Inter-Bank Network and Systemic Risk: Insight from Mean Field Game Model

Course Project
IE612

Submitted by

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Abstract

This project investigates systemic risk in interbank lending markets using tools from Mean Field Game (MFG) theory. Systemic risk—defined as the risk of disruption in capital flows leading to global economic slowdowns—arises from the interconnected dynamics of individual financial institutions. To study this, I first analyze a simplified model of bank reserve dynamics governed by mean-reverting drift and stochastic diffusion, providing a baseline for understanding uncontrolled interactions. In the second phase, I introduce optimal control through the Hamilton–Jacobi–Bellman (HJB) framework, allowing banks to minimize costs associated with deviations from the population mean and lending/borrowing activity. The resulting closed-loop equilibrium highlights how optimal strategies amplify effective interbank lending rates and generate additional liquidity. This controlled setting provides insight into how micro-level incentives influence macro-level financial stability, offering a mathematically tractable framework for studying contagion and coordination in financial networks.

1 Introduction

In modern financial systems, the health and stability of individual institutions are deeply interconnected. Localized failures — such as the default of a bank or a major financial entity — can propagate throughout the system, potentially leading to widespread crises. Understanding how such failures accumulate and interact is a central question in the study of **systemic risk**.

Systemic risk is defined as *"the risk of a disruption of the market's ability to facilitate the flows of capital that results in the reduction in the growth of the global GDP."* [2] This definition emphasizes that systemic risk is not merely about individual losses; it concerns the overall functioning of the financial ecosystem and its impact on the broader economy.

Parallel to these concerns, **Mean Field Game (MFG)** theory has emerged as a powerful framework to model the collective behavior of a large number of small interacting agents. Introduced independently by Lasry and Lions, and by Huang, Caines, and Malhamé, in 2006 [5],[6]. Mean Field Games provide mathematical tools to describe the limiting behavior of systems as the number of agents tends to infinity, where each agent optimizes its own cost while interacting with the aggregate effect of others. MFG theory has found broad applications across a variety of disciplines, including crowd dynamics, energy networks, traffic flow, opinion formation, and epidemic modeling—domains where decentralized decisions of a large population influence and are influenced by the collective state. MFG models have found significant applications in economics, finance, and systemic risk analysis, providing insights into how local optimization can lead to global phenomena.

This project studies systemic risk through the lens of MFGs in two stages. In the first part, I examined a simplified setup consisting only of mean-reversion and diffusion terms to understand the baseline behavior of bank reserves under stochastic dynamics without control. In the second part, I introduced control via a Hamilton–Jacobi–Bellman (HJB) framework to capture the optimal response of individual banks in a lending-borrowing game. By solving the resulting HJB equation, I derived optimal strategies and examined how they influence the collective behavior of the system, particularly in terms of effective interest rates and liquidity generation.

2 Model Description

Overview

We consider a finite population of N banks (or financial institutions), each indexed by $i = 1, \dots, N$. The state variable X_t^i denotes the monetary reserve of bank i at time t. The evolution of X_t^i is modeled by a stochastic differential equation that accounts for interaction between banks, individual controls for borrowing/lending, and random fluctuations. The model is designed to reflect how banks adjust their reserves based on peer influence, control incentives, and systemic randomness.

2.1 Bank Reserve Dynamics

The dynamics of each bank's reserve are given by:

$$dX_t^i = \frac{a}{N}\sum_{j=1}^{N}(X_t^j - X_t^i)dt + \alpha_t^i dt + \sigma d\widetilde{W}_t^i, \quad i = 1, \dots, N. \tag{1}$$

where:

- $a \geq 0$ is a parameter controlling the strength of mean-reversion — i.e., the tendency of bank i to align its reserves with the average of the population.
- α_t^i is the control rate, representing bank i's decision to borrow from or lend to a central authority (e.g., a central bank).
- $\sigma > 0$ is the volatility of the random noise.
- $\widetilde{W}_t^i$ are independent standard Brownian motions modeling idiosyncratic shocks to each bank.

