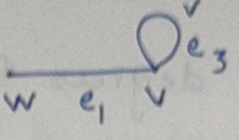
INTRODUCTION

11/08

Graph → set of vertices, edges & relations.

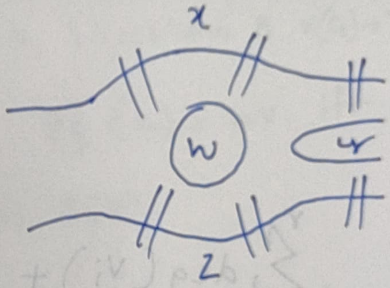
$G = (V, E, \psi)$

Loop → $\psi(e_3) = vv$

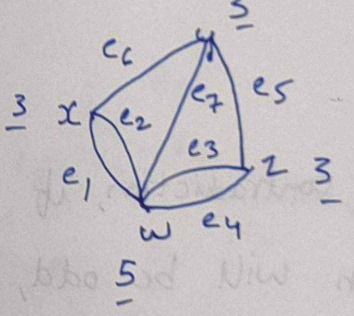


simple graph → no loops & no multiple edges

KONIGSBERG BRIDGE PROBLEM:



is a path possible such that, you has to leave & reach home such you cross every bridge exactly once?!



here, every vertex has odd degree, but there must be one edge that enters & one that leaves that vertex, so if degree is even it is possible.

↓

all odd vertices so not possible.

Degree of vertex (v): $v \in V$;

no. of edges incident at a vertex.

HANDSHAKING LEMMA:

→ for any finite graph $G = (V, E)$, the sum of degrees of all vertices is twice the no. of its edges.

$$\sum \deg(v) = 2|E|.$$

$\because$ each edge is incident to exactly 2 vertices.

> Sum of degrees of odd vertices in a finite graph is even.

Assume, vertices $\underbrace{v_1, v_2, \ldots\ldots}_{\text{odd degree}} \underbrace{v_r, \ldots\ldots v_n}_{\text{even degree}}$

$$\rightarrow \sum_{i=1}^{r} \deg(v_i) + \underbrace{\sum_{i=r+1}^{n} \deg(v_i)}_{\text{even}} = \underbrace{\sum_{i=1}^{n} \deg(v_i)}_{\text{even}}$$

by contradiction, if first term is odd, then odd + even will be odd, but result is even, so,

$$\sum_{i=1}^{r} \deg(v_i) \rightarrow \text{even}.$$

→ If G is simple graph, then $|E| \le {}^{n}C_2$,
$n \rightarrow$ no. of vertices.

planar → whose edges intersect only at their ends.

one vertex → trivial else non-trivial.

Sum of all edges (even) → Handshaking protocol.

Complete Graph: a simple graph with n-vertices and an edge b/n every two vertices.

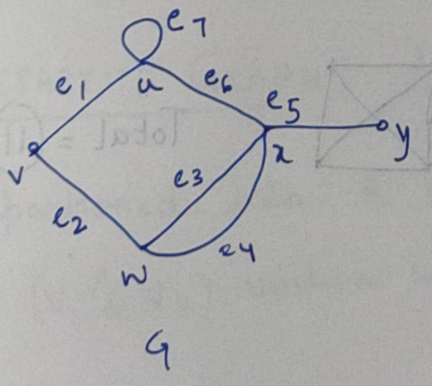
$deg(v) = n-1$ in K_n, $|E| = {}^nC_2 = \frac{n(n-1)}{2}$.

