Higher Algebra in Symmetry TQFT

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1 Introduction

It's a note for Higher Algebra and twist theory I studied in Amherst .

2 Topology Quantum Field Theory

Followings are the formal language I will use. (refer[2])

2.1 Basic structure

Definition 2.1 (TQFT)**.** *Topology Field Theory is QFT aligned with the generator of algebra* $\{P_\mu, Q, Q_\mu\}$*,satisfy:*

$$Q^2 = [Q_\mu, Q_\nu] = [Q_\mu, P_\nu] = [Q, P_\nu] = 0, \quad [Q, Q_\mu] = iP_\mu. \quad (2.1)$$

and observable are Q-cohomology.

Here,Q usually comes from twist of Supersymmetry charge.

Definition 2.2. $Op_{\delta,x}$ *:=all operators supported in* $Ball_\delta(x)$

Prop 2.2.1. *Since the theory is topological,* $Op_{\delta,x}$ *is independent on* δ *and we denoted its cohomology class as* $\mathcal{A}_x$*.*

What's more for any operator supported on B_δ,we can use
Operator $\Rightarrow$ *State* and *Operator* $\Leftrightarrow$ *State* (details in [4]) to regard it as a local operator on x.

2.2 Topological Algebra(First product)

Definition 2.3. *The first product is a family of embedding defined on $C^2(M)$ given by:*

$$\begin{aligned} *_{x1,x2} : A \otimes A \to Op_1 \\ O_1, O_2 \mapsto O_1(x_1)O_2(x_2). \end{aligned}$$

The first order product in fact have the factorization structure:

Prop 2.3.1. *$*_{x1,x2}$ is equal in the same connected component of $C_2(M)$, in R^d case,since $C_2(R^d)$ is trivial when $d \geq 2$,we can define $* := *_{x1,x2}$*

$$\begin{aligned} Pf : O_1 *_{x1,x2+a} O_2 - O_1 *_{x1,x2} O_2 &= O_1(x^1)O_2(x^2 + a) - O_1(x^1)O_2(x^2) \\ &= a \cdot O_1(x^1)\partial_\mu O_2(x^2) = a \cdot O_1(x^1)QQ_\mu O_2(x^2) \\ &= a \cdot (-1)^{F_1} Q(a \cdot O_1(x^1)Q_\mu O_2(x^2)) = 0 \end{aligned}$$

Prop 2.3.2. ** is associative*

Prop 2.3.3. *Obvious,* is graded-commutative.*

2.3 Secondary Product

Definition 2.4 (Descendant). $\mathcal{O}^{(k)}(x) = \frac{1}{k!}\left(Q_{\mu_1} \cdots Q_{\mu_k}\mathcal{O}\right)(x)\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}$
$\mathcal{O}^* = \sum_{k=0}^{\infty} \mathcal{O}^{(k)}$

Prop 2.4.1 (Descendant Equation). $Q\left(\mathcal{O}^{(k)}(x)\right) = d\mathcal{O}^{(k-1)}(x), \quad Q\mathcal{O}^* = d\mathcal{O}^*$

$$Pf : Q\mathcal{O}^{(k)}(x) = Q(\frac{1}{k!}Q_{\mu 1}...) = \frac{1}{k!} \cdot iP_{\mu 1}(...)dx^{\mu 1} = dO^{(k-1)}$$

Definition 2.5 (Homology Operator). *Denote $\mathcal{O}(\gamma) := \int_\gamma \mathcal{O}^*$,here $[\gamma]$ is homology class.*

Prop 2.5.1. *$O(\gamma)$ is operator supported on γ, and as Prop 2.2.1 $\mathcal{O}(\gamma) \in A$*

Definition 2.6 (Secondary Product). *The secondary product is a family of map defined on $C^1(M) = M$ given by:*

$$\begin{aligned} \{,\}_x : A \otimes A \to A_x \\ O_1, O_2 \mapsto \mathcal{O}_1\left(S_x^{d-1}\right)\mathcal{O}_2(x) \end{aligned}$$

Prop 2.6.1. *The same as the proof of* $*_{x1,x2}$ *is invariant under deformation,*$\{,\}_x$ *is invariant under deformation,so we can define* $\{,\}_x := \{,\}$ *in* R^d *so it's a product in* A.

