VELTECH HIGHTECH Dr.RANGARAJAN Dr.SAKUNTHALA ENGINEERING COLLEGE
DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING

GRAPH THEORY AND APPLICATIONS

CS6702 GRAPH THEORY AND APPLICATIONS L T P C 3 0 0 3

OBJECTIVES: The student should be made to:

- Be familiar with the most fundamental Graph Theory topics and results.
- Be exposed to the techniques of proofs and analysis.

UNIT I INTRODUCTION 9

Graphs – Introduction – Isomorphism – Sub graphs – Walks, Paths, Circuits – Connectedness – Components – Euler graphs – Hamiltonian paths and circuits – Trees – Properties of trees – Distance and centers in tree – Rooted and binary trees.

UNIT II TREES, CONNECTIVITY & PLANARITY 9

Spanning trees – Fundamental circuits – Spanning trees in a weighted graph – cut sets – Properties of cut set – All cut sets – Fundamental circuits and cut sets – Connectivity and separability – Network flows – 1-Isomorphism – 2-Isomorphism – Combinational and geometric graphs – Planer graphs – Different representation of a planer graph.

UNIT III MATRICES, COLOURING AND DIRECTED GRAPH 8

Chromatic number – Chromatic partitioning – Chromatic polynomial – Matching – Covering -Four color problem – Directed graphs – Types of directed graphs – Digraphs and binary relations – Directed paths and connectedness – Euler graphs.

UNIT IV PERMUTATIONS & COMBINATIONS 9

Fundamental principles of counting - Permutations and combinations - Binomial theorem - combinations with repetition - Combinatorial numbers - Principle of inclusion and exclusion - Derangements - Arrangements with forbidden positions.

UNIT V GENERATING FUNCTIONS 10

Generating functions - Partitions of integers - Exponential generating function – Summation operator - Recurrence relations - First order and second order – Non-homogeneous recurrence relations - Method of generating functions.

TOTAL: 45 PERIODS

OUTCOMES:

Upon Completion of the course, the students should be able to:

- Write precise and accurate mathematical definitions of objects in graph theory.
- Use mathematical definitions to identify and construct examples and to distinguish examples from non-examples.
- Validate and critically assess a mathematical proof.
- Use a combination of theoretical knowledge and independent mathematical thinking in creative investigation of questions in graph theory.
- Reason from definitions to construct mathematical proofs.

TEXT BOOKS:

1. Narsingh Deo, "Graph Theory: With Application to Engineering and Computer Science", Prentice Hall of India, 2003.
2. Grimaldi R.P. "Discrete and Combinatorial Mathematics: An Applied Introduction", Addison Wesley, 1994.

REFERENCES:

1. Clark J. and Holton D.A, "A First Look at Graph Theory", Allied Publishers, 1995.
2. Mott J.L., Kandel A. and Baker T.P. "Discrete Mathematics for Computer Scientists and Mathematicians" , Prentice Hall of India, 1996.
3. Liu C.L., "Elements of Discrete Mathematics", Mc Graw Hill, 1985.
4. Rosen K.H., "Discrete Mathematics and Its Applications", Mc Graw Hill, 2007.

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2 Marks Questions and Answers **74**

Question Bank **100**

Model Question paper **105**

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CS6702 GRAPH THEORY AND APPLICATIONS

UNIT I INTRODUCTION

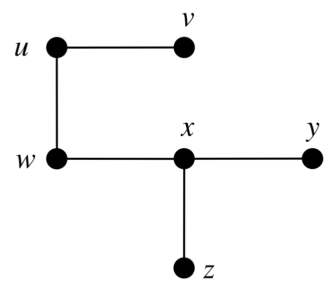
1.1 GRAPHS – INTRODUCTION

1.1.1 Introduction

A graph G = (V, E) consists of a set of objects V={v_1, v_2, v_3, … } called **vertices** (also called **points** or **nodes**) and other set E = {e_1, e_2, e_3, ...} whose elements are called **edges** (also called **lines** or **arcs**).

For example : A graph G is defined by the sets V(G) = {*u, v, w, x, y, z*} and E(G) = {*uv, uw, wx, xy, xz*}.

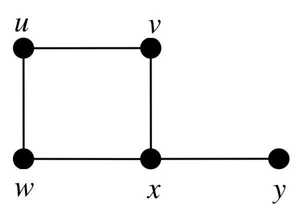
Graph G:



Graph G with 6 vertices and 5 edges

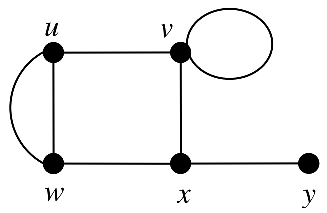
- The set V(G) is called the **vertex set** of G and E(G) is the **edge set** of G.
- A graph with p-vertices and q-edges is called a **(p, q) graph**.
- The (1, 0) graph is called **trivial graph**.
- An edge having the same vertex as its end vertices is called a **self-loop**.
- More than one edge associated a given pair of vertices called **parallel edges**.
- Intersection of any two edges is not a vertex.
- A graph that has neither self-loops nor parallel edges is called **simple graph**.

Graph G:



Simple Graph

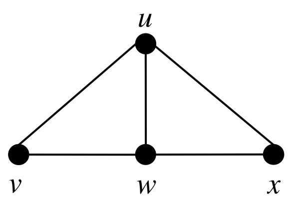
Graph H:



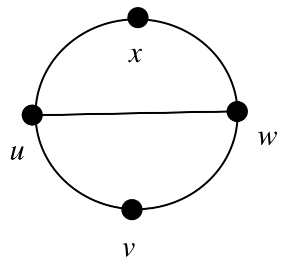
Pseudo Graph

- Same graph can be drawn in different ways.

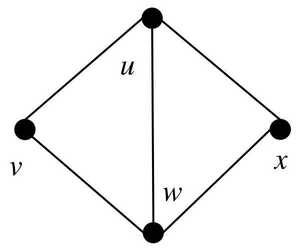
Graph G_1:



Graph G_2:



Graph G_3:



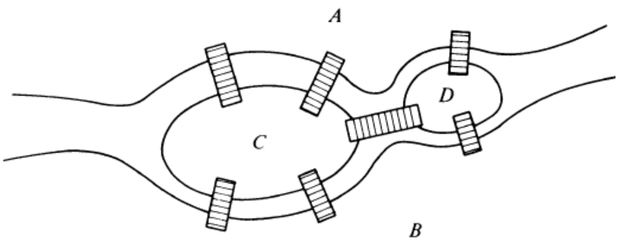
- A graph is also called a linear complex, a 1-complex, or a one-dimensional complex.
- A vertex is also referred to as a node, a junction, a point, O-cell, or an O-simplex.
- Other terms used for an edge are a branch, a line, an element, a 1-cell, an arc, and a 1-simplex.

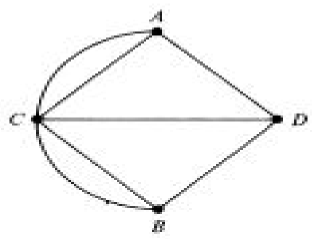
1.1.2 Applications of graph.

(i) Konigsberg bridge problem

The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on *both sides* (A and B) of the Pregel River, and included two *large islands* (C and D) which were connected to each other and the mainland by *seven bridges*. The problem was to devise a walk through the city that would cross each bridge once and only once, with the provisos that: the islands could only be reached by the bridges and every bridge once accessed must be crossed to its other end. The starting and ending points of the walk need not be the same.

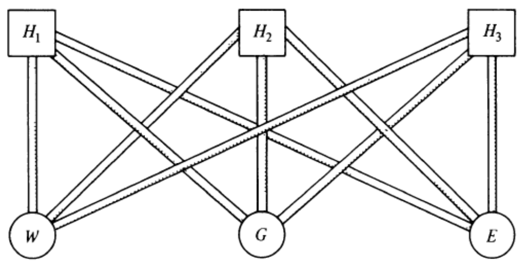
Euler proved that the problem has no solution. This problem can be represented by a graph as shown below.

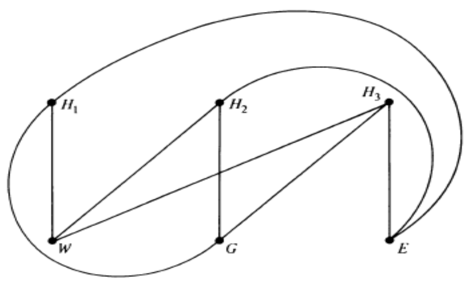




(ii) Utilities problem

There are three houses H1, H2 and H3, each to be connected to each of the three utilities water (W), gas (G) and electricity (E) by means of conduits. This problem can be represented by a graph as shown below.

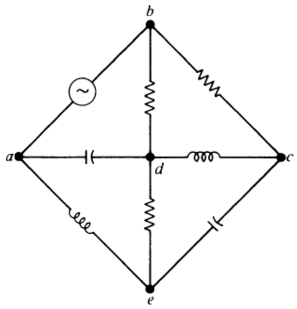


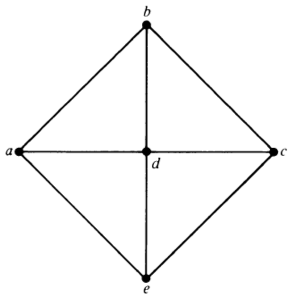


(iii) Electrical network problems

Every Electrical network has two factor.

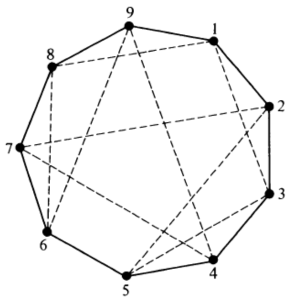
1. Elements such as resisters, inductors, transistors, and so on.
2. The way these elements are connected together (topology)





(iv) Seating problems

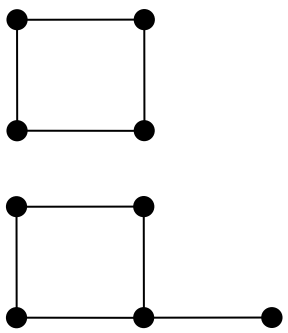
Nine members of a new club meet each day for lunch at a round table. They decide to sit such that every member has different neighbors at each lunch. How many days can this arrangement last?



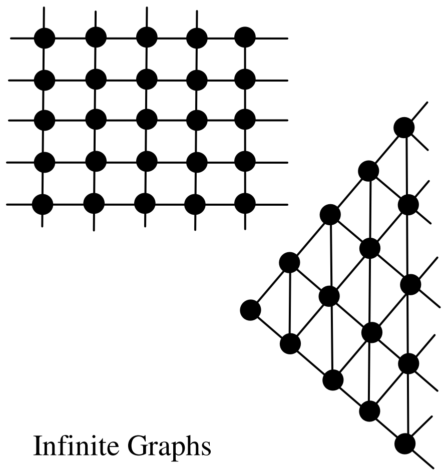
This situation can be represented by a graph with nine vertices such that each vertex represents a member, and an edge joining two vertices represents the relationship of sitting next to each other. Figure shows two possible seating arrangements—these are 1 2 3 4 5 6 7 8 9 1 (solid lines), and 1 3 5 2 7 4 9 6 8 1 (dashed lines). It can be shown by graph-theoretic considerations that there are more arrangements possible.

1.2.3 Finite and infinite graphs

A graph with a finite number off vertices as well as a finite number of edges is called a *finite* graph; otherwise, it is an *infinite* graph.



Finite Graphs

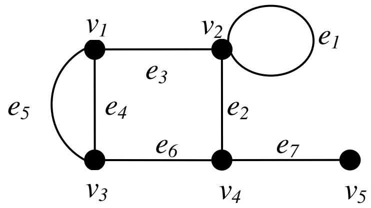


Infinite Graphs

1.1.4 Incidence, adjacent and degree.

When a vertex v_i is an end vertex of some edge e_j, v_i and e_j are said to be *incident* with each other. Two non parallel edges are said to be *adjacent* if they are incident on a common vertex. The number of edges incident on a vertex v_i, with self-loops counted twice, is called the *degree* (also called valency), $d(v_i)$, of the vertex v_i. A graph in which all vertices are of equal degree is called *regular graph.*

Graph G:



The edges e_2, e_6 and e_7 are incident with vertex v_4.

The edges e_2 and e_7 are adjacent.

The edges e_2 and e_4 are not adjacent.

The vertices v_4 and v_5 are adjacent.

The vertices v_1 and v_5 are not adjacent.

$d(v_1) = d(v_3) = d(v_4) = 3$. $d(v_2) = 4$. $d(v_5) = 1$.

Total degree $= d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5)$

$= 3 + 4 + 3 + 3 + 1 = 14$ = Twice the number of edges.

Theorem 1-1

The number of vertices of odd degree in a graph is always even.

Proof: Let us now consider a graph G with e edges and n vertices $v_1, v_2, ..., v_n$. Since each edge contributes two degrees, the sum of the degrees of all vertices in G is twice the number of edges in G. That is,

$$\sum_{i=1}^{n} d(v_i) = 2e.$$

If we consider the vertices with odd and even degrees separately, the quantity in the left side of the above equation can be expressed as the sum of two sums, each taken over vertices of even and odd degrees, respectively, as follows:

$$\sum_{i=1}^{n} d(v_i) = \sum_{even} d(v_j) + \sum_{odd} d(v_k)$$

Since the left-hand side in the above equation is even, and the first expression on the right-hand side is even (being a sum of even numbers), the second expression must also be even:

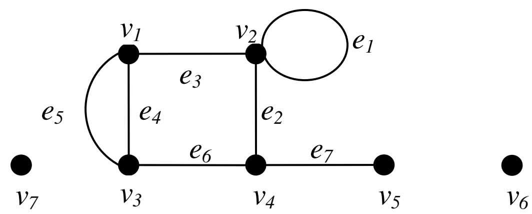
$$\sum_{odd} d(v_k) = an\ even\ number$$

Because in the above equation each $d(v_k)$ is odd, the total number of terms in the sum must be even to make the sum an even number. Hence the theorem. ■

1.1.5 Define Isolated and pendent vertex.

A vertex having no incident edge is called an *isolated vertex*. In other words, isolated vertices are vertices with zero degree. A vertex of degree one is called a *pendant vertex* or an *end vertex*.

Graph G:



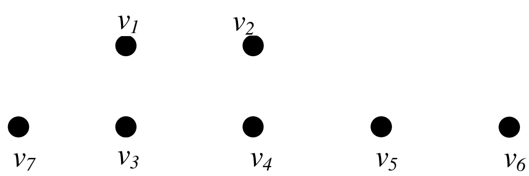
The vertices v_6 and v_7 are *isolated vertices*.

The vertex v_5 is a *pendant vertex*.

1.1.6 Null graph and Multigraph

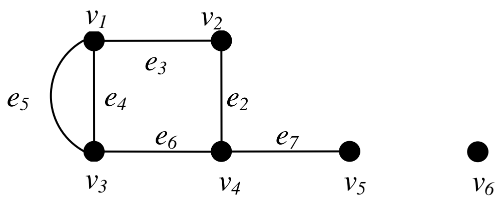
In a graph G=(V, E), If E is empty (Graph without any edges), then G is called a **null graph**.

Graph G:



In a multigraph, no loops are allowed but more than one edge can join two vertices, these edges are called **multiple edges** or parallel edges and a graph is called **multigraph.**

Graph G:



The edges e_5 and e_4 are **multiple (**parallel) edges.

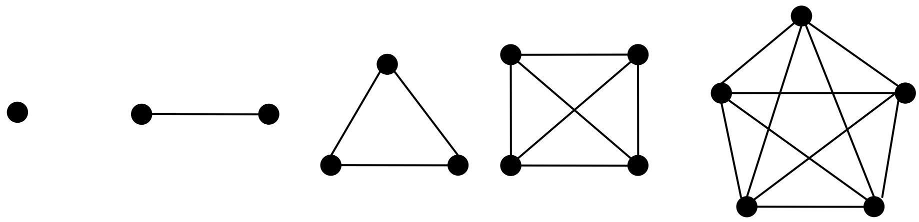
1.1.7 Complete graph and Regular graph

Complete graph

A simple graph G is said to be **complete** if every vertex in G is connected with every other vertex. *i.e.,* if G contains exactly one edge between each pair of distinct vertices.

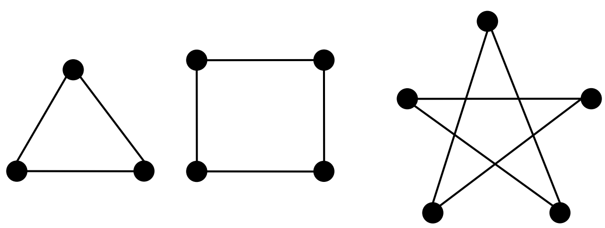
A complete graph is usually denoted by $\mathbf{K_n}$**.** It should be noted that Kn has exactly $n(n-1)/2$ edges.

The complete graphs K_n for n = 1, 2, 3, 4, 5 are show in the following Figure.



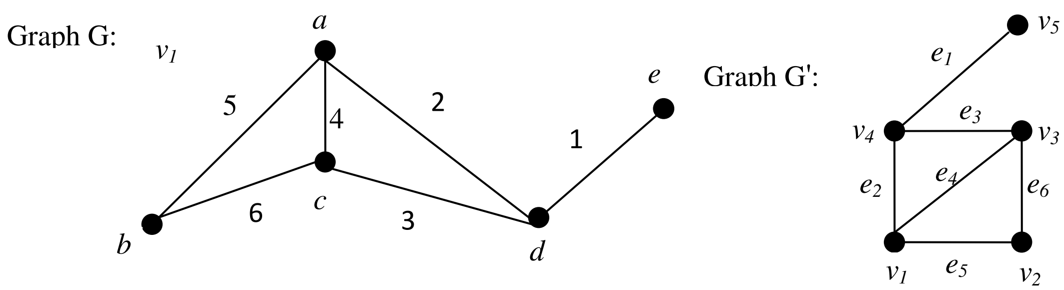
Regular graph

A graph, in which all vertices are of **equal degree,** is called a **regular graph.** If the degree of each vertex is r, then the graph is called a regular **graph of degree** $\boldsymbol{r}$**.**



1.2 ISOMORPHISM

Two graphs G and G' are said to be **isomorphic** to each other if there is a one-to-one correspondence (bijection) between their vertices and between their edges such that the incidence relationship is preserved.



Correspondence of vertices

$f(a) = v_1$

$f(b) = v_2$

$f(c) = v_3$

$f(d) = v_4$

$f(e) = v_5$

Correspondence of edges

$f(1) = e_1$

$f(2) = e_2$

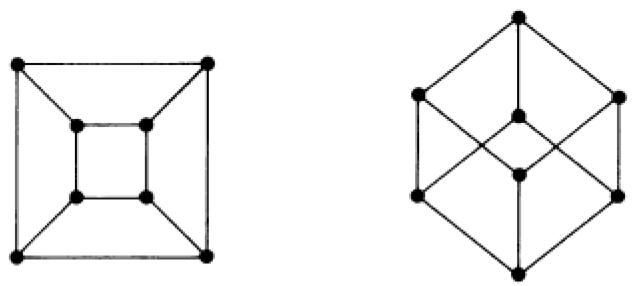
$f(3) = e_3$

$f(4) = e_4$

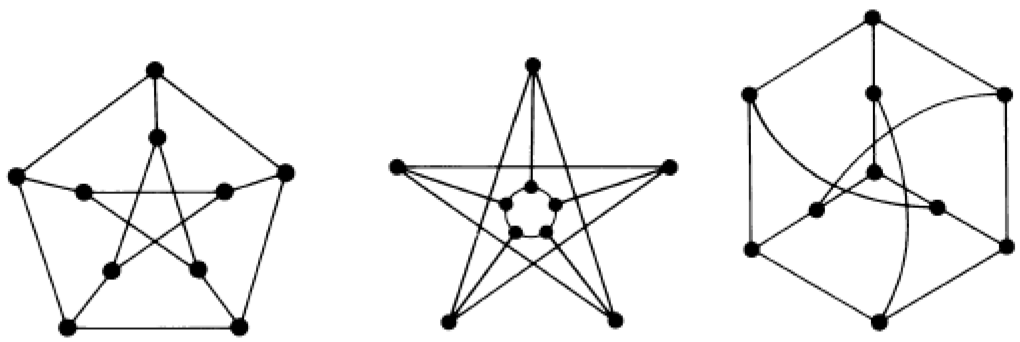
$f(5) = e_5$

Adjacency also preserved. Therefore G and G' are said to be isomorphic.

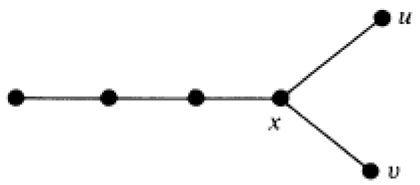
The following graphs are isomorphic to each other. i.e two different ways of drawing the same geaph.

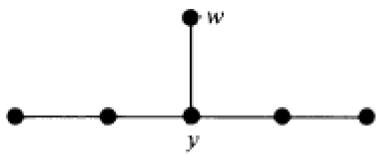


The following three graphs are isomorphic.



The following two graphs are not isomorphic, because x is adjacent to two pendent vertex is not preserved.

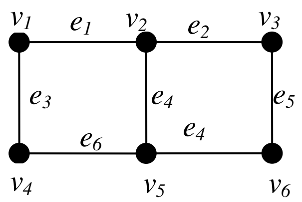




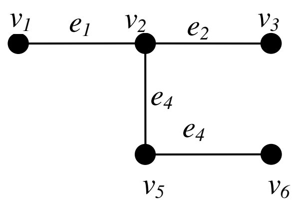
1.3 SUB GRAPHS

A graph G' is said to be a subgraph of a graph G, if all the vertices and all the edges of G' are in G, and each edge of G' has the same end vertices in G' as in G.

Graph G:



Subgraph G' of G:



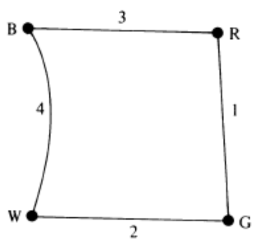
A subgraph can be thought of as being contained in (or a part of) another graph. The symbol from set theory, g ⊂ G, is used in stating "g is a subgraph of G".

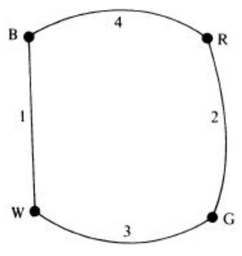
The following observations can be made immediately:

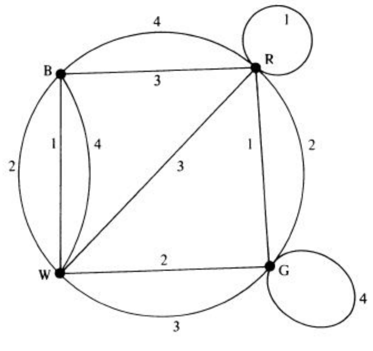
1. Every graph is its own subgraph.
2. A subgraph of a subgraph of G is a subgraph of G.
3. A single vertex in a graph C is a subgraph of G.
4. A single edge in G, together with its end vertices, is also a subgraph of G.

Edge-Disjoint Subgraphs: Two (or more) subgraphs g_1, and g_2 of a graph G are said to be edge disjoint if g_1, and g_2 do not have any edges in common.

For example, the following two graphs are edge-disjoint sub-graphs of the graph G.







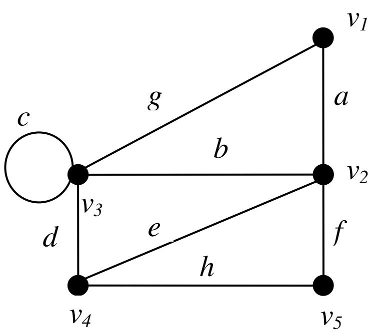
Note that although edge-disjoint graphs do not have any edge in common, they may have vertices in common. Sub-graphs that do not even have vertices in common are said to be vertex disjoint. (Obviously, graphs that have no vertices in common cannot possibly have edges in common.)

1.4 WALKS, PATHS, CIRCUITS

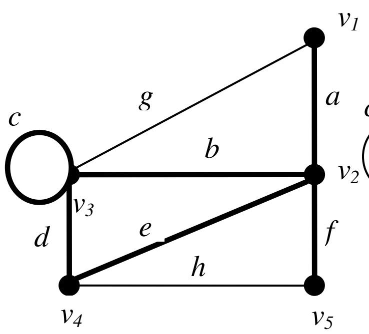
A **walk** is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices. No edge appears more than once. It is also called as an edge train or a chain.

An open walk in which no vertex appears more than once is called **path**. The number of edges in the path is called **length of a path**.

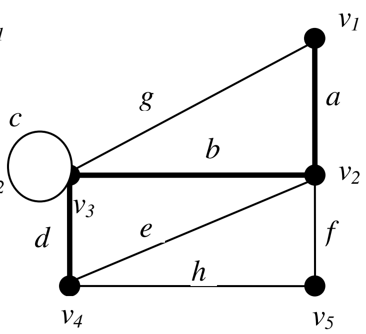
A closed walk in which no vertex (except initial and final vertex) appears more than once is called a **circuit**. That is, a circuit is a closed, nonintersecting walk.



Graph G:



Open walk



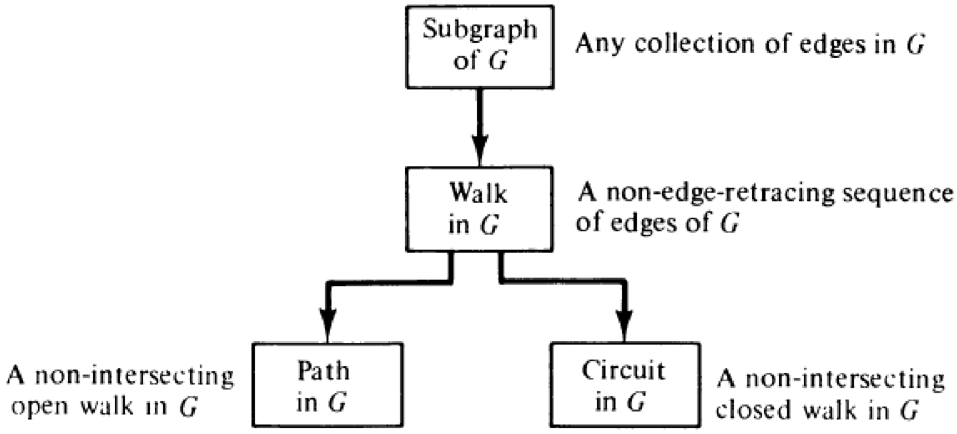
Path of length 3

$v_1\ a\ v_2\ b\ v_3\ c\ v_3\ d\ v_4\ e\ v_2\ f\ v_5$ is a walk. v_1 *and* v_5 are terminals of walk.

$v_1\ a\ v_2\ b\ v_3\ d\ v_4$ is a path. $a\ v_2\ b\ v_3\ c\ v_3\ d\ v_4\ e\ v_2\ f\ v_5$ is not a path.

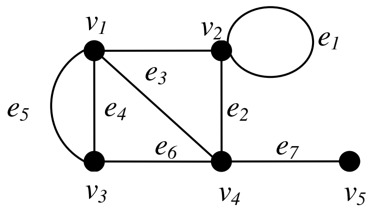
$v_2\ b\ v_3\ d\ v_4\ e\ v_2$ is a circuit.

The concept of walks, paths, and circuits are simple and tha relation is represented by the following figure.

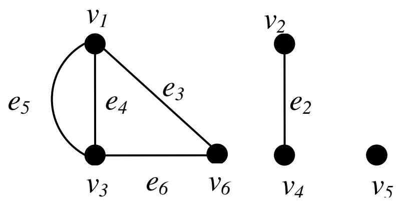


1.5 CONNECTEDNESS

A graph G is said to be **connected** if there is at least one path between every pair of vertices in G. Otherwise, G is disconnected.



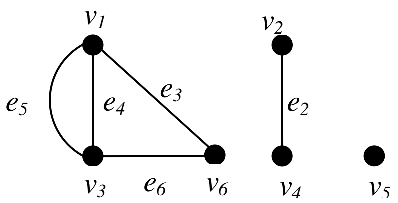
Connected Graph G



Disconnected Graph H

1.6 COMPONENTS

A disconnected graph consists of two or more connected graphs. Each of these connected subgraphs is called a component.



Disconnected Graph H with 3 components

THEOREM 1-2

A graph G is disconnected if and only if its vertex set V can be partitioned into two nonempty, disjoint subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in subset V_1 and the other in subset V_2.

Proof: Suppose that such a partitioning exists. Consider two arbitrary vertices a and b of G, such that $a \in V_1$ and $b \in V_2$. No path can exist between vertices a and b; otherwise, there would be at least one edge whose one end vertex would be in V_1 and the other in V_2. Hence, if a partition exists, G is not connected.

Conversely, let G be a disconnected graph. Consider a vertex a in G. Let V_1 be the set of all vertices that are joined by paths to a. Since G is disconnected, V_1 does not include all vertices of G. The remaining vertices will form a (nonempty) set V_2. No vertex in V_1 is joined to any in V_2 by an edge. Hence the partition. ■

THEOREM 1-3

If a graph (connected or disconnected) has exactly two vertices of odd degree, there must be a path joining these two vertices.

Proof: Let G be a graph with all even vertices except vertices v_1, and v_2, which are odd. From Theorem 1-2, which holds for every graph and therefore for every component of a disconnected graph, no graph can have an odd number of odd vertices. Therefore, in graph G, v_1 and v2 must belong to the same component, and hence must have a path between them. ■

THEOREM 1-4

A simple graph (i.e., a graph without parallel edges or self-loops) with n vertices and k components can have at most $(n - k)(n - k + 1)/2$ edges.

Proof: Let the number of vertices in each of the k components of a graph G be n_1, n_2, ..., n_k. Thus we have $n_1 + n_2 + \cdots + n_k = n, \quad n_k \geq 1$

The proof of the theorem depends on an algebraic inequality.

$$\sum_{i-1}^{k} n_i^2 \leq n^2 - (k-1)(2n-k)$$

Now the maximum number of edges in the ith component of G (which is a simple connected graph) is $\frac{1}{2}n_i(n_i - 1)$. Therefore, the maximum number of edges in G is

$$\frac{1}{2}\sum_{i-1}^{k}(n_i - 1)n_i = \frac{1}{2}\left(\sum_{i-1}^{k} n_i^2\right) - \frac{n}{2}$$

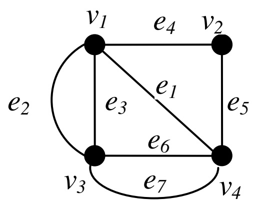
$$\leq \frac{1}{2}[n^2 - (k-1)(2n-k)] - \frac{n}{2}$$

$$= \frac{1}{2}(n-k)(n-k+1). \blacksquare$$

1.7 EULER GRAPHS

A path in a graph G is called Euler path if it includes every edges exactly once. Since the path
contains every edge exactly once, it is also called Euler trail / Euler line.