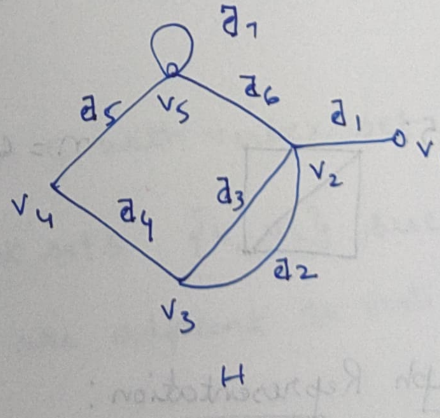
> Two graphs G & H are said to be identical if $V(G) = V(H)$ & $E(G) = E(H)$ & $\psi_G = \psi_H$

> Isomorphism: $G \cong H$ if there are bijections $\theta: V(G) \to V(H)$ and $\phi: E(G) \to E(H)$ such that $\psi_G(e) = uv$ iff

$\psi_H(\phi(e)) = \phi(u) \cdot \phi(v) = \theta(u) \cdot \theta(v)$

if there is an adjacency relation in G, then it has to be in H.





Mappings: $\theta(y) = v_1$; $\theta(x) = v_2$; $\theta(u) = v_5$; $\theta(v) = v_4$

$\theta(w) = v_3$

$\phi(e_1) = a_5$; $\phi(e_2) = a_4$; $\phi(e_3) = a_3$; $\phi(e_4) = a_2$; $\phi(e_5) = a_1$

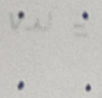
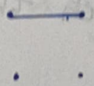
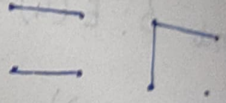
$\phi(e_6) = a_6$; $\phi(e_7) = a_7 = \theta(u) \cdot \theta(u) = v_5 \cdot v_5$

Δ If $G \cong H$, then two graphs same number of vertices, edges and same degree sequence [list all degrees in non-decreasing order] converse need not be true

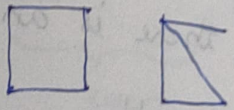
Δ degree of all vertices is $K \rightarrow K$- Regular Graph.

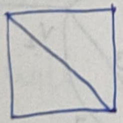
Δ Number of non-isomorphic graphs with 4 vertices

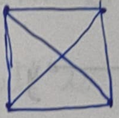
n (edges) for 4 vertices $= {}^4C_2 = 6$ So, max 6 edges

$n=0$:  $n=1$:  $n=2$: 

$n=3$: 

$n=4$: 

$n=5$: 

$n=6$: 

Total = 11

Graph Representation:

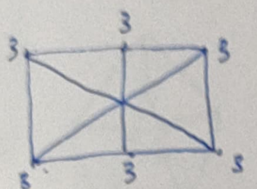
1) Picture of graph 2) list of vertices & edges

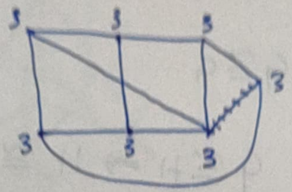
3) Adjacency Matrix : $|V| \times |V|$; $a_{vu} = 1$ if $uv \in E$
$= 0$, otherwise

4) Adjacency List

Δ Prove that two isomorphic graphs must have the same degree sequence:-

▷ How many non-isomorphic 3-regular graphs with 1) 6 vertices 2) 7 vertices.





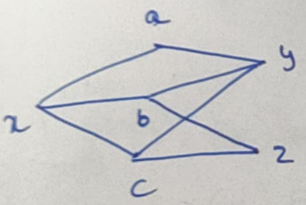
degree ↑

for vertices → $3 \times 7 = \boxed{21}$

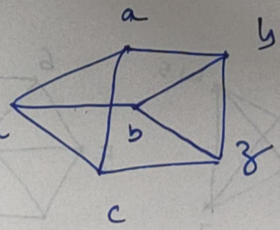
sum odd

$\boxed{0}$ (so not possible)

Case . I :



Case . II

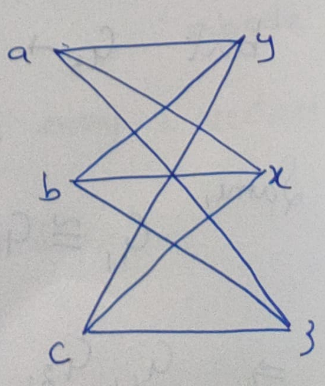


6 - vertices → 2

▷ BIPARTITE GRAPH : A graph G with vertex set V is partioned into 2 disjoint sets $\{V_1, V_2\}$ such that (V_1 & V_2) vertices in V_1 are adjacent to vertices in V_2.