Now we define a new product $\star$ defined on $C^n(M)$,and it captures the structure of Poisson structure of secondary product.

Definition 2.7 (Descendant on configuration space)**.** $(\mathcal{O}_1 \boxtimes \mathcal{O}_2 ... \boxtimes \mathcal{O}_n)^* = \mathcal{O}_1^* \wedge \sigma^{F_1} \mathcal{O}_2^* ... \wedge \sigma^{F_1...+F_{n-1}} \mathcal{O}_n^* .$, *where* σ *acts as* $(-1)^k$ *on the degree* k *part.*

Prop 2.7.1 (Descendant Equation)**.** $\mathcal{Q}\left(\mathcal{O}_1 ... \boxtimes \mathcal{O}_n\right)^* = d\left(\mathcal{O}_1 ... \boxtimes \mathcal{O}_n\right)^*.$

In particular,$\left(\mathcal{O}_1 \boxtimes \mathcal{O}_2\right)^{(k)}(x_1, x_2) = \sum_{n=0}^{k} (-1)^{(k-n)F_1} \mathcal{O}_1^{(n)}(x_1) \wedge \mathcal{O}_2^{(k-n)}(x_2)$
For $P \in H_\bullet(C^n(M), \mathbb{Z})$,we define

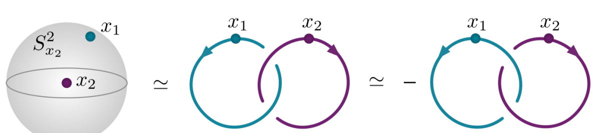
$$\star_P : A^{\otimes n} \to A$$
$$(O_1, ..., O_n) \mapsto \left(\mathcal{O}_1 \boxtimes \mathcal{O}_2 ... \boxtimes \mathcal{O}_n\right)(P) := \int_P \left(\mathcal{O}_1 \boxtimes \mathcal{O}_2 ... \boxtimes \mathcal{O}_n\right)^*$$

Then we prove some property of secondary product: .6.2

Prop 2.8.1 (Commutative property)**.** $\{,\}$ *is graded-commutative*

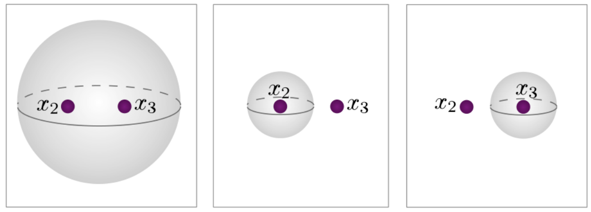
$$Pf : \{O_1, O_2\}_x = O_1(S_x^{d-1})O_2(x) = (O_1 \boxtimes O_2)(S_x^{d-1}, \{x\}) = O_1 \star_{[S_x^{d-1} \times \{x\}]} O_2$$
in R^d case, $C^2(M) \cong S^{d-1}$,
$[S_x^{d-1} \times \{x\}]$ *and* $[\{x\} \times S_x^{d-1}]$ related by antipodal map
$$\therefore O_1 \star_{[S_x^{d-1} \times \{x\}]} O_2 = (-1)^d O_1 \star_{[\{x\} \times S_x^{d-1}]} O_2 = (-1)^d O_1(x) O_2(S^{d-1})$$
$$= (-1)^{F_1 F_2 + d} \{O_2, O_1\}$$

In particular ,in d=3 case $\{,\}$ can be used to compute linking number. That is for Γ_1, Γ_2 two circles in R^3 , we have $[O_1 \boxtimes O_2]([\Gamma_1, \Gamma_2]) = l \ \cdot (O_1 \boxtimes O_2)(S_x^{d-1}, \{x\}) = l \ \cdot \{O_1, O_2\}$