A closed Euler path is called Euler circuit. A graph which contains an Eulerian circuit is called an Eulerian graph.



$v_4\ e_1\ v_1\ e_2\ v_3\ e_3\ v_1\ e_4\ v_2\ e_5\ v_4\ e_6\ v_3\ e_7\ v_4$ is an Euler circuit. So the above graph is Euler graph.

THEOREM 1-4

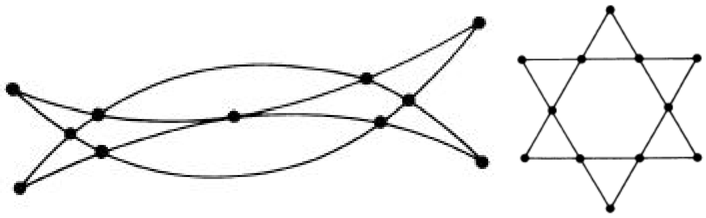
A given connected graph ***G*** is an Euler graph if and only if all vertices of ***G*** are of even degree.

Proof: Suppose that ***G*** is an Euler graph. It therefore contains an Euler line (which is a closed walk). In tracing this walk we observe that every time the walk meets a vertex *v* it goes through two "new" edges incident on *v* - with one we "entered" *v* and with the other "exited." This is true not only of all intermediate vertices of the walk but also of the terminal vertex, because we "exited" and "entered" the same vertex at the beginning and end of the walk, respectively. Thus if G is an Euler graph, the degree of every vertex is even.

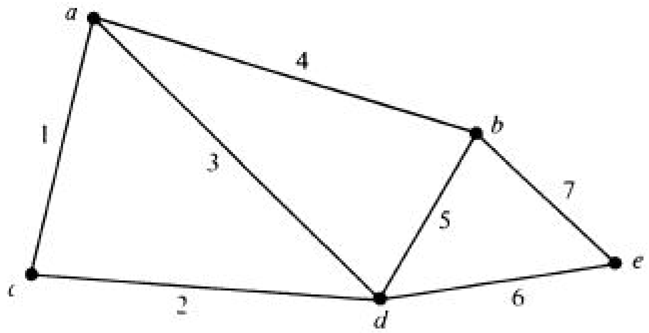
To prove the sufficiency of the condition, assume that all vertices of ***G*** are of even degree. Now we construct a walk starting at an arbitrary vertex *v* and going through the edges of ***G*** such that no edge is traced more than once. We continue tracing as far as possible. Since every vertex is of even degree, we can exit from every vertex we enter; the tracing cannot stop at any vertex but *v*. And since *v* is also of even degree, we shall eventually reach *v* when the tracing comes to an end. If this closed walk *h* we just traced includes all the edges of ***G, G*** is an Euler graph. If not, we remove from ***G*** all the edges in ***h*** and obtain a subgraph ***h'*** of ***G*** formed by the remaining edges. Since both ***G*** and ***h*** have all their vertices of even degree, the degrees of the vertices of ***h'*** are also even. Moreover, ***h'*** must touch ***h*** at least at one vertex ***a,*** because ***G*** is connected. Starting from ***a,*** we can again construct a new walk in graph ***h'***. Since all the vertices of ***h'*** are of even degree, this walk in h' must terminate at vertex a; but this walk in h' can be combined with h to form a new walk, which starts and ends at vertex v and has more edges than h. This process can be repeated until we obtain a closed walk that traverses all the edges of G. Thus G is an Euler graph. ■

Unicursal graph

An open walk that includes all the edges of a graph without retracing any edge is called ***unicrusal line*** or an ***open Euler line***. A (connected) graph that has a unicrusal line will be called a ***unicursal graph.***



Euler graphs (i) Mohammed's scimitars (ii) Star of david.



unicursal graph with a walk a 1 c 2 d 3 a 4 b 5 d 6 e 7 b.

THEOREM 1-5

In a connected graph G with exactly $2k$ odd vertices, there exist k edge-disjoint subgraphs such that they together contain all edges of G and that each is a unicursal graph.

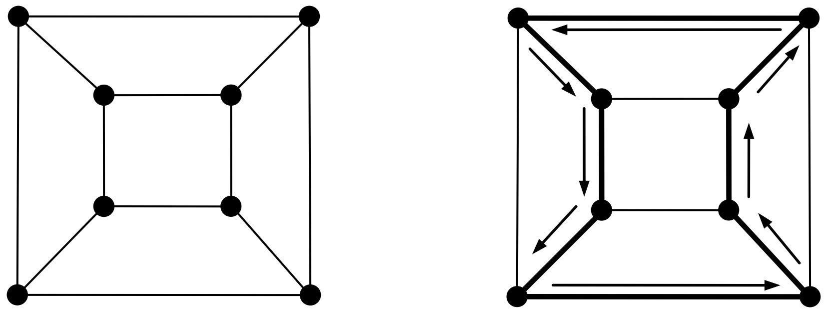
Proof: Let the odd vertices of the given graph G be named $v_1, v_2, \ldots, v_k$; $w_1, w_2, \ldots, w_k$ in any arbitrary order. Add k edges to G between the vertex pairs (v_1, w_1), $(v_2, w_2), \ldots, (v_k, w_k)$ to form a new graph G'.

Since every vertex of G' is of even degree, G' consists of an Euler line p. Now if we remove from p the k edges we just added (no two of these edges are incident on the same vertex), p will be split into k walks, each of which is a unicursal line: The first removal will leave a single unicursal line; the second removal will split that into two unicursal lines; and each successive removal will split a unicursal line into two unicursal lines, until there are k of them. Thus the theorem. ■

1.8 HAMILTONIAN PATHS AND CIRCUITS

A **Hamiltonian circuit** in a connected graph is defined as a closed walk that traverses every vertex of graph G exactly once except starting and terminal vertex.

Removal of any one edge from a Hamiltonian circuit generates a path. This path is called **Hamiltonian path**.

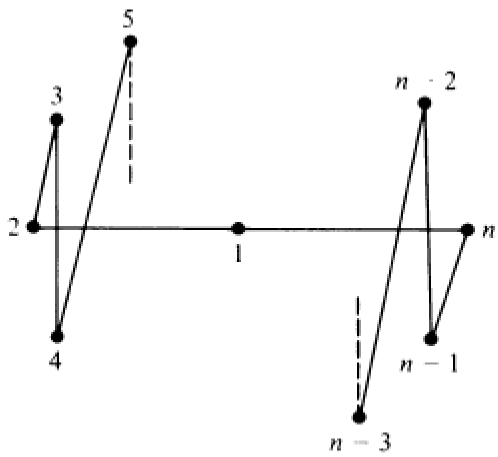


THEOREM 1-6

In a complete graph with n vertices there are $(n-1)/2$ edge-disjoint Hamiltonian circuits, if n is an odd number ≥ 3.

Proof: A complete graph G of n vertices has $n(n-1)/2$ edges, and a Hamiltonian circuit in G consists of n edges. Therefore, the number of edge-disjoint Hamiltonian circuits in G cannot exceed $(n-1)/2$. That there are $(n-1)/2$ edge-disjoint Hamiltonian circuits, when n is odd, can be shown as follows:

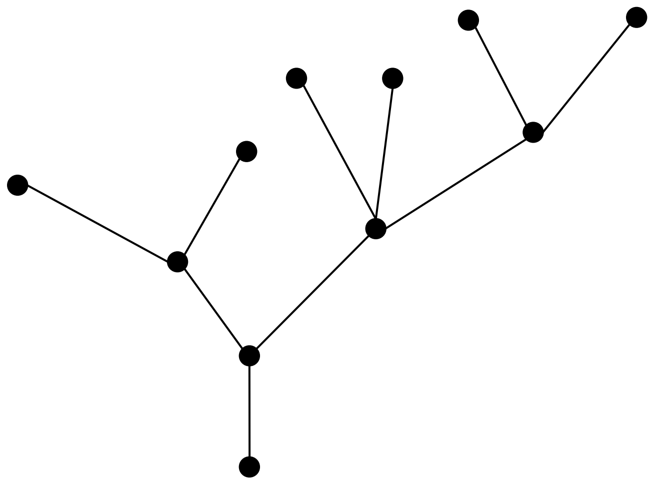
The subgraph (of a complete graph of n vertices) in Figure is a Hamiltonian circuit. Keeping the vertices fixed on a circle, rotate the polygonal pattern clockwise



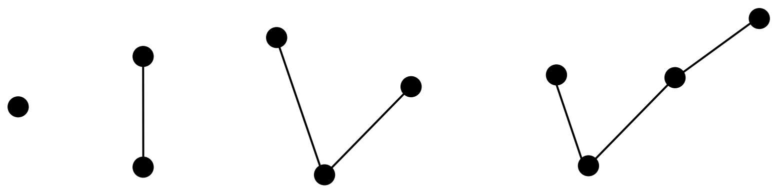
by $360/(n-1), 2 \cdot 360/(n-1), 3 \cdot 360/(n-1) \ldots\ldots (n-3)/2 \cdot 360/(n-1)$ degrees. Observe that each rotation produces a Hamiltonian circuit that has no edge in common with any of the previous ones. Thus we have ***(n*** **-** 3)/2 new Hamiltonian circuits, all edge disjoint from the one in Figure and also edge disjoint among themselves. Hence the theorem. ■

1.9 TREES

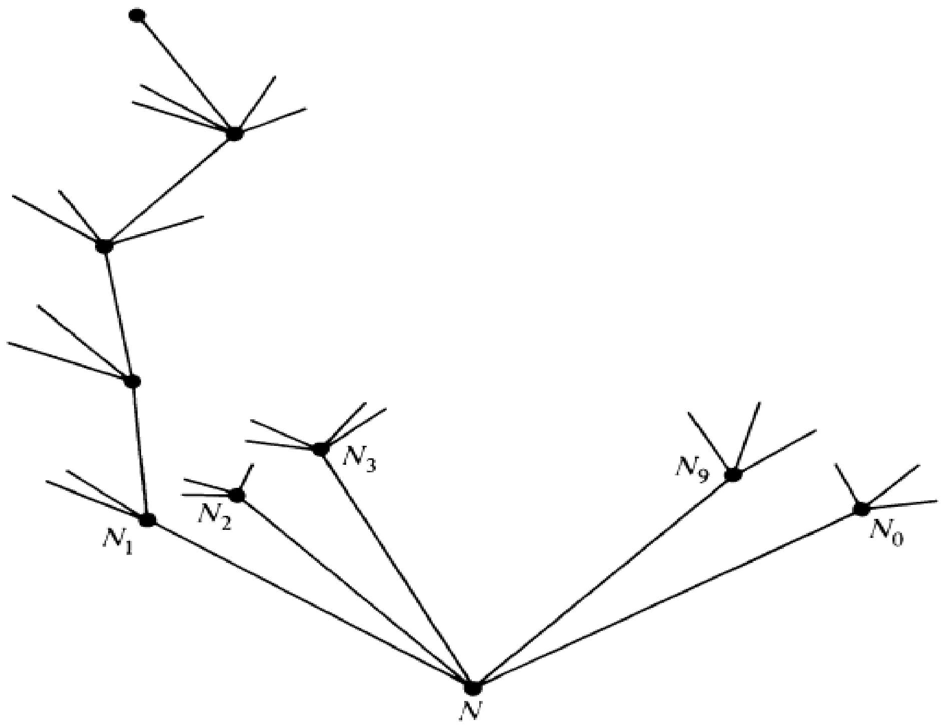
A tree is a connected graph without any circuits.



Trees with 1, 2, 3, and 4 vertices are shown in figure.



Decision tree shown in figure



1.10 PROPERTIES OF TREES

1. There is one and only one path between every pair of vertices in a tree T.
2. In a graph G there is one and only one path between every pair of vertices, G is a tree.
3. A tree with n vertices has n-1 edges.
4. Any connected graph with n vertices has n-1 edges is a tree.
5. A graph is a tree if and only if it is minimally connected.
6. A graph G with n vertices has n-1 edges and no circuits are connected.

THEOREM 1-7

There is one and only one path between every pair of vertices in a tree, T.

Proof: Since T is a connected graph, there must exist at least one path between every pair of vertices in T. Now suppose that between two vertices a and b of T there are two distinct paths. The union of these two paths will contain a circuit and T cannot be a tree. ■

Conversely:

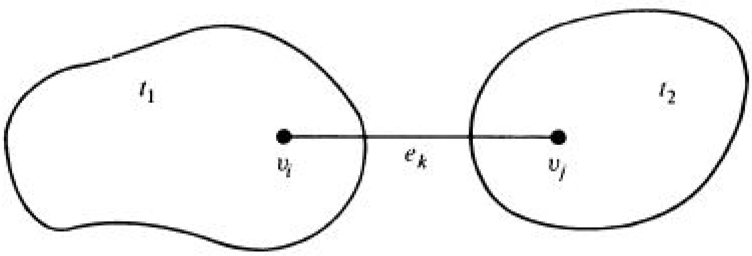
THEOREM 1-8

If in a graph G there is one and only one path between every pair of vertices, G is a tree.

Proof: Existence of a path between every pair of vertices assures that G is connected. A circuit in a graph (with two or more vertices) implies that there is at least one pair of vertices a, b such that there are two distinct paths between a and b. Since G has one and only one path between every pair of vertices, G can have no circuit. Therefore, G is a tree. Prepared by G. Appasami, Assistant professor, Dr. pauls Engineering College. ■

THEOREM 1-9

A tree with n vertices has $n - 1$ edges.



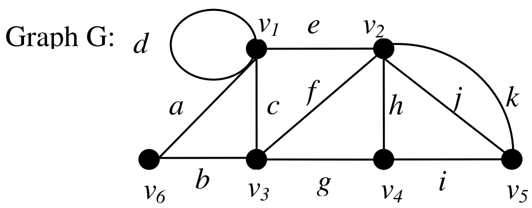
T

Proof: The theorem will be proved by induction on the number of vertices. It is easy to see that the theorem is true for n = 1, 2, and 3 (see Figure). Assume that the theorem holds for all trees with fewer than n vertices.

Let us now consider a tree T with n vertices. In T let e_k be an edge with end vertices v_i and v_j. According to Theorem 1-9, there is no other path between v_i and v_j, except e_k. Therefore, deletion of e_k from T will disconnect the graph, as shown in Figure. Furthermore, $T — e_k$ consists of exactly two components, and since there were no circuits in no begin with, each of these components is a tree. Both these trees, t_1 and t_2, have fewer than n vertices each, and therefore, by the induction hypothesis, each contains one less edge than the number of vertices in it. Thus $T — e_k$ consists of $n — 2$ edges (and n vertices). Hence T has exactly $n — 1$ edges. ■

1.11 DISTANCE AND CENTERS IN TREE

In a connected graph G, the distance $d(v_i , v_j)$ between two of its vertices v_i and v_j is the length of the shortest path.



Paths between vertices v_6 and v_2 are (a, e), (a, c, f), (b, c, e), (b, f), (b, g, h), and (b, g, i, k). The shortest paths between vertices v_6 and v_2 are (a, e) and (b, f), each of length two.

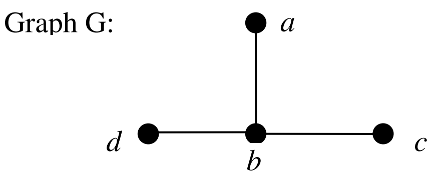
Hence $d(v_6 , v_2) = 2$

Define eccentricity and center.

The eccentricity E(v) of a vertex v in a graph G is the distance from v to the vertex farthest from v in G; that is,

$$E(v) = \max_{v_i \in G} d(v, v_i)$$

A vertex with minimum eccentricity in graph G is called a center of G

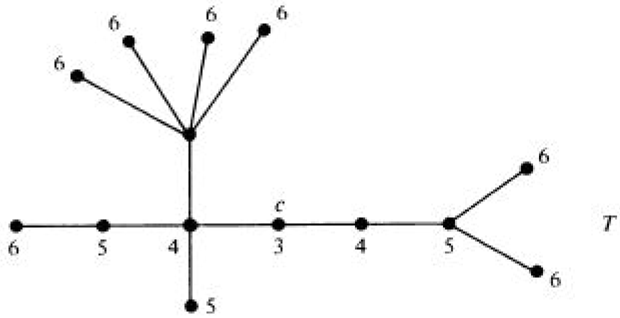


Distance d(a, b) = 1, d(a, c) =2, d(c, b)=1, and so on.

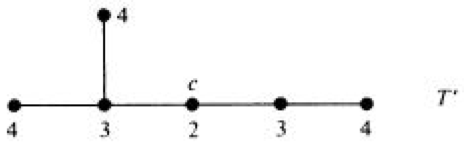
Eccentricity E(a) =2, E(b) =1, E(c) =2, and E(d) =2.

Center of G = A vertex with minimum eccentricity in graph G = b.

Finding Center of graph.



(a)



(b)

(c)

Distance metric.

The function $f\ (x,\ y)$ of two variables defines the distance between them. These function must satisfy certain requirements. They are

1. Non-negativity: $f\,(x, y) \geq 0$, and $f\,(x, y) = 0$ if and only if $x = y$.
2. Symmetry: $f\,(x, y) = f\,(x, y)$.
3. Triangle inequality: $f\,(x, y) \ \leq\ f\,(x, z) + f\,(z, y)$ for any z.

Radius and Diameter in a tree.

The eccentricity of a center in a tree is defined as the radius of tree.

The length of the longest path in a tree is called the diameter of tree.

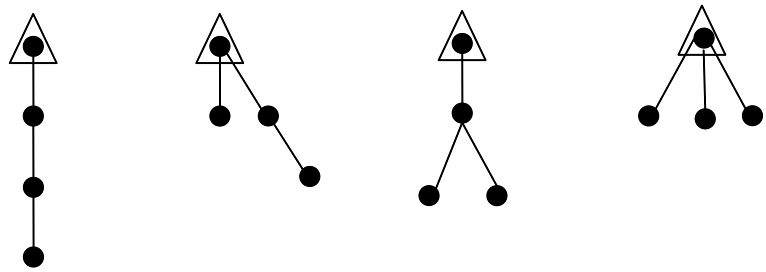
1.12 ROOTED AND BINARY TREES

Rooted tree

A tree in which one vertex (called the root) is distinguished from all the others is called a **rooted tree**.

In general tree means without any root. They are sometimes called as **free trees** (non rooted trees).

The root is enclosed in a small triangle. All rooted trees with four vertices are shown below.



Rooted binary tree

There is exactly one vertex of degree two (root) and each of remaining vertex of degree one or three. A binary rooted tree is special kind of rooted tree. Thus every binary tree is a rooted tree. A non pendent vertex in a tree is called an internal vertex.

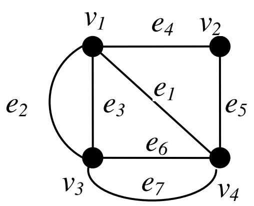
UNIT II TREES, CONNECTIVITY & PLANARITY

2. 1 SPANNING TREES

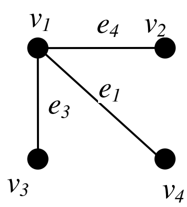
2.1.1 Spanning trees.

A tree T is said to be a spanning tree of a connected graph G if T is a subgraph of G and T contains all vertices (maximal tree subgraph).

Graph G:



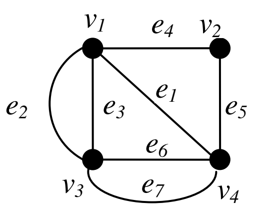
Spanning Tree T:



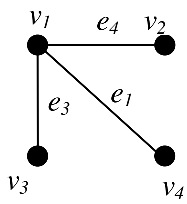
2.1.2. Branch and chord.

An edge in a spanning tree *T* is called a *branch* of *T*. An edge of G is not in a given spanning tree *T* is called a *chord* (*tie* or *link*).

Graph G:



Spanning Tree T:

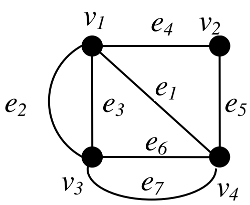


Edge e_1 is a branch of *T* *Edge e_5 is a chord* of *T*

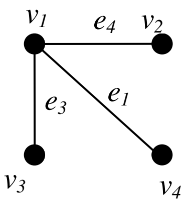
2.1.3. Complement of tree.

If T is a spanning tree of graph G, then the complement of T of G denoted by $\overline{T}$ is the collection of chords. It also called as *chord set* (*tie* set or *cotree*) of T

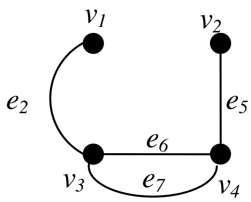
Graph G:



Spanning Tree T:



$\overline{\boldsymbol{T}}$:Complement of Tree T



$$\boldsymbol{T} \cup \overline{\boldsymbol{T}} = \boldsymbol{G}$$

2.1.4. Rank and Nullity:

A graph G with *n* number of vertices, *e* number of edges, and *k* number of components with the following constraints $n - k \geq 0$ and $e - n + k \geq 0$.

Rank $\quad r = n - k$

Nullity $\mu = e - n + k$ (Nullity also called as *Cyclomatic number* or *first betti number*)

Rank of G = number of branches in any spanning tree of G

Nullity of G = number of chords in G

Rank + Nullity = e = number of edges in G

2. 2 FUNDAMENTAL CIRCUITS

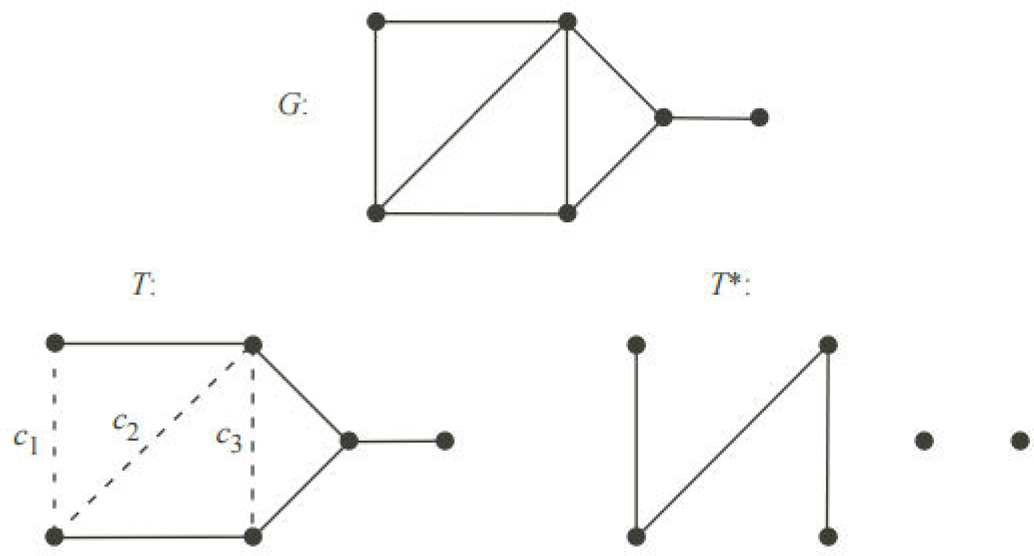
Addition of an edge between any two vertices of a tree creates a circuit. This is because there already exists a path between any two vertices of a tree.

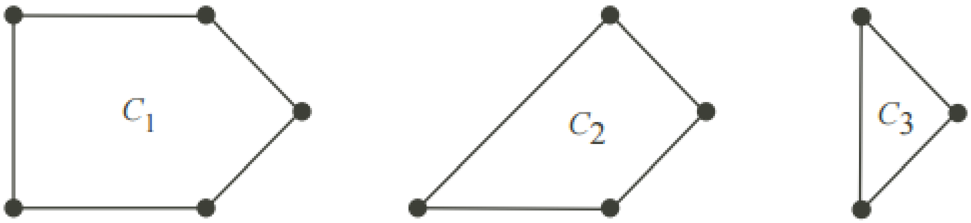
If the branches of the spanning tree T of a connected graph G are $b_1, \ldots, b_{n-1}$ and the corresponding links of the co spanning tree T $*$ are $c_1, \ldots, c_m-n+1$, then there exists one and only one circuit Ci in T + c_i (which is the subgraph of G induced by the branches of T and c_i)

Theorem: We call this circuit a fundamental circuit. Every spanning tree defines m − n + 1 fundamental circuits $C_1, \ldots, C_{m-n+1}$, which together form a fundamental set of circuits. Every fundamental circuit has exactly one link which is not in any other fundamental circuit in the fundamental set of circuits.

Therefore, we can not write any fundamental circuit as a ring sum of other fundamental circuits in the same set. In other words, the fundamental set of circuits is linearly independent under the ring sum operation.

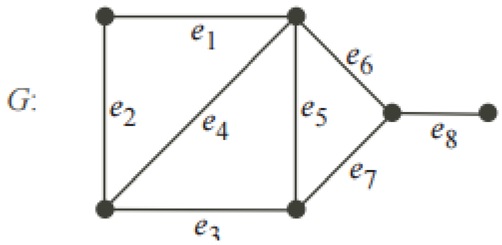
Example:

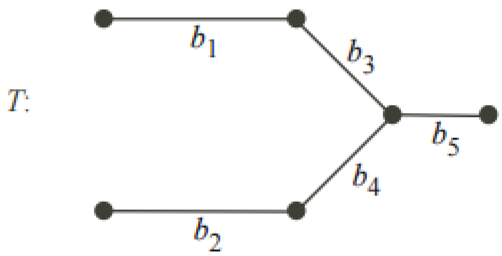




The graph T − b_i has two components T_1 and T_2. The corresponding vertex sets are V_1 and V_2. Then, (v_1,v_2) is a cut of G. It is also a cut set of G if we treat it as an edge set because G − hV1, V2i has two components . Thus, every branch bi of T has a corresponding cut set I_i . The cut sets I_1, . . . , I_{n-1} are also known as fundamental cut sets and they form a fundamental set of cut sets. Every fundamental cut set includes exactly one branch of T and every branch of T belongs to exactly one fundamental cut set. Therefore, every spanning tree defines a unique fundamental set of cut sets for G.

Example. (Continuing from the previous example) .





The graph has the spanning tree that defines these fundamental cut sets:

b1 : {e1, e2} b2 : {e2, e3, e4} b3 : {e2, e4, e5, e6} b4 : {e2, e4, e5, e7}b5 : {e8}

Next, we consider some properties of circuits and cut sets:

(a) Every cut set of a connected graph G includes at least one branch from every spanning tree of G. (Counter hypothesis: Some cut set F of G does not include any branches of a spanning tree T. Then, T is a subgraph of G − F and G − F is connected.

(b) Every circuit of a connected graph G includes at least one link from every co spanning tree of G. (Counter hypothesis: Some circuit C of G does not include any link of a cos panning tree T ∗ . Then, T = G − T ∗ has a circuit and T is not a tree.

2. 3 SPANNING TREES IN A WEIGHTED GRAPH

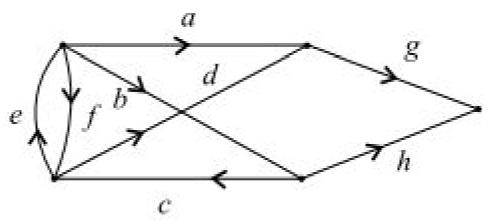
A spanning tree in a graph G is a minimal subgraph connecting all the vertices of G. If G is a weighted graph, then the weight of a spanning tree *T* of G is defined as the sum of the weights of all the branches in *T*.

A spanning tree with the smallest weight in a weighted graph is called a *shortest spanning tree (shortest-distance spanning tree* or *minimal spanning tree)*.

A shortest spanning tree T for a weighted connected graph G with a constraint $d(v_i) \leq k$ for all vertices in T. for k=2, the tree will be Hamiltonian path.

A spanning tree is an n-vertex connected digraph analogous to a spanning tree in an undirected graph and consists of n − 1 directed arcs.

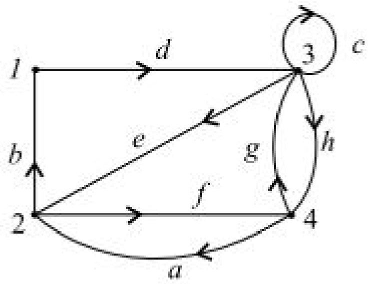
A spanning arborescence in a connected digraph is a spanning tree that is an arborescence. For example, {a, b, c, g} is a spanning arborescence in Figure .



Theorem: In a connected isograph D of n vertices and m arcs, let W = (a_1, a_2,..., a_m) be an Euler line, which starts and ends at a vertex v (that is, v is the initial vertex of a_1 and the terminal vertex of a_m). Among the m arcs in W there are n − 1 arcs that enter each of n−1 vertices, other than v, for the first time. The sub digraph D_1 of these n−1 arcs together with the n vertices is a spanning arborescence of D, rooted at vertex v. Prepared by G. Appasami, Assistant professor, Dr. pauls Engineering College.

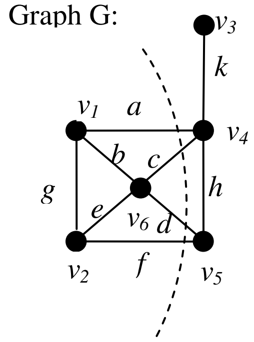
Proof : In the sub digraph D_1, vertex v is of in degree zero, and every other vertex is of indegree one, for D_1 includes exactly one arc going to each of the n−1 vertices and no arc going to v. Further, the way D_1 is defined in W, implies that D_1 is connected and contains n−1 arcs. Therefore D1 is a spanning arborescence in D and is rooted at v.

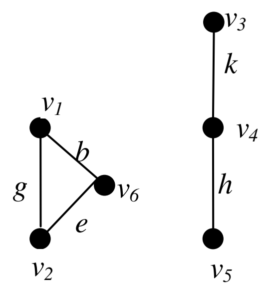
Illustration: In Figure, W = (b d c e f g h a) is an Euler line, starting and ending at vertex 2. The sub digraph {b, d, f } is a spanning arborescence rooted at vertex 2.



2. 4 CUT SETS

In a connected graph G, a cut-set is a set of edges whose removal from G leave the graph G disconnected.





Disconnected graph G with 2 components after removing cut set {a, c, d, f}

Possible cut sets are {a, c, d, f}, {a, b, e, f}, {a, b, g}, {d, h, f}, {k}, and so on.

{a, c, h, d} is not a cut set, because its proper subset {a, c, h} is a cut set.

{g, h} is not a cut set.

A minimal set of edges in a connected graph whose removal reduces the rank by one is called minimal cut set (simple cut-set or cocycle). Every edge of a tree is a cut set.

2. 5 PROPERTIES OF CUT SET

- Every cut-set in a connected graph G must contain at least one branch of every spanning tree of G.
- In a connected graph G, any minimal set of edges containing at least one branch of every spanning tree of G is a cut-set.
- Every circuit has an even number of edges in common with any cut set.

