Discrete Fourier Transforms (DFT)
Assignment 9

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Abstract

This report will discuss about finding DFT Discrete fourier transforms for periodic and Gaussian signals and using of fft which is used to find DFT of signal and fftshift which is used for centering phase response. Also analyse how to find DFT for Non Bandlimited frequencies Signals like gaussian.

1 Introduction

- We analyse and use the infamous DFT to find the Fourier transform of periodic signals and non periodic ones using fft and fftshift

```
In [1]: import writefile_run as writefile_run
```

```
In [2]: %%writefile_run ee16b031_assignment9.py
        # load libraries and set plot parameters
        %matplotlib inline
        from pylab import *

        from IPython.display import set_matplotlib_formats
        set_matplotlib_formats('pdf', 'png')
        plt.rcParams['savefig.dpi'] = 75

        plt.rcParams['figure.autolayout'] = False
        plt.rcParams['figure.figsize'] = 12,9
        plt.rcParams['axes.labelsize'] = 18
        plt.rcParams['axes.titlesize'] = 20
        plt.rcParams['font.size'] = 16
        plt.rcParams['lines.linewidth'] = 2.0
        plt.rcParams['lines.markersize'] = 6
        plt.rcParams['legend.fontsize'] = 18
        plt.rcParams['legend.numpoints'] = 2
        plt.rcParams['legend.loc'] = 'best'
        plt.rcParams['legend.fancybox'] = True
        plt.rcParams['legend.shadow'] = True
        plt.rcParams['text.usetex'] = True
        plt.rcParams['font.family'] = "serif"
        plt.rcParams['font.serif'] = "cm"
        plt.rcParams['text.latex.preamble'] = r"\usepackage{subdepth}, \usepackage{type1cm}"
```

2 Question 1:

- To find Discrete Fourier Transform DFT of $\sin(5t)$ and (AM) Amplitude Modulated signal given by $(1 + 0.1\cos(t))\cos(10t)$
- Plot and analyse the spectrum obtained for both the functions given above.
- Cross validate the spectrum obtained with what is expected.
- To compare the spectrum obtained for $\sin(5t)$, we use

$$\sin(5t) = \frac{1}{2j}e^{j5} - \frac{1}{2j}e^{-j5} \tag{1}$$

- So the fourier transform of $\sin(5t)$ using above relation is

$$\mathcal{F}(\sin(5t)) \rightarrow \frac{1}{2j}(\ \delta(\omega-5) - \delta(\omega+5)\ ) \tag{2}$$

- Similarly for finding Fourier Transform AM signal following relations are used

$$(1 + 0.1\cos(t))\cos(10t) \rightarrow \cos(10t) + 0.1\cos(10t)\cos(t) \tag{3}$$

$$0.1\cos(10t)\cos(t) \rightarrow 0.05(\ \cos(11t) + cos(9t)\ ) \tag{4}$$

$$0.1\cos(10t)\cos(t) \rightarrow 0.025(e^{j11t} + e^{j9t} + e^{-j11t} + e^{-j9t}) \tag{5}$$

- So we can find fourier transform from above relation
- So using this we compare the plots of Magnitude and phase spectrum obtained using DFT and analyse them.

In [3]:
```
%%writefile_run ee16b031_assignment9.py -a

'''
Function to select different functions
Arguments:
 t -> vector of time values
 n -> encoded from 1 to 6 to select function
'''

def f(t,n):
    if(n == 1):
        return sin(5*t)
    elif(n==2):
        return (1+0.1*cos(t))*cos(10*t)
    elif(n==3):
        return pow(sin(t),3)
    elif(n==4):
        return pow(cos(t),3)
    elif(n==5):
        return cos(20*t +5*cos(t))
    elif(n==6):
        return exp(-pow(t,2)/2)
    else:
        return sin(5*t)
```

In [4]: %%writefile_run ee16b031_assignment9.py -a

```
'''
Function to find Discrete Fourier Transform
Arguments:
 low_lim,up_lim -> lower & upper limit for time vector
 no_points      -> Sampling rate
 f              -> function to compute DFT for
 n              -> mapped value for a function ranges(1,6)
 norm_factor    -> default none, only for Gaussian function
                   it is given as parameter
'''

def findFFT(low_lim,up_lim,no_points,f,n,norm_Factor=None):
    t = linspace(low_lim,up_lim,no_points+1)[:-1]
    y = f(t,n)
    N = no_points

    # DFT for gaussian function
    # ifftshift is used to center the function to zero
    # norm_factor is multiplying constant to DFT

    if(norm_Factor!=None):
        Y = fftshift((fft(ifftshift(y)))*norm_Factor)
    else:
        #normal DFT for periodic functions
        Y = fftshift(fft(y))/(N)

    w_lim = (2*pi*N/((up_lim-low_lim)))
    w = linspace(-(w_lim/2),(w_lim/2),(no_points+1))[:-1]
    return t,Y,w
```