Complete bipartite : every vertex in V_1 adj to in V_2.

$K_{m,n}$ → complete bipartite graph



K_n → complete Graph.

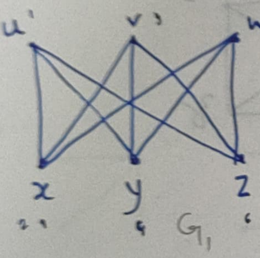
D An Isomorphic relation is as equivalence relation if

→ Reflexive : $G \cong G$

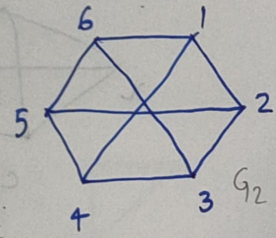
→ Symmetric : $G \cong H \rightarrow H \cong G$

→ Transitive : $G \cong H$ & $H \cong F \Rightarrow G \cong F$

Q: Determine which pairs of graphs below are isomorphic? Justify



$\cong$

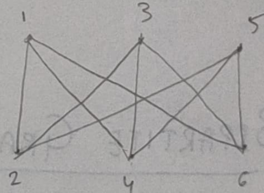


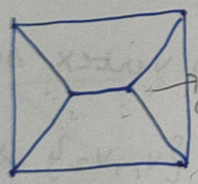
$\theta(u) = 1$ $\theta(y) = 4$

$\theta(x) = 2$ $\theta(w) = 5$

$\theta(v) = 3$ $\theta(z) = 6$

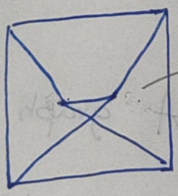
⇓





3 cycle

G_3 $\ncong$



→ no 3-cycle

G_4

$G_1 \rightarrow$ bi-partite graph $K_{3,3}$ → no odd cycle in bi-partite graph.

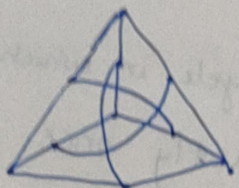
but $G_3 \rightarrow$ 3-cycle $\Rightarrow G_1 \ncong G_3$

Same,

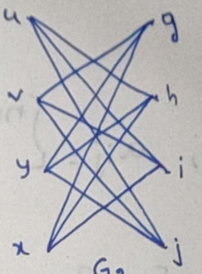
$G_1 \cong G_4$ (but G_3 not isomorphic)

$\Rightarrow G_1, G_2, G_4$ are isomorphic to each other.

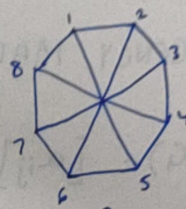
▷ Show that the following graphs are isomorphic.



G_1



G_2

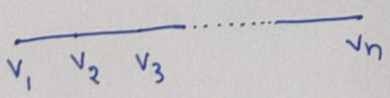


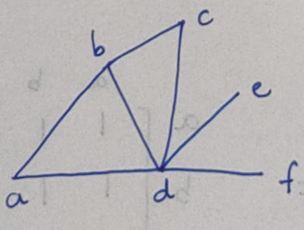
G_3

▷ A graph is bi-partite if & only if it has no odd cycle.

▷ <u>AUTOMORPHISM</u> : A graph G, isomorphic from G to itself.

▷ <u>WALK</u> : a sequence of vertices $v_1, v_2 \dots v_n$ such that v_j is adjacent to v_{j+1}, $\forall j$.