Properties of circuits and cut sets:

Every cut set of a connected graph G includes at least one branch from every spanning tree of G. (Counter hypothesis: Some cut set F of G does not include any branches of a spanning tree T. Then, T is a subgraph of G − F and G − F is connected.)

(b) Every circuit of a connected graph G includes at least one link from every co-spanning tree of G. (Counter hypothesis: Some circuit C of G does not include any link of a co-spanning tree T∗. Then, T = G − T∗ has a circuit and T is not a tree.)

Theorem : The edge set F of the connected graph G is a cut set of G if and only if

(i) F includes at least one branch from every spanning tree of G, and

(ii) if H ⊂ F, then there is a spanning tree none of whose branches is in H.

Proof. Let us first consider the case where F is a cut set. Then, (i) is true (previous proposition

(a). If H ⊂ F then G − H is connected and has a spanning tree T. This T is also a spanning tree of G. Hence, (ii) is true.

Let us next consider the case where both (i) and (ii) are true. Then G − F is disconnected.

If H ⊂ F there is a spanning tree T none of whose branches is in H. Thus T is a subgraph of G − H and G − H is connected. Hence, F is a cut set.

2. 6 ALL CUT SETS

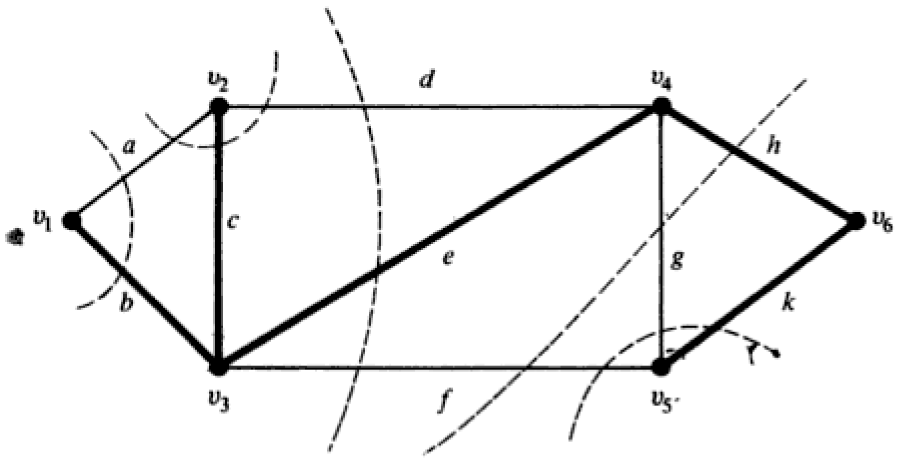
It was shown how cut-sets are used to identify weak spots in a communication net. For this purpose we list all cut-sets of the corresponding graph, and find which ones have the smallest number of edges. It must also have become apparent to you that even in a simple example, such as in Fig.

There is a large number of cut-sets, and we must have a systematic method of generating all relevant cut-sets. In the case of circuits, we solved a similar problem by the simple technique of finding a set of fundamental circuits and then realizing that other circuits in a graph are just combinations of two or more fundamental circuits.

We shall follow a similar strategy here. Just as a spanning tree is essential for defining a set of fundamental circuits, so is a spanning tree essential for a set of fundamental cut-sets. It will be beneficial for the reader to look for the parallelism between circuits and cut-sets. Fundamental Cut-Sets: Consider a spanning tree T of a connected graph G.

Take any branch b in T. Since {b) is a cut-set in T, (b) partitions all vertices of T into two disjoint sets—one at each end of b. Consider the same partition of vertices in G, and the cut set S in G that corresponds to this partition. Cut-set S will contain only one branch b of T, and the rest (if any) of the edges in S are chords with respect to T.

Such a cut-set S containing exactly one branch of a tree T is called a fundamental cut-set with respect to T. Sometimes a fundamental cut-set is also called a basic cut-set. Prepared by G. Appasami, Assistant professor, Dr. pauls Engineering College.



Fundamental cut sets of graph

T (in heavy lines) and all five of the fundamental cut-sets with respect to T are shown (broken lines "cutting" through each cut-set). Just as every chord of a spanning tree defines a unique fundamental cir-cuit, every branch of a spanning tree defines a unique fundamental cut-set.

It must also be kept in mind that the term fundamental cut-set (like the term fundamental circuit) has meaning only with respect to a given spanning tree. Now we shall show how other cut-sets of a graph can be obtained from a given set of cut-sets.

2. 7 FUNDAMENTAL CIRCUITS AND CUT SETS

Adding just one edge to a spanning tree will create a cycle; such a cycle is called a **fundamental cycle** (**Fundamental circuits**). There is a distinct fundamental cycle for each edge; thus, there is a one-to-one correspondence between fundamental cycles and edges not in the spanning tree. For a connected graph with V vertices, any spanning tree will have $V-1$ edges, and thus, a graph of E edges and one of its spanning trees will have $E-V+1$ fundamental cycles.

Dual to the notion of a fundamental cycle is the notion of a **fundamental cutset**. By deleting just one edge of the spanning tree, the vertices are partitioned into two disjoint sets. The fundamental cutset is defined as the set of edges that must be removed from the graph G to accomplish the same partition. Thus, each spanning tree defines a set of $V-1$ fundamental cutsets, one for each edge of the spanning tree.

Consider a spanning tree T in a given connected graph G. Let c, be a chord with respect to T, and let the fundamental circuit made by e_i be called r, con-sisting of k branches b_1, b_2, . . . , b_4 in addition to the chord 4; that is, r = {b_1, b_2, b_3 ..... B_4} is a fundamental circuit with respect to T. Every branch of any spanning tree has a fundamental cut-set associated with it.

Let S_i be the fundamental cut-set associated with by consisting of q chords in addition to the branch b_1; that is,

S_i = {b_1, c_1 c_2 ..... C_4} is a fundamental cut-set with respect to T.

Because of above Theorem , there must be an even number of edges common to and S_i. Edge b, is in both Γ and S_i and there is only one other edge in Γ (which is c,) that can possibly also be in S_i. Therefore, we must have two edges b, and c, common to S, and r. Thus the chord c, is one of the chords c_1, , C_4.

Exactly the same argument holds for fundamental cut-sets associated with b_2,b_3, . , and b_k. Therefore, the chord c, is contained in every fundamental cut-set associated with branches in Γ . Is it possible for the chord c, to be in any other fundamental cut-set S' (with respect to T, of course) besides those associated with b_2,b_3, . and b_k. The answer is no.

Otherwise (since none of the branches in r are in S'), there would be only one edge c, common to S' and Γ , a contradiction to Theorem. Thus we have an important result.

THEOREM With respect to a given spanning tree T, a chord c_i that determines a fundamental circuit Γ occurs in every fundamental cut-set associated with the branches in Γ and in no other.

As an example, consider the spanning tree (b, c, e, h, k}, shown in heavy lines, in Fig. The fundamental circuit made by chord f is {f, e, h, k}.

The three fundamental cut-sets determined by the three branches e, h, and k are

determined by branch e: {d, e, f},

determined by branch h: {f, g, h},

determined by branch k: {f, g, k}.

Chord f occurs in each of these three fundamental cut-sets, and there is no other fundamental cut-set that contains f. The converse of above Theorem is also true.

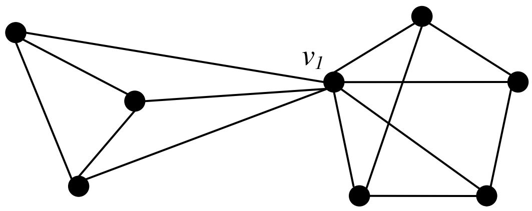
2. 8 CONNECTIVITY AND SEPARABILITY

edge Connectivity.

Each cut-set of a connected graph G consists of certain number of edges. The number of edges in the smallest cut-set is defined as the **edge Connectivity of G.**

The **edge Connectivity** of a connected graph G is defined as the minimum number of edges whose removal reduces the rank of graph by one.

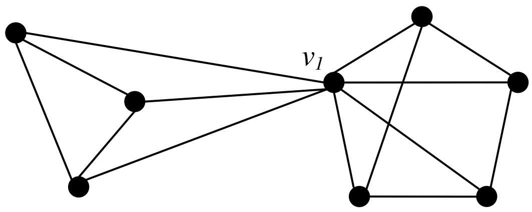
The edge Connectivity of a tree is one.



The edge Connectivity of the above graph G is three.

vertex Connectivity

The **vertex Connectivity** of a connected graph G is defined as the minimum number of vertices whose removal from G leaves the remaining graph disconnected. The vertex Connectivity of a tree is one.

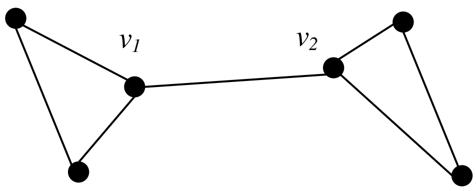


The vertex Connectivity of the above graph G is one.

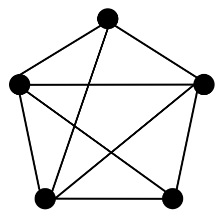
separable and non-separable graph.

A connected graph is said to be separable graph if its vertex connectivity is one. All other connected graphs are called non-separable graph.

Separable Graph G:

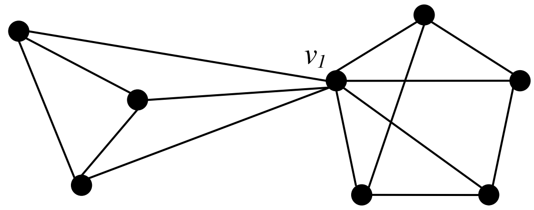


Non-Separable Graph H:



articulation point.

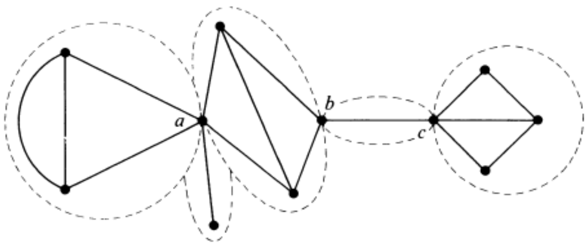
In a separable graph a vertex whose removal disconnects the graph is called *a cut-vertex, a cut-node, or an articulation point.*



v_1 is an articulation point.

component (or block) of graph.

A separable graph consists of two or more non separable subgraphs. Each of the largest nonseparable is called a block (or component).



The above graph has 5 blocks.

Lemmas:

If $S \subseteq V_G$ separates u and v, then every path P: u $\xrightarrow{\star}$ v visits a vertex of S.

If a connected graph G has no separating sets, then it is a complete graph.

Proof. If $v_G \leq 2$, then the claim is clear. For $V_G \geq 3$, assume that G is not complete,and let uv $\notin$ G. Now VG \ {u, v} is a separating set.

DEFINITION. The (vertex) connectivity number k(G) of G is defined as $k(G) = \min\{k \mid k = |S|, G-S \text{ disconnected or trivial}, S \subseteq V_G\}$.

A graph G is k-connected, if $k(G) \geq k$.

In other words,

• $k(G) = 0$, if G is disconnected,

• $k(G) = v_G - 1$, if G is a complete graph, and

• otherwise k(G) equals the minimum size of a separating set of G. Clearly, if G is connected, then it is 1-connected.

2. 9 NETWORK FLOWS

A **flow network** (also known as a transportation **network**) is a **graph** where each edge has a capacity and each edge receives a **flow**. The amount of **flow** on an edge cannot exceed the capacity of the edge.

The max. flow between two vertices = Min. of the capacities of all cut-sets.

Various transportation networks or water pipelines are conveniently represented by weighted directed graphs. These networks usually possess also some additional requirements.

Goods are transported from specific places (warehouses) to final locations (marketing places) through a network of roads.

In modeling a transportation network by a digraph, we must make sure that the number of goods remains the same at each crossing of the roads.

The problem setting for such networks was proposed by T.E. Harris in the 1950s. The connection to Kirchhoff's Current Law (1847) is immediate.

According to this law, in every electrical network the amount of current flowing in a vertex equals the amount flowing out that vertex.

Flows

A network N consists of

1) An **underlying digraph** D = (V, E),

2) Two distinct vertices s and r, called the source and the sink of N, and

3) A **capacity function** $\alpha : V \times V \rightarrow R_+$ (nonnegative real numbers), for which $\alpha(e) = 0$, if $e \notin E$.

Denote $V_N = V$ and $E_N = E$.

Let $A \subseteq V_N$ be a set of vertices, and $f : V_N \times V_N \rightarrow R$ any function such that $f(e) = 0$, if $e \notin N$. We adopt the following notations:

$[A, \overline{A}] = \{e \in D \mid e = uv, u \in A, v \notin A\}$,

$$f^+(A) = \sum_{e \in [A, \overline{A}]} f(e) \quad \text{and} \quad f^-(A) = \sum_{e \in [\overline{A}, A]} f(e) .$$

In particular,

$$f^+(u) = \sum_{v \in N} f(uv) \quad \text{and} \quad f^-(u) = \sum_{v \in N} f(vu) .$$

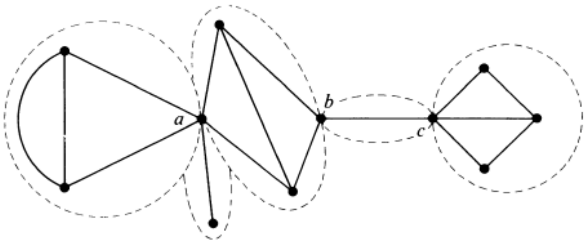
2. 10 1-ISOMORPHISM

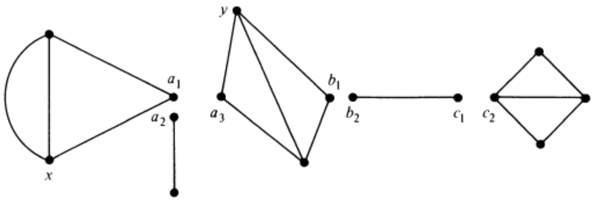
A graph G_1 was 1-Isomorphic to graph G_2 if the blocks of G_1 were isomorphic to the blocks of G_2.

Two graphs G_1 and G_2 are said to be 1-Isomorphic if they become isomorphic to each other under repeated application of the following operation.

Operation 1: “Split” a cut-vertex into two vertices to produce two disjoint subgraphs.

Graph G_1: Graph G_2:





Graph G_1 is 1-Isomorphism with Graph G_2.

A separable graph consists of two or more non separable subgraphs. Each of the largest non separable subgraphs is called a block.
(Some authors use the term component, but to avoid confusion with components of a disconnected graph, we shall use the term block.) The graph in Fig has two blocks. The graph in Fig has five blocks (and three cut-vertices a, b, and c); each block is shown enclosed by a broken line.
Note that a non separable connected graph consists of just one block. Visually compare the disconnected graph in Fig. with the one in Fig. These two graphs are certainly not isomorphic (they do not have the same number of vertices), but they are related by the fact that the blocks of the graph in Fig.are isomorphic to the components of the graph in above Fig. Such graphs are said to be I-isomorphic.
More formally: Two graphs G_1 and G_2 are said to be I-isomorphic if they become isomorphic to each other under repeated application of the following operation. Operation I: "Split" a cut-vertex into two vertices to produce two disjoint subgraphs. From this

definition it is apparent that two nonseparable graphs are 1-isomorphic if and only if they are isomorphic.

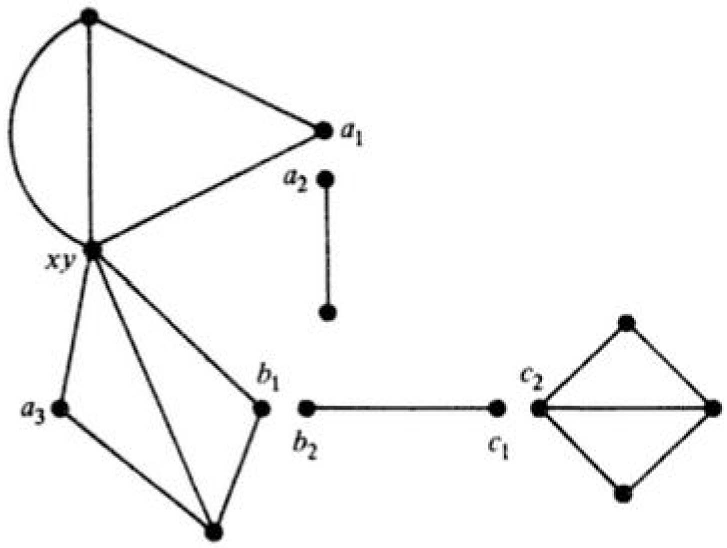
THEOREM If G_I and G_2 arc two 1-isomorphic graphs, the rank of GI equals the rank of C2 and the nullity of CI equals the nullity of G_2.

Proof: Under operation 1, whenever a cut-vertex in a graph G is "split" into two vertices, the number of components in G increases by one. Therefore, the rank of G which is number of vertices in C - number of components in G remains invariant under operation 1.

Also, since no edges are destroyed or new edges created by operation I, two 1-isomorphic graphs have the same number of edges.

Two graphs with equal rank and with equal numbers of edges must have the same nullity, because nullity = number of edges -- rank. What if we join two components of Fig by "gluing" together two vertices (say vertex x to y)? We obtain the graph shown in Fig. Clearly, the graph in Fig is 1-isomorphic to the graph in Fig.

Since the blocks of the graph in Fig are isomorphic to the blocks of the graph in Fig, these two graphs are also 1-isomorphic. Thus the three graphs in above Figs. are 1-isomorphic to one another.

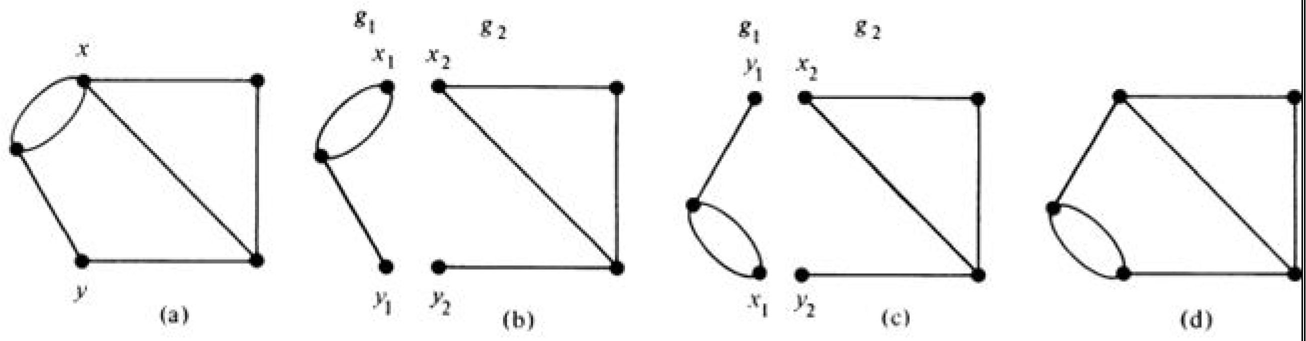


2. 11 2-ISOMORPHISM

Two graphs G_1 and G_2 are said to be **2-Isomorphic** if they become isomorphic after undergoing *operation 1* or *operation 2*, or both operations any number of times.

Operation 1: "Split" a cut-vertex into two vertices to produce two disjoint subgraphs.

Operation 2: "Split" the vertex x into x_1 and x_2 and the vertex y into y_1 and y_2 such that G is split into g_1 and g_2. Let vertices x_1 and y_1 go with g_1 and vertices x_2 and y_2 go with g_2. Now rejoin the graphs g_1 and g_2 by merging x_1 with y_2 and x_2 with y_1.



2. 12 COMBINATIONAL AND GEOMETRIC GRAPHS

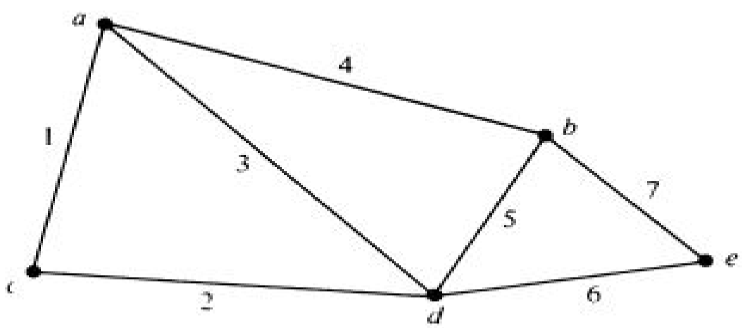
An abstract graph G can be defined as G = (V, E, Ψ)

Where the set V consists of five objects named *a, b, c, d*, and *e*, that is, $V = \{\ a, b, c, d, e\ \}$ and the set E consist of seven objects named 1, 2, 3, 4, 5, 6, and 7, that is, $E = \{\ 1, 2, 3, 4, 5, 6, 7\}$, and the relationship between the two sets is defined by the mapping Ψ, which consist of

Ψ= [1→(a, c), 2→(c, d) , 3→(a, d) , 4→(a, b) , 5→(b, d) , 6→(d, e) , 7→(b, e)].

Here the symbol 1→*(a, c)*, says that object 1 from set E is mapped onto the pair (a, c) of objects from set V.

This combinatorial abstract object G can also be represented by means of a geometric figure.



The figure is one such geometric representation of this graph G.

Any graph can be geometrically represented by means of such configuration in three dimensional Euclidian space.

2. 13 PLANER GRAPHS

A graph is said to be planar if it can be drawn in the plane in such a way that no two edges intersect each other. Drawing a graph in the plane without edge crossing is called embedding the graph in the plane (or planar embedding or planar representation).

Given a planar representation of a graph G, a face (also called a region) is a maximal section of the plane in which any two points can be joint by a curve that does not intersect any part of G.

When we trace around the boundary of a face in G, we encounter a sequence of vertices and edges, finally returning to our final position. Let v_1, e_1, v_2, e_2, . . . , v_d, e_d, v_1 be the sequence obtained by tracing around a face, then d is the degree of the face.

Some edges may be encountered twice because both sides of them are on the same face. A tree is an extreme example of this: each edge is encountered twice.

The following result is known as Euler's Formula.

PLANAR GRAPHS:

It has been indicated that a graph can be represented by more than one geometrical drawing. In some drawing representing graphs the edges intersect (cross over) at points which are not vertices of the graph and in some others the edges meet only at the vertices.

A graph which can be represented by at least one plane drawing in which the edges meet only at vertices is called a 'planar graph'.

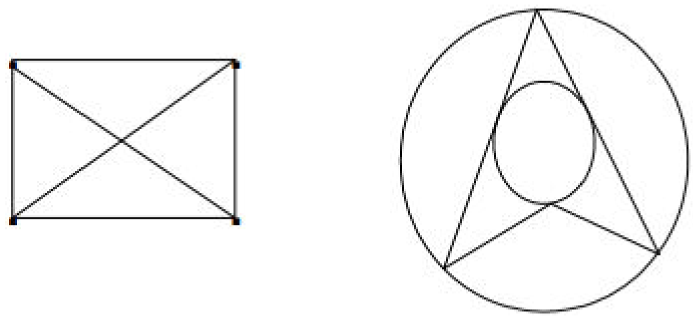
On the other hand, a graph which cannot be represented by a plane drawing in which the edges meet only at the vertices is called a non planar graph.

In other words, a non planar graph is a graph whose every possible plane drawing contains at least two edges which intersect each other at points other than vertices.

Example 1

Show that (a) a graph of order 5 and size 8, and (b) a graph of order 6 and size 12, are planar graphs.

Solution: A graph of order 5 and size 8 can be represented by a plane drawing.



Graph(a)Graph(b)

In which the edges of the graph meet only at the vertices, as shown in fig a. therefore, this graph is a planar graph.

Similarly, fig. b shows that a graph of order 6 and size 12 is a planar graph.

The plane representations of graphs are by no means unique. Indeed, a graph G can be drawn in arbitrarily many different ways.

Also, the properties of a graph are not necessarily immediate from one representation, but may be apparent from another.There are, however, important families of graphs, the surface graphs, that rely on the (topological or geometrical) properties of the drawings of graphs.

We restrict ourselves in this chapter to the most natural of these, the planar graphs. The geometry of the plane will be treated intuitively.

A planar graph will be a graph that can be drawn in the plane so that no two edges intersect with each other.

Such graphs are used, e.g., in the design of electrical (or similar) circuits, where one tries to (or has to) avoid crossing the wires or laser beams.

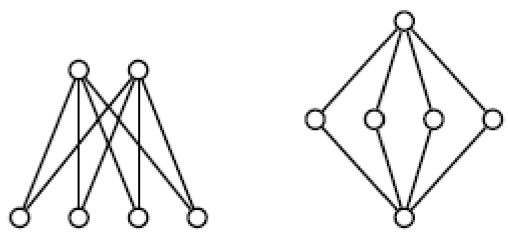
Planar graphs come into use also in some parts of mathematics, especially in group theory and topology.

There are fast algorithms (linear time algorithms) for testing whether a graph is planar or not. However, the algorithms are all rather difficult to implement. Most of them are based on an algorithm.

Definition

A graph G is a **planar graph**, if it has a plane figure P(G), called the **plane embedding** of G,where the lines (or continuous curves) corresponding to the edges do not intersect each other except at their ends.

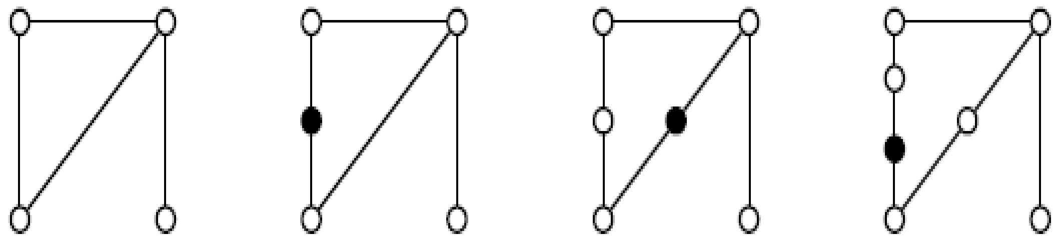
The complete bipartite graph $K_{2,4}$ is a planar graph.



Bipartite graph

An edge $e = uv \in G$ is **subdivided**, when it is replaced by a path $u \longrightarrow x \longrightarrow v$ of length two by introducing a new vertex x.

A **subdivision** H of a graph G is obtained from G by a sequence of subdivisions.



Graph

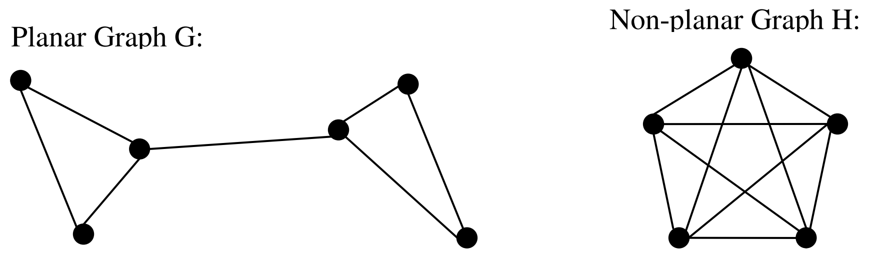
The following result is clear.

2. 14 DIFFERENT REPRESENTATION OF A PLANER GRAPH

between Planar and non-planar graphs

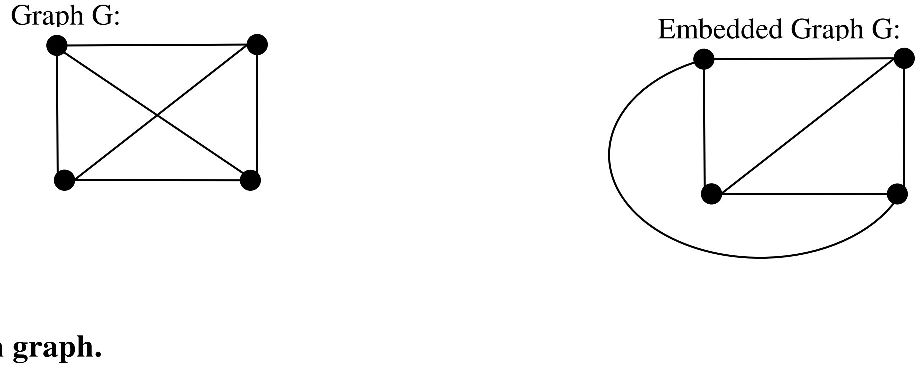
A graph G is said to be *planar* if there exists some geometric representation of G which can be drawn on a plan such that no two of its edges intersect.

A graph that cannot be drawn on a plan without crossover its edges is called *non-planar.*



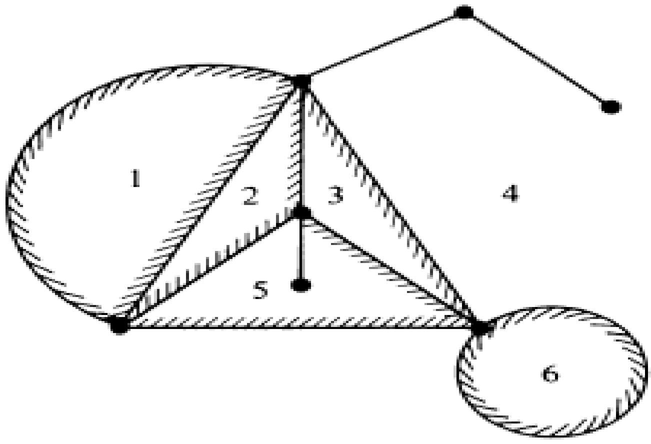
embedding graph.

A drawing of a geometric representation of a graph on any surface such that no edges intersect is called embedding.



region in graph.

In any planar graph, drawn with no intersections, the edges divide the planes into different **regions (windows, faces, or meshes)**. The regions enclosed by the planar graph are called **interior faces** of the graph. The region surrounding the planar graph is called the **exterior** (or infinite or unbounded) face of the graph. Prepared by G. Appasami, Assistant professor, Dr. pauls Engineering College.



The graph has 6 regions.

graph embedding on sphere.

To eliminate the distinction between finite and infinite regions, a planar graph is often embedded in the surface of sphere. This is done by stereographic projection.