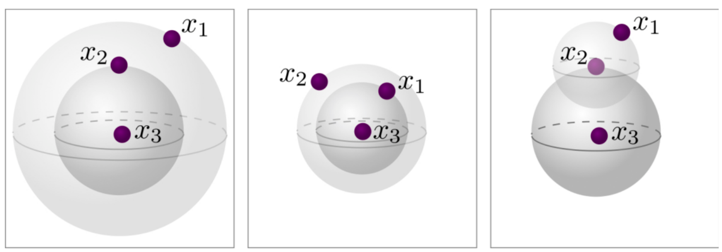
Prop 2.8.2 (Derivation property)**.** $\{[\mathcal{O}_1],\ [\mathcal{O}_2] * [\mathcal{O}_3]\} = \{[\mathcal{O}_1],\ [\mathcal{O}_2]\} * [\mathcal{O}_3] + (-1)^{(F_1+d-1)F_2} [\mathcal{O}_2] * \{[\mathcal{O}_1],\ [\mathcal{O}_3]\}$

The proof is directly,$(-1)^{F_1+d-1}$ from the order (d-1) form.And notice that two parts of the equation correspond to these graphs,and $[\Gamma_1]=[\Gamma_2]+[\Gamma_3]$



Prop 2.8.3 (Jacobi identity)**.**

$$\{[\![\mathcal{O}_1]\!],\{[\![\mathcal{O}_2]\!],[\![\mathcal{O}_3]\!]\}\}-(-1)^{(F_1+d-1)(F_2+d-1)}\{[\![\mathcal{O}_2]\!],\{[\![\mathcal{O}_1]\!],[\![\mathcal{O}_3]\!]\}\}$$
$$=(-1)^{(d-1)(F_1+d-1)}\{\{[\![\mathcal{O}_1]\!],[\![\mathcal{O}_2]\!]\},[\![\mathcal{O}_3]\!]\}$$



2.4 Example:RW-twist of 3d N=4

Now,I give an example to illustrate what the component of A looks like and what the $\{,\}$ exactly is.

2.4.1 3d N=4 Super Algebra

Before this section,I highly commend reader skip to 3.1 part to study representation of Super-Poincare algebra.

Definition 2.9. *3d N=4 Super Algebra is given by* $\{P_\mu, Q_\alpha^{a,b}\}$*,satisfy:*

$$[Q_\alpha^{ab}, Q_\beta^{cd}] = \epsilon^{ac}\epsilon^{bd}\sigma^\mu_{\alpha\beta}P_\mu$$

the index α *means* Q *is Majorana spinor,index* a *is the spinor representation of* $SU(2)_H$ *(Higgs rotation),index* b *is spinor representation of* $SU(2)_C$ *(Coulomb rotation)*

Without proof,we claim 3d N=4 have a hyper-multiplet representation,consist of $4\phi+2\psi$,we denote them $\phi^{a1},\phi^{a2},\psi^{b1}_{\alpha},\psi^{b2}_{\alpha}$, the index "a" in ϕ^{a1} is the same as in Q, means it's $SU(2)_H$ representation,the indexes in ψ^b is similar.

2.4.2 Twist

Now,we will play twist on this super theory to make it be a TQFT.We define:

$$Q := \delta^{\alpha}_{b} Q^{1b}_{\alpha} = Q^{11}_{1} + Q^{12}_{2} \quad Q_{\mu} := -\frac{i}{2}(\sigma^{\mu})^{\alpha}_{b} \cdot Q^{2b}_{\alpha}$$

The new operator satisfy definition 2.1 ,so this is a twist CTQFT.

And now,we consider the pure hypermultiplets Lagrangian,as I have mentioned it's consist of $\phi^{aA},\psi^{bA}_{\alpha}\ \ A \in [1,2N_f]$,(N_f) is the number of hypermultiplets.