'abdbc'

▷ <u>TRAIL</u>: a walk with distinct edges. 'abdcb'

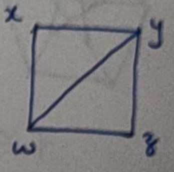
▷ <u>PATH</u>: a path is a walk with all distinct vertices.
- 'abcde'

▷ <u>Connected Graph</u>: There is a path b/n every 2 vertices.

▷ <u>Cycle</u>: a path with starting point & ending point same.

▷ EULAR TRAIL: a trail in which every edge of the graph appears exactly once.

▷ Hamilton Cycle: a cycle in which every vertex of the graph appears exactly once.



xyzw

▷ ADJACENCY MATRIX:

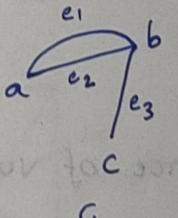
$$A(G) = [a_{ij}]_{n\times n} \;;\; a_{ij} = \begin{cases} \text{no. of edges with } \{v_i, v_j\} \end{cases}$$

Incident matrix $M(G) = [m_{ij}]_{n\times m}$; $m_{ij} = \begin{cases} 1 \;; & v_i \text{ is endpoint of } e_j \\ 0 \;; & \text{else} \end{cases}$

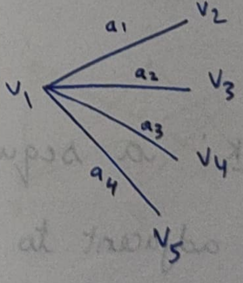
edges → m

vertices → n

$$A(G_2) = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$



G_2



G_1

$$M(G_2) = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Sum of columns of A(G) is degree of resp. vertex.

▷ If G & H are isomorphic, then there adjacency matrices have the same eign values. [converse need not be true].

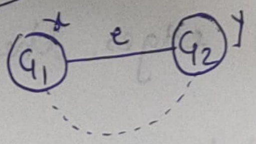
PROOF : If G is a tree, then $|E| = |V| - 1$

① If G is a tree : → G is ascyclic & for every pair of vertices (x, y) ∃ only one path.

ASSUME TRUE FOR $n-1$,

Suppose,
we split the graph G into 2 components G_1 & G_2 by removing an edge

Suppose, the edge doesn't divide the graph, then it means that there is an other edge b/n G_1 & G_2, which implies there's cycle b/n G_1 & G_2, which is not possible, So, it is cut-edge. (ascyclic)



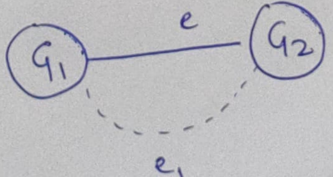
i.e., all edges are cut-edge

$G + e \to$ cycle

Now, G_1 & G_2 have vertices less than n, So as per our assumption, since it is true for $n-1$, It is also true for n.

PROOF : An edge e of G is a cut edge iff e is contained in no cycle of G.

Assume, that an edge e of G is in a cycle of G.



suppose, here if we remove the edge, it

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doesn't ~~remove~~ divide the graph G into two components because, being cycle, there is an other edge. So, it is not cut-edge.

↳ CONTRADICTION

So, that edge should not be in a cycle.

[Here, G is connected graph]

<u>PROOF</u>: If G is connected with n-vertices and n-1 edges, then G is tree.

By induction on vertices:

① Let $|V| = n$,

$n = 0$ (trivial case) $n = 1$, 1 vertex 0 edges True

$n = 2$ ——— true

② by hypothesis for $|V| = n-1$, the graph having $|V|-1$ edges is true → Assume

G: connected

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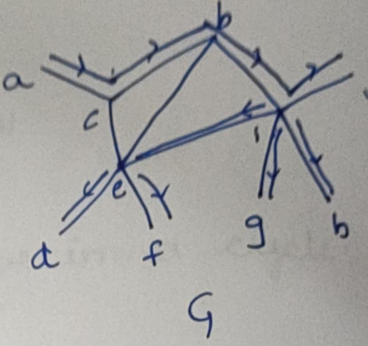
SPANNING TREE: If $T \subseteq G$ is a tree and $V(T) = V(G)$ then we called T is a spanning tree.