UNIT III MATRICES, COLOURING AND DIRECTED GRAPH

3. 1 CHROMATIC NUMBER

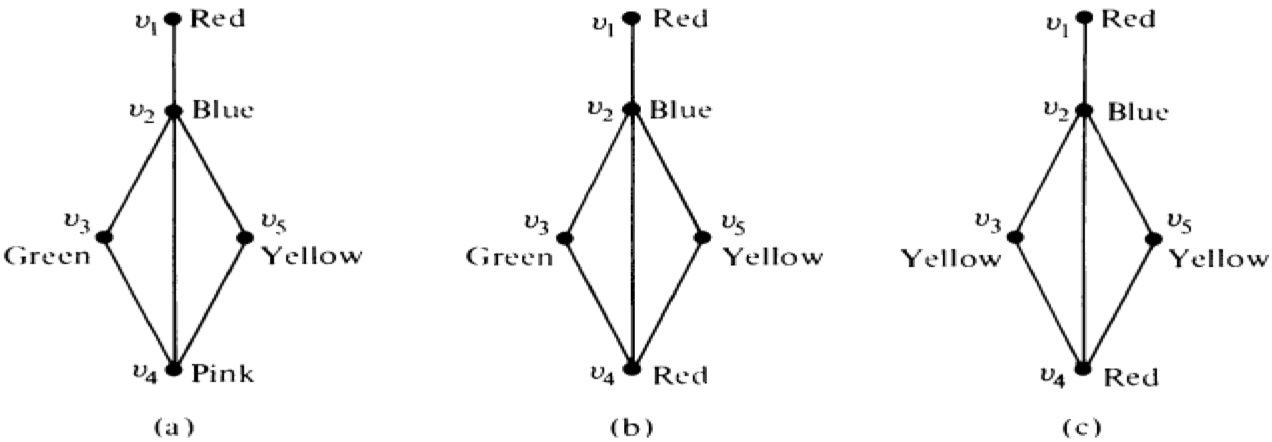
1. proper coloring

Painting all the vertices of a graph with colors such that no two adjacent vertices have the same color is called the *proper coloring* (simply *coloring*) of a graph. A graph in which every vertex has been assigned a color according to a proper coloring is called a *properly colored graph.*

2. Chromatic number

A graph G that requires k different colors for its proper coloring, and no less, is called *k-chromatic* graph, and the number k is called the *chromatic number* of G.

The minimum number of colors required for the proper coloring of a graph is called *Chromatic number.*



The above graph initially colored with 5 different colors, then 4, and finally 3. So the chromatic number is 3. i.e., The graph is 3-chromatic

3. properties of chromatic numbers (observations).

- A graph consisting of only isolated vertices is 1-chromatic.
- Every tree with two or more vertices is 2-chromatic.
- A graph with one or more vertices is at least 2-chromatic.
- A graph consisting of simply one circuit with $n \geq 3$ vertices is 2-chromatic if n is even and 3-chromatic if n is odd.
- A complete graph consisting of n vertices is n-chromatic.

SOME RESULTS:

i) A graph consisting of only isolated vertices (ie., Null graph) is 1–Chromatic (Because no two vertices of such a graph are adjacent and therefore we can assign the same color to all vertices).

ii) A graph with one or more edges is at least 2 -chromatic (Because such a graph has at least one pair of adjacent vertices which should have different colors).

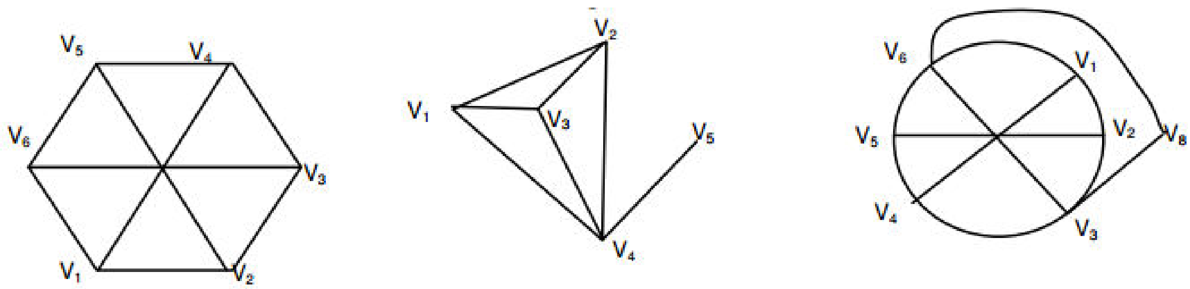
iii) If a graph G contains a graph G1 as a subgraph, then $\chi(G) \geq \chi(G_1)$.

iv. If G is a graph of n vertices, then $\chi(G) \leq n$.

v. (Kn) = n, for all n 1. (Because, in Kn, every two vertices are adjacent and as such all the n vertices should have different colors)

vi. If a graph G contains Kn as a subgraph, then $\chi(G) \geq n$.

Example 1: Find the chromatic number of each of the following graphs.



Solution

i) For the graph (a), let us assign a color a to the vertex V_1, then for a proper coloring, we have to assign a different color to its neighbors V_2,V_4,V_6, since V_2, V_4, V6are mutually non-adjacent vertices, they can have the same color as V_1, namely α.

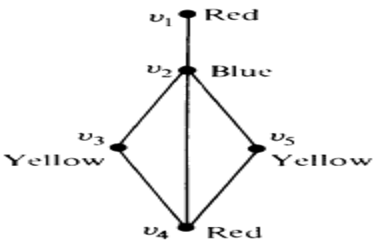
Thus, the graph can be properly colored with at lest two colors, with the vertices V_1,V_3,V_5 having one color α and V_2,V_4,V_6 having a different color β. Hence, the chromatic number of the graph is 2.

ii) For the graph (b) , let us assign the color α to the vertex V_1. Then for a proper coloring its neighbors V_2,V_3 & V_4 cannot have the color α. Further more, V_2, V_3,V_4 must have different colors, say β, γ , δ .Thus, at least four colors are required for a proper coloring of the graph. Hence the chromatic number of the graph is 4.

iii) For the graph (c) , we can assign the same color, say α, to the non-adjacent vertices V_1,V_3, V_5. Then the vertices V_2,V_4,V_6consequently $V_7$and V_8 can be assigned the same color which is different from both α and β. Thus, a minimum of three colors are needed for a proper coloring of the graph. Hence its chromatic number is 3. Prepared by G. Appasami, Assistant professor, Dr. pauls Engineering College.

3. 2 CHROMATIC PARTITIONING

A proper coloring of a graph naturally induces a partitioning of the vertices into different subsets based on colors.



For example, the coloring of the above graph produces the portioning $\{v_1, v_4\}$, $\{v_2\}$, and $\{v_3, v_5\}$.

A proper coloring of a graph naturally induces a partitioning of the vertices into different subsets. For example, the coloring in Fig. produces the partitioning

$$\{v_1, v_4\}, \quad \{v_2\}, \quad \text{and} \quad \{v_3, v_5\}.$$

No two vertices in any of these three subsets are adjacent. Such a subset of vertices is called an independent set; more formally:

A set of vertices in a graph is said to be an independent set of vertices or simply an independent set (or an internally stable set) if no two vertices in the set arc adjacent.

For example: in Fig., {a, c, d} is an independent set. A single vertex in any graph constitutes an independent set. A maximal independent set (or maximal internally stable set) is an independent set to which no other vertex can be added without destroying its independence property.

The set {a, c, d, f} in Fig. is a maximal independent set.

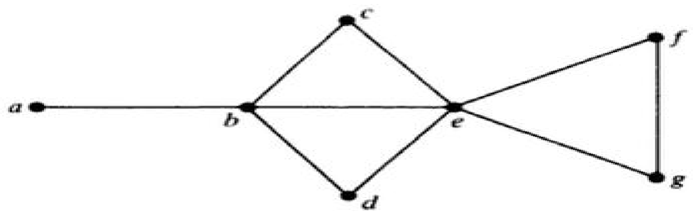
The set lb,/ I is another maximal independent set.

The set {b, g} is a third one. From the preceding example, it is clear that a graph, in general, has many maximal independent sets; and they may be of different sizes.

Among all maximal independent sets, one with the largest number of vertices is often of particular interest . Suppose that the graph in Fig. describes the following problem.

Each of the seven vertices of the graph is a possible code word to be used in some communication. Some words are so close (say, in sound) to others that they might be confused for each other. Pairs of such words that may be mistaken for one another are joined by edges. Find a largest set of code words for a reliable communication.

This is a problem of finding a maximal independent set with largest number of vertices. In this simple example, {a, c, d, f} is an answer.



Chromatic partitioning graph

The number of vertices in the largest independent set of a graph G is called the independence number (or coefficient of internal stability), β(G).

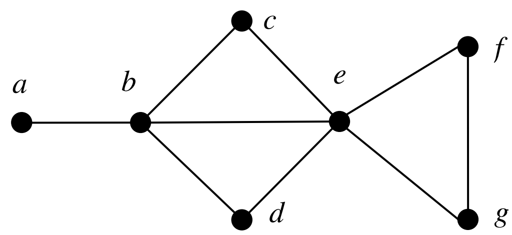
Consider a K-chromatic graph G of n vertices properly colored with different colors.

Since the largest number of vertices in G with the same color cannot exceed the independence number β(G), we have the inequality

$$\beta(G) \geq \frac{n}{\kappa}.$$

3. 3 CHROMATIC POLYNOMIAL

A set of vertices in a graph is said to be an *independent set* of vertices or simply independent set (or an internally stable set) if two vertices in the set are adjacent.



For example, in the above graph produces {a, c, d} is an independent set.

A single vertex in any graph constitutes an independent set.

A maximal independent set is an independent set to which no other vertex can be added without destroying its independence property.

{a, c, d, f} is one of the maximal independent set. {b, f} is one of the maximal independent set.

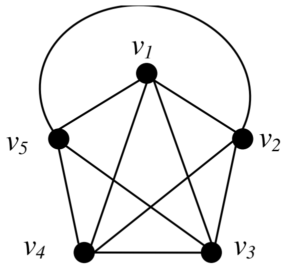
The number of vertices in the largest independent set of a graph G is called the *independence number* (or coefficients of internal stability), denoted by β(G).

For a K-chromatic graph of n vertices, the independence number β(G) $\geq \frac{n}{k}$.

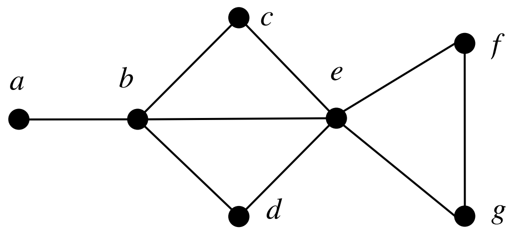
Uniquely colorable graph.

A graph that has only one chromatic partition is called a uniquely colorable graph. For example,

Uniquely colorable graph G:

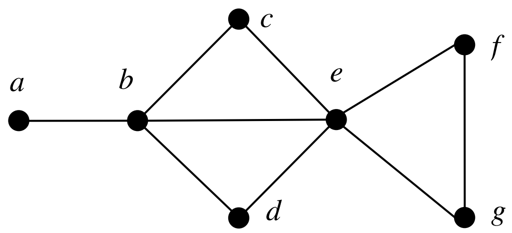


Not uniquely colorable graph H:



Dominating set.

A dominating set (or an externally stable set) in a graph G is a set of vertices that dominates every vertex v in G in the following sense: Either v is included in the dominating set or is adjacent to one or more vertices included in the dominating set.



{b, g} is a dominating set, {a, b, c, d, f} is a dominating set. A is a dominating set need not be independent set. Set of all vertices is a dominating set.

A minimal dominating set is a dominating set from which no vertex can be removed without destroying its dominance property.

{b, e} is a minimal dominating set.

Chromatic polynomial.

A graph G of n vertices can be properly colored in many different ways using a sufficiently large number of colors. This property of a graph is expressed elegantly by means of polynomial. This polynomial is called the Chromatic polynomial of G.

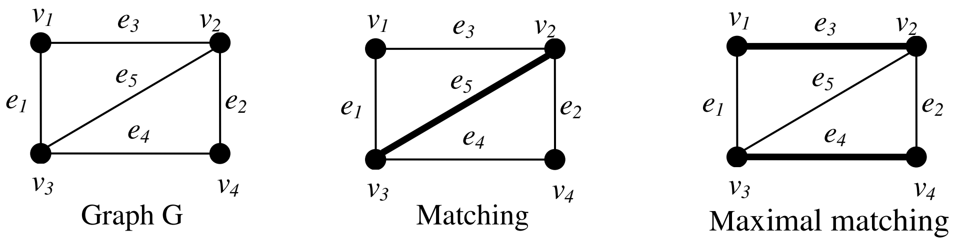
The value of the Chromatic polynomial $P_n(\lambda)$ of a graph with n vertices the number of ways of properly coloring the graph , using λ or fewer colors.

3. 4 MATCHING

A *matching* in a graph is a subset of edges in which no two edges are adjacent. A single edge in a graph is a matching.

A *maximal matching* is a matching to which no edge in the graph can be added.

The maximal matching with the largest number of edges are called the *largest maximal matching*.



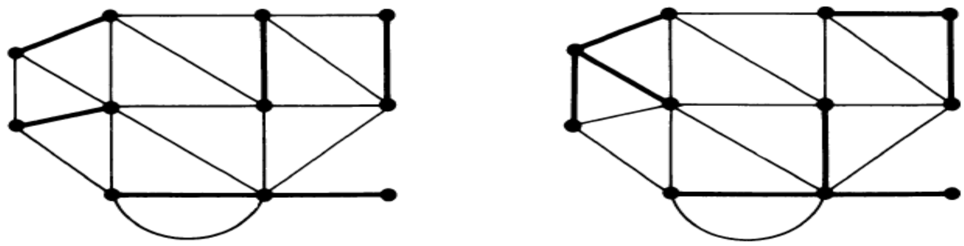
3. 5 COVERING

A set g of edges in a graph G is said to be cover og G if every vertex in G is incident on at least one edge in g. A set of edges that covers a graph G is said to be a covering (or an edge covering, or a coverring subgraph) of G.

Every graph is its own covering.

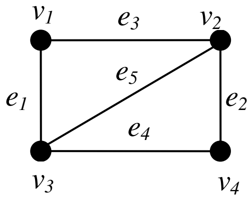
A spanning tree in a connected graph is a covering.

A Hamiltonian circuit in a graph is also a covering.

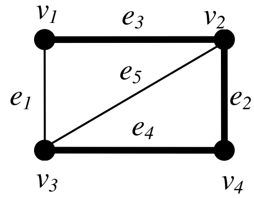


Minimal cover.

A **minimal covering** is a covering from which no edge can be removed without destroying it ability to cover the graph G.



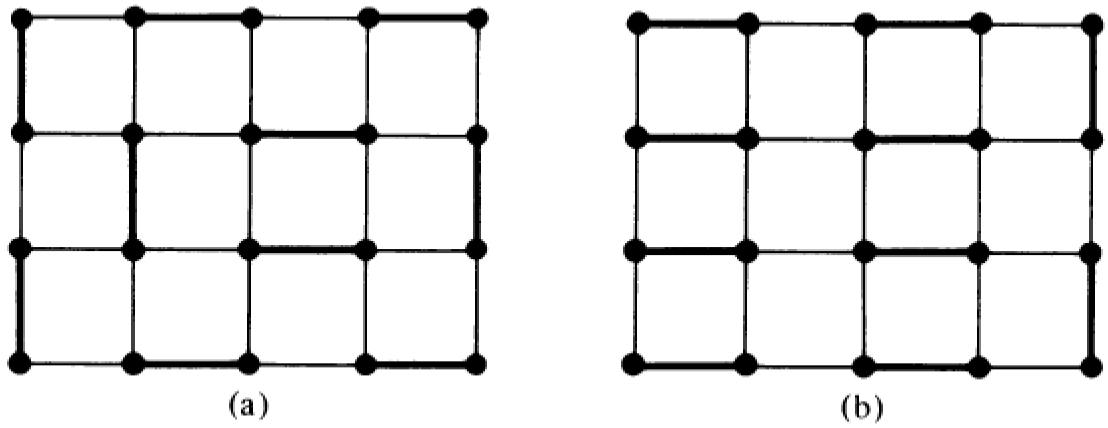
Graph G



Minimal cover

Dimer covering

A covering in which every vertex is of degree one is called a *dimer covering* or a *1-factor*. A dimmer covering is a maximal matching because no two edges in it are adjacent.



Two dimmer coverings.

Let G = (V, E) be a graph. A stable set is a subset C of V such that e ⊆ C for each edge e of G. A vertex cover is a subset W of V such that e ∩ W = ∅ for each edge e of G. It is not difficult to show that for each U ⊆ V :

U is a stable set ⇐⇒ V \ U is a vertex cover.

A matching is a subset M of E such that e∩e ′ = ∅ for all e, e′ ∈ M with e = e ′ . A matching is called perfect if it covers all vertices (that is, has size 1 2 |V |). An edge cover is a subset F of E such that for each vertex v there exists e ∈ F satisfying v ∈ e. Note that an edge cover can exist only if G has no isolated vertices.

Define:

α(G) := max{|C| | C is a stable set},

τ (G) := min{|W| | W is a vertex cover},

ν(G) := max{|M| | M is a matching},

ρ(G) := min{|F| | F is an edge cover}.

These numbers are called the stable set number, the vertex cover number, the matching number, and the edge cover number of G, respectively.

It is not difficult to show that: (3) $\alpha(G) \leq \rho(G)$ and $\nu(G) \leq \tau(G)$. The triangle K3 shows that strict inequalities are possible.

Theorem 1(Gallai's theorem). If G = (V, E) is a graph without isolated vertices, then
$\alpha(G) + \tau(G) = |V| = \nu(G) + \rho(G)$.

Proof.

The first equality follows directly from (1).

To see the second equality, first let M be a matching of size $\nu(G)$. For each of the $|V| - 2|M|$ vertices v missed by M, add to M an edge covering v. We obtain an edge cover F of size $|M|+(|V|-2|M|) = |V|-|M|$. Hence $\rho(G) \leq |F| = |V|-|M| = |V|-\nu(G)$.

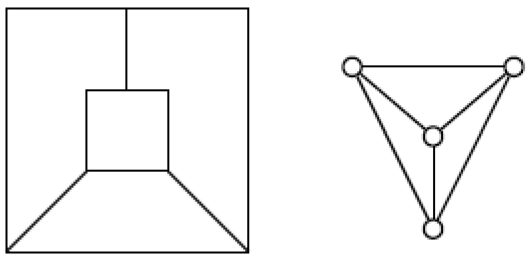
Second, let F be an edge cover of size $\rho(G)$. Choose from each component of the graph (V, F) one edge, to obtain a matching M. As (V, F) has at least $|V| - |F|$ components , we have $\nu(G) \geq |M| \geq |V| - |F| = |V| - \rho(G)$.

This proof also shows that if we have a matching of maximum cardinality in any graph G, then we can derive from it a minimum cardinality edge cover, and conversely.

3. 6 FOUR COLOR PROBLEM

- Every planar graph has a chromatic number of four or less.
- Every triangular planar graph has a chromatic number of four or less.
- The regions of every planar, regular graph of degree three can be colored properly with four colors.
- **4-Colour Theorem:**
- If G is a planar graph, then $\chi(G) \leq 4$. By the following theorem, each planar graph can be decomposed into two bipartite graphs.
- Let G = (V, E) be a 4-chromatic graph, $\chi(G) \leq 4$.
- Then the edges of G can be partitioned into two subsets E_1 and E_2 such that (V, E_1) and (V, E_2) are both bipartite.
- **Proof.** Let $V_i = \alpha^{-1}(i)$ be the set of vertices coloured by i in a proper 4-colouring a of G.
- The define E_1 as the subset of the edges of G that are between the sets $V_1$and V_2;V_1 and V_4; V_3 and V_4.
- Let E_2 be the rest of the edges, that is, they are between the sets V_1 and V_3; V_2 and V_3; V_2 and V_4. It is clear that (V, E_1) and (V, E_2) are bipartite, since the sets V_i are stable.

- **Map colouring***
- The 4-Colour Conjecture was originally stated for maps.
- In the map-colouring problem we are given several countries with common borders and we wish to colour each country so that no neighboring countries obtain the same colour.
- How many colors are needed?
- A border between two countries is assumed to have a positive length in particular,countries that have only one point in common are not allowed in the map colouring.
- Formally, we define a map as a connected planar (embedding of a) graph with no bridges. The edges of this graph represent the boundaries between countries.
- Hence a country is a face of the map, and two neighbouring countries share a common edge (not just a single vertex). We deny bridges, because a bridge in such a map would be a boundary inside a country.
- The map-colouring problem is restated as follows:
- How many colours are needed for the faces of a plane embedding so that no adjacent faces obtain the same colour.
- The illustrated map can be 4-coloured, and it cannot be coloured using only 3 colours, because every two faces have a common border.



-
- Colour map
- Let $F_1, F_2, \ldots, F_n$ be the countries of a map M, and define a graph G with $V_G = \{v_1, v_2, \ldots, v_n\}$ such that $v_iv_j \in G$ if and only if the countries F_i and $F_j$are neighbour's.
- It is easy to see that G is a planar graph. Using this notion of a dual graph, we can state the map-colouring problem in new form:
- What is the chromatic number of a planar graph? By the 4-Colour Theorem it is at most four.
- Map-colouring can be used in rather generic topological setting, where the maps are defined by curves in the plane.

- As an example, consider finitely many simple closed curves in the plane. These curves divide the plane into regions. The regions are 2-colourable.
- That is, the graph where the vertices correspond to the regions, and the edges correspond to the neighbourhood relation, is bipartite.
- To see this, colour a region by 1, if the region is inside an odd number of curves, and, otherwise, colour it by 2.

State five color theorem

Every planar map can be properly colored with five colors.

i.e., the vertices of every plannar graph can be properly colored with five colors.

Vertex coloring and region coloring.

A graph has a dual if and only if it is planar. Therefore, coloring the regions of a planar graph G is equivalent to coloring the vertices of its dual G* and vice versa.

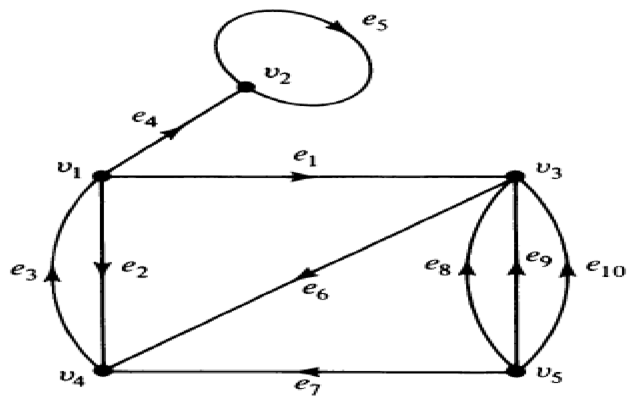
Regularization of a planar graph

- Remove every vertex of degree one from the graph G does not affect the regions of a plannar graph.
- Remove every vertex of degree two and merge the two edges in series from the graph G.
- Such a transformation may be called regularization of a planar graph.

3. 7 DIRECTED GRAPHS

A *directed graph* (or a *digraph, or an oriented graph*) G consists of a set of vertices $V = \{ v_1, v_2, \ldots \}$, a set of edges $E = \{ e_1, e_2, \ldots \}$, and a mapping Ψ that maps every edge onto some ordered pair of vertices (v_i, v_j).

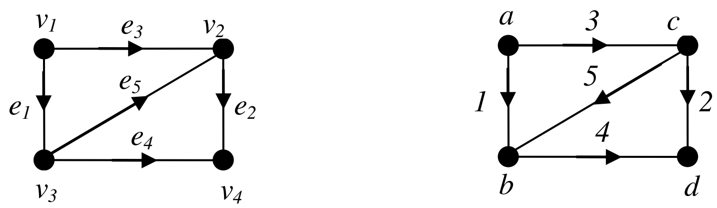
For example,



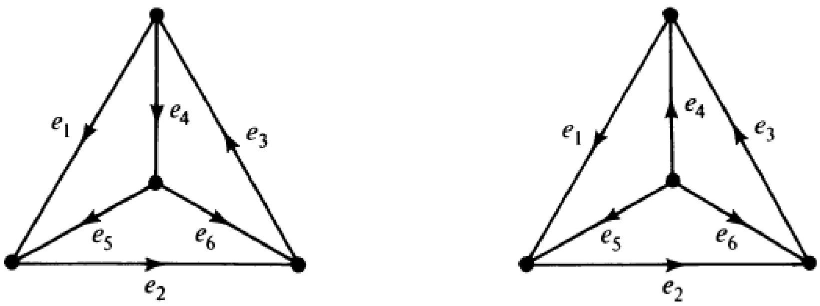
isomorphic digraph.

Among directed graphs, if their labels are removed, two isomorphic graphs are indistinguishable then these graphs are **isomorphic digraph**.

For example,



Two isomorphic digraphs.



Two non-isomorphic digraphs.

3. 8 TYPES OF DIRECTED GRAPHS

Like undirected graphs , digraphs are also has so many verities. In fact, due to the choice of assigning a direction to each edge, directed graphs have more varieties than undirected ones.

Simple Digraphs:

A digraph that has no self-loop or parallel edges is called a simple digraph .

Asymmetric Digraphs:

Digraphs that have at most one directed edge between a pair of vertices, but are allowed to have self-loops, are called asymmetric or antisymmetric.

Symmetric Digraphs:

Digraphs in which for every edge (a, b) (i.e., from vertex a to b) there is also an edge (b, a).

A digraph that is both simple and symmetric is called a simple symmetric digraph. Similarly, a digraph that is both simple and asymmetric is simple asymmetric.

The reason for the terms symmetric and asymmetric will be apparent in the context of binary relations.

Complete Digraphs:

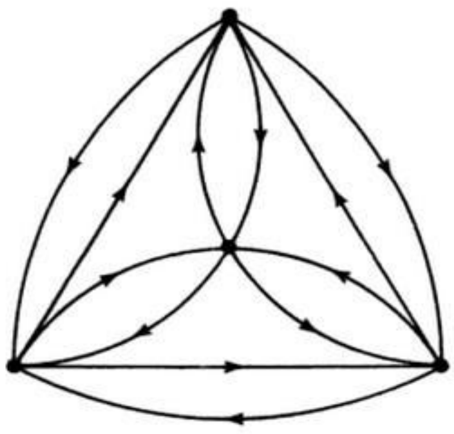
A complete undirected graph was defined as a simple graph in which every vertex is joined to every other vertex exactly by one edge.

For digraphs we have two types of complete graphs.

A complete symmetric digraph is a simple digraph in which there is exactly one edge directed from every vertex to every other vertex, and a complete asymmetric digraph is an asymmetric digraph in which there is exactly one edge between every pair of vertices.

A complete asymmetric digraph of n vertices contains n(n - 1)/2 edges, but a complete symmetric digraph of n vertices contains n(n - 1) edges. A complete asymmetric digraph is also called a tournament or a complete tournament (the reason for this term will be made clear).

A digraph is said to be balanced if for every vertex v, the in-degree equals the out-degree; that is, d+(vi) = div,). (A balanced digraph is also referred to as a pseudo symmetric digraph. or an isograph.) A balanced digraph is said to be regular if every vertex has the same in-degree and out-degree as every other vertex.



Complete symmetric digraph of four vertices

3. 9 DIGRAPHS AND BINARY RELATIONS

In a set of objects, X, where X={x_1, x_2, …}, A *binary relation R* between pairs (x_i, x_j) can be written as x_i *R* x_j and say that x_i has relation *R* to x_j.

If the binary relation *R* is reflexive, symmetric, and transitive then *R* is an equivalence relation. This produces equivalence classes.

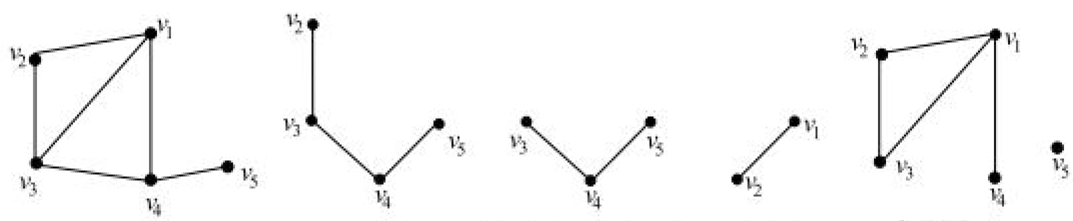
Let A and B be nonempty sets. A (binary) relation R from A to B is a subset of A x B. If R $\subseteq$ AxB and(a,b) $\in$ R, where a $\in$ A, b $\in$ B, we say a "is related to" b by R, and we write aRb. If a is not related to b by R, we write a 0 b. A relation R defined on a set X is a subset of X x X.

For example, less than, greater than and equality are the relations in the set of real numbers. The property "is congruent to" defines a relation in the set of all triangles in a plane. Also, parallelism defines a relation in the set of all lines in a plane.

Let R define a relation on a non empty set X. If R relates every element of X to itself, the relation R is said to be reflexive. A relation R is said to be symmetric if for all x_i x_j $\in$ X, x_i R xj implies x_j R x_i. A relation R is said to be transitive if for any three elements x_i, x_j and x_k in X, x,Rx, and x_j R x_k imply xi /2 x_k. A binary relation is called an equivalence relation if it is reflexive, symmetric and transitive.

A binary relation Ron a set X can always be represented by a digraph. In such a representation, each x_j E X is represented by a vertex xi and whenever there is a relation R from xi to x_j, an arc is drawn from xi to x_j, for every pair (x_i, x_j). The digraph in Figure represents the relation is less than, on a set consisting of four numbers 2, 3, 4, 6.

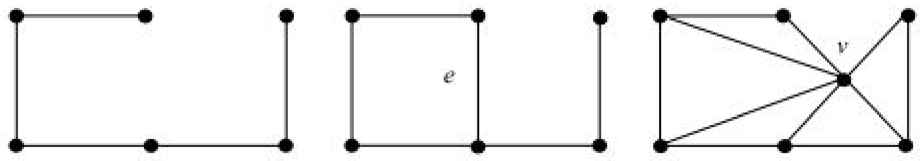
We note that every binary relation on a finite set can be represented by a digraph without parallel edges and vice versa. Clearly, the digraph of a reflexive relation contains a loop at every vertex Fig. A digraph representing a reflexive binary relation is called a reflexive digraph.



G G-v_1 H (Subgraph of G) G-v(H) G-E(H)

Example

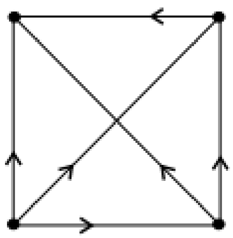
The digraph of a symmetric relation is a symmetric digraph because for every arc from x_i to x_j , there is an arc from x_j to x_i . Figure shows the digraph of an irreflexive and symmetric relation on a set of three elements.



Graph G (a) $G+e$ (b) $G+v$ (c)

Example

A digraph representing a transitive relation on its vertex set is called a transitive digraph. Figure shows the digraph of a transitive, which is neither reflexive, nor symmetric.



Example

A binary relation R on a set M can also be represented by a matrix, called a relation matrix. This is a (0, 1), n×n matrix $M_R = [m_{i\,j}]$, where n is the number of elements in M, and is defined by

$$m_{ij} = \begin{cases} 1 & if\ x_i\ R\ x_j\ is\ true, \\ 0, & otherwise. \end{cases}$$

In some problems the relation between the objects is not symmetric. For these cases we need directed graphs, where the edges are oriented from one vertex to another.

