HOMEWORK -2.

(1) GIVEN: $f = 2x_1^2 - 4x_1x_2 + 1.5x_2^2 + x_2$

$\longrightarrow$ $f_{x_1} = 4x_1 - 4x_2$, $f_{x_2} = -4x_1 + 3x_2 + 1$, $f_{x_1x_1} = 4$, $f_{x_2x_2} = 3$, $f_{x_1x_2} = -4$

$\therefore$ $4x_1 - 4x_2 = 0$

$\Rightarrow \boxed{x_1 = x_2}$ -(i)

Considering f_{x_2},

$-4x_1 + 3x_2 + 1 = 0$

from (i),

$-4x_1 + 3x_1 + 1 = 0$

$-x_1 = -1$

$\Rightarrow \boxed{x_1 = x_2 = 1}$

Verification of saddle point.

$D = f_{x_1x_1} \cdot f_{x_2x_2} - [f_{x_1x_2}]^2$

$D = 12 - 16$

$\boxed{D = -4}$

$\therefore$ The saddle point is $[1, 1] \Rightarrow (S_0 = [1, 1])$

$\therefore$ Gradient is given by:

- Gradient for $f = 2x_1^2 - 4x_1x_2 + 1.5x_2^2 + x_2$

$$g = \begin{bmatrix} 4x_1 - 4x_2 \\ -4x_1 + 3x_2 + 1 \end{bmatrix} \Rightarrow g(S_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore H = \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}, \quad f(S_0) = 0.5$$

* To get Eigen values:

$$|H - I\lambda| = \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$|H - I\lambda| = \begin{bmatrix} 4-\lambda & -4 \\ -4 & 3-\lambda \end{bmatrix}$$

$$= (\lambda^2 - 7\lambda + 12) - 16$$

$$= \lambda^2 - 7\lambda - 4$$

$$\Rightarrow \boxed{\lambda = \frac{7 \pm \sqrt{65}}{2}} \quad \text{[Eigen values]}$$

$\therefore$ Eigen vector $= \begin{bmatrix} 4 - \frac{7+\sqrt{65}}{2} & -4 \\ -4 & 3 - \frac{7+\sqrt{65}}{2} \end{bmatrix}$ for $\lambda = \frac{7+\sqrt{65}}{2}$.

$$V_1 = \begin{bmatrix} -\left(\frac{1+\sqrt{65}}{8}\right) \\ 1 \end{bmatrix}, \quad V_2 = \begin{bmatrix} \left(\frac{-1+\sqrt{65}}{8}\right) \\ 1 \end{bmatrix}$$

As the eigen values are of same magnitude but opposite in direction, hence, It is an Indefinite Hessian function.

Applying taylor's equation at [1,1], we have,

$$g_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad H_0 = \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}, \quad f_0 = 0.5$$

$$f(x) = f_0 + g_0^T(x - x_0) + \frac{1}{2}(x - x_0)^T H (x - x_0).$$

but, After substituting g_0, H_0 & f_0, we have.

$$\frac{1}{2}[x_1, x_2]\begin{bmatrix}4 & -4\\ -4 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = f(x) - 0.5 < 0$$

$$\Rightarrow \frac{1}{2}\left(4x_1^2 - 8x_1x_2 + 3x_2^2\right) < 0$$

$\therefore$ $2x_1 - x_2 < 0$ and $2x_2 - 3x_2 > 0$

OR

$2x_1 - x_2 > 0$ and $2x_2 - 3x_2 < 0$.

(2) GIVEN: Plane $x_1 + 2x_2 + 3x_3 = 1$, Point on the plane is $(-1,0,1)^T$

→ ∴ The distance between the point on the plane (x_1, x_2, x_3) and $(-1,0,1)$ is

$$= \sqrt{(x_1+1)^2 + x_2^2 + (x_3-1)^2}$$

In this case, we know that $\min(f(x)) \Rightarrow \min(\sqrt{f(x)})$.

Hence, we can minimize $f(x)$, which is

$$\min\left[(x_1+1)^2 + x_2^2 + (x_3-1)^2\right] \text{ —(i)}$$

Subject to: $x_1 + 2x_2 + 3x_3 = 1$ (plane equation) —(ii)

$\Rightarrow x_1 = 1 - 2x_2 - 3x_3$ —(ii)

from (i) & (ii),

$$\min\left[(1 - 2x_2 - 3x_3 + 1)^2 + x_2^2 + (x_3-1)^2\right]$$

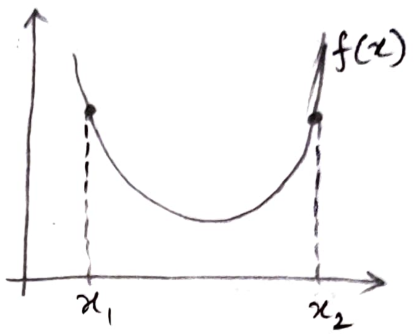
$$\Rightarrow f(x,y) = (2 - 2x - 3y)^2 + x^2 + (y-1)^2.$$

Gradient $g = \begin{bmatrix} 10x + 12y - 8 \\ 20y + 12x - 14 \end{bmatrix}$

Hessian $H = \begin{bmatrix} 10 & 12 \\ 12 & 20 \end{bmatrix}$

∴ All the quantities in the Hessian matrix are positive, hence we can say that, $f(x,y)$ is a convex function.