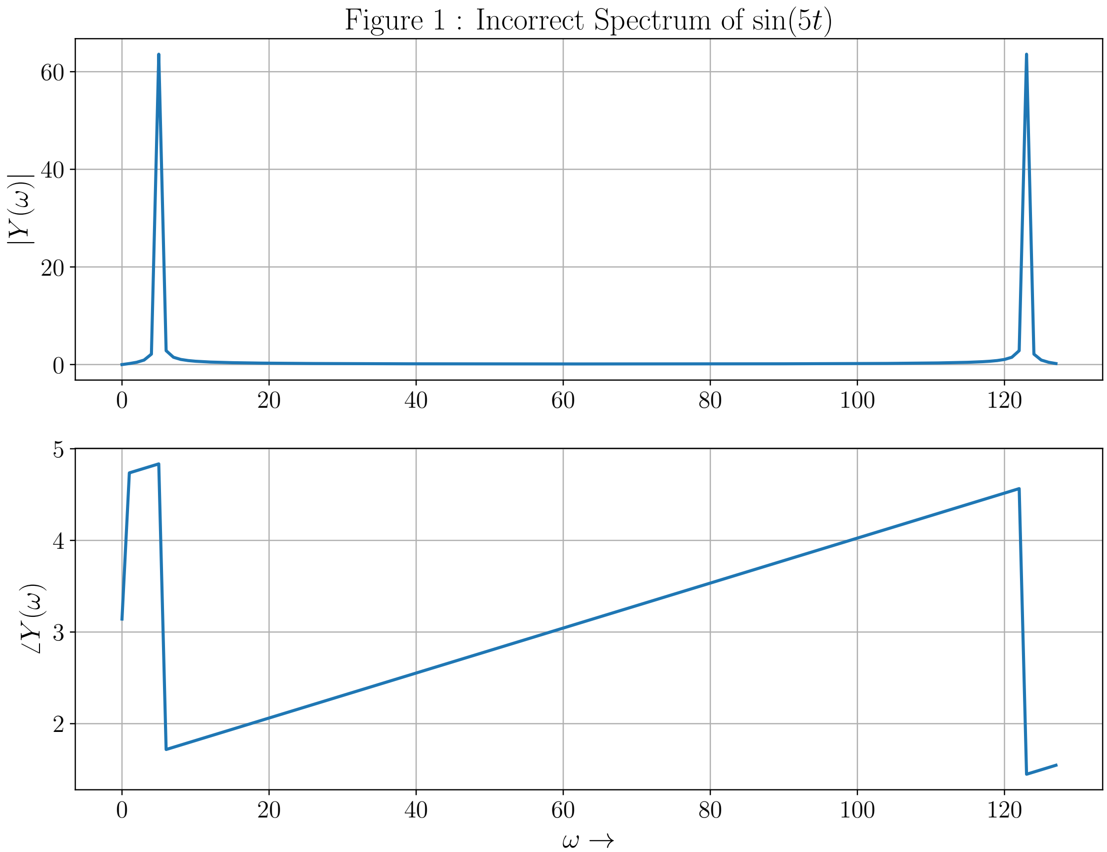
In [5]: %%writefile_run ee16b031_assignment9.py -a

```
'''
Function to plot Magnitude and Phase spectrum for given function
Arguments:
 t              -> time vector
 Y              -> DFT computed
 w              -> frequency vector
 threshold      -> value above which phase is made zero
 Xlims,Ylims    -> limits for x&y axis for spectrum
 plot_title,fig_no -> title of plot and figure no
'''

def plot_FFT(t,Y,w,threshold,Xlims,plot_title,fig_no,Ylims=None):

    figure()
    subplot(2,1,1)
    plot(w,abs(Y),lw=2)
    xlim(Xlims)
    if(Ylims!=None):
        ylim(Ylims)
```

```
    ylabel(r"$|Y(\omega)| \to$")
    title(plot_title)
    grid(True)

    ax = subplot(2,1,2)
    ii=where(abs(Y)>threshold)
    plot(w[ii],angle(Y[ii]),'go',lw=2)

    if(Ylims!=None):
        ylim(Ylims)

    xlim(Xlims)
    ylabel(r"$\angle Y(j\omega) \to$")
    xlabel(r"$\omega \to$")
    grid(True)
    savefig("fig9-"+fig_no+".png")
    show()
```