Definitions

A digraph (or a directed graph) $D = (V_D, E_D)$ consists of the vertices V_D and (directed) edges $E_D \subseteq V_D \times V_D$ (without loops vv).

We still write uv for (u, v), but note that now uv $\neq$ vu.

For each pair e = uv define the inverse of e as e^{-1} = vu (= (v, u)).

Note that $e \in D$ does not imply $e^{-1} \in D$.

Let D be a digraph. Then A is its further classified into:

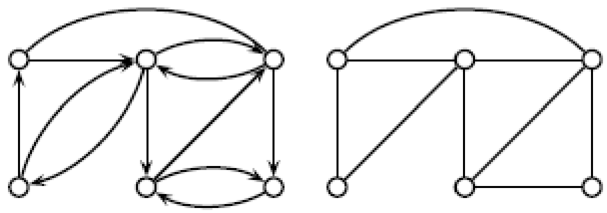
Subdigraph, if $V_A \subseteq V_D$ and $E_A \subseteq E_D$,

Induced subdigraph, A = D[X], if VA = X and $E_A = E_D \cap (X \times X)$.

The **underlying graph** U(D) of a digraph D is the graph on V_D such that if e ∈ D, then the undirected edge with the same ends is in U(D).

A digraph D is an **orientation** of a graph G, if G = U(D) and e ∈ D implies $e^{-1} \notin D$.

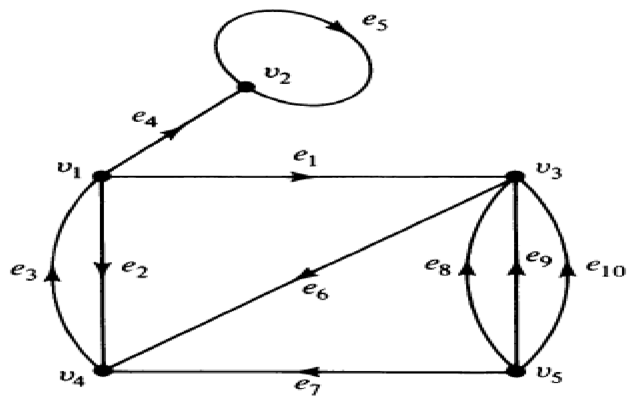
In this case, D is said to be an **oriented graph**.



Example

3. 10 DIRECTED PATHS AND CONNECTEDNESS

A path in a directed graph is called Directed path.



$v_5\ e_8\ v_3\ e_6\ v_4\ e_3\ v_1$ is a directed path from v_5 to v_1.

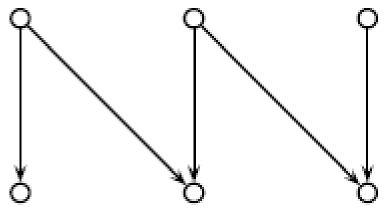
Whereas $v_5\ e_7\ v_4\ e_6\ v_3\ e_1\ v_1$ is a semi-path from v_5 to v.

- **Strongly connected digraph:** A digraph G is said to be strongly connected if there is at least one directed path from every vertex to every other vertex.

Weakly connected digraph: A digraph G is said to be weakly connected if its corresponding undirected graph is connected. But G is not strongly connected.

The relationship between paths and directed paths is in general rather complicated. This digraph has a path of length five, but its directed paths are of length one.

There is a nice connection between the lengths of directed paths and the chromatic number χ(D) = χ(U(D)).



Example

Theorem : A digraph D has a directed path of length χ(D) −1.

Proof. Let A ⊆ E_D be a minimal set of edges such that the subdigraph D−A contains no directed cycles.

Let k be the length of the longest directed path in D−A.

For each vertex v ∈ D, assign a colour α(v) = i, if a longest directed path from v has length i −1 in D−A. Here 1 ≤ i ≤ k +1.

First we observe that if P = $e_1e_2 \ldots e_r$ (r ≥ 1) is any directed path u $\xrightarrow{*}$ v in D−A, then α(u) ≠ α(v).

Indeed, if a(v) = i, then there exists a directed path Q: v $\xrightarrow{*}$ w of length i − 1, and PQ is a directed path, since D−A does not contain directed cycles.

Since PQ: u $\xrightarrow{*}$ w, a(u) ≠ i = a(v). In particular, if e = uv ∈ D−A, then α(u) ≠ α(v).

Consider then an edge e = vu ∈ A. By the minimality of A, (D−A)+ e contains a directed cycle C: u $\xrightarrow{*}$ v −→ u, where the part u $\xrightarrow{*}$ v is a directed path in D−A, and hence a(u) ≠ a(v).

This shows that a is a proper colouring of U(D), and therefore χ(D) ≤ k + 1, that is, k ≥ χ (D) − 1.

The bound χ(D) − 1 is the best possible in the following sense.

Connectedness:

A digraph is said to be disconnected if it is not even weak. A digraph is said to be strictly weak if it is weak, but not unilateral.

It is strictly unilateral, if it is unilateral but not strong. Two vertices of a digraph D are said to be

i. 0-connected if there is no semi path joining them,

ii. 1-connected if there is a semi path joining them, but there is no u−v path or v−u path,

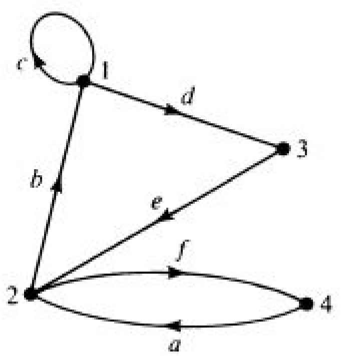
iii. 2-connected if there is a u−v or a v−u path, but not both,

iv. 3-connected if there is u−v path and a v−u path.

3. 11 EULER GRAPHS

In a digraph G, a closed directed walk which traverses every edge of G exactly once is called a *directed Euler line*. A digraph containing a directed Euler line is called an **Euler digraphs**

For example,



It contains directed Euler line **a b c d e f.**

teleprinter's problem.

Constructing a longest circular sequence of 1's and 0's such that no subsequence of r bits appears more than once in the sequence.

Teleprinter's problem was solved in 1940 by I.G. Good using digraph.

Theorem:

A connected graph G is an Euler graph if and only if all vertices of G are of even degree.

Proof : ***Necessity*** Let G(V, E) be an Euler graph.

Thus G contains an Euler line Z, which is a closed walk. Let this walk start and end at the vertex $u \in V$. Since each visit of Z to an intermediate vertex v of Z contributes two to the degree of v and since Z traverses each edge exactly once, d(v) is even for every such vertex. Each intermediate visit to u contributes two to the degree of u, and also the initial and final edges of Z contribute one each to the degree of u. So the degree d(u) of u is also even.

Sufficiency Let G be a connected graph and let degree of each vertex of G be even.

Assume G is not Eulerian and let G contain least number of edges. Since $\delta \geq 2$, G has a cycle. Let Z be a closed walk in G of maximum length.

Clearly, G−E(Z) is an even degree graph. Let C1 be one of the components of G−E(Z). As C1 has less number of edges than G, it is Eulerian and has a vertex v in common with Z.

Let Z' be an Euler line in C1. Then Z' ∪Z is closed in G, starting and ending at v. Since it is longer than Z, the choice of Z is contradicted. Hence G is Eulerian.

Second proof for sufficiency Assume that all vertices of G are of even degree.

We construct a walk starting at an arbitrary vertex v and going through the edges of G such that no edge of G is traced more than once. The tracing is continued as far as possible.

Since every vertex is of even degree, we exit from the vertex we enter and the tracing clearly cannot stop at any vertex but v. As v is also of even degree, we reach v when the tracing comes to an end.

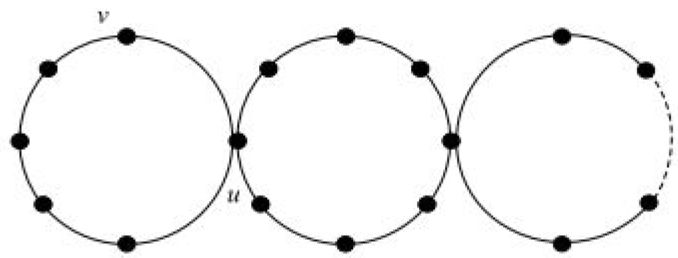
If this closed walk Z we just traced includes all the edges of G, then G is an Euler graph.

If not, we remove from G all the edges in Z and obtain a subgraph Z' of G formed by the remaining edges. Since both G and Z have all their vertices of even degree, the degrees of the vertices of Z' are also even.

Also, Z' touches Z' at least at one vertex say u, because G is connected. Starting from u, we again construct a new walk in Z' .

As all the vertices of Z' are of even degree, therefore this walk in Z' terminates at vertex u. This walk in Z'combined with Z forms a new walk, which starts and ends at the vertex v and has more edges than Z. Prepared by G. Appasami, Assistant professor, Dr. pauls Engineering College.

This process is repeated till we obtain a closed walk that traces all the edges of G. Hence G is an Euler graph.



Euler graph

UNIT IV PERMUTATIONS & COMBINATIONS

4. 1 FUNDAMENTAL PRINCIPLES OF COUNTING

The Fundamental Counting Principle is a way to figure out the total number of ways different events can occur.

1. rule of sum.

If the first task can be performed in *m* ways, while a second task can be performed in *n* ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in any one of $m + n$ ways.

Example: A college library has 40 books on C++ and 50 books on Java. A student at this college can select 40+50=90 books to learn programming language.

2. Define rule of Product

If a procedure can be broken into first and second stages, and if there are *m* possible outcomes for the first stage and if, for each of these outcomes, there are *n* possible outcomes for the second stage, then the total procedure can be carried out, in the designed order, in *mn* ways.

Example: A drama club with six men and eight can select male and female role in 6 x 8 = 48 ways.

4. 2 PERMUTATIONS AND COMBINATIONS

Permutations

For a given collection of *n* objects, any linear arrangement of these objects is called a permutation of the collection. Counting the linear arrangement of objects can be done by rule of product.

For a given collection of *n* distinct objects, and *r* is an integer, with $1 \leq r \leq n$, then by rule of product, the number of permutations of size r for the n objects is

$$P(n,r) = n \times (n-1) \times (n-2) \times \ldots \times (n-r+1) = \frac{n!}{(n-r)!}, \quad 0 \leq r \leq n$$

Example: In a class of 10 students, five are to be chosen and seated in a row for a picture.

The total number of arrangements = 10 x 9 x 8 x 7 x 6 = 30240.

Define combinations

For a given collection of *n* objects, each selection, or combination, of *r* of these objects, with no reference to order, corresponds to *r*! (Permutations of size *r* from the *n* objects). Thus the number of combinations of size *r* from a collection of size *n* is

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!\,(n-r)!}, \quad 0 \le r \le n$$

Example: In a test, students are directed to answer 7 questions out of 10. The student can answer the examination in

$$C(n,r) = C(10,7) = \frac{10!}{7!\,(10-7)!} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120\, ways$$

4. 3 BINOMIAL THEOREM

The Binomial theorem: If *x* and *y* are variables and n is a positive integer, then

$$(x+y)^n = \binom{n}{0} x^0 y^n + \binom{n}{1} x^1 y^{n-1} + \binom{n}{2} x^2 y^{n-2} + \cdots$$

$$\binom{n}{n-1} x^{n-1} y^1 + \binom{n}{n} x^n y^0 = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$\binom{n}{k}$ is referred as Binomial coefficient.

Application of permutation and combinations:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$(1+x)^3 = \overbrace{(1+x)}^{1}\overbrace{(1+x)}^{2}\overbrace{(1+x)}^{3} = 1 + 3x + 3x^2 + x^3 = \binom{3}{0} + \binom{3}{1}x + \binom{3}{2}x^2 + \binom{3}{3}x^3$$

Proof.

$$(1+x)^n = \overbrace{(1+x)}^{1}\overbrace{(1+x)}^{2}\cdots\overbrace{(1+x)}^{n}$$

In order to get x^k, we need to choose x in k of {1, . . . , n}. There are $\binom{n}{k}$ ways of doing this.

Theorem: 1

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}y^n = \sum_{i=0}^{n}\binom{n}{i}x^{n-i}y^i$$

Proof.

We prove this by induction on n. It is easy to check the first few, say for n=0,1,2, which form the base case. Now suppose the theorem is true for n−1, that is,

$$(x+y)^{n-1} = \sum_{i=0}^{n-1}\binom{n-1}{i}x^{n-1-i}y^i.$$

Then

$$(x+y)^n = (x+y)(x+y)^{n-1} = (x+y)\sum_{i=0}^{n-1}\binom{n-1}{i}x^{n-1-i}y^i.$$

Using the distributive property,this becomes

$$x\sum_{i=0}^{n-1}\binom{n-1}{i}x^{n-1-i}y^i + y\sum_{i=0}^{n-1}\binom{n-1}{i}x^{n-1-i}y^i$$
$$= \sum_{i=0}^{n-1}\binom{n-1}{i}x^{n-i}y^i + \sum_{i=0}^{n-1}\binom{n-1}{i}x^{n-1-i}y^{i+1}.$$

These two sums have much in common, but it is slightly disguised by an "offset": the first sum starts with an x^ny^0 term and ends with an x^1y^n-1 term, while the corresponding terms in the second sum are $x^{n-1}y^1$ and x^0y^n.

Let's rewrite the second sum so that they match:

$$\sum_{i=0}^{n-1}\binom{n-1}{i}\,x^{n-i}y^i + \sum_{i=0}^{n-1}\binom{n-1}{i}x^{n-1-i}y^{i+1}$$
$$= \sum_{i=0}^{n-1}\binom{n-1}{i}x^{n-i}y^i + \sum_{i=1}^{n}\binom{n-1}{i-1}x^{n-i}y^i$$
$$\binom{n-1}{0}x^n + \sum_{i=1}^{n-1}(\binom{n-1}{i} + \binom{n-1}{i-1})x^{n-i}y^i + \binom{n-1}{n-1}y^n$$
$$= \binom{n-1}{0}x^n + \sum_{i=1}^{n-1}\binom{n}{i}x^{n-i}y^i + \binom{n-1}{n-1}y^n.$$
$$= \binom{n}{0}x^n + \sum_{i=1}^{n-1}\binom{n}{i}x^{n-i}y^i + \binom{n}{n}y^n$$
$$= \sum_{i=0}^{n}\binom{n}{i}x^{n-i}y^i.$$

At the next to last step we used the facts that $\binom{n-1}{0} = \binom{n}{0}$ and $\binom{n-1}{n-1} = \binom{n}{n}$.

Here is an interesting consequence of this theorem: Consider

$$(x+y)^n = (x+y)(x+y)\cdots(x+y).$$

One way we might think of attempting to multiply this out is this: Go through the n factors (x+y) and in each factor choose either the x or the y; at the end, multiply your choices together, getting some term like $xxyxyy\cdots yx = x^i y^j$, where of course $i+j=n$.

If we do this in all possible ways and then collect like terms, we will clearly get

$$\sum_{i=0}^{n} a_i x^{n-i} y^i.$$

We know that the correct expansion has $\binom{n}{i} = a_i$; is that in fact what we will get by this method? Yes: consider $x^{n-i}y^i$.

How many times will we get this term using the given method? It will be the number of times we end up with i y-factors.

Since there are n factors (x+y), the number of times we get i y-factors must be the number of ways to pick i of the (x+y) factors to contribute a y, namely $\binom{n}{i}$. This is probably not a useful method in practice, but it is interesting and occasionally useful.

4. 4 COMBINATIONS WITH REPETITION

If there is a selection with repetition, r of n distinct objects, then the combinations with of n objects taken r at a time with repetition is $C(n+r-1, r)$.

$$C(n+r-1, r) = \frac{(n+r-1)!}{r!\,(n-1)!} = \binom{n+r-1}{r}$$

Example: A donut shop offers 20 kinds of donuts. Assuming that there are at least a dozen of each kind when we enter the shop. We can select a dozen donuts in $C(20+12-1, 12)$= $C(31, 12)$ = 141120525 ways

We can also have an r-combination of n items with repetition.

Same as other combinations: order doesn't matter.

Same as permutations with repetition: we can select the same thing multiple times.

Example: You walk into a candy store and have enough money for 6 pieces of candy. The store has chocolate (C), gummies (G), and horrible Chinese candy (H). How many different selections can you make?

Here are some possible selections you might make:

C C C G G H

C G G G G H

C C C C G G

H H H H H H

Since order doesn't matter, we'll list all of our selections in the same order: C then G then H.

We don't want our candy to mix: let's separate the types.

C C C | G G | H

C | G G G G | H

C C C C | G G |

| | H H H H H H

Now we don't need the actual identities in the diagram to know what's there:

- - - | - - | -

- | - - - - | -

- - - - | - - |

| | - - - - - -

Now the answer becomes obvious: we have 8 slots there and just have to decide where to put the two dividers.

There are C(8,2) ways to do that, so C(8,2)=28 possible selections. Or equivalently, there are C(8,6)=28 ways to place the candy selections.

If we are selecting an r-combination from n elements with repetition, there are $C(n+r-1,r)=C(n+r-1,n-1)$ ways to do so.

Proof: like with the candy, but not specific to r=6 and n=3.

Example: How many solutions does this equation have in the non-negative integers?$a+b+c=100$

In order to satisfy the equation, we have to select 100 “ones”, some that will contribute to a, some to b, some to c. In other words, we have balls labeled a, b, and c. We select 100 of them (with repetition) and that gives us a solution to the equation. C(102,2) solutions.

In summary we have these ways to select r things from n possibilities: Prepared by G. Appasami, Assistant professor, Dr. pauls Engineering College.

Order?	Repetition?	Formula
Yes (permutation)	No	$P(n,r)=\frac{n!}{(n-r)!}$
No (combination)	No	$C(n,r)=\frac{n!}{r!(n-r)!}$
Yes (permutation)	Yes	n^r
No (combination)	Yes	$C(n+r-1,r)=\frac{(n+r-1)!}{r!(n-1)!}$

4. 5 COMBINATORIAL NUMBERS

The Catalan numbers form a sequence of natural numbers that occur in various counting problems, often involving recursively-defined objects. They are named after the Belgian mathematician Eugène Charles Catalan. the nth Catalan number is given directly in terms of binomial coefficients by

$$C_n = \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{(n+1)!\,n!} = \frac{1}{n+1}\binom{2n}{n} \qquad \text{for } n \geq 0$$

4. 6 PRINCIPLE OF INCLUSION AND EXCLUSION

Very often, we need to calculate the number of elements in the union of certain sets.
Assuming that we know the sizes of these sets, and their mutual intersections, the principle of inclusion and exclusion allows us to do exactly that.
Suppose that we have two sets A, B.
The size of the union is certainly at most |A| + |B|.
This way, however, we are counting twice all elements in A ∩ B, the intersection of the two sets. To correct for this, we subtract |A ∩ B| to obtain the following formula:
|A ∪ B| = |A| + |B| − |A ∩ B|.
In general, the formula gets more complicated because we have to take into account intersections of multiple sets. The following formula is what we call the principle of inclusion and exclusion

$$\begin{aligned} N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = N &- [N(c_1)+N(c_2)+N(c_3)+N(c_4)] \\ &+ [N(c_1c_2)+N(c_1c_3)+N(c_1c_4)+N(c_2c_3)+N(c_2c_4)+N(c_3c_4)] \\ &- [N(c_1c_2c_3)+N(c_1c_2c_4)+N(c_1c_3c_4)+N(c_2c_3c_4)] \\ &+ N(c_1c_2c_3c_4). \end{aligned}$$

4. 7 DERANGEMENTS

A derangement is a permutation of the elements of a set, such that no element appears in its original position.

The number of derangements of a set of size n, usually written D_n, d_n, or !n, is called the "derangement number" or "de Montmort number".

Example: The number of derangements of 1, 2, 3, 4 is

$d_4 = 4!\ [1 - 1 + (1/2!)-(1/3!)+(1/4!)] = 9.$

4. 8 ARRANGEMENTS WITH FORBIDDEN POSITIONS

The number of acceptable assignments is equal to the number of ways of placing nontaking rooks on this chessboard so that none of the rooks is in a forbidden position. The key to determining this number of arrangements is the inclusion- exclusion principle.

Suppose we shuffle a deck of cards; what is the probability that no card is in its original location?

More generally, how many permutations of $[n]=\{1,2,3,\ldots,n\}$ have none of the integers in their "correct" locations? That is, 1 is not first, 2 is not second, and so on. Such a permutation is called a derangement of [n].

Let S be the set of all permutations of [n] and Ai be the permutations of [n] in which i is in the correct place. Then we want to know $|\bigcap n_i=1A^c{}_i|$.

For any i, $|A_i|=(n-1)!$: once i is fixed in position i, the remaining $n-1$ integers can be placed in any locations.

What about $|A_i \cap A_j|$? If both i and j are in the correct position, the remaining $n-2$integers can be placed anywhere, so $|A_i \cap A_j|=(n-2)!$.

In the same way, we see that $|A_{i1} \cap A_{i2} \cap \cdots \cap A_{ik}|=(n-k)!$.

$$\begin{aligned}
|\bigcap_{i=1}^{n} A_i^c| &= |S| + \sum_{k=1}^{n}(-1)^k \binom{n}{k}(n-k)! \\
&= n! + \sum_{k=1}^{n}(-1)^k \frac{n!}{k!(n-k)!}(n-k)! \\
&= n! + \sum_{k=1}^{n}(-1)^k \frac{n!}{k!} \\
&= n! + n!\sum_{k=1}^{n}(-1)^k \frac{1}{k!} \\
&= n!\left(1 + \sum_{k=1}^{n}(-1)^k \frac{1}{k!}\right) \\
&= n!\sum_{k=0}^{n}(-1)^k \frac{1}{k!}.
\end{aligned}$$

The last sum should look familiar:

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

Substituting x=−1 gives

$$e^{-1} = \sum_{k=0}^{\infty} \frac{1}{k!}(-1)^k.$$

The probability of getting a derangement by chance is then

$$\frac{1}{n!} n! \sum_{k=0}^{n}(-1)^k \frac{1}{k!} = \sum_{k=0}^{n}(-1)^k \frac{1}{k!},$$

and when n is bigger than 6, this is quite close to$e^{-1}\approx 0.3678$.

So in the case of a deck of cards, the probability of a derangement is about 37%.

Let $D_n = n!\sum_{k=0}^{n}(-1)^k \frac{1}{k!}$ These derangement numbers have some interesting properties.

The derangements of [n] may be produced as follows: For each i∈{2,3,…,n}, put i in position 1 and 1 in position i.

Then permute the numbers {2,3,…,i−1,i+1,…n} in all possible ways so that none of these n−2 numbers is in the correct place.There are D_{n-2} ways to do this.

Then, keeping 1 in position i, derange the numbers {i,2,3,…,i−1,i+1,…n}, with the "correct" position of i now considered to be position 1.

There are D_{n-1} ways to do this. Thus, $D_n=(n-1)(D_{n-1}+D_{n-2})$.

We explore this recurrence relation a bit:

$D_n=nD_{n-1}-D_{n-1}+(n-1)D_{n-2}$

$=nD_{n-1}-(n-2)(D_{n-2}+D_{n-3})+(n-1)D_{n-2}$

$=nD_{n-1}-(n-2)D_{n-2}-(n-2)D_{n-3}+(n-1)D_{n-2}$

$=nD_{n-1}+D_{n-2}-(n-2)D_{n-3}$

$=nD_{n-1}+(n-3)(D_{n-3}+D_{n-4})-(n-2)D_{n-3}$

$=nD_{n-1}+(n-3)D_n-3+(n-3)D_{n-4}-(n-2)D_{n-3}$

$=nD_{n-1}-D_{n-3}+(n-3)D_{n-4}$

$=nD_{n-1}-(n-4)(D_n-4+D_n-5)+(n-3)D_{n-4}$

$=nD_{n-1}-(n-4)D_{n-4}-(n-4)D_{n-5}+(n-3)D_{n-4}$

$=nD_{n-1}+D_{n-4}-(n-4)D_{n-5}$.

It appears from the starred lines that the pattern here is that

$D_n=nD_{n-1}+(-1)^kD_{n-k}+(-1)^{k+1}(n-k)D_{n-k-1}$.

If this continues, we should get to

$D_n=nD_{n-1}+(-1)^{n-2}D_2+(-1)^{n-1}(2)D_1$.

Since $D_2=1$ and $D_1=0$, this would give $D_n=nD_{n-1}+(-1)^n$,

Since $(-1)^n=(-1)^{n-2}$. Indeed this is true, and can be proved by induction. This gives a somewhat simpler recurrence relation, making it quite easy to compute D_n.

UNIT V GENERATING FUNCTIONS

5. 1 GENERATING FUNCTIONS

A **generating function** describes an infinite sequence of numbers (a_n) by treating them like the coefficients of a series expansion. The sum of this infinite series is the generating function. Unlike an ordinary series, this formal series is allowed to diverge, meaning that the generating function is not always a true function and the "variable" is actually an indeterminate.

The generating function for 1, 1, 1, 1, 1, 1, 1, 1, 1, ..., whose ordinary generating function is

$$\sum_{n=0}^{\infty}(x)^n = \frac{1}{1-x}$$

The generating function for the geometric sequence 1, a, a^2, a^3, ... for any constant a:

$$\boldsymbol{\sum_{n=0}^{\infty}(ax)^n = \frac{1}{1-ax}}$$

5. 2 PARTITIONS OF INTEGERS

Partitioning a positive n into positive summands and seeking the number of such partitions without regard to order is called Partitions of integer.

This number is denoted by $p(n)$. For example

P(1) = 1: 1

P(2) = 2: 2 = 1 + 1

P(3) = 3: 3 = 2 +1 = 1 + 1 +1

P(4) = 5: 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1

P(5) = 7: 5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1+ 1 = 1 + 1 + 1 + 1 + 1

There is no simple formula for p_n, but it is not hard to find a generating function for them. As with some previous examples, we seek a product of factors so that when the factors are multiplied out, the coefficient of x_n is p_n.

We would like each x_n term to represent a single partition, before like terms are collected. A partition is uniquely described by the number of 1s, number of 2s, and so on, that is, by the repetition numbers of the multi-set. We devote one factor to each integer:

$$(1+x+x^2+x^3+\cdots)(1+x^2+x^4+x^6+\cdots)\cdots(1+x^k+x^{2k}+x^{3k}+\cdots)\cdots=\prod_{k=1}^{\infty}\sum_{i=0}^{\infty}x^{ik}$$

When this product is expanded, we pick one term from each factor in all possible ways, with the further condition that we only pick a finite number of "non-1" terms. For example, if we pick x^3 from the first factor, x^3 from the third factor, x^{15} from the fifth factor, and 1s from all other factors, we get x^{21}.

In the context of the product, this represents 3.1+1.3+3.5, corresponding to the partition 1+1+1+3+5+5+5, that is, three 1s, one 3, and three 5s. Each factor is a geometric series; the kth factor is

$$1+x^k+(x^k)^2+(x^k)^3+\cdots=\frac{1}{1-x^k},$$

.so the generating function can be written

$$\prod_{k=1}^{\infty}\frac{1}{1-x^k}.$$

Note that if we are interested in some particular pn, we do not need the entire infinite product, or even any complete factor, since no partition of n can use any integer greater than n, and also cannot use more than n/k copies of k.

Example 2:Find p8

We expand

$(1+x(1+x^2+x^3+x^4+x^5+x^6+x^7+x^8)(1+x^2+x^4+x^6+x^8)(1+x^3+x^6)+x^4+x^8)(1+x^5)(1+x^6)(1+x^7)(1+x^8$
)

$=1+x+2x^2+3x^3+5x^4+7x^5+11x^6+15x^7+22x^8+\cdots+x^{56}$,

so $p_8=22$. Note that all of the coefficients prior to this are also correct, but the following coefficients are not necessarily the corresponding partition numbers.

Partitions of integers have some interesting properties. Let pd(n) be the number of partitions of n into distinct parts; let po(n) be the number of partitions into odd parts.

5. 3 EXPONENTIAL GENERATING FUNCTION

For a sequence $a_0, a_1, a_2, a_3, \ldots$ of real numbers.

$$f(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} a_i \frac{x^{i^i}}{i!}$$

is called the exponential generating function for the given sequence.

Ordinary generating functions arise when we have a (finite or countably in- finite) set of objects S and a weight function ω : S → N ^{r} .

Then the ordinary generating function Φω S (x) is defined and we can proceed with calculations. Exponential generating functions arise in a somewhat more complicated situation. The basic idea is that they are used to enumerate "combinatorial structures on finite sets".

Definition 1:(Exponential Generating Functions). Let A be a class of structures. The exponential generating function of A is

$$A(x) := \sum_{n=0}^{\infty} (\#\mathcal{A}_n) \frac{x^n}{n!}.$$

Let's illustrate this with a few examples for which we already know the answer.

Example:

First, consider the class S of permutations: to each finite set X it associates the finite set SX of all bijections σ : X → X from X to X.

Condition (i) is easy, and condition (ii) follows from Examples, so that S satisfies above definition . Way back in Theorem 2 we saw that #Sn = n! for all n ∈ N, so that the exponential generating function for the class of permutations is

$$S(x) := \sum_{n=0}^{\infty} (\#\mathcal{S}_n) \frac{x^n}{n!} = \sum_{n=0}^{\infty} n! \frac{x^n}{n!} = \frac{1}{1-x}.$$

5. 4 SUMMATION OPERATOR

1. aclaurin series expansion of e^x and e^{-x}.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$$

Adding these two series together, we get,

$$e^x + e^{-x} = 2(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots)$$

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

Generating function for a sequence $a_0, a_0 + a_1, a_0 + a_1 + a_2, a_0 + a_1 + a_2 + a_3, \ldots$.

For $(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$,, consider the function *f(x)/(1-x)*

$$\frac{f(x)}{1-x} = f(x).\frac{1}{1-x} = [a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots][1 + x + x^2 + x^3 + \cdots]$$

$$= a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + +(a_0 + a_1 + a_2 + a_3)x^3 + \cdots$$

So *f(x)/(1-x)* generates the sequence of sums $a_0, a_0 + a_1, a_0 + a_1 + a_2, a_0 + a_1 + a_2 + a_3$,,

1/(1-*x*) is called the summation operator

5. 5 RECURRENCE RELATIONS

A **recurrence relation** is an equation that recursively defines a sequence or multidimensional array of values, once one or more initial terms are given: each further term of the sequence or array is defined as a function of the preceding terms.

The term **difference equation** sometimes (and for the purposes of this article) refers to a specific type of recurrence relation. However, "difference equation" is frequently used to refer to *any* recurrence relation.

