KZ equation and correspondence in CS

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Abstract

A sketch from the basics of the WZW model to the KZ equation and its relation to Chern-Simons theory.

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1 Frame of WZW model

WZW model is a nonlinear sigma model with the target space being a Lie group,in this sketch,we choose the map form Σ to SU(2) as a model.In fact ,this map describes the gauge transformation over the SU(2) principle bundle and relate to Chern-Simons theory.

1.1 Action and Conservation Current

In WZW model,the action is:

$$S^{\mathrm{WZW}} = \frac{k}{16\pi} \int \mathrm{d}^2 x \, \mathrm{Tr}' \left(\partial^\mu g^{-1} \partial_\mu g \right) + k\Gamma$$

with

$$\Gamma = \frac{-i}{24\pi} \int_B \mathrm{d}^3 y \, \epsilon_{\alpha\beta\gamma} \, \mathrm{Tr}' \left(\tilde{g}^{-1} \partial^\alpha \tilde{g} \, \tilde{g}^{-1} \partial^\beta \tilde{g} \, \tilde{g}^{-1} \partial^\gamma \tilde{g} \right)$$

the kinetic term is nature,since $\partial^\mu g^{-1} = -(g^{-1}\partial_\mu g)(g^{-1}\partial_\mu g)$ and $U = g^{-1}\partial_\mu g$ is the lie algebra,so it's the contraction of killing form with derivation as common kinetic term.But if calculate its emotion function,we get :

$$\delta\mathcal{L}_{\prime} = \frac{\partial \mathrm{Tr}' \left(U^\dagger U \right)}{\partial U} \cdot \delta U = \mathrm{Tr}' \left(\delta g \cdot \partial_\mu U \right) = \mathrm{Tr}' \left(\delta g \cdot \partial^\mu \left(g^{-1} \partial_\mu g \right) \right)$$

which results in the following equation of motion:

$$\partial^\mu \left(g^{-1} \partial_\mu g \right) = 0.$$

but if we translate this to complex coordinate,it is:

$$\partial_z \tilde{J}_{\bar{z}} + \partial_{\bar{z}} \tilde{J}_z = 0.$$

with $\tilde{J}_z = g^{-1}\partial_z g$ and $\tilde{J}_{\bar{z}} = g^{-1}\partial_{\bar{z}} g$. yet in classical CFT,we know that the separation of homology and anti-homology(i.e $\partial_{\bar{z}} T = 0$) plays an important role in OPE, and corresponds to conformal invariance under renormalization. So we introduce the term Γ (it can also be considered as the pull-back of the wedge 3-form of the Maurer-Cartan form) ,here $\tilde{g}$ means extension g to a 3D-manifold with boundary Σ ,since the extension is not unique ,k has quantized to some positive integer.

Check its motion equation,we have: $\partial_z\left(g^{-1}\partial_{\bar{z}}g\right)=0$ and let

$$J_z=-k\partial_z g g^{-1},\quad J_{\bar{z}}=kg^{-1}\partial_{\bar{z}}g$$

,we get the expected property.

Analog with energy-momentum tensor as conservation current as translation,we define the corresponding infinitesimal transform:

$$\delta_\omega g=\omega g\quad \delta_{\bar{\omega}}g=-g\bar{\omega}\quad (\omega\ belongs\ to\ lie\ algebra)$$

check $\delta\mathcal{L}$,this is a local symmetry transformation

$$G(z)\times G(\bar{z})$$

1.2 Affine Lie algebra

We extend J and w:

$$J=\sum_a J^a t^a,\quad \omega=\sum_a \omega^a t^a$$

by directly calculation

$$\delta_\omega J=-k\left[\partial_z(\delta_\omega g)g^{-1}-\partial_z g g^{-1}\delta_\omega g^{-1}\right]=-k(\partial_z\omega g+\omega\partial_z g)g^{-1}+k\partial_z g g^{-1}\omega=[\omega,J]-k\partial_z\omega$$

