

Notes on Seiberg–Witten Theory

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1 Introduction

It's a note for Vafa-Witten theory I studied in Amherst .

2 Cohomological Topology Field Theory

The basic construction and first example can refer[1]

2.1 Basic structure

Definition 2.1 (CTQFT). *Cohomological Topology Field Theory is TQFT(Atiyah definite), with the generator of algebra $\{P_\mu, Q, Q_\mu\}$, satisfy*

$$Q^2 = [Q_\mu, Q_\nu] = [Q_\mu, P_\nu] = [Q, P_\nu] = 0, \quad [Q, Q_\mu] = iP_\mu. \quad (2.1)$$

and observable are Q -cohomology.

Here, Q usually comes from twist of Supersymmetry charge.

Definition 2.2. $Op_{\delta,x} :=$ all operators supported in $Ball_\delta(x)$

Prop 2.2.1. *Since the theory is topological, $Op_{\delta,x}$ is independent on δ and we denoted its cohomology class as \mathcal{A}_x .*

What's more for any operator supported on B_δ , we can use

Operator \Rightarrow State and Operator \Leftrightarrow State (details in [3]) to regard it as a local operator on x .

2.2 Topological Algebra(First product)

Definition 2.3. *The first product is a family of embedding defined on $C^2(M)$ given by:*

$$\begin{aligned} *_{x1,x2} : A \otimes A &\rightarrow Op_1 \\ O_1, O_2 &\mapsto O_1(x_1)O_2(x_2). \end{aligned}$$

The first order product in fact have the factorization structure:

Prop 2.3.1. *$*_{x1,x2}$ is equal in the same connected component of $C_2(M)$, in R^d case,since $C_2(R^d)$ is trivial when $d \geq 2$,we can define $*$:= $*_{x1,x2}$*

$$\begin{aligned} Pf : O_1 *_{x1,x2+a} O_2 - O_1 *_{x1,x2} O_2 &= O_1(x^1)O_2(x^2 + a) - O_1(x^1)O_2(x^2) \\ &= a \cdot O_1(x^1)\partial_\mu O_2(x^2) = a \cdot O_1(x^1)Q Q_\mu O_2(x^2) \\ &= a \cdot (-1)^{F_1} Q(a \cdot O_1(x^1)Q_\mu O_2(x^2)) = 0 \end{aligned}$$

Prop 2.3.2. *$*$ is associative*

Prop 2.3.3. *Obvious, $*$ is graded-commutative.*

2.3 Secondary Product

Definition 2.4 (Descendant). $\mathcal{O}^{(k)}(x) = \frac{1}{k!} (Q_{\mu_1} \cdots Q_{\mu_k} \mathcal{O})(x) dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}$
 $\mathcal{O}^* = \sum_{k=0}^{\infty} \mathcal{O}^{(k)}$

Prop 2.4.1 (Descendant Equation). $Q(\mathcal{O}^{(k)}(x)) = d\mathcal{O}^{(k-1)}(x), \quad Q\mathcal{O}^* = d\mathcal{O}^*$

$$Pf : Q\mathcal{O}^{(k)}(x) = Q(\frac{1}{k!} Q_{\mu_1} \dots) = \frac{1}{k!} \cdot iP_{\mu_1}(\dots) dx^{\mu_1} = d\mathcal{O}^{(k-1)}$$

Definition 2.5 (Homology Operator). Denote $\mathcal{O}(\gamma) := \int_\gamma \mathcal{O}^*$, here $[\gamma]$ is homology class.

Prop 2.5.1. $\mathcal{O}(\gamma)$ is operator supported on γ , and as Prop 2.2.1 $\mathcal{O}(\gamma) \in A$

Definition 2.6 (Secondary Product). *The first product is a family of map defined on $C^1(M) = M$ given by:*

$$\begin{aligned} \{, \}_x : A \otimes A &\rightarrow A_x \\ O_1, O_2 &\mapsto \mathcal{O}_1(S_x^{d-1}) \mathcal{O}_2(x) \end{aligned}$$

Prop 2.6.1. *The same as the proof of \star_{x_1, x_2} is invariant under deformation, $\{, \}_x$ is invariant under deformation, so we can define $\{, \}_x := \{, \}$ in R^d so it's a product in A .*

Now we define a new product \star defined on $C^n(M)$, and it captures the structure of Poisson structure of secondary product.

Definition 2.7 (Descendant on configuration space). $(\mathcal{O}_1 \boxtimes \mathcal{O}_2 \dots \boxtimes \mathcal{O}_n)^* = \mathcal{O}_1^* \wedge \sigma^{F_1} \mathcal{O}_2^* \dots \wedge \sigma^{F_1 \dots + F_{n-1}} \mathcal{O}_n^*$, where σ acts as $(-1)^k$ on the degree k part.