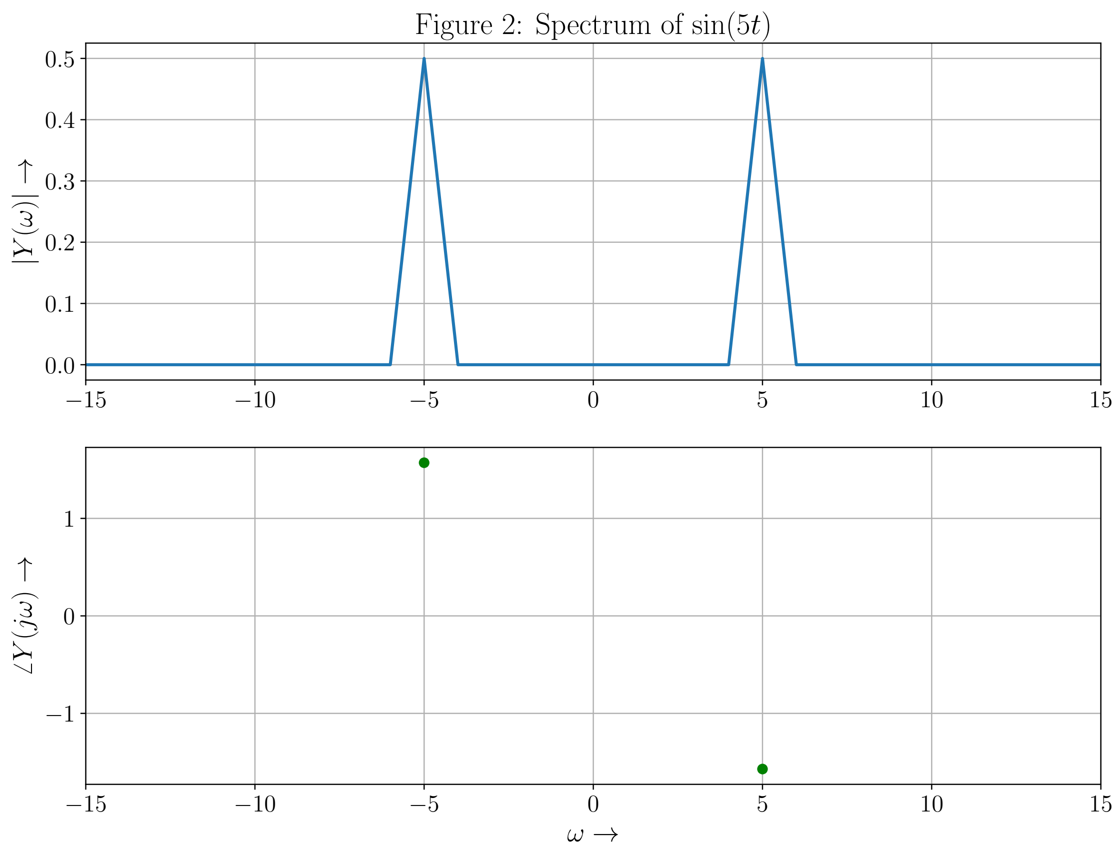
2.1 Imperfect Calculation of DFT

```
In [6]: %%writefile_run ee16b031_assignment9.py -a

        '''
        DFT for sin(5t) computed in incorrect way
        * like without normalizing  factor
        * without centering fft of function to zero
        '''

        x=linspace(0,2*pi,128)
        y=sin(5*x)
        Y=fft(y)
        figure()
        subplot(2,1,1)
        plot(abs(Y),lw=2)
        title(r"Figure 1 : Incorrect Spectrum of $\sin(5t)$")
        ylabel("$|Y(\omega)|$")
        grid(True)
        subplot(2,1,2)
        plot(unwrap(angle(Y)),lw=2)
        xlabel(r"$\omega \to $")
        ylabel(r"$\angle Y(\omega)$")
        grid(True)
        savefig("fig9-1.png")
        show()
```



2.2 Corrected Method to compute DFT for Periodic signals

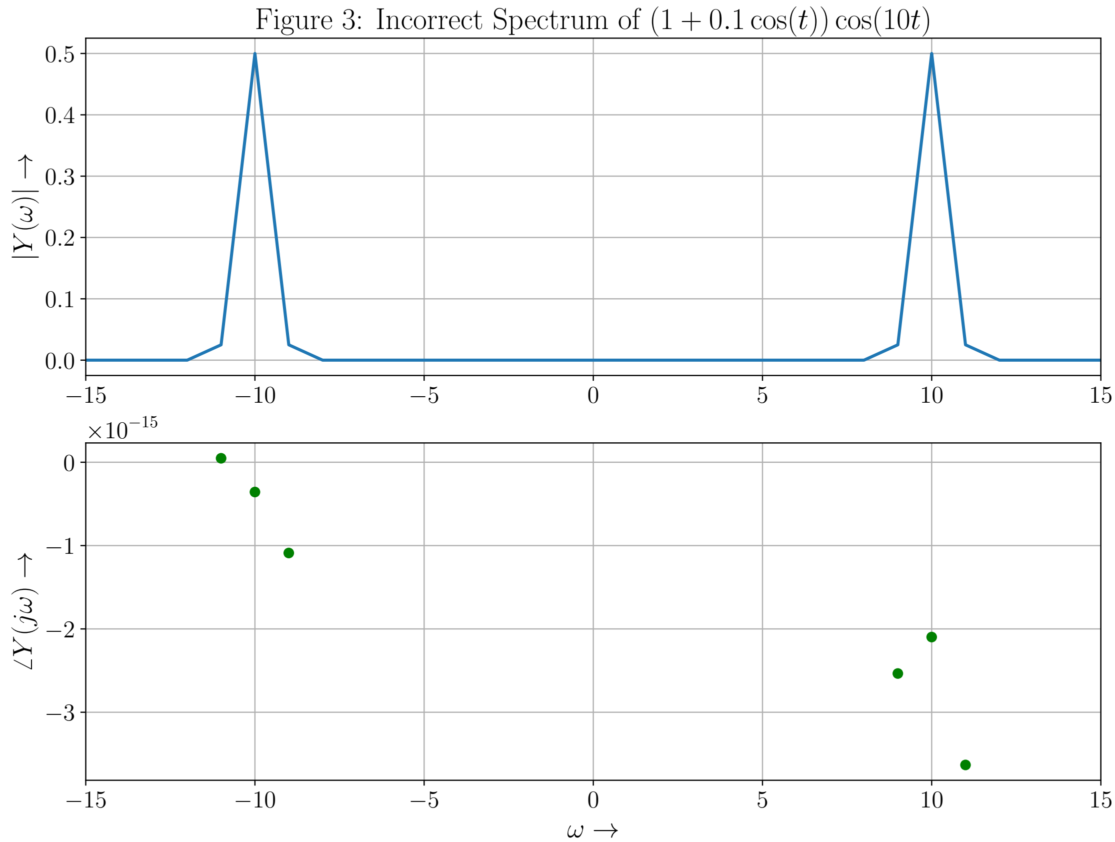
In [7]:
```
%%writefile_run ee16b031_assignment9.py -a

t,Y,w = findFFT(0,2*pi,128,f,1)
Xlims = [-15,15]
plot_FFT(t,Y,w,1e-3,Xlims,r"Figure 2: Spectrum of $\sin(5t)$","2")
```



