Digital Signal Processing

Rohith*

Contents

***Abstract*—This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://raw.githubusercontent.com/
    gadepall/
EE1310/master/filter/codes/Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem **??** in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```
#pip install soundfile
import soundfile as sf
from scipy import signal
#read .wav file
input_signal,fs = sf.read("Sound␣Noise.wav"
)
#sampling frequency of Input signal
samp_freq=fs
#order of the filter
order=4
#cutoff frquency 4kHz
cutoff_freq=4000.0
#digital frequency
Wn=2*cutoff_freq/sampl_freq
# b and a are numerator and denominator
    polynomials respectively
b, a = signal.butter(order,Wn, "low")
#filter the input signal with butterworth filter
output_signal = signal.filtfilt(b, a,
input_signal)
#output signal = signal.lfilter(b, a,
    input_signal)
#write the output signal into .wav file
sf.write("Sound␣With␣ReducedNoise.wav",
    output_signal, fs)
```

2.4 The output of the python script in Problem **??** is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem **??**. What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$.

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.
Solution: The following code yields Fig. **??**.

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

```
wget https://github.com/gadepall/EE1310/raw/
    master/filter/codes/xnyn.py
```

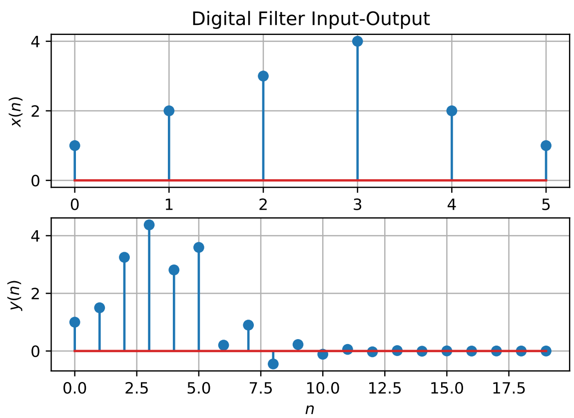


Fig. 3.2

3.3 Repeat the above exercise using a C code.

4 Z-TRANSFORM

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (**??**),

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.5}$$

resulting in (**??**). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.6}$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem **??**.
Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = 1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \frac{2}{z^4} + \frac{1}{z^5} \tag{4.7}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.8}$$

from (**??**) assuming that the Z-transform is a linear operation.
Solution: Applying (**??**) in (**??**),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.9}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{4.10}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.11}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.12}$$

is

$$U(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1 \tag{4.13}$$

Solution: It is easy to show that

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \tag{4.14}$$

and from (**??**),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.15}$$

$$= \frac{1}{1-z^{-1}}, \quad |z| > 1 \tag{4.16}$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1-az^{-1}} \quad |z| > |a| \tag{4.17}$$

Solution: On applying Z-transform:

$$A(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} \tag{4.18}$$

$$= \sum_{n=0}^{\infty} \frac{a^n}{z^n}, \quad |z| > |a| \tag{4.19}$$

$$= \frac{1}{1-\frac{a}{z}} \tag{4.20}$$

Using the formula for infinite geometric progression with $r = \frac{a}{z}$

4.6 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.21}$$

Plot $\left|H\left(e^{\jmath\omega}\right)\right|$. Is it periodic? If so, find the period. $H(e^{\jmath\omega})$ is known as the *Discrete Time Fourier Transform* (DTFT) of $h(n)$.
Solution: The following code plots Fig. **??**.

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/dtft.
    ipynb
```

$$H\left(z\right) = \frac{1 + z^2}{z^2 + \frac{z}{2}} \tag{4.22}$$

$$H(e^{\jmath\omega}) = \frac{1 + e^{2\jmath\omega}}{e^{2\jmath\omega} + \frac{e^{\jmath\omega}}{2}} \tag{4.23}$$

$$H(e^{\jmath\omega}) = \frac{2e^{\jmath\omega}\left(e^{\jmath\omega} + e^{-\jmath\omega}\right)}{2e^{2\jmath\omega} + e^{\jmath\omega}} \tag{4.24}$$

$$H(e^{\jmath\omega}) = \frac{4cos(w)}{1 + 2e^{\jmath\omega}} \tag{4.25}$$

$$H\left(e^{\jmath\omega}\right) = \frac{cos(w)}{1 + 2cos(w) + 2jsin(w)} \tag{4.26}$$

$$\left|H\left(e^{\jmath\omega}\right)\right| = \frac{|cos(w)|}{\sqrt{5 + 4cos(w)}} \tag{4.27}$$