This model reflects a peer-influenced adjustment mechanism: the term involving a drives banks toward the mean reserve level, while α_t^i allows each bank to optimize its position individually.

2.2 Mean-Field Formulation

Define the empirical mean reserve level:

$$\bar{X}_t = \frac{1}{N}\sum_{i=1}^{N} X_t^i.$$

Then the dynamics of X_t^i can be rewritten as:

$$dX_t^i = \left[a(\bar{X}_t - X_t^i) + \alpha_t^i\right]dt + \sigma d\widetilde{W}_t^i. \tag{2}$$

This form emphasizes the individual deviation from the population average. The term $a(\bar{X}_t - X_t^i)$ enforces convergence to the mean, and α_t^i represents optimal control actions in the presence of this interaction.

2.3 Control Objective and Cost Function

Each bank i seeks to choose a control process $\alpha^i = (\alpha_t^i)_{t\in[0,T]}$ to minimize the expected total cost:

$$J^i(\alpha^1,\ldots,\alpha^N) = \mathbb{E}\left[\int_0^T f_i(X_t,\alpha_t^i)\,dt + g_i(X_T^i)\right], \tag{3}$$

where:

- $f_i(x,\alpha^i)$ is the running cost function.
- $g_i(x)$ is the terminal cost function.

The running cost is given by:

$$f_i(x,\alpha^i) = \frac{1}{2}(\alpha^i)^2 - q\alpha^i(\bar{x}-x^i) + \frac{\varepsilon}{2}(\bar{x}-x^i)^2, \tag{4}$$

where:

- The term $\frac{1}{2}(\alpha^i)^2$ penalizes the use of control — large borrowing/lending incurs higher cost.
- The term $-q\alpha^i(\bar{x}-x^i)$ creates an incentive for alignment: it rewards borrowing/lending that reduces deviation from the mean. Here, $q>0$ is a parameter tuning this incentive.
- The term $\frac{\varepsilon}{2}(\bar{x}-x^i)^2$ penalizes deviation from the population mean directly, with $\varepsilon>0$ reflecting the cost of systemic risk or illiquidity.

The terminal cost is given by:

$$g_i(x) = \frac{c}{2}(\bar{x}-x^i)^2, \tag{5}$$

penalizing the deviation of the final reserve level from the average, with $c>0$.

2.4 Interpretation of Parameters

- **Mean-Reversion Parameter** (a): Governs how strongly banks adjust towards the empirical mean. A larger a implies tighter coupling between banks' behaviors.
- **Control Cost (Quadratic)**: $\frac{1}{2}(\alpha^i)^2$ discourages excessive intervention.
- **Alignment Incentive** (q): Encourages banks to act in a way that reduces their deviation from the average, indirectly promoting systemic stability.
- **Systemic Risk Penalty** (ε): Directly penalizes states that are far from the mean, representing regulation or financial stability concerns.
- **Terminal Penalty** (c): Ensures that reserves remain close to the mean at maturity T, reducing end-time risk imbalances.

3 Stability Study of a Simplified Model

Before solving the full control problem, it is instructive to first study the behavior of a simpler system. This simplified model omits the control term and retains only the stochastic fluctuations and a mean-reverting interaction among agents. Analyzing this case allows us to better understand the role of each component in the dynamics and their influence on the system's stability and risk of default.

The simplified stochastic differential equation (SDE) governing the dynamics of the i-th agent is given by:

$$dX_t^i = a\left(\bar{X}_t - X_t^i\right)dt + \sigma dW_t^i, \tag{6}$$

A bank is considered to have *defaulted* if its state X_T^i at final time T falls below the default threshold D.