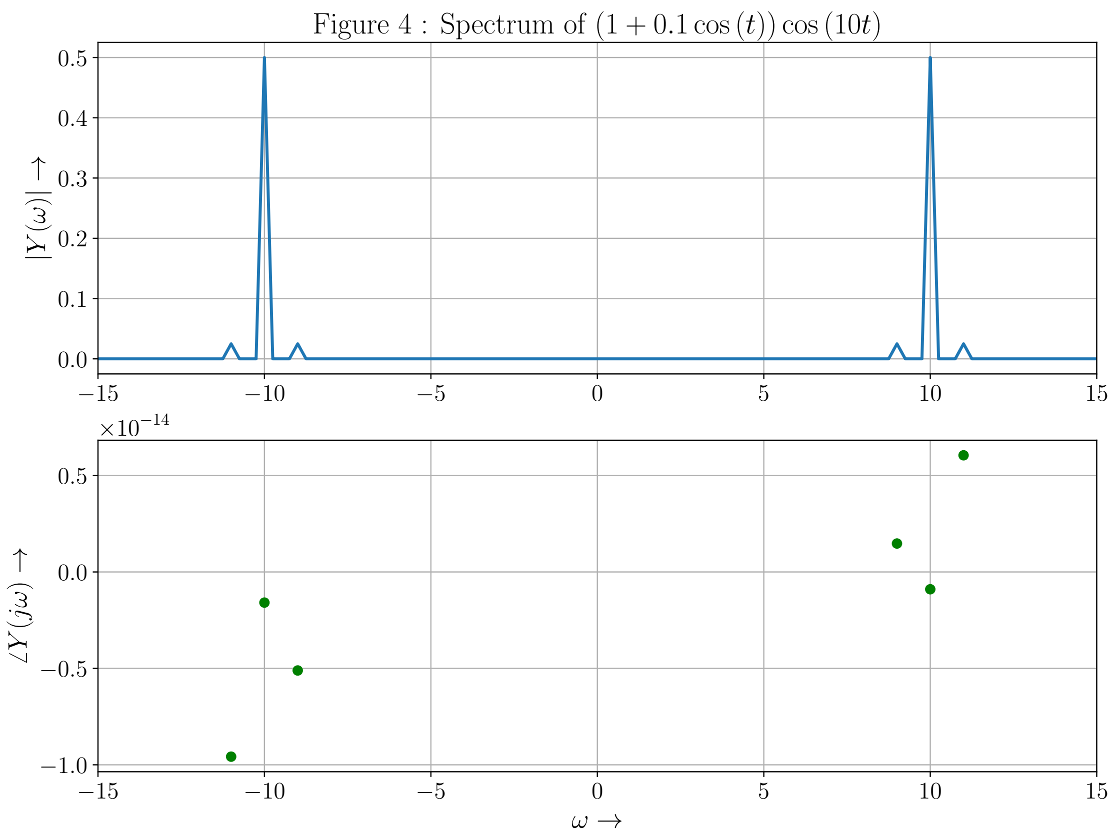
2.2.1 Results and Discussion :

- As we observe the plot frequency contents are of $\omega = 5, -5$
- Since everything consists of sin terms so phase is zero and π alternatively.

In [8]:
```
%%writefile_run ee16b031_assignment9.py -a

t,Y,w = findFFT(0,2*pi,128,f,2)
Xlims = [-15,15]
Ylims = []
plot_FFT(t,Y,w,1e-4,Xlims,r"Figure 3: Incorrect Spectrum of $(1+0.1\cos(t))\cos(10t)$","3")
```




```
In [9]: %%writefile_run ee16b031_assignment9.py -a

        t,Y,w = findFFT(-4*pi,4*pi,512,f,2)
        Xlims = [-15,15]
        Ylims = []
        plot_FFT(t,Y,w,1e-4,Xlims,r"Figure 4 : Spectrum of AM signal)
```



2.2.2 Results and Discussion :

- As we observe the plot it has center frequencies of $\omega = 10, -10$ and as expected we get side band frequencies due to amplitude modulation with phase of 0 since only cos terms

3 Question 2:

- To find Discrete Fourier Transform DFT of $\sin^3(t)$ and $\cos^3(t)$
- Plot and analyse the spectrum obtained for both the functions given above.
- Cross validate the spectrum obtained with what is expected.
- To compare the spectrum obtained for $\sin^3(t)$, we use

$$\sin^3(t) = \frac{3}{4}\sin(t) - \frac{1}{4}\sin(3t) \tag{6}$$

- So the fourier transform of $\sin^3(t)$ using above relation is

$$\mathcal{F}(\sin^3(t)) \rightarrow \frac{3}{8j}(\ \delta(\omega-1) - \delta(\omega+1)\ ) - \frac{1}{8j}(\ \delta(\omega-3) - \delta(\omega+3)\ ) \tag{7}$$