Therefore, the period is 2π

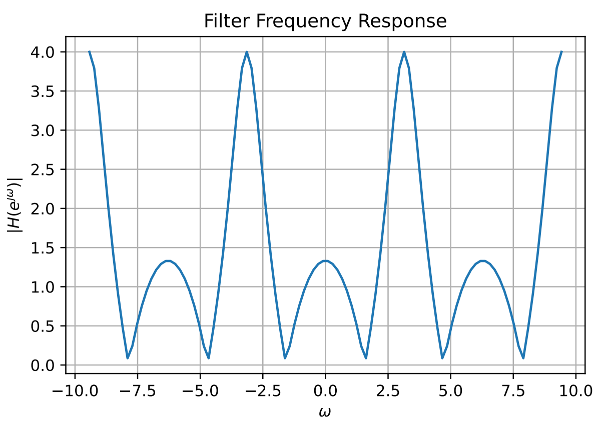


Fig. 4.6: $\left|H\left(e^{\jmath\omega}\right)\right|$

4.7 Express $h(n)$ in terms of $H\left(e^{\jmath\omega}\right)$.

We know that,

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)e^{-\jmath nz} \tag{4.28}$$

Now,

$$\int_{-\pi}^{\pi} H(z)e^{\jmath kz}\, dz \tag{4.29}$$

$$= \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} h(n)e^{-\jmath nz}e^{\jmath kz}\, dz \tag{4.30}$$

If n = k, the above integral evaluates to:

$$2\pi h(k) = 2\pi h(n) \tag{4.31}$$

If n ≠ k, the above integral evaluates to:

$$\sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} h(n)e^{\jmath z(k-n)}\, dz \tag{4.32}$$

$$= \sum_{n=-\infty}^{\infty} h(n) \int_{-\pi}^{\pi} e^{\jmath z(k-n)}\, dz \tag{4.33}$$

$$= \sum_{n=-\infty}^{\infty} h(n) \int_{-\pi}^{\pi} cos(z(k-n)) + \jmath sin(z(k-n))\, dz \tag{4.34}$$

Taking z(k-n) = t,

$$\sum_{n=-\infty}^{\infty} h(n) \int_{-\pi(k-n)}^{\pi(k-n)} cos(t) + \jmath sin(t)\, dt \tag{4.35}$$

$$= 0 \tag{4.36}$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in (**??**).
Solution:

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{z^{-1}}{2}} \tag{5.2}$$

Let

$$z^{-1} = t \tag{5.3}$$

Hence,

$$1+t^2 = (2t-4)(1+\frac{t}{2})+5 \tag{5.4}$$

$$H(z) = (2z^{-1}-4)+\frac{5}{1+\frac{1}{2z}} \tag{5.5}$$

$$= \frac{2}{z}-4+5-\frac{5}{2z}+\frac{5}{4z^2}+... \tag{5.6}$$

$$= 1-\frac{1}{2z}+\frac{5}{4z^2}-\frac{5}{8z^3}+\frac{5}{16z^4} \tag{5.7}$$

Hence, by comparing with the coefficients with the definition of z-transform

$$h(0) = 1 \tag{5.8}$$

$$h(1) = -\frac{1}{2} \tag{5.9}$$

$$h(2) = \frac{5}{4} \tag{5.10}$$

$$h(3) = -\frac{5}{8} \tag{5.11}$$

$$h(4) = \frac{5}{16} \tag{5.12}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.13}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (**??**).
Solution: From (**??**),

$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}}+\frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.14}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^n u(n)+\left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.15}$$

For $\left|\frac{1}{2z}\right| < 1$ or $|z| > \frac{1}{2}$ using (**??**) and (**??**).