Fibonacci numbers and relation

The recurrence satisfied by the Fibonacci numbers is the archetype of a homogeneous linear recurrence relation with constant coefficients (see below). The Fibonacci sequence is defined using the recurrence

$F_n = F_{n-1} + F_{n-2}$

with seed values $F_0 = 0$ and $F_1 = 1$

We obtain the sequence of Fibonacci numbers, which begins

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Recurrence Relation (Epp)

A **recurrence relation** for a sequence a_0, a_1, a_2, … is a formula that relates each term a_k to certain of its predecessors a_{k-1}, …, a_{k-i}, where i is fixed and $k \geq i$. The initial conditions specify the fixed values of a_0, …, a_{i-1}.

Most of the time, though, there is only one fixed value or base value.

However, nothing prevents us from defining a sequence with multiple base values (consider the two 1s in the Fibonacci sequence), hence the generality with i in the definition above.

There are also two forms of induction to handle this generality. The one we looked at earlier works when we have one base case. There's another form of induction called *strong induction* that proves claims where there are multiple base cases.

We could spend quite a bit of time studying recurrences by themselves. We'll just scratch the surface and use them a few times in the remainder of the course.

Question: If we have a recurrence relation for a sequence, is it possible to express the sequence in a way that does *not* use recursion?

Answer: Sometimes. When we are able to do so, we find what is called the **closed form** of the recurrence. It is an algebraic formula or a definition that tells us how to find the nth term without needing to know any of the preceding terms. The process of finding the closed form is called **solving a recurrence**.

There are various methods to "solving" recurrence that are used in practice. Each has its place, each has a difference sort of output.

Three Methods to Solving Recurrences:

Iteration: Start with the recurrence and keep applying the recurrence equation until we get a pattern. The result is a *guess* at the closed form.

Substitution: Guess the solution; prove it using induction. The result here is a proven closed form. It's often difficult to come up the guess so, in practice, iteration and substitution are used hand-in-hand.

Master Theorem: Plugging into a formula that gives an approximate bound on the solution. The result here is only a *bound* on the closed form. It is not an exact solution.

5. 6 FIRST ORDER

The general form of First order linear homogeneous recurrence relation can be written as

$a_{n+1} = d\, a_n$, n ≥ 0, where d is a constant. The relation is first order since a_{n+1} depends on a_n.

a_0 *or* a_1 are called boundary conditions.

5. 7 SECOND ORDER

Let $k \in \mathbf{Z}^+$ and C_0 $(\neq 0)$, $C_1, C_2, \ldots, C_k$ $(\neq 0)$ be real numbers. If a_n, for $n \geq 0$, is a discrete function, then

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = f(n), \qquad n \geq k,$$

is a linear recurrence relation (with constant coefficients) of *order* k. When $f(n) = 0$ for all $n \geq 0$, the relation is called *homogeneous*; otherwise, it is called *nonhomogeneous*.

In this section we shall concentrate on the homogeneous relation of order two:

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} = 0, \qquad n \geq 2.$$

On the basis of our work in Section 10.1, we seek a solution of the form $a_n = cr^n$, where $c \neq 0$ and $r \neq 0$.

Substituting $a_n = cr^n$ into $C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} = 0$, we obtain

$$C_0 cr^n + C_1 cr^{n-1} + C_2 cr^{n-2} = 0.$$

With c, $r \neq 0$, this becomes $C_0 r^2 + C_1 r + C_2 = 0$, a quadratic equation which is called the *characteristic equation*. The roots r_1, r_2 of this equation determine the following three cases: (a) r_1, r_2 are distinct real numbers; (b) r_1, r_2 form a complex conjugate pair; or (c) r_1, r_2 are real, but $r_1 = r_2$. In all cases, r_1 and r_2 are called the *characteristic roots*.

5. 8 NON-HOMOGENEOUS RECURRENCE RELATIONS

We now turn to the recurrence relations

$$a_n + C_1 a_{n-1} = f(n), \qquad n \geq 1, \tag{1}$$

$$a_n + C_1 a_{n-1} + C_2 a_{n-2} = f(n), \qquad n \geq 2, \tag{2}$$

where C_1 and C_2 are constants, $C_1 \neq 0$ in Eq. (1), $C_2 \neq 0$, and $f(n)$ is not identically 0. Although there is no general method for solving all nonhomogeneous relations, for certain functions $f(n)$ we shall find a successful technique.

We start with the special case for Eq. (1), when $C_1 = -1$. For the nonhomogeneous relation $a_n - a_{n-1} = f(n)$, we have

$$\begin{aligned}
a_1 &= a_0 + f(1)\\
a_2 &= a_1 + f(2) = a_0 + f(1) + f(2)\\
a_3 &= a_2 + f(3) = a_0 + f(1) + f(2) + f(3)\\
&\vdots\\
a_n &= a_{n-1} + f(n) = a_0 + f(1) + \cdots + f(n) = a_0 + \sum_{i=1}^{n} f(i).
\end{aligned}$$

We can solve this type of relation in terms of n, if we can find a suitable summation formula for $\sum_{i=1}^{n} f(i)$.

Definition 1. A sequence $\{a_n\}$ is given by a linear non homogeneous recurrence relation of order k if an = $c_1a_{n-1} + c_2a_{n-2} + c_3a_{n-3} + \cdots + c_ka_{n-k} + p(n)$ for all $n \geq k$. The recurrence relation $b_n = c_1b_{n-1} + c_2b_{n-2} + c_3b_{n-3} + \cdots + c_kb_{n-k}$ is referred to as the associated linear homogeneous recurrence relation One result is as easy to show for LNRRs as for LHRRs; the following can be proven as a very slight variation of the similar proof for LHRRs

Proposition 1. A sequence is uniquely determined by an LNRR of order k and the initial values $a_0, a_1, a_2, \ldots, a_{k-1}$. However, there is one very important difference between LNRRs and LHRRs: linear combinations of LNRR-satisfying sequences do no, in general, satisfy the LNRR. However, we do have the result:

Proposition 2. If $\{a_n\}$ satisfies an LNRR, and $\{b_n\}$ satisfies the associated LHRR, then $\{a_n+b_n\}$ satisfies the LNRR.

Proof:We know that

$a_n = c_1a_{n-1} + c_2a_{n-2} + c_3a_{n-3} + \cdots + c_ka_{n-k} + p(n)$

And $b_n = c_1b_{n-1} + c_2b_{n-2} + c_3b_{n-3} + \cdots + c_kb_{n-k}$

Adding these two equations will give $(a_n + b_n) = c_1(a_{n-1} + b_{n-1}) + c_2(a_{n-2} + b_{n-2}) + \cdots + c_k(a_{n-k} + b_{n-k}) + p(n)$ and thus $\{a_n + b_n\}$ satisfies the LNRR.

This means that if we have a specific LNRR solution, then we can get a wide range of LNRR solutions simply by adding the associated LHRR solution. Note that this is very similar to the method used to solve non homogeneous linear differential equations.

The tricky part of this is, of course, coming up with a solution to the LNRR in the first place. Let's try doing that for the example above, where $a_n = 3a_{n-1} + 5^{n-1}$. We might do this by inspired guesswork: since the in homogeneous term is 5^{n-1} , we might think some multiple of 5^n will do the trick, so suppose $a_n = C5^n$. Then, the recurrence relation gives us $C5^n = 3C \cdot 5^{n-1} + 5^{n-1} = (3C + 1)5^{n-1}$

The solution method for solving an LNRR with initial conditions is a very minor variation on the LHRR solution method. Given a LNRR $an = c_1a_{n-1}+c_2a_{n-2} +c_3a_{n-3} +\cdot\cdot\cdot+c_ka_{n-k} +p(n)$ with initial conditions $a0, \ldots, a_{k-1}$, this is our process:

1.Find a single sequence $\{a_n{}^P\}$ to the LNRR.

Find the general solution $\{b_n\}$ to the associated LHRR. By the nature of its construction, $\{bn\}$ will have k undetermined constants.

The general solution to the LNRR will be $\{a_n\} = \{a_n{}^P + b_n\}$. Like $\{b_n\}$, the sequence $\{a_n\}$ will have k undetermined constants in its expression.

4. Setting the known values of $a_0, a_1, \ldots, a_k$ equal to the general-form expressions will yield k equations in k unknowns. Solve for the unknowns to determine the constants in the formula for $\{a_n\}$.

5. 9 METHOD OF GENERATING FUNCTIONS.

On solving a recurrence relation, we have the solution in the form of a sequence. Instead of solving in the form of a sequence, we can also determine the generating function of the sequence from the recurrence relation. One of the uses of generating function method is to find the closed form formula for a recurrence relation.

Once the generating function is known, an expression for the value of sequence can easily be obtained. Before using this method, ensure that the given recurrence equation is in linear form. A non-linear recurrence equation cannot be solved by the generating method.

We use substitution of variable technique to convert a non-linear recurrence relation into linear equation. We explain this method by means of the examples.

Example: Solve the recurrence relation

$$a_r - 3a_{r-1} + 2a_{r-2} = 0 \quad r \geq 2$$

by the generating function with the initial conditions $a_0 = 2$ *and* $a_1 = 3$.

SOLUTION. Let $A(Z)$ be the generating function of the sequence $<a_n>$ that is

$$A(Z) = \sum_{r=0}^{\infty} a_r Z^r$$

Multiply the given recurrence relation by Z^r, we get

$$a_r Z^r - 3a_{r-1} Z^r + 2a_{r-2} Z^r = 0$$

Summing from $r = 2$ to ∞, we obtain

$$\sum_{r=2}^{\infty} a_r Z^r - 3\sum_{r=2}^{\infty} a_{r-1} Z^r + 2\sum_{r=2}^{\infty} a_{r-2} Z^r = 0$$

$$(A(Z) - a_0 - a_1 Z) - 3Z(A(Z) - a_0) + 2Z^2 A(Z) = 0$$

$$(2Z^2 - 3Z + 1)\, A(Z) - a_0 - a_1 Z + 3a_0 Z = 0$$

Now using the given conditions *i.e.*, $a_0 = 2$, $a_1 = 3$, we get

$$(2Z^2 - 3Z + 1)\, A(Z) - 2 - 3Z + 6Z = 0$$

$$A(Z) = \frac{2-3Z}{2Z^2-3Z+1}$$

$$= \frac{1}{(1-Z)} + \frac{1}{(1-2z)}$$

Thus, $$a_r = 1 + 2^r$$

CS6702 GRAPH THEORY AND APPLICATIONS
2 MARKS QUESTIONS AND ANSWERS

UNIT I INTRODUCTION

1. Define Graph.

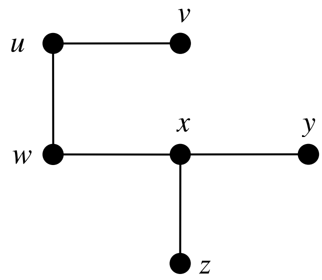
A graph G = (V, E) consists of a set of objects V={v_1, v_2, v_3, … } called **vertices** (also called **points** or **nodes**) and other set E = {e_1, e_2, e_3, ...} whose elements are called **edges** (also called **lines** or **arcs**).

The set V(G) is called the **vertex set** of G and E(G) is the **edge set** of G.

For example :

A graph G is defined by the sets V(G) = {*u, v, w, x, y, z*} and E(G) = {*uv, uw, wx, xy, xz*}.

Graph G:

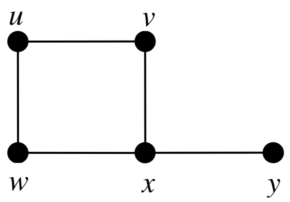


A graph with *p*-vertices and *q*-edges is called a **(*p, q*) graph.** The (1, 0) graph is called **trivial graph.**

2. Define Simple graph.

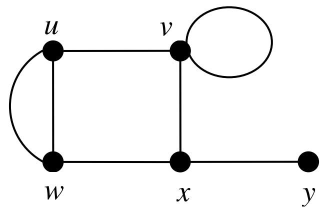
- An edge having the same vertex as its end vertices is called a self-loop.
- More than one edge associated a given pair of vertices called parallel edges.
- A graph that has neither self-loops nor parallel edges is called simple graph.

Graph G:



Simple Graph

Graph H:



Pseudo Graph

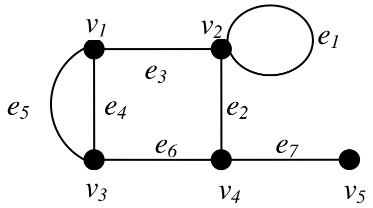
3. Write few problems solved by the applications of graph theory.

Konigsberg bridge problem
Utilities problem
Electrical network problems
Seating problems

4. Define incidence, adjacent and degree.

When a vertex v_i is an end vertex of some edge e_j, v_i and e_j are said to be *incident* with each other. Two non parallel edges are said to be *adjacent* if they are incident on a common vertex. The number of edges incident on a vertex v_i, with self-loops counted twice, is called the *degree* (also called valency), $d(v_i)$, of the vertex v_i. A graph in which all vertices are of equal degree is called *regular graph.*

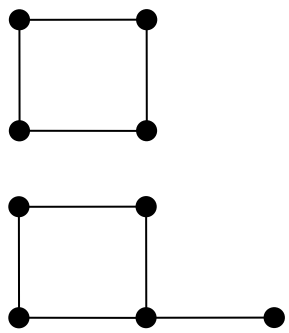
Graph G:



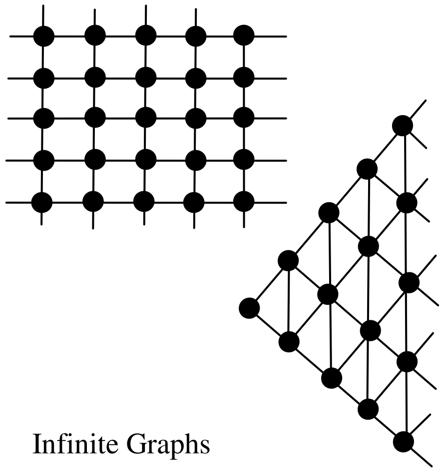
The edges e_2, e_6 and e_7 are incident with vertex v_4.
The edges e_2 and e_7 are adjacent.
The edges e_2 and e_4 are not adjacent.
The vertices v_4 and v_5 are adjacent.
The vertices v_1 and v_5 are not adjacent.
$d(v_1) = d(v_3) = d(v_4) = 3$. $d(v_2) = 4$. $d(v_5) = 1$.

5. What are finite and infinite graphs?

A graph with a finite number off vertices as well as a finite number of edges is called a *finite* graph; otherwise, it is an *infinite* graph.



Finite Graphs

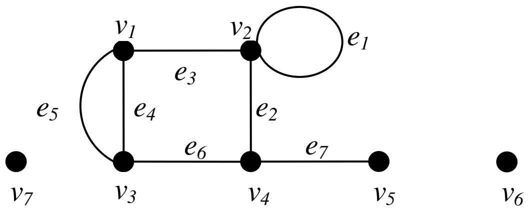


Infinite Graphs

6. Define Isolated and pendent vertex.

A vertex having no incident edge is called an *isolated vertex*. In other words, isolated vertices are vertices with zero degree. A vertex of degree one is called a *pendant vertex* or an *end vertex*.

Graph G:

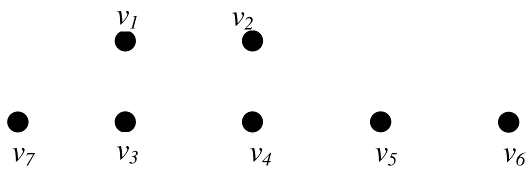


The vertices v_6 and v_7 are *isolated vertices*.
The vertex v_5 is a *pendant vertex*.

7. Define null graph.

In a graph G=(V, E), If E is empty (Graph without any edges) Then G is called a *null graph*.

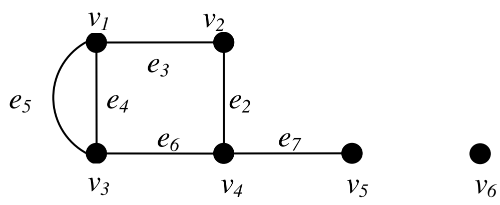
Graph G:



8. Define Multigraph

In a multigraph, no loops are allowed but more than one edge can join two vertices, these edges are called **multiple edges** or parallel edges and a graph is called **multigraph.**

Graph G:



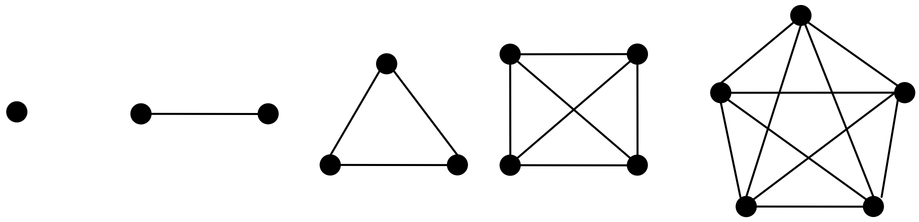
The edges e_5 and e_4 are **multiple** (parallel) edges.

9. Define complete graph

A simple graph G is said to be **complete** if every vertex in G is connected with every other vertex. *i.e.,* if G contains exactly one edge between each pair of distinct vertices.

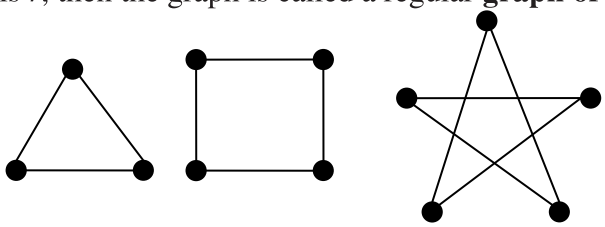
A complete graph is usually denoted by $\mathbf{K_n}$. It should be noted that K_n has exactly $n(n-1)/2$ edges.

The complete graphs K_n for $n = 1, 2, 3, 4, 5$ are show in the following Figure.



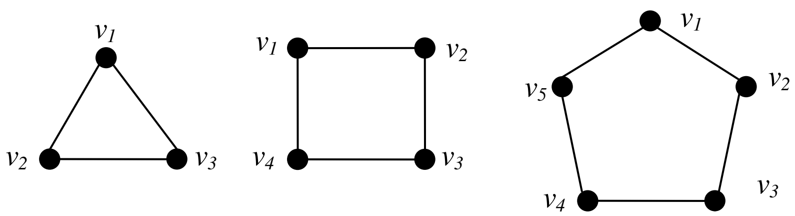
10. Define Regular graph

A graph in which all vertices are of **equal degree,** is called a **regular graph.**
If the degree of each vertex is r, then the graph is called a regular **graph of degree *r*.**



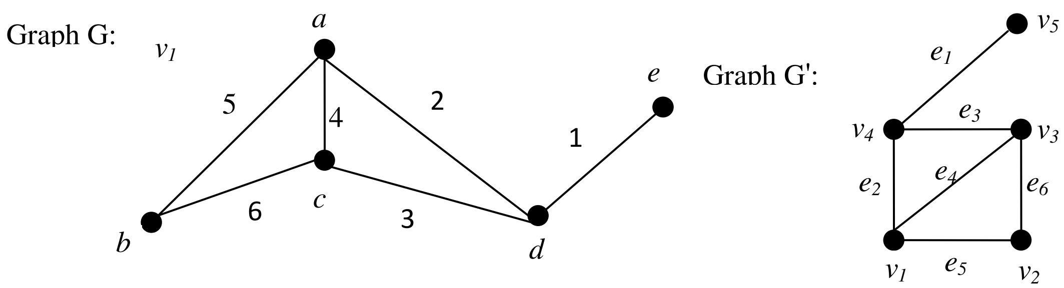
11. Define Cycles

The cycle C_n, $n \geq 3$, consists of n vertices $v_1, v_2, ..., v_n$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, ..., $\{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$.
The cyles c_3, c_4 and c_5 are shown in the following Figures



12. Define Isomorphism.

Two graphs G and G' are said to be **isomorphic** to each other if there is a one-to-one correspondence between their vertices and between their edges such that the incidence relationship is preserved.



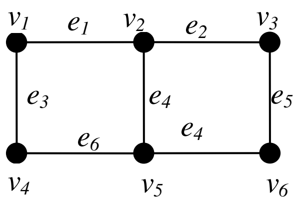
Correspondence of vertices	Correspondence of edges
$f(a) = v_1$	$f(1) = e_1$
$f(b) = v_2$	$f(2) = e_2$
$f(c) = v_3$	$f(3) = e_3$
$f(d) = v_4$	$f(4) = e_4$
$f(e) = v_5$	$f(5) = e_5$

Adjacency also preserved. Therefore G and G' are said to be isomorphic.

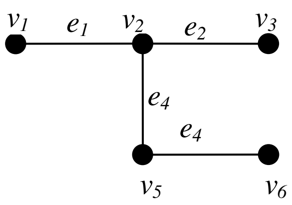
13. What is Subgraph?

A graph G' is said to be a subgraph of a graph G, if all the vertices and all the edges of G' are in G, and each edge of G' has the same end vertices in G' as in G.

Graph G:



Subgraph G' of G:

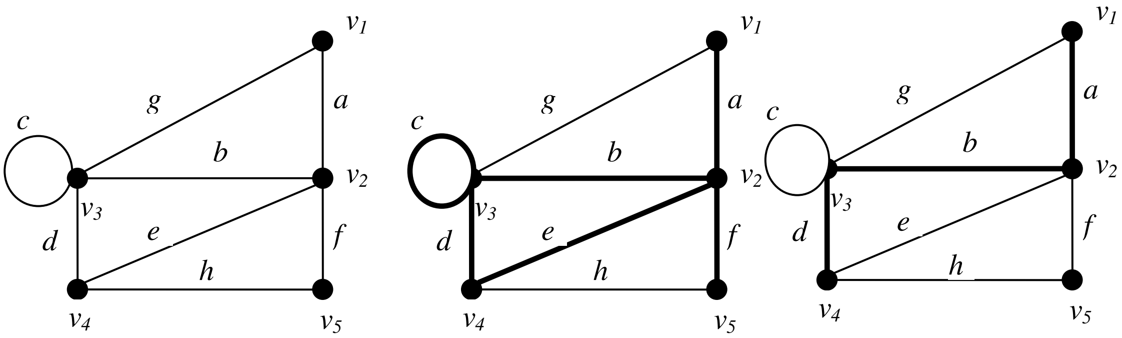


14. Define Walk, Path and Circuit.

A **walk** is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices. No edge appears more than once. It is also called as an edge train or a chain.

An open walk in which no vertex appears more than once is called **path**. The number of edges in the path is called **length of a path**.

A closed walk in which no vertex (except initial and final vertex) appears more than once is called a circuit. That is, a circuit is a closed, nonintersecting walk.



Graph G: Open walk Path of length 3

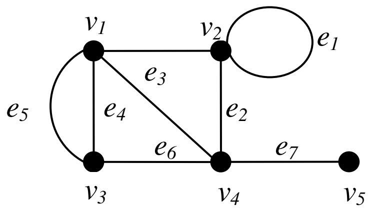
$v_1\ a\ v_2\ b\ v_3\ c\ v_3\ d\ v_4\ e\ v_2 f\ v_5$ is a walk. v_1 *and* v_5 are terminals of walk.

$v_1\ a\ v_2\ b\ v_3\ d\ v_4$ is a path. $a\ v_2\ b\ v_3\ c\ v_3\ d\ v_4\ e\ v_2 f\ v_5$ is not a path.

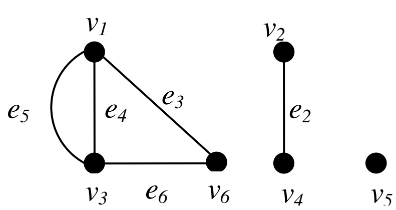
$v_2\ b\ v_3\ d\ v_4\ e\ v_2$ is a circuit.

15. Define connected graph. What is Connectedness?

A graph G is said to be **connected** if there is at least one path between every pair of vertices in G. Otherwise, G is disconnected.



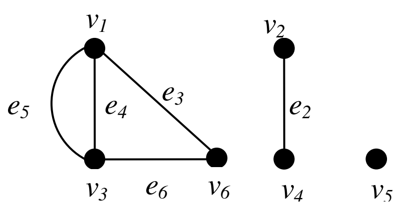
Connected Graph G



Disconnected Graph H

16. Define Components of graph.

A disconnected graph consists of two or more connected graphs. Each of these connected subgraphs is called a component.

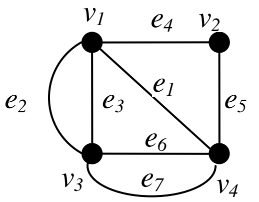


Disconnected Graph H with 3 components

17. Define Euler graph.

A path in a graph G is called Euler path if it includes every edges exactly once. Since the path contains every edge exactly once, it is also called Euler trail / Euler line.

A closed Euler path is called Euler circuit. A graph which contains an Eulerian circuit is called an Eulerian graph.

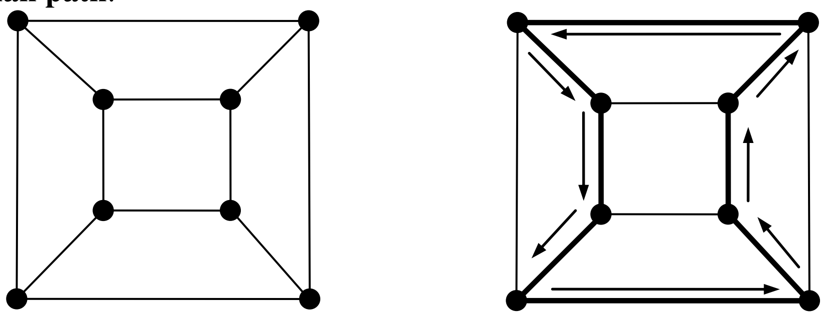


$v_4\ e_1\ v_1\ e_2\ v_3\ e_3\ v_1\ e_4\ v_2\ e_5\ v_4\ e_6\ v_3\ e_7\ v_4$ is an Euler circuit. So the above graph is Euler graph.

18. Define Hamiltonian circuits and paths

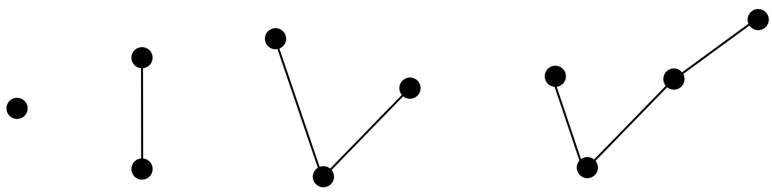
A **Hamiltonian circuit** in a connected graph is defined as a closed walk that traverses every vertex of graph G exactly once except starting and terminal vertex.

Removal of any one edge from a Hamiltonian circuit generates a path. This path is called **Hamiltonian path**.



19. Define Tree

A tree is a connected graph without any circuits. Trees with 1, 2, 3, and 4 vertices are shown in figure.



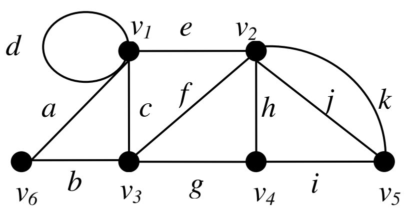
20. List out few Properties of trees.

1. There is one and only one path between every pair of vertices in a tree T.
2. In a graph G there is one and only one path between every pair of vertices, G is a tree.
3. A tree with *n* vertices has *n*-1 edges.
4. Any connected graph with *n* vertices has *n*-1 edges is a tree.
5. A graph is a tree if and only if it is minimally connected.
6. A graph G with *n* vertices has *n*-1 edges and no circuits are connected.

21. What is Distance in a tree?

In a connected graph G, the distance $d(v_i, v_j)$ between two of its vertices v_i and v_j is the length of the shortest path.

Graph G:



Paths between vertices v_6 and v_2 are (a, e), (a, c, f), (b, c, e), (b, f), (b, g, h), and (b, g, i, k).
The shortest paths between vertices v_6 and v_2 are (a, e) and (b, f), each of length two.
Hence $d(v_6, v_2) = 2$

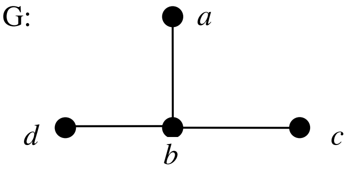
22. Define eccentricity and center.

The eccentricity E(v) of a vertex v in a graph G is the distance from v to the vertex farthest from v in G; that is,

$$E(v) = \max_{v_i \in G} d(v, v_i)$$

A vertex with minimum eccentricity in graph G is called a center of G

Graph G:



Distance d(a, b) = 1, d(a, c) =2, d(c, b)=1, and so on.
Eccentricity E(a) =2, E(b) =1, E(c) =2, and E(d) =2.
Center of G = A vertex with minimum eccentricity in graph G = b.

23. Define distance metric.

The function $f(x, y)$ of two variables defines the distance between them. These function must satisfy certain requirements. They are

1. Non-negativity: $f(x, y) \geq 0$, and $f(x, y) = 0$ if and only if $x = y$.
2. Symmetry: $f(x, y) = f(x, y)$.
3. Triangle inequality: $f(x, y) \leq f(x, z) + f(z, y)$ for any z.

24. What are the Radius and Diameter in a tree.

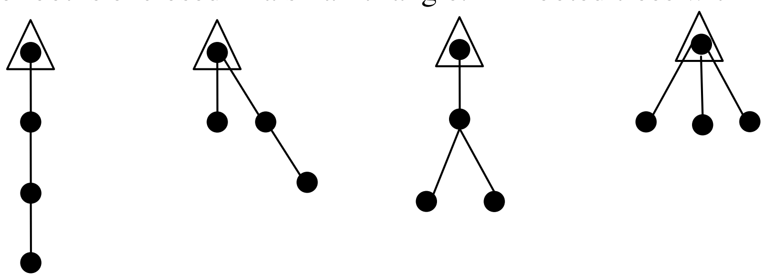
The eccentricity of a center in a tree is defined as the radius of tree.
The length of the longest path in a tree is called the diameter of tree.

25. Define Rooted tree

A tree in which one vertex (called the root) is distinguished from all the others is called a **rooted tree**.

In general tree means without any root. They are sometimes called as **free trees** (non rooted trees).

The root is enclosed in a small triangle. All rooted trees with four vertices are shown below.



26. Define Rooted binary tree

There is exactly one vertex of degree two (root) and each of remaining vertex of degree one or three.

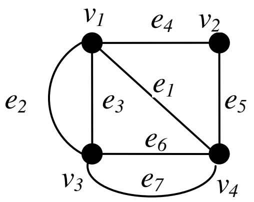
A binary rooted tree is special kind of rooted tree. Thus every binary tree is a rooted tree. A non pendent vertex in a tree is called an internal vertex. Prepared by G. Appasami, Assistant professor, Dr. pauls Engineering College.

UNIT II TREES, CONNECTIVITY & PLANARITY

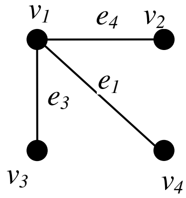
1. Define Spanning trees.