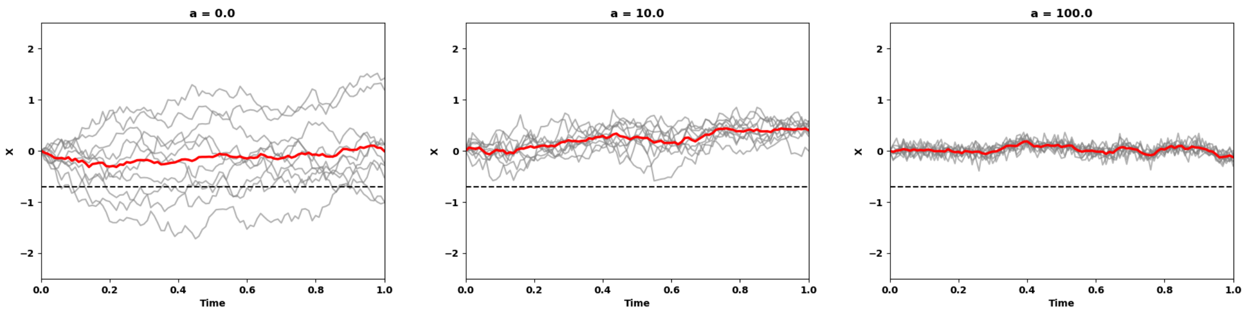


Figure 1: One realization of the N=10 trajectories for different values of the mean-reversion parameter a. As a increases, the distribution becomes more sharply peaked around the mean. The bold red line is the mean of all the trajectories. The dashed horizontal line is the default level, $D = -0.7$

3.1 Large deviations and systemic risk

Independent Case $(a = 0)$**:**

In the case of independent banks ($a = 0$), each bank follows a simple Brownian motion:

$$dX_t^i = \sigma dW_t^i. \tag{7}$$

The probability that a single bank defaults is given by the cumulative distribution function (CDF) of a normal random variable. Since $X_T^i \sim \mathcal{N}(0, \sigma^2 T)$, the probability of default is:

$$p = \mathbb{P}\left(\min_{0\le t\le T}(\sigma W_t) \le D\right) = 2\Phi\left(\frac{D}{\sigma\sqrt{T}}\right) \tag{8}$$

where $\Phi(\cdot)$ denotes the standard normal CDF. Here, p is the probability that the Brownian motion hits the barrier D at any time before time T. The factor of 2 in the LHS comes from the reflection principle in probability theory.

Dependent Case $(a > 0)$**:**

If we sum over all i's on both sides of equation 6 and scale it by N, we will get,

$$\bar{X}_t = \frac{\sigma}{N}\sum_{i=1}^{N} W_t^i \tag{9}$$

To study the systemic risk and large deviation we focus on the event when the mean drops below the default value. The probability of this event is given as

$$\begin{aligned} p = \mathbb{P}\left(\min_{0\le t\le T}(\bar{X}_t) \le D\right) &= \mathbb{P}\left(\min_{0\le t\le T}\left(\frac{\sigma}{N}\sum_{i=1}^{N} W_t^i\right) \le D\right) \\ &= \mathbb{P}\left(\min_{0\le t\le T}\tilde{W}_t \le \frac{D\sqrt{N}}{\sigma}\right) \\ &= 2\Phi\left(\frac{D\sqrt{N}}{\sigma\sqrt{T}}\right) \end{aligned} \tag{11}$$

As, D is negative, the probability of default for dependent case goes to zero as $N \to \infty$. So, increasing the number of banks, we can reduce the possibility of a systemic event. From equation 11, we can see that this event does not depend on $a > 0$. Increasing a increases the stability, but it does not prevent the systemic event. On the other hand, for a large value of a, the log-monetary reserves of all individual banks stay very close to the mean value (see figure **??**). This will create a situation where all banks will default together.

In summary, this simple model shows that lending and borrowing increase stability, but also contribute to systemic risk.