5.3 Sketch $h(n)$. Is it bounded? Justify theoretically.
Solution: The following code plots Fig. **??**.

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/hn.py
```

h(n) is bounded as it tends to 0 as n tends to 12

5.4 Convergent? Justify using the ratio test.
Solution:

$$h(n) = (\frac{-1}{2})^n u(n)+(\frac{-1}{2})^{n-2}u(n-2) \tag{5.16}$$

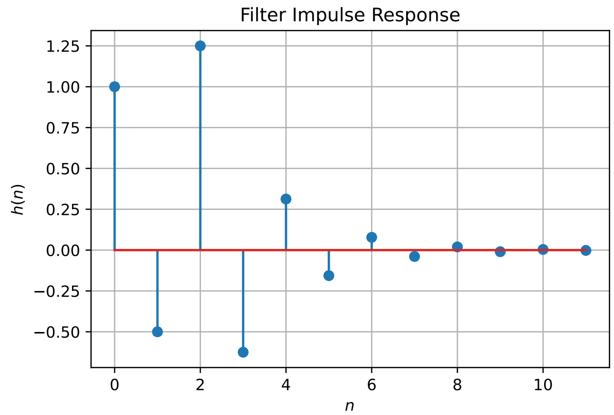


Fig. 5.3: $h(n)$ as the inverse of $H(z)$

As n $\to \infty$,

$$h(n)/h(n-1) = \frac{(\frac{-1}{2})^n+(\frac{-1}{2})^{n-2}}{(\frac{-1}{2})^{n-1}+(\frac{-1}{2})^{n-3}} = -\frac{1}{2} \tag{5.17}$$

As $\left|\frac{h(n)}{h(n-1)}\right|<1$, the series is convergent due to which it is bounded.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.18}$$

Is the system defined by (**??**) stable for the impulse response in (**??**)?
Solution: By using h(n) from 5.3

$$h(n) = \left(-\frac{1}{2}\right)^n u(n)+\left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.19}$$

$$= \sum_{n=-\infty}^{\infty}\left(-\frac{1}{2}\right)^n u(n)+\left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.20}$$

$$= \sum_{n=-\infty}^{\infty}\left(-\frac{1}{2}\right)^n u(n)+\sum_{n=-\infty}^{\infty}\left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.21}$$

$$= \sum_{n=-\infty}^{\infty}\left(-\frac{1}{2}\right)^n+\sum_{n=-\infty}^{\infty}\left(-\frac{1}{2}\right)^{n-2} \tag{5.22}$$

$$\tag{5.23}$$

$$= \frac{2}{3}+\frac{2}{3} < \infty \tag{5.24}$$

5.6 Verify the above result using a python code.

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.25}$$

This is the definition of $h(n)$.
Solution: The following code plots Fig. **??**. Note that this is the same as Fig. **??**.

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/hndef
    .ipynb
```

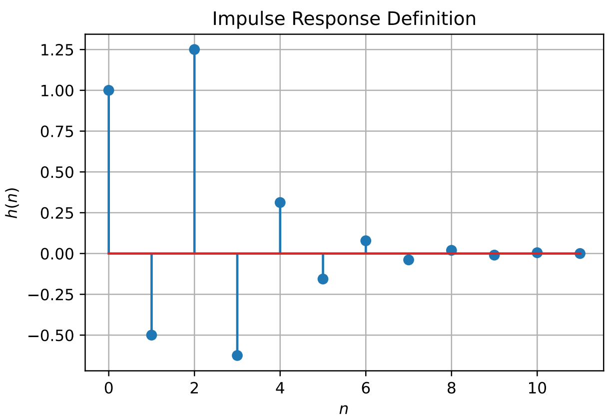


Fig. 5.7: $h(n)$ from the definition

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \tag{5.26}$$

Comment. The operation in (**??**) is known as *convolution*.
Solution: The following code plots Fig. **??**. Note that this is the same as $y(n)$ in Fig. **??**.

