1) a) krichoff's law

$V_S = V_R + V_C$

$V_C = V_S - V_R$

$V_C = V_S - I(t) R$

$V_C = V_S - RC\frac{dv}{dt}$

$y(t) + RC\frac{d\,y(t)}{dt} = x(t)$

$I(t) = \frac{dq}{dt}$

$= \frac{d(cV_c(t))}{dt}$

$= c\frac{d\,V_c(t)}{dt}$

b) ~~$x(t) = v(t)$~~

~~By KVL~~

~~$V_S = V_R + V_C$~~

~~$V = IR$,, $Q = CV$~~

~~$V_S(I) = IR + C\frac{dv}{dt}$~~

~~$\frac{dq}{dt} = C\frac{dV_s(t)}{dt}$~~

~~$i(t) = C\frac{dV_c(t)}{dt}$~~

b) $V_S(t) = V_C(t) + i(t) R$

$x(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)\,dt + i(t) R$

$i(t) = \frac{dq(t)}{dt} = C\frac{dV_c}{dt}$

$\frac{d}{dt}\left(\int_{g(x)}^{h(x)} F(t)\,dt\right) = F(h(x))h'(x) - F(g(x))g'(x)$

$i(t)\,dt = c\,dV_c$

$\frac{1}{C}\int_{-\infty}^{t} i(t)\,dt = \int_{0}^{V_c(t)} dV_c$

$V_c(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)\,dt\ (-1)$

7 a) $i(t) = \frac{dq(t)}{dt} = \frac{d}{dt}(Cv_c(t)) = C\frac{dv_c(t)}{dt}$

$x(t) = C\frac{dy(t)}{dt}$

$= \frac{1}{C}\int x(t)\,dt = \int dy(t)$

$\frac{1}{C}\int_{-\infty}^{t} x(t)\,dt = y(t)$

b) (i) Memoryless: The system is not memoryless because it depends on previous values of t.

(ii) Causal: The system is causal as it only depends on input at time less than or equal to t.

(iii) Linear:

Additivity:

$y_1(t) = \frac{1}{C}\int_{-\infty}^{t} x_1(\tau)\,d\tau$

$y_2(t) = \frac{1}{C}\int_{-\infty}^{t} x_2(\tau)\,d\tau$

for $x_1(t) + x_2(t)$

$y_{add}(t) = \frac{1}{C}\int_{-\infty}^{t}[x_1(\tau)+x_2(\tau)]\,d\tau$

$= \frac{1}{C}\int_{-\infty}^{t} x_1(\tau)\,d\tau + \frac{1}{C}\int_{-\infty}^{t} x_2(\tau)\,d\tau = y_1(t) + y_2(t)$

Homogeneity:

$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau$

$y_{hom}(t) = \frac{1}{C}\int_{-\infty}^{t}[a x(\tau)]\,d\tau = a\left[\frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau\right] = a y(t)$

$\Rightarrow$ system is Linear

iv) Time-Invariant:

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau$$

$$x_d(t) = x(t-T)$$

$$y_d(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau - T)\,d\tau \qquad \text{let } \sigma = \tau - T$$

$$= \frac{1}{C}\int_{-\infty}^{t-T} x(\sigma)\,d\sigma$$

$\Rightarrow y(t-T)$

$$y(t-T) = \frac{1}{C}\int_{-\infty}^{t-T} x(\tau)\,d\tau.$$

$$y_d(t) = y(t-T)$$

so system is time-invariant

v) stable:

Let Input is bounded $|x(t)| \leq M$

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau.$$

the output is $y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau$

let $x(t) = A \qquad t \geq 0$

$x(t) = 0 \qquad t < 0$

$$y(t) = \frac{1}{C}\int_{0}^{t} A\,d\tau = \frac{A}{C}\,t.$$

3) $y(t) = x(t)\cos(\omega_0 t)$

a) Memoryless:

The system is memoryless as $y(t)$ depends on only the current time t

b) Causal:

The system is causal as it depends only on current time.

c) Linear:

Additivity:

$y_{add}(t) = (x_1(t) + x_2(t))\cos(\omega_0 t) = x_1(t)\cos(\omega_0 t) + x_2(t)\cos(\omega_0 t)$