3.2 Monte Carlo Estimation of Loss Distributions

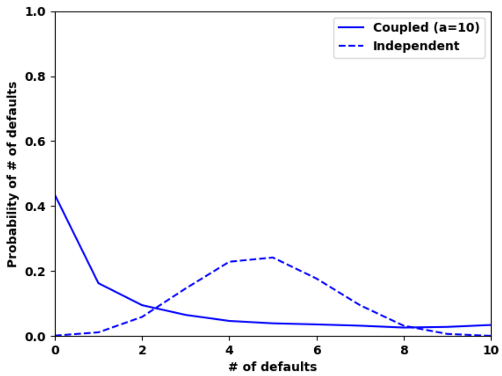
To compute the loss distributions for different values of a, we perform 10^4 Monte Carlo simulations of the system. At each simulation, we count the number of banks that have defaulted at time $t \in (0,T)$. The resulting histograms allow us to observe how increasing the mean-reversion strength a influences the distribution of defaults. The whole algorithm is given in the Appendix (A1).

This preliminary study provides crucial intuition for understanding the stabilizing role of inter-agent interactions and the importance of diffusion effects in the full control problem considered later.

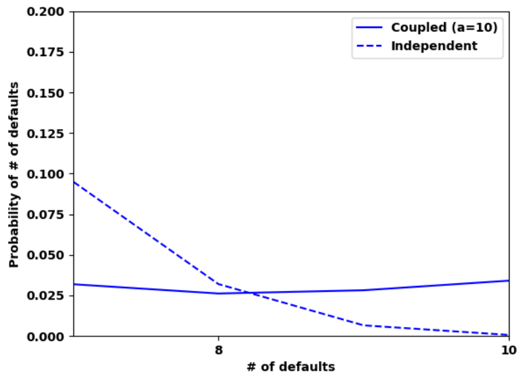
4 HJB Formalism and Its Application to Our Mean Field Game Problem

4.1 HJB Formalism for stochastic control Problems

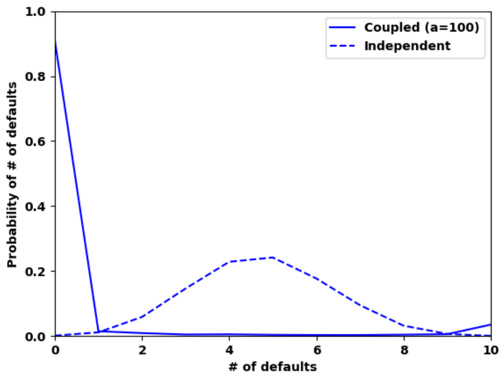
Before solving our specific problem, we briefly review the Hamilton–Jacobi–Bellman (HJB) formalism for a general stochastic control problem stated below.



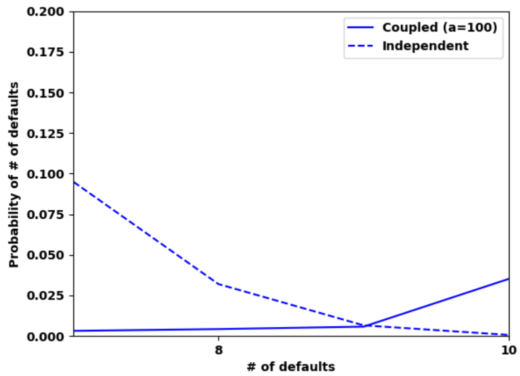
(a) Loss distribution for $a = 10$



(b) Tail of loss distribution for $a = 10$



(c) Loss distribution for $a = 100$



(d) Tail of loss distribution for $a = 100$

Figure 2: Comparison of loss distributions and their tails for different values of a.

The objective is to find a control α_t that minimizes the expected total cost:

$$J(\alpha) = \mathbb{E}\left[\int_0^T f(X_t, \alpha_t)\, dt + g(X_T)\right], \tag{10}$$

subject to the dynamics:

$$d\vec{X}_t = \vec{b}(X_t, \alpha_t, t)\, dt + \sigma(X_t, \alpha_t, t)\, d\vec{W}_t, \tag{11}$$

where X_t is the state, α_t is the control, and W_t is a standard Brownian motion.