- Similarly $\cos^3(t)$ is given by

$$\cos^3(t) = \frac{3}{4}\cos(t) + \frac{1}{4}\cos(3t) \tag{8}$$

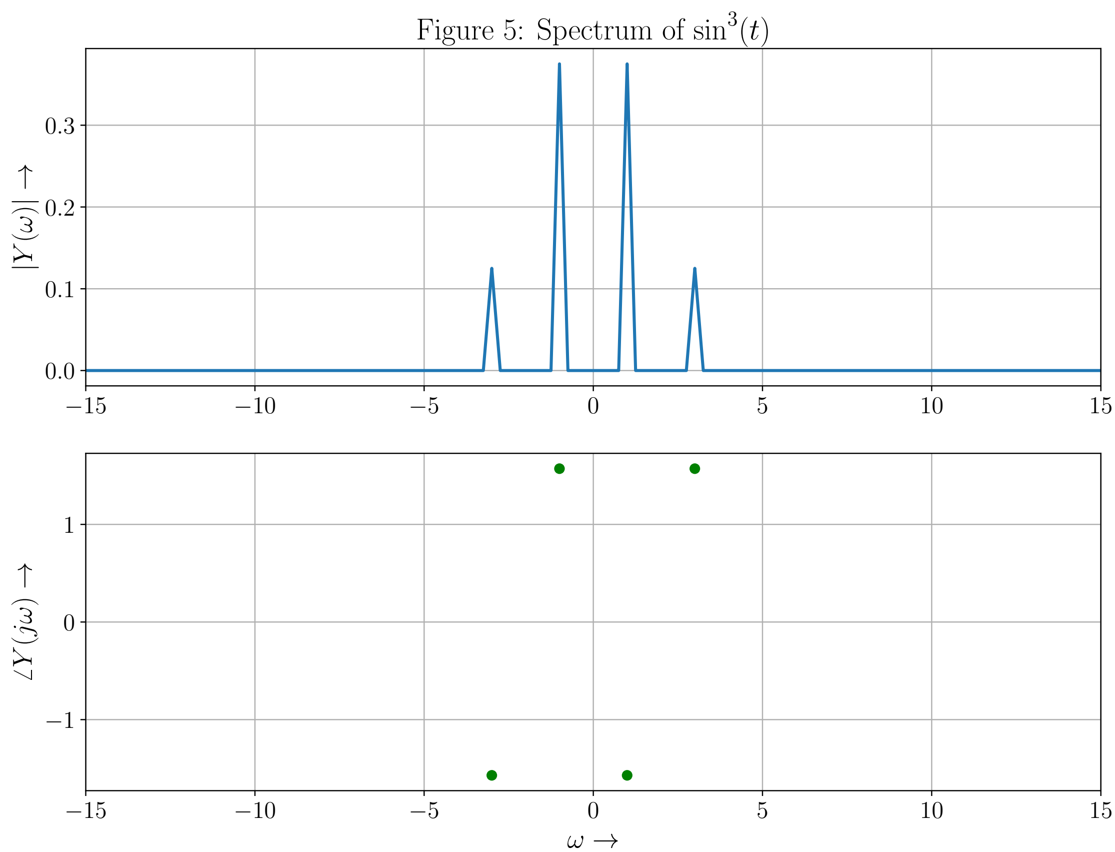
- So the fourier transform of $\sin^3(t)$ using above relation is

$$\mathcal{F}(\cos^3(t)) \rightarrow \frac{3}{8j}(\ \delta(\omega-1)+\delta(\omega+1)\ ) + \frac{1}{8j}(\ \delta(\omega-3)+\delta(\omega+3)\ ) \tag{9}$$

- So using this we compare the plots of Magnitude and phase spectrum obtained using *DFT* and analyse them.

In [10]: %%writefile_run ee16b031_assignment9.py -a

```
t,Y,w = findFFT(-4*pi,4*pi,512,f,3)
Xlims = [-15,15]
Ylims = []
plot_FFT(t,Y,w,1e-4,Xlims,r"Figure 5: Spectrum of $\sin ^{3}(t)$","5")
```