It can be rewritten as

$$\delta_\omega J^a=\sum_{b,c}f_{abc}\omega^b J^c-k\partial_z\omega^a$$

by Ward identity in integration form,

$$\langle\delta X\rangle=\int dx\,\partial_\mu(j_a^\mu(x)X)\omega_a(x)=\delta_{\omega,\bar{\omega}}\langle X\rangle=-\frac{1}{2\pi i}\oint dz\,\omega^a\langle J^a X\rangle$$

by conformal Ward identity,

$$[Q_a,\Phi]=-i\delta_a\Phi$$

we get

$$[J^a_{n'},J^b_m]=\sum_c if_{abc}J^c_{n+m}+kn\delta_{ab}\delta_{n+m,0}\ (here\ the\ low\ index\ n\ means\ laurent\ expansion)$$

it's a affine lie algebra structure denoted as

$$\hat{g}=g\otimes\mathbb{C}[m,m^{-1}]\oplus C\hat{k}\oplus CL_0$$

its irreducible representation can be calculate in wight vector space. For the case of $su(2)$. The generator of lie algebra

$$t_1=\frac{1}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix},$$
$$t_2=\frac{1}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix},$$
$$t_3=\frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$[T_a, T_b] = i\epsilon_{abc}T_c \quad (a,b,c=1,2,3)$$

its H_0^1 chosen as $-t^3 \otimes m^0$ then the affine weight of a state is marked by the eigenvalue of $(H_0^1, \hat{k}, -L_0)$.

We can also define highest weight through primary field,where we extend

$$J^a(z) = \sum_i \frac{J_i^a}{z^i+1}$$

and define primary field as:

$$J_0^a \phi_\lambda = -t_\lambda^a \phi_\lambda \quad J_n^a \phi_\lambda = 0$$

Here ,we claim that state defined by representation is equivalent to primary and secondary field, since for highest state,

$$J_0^a|\lambda\rangle = J_0^a \phi_\lambda|0\rangle = -t_\lambda^a \phi_\lambda|0\rangle = -t_\lambda^a|\lambda\rangle = \lambda|\lambda\rangle$$

the condition $J_n^a \phi_\lambda = 0$ can be proved in similar.

To $\hat{su(2)}$, its representation is limited by Weyl chambers:

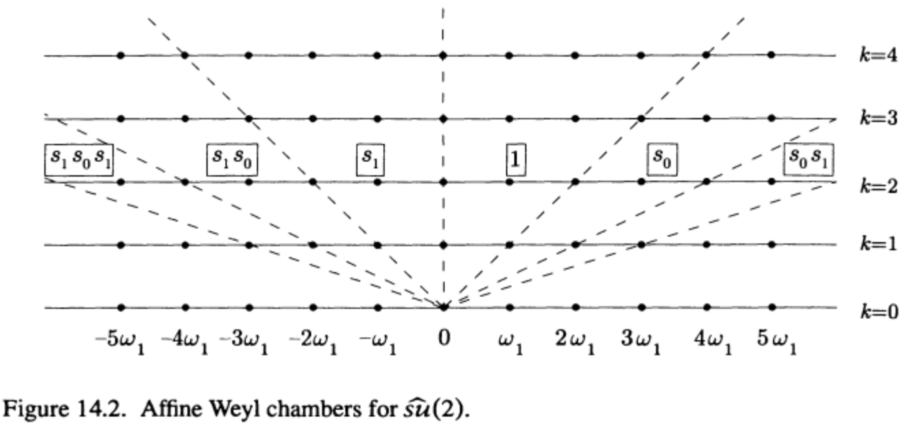

Figure 1: Weyl chambers for $\hat{su(2)}$

when $k > 0$,all its representation are integrable.

