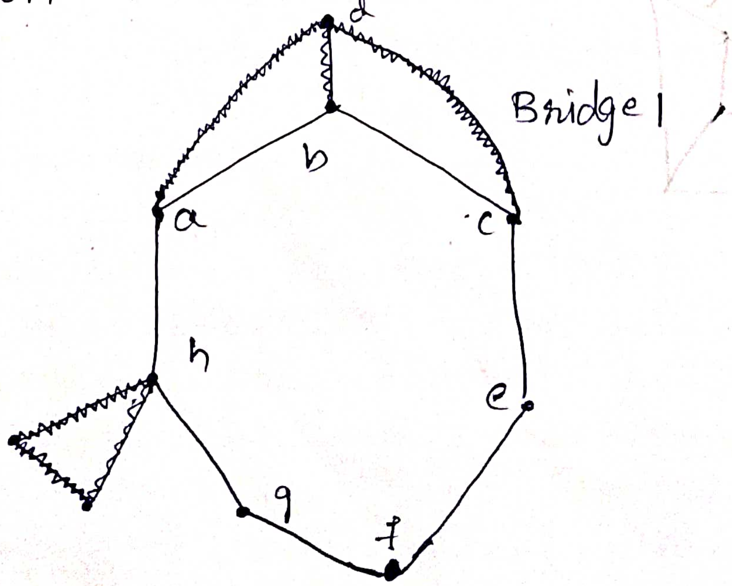
Bridges:-

* Let H be a given subgraph of a graph G we define a relation $\sim$ on $E(G)\setminus E(H)$ by the condition that $e_1 \sim e_2$ if there exist a walk W such that

i) first & last edges of W are e_1 & e_2, respectively that is internally-disjoint from H that is no internal vertices of W is vertex of H.

Note:-

① This relation $\sim$ is equivalence
② A subgraph $G-E(H)$ induced by an equivalence relation $\sim$ is called a bridge of H in G

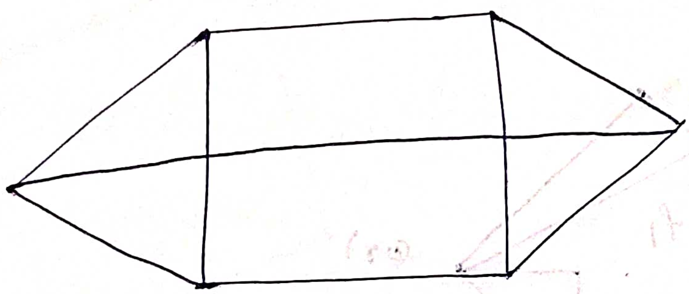


③ $V(B)\cap V(H) = V(B,H)$
Set of vertices of attachment of B to H

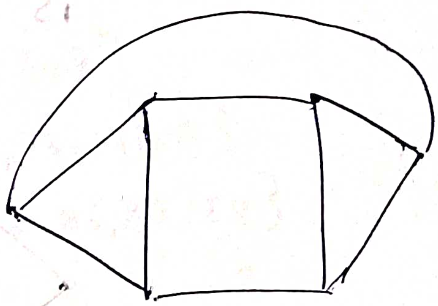
G- admissible :-

let H be a planar subgraph of a graph G
$\hat{H}$ be an embeding of H in the plane
we say that $\hat{H}$ is G- admissible if G is planar
and there is planar embeding $\hat{G}$ of G such that
$\hat{H} \subseteq \hat{G}$.