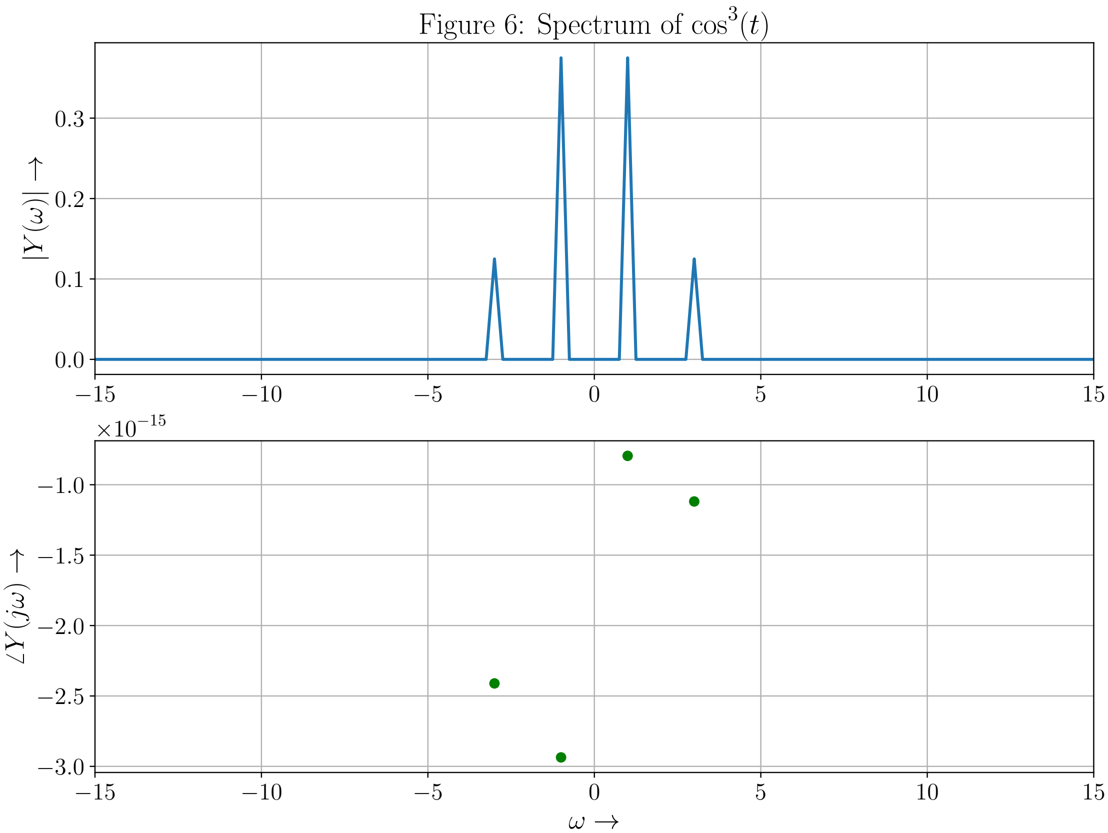


3.0.3 Results and Discussion :

- As we observe the plot frequency contents are of $\omega = 1, -1, 3, -3$ and with their amplitude in 1:3 ratio
- Since everything consists of sin terms so phase is zero and π alternatively.

In [11]: %%writefile_run ee16b031_assignment9.py -a

```
t,Y,w = findFFT(-4*pi,4*pi,512,f,4)
Xlims = [-15,15]
plot_FFT(t,Y,w,1e-4,Xlims,r"Figure 6: Spectrum of $\cos^{3}(t)$","6")
```



3.0.4 Results and Discussion :

- As we observe the plot frequency contents are of $\omega = 1, -1, 3, -3$ and with their amplitude in 1:3 ratio
- Since everything consists of cos terms so phase is zero. But due to lack of infinite computing power they are nearly zero in the order of 10^{-15}

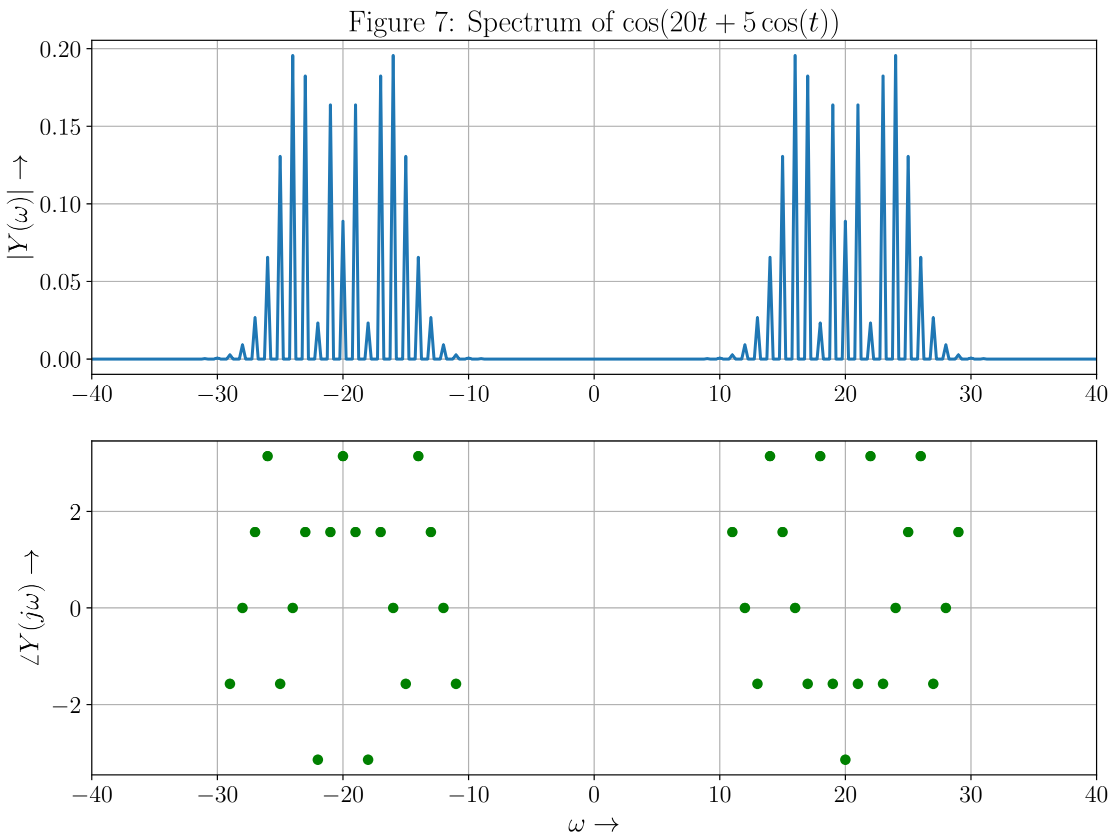
4 Question 3:

- To generate the spectrum of $\cos(20t + 5\cos(t))$.
- Plot phase points only where the magnitude is significant ($> 10^{-3}$).
- Analyse the spectrums obtained.