A tree T is said to be a spanning tree of a connected graph G if T is a subgraph of G and T contains all vertices (maximal tree subgraph).

Graph G:



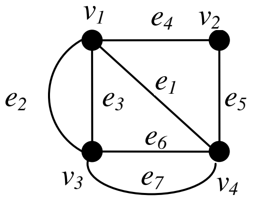
Spanning Tree T:



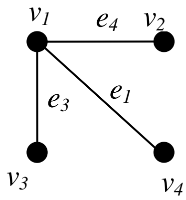
2. Define Branch and chord.

An edge in a spanning tree *T* is called a *branch* of *T*. An edge of G is not in a given spanning tree *T* is called a *chord* (*tie* or *link*).

Graph G:



Spanning Tree T:

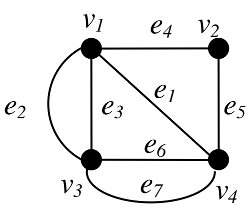


Edge e_1 is a branch of *T* *Edge e_5 is a chord* of *T*

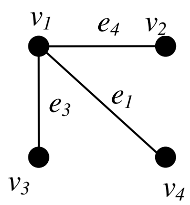
3. Define complement of tree.

If T is a spanning tree of graph G, then the complement of T of G denoted by $\bar{T}$ is the collection of chords. It also called as *chord set* (*tie* set or *cotree*) of T

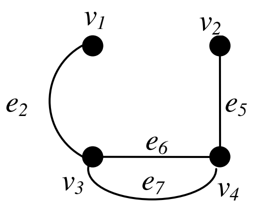
Graph G:



Spanning Tree T:



$\bar{T}$:Complement of Tree T



$$T \cup \bar{T} = G$$

4. Define Rank and Nullity:

A graph G with *n* number of vertices, *e* number of edges, and *k* number of components with the following constraints $n - k \geq 0$ and $e - n + k \geq 0$.

Rank $r = n - k$

Nullity $\mu = e - n + k$ (Nullity also called as *Cyclomatic number* or *first betti number*)

Rank of G = number of branches in any spanning tree of G

Nullity of G = number of chords in G

Rank + Nullity = e = number of edges in G

5. How Fundamental circuits created?

Addition of an edge between any two vertices of a tree creates a circuit. This is because there already exists a path between any two vertices of a tree.

6. Define Spanning trees in a weighted graph

A spanning tree in a graph G is a minimal subgraph connecting all the vertices of G. If G is a weighted graph, then the weight of a spanning tree T of G is defined as the sum of the weights of all the branches in T.

A spanning tree with the smallest weight in a weighted graph is called a *shortest spanning tree (shortest-distance spanning tree* or *minimal spanning tree).*

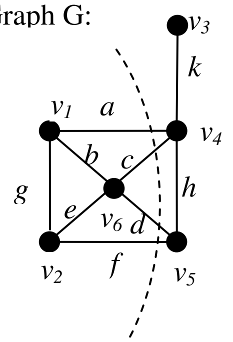
7. Define degree-constrained shortest spanning tree.

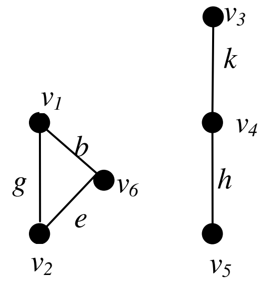
A shortest spanning tree T for a weighted connected graph G with a constraint $d(v_i) \leq k$ for all vertices in T. for k=2, the tree will be Hamiltonian path.

8. Define cut sets and give example.

In a connected graph G, a cut-set is a set of edges whose removal from G leave the graph G disconnected.

Graph G:





Disconnected graph G with 2 components
after removing cut set {a, c, d, f}

Possible cut sets are {a, c, d, f}, {a, b, e, f}, {a, b, g}, {d, h, f}, {k}, and so on.
{a, c, h, d} is not a cut set, because its proper subset {a, c, h} is a cut set.
{g, h} is not a cut set.

A minimal set of edges in a connected graph whose removal reduces the rank by one is called minimal cut set (simple cut-set or cocycle). Every edge of a tree is a cut set.

9. Write the Properties of cut set

- Every cut-set in a connected graph G must contain at least one branch of every spanning tree of G.
- In a connected graph G, any minimal set of edges containing at least one branch of every spanning tree of G is a cut-set.
- Every circuit has an even number of edges in common with any cut set.

10. Define Fundamental circuits

Adding just one edge to a spanning tree will create a cycle; such a cycle is called a **fundamental cycle (Fundamental circuits)**. There is a distinct fundamental cycle for each edge; thus, there is a one-to-one correspondence between fundamental cycles and edges not in the spanning tree. For a connected graph with V vertices, any spanning tree will have $V - 1$ edges, and thus, a graph of E edges and one of its spanning trees will have $E - V + 1$ fundamental cycles.

11. Define Fundamental cut sets

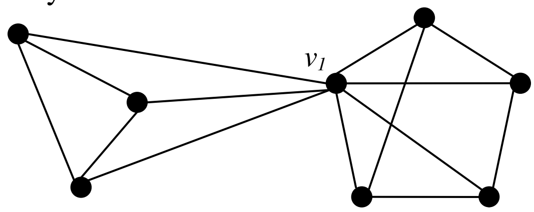
Dual to the notion of a fundamental cycle is the notion of a **fundamental cutset**. By deleting just one edge of the spanning tree, the vertices are partitioned into two disjoint sets. The fundamental cutset is defined as the set of edges that must be removed from the graph G to accomplish the same partition. Thus, each spanning tree defines a set of $V - 1$ fundamental cutsets, one for each edge of the spanning tree.

12. Define edge Connectivity.

Each cut-set of a connected graph G consists of certain number of edges. The number of edges in the smallest cut-set is defined as the **edge Connectivity of G.**

The **edge Connectivity** of a connected graph G is defined as the minimum number of edges whose removal reduces the rank of graph by one.

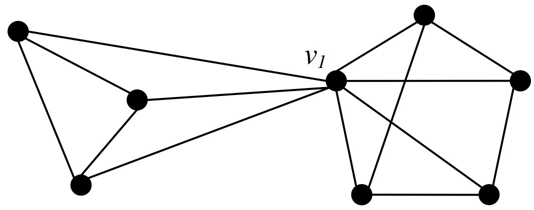
The edge Connectivity of a tree is one.



The edge Connectivity of the above graph G is three.

13. Define vertex Connectivity

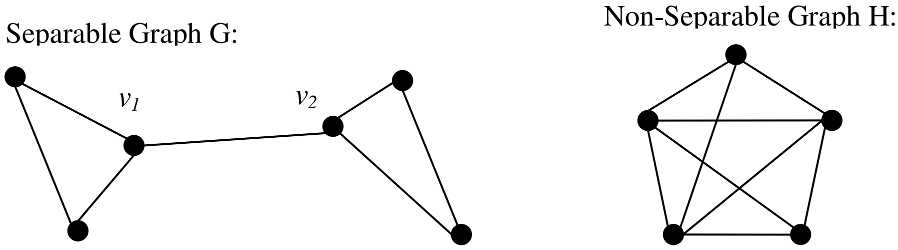
The **vertex Connectivity** of a connected graph G is defined as the minimum number of vertices whose removal from G leaves the remaining graph disconnected. The vertex Connectivity of a tree is one.



The vertex Connectivity of the above graph G is one.

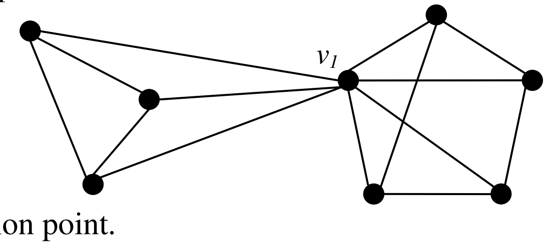
14. Define separable and non-separable graph.

A connected graph is said to be separable graph if its vertex connectivity is one. All other connected graphs are called non-separable graph.



15. Define articulation point.

In a separable graph a vertex whose removal disconnects the graph is called *a cut-vertex, a cut-node, or an articulation point.*



v_1 is an articulation point.

16. What is Network flows

A **flow network** (also known as a transportation **network**) is a **graph** where each edge has a capacity and each edge receives a **flow**. The amount of **flow** on an edge cannot exceed the capacity of the edge.

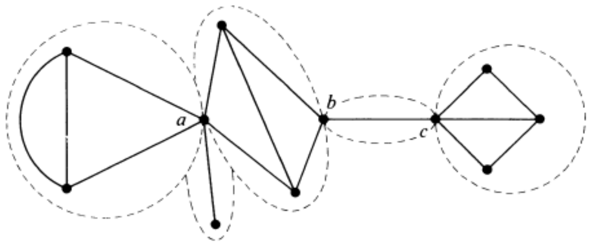
17. Define max-flow and min-cut theorem (equation).

The maximum flow between two vertices a and b in a flow network is equal to the minimum of the capacities of all cut-sets with respect to a and b.

The max. flow between two vertices = Min. of the capacities of all cut-sets.

18. Define component (or block) of graph.

A separable graph consists of two or more non separable subgraphs. Each of the largest nonseparable is called a block (or component).



The above graph has 5 blocks.

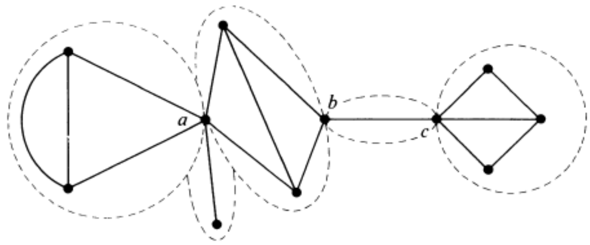
19. Define 1-Isomorphism

A graph G_1 was 1-Isomorphic to graph G_2 if the blocks of G_1 were isomorphic to the blocks of G_2.

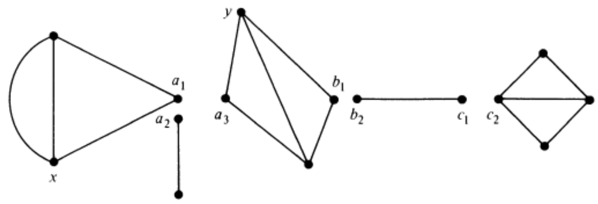
Two graphs G_1 and G_2 are said to be 1-Isomorphic if they become isomorphic to each other under repeated application of the following operation.

Operation 1: "Split" a cut-vertex into two vertices to produce two disjoint subgraphs.

Graph G_1:



Graph G_2:



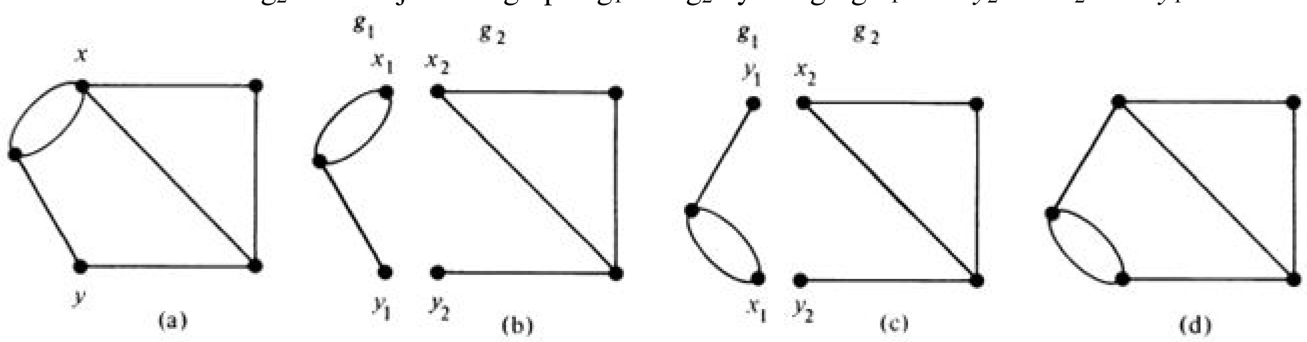
Graph G_1 is 1-Isomorphism with Graph G_2.

20. Define 2-Isomorphism

Two graphs G_1 and G_2 are said to be **2-Isomorphic** if they become isomorphic after undergoing *operation 1* or *operation 2*, or both operations any number of times.

Operation 1: "Split" a cut-vertex into two vertices to produce two disjoint subgraphs.

Operation 2: "Split" the vertex x into x_1 and x_2 and the vertex y into y_1 and y_2 such that G is split into g_1 and g_2. Let vertices x_1 and y_1 go with g_1 and vertices x_2 and y_2 go with g_2. Now rejoin the graphs g_1 and g_2 by merging x_1 with y_2 and x_2 with y_1.



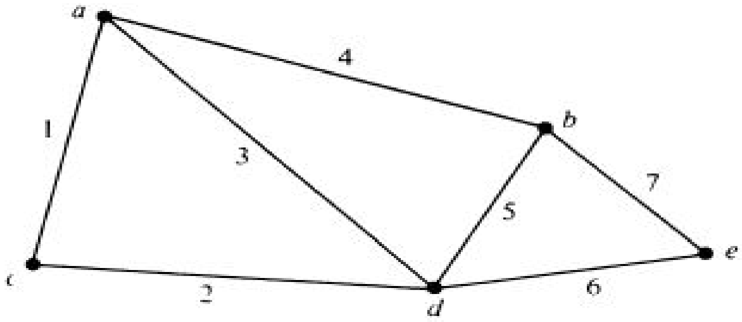
21. Briefly explain Combinational and geometric graphs

An abstract graph G can be defined as G = (V, E, Ψ)

Where the set V consists of five objects named *a, b, c, d*, and *e*, that is, $V = \{ a, b, c, d, e \}$ and the set E consist of seven objects named 1, 2, 3, 4, 5, 6, and 7, that is, $E = \{ 1, 2, 3, 4, 5, 6, 7\}$, and the relationship between the two sets is defined by the mapping Ψ, which consist of

Ψ= [1→(a, c), 2→(c, d) , 3→(a, d) , 4→(a, b) , 5→(b, d) , 6→(d, e) , 7→(b, e)].
Here the symbol 1→*(a, c)*, says that object 1 from set E is mapped onto the pair (a, c) of objects from set V.

This combinatorial abstract object G can also be represented by means of a geometric figure.



The figure is one such geometric representation of this graph G.

Any graph can be geometrically represented by means of such configuration in three dimensional Euclidian space. Prepared by G. Appasami, Assistant professor, Dr. pauls Engineering College.

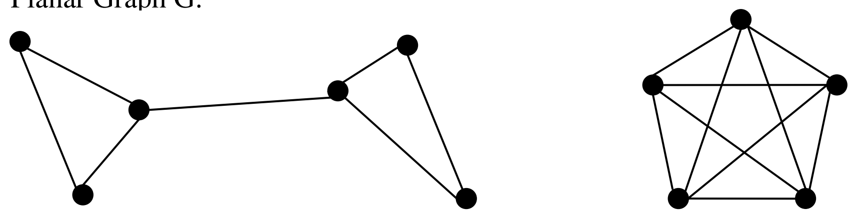
22. Distinguish between Planar and non-planar graphs

A graph G is said to be *planar* if there exists some geometric representation of G which can be drawn on a plan such that no two of its edges intersect.

A graph that cannot be drawn on a plan without crossover its edges is called *non-planar.*

Planar Graph G:

Non-planar Graph H:



23. Define embedding graph.

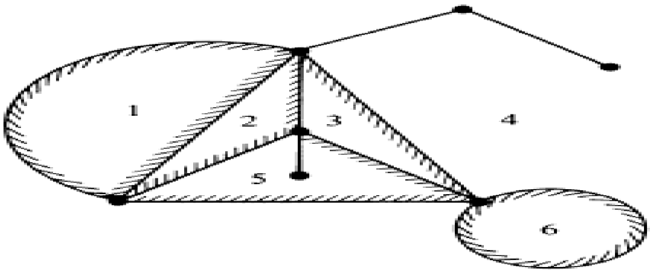
A drawing of a geometric representation of a graph on any surface such that no edges intersect is called embedding.

Graph G:

Embedded Graph G:

24. Define region in graph.

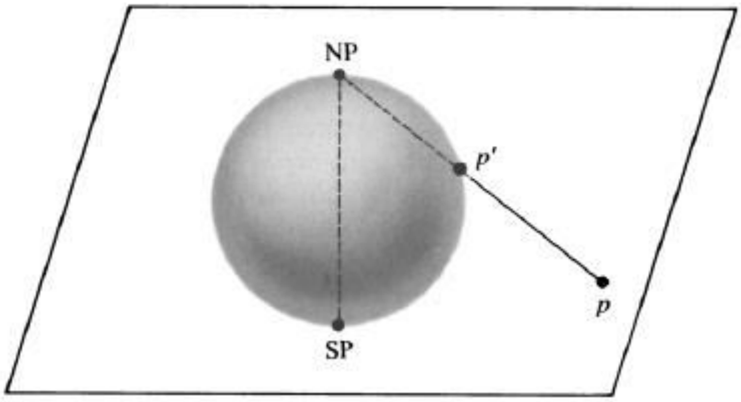
In any planar graph, drawn with no intersections, the edges divide the planes into different **regions (windows, faces, or meshes)**. The regions enclosed by the planar graph are called **interior faces** of the graph. The region surrounding the planar graph is called the **exterior** (or infinite or unbounded) face of the graph. Prepared by G. Appasami, Assistant professor, Dr. pauls Engineering College.



The graph has 6 regions.

25. Why the graph is embedding on sphere.

To eliminate the distinction between finite and infinite regions, a planar graph is often embedded in the surface of sphere. This is done by stereographic projection.



UNIT III MATRICES, COLOURING AND DIRECTED GRAPH

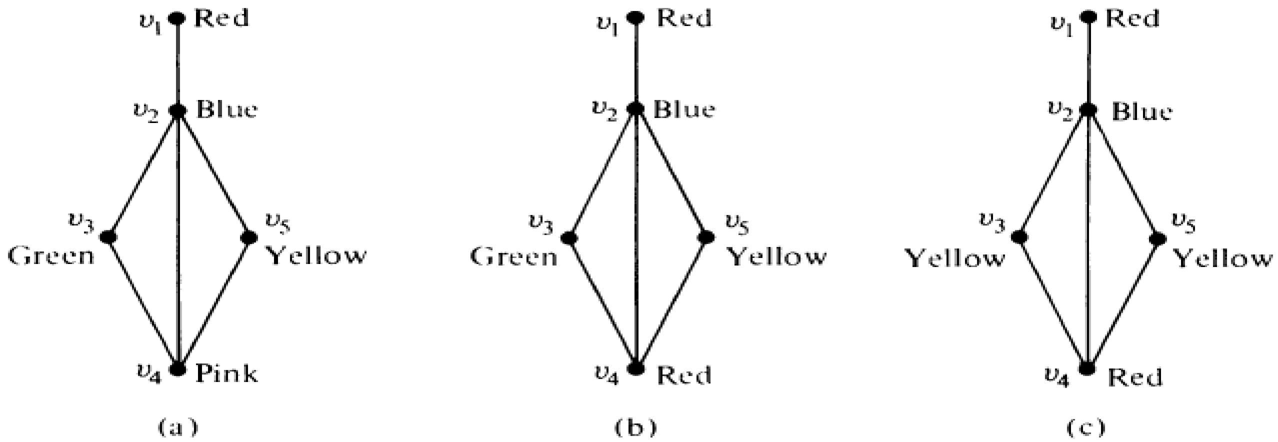
1. What is proper coloring?

Painting all the vertices of a graph with colors such that no two adjacent vertices have the same color is called the *proper coloring* (simply *coloring*) of a graph. A graph in which every vertex has been assigned a color according to a proper coloring is called a *properly colored graph.*

2. Define Chromatic number

A graph G that requires k different colors for its proper coloring, and no less, is called *k-chromatic* graph, and the number k is called the *chromatic number* of G.

The minimum number of colors required for the proper coloring of a graph is called *Chromatic number.*



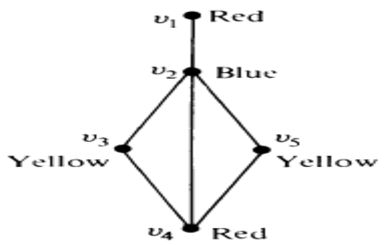
The above graph initially colored with 5 different colors, then 4, and finally 3. So the chromatic number is 3. i.e., The graph is 3-chromatic.

3. Write the properties of chromatic numbers (observations).

- A graph consisting of only isolated vertices is 1-chromatic.
- Every tree with two or more vertices is 2-chromatic.
- A graph with one or more vertices is at least 2-chromatic.
- A graph consisting of simply one circuit with $n \geq 3$ vertices is 2-chromatic if n is even and 3-chromatic if n is odd.
- A complete graph consisting of n vertices is n-chromatic.

4. Define Chromatic partitioning

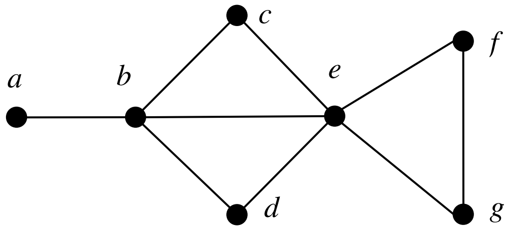
A proper coloring of a graph naturally induces a partitioning of the vertices into different subsets based on colors.



For example, the coloring of the above graph produces the portioning $\{v_1, v_4\}$, $\{v_2\}$, and $\{v_3, v_5\}$.

5. Define independent set and maximal independent set.

A set of vertices in a graph is said to be an *independent set* of vertices or simply independent set (or an internally stable set) if two vertices in the set are adjacent.



For example, in the above graph produces {a, c, d} is an independent set.

A single vertex in any graph constitutes an independent set.

A maximal independent set is an independent set to which no other vertex can be added without destroying its independence property.

{a, c, d, f} is one of the maximal independent set. {b, f} is one of the maximal independent set.

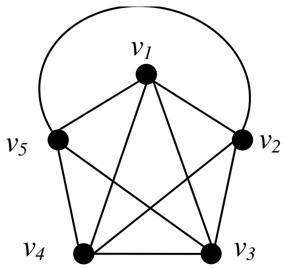
The number of vertices in the largest independent set of a graph G is called the *independence number* (or coefficients of internal stability), denoted by β(G).

For a K-chromatic graph of n vertices, the independence number $\beta(G) \geq \frac{n}{k}$.

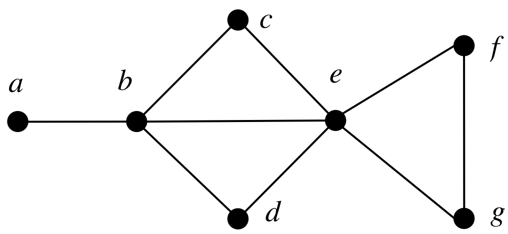
6. Define uniquely colorable graph.

A graph that has only one chromatic partition is called a uniquely colorable graph. For example,

Uniquely colorable graph G:

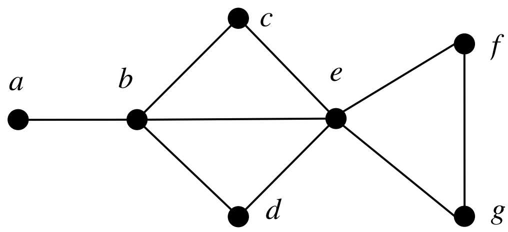


Not uniquely colorable graph H:



7. Define dominating set.

A dominating set (or an externally stable set) in a graph G is a set of vertices that dominates every vertex v in G in the following sense: Either v is included in the dominating set or is adjacent to one or more vertices included in the dominating set.



{b, g} is a dominating set, {a, b, c, d, f} is a dominating set. A is a dominating set need not be independent set. Set of all vertices is a dominating set.

A minimal dominating set is a dominating set from which no vertex can be removed without destroying its dominance property.

{b, e} is a minimal dominating set.

8. Define Chromatic polynomial.

A graph G of n vertices can be properly colored in many different ways using a sufficiently large number of colors. This property of a graph is expressed elegantly by means of polynomial. This polynomial is called the Chromatic polynomial of G.

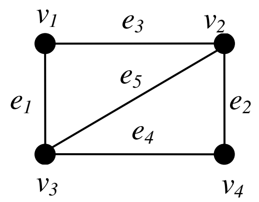
The value of the Chromatic polynomial $P_n(\lambda)$ of a graph with n vertices the number of ways of properly coloring the graph , using λ or fewer colors.

9. Define Matching (Assignment).

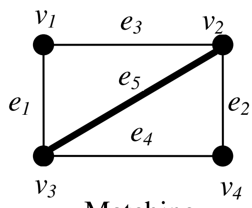
A *matching* in a graph is a subset of edges in which no two edges are adjacent. A single edge in a graph is a matching.

A *maximal matching* is a matching to which no edge in the graph can be added.

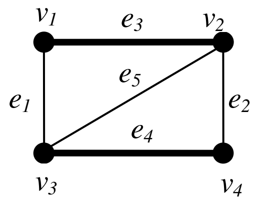
The maximal matching with the largest number of edges are called the *largest maximal matching*.



Graph G



Matching



Maximal matching

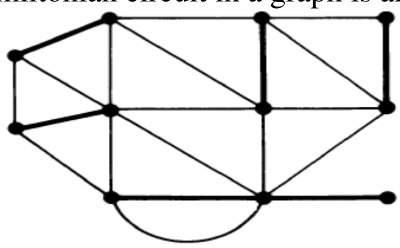
10. What is Covering?

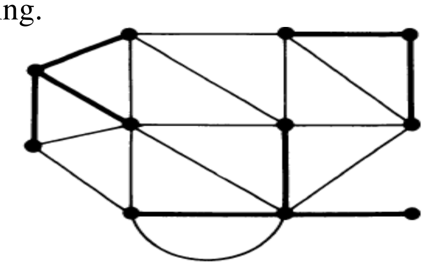
A set g of edges in a graph G is said to be cover og G if every vertex in G is incident on at least one edge in g. A set of edges that covers a graph G is said to be a covering (or an edge covering, or a coverring subgraph) of G.

Every graph is its own covering.

A spanning tree in a connected graph is a covering.

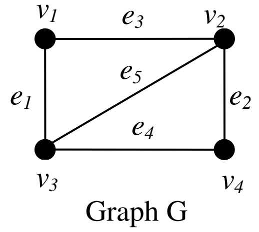
A Hamiltonian circuit in a graph is also a covering.



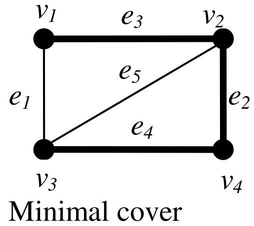


11. Define minimal cover.

A **minimal covering** is a covering from which no edge can be removed without destroying it ability to cover the graph G.



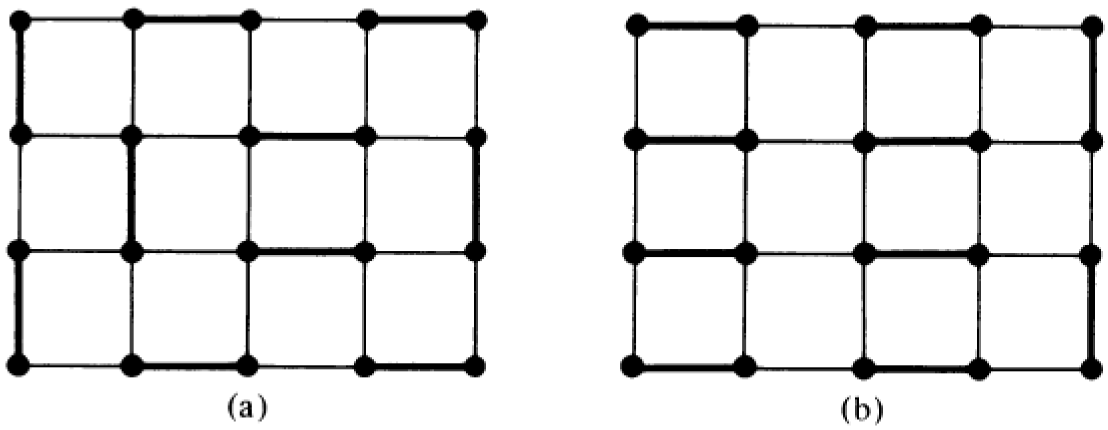
Graph G



Minimal cover

12. What is dimer covering?

A covering in which every vertex is of degree one is called a *dimer covering* or a *1-factor*. A dimmer covering is a maximal matching because no two edges in it are adjacent. Prepared by G. Appasami, Assistant professor, Dr. pauls Engineering College.



Two dimmer coverings.

13. Define four color problem / conjecture.

- Every planar graph has a chromatic number of four or less.
- Every triangular planar graph has a chromatic number of four or less.
- The regions of every planar, regular graph of degree three can be colored properly with four colors.

14. State five color theorem

Every planar map can be properly colored with five colors.
i.e., the vertices of every plannar graph can be properly colored with five colors.

15. Write about vertex coloring and region coloring.

A graph has a dual if and only if it is planar. Therefore, coloring the regions of a planar graph G is equivalent to coloring the vertices of its dual G* and vice versa.

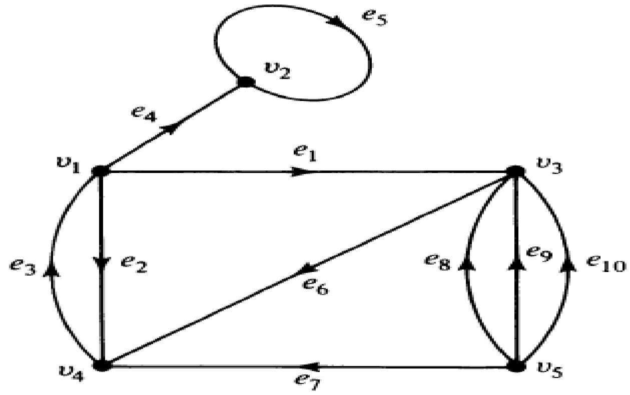
What is meant by regularization of a planar graph?

- Remove every vertex of degree one from the graph G does not affect the regions of a plannar graph.
- Remove every vertex of degree two and merge the two edges in series from the graph G.
- Such a transformation may be called regularization of a planar graph.

16. Directed graphs

A *directed graph* (or a *digraph, or an oriented graph*) G consists of a set of vertices $V = \{ v_1, v_2, \dots \}$, a set of edges $E = \{ e_1, e_2, \dots \}$, and a mapping Ψ that maps every edge onto some ordered pair of vertices (v_i, v_j).