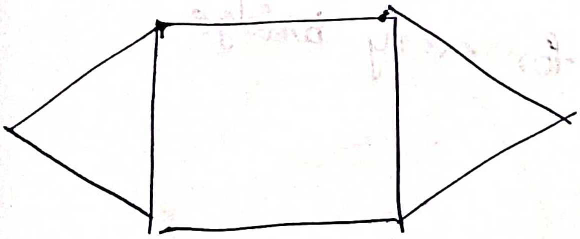
eg:-



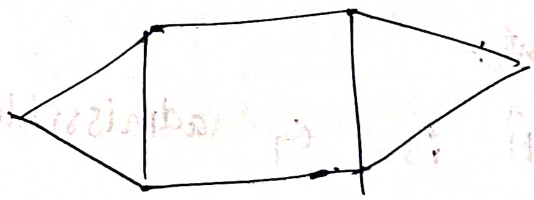
G



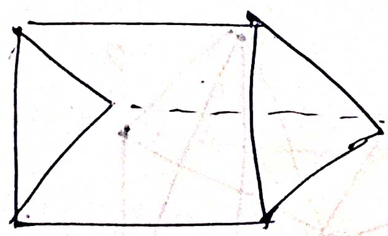
$\hat{G}$



H sub planar graph so also $\hat{H}$



$\hat{H}$ G admisenible since $\hat{H}$ is subgraph of $\hat{G}$



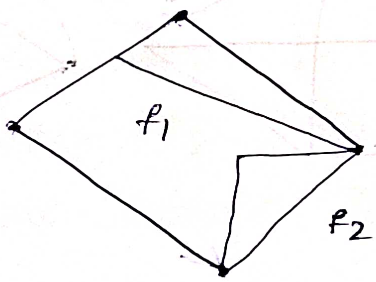
$\hat{H}$

Not G admissible
Since $\hat{H}$ is not subgraph to $\hat{G}$

Statement:-

① If B is a Bridge of H (in G) then B is said to be drawable in a face f of $\hat{H}$ if the vertices of attachment of B to H are contained in the boundary of f.

* $F(B, \hat{H})$ - Set of faces of $\hat{H}$ in which B is drawable.



(or)

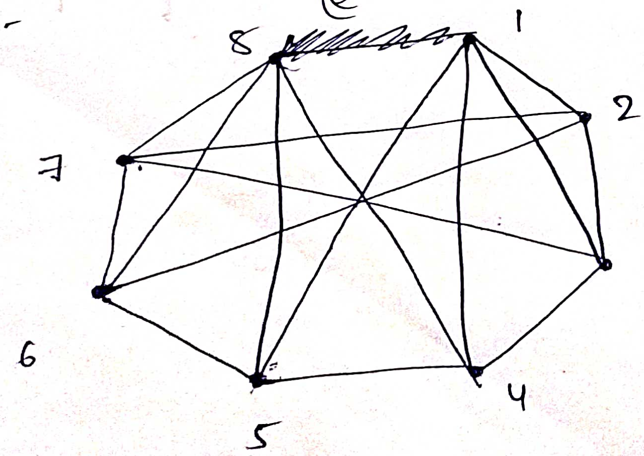
Statement:-

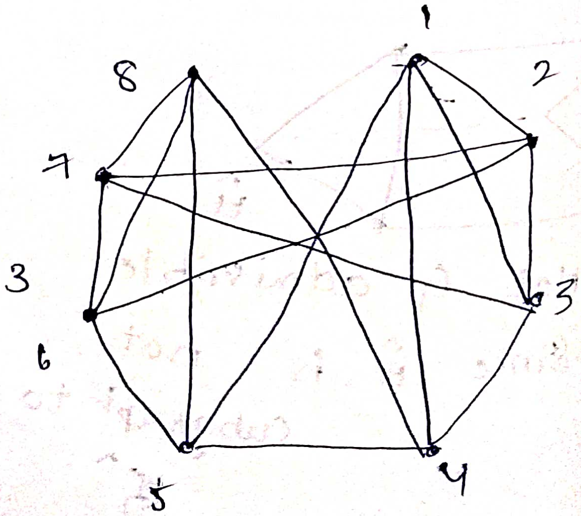
If $\hat{H}$ is G admissible, then for every bridge B of H