Edges of S.T = {ac, cb, bi, ie, cd, ef, ig, ih, ij}



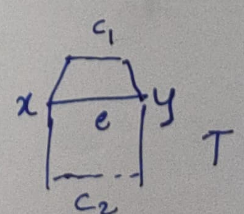
◇ Let T be a spanning tree of a connected graph G & e be an edge of G not in T, then $T+e$ contains a unique cycle.

Proof: Given that, T is a spanning tree & e is an edge of G. Add an edge e to ~~G~~ T.

Assume, there are 2 cycles C_1, C_2 in T containing e, then, it violates the tree property [unique path]

↳ Contradiction, So there must be a unique cycle.

here 2 cycles are formed.



◇ Prove by induction that every tree is bipartite graph.

FOREST : A graph with no cycles.

⇒ If G is a forest, then components of G $comp(G)$ is same as no. of vertices − no. of edges.

$$comp(G) = |V(G)| - |E(G)|$$

Proof: in G, every component is a tree, so every edge is a cut edge, which means, after removing an edge, one component is increased.

$$comp(G-e) = comp(G) + 1$$

So, no. of edges in $G-e$ +1 is number of edges in G.

$$|E(G-e)|+1 = |E(G)|$$

$$\text{comp}(G-e) = \text{comp}(G)+1$$

$$-|E(G-e)| + |V(G-e)| - 1 = \text{comp}(G)$$

(no. of vertices are same in G, $G-e$)

So $|V(G-e)| = |V(G)|$

$$\Rightarrow |V(G)| - |E(G)| = \text{comp}(G)$$

Hence Proved.

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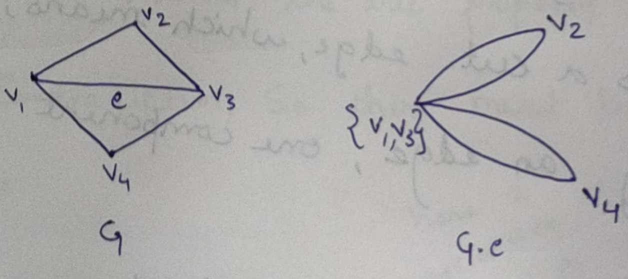
▷ CONTRACTED EDGE : an edge e of G is said to be contracted, if it is deleted & its ends are identified. Then, the resultant graph is denoted by $G.e$

ex



If e is a link of G, then, $|V(G.e)| = |V(G)| - 1$

$|E(G.e)| = |E(G)| - 1$; $\text{comp}(G.e) = \text{comp}(G)$.

(an edge with different end points → link)

Suppose, $\tau(G)$ denotes no. of spanning trees of G.

Theorem: If e is a link of G then,

$$\tau(G) = \tau(G-e) + \tau(G \cdot e)$$

proof: G - graph; e - edge of G, $T_1 \rightarrow$ spanning tree of G not containing edge 'e'.

$\Rightarrow T_1$ also a spanning tree of $(G-e)$

$$\tau(G-e) = \tau(G) \text{ where e is not}$$

contained in the spanning tree.

T_2 - spanning tree of G that contains e

$\Rightarrow$ there is a corresponding spanning tree $T \cdot e$ in $G \cdot e$.

$\Rightarrow$ This correspondence is bijection

i.e., $\tau(G \cdot e)$ is the no. of spanning trees contains 'e'.

$$\Rightarrow \tau(G) = \tau(G-e) + \tau(G \cdot e)$$

Theorem: $\tau(K_n) = n^{n-2}$ $\quad$ $K \rightarrow$ complete graph

proof: $N = \{1, 2, 3, \ldots n\}$

assume, that n^{n-2} sequences of length $n-2$ are formed from N.