3(a) for any points x_1 & x_2 which belong to $\mathcal{X}$, The Convex function is based on Convex Set



$$\lambda x_1 + (1-\lambda)x_2$$

function $f: X \to R$ is a Convex function, If '$\mathcal{X}$' is a Convex Set and 'λ' belongs to $[0,1]$

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

Similarly,

$$g(\lambda x_1 + (1-\lambda)x_2) \leq \lambda g(x_1) + (1-\lambda) g(x_2)$$

$$af(\lambda x_1 + (1-\lambda)x_2) + bg(\lambda x_1 + (1-\lambda)x_2) \leq \lambda(af(x_1) + bg(x_1)) + (1-\lambda)(af(x_2) + bg(x_2))$$

$\therefore$ As the above function is a Convex function, As a result, $af(x) + bg(x)$ is Convex, For all values of $a, b > 0$.

3(b). For $f(g(x))$ to be convex,

Let $h(x) = f(g(x))$ $\quad$ $\text{dom } h = \{x \in \text{dom } g \mid g(x) \in \text{dom } f\}$

Hence the second derivative of $h = f \circ g$

$$h''(x) = f''(g(x))\, g'(x)^2 + f'(g(x))\, g''(x)$$

Now if g is convex $(g'' > 0)$

f is convex and increasing $(f'' \geq 0 \;\&\; f' \geq 0)$

$h'' \geq 0$.

Hence, 'h' is convex.

Similarly,

'h' is convex when,

- 'f' is convex and increasing and 'g' is convex
- 'f' is convex and increasing and 'g' is concave

(4)

$$F(x_1) \geq F(x_0) + g_{x_0}^{T}(x_1 - x_0)$$

For convex function $F(x): \mathcal{X} \to R$ and $x_0, x_1 \in x$.

$$F(\lambda y + (1-\lambda)x) \leq \lambda f(y) + (1-\lambda)f(x), \ \forall \lambda \in [0,1]$$

$$F(x + \lambda(y-x) \leq (1-\lambda)f(x) + y$$

(i) 'F' is Convex.

(ii) $f(y) \geq f(x) \nabla f(x)^{T}(y-x)$. [First order taylor expansion]

(iii) $\nabla^2 f(x) \geq 0$.

$$f(x + \lambda(y-x) \leq f(x) + \lambda(f(y) - f(x))$$

$$f(y) - f(x) \geq \frac{f(x+\lambda(y-x)) - f(x)}{\lambda}, \ \forall \lambda \in [0,1]$$

$$f(y) \geq f(x) + \frac{f(x+\lambda(y-x)) - f(x)}{\lambda}$$

$$\lim_{t \to 0} \frac{f(x+\lambda(y-x)) - f(x)}{\lambda} = f'(x)(y-x)$$

$$f(y) \geq f(x) + f'(x)(y-x)$$

$$w = \lambda x + (1-\lambda)y \ \forall \lambda \in [0,1]$$

$$f(x) \geq f(w) + f'(w)(x-w) \Rightarrow 2f(x) + (1-R)f(y) \geq f(w).$$

$$f(y) \geq f(w) + f'(w)(y-w)$$

$$g(t) = f(\alpha y + (1-\alpha)x)$$

$$g'(t) = \nabla f(\alpha y + (1-\alpha)x)^T (y-x)$$

As 'f' is convex which implies 'g' is convex,

$$g(1) \geq g(0) + g'(0)$$

$$f(y) \geq f(x) + \nabla f(x)^T (y-x)$$

$$\boxed{f(x_1) \geq f(x_0) + g_{x_0}^T (x_1 - x_0)}$$

5 (a) We know that, we should minimize the error between I and $a_k^T P$, we should be tweak the values of 'P'. The unconstraint optimization problem is.

$$\min_P \sum_{k=1}^{m} (a_k^T P_i - I)^2$$

$$0 \leq P_i < P_{max} \quad \forall i = 1, \ldots\ldots n.$$

Thus, $f(P) = (a^T P - I)^2$.

$$g = 2(a^T P - I)a.$$

$$H = 2aa^T \geq 0.$$

5(b). We know that,

if $d^T H d \geq 0$.

Here $d \neq 0$.

This means that H is positive semi definite

$\Rightarrow d^T H d = 2 d^T [a_k][a_k]^T d$

$= 2 a_k^2 \geq 0$

Hence, Hessian is positive semi definite.

$\therefore$ The function is convex.