P1) $f(x,y) = (x+y-1)^4 + (x-y)^2$

$x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$H_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\alpha = 1$

First update step (k=0)

Gradient at x_0:

$$\nabla f(x_0) = \begin{bmatrix} 4(0+0-1)^3 + 2(0-0) \\ 4(0+0-1)^3 - 2(0-0) \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$$

calculate x_1:

$$x_1 = x_0 - \alpha H_0 \nabla f(x_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -4 \\ -4 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$s_0 = x_1 - x_0 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$\nabla f(x_1) = \begin{bmatrix} 4(4+4-1)^3 + 2(4-4) \\ 4(4+4-1)^3 - 2(4-4) \end{bmatrix} = \begin{bmatrix} 1372 \\ 1372 \end{bmatrix}$$

$$y_0 = \nabla f(x_1) - \nabla f(x_0) = \begin{bmatrix} 1372 \\ 1372 \end{bmatrix} - \begin{bmatrix} -4 \\ -4 \end{bmatrix} = \begin{bmatrix} 1376 \\ 1376 \end{bmatrix}$$

H_1 (BFGS & DFP are identical)

$$H_1 = \frac{1}{688} \begin{bmatrix} 345 & -343 \\ -343 & 345 \end{bmatrix}$$

second update (k=1)

calculate x_2:

$$x_2 = x_1 - \alpha H_1 \nabla f(x_1) = \begin{bmatrix} 4 \\ 4 \end{bmatrix} - \frac{1}{688} \begin{bmatrix} 345 & -343 \\ -343 & 345 \end{bmatrix} \begin{bmatrix} 1372 \\ 1372 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$s_1 = x_2 - x_1 = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$$

$$\nabla f(x_2) = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$$

$$y_1 = \nabla f(x_2) - \nabla f(x_1) = \begin{bmatrix} -4 \\ -4 \end{bmatrix} - \begin{bmatrix} 1372 \\ 1372 \end{bmatrix} = \begin{bmatrix} -1376 \\ -1376 \end{bmatrix}$$

i) BFGS Final Inverse Hessian (H_2^{BFGS}):

$$H_2^{BFGS} = \left(I - \frac{s_1 y_1^T}{y_1^T s_1}\right) H_1 \left(I - \frac{y_1 s_1^T}{y_1^T s_1}\right) + \frac{s_1 s_1^T}{y_1^T s_1} = \frac{1}{688}\begin{bmatrix} 689 & -687 \\ -687 & 689 \end{bmatrix}$$

DFP Final Inverse Hessian (H_2^{DFP}):

$$H_2^{DFP} = H_1 + \frac{s_1 s_1^T}{s_1^T y_1} - \frac{H_1 y_1 y_1^T H_1}{y_1^T H_1 y_1} = \frac{1}{688}\begin{bmatrix} 395 & -393 \\ -393 & 395 \end{bmatrix}$$

True Inverse Hessian at x_2:

The Hessian at $x_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is $J(0,0) = \begin{bmatrix} 19 & 10 \\ 10 & 19 \end{bmatrix}$

$$J^{-1}(0,0) = \frac{1}{96}\begin{bmatrix} 14 & -10 \\ -10 & 19 \end{bmatrix} = \begin{bmatrix} 0.1458 & -0.1042 \\ -0.1042 & 0.1458 \end{bmatrix}$$

H_2^{DFP} is significantly closer than th BFGS.