Prop 2.7.1 (Descendant Equation). $\mathcal{Q}(\mathcal{O}_1 \dots \boxtimes \mathcal{O}_n)^* = d(\mathcal{O}_1 \dots \boxtimes \mathcal{O}_n)^*$.

In particular, $(\mathcal{O}_1 \boxtimes \mathcal{O}_2)^{(k)}(x_1, x_2) = \sum_{n=0}^k (-1)^{(k-n)F_1} \mathcal{O}_1^{(n)}(x_1) \wedge \mathcal{O}_2^{(k-n)}(x_2)$

For $P \in H_\bullet(C^n(M), \mathbb{Z})$, we define

$$\star_P : A^{\otimes n} \rightarrow A$$

$$(\mathcal{O}_1, \dots, \mathcal{O}_n) \mapsto (\mathcal{O}_1 \boxtimes \mathcal{O}_2 \dots \boxtimes \mathcal{O}_n)(P) := \int_P (\mathcal{O}_1 \boxtimes \mathcal{O}_2 \dots \boxtimes \mathcal{O}_n)^*$$

Then we prove some property of secondary product: .6.2

Prop 2.8.1 (Commutative property). $\{, \}_x$ is graded-commutative

$$Pf : \{O_1, O_2\}_x = O_1(S_x^{d-1})O_2(x) = (O_1 \boxtimes O_2)(S_x^{d-1}, \{x\}) = O_1 \star_{[S_x^{d-1} \times \{x\}]} O_2$$

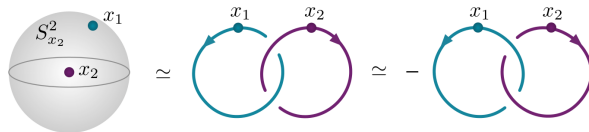
in R^d case, $C^2(M) \cong S^{d-1}$,

$[S_x^{d-1} \times \{x\}]$ and $\{\{x\} \times S_x^{d-1}\}$ related by antipodal map

$$\therefore O_1 \star_{[S_x^{d-1} \times \{x\}]} O_2 = (-1)^d O_1 \star_{[\{x\} \times S_x^{d-1}]} O_2 = (-1)^d O_1(x) O_2(S^{d-1})$$

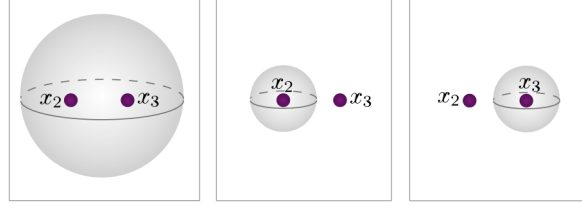
$$= (-1)^{F_1 F_2 + d} \{O_2, O_1\}$$

In particular, in $d=3$ case $\{, \}_x$ can be used to compute linking number. That is for Γ_1, Γ_2 two circles in R^3 , we have $[O_1 \boxtimes O_2](\Gamma_1, \Gamma_2) = l \cdot (O_1 \boxtimes O_2)(S_x^{d-1}, \{x\}) = l \cdot \{O_1, O_2\}$



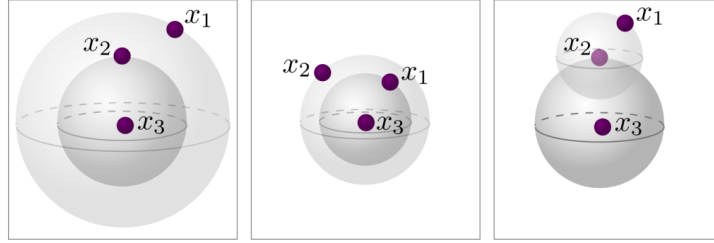
Prop 2.8.2 (Derivation property). $\{[O_1], [O_2] * [O_3]\} = \{[O_1], [O_2]\} * \{[O_1], [O_3]\} + (-1)^{(F_1 + d - 1)F_2} [O_2] * \{[O_1], [O_3]\}$

The proof is directly, $(-1)^{F_1+d-1}$ from the order (d-1) form. And notice that two parts of the equation correspond to these graphs, and $[\Gamma_1] = [\Gamma_2] + [\Gamma_3]$



Prop 2.8.3 (Jacobi identity).

$$\begin{aligned} \{[\mathcal{O}_1], \{[\mathcal{O}_2], [\mathcal{O}_3]\}\} - (-1)^{(F_1+d-1)(F_2+d-1)} \{[\mathcal{O}_2], \{[\mathcal{O}_1], [\mathcal{O}_3]\}\} \\ = (-1)^{(d-1)(F_1+d-1)} \{\{[\mathcal{O}_1], [\mathcal{O}_2]\}, [\mathcal{O}_3]\} \end{aligned}$$



2.4 Example:RW-twist of 3d N=4

Now, I give an example to illustrate what the component of A looks like and what the $\{, \}$ exactly is.