The **value function** is defined as:

$$V(t,x) = \inf_{\alpha} \mathbb{E}\left[\int_t^T f(X_s, \alpha_s)\, ds + g(X_T) \,\middle|\, X_t = x\right]. \tag{12}$$

The associated **HJB equation** is:

$$\partial_t V + \inf_{\alpha}\left\{ f(x,\alpha,t) + \vec{\nabla} V . \vec{b}(x,\alpha,t) + \frac{1}{2} Tr(\sigma\sigma^T \nabla^2 V)\right\} = 0, \tag{13}$$

with the terminal condition $V(T,x) = g(x)$.

4.2 Application to Our MFG Problem

Comparing with our Mean Field Game (MFG) formulation in equations (1)-(5), we got,

$$b_i(\bar{X}_t^i, \alpha_t^i, t) = \left[a(\bar{X}_t - X_t^i) + \alpha_t^i\right]$$

All other functions are same as stated in equation (1)-(5). Using all those things, the resultant HJB equation is,

$$\partial_t V^i + \inf_{\alpha}\left\{ \sum_{j=1}^{N} \left[\alpha^j(\overline{X} - X^j) + \alpha^j\right] \partial_{x^j} V^i + \frac{\sigma^2}{2} \sum_{j=1}^{N}\sum_{k=1}^{N} (\beta^2 + \delta_{jk}(1-\beta^2))\partial^2_{x^j x^k} V^i \right. \tag{14}$$

$$\left. + \frac{(\alpha^i)^2}{2} - q^i \alpha^i (\overline{X} - X^i) + \frac{\varepsilon}{2}(\overline{X} - X^i)^2 \right\} = 0,$$

with terminal condition:

$$V^i(T, \vec{x}) = \frac{c}{2}(\overline{X} - X^i)^2. \tag{15}$$

To Solve the infimum over α^i, we have to solve,

$$\frac{\partial}{\partial \alpha^i}\left[\alpha^i \partial_{x^i} V^i + \frac{(\alpha^i)^2}{2} - q^i \alpha^i (\overline{X} - X^i)\right] = 0,$$

which gives the optimal feedback control:

$$\alpha^{i*} = q^i(\overline{X} - X^i) - \partial_{x^i} V^i. \tag{16}$$

Inspired from the optimal solution of α^i (equation 16), a solution ansatz for $V^i(t,\vec{x})$ is typically quadratic in $(\overline{X} - X^i)$,

$$V^i(t,x) = \frac{\eta_t}{2}(\bar{x} - x^i)^2 + \mu_t \tag{17}$$

here, η_t and μ_t are deterministic funtion of time only with terminal condition, $\eta_T = c$ and $\mu_T = 0$, such that, $V^i(T,x) = \frac{c}{2}(\bar{x} - x^i)^2$.

If put this ansatz into equation 16, we got the resultant optimal strategies as,

$$\alpha_t^{i*} = \left[q + \left(1 - \frac{1}{N}\right)\eta_t\right]\left(\overline{X}_t - X_t^i\right), \tag{18}$$

with the coefficient satisfying,

$$\dot{\eta}_t = 2(a+q)\eta_t + (1 - \frac{1}{N^2})\eta_t^2 - (\varepsilon - q^2), \tag{19}$$

with the terminal condition, $\eta_T = c$. The above equation is called Riccati equation. One can get this equation by simply puttin the optimal strategies and the ansatz for the value function into the HJB equation. The detailed derivation of this equation is given in the Appendix B.