2 Relation to Chern-Simons Theory

The action of Chern-Simons Theory is

$$CS(A) = \frac{k}{4\pi}\int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right),$$

here we use the following model:

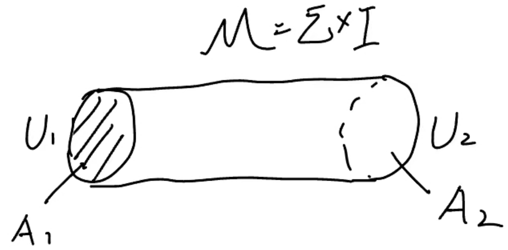

Figure 2: Boundary model

here, M is a compact oriented 3-D manifold with boundary Σ. We assume

$$M = \Sigma \times I, I\ close\ in\ R, \Sigma = U_1 \cup U_2$$

P is the principal SU(2) bundle over M, A_M is the space of connections on $P, \mathcal{G}$ is the gauge group of P so the corresponding path-integral is:

$$Z_{CS} = \int_{\mathcal{A}_{\mathcal{M}}} e^{iS_{CS}(A)}$$

but for manifolds with boundary, it depends on the configuration of A limit on the boundary, we denote A_1, A_2, so the path integral in fact is a functional of A_1, A_2,so we denote $\mathcal{A}_\Sigma$, the space of connection of P limit on Σ.Now,the path-integral is

$$Z_{CS}(A_1, A_2) = \int_{A|_\Sigma=(A_1,A_2)} \mathcal{D}A\, e^{iS_{CS}(A)}$$

a function on $\mathcal{A}_\Sigma$. It can be seemed as section of a line bundle over $\mathcal{A}_\Sigma$ with nature hermitian metric bu fusion rule.

In category language,this means "a 3-D QFT assign to 3-D manifold gives a partition function and assign to 2-D manifold gives a quantum Hilbert space".In particular,since Chern-Simons is topology field theory,and observable depends on gauge field, we should process geometric quantization on moduli space of flat connections on boundary modulo gauge transformations,denoted $\mathcal{M}$.

To realize Kahler polarization on $\mathcal{M}$, we select a complex structure to Σ (this also means selecting a conformal structure),and we can construct a symplectic form on $\mathcal{M}$, denoted the space $\mathcal{M}_J$ and get a Quantum Hilbert space $\mathcal{H}_\Sigma^{(J)}$, and we regard it as a modular functor , since we can construct a flat connection to identify complex structure, at last we get functor

$$\mathcal{H} : \Sigma \to \mathcal{H}_\Sigma$$

In this sense ,if we isomorphic $\mathcal{H}_\Sigma$ with its dual space.The correlation function can be seen as operator from state on U1 to state on U2.

3 KZ equation

3.1 Sugawara Construction

By directly calculation,the energy-momentum tensor of $\mathcal{L}_{wzw}$ is:

$$T(z) = \frac{k}{2} Tr(\partial_z g g^{-1} \partial_z g g^{-1}) = \frac{k}{2} Tr(JJ)$$

and by the orthonormal relation of generator we have

$$T(z) = \frac{k}{2}\Sigma_a (J^a J^a)$$

but there is a abnormal that

$$J^a(w)T(z) \neq \overline{J^a(w)(J^b(z)}J^b(z))$$

here the overline means the contraction in OPE. So we modify it to

$T(z) = \frac{1}{2(k+g)}\sum_a (J^a J^a)(z)$ (g is the dual Coxeter number relate to structure constant), $for\ SU(2)\ g = 2$

and under Laurent extension,we have the KZ Equation,which is a necessary and sufficient condition to highest weight state:

$$L_{-1}|\varphi_i\rangle = \frac{1}{k+g}\sum_a \left(J^a_{-1}J^a_0\right)|\varphi_i\rangle = -\frac{1}{k+g}\sum_a \left(J^a_{-1}t^a_i\right)|\varphi_i\rangle$$