But now, what we did in twist is mix $SU(2)_C \times SU(2)_E$ diagonally to new Euclidean rotation group $SU(2)'_E$. So in twist theory, we don't have to consider ψ but the twisted one,so we define:

$$\eta_A := -\delta^{\alpha}_{b}\psi^{bA}_{\alpha} \quad \chi^{A}_{\mu} := \frac{i}{2}(\sigma^{\mu})^{\alpha}_{b}\psi^{bA}_{\alpha}$$

is the $1 \otimes 3$ representation of $SU(2)'$.

3 Higher Algebra

3.1 Operad(ref[5])

Definition 3.1. *Operad means a category $(\mathcal{O},\circ_i)$ consisting of objects $\{\mathcal{O}(n)\}$ and the products:*

$$\circ_i : \mathcal{O}(m) \times O(n) \to O(m+n-1)$$

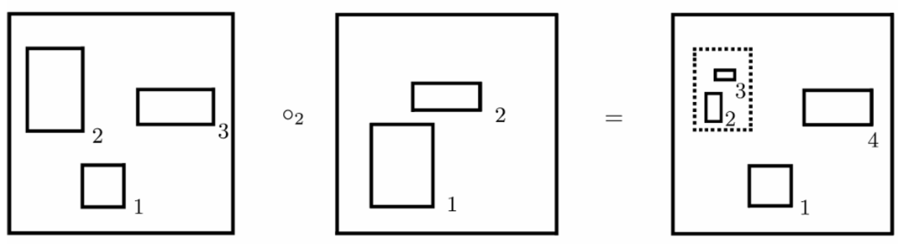
Example 3.1.1 (End_X)**.** *For a set X ,we define Operad End_X as:*

$$O(n) := Map(X^n, X) \quad (f_m \circ_i g_n)(x_1,...,x_{m+n-1}) = f(x_1,...,x_{i-1}, g(x_i,...,x_{i+n-1}),...,x_{m+n-1})$$

This codes the product structure of the space X.

Example 3.1.2 (little n-cubes operad(E_d))**.**

$\mathcal{O}(j) := C_n(j) :=$ *configuration space of j-little cubes embeded in n-unit cube I^n with ordering*



The purpose of little n-cubes operad is to describe the iterated loop space. For example, in $\mathbb{C}\setminus\{0\}$,the loops look like:

$$S_1 = e^{2\pi i t} \quad S_2 = 1 \in \Omega X$$

And the homotopy map between loops for example:

$$F_1 : S_1 \circ S_2 \to S_2 \circ S_1$$

$$F_1(t,s) = \begin{cases} 1 & 0 \le s \le \frac{t}{2} \\ e^{2\pi i(2s-t)} & \frac{s}{2} \le s \le \frac{1+t}{2} \\ 1 & \frac{1+t}{2} \le s \le 1 \end{cases}$$

Also:

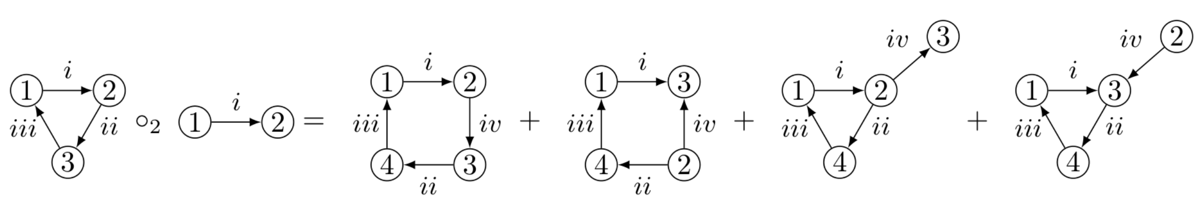
$$F_2 : S_2 \circ S_1 \to S_1 \circ S_2$$

Here $F_1, F_2 \in \Omega^2 X$,and in fact F_1 is homotopy to F_2.

And in general case,for $F_1, F_2 \in \Omega^2 X$ we have $F_1 \circ F_2 = F_2 \circ F_1$ and associative property. Which means the product structure of $\Omega^2(X)$ is commutative and has E_2 algebra structure. Here E_2 algebra means a map:$E_2 \to End_X$,just like insert two operators in I_2 and collide them.