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/
    ynconv.ipynb
```

5.9 Express the above convolution using a Teoplitz matrix.

5.10 Show that

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \tag{5.27}$$

Solution: We know that,

$$y(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \tag{5.28}$$

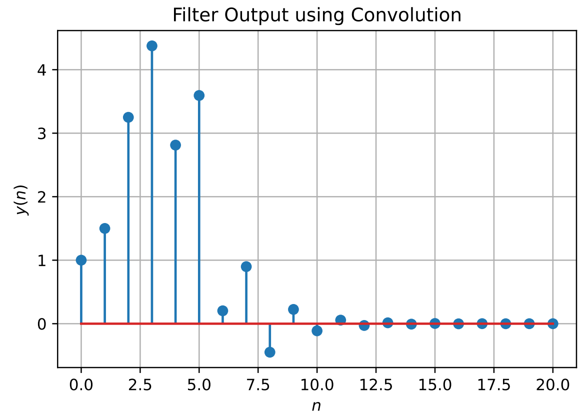


Fig. 5.8: $y(n)$ from the definition of convolution

Taking k = n-k

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \tag{5.29}$$

5.11 Oppenheimer question

$$x[n] = u[n+10] - u[n+5] \tag{5.30}$$

$$= \begin{cases} 1 & -10 <= n < -6 \\ 0 & otherwise \end{cases} \tag{5.31}$$

x[n] is finite length and has only positive powers of z, hence, its ROC is $|z| < \infty$

6 DFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

6.3 Compute

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k) \cdot e^{\jmath 2\pi kn/N}, \quad n = 0, 1, \dots, N-1 \tag{6.3}$$

Solution: The following code plots Fig. **??**. Note that this is the same as $y(n)$ in Fig. **??**.

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/yndft.
    ipynb
```

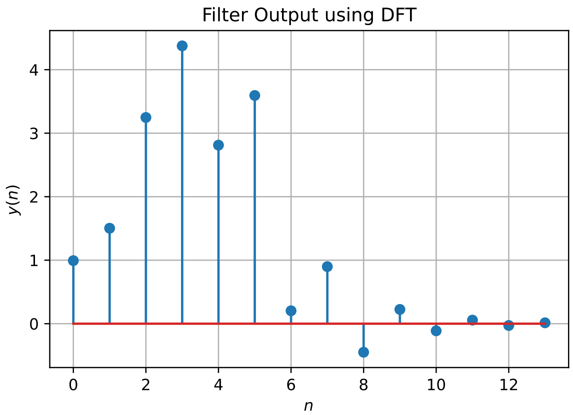


Fig. 6.3: $y(n)$ from the DFT

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.

7 FFT

1. The DFT of $x(n)$ is given by
$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{7.1}$$
2. Let
$$W_N = e^{-j2\pi/N} \tag{7.2}$$
Then the N-point *DFT matrix* is defined as
$$\vec{F}_N = [W_N^{mn}], \quad 0 \le m, n \le N-1 \tag{7.3}$$
where W_N^{mn} are the elements of $\vec{F}_N$.
3. Let
$$\vec{I}_4 = \vec{e}_4^1 \vec{e}_4^2 \qquad \vec{e}_4^3 \vec{e}_4^4 \tag{7.4f}$$
be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as
$$\vec{P}_4 = \vec{e}_4^1 \vec{e}_4^3 \qquad \vec{e}_4^2 \vec{e}_4^4 \tag{7.5f}$$
4. The 4 point *DFT diagonal matrix* is defined as
$$\vec{D}_4 = diag W_8^0 W_8^1 \qquad W_8^2 W_8^3 \qquad ref(7.6)$$
5. Show that
$$W_N^2 = W_{N/2} \tag{7.7}$$
Solution:
$$W_N^2 = e^{-j*2*2\pi/N} \tag{7.8}$$
$$= e^{-j2\pi/(N/2)} \tag{7.9}$$
$$= W_{N/2} \tag{7.10}$$
6. Show that
$$\vec{F}_4 = \begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} \vec{P}_4 \tag{7.11}$$
Solution:
$$\vec{F}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \vec{W}_N & \vec{W}_N^2 & \vec{W}_N^3 \\ 1 & \vec{W}_N^2 & \vec{W}_N^4 & \vec{W}_N^6 \\ 1 & \vec{W}_N^3 & \vec{W}_N^6 & \vec{W}_N^9 \end{bmatrix} \tag{7.12}$$
On post multiplying F_4 with P_4 and using $W_N^4 = 1$ we get,
$$\vec{F}_4\vec{P}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \vec{W}_N^2 & \vec{W}_N & \vec{W}_N^3 \\ 1 & 1 & \vec{W}_N^2 & \vec{W}_N^6 \\ 1 & \vec{W}_N^2 & \vec{W}_N^3 & \vec{W}_N^9 \end{bmatrix} \tag{7.13}$$
Using $W_N^2 = W_{N/2}$, upper left and lower left $2*2$ matrices become equal to F_2. The top right matrix =
$$\begin{bmatrix} 1 & 1 \\ \vec{W}_N & \vec{W}_N^3 \end{bmatrix} \tag{7.14}$$
=
$$\begin{bmatrix} 1 & 0 \\ 0 & \vec{W}_N \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & \vec{W}_N^2 \end{bmatrix} \tag{7.15}$$
$= D_2F_2$
Similarly, bottom right matrix = = $-D_2F_2$
Hence,
F_4P_4 =
$$\begin{bmatrix} \vec{F}_2 & D_2\vec{F}_2 \\ \vec{F}_2 & -D_2\vec{F}_2 \end{bmatrix} \tag{7.16}$$
=
$$\begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} \tag{7.17}$$
7. Show that
$$\vec{F}_N = \begin{bmatrix} \vec{I}_{N/2} & \vec{D}_{N/2} \\ \vec{I}_{N/2} & -\vec{D}_{N/2} \end{bmatrix} \begin{bmatrix} \vec{F}_{N/2} & 0 \\ 0 & \vec{F}_{N/2} \end{bmatrix} \vec{P}_N \tag{7.18}$$
Solution: Similar to the above, we post multi-