In [12]: %%writefile_run ee16b031_assignment9.py -a

```
t,Y,w = findFFT(-4*pi,4*pi,512,f,5)
```

```
Xlims = [-40,40]
plot_FFT(t,Y,w,1e-3,Xlims,r"Figure 7: Spectrum of $\cos(20t+5\cos(t))$","7")
```



4.0.5 Results and Discussion :

- As we observe the plot that its a Frequency modulation since center frequency being $\omega = 20$ and side band frequencies which are produced by $5\cos t$.
- But the more detailed reason we will be learning in Communication systems next semester.

5 Question 4:

- To generate the spectrum of the Gaussian $e^{-\frac{t^2}{2}}$ which is not *bandlimited* in frequency and find Fourier transform of it using DFT.

$$\mathcal{F}(\ e^{-\frac{t^2}{2}}\ ) \rightarrow \frac{1}{\sqrt{2\pi}} e^{\frac{-\omega^2}{2}} \tag{10}$$

- To find the normalising constant for DFT obtained we use following steps to derive it :

- window the signal $e^{-\frac{t^2}{2}}$ by rectangular function with gain 1 and window_size 'T' which is equivalent to convolving with $Tsinc(\omega T)$ in frequency domain. So As T is very large the $sinc(\omega T)$ shrinks , we can approximate that as $\delta(\omega)$. So convolving with that we get same thing.

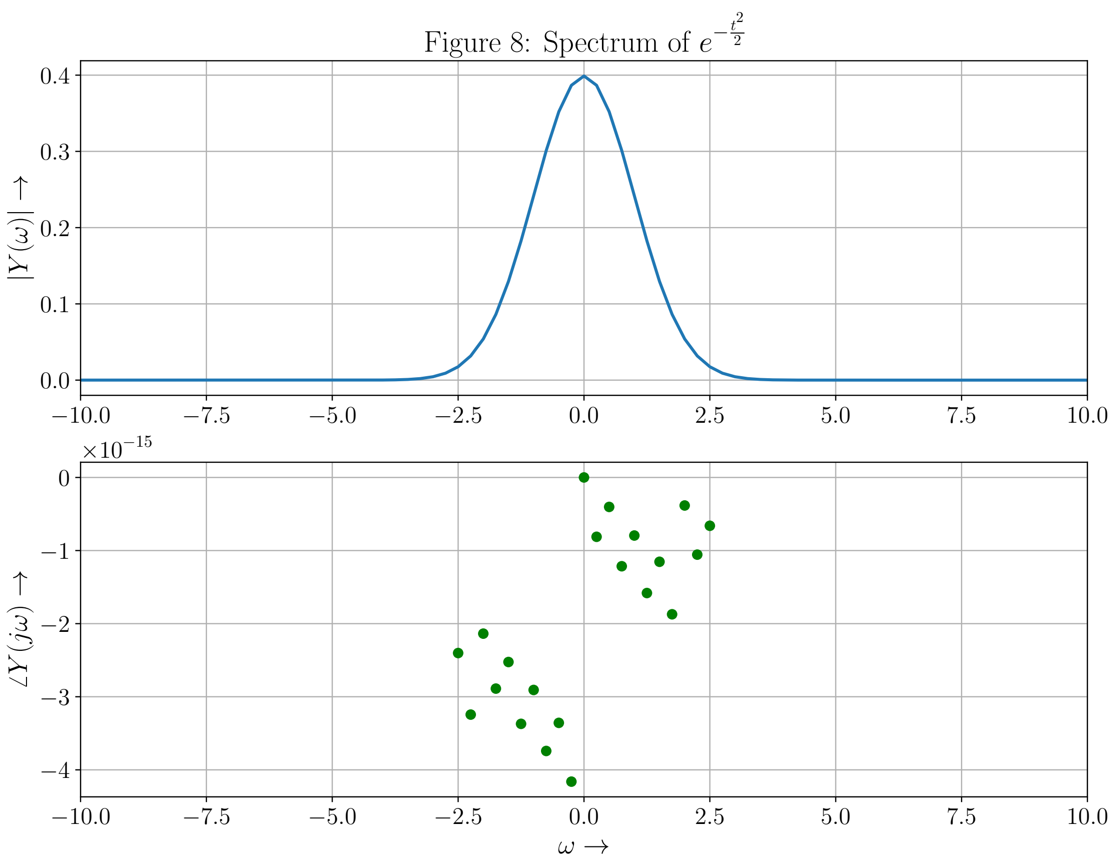
- Windowing done because finite computing power and so we cant represent infinetly wide signal .
- Now we sample the signal with sampling rate N,which is equivalent to convolving impulse train in frequency domain
- And finally for DFT we create periodic copies of the windowed sampled signal and make it periodic and then take one period of its Fourier transform i.e is DFT of gaussian.
- Following these steps we get normalising factor of **Window_size/(**2π **Sampling_rate)**
- To find the Discrete Fourier transform equivalent for Continous Fourier transform of Gaussian function by finding absolute error between the DFT obtained using the normalising factor obtained with exact Fourier transform and find the parameters such as Window_size and sampling rate by minimising the error obtained with tolerance of 10^{-15}
- To generate the spectrum of