$= y_1(t) + y_2(t)$

Homogenity: $ax(t)$

$y_{hom}(t) = (ax(t))\cos(\omega_0 t) = a(x(t)\cos(\omega_0 t)) = ay(t)$

the system is Linear

d) Time Invariant

- shift input by t_0

$y_0(t) = x(t-t_0)\cos(\omega_0 t) \rightarrow ①$

$y(t-t_0) = x(t-t_0)\cos(\omega_0(t-t_0)) - ②$

as both are not same it is not time-invariant

e) stable:

consider bounded Input $x(t)$

we know $\cos(\omega_0 t)$ is bounded $[-1, 1]$

$y(t) = x(t)\cos(\omega_0 t)$ is bounded

$\therefore$ system is stable.

1) a) $x(t) = \cos\left(t + \frac{\pi}{4}\right)$

$T_0 = \frac{2\pi}{\omega} = \frac{2\pi}{1} = 2\pi$

periodic
$T_0 = 2\pi$

(b) $x(t) = \sin\left(\frac{2\pi}{3} t\right)$

$T_0 = \frac{2\pi}{\omega} = \frac{2\pi}{\left(\frac{2\pi}{3}\right)} = 3$

Periodic
$T_0 = 3$

(c) $x(t) = \cos\left(\frac{\pi}{3} t\right) + \sin\left(\frac{\pi}{4} t\right)$

$\omega_1 = \frac{\pi}{3}$ $\quad T_1 = \frac{2\pi}{\pi/3} = 6$

$\omega_2 = \frac{\pi}{4} \Rightarrow T_2 = \frac{2\pi}{\frac{\pi}{4}} = 8$

$T_0 = LCM(T_1, T_2) = 24$

periodic
$T_0 = 24$

(d) $x(t) = \cos(t) + \sin(\sqrt{2} t)$

$\omega_1 = 1 \Rightarrow T_1 = 2\pi$

$\omega_2 = \sqrt{2} = T_2 = \frac{2\pi}{\sqrt{2}} = \pi\sqrt{2}$

$\frac{T_1}{T_2} = \frac{2\pi}{\pi/\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

Not periodic

(e) $x(t) = \sin^2(t)$

$\sin^2(t) = \frac{1 - \cos(2t)}{2}$

$x(t) = \frac{1}{2} - \frac{1}{2}\cos(2t)$

$T_0 = \frac{2\pi}{\omega} = \pi$

periodic
$T_0 = \pi$

f) $x(t) = e^{j\left(\frac{\pi}{2}t - 1\right)}$

$x(t) = e^{-j} \cdot e^{j\left(\frac{\pi}{2}\right)t}$

$\omega_0 = \frac{\pi}{2}$

$T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{\pi/2} = 4$

periodic

$T_0 = 4$

(g) $x[n] = e^{j\pi n/4}$

$\Omega = \frac{\pi}{4}$

$\frac{\Omega}{2\pi} = \frac{\pi/4}{2\pi} = \frac{1}{8}$

$N_0 = 8$

periodic

$N_0 = 8$

(h) $x[n] = \cos\frac{n}{4}$

$\Omega = \frac{1}{4}$

$\frac{\Omega}{2\pi} = \frac{1/4}{2\pi} = \frac{1}{8\pi}$

since irrational

$\therefore$ Not periodic

(i) $x[n] = \cos\left(\frac{\pi}{3}n\right) + \sin\left(\frac{\pi}{4}n\right)$

$\Omega_1 = \frac{\pi}{3} \Rightarrow \frac{\Omega_1}{2\pi} = \frac{\pi}{3(2\pi)} = \frac{1}{6}$ $\boxed{N_1 = 6}$

$\Omega_2 = \frac{\pi}{4} \Rightarrow \frac{\Omega_2}{2\pi} = \frac{\pi/4}{2\pi} = \frac{1}{8}$ $\boxed{N_2 = 8}$