Example 3.1.3 (Graph Operad(Gra_d)). *As figure,$gra_{N,\ j}$ means graphs with N vertex,j side.Index d means equipping each side with weight (d-1).The figure describe:*

$$\circ_i : gra_{3,3} \otimes gra_{2,1} \to Gra_d(4)$$



3.2 Graph complex and stable structure(ref.[3])

3.2.1 Graph complex

Definition 3.2 (Gra_d)**.**

$$d \ \ even : Gra_d(N) := \prod_{k\geq 0}\left(\mathbb{K}\left\langle gra_{N,k}\right\rangle \otimes_{\mathbb{S}_k\times\mathbb{S}_2^{\times k}} sgn_k\right)[k(d-1)]$$

$$d \ \ odd : Gra_d(N) := \prod_{k\geq 0}\left(\mathbb{K}\left\langle gra_{N,k}\right\rangle \otimes_{\mathbb{S}_k\times\mathbb{S}_2^{\times k}} sgn_2^{\otimes k}\right)[k(d-1)]$$

Usually, the elements are just the equivalence class of graphs. For instance, in even case, flipping the direction of edges are equivalent, and changes of vertices will lead to minus. And the shift means re-defining the weight of graph as: $|\gamma| = k(1-d)$

Definition 3.3 (fGC_d)**.**

$$d \ \ even : fGC_d := \prod_{N\geq 1}\left(\mathrm{Gra}_d(N)[d(1-N)]\right)^{S_N}$$

$$d \ \ odd : fGC_d := \prod_{N\geq 1}\left(\mathrm{Gra}_d(N)\otimes \mathrm{sgn}_N[d(1-N)]\right)^{S_N}$$

The full graph complex (fGC$_d$) is a subcategory of Gra_d, which focus on the S_N invariant linear combination, and re-defining the weight of graph: $|\gamma| := d(N-1) + k(1-d)$.

Prop 3.3.1. *fGC_d is a dgLa:*

$$\gamma\circ\gamma' = \sum_{\sigma\in Sh^{-1}(N',N-1)} (-1)^{d|\sigma|}\Sigma_{N+N'-1}\left(\gamma\circ_1\gamma'|\sigma\right)$$

$$[\gamma_1,\gamma_2] = \gamma_1\circ\gamma_2 - (-1)^{|\gamma_1||\gamma_2|}\gamma_2\circ\gamma_1$$

$$\delta := [\Gamma_{\bullet-\bullet}, \_]$$

Definition 3.4 ($dGra_d$)**.**

$$d \ \ even : dG_{\mathrm{rad}}(N) := \prod_{k\geq 0}\left(\mathbb{K}\left\langle \mathrm{gra}_{N,k}\right\rangle \otimes_{\mathbb{S}_k} \mathrm{sgn}_k\right)[k(d-1)]$$

$$d \ \ odd : dG_{\mathrm{rad}}(N) := \prod_{k\geq 0}\left(\mathbb{K}\left\langle \mathrm{gra}_{N,k}\right\rangle_{\mathbb{S}_k}\right)[k(d-1)]$$

As you see, the difference with Gra_d is the flipping of direction of side is not equivalent, and there is a map:

$$Or : Gra_d \hookrightarrow dGra_d$$

$$\bullet \longrightarrow \bullet := ① \longrightarrow ② + (-1)^d\, ② \longrightarrow ①$$

In fact, we can define $dfGC_d$ in a similar way endowed with dgLa structure, as a result, the inducing map:

$$Or* : fGra_d \hookrightarrow dfGra_d$$

is a quasi-isomorphism of dgLa.