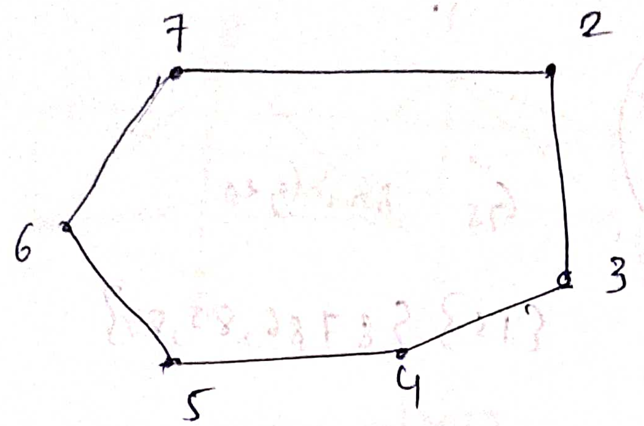
$F(B, \hat{H}) \neq \Phi$ → atleast one face such that we can draw a bridge

(not there)

ex:-



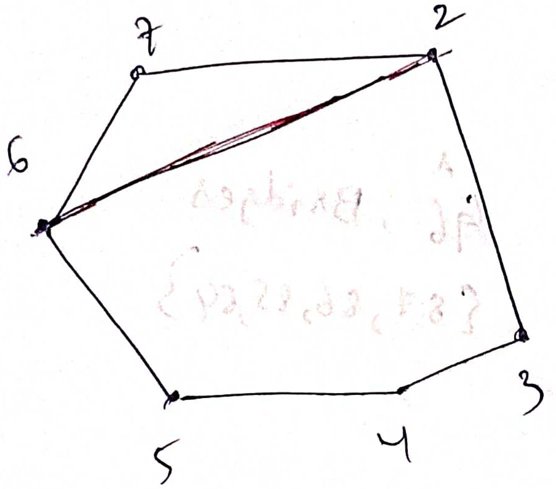




$\hat{G}$ list all bridges.

{26}, {37}

Bridge {12, 13, 14, 15} {87, 86, 85, 84}

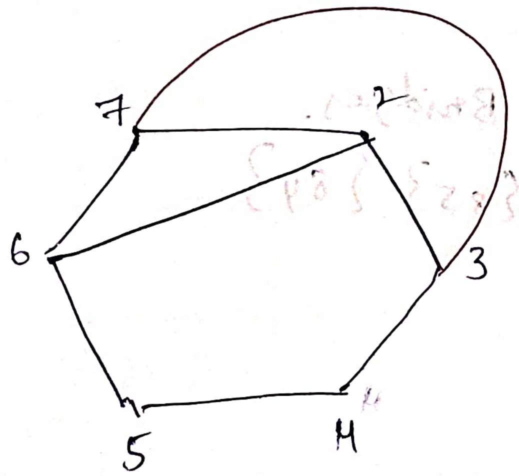


$\hat{G}_2$ list all bridges

{3, 7}

{12, 13, 14, 15}

{87, 86, 85, 84}

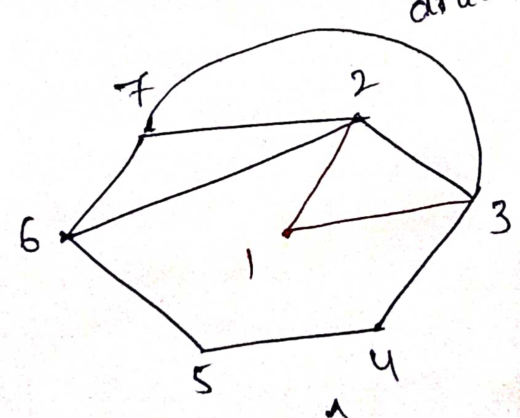


$\hat{G}_3$, {12, 13, 14, 15}

{87, 86, 85, 84}

check if it's drawable

for drawing {12, 13}.