For example,

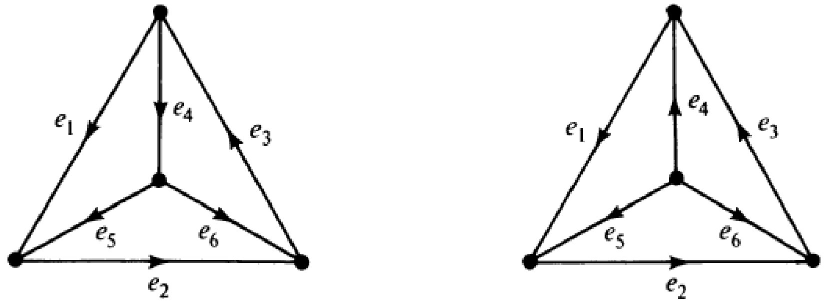


17. Define isomorphic digraph.

Among directed graphs, if their labels are removed, two isomorphic graphs are indistinguishable then these graphs are **isomorphic digraph**.

For example,

Two isomorphic digraphs.



Two non-isomorphic digraphs.

18. List out some types of directed graphs

- Simple Digraphs
- Asymmetric Digraphs (Anti-symmetric)
- Symmetric Digraphs
- Simple Symmetric Digraphs
- Simple Asymmetric Digraphs
- Complete Digraphs
- Complete Symmetric Digraphs
- Complete Asymmetric Digraphs (tournament)
- Balance digraph (a pseudo symmetric digraph or an isograph)

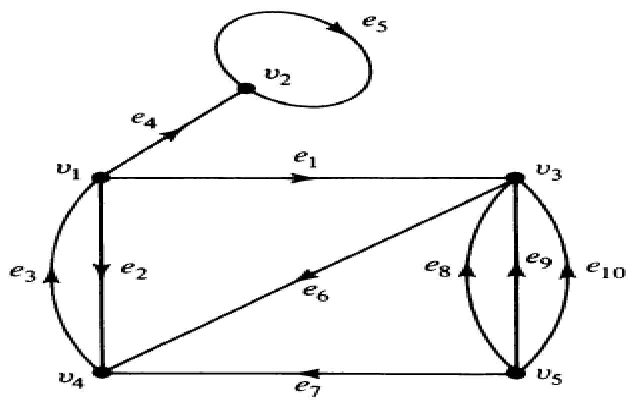
19. Define binary relations.

In a set of objects, X, where X={x_1, x_2, …}, A *binary relation R* between pairs (x_i, x_j) can be written as $x_i \, R \, x_j$ and say that x_i has relation R to x_j.

If the binary relation R is reflexive, symmetric, and transitive then R is an equivalence relation. This produces equivalence classes.

20. What is Directed path?

A path in a directed graph is called Directed path.



$v_5\ e_8\ v_3\ e_6\ v_4\ e_3\ v_1$ is a directed path from v_5 to v_1.

Whereas $v_5\ e_7\ v_4\ e_6\ v_3\ e_1\ v_1$ is a semi-path from v_5 to v.

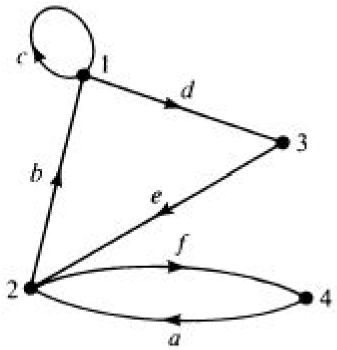
21. Write the types of connected digraphs

- **Strongly connected digraph:** A digraph G is said to be strongly connected if there is at least one directed path from every vertex to every other vertex.
- **Weakly connected digraph**: A digraph G is said to be weakly connected if its corresponding undirected graph is connected. But G is not strongly connected.

22. Define Euler digraphs

In a digraph G, a closed directed walk which traverses every edge of G exactly once is called a *directed Euler line*. A digraph containing a directed Euler line is called an **Euler digraphs**

For example,



It contains directed Euler line **a b c d e f.**

23. What is teleprinter's problem.

Constructing a longest circular sequence of 1's and 0's such that no subsequence of r bits appears more than once in the sequence.

Teleprinter's problem was solved in 1940 by I.G. Good using digraph.

UNIT IV PERMUTATIONS & COMBINATIONS

1. Define Fundamental principles of counting

The Fundamental Counting Principle is a way to figure out the total number of ways different events can occur.

If the first task can be performed in *m* ways, while a second task can be performed in *n* ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in any one of $m + n$ ways.

If a procedure can be broken into first and second stages, and if there are *m* possible outcomes for the first stage and if, for each of these outcomes, there are *n* possible outcomes for the second stage, then the total procedure can be carried out, in the designed order, in *mn* ways.

2. Define rule of sum.

If the first task can be performed in *m* ways, while a second task can be performed in *n* ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in any one of $m + n$ ways.

Example: A college library has 40 books on C++ and 50 books on Java. A student at this college can select 40+50=90 books to learn programming language.

3. Define rule of Product

If a procedure can be broken into first and second stages, and if there are *m* possible outcomes for the first stage and if, for each of these outcomes, there are *n* possible outcomes for the second stage, then the total procedure can be carried out, in the designed order, in *mn* ways.

Example: A drama club with six men and eight can select male and female role in 6 x 8 = 48 ways.

4. Define Permutations

For a given collection of n objects, any linear arrangement of these objects is called a permutation of the collection. Counting the linear arrangement of objects can be done by rule of product.

For a given collection of *n* distinct objects, and *r* is an integer, with $1 \leq r \leq n$, then by rule of product, the number of permutations of size r for the n objects is

$$P(n,r) = n \times (n-1) \times (n-2) \times \ldots \times (n-r+1) = \frac{n!}{(n-r)!}, \quad 0 \leq r \leq n$$

Example: In a class of 10 students, five are to be chosen and seated in a row for a picture.

The total number of arrangements = 10 x 9 x 8 x 7 x 6 = 30240.

5. Define combinations

For a given collection of *n* objects, each selection, or combination, of *r* of these objects, with no reference to order, corresponds to *r*! (Permutations of size *r* from the *n* objects). Thus the number of combinations of size *r* from a collection of size *n* is

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!\,(n-r)!}, \quad 0 \leq r \leq n$$

Example: In a test, students are directed to answer 7 questions out of 10. The student can answer the examination in

$$C(n,r) = C(10,7) = \frac{10!}{7!\,(10-7)!} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120\ ways$$

6. State Binomial theorem

The Binomial theorem: If *x* and *y* are variables and n is a positive integer, then

$$(x+y)^n = \binom{n}{0} x^0 y^n + \binom{n}{1} x^1 y^{n-1} + \binom{n}{2} x^2 y^{n-2} + \cdots$$
$$\binom{n}{n-1} x^{n-1} y^1 + \binom{n}{n} x^n y^0 = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$\binom{n}{k}$ is referred as Binomial coefficient.

7. Define combinations with repetition

If there is a selection with repetition, r of n distinct objects, then the combinations with of n objects taken r at a time with repetition is $C(n+r-1,r)$.

$$C(n+r-1,r) = \frac{(n+r-1)!}{r!\,(n-1)!} = \binom{n+r-1}{r}$$

Example: A donut shop offers 20 kinds of donuts. Assuming that there are at least a dozen of each kind when we enter the shop. We can select a dozen donuts in $C(20+12-1,12)$= $C(31,12)$ = 141120525 ways

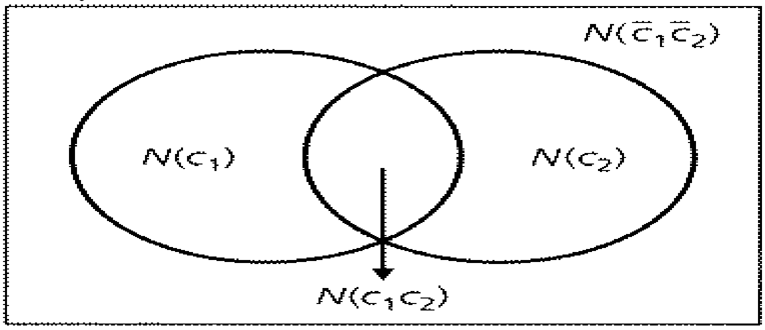
8. Define Catalan numbers

The Catalan numbers form a sequence of natural numbers that occur in various counting problems, often involving recursively-defined objects. They are named after the Belgian mathematician Eugène Charles Catalan. the nth Catalan number is given directly in terms of binomial coefficients by Prepared by G. Appasami, Assistant professor, Dr. pauls Engineering College.

$$C_n = \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{(n+1)!\,n!} = \frac{1}{n+1}\binom{2n}{n} \qquad \text{for } n \geq 0$$

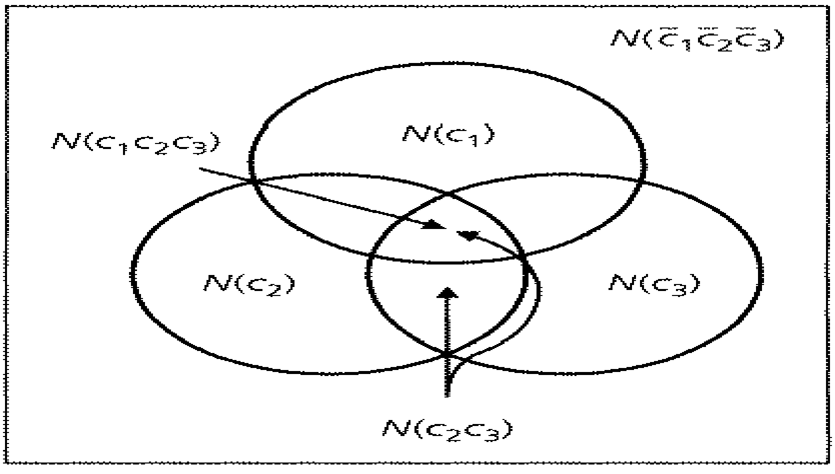
9. Write the Principle of inclusion and exclusion formula.

For any 2 sets, C_1 and C_2,



$$N(\bar{c}_1\bar{c}_2) = N - [N(c_1) + N(c_2)] + N(c_1c_2)$$

For any 3 sets, C_1, C_2 and C_3,



$$N(\bar{c}_1\bar{c}_2\bar{c}_3) = N - [N(c_1) + N(c_2) + N(c_3)] + [N(c_1c_2) + N(c_1c_3) + N(c_2c_3)] - N(c_1c_2c_3).$$

For any 4 sets, C_1, C_2 , C_2 and C_4,

$$\begin{aligned}N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = N &- [N(c_1) + N(c_2) + N(c_3) + N(c_4)] \\ &+ [N(c_1c_2) + N(c_1c_3) + N(c_1c_4) + N(c_2c_3) + N(c_2c_4) + N(c_3c_4)] \\ &- [N(c_1c_2c_3) + N(c_1c_2c_4) + N(c_1c_3c_4) + N(c_2c_3c_4)] \\ &+ N(c_1c_2c_3c_4).\end{aligned}$$

10. Define Derangements

A derangement is a permutation of the elements of a set, such that no element appears in its original position.

The number of derangements of a set of size n, usually written D_n, d_n, or !n, is called the "derangement number" or "de Montmort number".

Example: The number of derangements of 1, 2, 3, 4 is

$d_4 = 4!\ [1 - 1 + (1/2!)-(1/3!)+(1/4!)] = 9.$

11. What is meant by Arrangements with forbidden (banned) positions.

The number of acceptable assignments is equal to the number of ways of placing nontaking rooks on this chessboard so that none of the rooks is in a forbidden position. The key to determining this number of arrangements is the inclusion- exclusion principle.

UNIT V GENERATING FUNCTIONS

1. Define Generating function.

A **generating function** describes an infinite sequence of numbers (a_n) by treating them like the coefficients of a series expansion. The sum of this infinite series is the generating function. Unlike an ordinary series, this formal series is allowed to diverge, meaning that the generating function is not always a true function and the "variable" is actually an indeterminate.

The generating function for 1, 1, 1, 1, 1, 1, 1, 1, 1, ..., whose ordinary generating function is

$$\sum_{n=0}^{\infty}(x)^n = \frac{1}{1-x}$$

The generating function for the geometric sequence 1, a, a^2, a^3, ... for any constant a:

$$\sum_{n=0}^{\infty}(ax)^n = \frac{1}{1-ax}$$

2. What is Partitions of integer?

Partitioning a positive n into positive summands and seeking the number of such partitions without regard to order is called Partitions of integer.

This number is denoted by $p(n)$. For example

P(1) = 1: 1

P(2) = 2: 2 = 1 + 1

P(3) = 3: 3 = 2 +1 = 1 + 1 +1

P(4) = 5: 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1

P(5) = 7: 5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1+ 1 = 1 + 1 + 1 + 1 + 1

3. Define Exponential generating function

For a sequence $a_0, a_1, a_2, a_3,, \ldots$ of real numbers.

$$f(x) = a_0 + a_1 x + a_2\frac{x^2}{2!} + a_3\frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} a_i \frac{x^i{}^i}{i!}$$

is called the exponential generating function for the given sequence.

4. Define Maclaurin series expansion of e^x and e^{-x}.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$$

Adding these two series together, we get,

$$e^x + e^{-x} = 2(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots)$$

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

5. Define Summation operator

Generating function for a sequence $a_0, a_0 + a_1, a_0 + a_1 + a_2, a_0 + a_1 + a_2 + a_3,, \ldots$.

For $(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$,, consider the function $f(x)/(1-x)$

$$\frac{f(x)}{1-x} = f(x).\frac{1}{1-x} = [a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots][1 + x + x^2 + x^3 + \cdots]$$

$$= a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + +(a_0 + a_1 + a_2 + a_3)x^3 + \cdots$$

So $f(x)/(1-x)$ generates the sequence of sums $a_0, a_0 + a_1, a_0 + a_1 + a_2, a_0 + a_1 + a_2 + a_3,$,

$1/(1-x)$ is called the summation operator.

6. What is Recurrence relation?

A **recurrence relation** is an equation that recursively defines a sequence or multidimensional array of values, once one or more initial terms are given: each further term of the sequence or array is defined as a function of the preceding terms.

The term **difference equation** sometimes (and for the purposes of this article) refers to a specific type of recurrence relation. However, "difference equation" is frequently used to refer to *any* recurrence relation.

7. Write Fibonacci numbers and relation

The recurrence satisfied by the Fibonacci numbers is the archetype of a homogeneous linear recurrence relation with constant coefficients (see below). The Fibonacci sequence is defined using the recurrence

$F_n = F_{n-1} + F_{n-2}$

with seed values $F_0 = 0$ and $F_1 = 1$

We obtain the sequence of Fibonacci numbers, which begins

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

8. Define First order linear recurrence relation

The general form of First order linear homogeneous recurrence relation can be written as

$a_{n+1} = d\, a_n$, $n \geq 0$, where d is a constant. The relation is first order since a_{n+1} depends on a_n.

a_0 *or* a_1 are called boundary conditions.

9. Define Second order recurrence relation

Let $k \in \mathbf{Z}^+$ and $C_0\ (\neq 0), C_1, C_2, \ldots, C_k\ (\neq 0)$ be real numbers. If a_n, for $n \geq 0$, is a discrete function, then

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = f(n), \qquad n \geq k,$$

is a linear recurrence relation (with constant coefficients) of *order k*. When $f(n) = 0$ for all $n \geq 0$, the relation is called *homogeneous*; otherwise, it is called *nonhomogeneous*.

In this section we shall concentrate on the homogeneous relation of order two:

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} = 0, \qquad n \geq 2.$$

On the basis of our work in Section 10.1, we seek a solution of the form $a_n = cr^n$, where $c \neq 0$ and $r \neq 0$.

Substituting $a_n = cr^n$ into $C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} = 0$, we obtain

$$C_0 cr^n + C_1 cr^{n-1} + C_2 cr^{n-2} = 0.$$

With $c, r \neq 0$, this becomes $C_0 r^2 + C_1 r + C_2 = 0$, a quadratic equation which is called the *characteristic equation*. The roots r_1, r_2 of this equation determine the following three cases: (a) r_1, r_2 are distinct real numbers; (b) r_1, r_2 form a complex conjugate pair; or (c) r_1, r_2 are real, but $r_1 = r_2$. In all cases, r_1 and r_2 are called the *characteristic roots*.

10. Briefly explain Non-homogeneous recurrence relation.

We now turn to the recurrence relations

$$a_n + C_1 a_{n-1} = f(n), \qquad n \geq 1, \tag{1}$$

$$a_n + C_1 a_{n-1} + C_2 a_{n-2} = f(n), \qquad n \geq 2, \tag{2}$$

where C_1 and C_2 are constants, $C_1 \neq 0$ in Eq. (1), $C_2 \neq 0$, and $f(n)$ is not identically 0. Although there is no general method for solving all nonhomogeneous relations, for certain functions $f(n)$ we shall find a successful technique.

We start with the special case for Eq. (1), when $C_1 = -1$. For the nonhomogeneous relation $a_n - a_{n-1} = f(n)$, we have

$$\begin{aligned}
a_1 &= a_0 + f(1)\\
a_2 &= a_1 + f(2) = a_0 + f(1) + f(2)\\
a_3 &= a_2 + f(3) = a_0 + f(1) + f(2) + f(3)\\
&\vdots\\
a_n &= a_{n-1} + f(n) = a_0 + f(1) + \cdots + f(n) = a_0 + \sum_{i=1}^{n} f(i).
\end{aligned}$$

We can solve this type of relation in terms of n, if we can find a suitable summation formula for $\sum_{i=1}^{n} f(i)$.

CS6702 GRAPH THEORY AND APPLICATIONS
QUESTION BANK

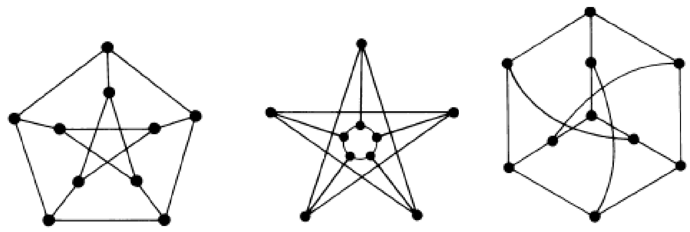
UNIT I INTRODUCTION

PART – A

1. Define Graph.
2. Define Simple graph.
3. Write few problems solved by the applications of graph theory.
4. Define incidence, adjacent and degree.
5. What are finite and infinite graphs?
6. Define Isolated and pendent vertex.
7. Define null graph.
8. Define Multigraph
9. Define complete graph
10. Define Regular graph
11. Define Cycles
12. Define Isomorphism.
13. What is Subgraph?
14. Define Walk, Path and Circuit.
15. Define connected graph. What is Connectedness?
16. Define Euler graph.
17. Define Hamiltonian circuits and paths
18. Define Tree
19. List out few Properties of trees.
20. What is Distance in a tree?
21. Define eccentricity and center.
22. Define distance metric.
23. What are the Radius and Diameter in a tree.
24. Define Rooted tree
25. Define Rooted binary tree

PART – B

1. Explain various applications of graph.
2. Define the following kn, cn, kn,n, dn, trail, walk, path, circuit with an example.
3. Show that a connected graph G is an Euler graph iff all vertices are even degree.
4. Prove that a simple graph with n vertices and k components can have at most (n-k)(n-k+1)/2 edges.



5. Are they isomorphic?
6. Prove that in a complete graph with n vertices there are (n-1)/2 edges-disjoint Hamiltonian circuits, if n is odd number ≥3.
7. Prove that, there is one and only one path between every pair of vertices in a tree T.
8. Prove the given statement, "A tree with *n* vertices has *n*-1 edges".
9. Prove that, any connected graph with *n* vertices has *n*-1 edges is a tree.
10. Show that a graph is a tree if and only if it is minimally connected.
11. Prove that, a graph G with *n* vertices has *n*-1 edges and no circuits are connected.

UNIT II TREES, CONNECTIVITY & PLANARITY

PART – A

1. Define Spanning trees.
2. Define Branch and chord.
3. Define complement of tree.
4. Define Rank and Nullity.
5. How Fundamental circuits created?
6. Define Spanning trees in a weighted graph.
7. Define degree-constrained shortest spanning tree.
8. Define cut sets and give example.
9. Write the Properties of cut set
10. Define Fundamental circuits
11. Define Fundamental cut sets
12. Define edge Connectivity.
13. Define vertex Connectivity.
14. Define separable and non-separable graph.
15. Define articulation point.
16. What is Network flows.
17. Define max-flow and min-cut theorem (equation).
18. Define component (or block) of graph.
19. Define 1-Isomorphism.
20. Define 2-Isomorphism.
21. Briefly explain Combinational and geometric graphs.
22. Distinguish between Planar and non-planar graphs.
23. Define embedding graph.
24. Define region in graph.
25. Why the graph is embedding on sphere.

PART – B

1. Find the shortest spanning tree for the following graph.

$$\begin{array}{c c}
 & \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\
\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} &
\begin{bmatrix}
- & 10 & 16 & 11 & 10 & 17 \\
10 & - & 9.5 & \infty & \infty & 19.5 \\
16 & 9\ 5 & - & 7 & \infty & 12 \\
11 & \infty & 7 & - & 8 & 7 \\
10 & \infty & \infty & 8 & - & 9 \\
17 & 19.5 & 12 & 7 & 9 & -
\end{bmatrix}
\end{array}$$

2. Explain 1 - isomarphism and 2 - isomarphism of graphs with your own example.
3. Prove that a connected graph G with *n* vertices and *e* edges has *e-n+2* regions.
4. Write all possible spanning tree for K5.
5. Prove that every cut-set in a connected graph G must contain at least one branch of every spanning tree of G.
6. Prove that the every circuit which has even number of edges in common with any cut-set.
7. Show that the ring sum of any two cut-sets in a graph is either a third cut set or en edge disjoint union of cut sets.
8. Explain network flow problem in detail.
9. If G_1 and G_2 are two 1-isomorphic graphs, the rank of G_1 equals the rank of G_2 and the nullity of G_1 equals the nullity of G_2, prove this.
10. Prove that any two graphs are 2-isomorphic if and only if they have circuit correspondence.

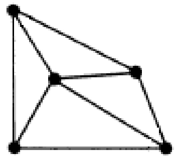
UNIT III MATRICES, COLOURING AND DIRECTED GRAPH

PART – A

1. What is proper coloring?
2. Define Chromatic number
3. Write the properties of chromatic numbers (observations).
4. Define Chromatic partitioning
5. Define independent set and maximal independent set.
6. Define uniquely colorable graph.
7. Define dominating set.
8. Define Chromatic polynomial.
9. Define Matching (Assignment).
10. What is Covering?
11. Define minimal cover.
12. What is dimer covering?
13. Define four color problem / conjecture.
14. State five color theorem
15. Write about vertex coloring and region coloring.
16. What is meant by regularization of a planar graph?
17. Define Directed graphs .
18. Define isomorphic digraph.
19. List out some types of directed graphs.
20. Define Simple Digraphs.
21. Define Asymmetric Digraphs (Anti-symmetric).
22. What is meant by Symmetric Digraphs?
23. Define Simple Symmetric Digraphs.
24. Define Simple Asymmetric Digraphs.
25. Give example for Complete Digraphs.
26. Define Complete Symmetric Digraphs.
27. Define Complete Asymmetric Digraphs (tournament).
28. Define Balance digraph (a pseudo symmetric digraph or an isograph).
29. Define binary relations.
30. What is Directed path?
31. Write the types of connected digraphs.
32. Define Euler graphs.

PART – B

1. Prove that any simple planar graph can be embedded in a plane such that every edge is drawn as a straight line.
2. Show that a connected planar graph with n vertices and e edges has e-n+2 regions.
3. Define chromatic polynomial. Find the chromatic polynomial for the following graph.



4. Explain matching and bipartite graph in detail.
5. Write the observations of minimal covering of a graph.
6. Prove that the vertices of every planar graph can be properly colored with five colors.
7. Explain matching in detail.
8. Prove that a covering g of graph G is minimal iff g contains no path of length three or more.
9. Illustrate four-color problem.
10. Explain Euler digraphs in detail.

UNIT IV PERMUTATIONS & COMBINATIONS

PART – A

1. Define Fundamental principles of counting
2. Define rule of sum.
3. Define rule of Product
4. Define Permutations
5. Define combinations
6. State Binomial theorem
7. Define combinations with repetition
8. Define Catalan numbers
9. Write the Principle of inclusion and exclusion formula.
10. Define Derangements
11. What is meant by Arrangements with forbidden (banned) positions.

PART – B

1. Explain the Fundamental principles of counting.
2. Find the number of ways of ways of arranging the word APPASAMIAP and out of it how many arrangements have all A's together.
3. Discuss the rules of sum and product with example.
4. Determine the number of (staircase) paths in the *xy*-plane from (2, 1) to (7, 4), where each path is made up of individual steps going 1 unit to the right (R) or one unit upward (U). iv. Find the coefficient of a^5b^2 in the expansion of $(2a - 3b)^7$.
5. State and prove binomial theorem.
6. How many times the print statement executed in this program segment?

```
for i := 1 to 20 do
  for j := 1 to i do
    for k := 1 to j do
      print (i * j + k)
```

7. Discuss the Principle of inclusion and exclusion.
8. How many integers between 1 and 300 (inc.) are not divisible by at least one of 5, 6, 8?
9. How 32 bit processors address the content? How many address are possible?
10. Explain the Arrangements with forbidden positions.

UNIT V GENERATING FUNCTIONS

PART – A

1. Define Generating function.
2. What is Partitions of integer?
3. Define Exponential generating function
4. Define Maclaurin series expansion of e^x and e^{-x}.
5. Define Summation operator
6. What is Recurrence relation?
7. Write Fibonacci numbers and relation
8. Define First order linear recurrence relation
9. Define Second order recurrence relation
10. Briefly explain Non-homogeneous recurrence relation.

PART – B

1. Explain Generating functions
2. Find the convolution of the sequences 1, 1, 1, 1, ….. and 1,-1,1,-1,1,-1.
3. Find the number of non negative & positive integer solutions of for x1+x2+x3+x4=25.
4. Find the coefficient of x5 in(1-2x)7.
5. The number of virus affected files in a system is 1000 and increases 250% every 2 hours.
6. Explain Partitions of integers
7. Use a recurrence relation to find the number of viruses after one day.
8. Explain First order homogeneous recurrence relations.
9. Solve the recurrence relation an+2-4an+1+3an=-200 with a0=3000 and a1=3300.
10. Solve the Fibonacci relation Fn = Fn-1+Fn-2.
11. Find the recurrence relation from the sequence 0, 2, 6, 12, 20, 30, 42, … .
12. Determine (1+√3i)10.
13. Discuss Method of generating functions.

All the Best – No substitution for hard work.

Mr. G. Appasami SET, NET, GATE, M.Sc., M.C.A., M.Phil., M.Tech., (P.hd.), MISTE, MIETE, AMIE, MCSI, MIAE, MIACSIT, MASCAP, MIARCS, MISIAM, MISCE, MIMS, MISCA, MISAET, PGDTS, PGDBA, PGDHN, PGDST, (AP / CSE / DRPEC).

Reg. No. | | | | | | | | | | | | |

MODEL EXAMINATION

Department of Computer Science and Engineering

Subject Name : Graph Theory and Applications **Date : 30/09/2016**

Subject Code : CS6702 **Duration : 3 Hours**

Year / Sem. : IV / VII **Marks : 100**

Part-A (10x2=20)

Answer All Questions

1. What are finite and infinite graphs? Give an example.
2. Differentiate regular and complete graph.
3. Define cut sets and give example.
4. Briefly explain Combinational and geometric graphs.
5. Define Chromatic number. Find Chromatic number for K_5.
6. Define Matching (Assignment).
7. A donut shop offers 20 kinds of donuts. Assuming that there is at least a dozen of each kind when we enter the shop. How many ways we can select a dozen donuts?
8. Define Derangements. Find the derangements of 1, 2, 3, 4.
9. What is Partitions of integer?
10. Define Second order recurrence relation. Give an example.

Part-B (5x16=80)

Answer All Questions

11. a. i. Explain various applications of graph. (8)

ii. Define the following k_n, c_n, $k_{n,n}$, d_n, trail, walk, path, circuit with an example. (8)

Or

b. i. Show that a connected graph G is an Euler graph iff all vertices are even degree. (8)

ii. Prove that a simple graph with *n* vertices and *k* components can have at most $(n-k)(n-k+1)/2$ edges. (8)

12. a. i. Find the shortest spanning tree for the following graph. (8)

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} - & 10 & 16 & 11 & 10 & 17 \\ 10 & - & 9.5 & \infty & \infty & 19.5 \\ 16 & 9.5 & - & 7 & \infty & 12 \\ 11 & \infty & 7 & - & 8 & 7 \\ 10 & \infty & \infty & 8 & - & 9 \\ 17 & 19.5 & 12 & 7 & 9 & - \end{bmatrix} \end{array}$$

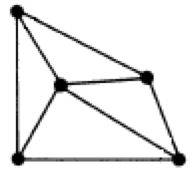
ii. Prove that the every circuit which has even number of edges in common with any cut-set. (8)

Or

b. i. Explain 1 - isomarphism and 2 - isomarphism of graphs with your own example. (8)

ii. Prove that a connected graph G with *n* vertices and *e* edges has *e-n+2* regions (8)

13. a. i. Define chromatic polynomial. Find the chromatic polynomial for the following graph. (8)



ii. Explain matching and bipartite graph in detail. (8)

Or

b. i. Write the observations of minimal covering of a graph. (8)

ii. Prove that the vertices of every planar graph can be properly colored with five colors. (8)

14. a. i. Find the number of ways of ways of arranging the word MASSASAUGA and out of it how many arrangements have all A's together. (4)

ii. Discuss the rules of sum and product with example. (4)

iii. Determine the number of (staircase) paths in the *xy*-plane from (2, 1) to (7, 4), where each path is made up of individual steps going 1 unit to the right (R) or one unit upward (U). (4)

iv. Find the coefficient of a^5b^2 in the expansion of $(2a - 3b)^7$. (4)

Or

b. i. State and prove binomial theorem. (8)

ii. How many times the print statement executed in this program segment? (4)

```
for i := 1 to 20 do
  for j := 1 to i do
    for k := 1 to j do
      print (i * j + k)
```

iii. How many integers between 1 and 300 (inc.) are not divisible by at least one of 5, 6, 8? (4)

15. a. i. Find the convolution of the sequences 1, 1, 1, 1, ….. and 1,-1,1,-1,1,-1. (4)

ii. Find the number of non negative & positive integer solutions of for $x_1+x_2+x_3+x_4=25$. (4)

iii. Find the coefficient of x^5 in$(1-2x)^7$. (4)

iv. The number of virus affected files in a system is 1000 and increases 250% every 2 hours. (4) Use a recurrence relation to find the number of viruses after one day.

Or

b. i. Solve the recurrence relation $a_{n+2}-4a_{n+1}+3a_n=-200$ with $a_0=3000$ and $a_1=3300$. (4)

ii. Solve the Fibonacci relation $F_n = F_{n-1}+F_{n-2}$. (4)

iii. Find the recurrence relation from the sequence 0, 2, 6, 12, 20, 30, 42, … . (4)

iv. Determine $(1+\sqrt{3}i)^{10}$. (4)

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