Effective Dynamics

Substituting the optimal control into the SDE for X_t^i yields the forward dynamics of each agent:

$$dX_t^i = \left[a + q + \left(1 - \frac{1}{N}\right)\eta_t\right]\left(\overline{X}_t - X_t^i\right)dt + \sigma\left(\sqrt{1-\rho^2}dW_t^i + \rho dW_t^0\right), \tag{20}$$

The Riccati equation (equation 19) is explicitly solvable and the solution is given as,

$$\eta(t) = \frac{-(\varepsilon - q^2)\left(e^{(\delta_+ - \delta_-)(T-t)} - 1\right) - c\left(\delta_+ e^{(\delta_+ - \delta_-)(T-t)} - \delta_-\right)}{\left(\delta_- e^{(\delta_+ - \delta_-)(T-t)} - \delta_+\right) - c\left(1 - \frac{1}{N^2}\right)\left(e^{(\delta_+ - \delta_-)(T-t)} - 1\right)}$$

Here,

$$\delta^{\pm} = -(a+q) \pm \sqrt{R}$$

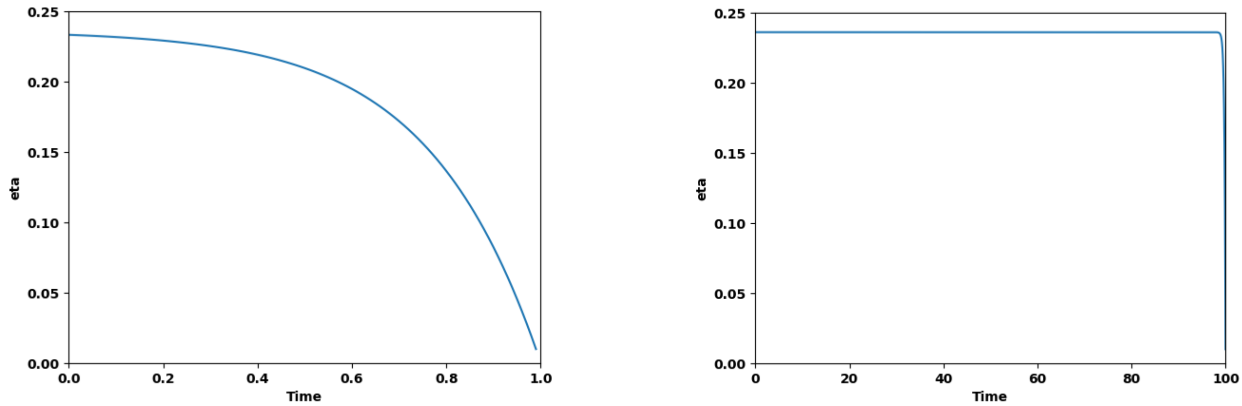
with

$$R = (a+q)^2 + (1 - \frac{1}{N})(\varepsilon - q^2)$$

5 Conclusion and Financial Implications

In this work, we investigated the systemic risk in interbank lending and borrowing through the lens of a mean field game (MFG) formulation, with particular focus on solving the Hamilton-Jacobi-Bellman (HJB) equation that governs the optimal control strategy for individual banks. Our model assumed the presence of a central authority that provides aggregate information (mean reserve level $\bar{X}_t$) to all banks, allowing them to independently compute their optimal controls based on local and aggregate states.

By solving the HJB equation, we derived the optimal control strategy $\hat{\alpha}_t^i$ for a representative bank, and studied the resulting dynamics of the log-reserve process X_t^i. The key insight from our solution is that the optimal feedback control modifies the drift term in the stochastic differential



(a) η_t as a function of time for $T = 1$

(b) η_t as a function of time for $T = 100$

Figure 3: Time evolution of η_t for two different horizons: $T = 1$ and $T = 100$, with $N = 10$, $a = 1$, $q = 1$, $\varepsilon = 2$, and $c = 0$.

equation for X_t^i, effectively enhancing the rate at which banks adjust their reserves relative to the average. The modified drift includes the term:

$$A_t := a + q + \left(1 - \frac{1}{N}\right)\eta_t,$$

where η_t arises from the solution of the HJB equation and can be interpreted as a time-varying incentive or pressure toward conformity among banks.

This adjustment has notable financial implications. In equilibrium, the presence of the control policy leads to enhanced interbank interactions—either encouraging lending and borrowing when $a > 0$, or effectively *creating* such a market dynamic when $a = 0$. Even in the absence of a traditional market for interbank borrowing and lending, the control term drives banks toward mean behavior, which mimics the function of such a market. The central authority, acting as a passive clearinghouse, merely provides aggregate information without enforcing trades, yet its presence is essential for the decentralized implementation of optimal controls.