It is sufficient to establish a one-one correspondence b/n the set of sequences of length (n-2) and $\tau(K_n)$.

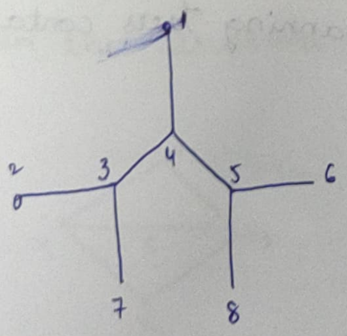
Suppose, $(t_1, t_2 \cdots t_{n-2})$ is a unique sequence associated with a spanning tree T of K_n.

Consider, N is ordered sequence:

1) $s_1 \rightarrow$ 1st vertex of degree 1 in T

2) t_1 - adjacent to s_1 3) delete s_1 from T, then denote by s_2, the 1st vertex of degree 1 in $T - s_1$

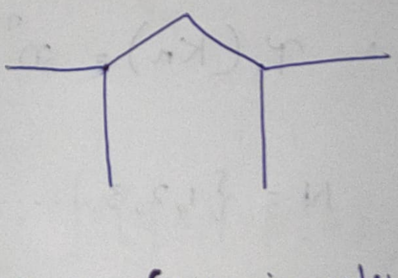
4) take vertex t_2 - adjacent to s_2 5) Repeat from step (2) until t_{n-2} has been defined. and a tree with 2 vertices remain.



s : 1 (1st vertex with degree 1)

t : 4

now remove it from graph



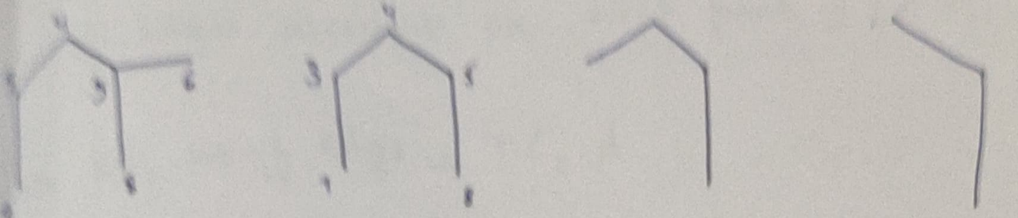
one spanning tree of K_8

(1, 2, 3, 4, 5, 6, 7, 8)

$\underbrace{(1, 1, 3, 3, 3, 1, 1, 1)}_{\text{deg. Seq}}$

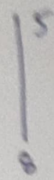
S : 2 (take in order of degree)

t : 3 (now remove this edge)



S: 6 S: 7 S: 3 S: 4

t: 5 t: 8 t: 4 t: 5



S : 1 2 6 7 3 4 [at 2

t : 4 3 5 3 4 5 vertices, stop]

[here after deleting every vertex, you need to update deg. seq, then check for least degree and go on]

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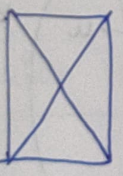
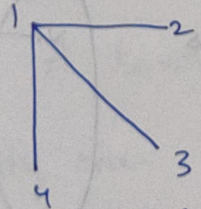
▷ Draw all spanning trees of K_4 by mentioning the sequence generated in caylès theorem.



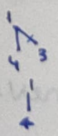
K_4



(1, 2, 3, 4)

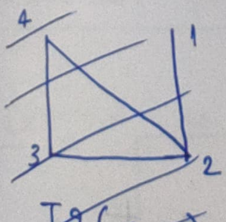
T1: 3, 1, 1, 1

T1 (1, 1) (1, 1)

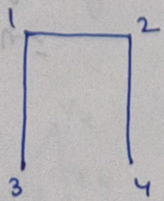


(?, ?, ?)