We consider the insertion of the zero vector

$$|x\rangle = \left[L_{-1} + \frac{1}{k+g}\sum_a (J^a_{-1}t^a_i)\right]|\varphi_i\rangle = 0 \tag{15.96}$$

inside the correlation function of a set of primary fields. We note that the insertion of the operator J^a_{-1} in the correlator can be expressed through Laurent extension at each z_i

$$\begin{aligned}\langle\varphi_1(z_1)\cdots(J^a_{-1}\varphi_i)(z_i)\cdots\varphi_n(z_n)\rangle &= \frac{1}{2\pi i}\oint_{z_i}\frac{dz}{z-z_i}\langle J^a(z)\varphi_1(z_1)\cdots\varphi_n(z_n)\rangle \\ &= \frac{1}{2\pi i}\oint_{z_{j\neq i}}\frac{dz}{z-z_i}\sum_{j\neq i}\frac{t^a_j}{z-z_j}(\varphi_1(z_1)\cdots\varphi_n(z_n)) \\ &= \sum_{j\neq i}\frac{t^a_j}{z_i-z_j}\langle\varphi_1(z_1)\cdots\varphi_n(z_n)\rangle\end{aligned}$$

At last we have:

$$\langle\varphi_1(z_1)\cdots\chi(z_i)\cdots\varphi_n(z_n)\rangle = \left[\partial_{z_i} + \frac{1}{k+g}\sum_{j\neq i}\frac{\sum_a t^a_i\otimes t^a_j}{z_i-z_j}\right]\langle\varphi_1(z_1)\cdots\varphi_n(z_n)\rangle = 0$$

this is KZ equation for conformal block and also necessary and sufficient

3.2 KZ connection

To relate with Chern-Simons, we give a geometry description of n-point correlation function.

We first divide n-point correlation function to conformal block by operator algebra:

$$\langle\varphi_1(x_1)\cdots\varphi_i(x_i)\rangle = \sum_{p,q}C_{p,q}F_p(x_i)\bar{F}_q(\bar{x}_i),$$

so the we get the KZ equation for conformal block:

$$\frac{\partial F}{\partial z_i} = \frac{1}{k+h^\vee}\sum_{j\neq i}\frac{\sum_a t^a_i\otimes t^a_j}{z_i-z_j}F,$$

here we can write the correlation function as $\langle\varphi_1(x_1)\cdots\varphi_i(x_i)\rangle = \langle 0|\lambda_1\cdots\lambda_i\rangle$,where we use the equivalent between primary field with state,in this case conformal block can be seen as a "channel" from $|\lambda_1\cdots\lambda_i\rangle$ go through integrable weight $|p\rangle$ to 0. So we can define conformal block as:

$$\Psi : H_{\lambda_1}\otimes\cdots\otimes H_{\lambda_n}\to\mathbb{C}$$

and the space of all the "channel" as $\mathcal{H}(p_1,\cdots,p_n;\lambda_1,\cdots,\lambda_n)$.

Because of the dependence on position and state,we define

$$\mathcal{E}_{\lambda_1,\ldots,\lambda_n} = \bigcup_{(p_1,\ldots,p_n)\in\mathrm{Conf}_n(\Sigma)} \mathcal{H}(p_1,\ldots,p_n;\lambda_1,\ldots,\lambda_n)$$

as a subbundle of E,which defined as:

$$E = \mathrm{Conf}_n(\mathbb{CP}^1)\times\mathrm{Hom}_{\mathbb{C}}\left(\bigotimes_{j=1}^{n} H_{\lambda_j},\mathbb{C}\right)$$

is a vector bundle over $Conf_n(\Sigma)$. To select the subbundle,we define KZ connection as:

$$\nabla_{\frac{\partial}{\partial z_j}}\Psi = \frac{\partial\Psi}{\partial z_j} - L_{(j)_{-1}}\Psi.$$

by KZ equation,we realize conformal block as horizontal section of E. If we regard $\mathcal{H}(p_1,\cdots,p_n;\lambda_1,\cdots,\lambda_n)$ as functor, in fact it's the same with the functor $\mathcal{H}$ (last part of section 2) act on Σ equip with wilson line[2].