Furthermore, although our HJB formulation does not explicitly depend on the common noise term (parameterized by ρ), this common noise still plays a significant role in the system's risk profile. As observed in earlier sections of the paper, such noise does not influence the form of the optimal control but does impact the variance of system-wide reserve dynamics, and hence the magnitude of systemic risk.

In the infinite-horizon limit, we observed that the optimal control converges to a steady state value $\bar{\eta}$,(see Figure 3) leading to a constant effective interbank rate:

$$A := a + q + \left(1 - \frac{1}{N}\right)\bar{\eta}.$$

This result aligns with the steady-state behavior observed in previous literature, despite the absence of an explicit open-loop vs. closed-loop comparison in our study. It reinforces the idea that under the mean field interaction and optimal feedback control, the banking system reaches a dynamic equilibrium characterized by increased liquidity and coordinated behavior.

In summary, our study highlights the utility of decentralized optimal control in modeling interbank dynamics, and shows how regulatory insight into aggregate states can facilitate coordination without active intervention. This has practical implications for financial regulation, suggesting that transparency and information sharing—rather than heavy-handed control—can be sufficient to stabilize systemic behavior and mitigate the risks of large deviations in reserve levels.

6 Model Extensions

The current model assumes homogeneity among banks, particularly through the use of a common volatility constant σ in the stochastic dynamics of reserves. A natural extension would be to introduce **heterogeneous volatility** across banks by assigning a distinct σ_i to each bank i. This would capture differences in risk profiles or portfolio compositions, and would result in a modified HJB formulation where the diffusion term in the dynamics becomes bank-specific. Analyzing how such heterogeneity affects equilibrium behavior and systemic risk propagation would be an important next step.

Another promising direction involves the **inclusion of a large (central) bank** in the model, as explored in recent work [3]. In this framework, the central bank interacts asymmetrically with smaller banks—borrowing from and lending to them—while itself being influenced only marginally by their behavior. This modification leads to a leader-follower structure in the mean field game, often modeled using Stackelberg strategies or through coupling a McKean-Vlasov control problem for the central bank with MFGs for the smaller banks. Introducing such a player allows the study of explicit monetary policy interventions and their stabilizing effects on systemic liquidity and volatility.

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Appendix

Appendix A: Simulation Algorithm

Algorithm 1 Monte Carlo Simulation of Bank Defaults

Input: Number of banks N, time horizon T, time step Δt, mean-reversion strength a, noise level σ, default level D, number of simulations n_{sim}
Initialize $steps = T/\Delta t$
for each simulation $s = 1$ to n_{sim} **do**
    Initialize positions $X_i = 0$ for all banks $i = 1$ to N
    if uncoupled dynamics **then**
        Initialize $min\_X_i = 0$ for all banks
    end if
    for each time step t **do**
        Generate independent Gaussian noise $dW_i \sim \mathcal{N}(0, \sigma\sqrt{\Delta t})$ for all banks
        if coupled dynamics **then**
            Compute average state $\bar{X} = \frac{1}{N}\sum X_i$
            Update: $X_i \leftarrow X_i + a(\bar{X} - X_i)\Delta t + \sigma dW_i$
            Update running minimum: $min\_X_i \leftarrow \min(min\_X_i, X_i)$
        else
            Update: $X_i \leftarrow X_i + \sigma dW_i$
            Update running minimum: $min\_X_i \leftarrow \min(min\_X_i, X_i)$
        end if
    end for
    if coupled dynamics **then**
        Count defaulted banks: $n_{\text{defaults}} = \#\{i : X_i < D\}$
    else
        Count defaulted banks: $n_{\text{defaults}} = \#\{i : min\_X_i < D\}$
    end if
    Record n_{defaults}
end for
Generate histogram of defaults from all simulations
Plot number of defaults vs. empirical probability