3.2.2 Stable structure

In short, an operad P is called stable-P algebra, if its algebra goes through $dGra_d$ operad:

$$P \to dGra_d \xrightarrow{dRep_d} End_X$$

In usual case(BV graded manifold), X is function space $\mathcal{C}^\infty(M)$ endowed with Poisson structure $\{,\}$, and $dRap_d$ is the canonical map:

$$\mathrm{dRep}_N^{(d)}(\gamma)(f_1 \otimes \cdots \otimes f_N) = \mu_N \left(\prod_{(i,j)\in E_\gamma} \overline{\Delta}_{ij}(f_1 \otimes \cdots \otimes f_N) \right)$$

For instance, if Poisson structure is $\frac{\partial}{\partial x^a} \wedge \frac{\partial}{\partial x^b}$, then

$$\mathrm{dRep}_N^{(d)}(\gamma)(f_1 \otimes \cdots \otimes f_N) = \sum_{(i,j)\in E_\gamma} f_1 \cdot ... \cdot \frac{\partial f^i}{\partial x^a} \cdot ... \frac{\partial f^j}{\partial x^b}$$

can be regarded as contracting by Poisson structure.

So, get down to the name, in which sense the algebra is stable? In BV setting(ref to my another note on BV), $T^*[-1]R^n$ is a graded manifold with BV algebra structure on its function ring $\mathcal{T}_{poly}R^n$. But in fact, there is nature embedding:

$$\mathcal{T}_{poly}(R^n) \hookrightarrow \mathcal{T}_{poly}(R^{n+1})$$

$$\mathbf{k}\left[x^1, \ldots, x^d, \eta_1, \ldots, \eta_d\right] \longrightarrow \mathbf{k}\left[x^1, \ldots, x^d, \eta_1, \ldots, \eta_d\right] \otimes \mathbf{k}\left[x^{d+1}, \eta_{d+1}\right]$$

So this inducing the map of $End_{\mathcal{T}_{poly}(R^n)}$ to $End_{\mathcal{T}_{poly}(R^{n+1})}$, define:

$G(n) := \lim_{d\to\infty} \mathrm{Map}\left(T_{\mathrm{poly}}\left(\mathbf{k}^d\right)^{\otimes n},\ T_{\mathrm{poly}}\left(\mathbf{k}^d\right)\right)^{A_d}$ as the limit. Here, A_d means invariant under coordinate rotation and translation.

And in fact, $G(n) \cong dGra_d$ (ref.[1]). So the stable structure captures the algebra independent of dimension.

4 Physics basement in SymTQFT

4.1 Anomaly TFT

Thought of SymTQFT comes from the attempt to explain 't Hooft anomaly of a QFT theory S_d on d-dimensional manifold M_d through lifting it to a d+1-dimensional theory S_{d+1} on M_{d+1}.

4.1.1 Higher Symmetry

Definition 4.1. *A p-form symmetry is a (n-p-1)-dimensional topological operator, acting on a p-dimensional object.*

Literally, I will use $W(\psi)(M_d) := exp(i \int_{M_d} \psi)$ to denote the n-dimensional operator generated by the n-form operator ψ; For a one-parameter d-form symmetry group G, suppose its topological operator is:

$$G(M_{n-d-1})(\alpha) := exp(i\alpha \int_{M_{n-d-1}} \phi)$$

I mean this d-form symmetry corresponds to n-1-d form ϕ

The canonical way of G-action on d-form operator is:

$$G(\alpha)(W) = G(\alpha)(S_{n-d-1})W = exp(i\alpha \int_{S_{n-d-1}} \phi)exp(i \int_{M_d} \psi)$$

Here S_{n-p-1} is a (n-p-1)-dimensional sphere s.t $S_{n-1-d} \cap M_d = x_0$

Prop 4.1.1. $\langle G(\alpha)W \rangle = exp(i\alpha \langle D_{n-p}, M_d \rangle) \langle W \rangle$