(1, 1)



~~T2~~ (~~2~~, 3)



T3 (1, 2)

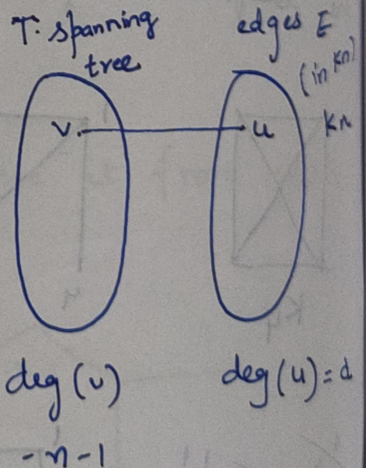
△ Show that, if e is an edge of K_n, then the no. of $\tau(K_n - e) = (n-2) \cdot n^{n-3}$

Proof: $\tau(K_n) = n^{n-2}$

→ T is spanning tree → no. of edges $(n-1)$

1) establish a relation as:

draw an edge b/n 2 vertices whenever a tree contains edge



• tree having $(n-1)$ edges, so the degree of every tree vertice is $n-1$.

• every edge in K_n belongs to same no. of trees say d.

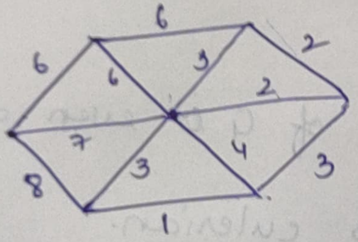
no. of edges incident on first part = $(n-1)\cdot n^{n-2}$

for K_n, no. of edges = ${}^{n}C_2 \cdot d$

$$d = \frac{(n-1)\cdot n^{n-2}}{\frac{(\cancel{n}-1)(n-2)}{2}} = \frac{2\cdot \cancel{n}^{n-2}}{\cancel{(n}-2)} = 2n^{n-3}$$

$$\tau(K_n - e) = n^{n-2} - d = n^{n-3}(n-2)$$

MINIMAL SPANNING TREE :



$1-3-2-2-6-6$

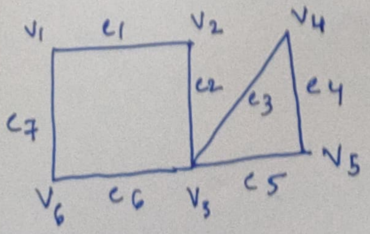
(Kruskals)

02/9

Euler Tour: A tour which traverse every edge exactly one.

Eulerian Graph: if graph contains euler tour.

Ex:



$v_1 \to v_2 \to v_3 \to v_4 \to v_5 \to v_3 \to v_6 \to v_1$

$e_1 \quad e_2 \quad e_3 \quad e_4 \quad e_5 \quad e_6 \quad e_7$

(odd degree vertices × N.P)

Theorem: A connected graph G is eulerian iff all the vertices of G are are of degree even.

proof: Assume, G is euler graph. $\Rightarrow \exists$ euler tour T

$$u v_1 v_2 v_3 \ldots\ldots v_n u \quad (T)$$

$\Rightarrow$ degree of all internal vertices,

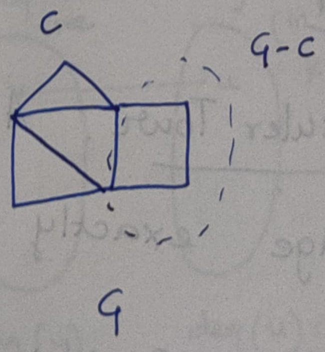
$$\deg(v_i) = 2 \text{ for } i = 1 \ldots n \text{ (at a time)}$$

$\rightarrow \deg(v) \geq 2 \quad \forall v \in V(G) - \{u\}$

$\therefore \deg(v)$ is even

II. Assume every vertex of G is even degree, we have to prove G is eulerian.

Assume G is not eulerian



<u>Theorem</u>: A connected graph has euler trail iff it has atmost 2 vertices of odd degree.

— (H.W) —

▷ Which of the following figures can be drawn without lifting pen from paper more than once.