ply F_N with P_N so that the resultant matrix can be split into 4 parts: top-left, top-right, bottom-left and bottom-right: all being $n/2*n/2$ square matrices.
Top left matrix = Bottom left matrix = $F_{N/2}$
Top right matrix = $D_{N/2}F_{N/2}$
Bottom right matrix = -$D_{N/2}F_{N/2}$ Hence,
$F_N P_N$ =

$$\begin{bmatrix} \vec{F_{N/2}} & D_{N/2}\vec{F_{N/2}} \\ \vec{F_{N/2}} & -D_{N/2}\vec{F_{N/2}} \end{bmatrix} \tag{7.19}$$

8. Find

$$\vec{P_4}\vec{x} \tag{7.20}$$

Solution:
$\vec{x}$ =

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{bmatrix} \tag{7.21}$$

So, $\vec{P_4}\vec{x}$ =

$$\begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \\ 0 \\ 0 \end{bmatrix} \tag{7.22}$$

9. Show that

$$\vec{X} = \vec{F_N}\vec{x} \tag{7.23}$$

where $\vec{x}, \vec{X}$ are the vector representations of $x(n), X(k)$ respectively.
Solution:
$\vec{x}$ =

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ . \\ . \\ . \\ x[n-1] \end{bmatrix} \tag{7.24}$$

F_N =

$$\begin{bmatrix} 1 & 1 & 1 & . & . & .1 \\ 1 & W_N & W_N^2 & . & . & .W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & . & . & .W_N^{2N-2} \\ . & . & . & . & . & .W_N^{3N-3} \\ . & & & & & \\ . & & & & & \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & . & . & .W_N^{N(N-1)} \end{bmatrix} \tag{7.25}$$

$\vec{X}$ =

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ . \\ . \\ . \\ X[n-1] \end{bmatrix} \tag{7.26}$$

By definition, $X[k] = \sum_{n=0}^{N-1} x(n)e^{-2*\pi*j*k*n}$
Hence, by the definition of matrix multiplication:
$\vec{X} = \vec{F_N}\vec{x}$ =

$$\begin{bmatrix} 1 & 1 & 1 & . & . & .1 \\ 1 & W_N & W_N^2 & . & . & .W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & . & . & .W_N^{2N-2} \\ . & . & . & . & . & .W_N^{3N-3} \\ . & & & & & \\ . & & & & & \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & . & . & .W_N^{N(N-1)} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ . \\ . \\ . \\ x[N-1] \end{bmatrix} \tag{7.27}$$

10. Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.28}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.29}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.30}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.31}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \quad (7.32)$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \quad (7.33)$$

$$P_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \quad (7.34)$$

$$P_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \quad (7.35)$$

$$P_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \quad (7.36)$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \quad (7.37)$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \quad (7.38)$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \quad (7.39)$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \quad (7.40)$$

Solution:
For every k, X[k] =

$$\sum_0^7 x(n)e^{-2jkn\pi/8} = \sum_0^3 x(2n)e^{-2jk(2n)\pi/8} + \sum_0^3 x(2n+1)e^{-2jk(2n+1)\pi/8} \quad (7.41)$$

=

$$= \sum_0^3 x(2n)e^{-2jk(n)\pi/4} + e^{-2kj\pi/8}\left(\sum_0^3 x(2n+1)e^{-2jk(n)\pi/4}\right) \quad (7.42)$$

= X_1[k]+$e^{-2kj\pi/8}X_2[k]$, where X_1[k] is FFT of even terms and X_2[k] is FFT of odd terms.

Also, if k>=4,
$e^{-2jk\pi/8}$ = -$e^{-2j(k-4)\pi/8}$
Hence,

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \quad (7.43)$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \quad (7.44)$$

Decomposing into 2 point dft matrices:

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \quad (7.45)$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \quad (7.46)$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \quad (7.47)$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \quad (7.48)$$

11. For

$$\vec{x} = 1 \quad (7.49)$$
$$2 \quad (7.50)$$
$$3 \quad (7.51)$$
$$4 \quad (7.52)$$
$$2 \quad (7.53)$$
$$1 \quad (7.54)$$

compte the DFT using (**??**)
Solution:
$\vec{F}_6 = \left[W_6^{MN}\right]$
Where M is the index of row of matrix and N the index of column of matrix
$\vec{F}_6\vec{x}$
=

$$\begin{bmatrix} 13 \\ -4 - \sqrt{3}j \\ 1 \\ -1 \\ 1 \\ -4 + \sqrt{3}j \end{bmatrix} \quad (7.55)$$

12. Repeat the above exercise using the FFT after zero padding $\vec{x}$.

Solution:
$\vec{x}$ =

$$\begin{bmatrix}1\\2\\3\\4\\2\\1\\0\\0\end{bmatrix} \tag{7.56}$$

Using ,

$$\vec{F}_8 = \begin{bmatrix}\vec{I}_4 & \vec{D}_4\\ \vec{I}_4 & -\vec{D}_4\end{bmatrix}\begin{bmatrix}\vec{F}_4 & 0\\ 0 & \vec{F}_4\end{bmatrix}\vec{P}_8 \tag{7.57}$$

$$\vec{F}_4 = \begin{bmatrix}\vec{I}_2 & \vec{D}_2\\ \vec{I}_2 & -\vec{D}_2\end{bmatrix}\begin{bmatrix}\vec{F}_2 & 0\\ 0 & \vec{F}_2\end{bmatrix}\vec{P}_4 \tag{7.58}$$

$$\vec{F}_2 = \begin{bmatrix}\vec{I}_1 & \vec{D}_1\\ \vec{I}_1 & -\vec{D}_1\end{bmatrix}\begin{bmatrix}\vec{F}_1 & 0\\ 0 & \vec{F}_1\end{bmatrix}\vec{P}_2 \tag{7.59}$$

$$\vec{F}_1 = [1] \tag{7.60}$$

Calculating $\vec{F}_2$,

$$\vec{F}_2 = \begin{bmatrix}\vec{F}_1 & D_1\vec{F}_1\\ \vec{F}_1 & -D_1\vec{F}_1\end{bmatrix}\vec{P}_2 \tag{7.61}$$

$$= \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \tag{7.62}$$

Calculating $\vec{F}_4$,

$$\vec{D}_2 = diag(1, W_4) = \begin{bmatrix}1 & 0\\ 0 & -j\end{bmatrix} \tag{7.63}$$

$$D_2\vec{F}_2 = \begin{bmatrix}1 & 0\\ 0 & -j\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \tag{7.64}$$

$$= \begin{bmatrix}1 & 1\\ -j & j\end{bmatrix} \tag{7.65}$$

$$\vec{F}_4 = \begin{bmatrix}\vec{F}_2 & D_2\vec{F}_2\\ \vec{F}_2 & -D_2\vec{F}_2\end{bmatrix}\vec{P}_4 \tag{7.66}$$

$$\vec{F}_4 = \begin{bmatrix}1 & 0 & 1 & 1\\ 0 & 1 & -j & j\\ 1 & 0 & -1 & -1\\ 0 & 1 & j & -j\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{7.67}$$

$$= \begin{bmatrix}1 & 1 & 0 & 1\\ 0 & -j & 1 & j\\ 1 & -1 & 0 & j\\ 0 & j & 1 & -j\end{bmatrix} \tag{7.68}$$