3.3 Solution to four point conformal block on SU(2)

Here we choose SU(2),K=1 to show the solution of conformal block.In K=1 case, by Weyl chambers there are only two holomorphic conformal block,denoted F_0 and F_1 ,corresponds to 1 and 3 dimension integrable representation. Since conformal block satisfy conformal invariance,it must have this form:

$$F(z_i) = [(z_1-z_4)(z_2-z_3)(\bar{z}_1-\bar{z}_4)(\bar{z}_2-\bar{z}_3)]^{-2h}\, F(x)$$

where h is the conformal dimension and

$$x = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$$

In the following, we set $z_{ij} = z_i - z_j$. so the Knizhnik-Zamolodchikov equation (15.99) becomes

$$\left(\partial_i + \frac{1}{k+N}\sum_{j\neq i} t_i^a\otimes t_j^a\frac{1}{z_i-z_j}\right)[z_{14}z_{23}]^{-2h}F = 0$$

The next step is to transform these partial differential equations into an ordinary differential equation in the variable x. Consider the case $i=1$; since

$$\partial_1 = \left(\frac{x}{z_{12}} - \frac{x}{z_{14}}\right)\partial_x$$

the equation takes the form

$$\left(-\frac{2h}{z_{14}} + \left(\frac{x}{z_{12}} - \frac{x}{z_{14}}\right)\partial_x + \frac{1}{k+N}\sum_{j=2,3,4}\frac{\sum_a t_j^a\otimes t_j^a}{z_1-z_j}\right)F = 0$$

Then we choose:

$$z_1 = x, \quad z_2 = 0, \quad z_3 = 1, \quad z_4 = \infty$$

The last equation is:

$$\left\{ \partial_x + \frac{1}{k+N} \sum_a t_1^a \otimes t_2^a \frac{1}{x} + \frac{1}{k+N} \sum_a t_1^a \otimes t_3^a \frac{1}{x-1} \right\} F = 0$$

so we have to determine the representation of tensor of generator on the singlet and triplet state,here we use Casimir Operator:

$$C = \sum_a J^a J^a$$

it's the unique commute with all the generators for SU(2).And for two representations of the tensor product $V_{j_1} \otimes V_{j_2}$, the total Casimir operator is given by:

$$C_{\text{total}} = \sum_a (t_1^a + t_2^a)^2 = C_1 + C_2 + 2 \sum_a t_1^a t_2^a.$$

Thus, the generator factor can be represented as:

$$\sum_a t_1^a \otimes t_2^a = \frac{1}{2}(C_{\text{total}} - C_1 - C_2).$$

for the case the weight of $\phi_1, \phi_2, \phi_3, \phi_4$ is (1/2,1/2,1/2,1/2)

$$C = (\lambda, \lambda + 2\rho)$$

λ is weight and ρ is root.So, $C_1 = -\frac{3}{4} = C_2$

For singlet state:

$$\sum_a t_1^a \otimes t_2^a = -\frac{3}{4} \quad \sum_a t_1^a \otimes t_3^a = -\frac{3}{4}$$

here to calculate cross term we have use :

$$\sum_a t_{ij}^a t_{kl}^a = \frac{1}{2} \delta_{il} \delta_{jk} - \frac{1}{2N} \delta_{ij} \delta_{kl}.$$

and notice the first tern is permutation which is (-1) for singlet the later is id. now the equation is:

$$\partial_x F_0(x) = -\frac{1}{4} \left(\frac{1}{x} + \frac{1}{x-1} \right) F_0(x),$$

with solution:

$$\mathcal{F}_0(x) = C \cdot x^{-\frac{1}{4}} (1-x)^{-\frac{1}{4}}.$$

Similar,for triplet state,

$$\sum_a t_1^a \otimes t_2^a = \frac{1}{4} \quad \sum_a t_1^a \otimes t_3^a = \frac{1}{4}$$

the solution is:

$$\mathcal{F}_1(x) = C \cdot x^{\frac{1}{4}} (1-x)^{\frac{1}{4}}.$$

References

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