$N_0 = LCM(6, 8) = 24$

Periodic

$N_0 = 24$

(i) $x[n] = \cos^2\left(\frac{\pi}{8}n\right)$

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$$

$$x[n] = \frac{1}{2} + \frac{1}{2}\cos\left(2\cdot\frac{\pi}{8}n\right) = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi}{4}n\right)$$

$$\Omega = \frac{\pi}{4}$$

$$\frac{\Omega}{2\pi} = \frac{\pi/4}{2\pi} = \frac{1}{8}$$

$N_0 = 8$

$\therefore$ Periodic

$N_0 = 8$

5) $\Rightarrow \int_{\alpha}^{\beta} x(t)\,dt = \int_{\alpha+T}^{\beta+T} x(t)\,dt$

$$I_1 = \int_{\alpha+T}^{\beta+T} x(t)\,dt.$$

$\tau = t - T$

$t = \tau + T$. $\quad dt = d\tau$

when $t = \alpha + T \quad \tau = (\alpha+T) - T = \alpha$

when $t = \beta + T \quad \tau = (\beta+T) - T = \beta$

$$I_1 = \int_{\alpha}^{\beta} x(\tau+T)\,d\tau$$

$x(t+T) = x(t)$

$\Rightarrow x(\tau+T) = x(\tau)$

$$I_1 = \int_{\alpha}^{\beta} x(\tau)\,d\tau$$

$$\boxed{I_1 = \int_{\alpha}^{\beta} x(t)\,dt}$$

$\rightarrow \int_0^T x(t)dt = \int_a^{a+T} x(t)dt$

$$= \int_a^T x(t)dt + \int_T^{a+T} x(t)dt \rightarrow (1)$$

$\int_T^{a+T} x(t)dt$

$\tau = t - T$

$t = \tau + T \quad dt = d\tau$

$t = T, \ \tau = T - T = 0$

$t = a + T, \ \tau = (a+T) - T = a$

$\int_T^{a+T} x(t)dt = \int_0^a x(\tau + T)d\tau$

$\therefore x(\tau + T) = x(\tau)$

$\Rightarrow \int_T^{a+T} x(t)dt = \int_0^a x(\tau)d\tau = \int_0^a x(t)dt$

sub in (1)

$\int_a^{a+T} x(t)dt = \int_a^T x(t)dt + \int_0^a x(t)dt$

$\int_a^T x(t)dt + \int_0^a x(t)dt = \int_0^T x(t)dt$

$\int_a^{a+T} x(t)dt = \int_0^T x(t)dt$

6) $P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$

$x(t+T_0) = x(t)$

let $T = NT_0$

$$\int_{-\frac{NT_0}{2}}^{NT_0/2} |x(t)|^2 dt = \sum_{K=-N/2}^{N/2-1} \int_{KT_0}^{(K+1)T_0} |x(t)|^2 dt$$

By periodicity each period Integral is same

$$\int_{-\frac{NT_0}{2}}^{NT_0/2} |x(t)|^2 dt = N \int_0^{T_0} |x(t)|^2 dt$$

for T,

$$\frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \frac{1}{NT_0} N \int_0^{T_0} |x(t)|^2 dt = \frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt$$

for general T write $T = NT_0 + r$ with integer N

$0 \le r < T_0$

$$\int_{-\frac{T}{2}}^{T/2} |x(t)|^2 dt = N \int_0^{T_0} |x(t)|^2 dt + R(T)$$

$|R(T)| \le r \cdot M$ where $M = \max_{t \in R} |x(t)|^2$

$$\lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{T/2} |x(t)|^2 dt = \lim_{T \to \infty} \frac{N}{NT_0 + r} \int_0^{T_0} |x(t)|^2 dt + \lim_{T \to \infty} \frac{R(T)}{T}$$

$\frac{N}{NT_0 + r} \to \frac{1}{T_0}$ $\quad \frac{|R(T)|}{T} \le \frac{rM}{T} \to 0$

$$P = \frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt$$

$$\int_{a}^{a+T_0} |x(t)|^2 dt = \int_{0}^{T_0} |x(t+a)|^2 dt = \int_{0}^{T_0} |x(u)|^2 du$$

so rHS do not depend on choice of Interval

$$\text{so,} \quad P = \frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt$$

~~is equal~~

is equal to average power