Calculating $\vec{F}_8$,

$$\vec{D}_4 = diag\left(1, W_8, W_8^2, W_8^3\right) \tag{7.69}$$

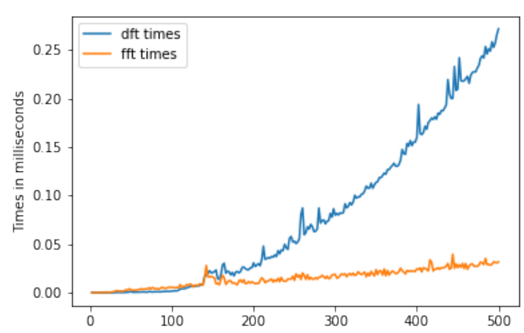
$$= \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \frac{1-j}{\sqrt{2}} & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & \frac{-1-j}{\sqrt{2}}\end{bmatrix} \tag{7.70}$$

$$D_4\vec{F}_4 = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \frac{1-j}{\sqrt{2}} & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & \frac{-1-j}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1 & 1 & 0 & 1\\ 0 & -j & 1 & j\\ 1 & -1 & 0 & j\\ 0 & j & 1 & -j\end{bmatrix} \tag{7.71}$$

$$= \begin{bmatrix}1 & 1 & 0 & 1\\ 0 & \frac{-1-j}{\sqrt{2}} & \frac{1-j}{\sqrt{2}} & \frac{1+j}{\sqrt{2}}\\ -1 & 1 & 0 & -j\\ 0 & \frac{1-j}{\sqrt{2}} & \frac{-1-j}{\sqrt{2}} & \frac{-1+j}{\sqrt{2}}\end{bmatrix} \tag{7.72}$$

Therefore, from F_8,

$$\vec{X} = \begin{bmatrix}13\\ -3.12-6.53j\\ j\\ 1.12-0.53j\\ -1\\ 1.12+0.53j\\ -j\\ -3.12+6.5355\end{bmatrix} \tag{7.73}$$



13. Write a C program to compute the 8-point FFT.
Solution:

```
#include <stdio.h>
#include <stdbool.h>
#include <math.h>
```

```
#include <stdlib.h>
#include <complex.h>
#include <time.h>
#define EPS 1e−6


complex ∗myfft(int n, complex ∗a)
{
        if (n == 1) return a;
        complex ∗g = (complex ∗)malloc(n
            /2∗sizeof(complex));
        complex ∗h = (complex ∗)malloc(n
            /2∗sizeof(complex));
        for (int i = 0; i < n; i++)
    {
                if (i%2) h[i/2] = a[i];
                else g[i/2] = a[i];
        }
        g = myfft(n/2, g);
        h = myfft(n/2, h);
        for (int i = 0; i < n; i++) a[i] = g[i
            %(n/2)] + cexp(−I∗2∗M_PI∗i/n)
            ∗h[i%(n/2)];
        free(g);
        free(h);
        return a;
}

int main()
{
        int n = 8;
        complex ∗a;
        a = (complex ∗)malloc(sizeof(
            complex)∗n);
        ∗a = 1.0, ∗(a+1) = 2.0, ∗(a+2) =
            3.0, ∗(a+3) = 4.0, ∗(a+4) = 2.0,
            ∗(a+5) = 1.0, ∗(a+6) = 0.0, ∗(a
            +7) = 0.0;
        complex∗ b;
        b = myfft(n, a);
        for (int i = 0; i < n; i++) printf("X
            (%d) = %lf + %lfj\n", i, creal
            (∗(b+i)), cimag(∗(b+i)));
        free(b);
        return 0;

}
```