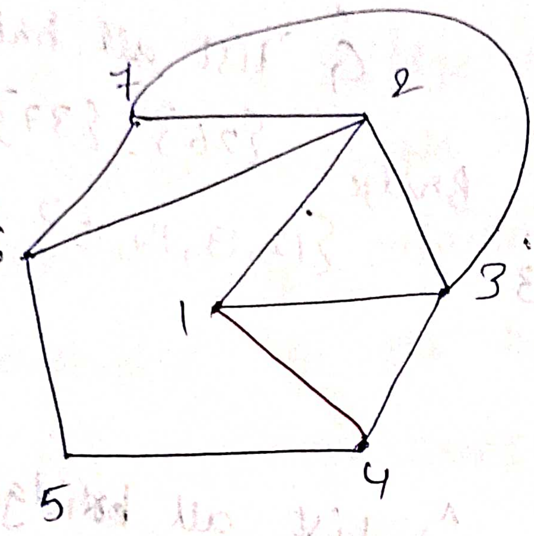


$\hat{G}_4$ Bridges

{87, 86, 85, 84}

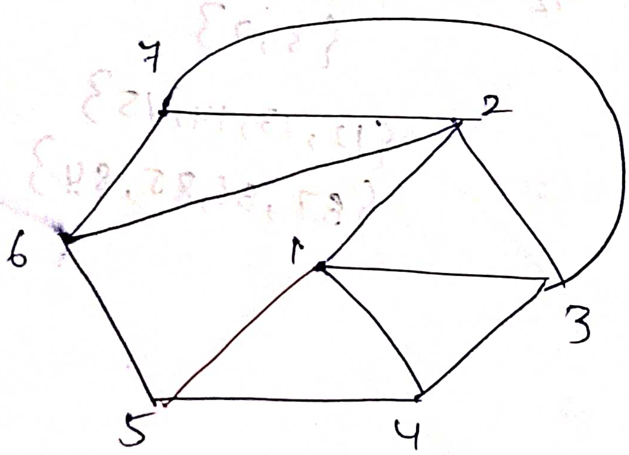
{1, 4} {1, 5}

$\hat{G}_4$ bridges.



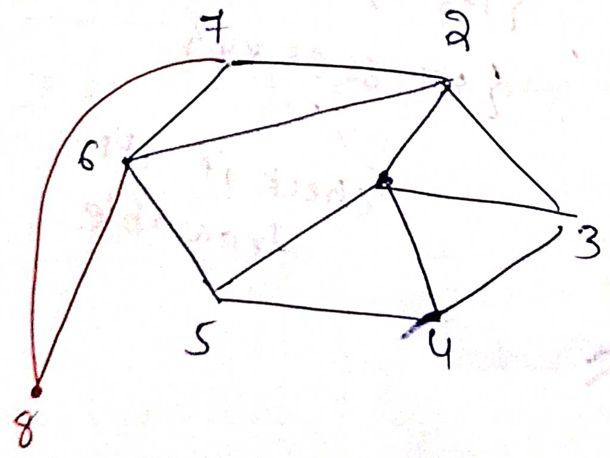
$\hat{G}_5$ Bridges

{15} {87, 86, 85, 84}



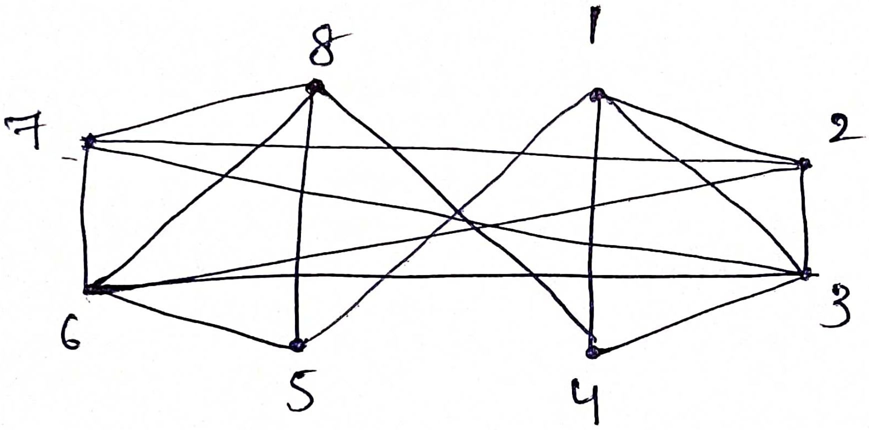
$\hat{G}_6$, Bridges

{87, 86, 85, 84}



$\hat{G}_7$ Bridges.

{85}, {84}



Non planar