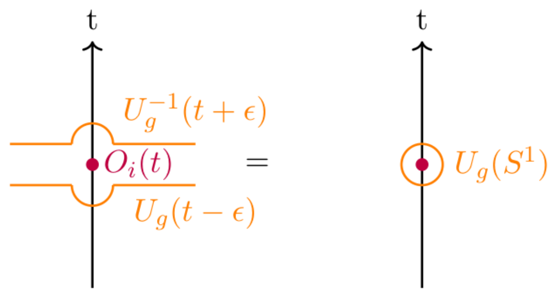
Prop 4.1.2. *When d=0, such a definition collapses to the Noether law.*

Pf:suppose $G(\alpha)(\psi) = \psi^{\alpha} \quad G(\alpha)(W) = exp(i\psi^{\alpha})$

By Noether Law, the conservation flow $J_{\mu} = \dfrac{\partial \mathcal{L}}{\partial(\partial\mu\psi)} \cdot \dfrac{\delta\psi}{\delta\alpha}$

correspond to closed (n-1)-form : $\phi := (-1)^{\mu-1} J_{\mu} dx^0 ... d\bar{x}^{\mu} ... dx^{n-1}$

Then by definition :$G(\alpha)(W) = \lim_{\epsilon\to 0} exp(i\alpha Q) exp(i\psi) exp(-i\alpha Q)) = exp(i\psi^{\alpha})$



4.1.2 t' Hooft anomaly

Now consider any QFT on M_d with global G-symmetry:

$$Z = \int \mathcal{D}\phi \exp(iS[\phi]) \quad S[\phi] = S[\phi^g] \text{ and } \mathcal{D}\phi = \mathcal{D}\phi^g, \quad \forall g \in G \quad s.t\ Z = Z^g$$

Anomaly appears when you try to gauge it:

$$Z[A] = \int \mathcal{D}\phi e^{iS[A,\phi]} \quad Z[A^g] = Z[A] \cdot Z_{\text{anomaly}}[A, g]$$

In particular,

$$Z_{\text{anomaly}}[A, g] = e^{2\pi i \int_{M_d} f(A,g)}$$

Because of unitarity, it must be a phase; because of localism, the functional must be integration. Now, we are searching for a d+1-dimensional theory S_{n+1} which captures S_n and $f(A,g)$. Manifestly, the $S_{d+1} = S_d + 2\pi \int_{M_{d+1}} w(A,g)$, satisfied $\delta w(A,g) = -df(A,g)$, properly invariant under G-action, so there is no 't-Hooft anomaly as n+1-dimensional theory.

Example 4.1.1.2:Maxwell theory with background B_E, B_M

$$S = \frac{1}{2e^2} \int (F - B_E) \wedge *(F - B_E) + \frac{1}{2\pi} \int (F - B_E) \wedge B_M$$
$$A \mapsto A + \lambda_E \,,\ B_E \mapsto B_E + \mathrm{d}\lambda_E \,,\ B_M \mapsto B_M + \mathrm{d}\lambda_M$$

Such a pure-gauge theory is born with a G_E 1-form symmetry, since the operator $Q_E(S_{d-2}) := \int_{M_{d-2}} *(F - ...) \quad G_E(T_a)(S_{d-2}) := exp(iT_a Q_E(S_{d-2})_a)$ is topological given by the motion equation; and a (d-3)-form symmetry U_M given by $dF = 0$.

Prop 4.1.3. $\langle U_E(T_a)(S)W_n(\gamma)\rangle = \exp(inT_a\langle S, \gamma\rangle)\langle W_n(\gamma)\rangle$

References

[1] Johan Alm. *Universal algebraic structures on polyvector fields.* PhD thesis, Department of Mathematics, Stockholm University, 2014.

[2] Christopher Beem, David Ben-Zvi, Mathew Bullimore, Tudor Dimofte, and Andrew Neitzke. Secondary products in supersymmetric field theory. *Annales Henri Poincaré*, 21(4):1235–1310, February 2020.

[3] Kevin Morand. M. kontsevich's graph complexes and universal structures on graded symplectic manifolds. *Higher Structures*, 7(1), 2023.

[4] David Simmons-Duffin. Tasi lectures on the conformal bootstrap, 2016.

[5] Jim Stasheff. W h a t i s . . . an operad?, 2004.