Appendix B: Derivation of the HJB solution

We consider the Hamilton–Jacobi–Bellman (HJB) equation derived in the main text for player i in the finite-agent Mean Field Game system:

$$\begin{aligned} \partial_t V^i + \inf_{\alpha^i} \Bigg\{ & \sum_{j=1}^{N} \left[a(\bar{X} - X^j) + \alpha^j \right] \partial_{x^j} V^i + \frac{\sigma^2}{2} \sum_{j,k=1}^{N} \left(\beta^2 + \delta_{jk}(1-\beta^2) \right) \partial^2_{x^j x^k} V^i \\ & + \frac{1}{2}(\alpha^i)^2 - q^i \alpha^i (\bar{X} - X^i) + \frac{\varepsilon}{2}(\bar{X} - X^i)^2 \Bigg\} = 0, \end{aligned} \tag{21}$$

with terminal condition:

$$V^i(T,\vec{x}) = \frac{c}{2}(\bar{X} - X^i)^2.$$

Optimal Control via First-Order Conditions

To solve the infimum over α^i, isolate the terms that depend on α^i:

$$\alpha^i \partial_{x^i} V^i + \frac{1}{2}(\alpha^i)^2 - q^i \alpha^i (\bar{X} - X^i).$$

Taking the derivative with respect to α^i and setting it to zero gives:

$$\frac{\partial}{\partial \alpha^i}\left(\alpha^i \partial_{x^i} V^i + \frac{1}{2}(\alpha^i)^2 - q^i \alpha^i (\bar{X} - X^i)\right) = 0,$$

which yields the optimal feedback control:

$$\alpha^{i*} = q^i(\bar{X} - X^i) - \partial_{x^i} V^i.$$

Ansatz for Value Function

Motivated by the quadratic terminal cost and the structure of the optimal control, we adopt a quadratic ansatz:

$$V^i(t,x) = \frac{\eta_t}{2}(\bar{x} - x^i)^2 + \mu_t, \tag{22}$$

where η_t and μ_t are deterministic functions of time, satisfying terminal conditions $\eta_T = c$, $\mu_T = 0$.

Let us compute the necessary derivatives:

$$\partial_{x^j} V^i = -\eta_t(\bar{x} - x^i)\left(\delta_{ij} - \frac{1}{N}\right),$$
$$\partial^2_{x^j x^k} V^i = -\left(\delta_{ij} - \frac{1}{N}\right)\left(\delta_{ij} - \frac{1}{N}\right)\eta_t$$

Plug Back into the HJB Equation

Insert the ansatz and optimal control into the HJB equation. After simplification, we equate coefficients of $(\bar{X} - X^i)^2$ on both sides. This yields the Riccati equation for η_t:

$$\dot{\eta}_t = 2a\left(1 - \frac{1}{N}\right)\eta_t - \left(1 - \frac{1}{N}\right)^2 \eta_t^2 + 2q\left(1 - \frac{1}{N}\right)\eta_t - \varepsilon, \tag{23}$$

with terminal condition $\eta_T = c$.

Similarly, collecting the constant terms leads to the ODE for μ_t:

$$\dot{\mu}_t = \sigma^2 \eta_t \left[\beta^2\left(1 - \frac{1}{N}\right) + \frac{1}{N}\right], \tag{24}$$

with terminal condition $\mu_T = 0$.

Final Form of Optimal Control

Using the value gradient:

$$\partial_{x^i} V^i = -\eta_t(\bar{X} - X^i)\left(1 - \frac{1}{N}\right),$$

the optimal control becomes:

$$\alpha_t^{i*} = q(\bar{X}_t - X_t^i) + \eta_t(\bar{X}_t - X_t^i)\left(1 - \frac{1}{N}\right) = \left[q + \left(1 - \frac{1}{N}\right)\eta_t\right](\bar{X}_t - X_t^i).$$

This completes the derivation.