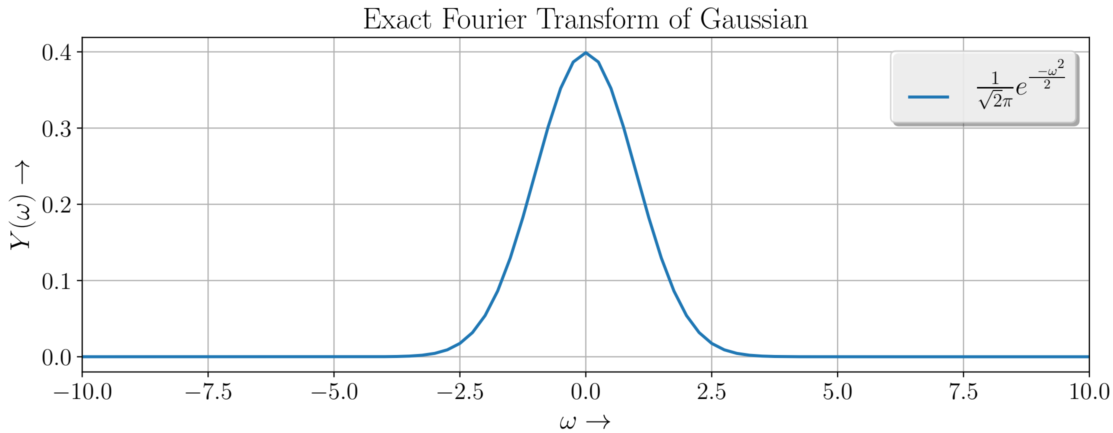
- Plot phase points only where the magnitude is significant ($> 10^{-2}$).
- Analyse the spectrums obtained.

In [13]:
```
%%writefile_run ee16b031_assignment9.py -a

# initial window_size and sampling rate defined
window_size = 2*pi
sampling_rate = 128
# tolerance for error
tol = 1e-15

#normalisation factor derived
norm_factor = (window_size)/(2*pi*(sampling_rate))


'''
For loop to minimize the error by increasing
both window_size and sampling rate as we made assumption that
when Window_size is large the sinc(w) acts like impulse, so we
increase window_size, similarly sampling rate increased to
overcome aliasing problems
'''

for i in range(1,10):

    t,Y,w = findFFT(-window_size/2,window_size/2,sampling_rate,f,6,norm_factor)

    #actual Y
    actual_Y = (1/sqrt(2*pi))*exp(-pow(w,2)/2)
    error = (np.mean(np.abs(np.abs(Y)-actual_Y)))
    print("Absolute error at Iteration - %g is : %g"%((i,error)))

    if(error < tol):
        print("\nAccuracy of the DFT is: %g and Iterations took: %g"%((error,i)))
        print("Best Window_size: %g , Sampling_rate: %g"%((window_size,sampling_rate)))
        break
    else:
        window_size   = window_size*2
```

```
        sampling_rate = (sampling_rate)*2
        norm_factor = (window_size)/(2*pi*(sampling_rate))


    Xlims = [-10,10]
    plot_FFT(t,Y,w,1e-2,Xlims,r"Figure 8: Spectrum of $e^{-\frac{t^{2}}{ 2}}$","8")

    #Plotting actual DFT of Gaussian
    subplot(2,1,2)
    plot(w,abs(actual_Y),label=r"$\frac{1}{\sqrt 2\pi} e^{\frac{\ - \omega ^{2}}{2}}$")
    title("Exact Fourier Transform of Gaussian")
    xlim([-10,10])
    ylabel(r"$Y(\omega) \to$")
    xlabel(r"$\omega \to$")
    grid()
    legend()
    show()
```

```
Absolute error at Iteration - 1 is : 5.20042e-05
Absolute error at Iteration - 2 is : 2.07579e-11
Absolute error at Iteration - 3 is : 4.14035e-17

Accuracy of the DFT is: 4.14035e-17 and Iterations took: 3
Best Window_size: 25.1327 , Sampling_rate: 512
```





5.0.6 Results and Discussion :

- As we observe the magnitude spectrum of $e^{-\frac{t^2}{2}}$ we see that it almost coincides with exact Fourier Transform plotted below with accuracy of $4.14035e^{-17}$
- To find the correct Window size and sampling rate,For loop is used to minimize the error by increasing both window_size and sampling rate as we made assumption that when Window_size is large the sinc(wT) acts like impulse $\delta(\omega)$
- so we increase window_size, similarly sampling rate is increased to overcome aliasing problems when sampling the signal in time domain.
- Similarly we observe the phase plot ,$\angle(Y(\omega) \approx 0$ in the order of 10^{-15}

5.1 Conclusion :

- Hence we analysed the how to find DFT for various types of signals and how to estimate normalising factors for Gaussian functions ,also to find parameters like window_size and sampling rate by minimizing the error with tolerance upto 10^{-15}!!