2.4.1 3d N=4 Super Algebra

Before this section, I highly commend reader skip to 3.1 part to study representation of Super-Poincare algebra.

Definition 2.9. 3d N=4 Super Algebra is given by $\{P_\mu, Q_\alpha^{a,b}\}$, satisfy:

$$[Q_\alpha^{ab}, Q_\beta^{cd}] = \epsilon^{ac}\epsilon^{bd}\sigma_{\alpha\beta}^\mu P_\mu$$

the index α means Q is Majorana spinor, index a is the spinor representation of $SU(2)_H$ (Higgs rotation), index b is spinor representation of $SU(2)_C$ (Coulomb rotation)

Without proof, we claim 3d N=4 have a hyper-multiplet representation, consist of $4\phi + 2\psi$, we denote them $\phi^{a1}, \phi^{a2}, \psi_\alpha^{b1}, \psi_\alpha^{b2}$, the index "a" in ϕ^{a1} is the same as in Q , means it's $SU(2)_H$ representation, the indexes in ψ^b is similar.

2.4.2 Twist

Now, we will play twist on this super theory to make it be a CTQFT. We define:

$$Q := \delta_b^\alpha Q_\alpha^{1b} = Q_1^{11} + Q_2^{12} \quad Q_\mu := -\frac{i}{2}(\sigma^\mu)_b^\alpha \cdot Q_\alpha^{2b}$$

The new operator satisfy definition 2.1, so this is a twist CTQFT.

And now, we consider the pure hypermultiplets Lagrangian, as I have mentioned it's consist of $\phi^{aA}, \psi_\alpha^{bA}$ $A \in [1, 2N_f]$, (N_f) is the number of hypermultiplets.

But now, what we did in twist is mix $SU(2)_C \times SU(2)_E$ diagonally to new Euclidean rotation group $SU(2)'_E$. So in twist theory, we don't have to consider ψ but the twisted one, so we define:

$$\eta_A := -\delta_b^\alpha \psi_\alpha^{bA} \quad \chi_\mu^A := \frac{i}{2}(\sigma^\mu)_b^\alpha \psi_\alpha^{bA}$$

is the $1 \otimes 3$ representation of $SU(2)'$.

3 Physics describe

3.1 Representation of Super-Pincare algebra

Poincare algebra is generated by $\{P_\mu, Q_a^A, \bar{Q}_{\dot{a}A}\}$, satisfied:

$$\{Q_\alpha^A, \bar{Q}_{\beta B}\} = 2\delta_B^A \sigma_{\alpha\beta}^\mu \mathbf{P}_\mu, \quad \{Q_\alpha^A, Q_\beta^B\} = \varepsilon_{\alpha\beta} Z^{AB}$$

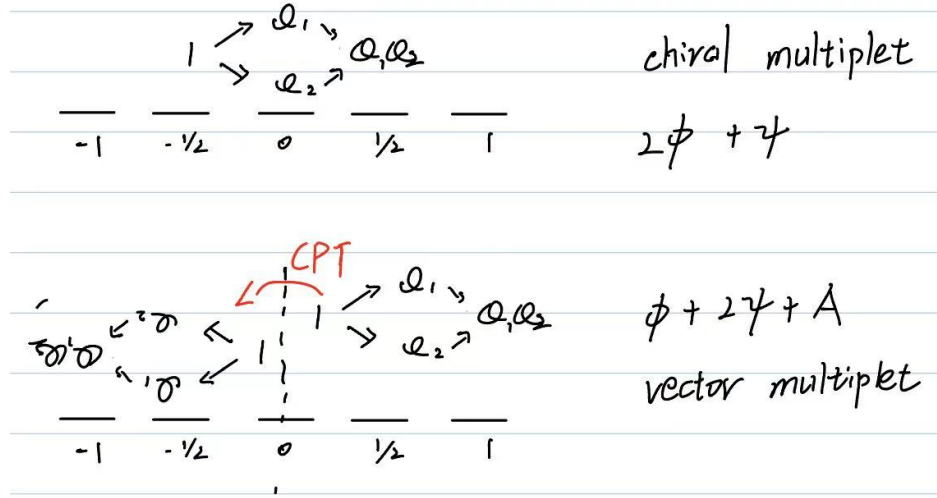
We claim it's the unique non-trivial extension of Pincare algebra.

3.1.1 N=1 massive

In the rest frame of a particle, $p_\mu = (m, 0, 0, 0)$ so

$$\{Q_\alpha, \bar{Q}_\beta\} = 2m\delta_{\alpha\beta}, \quad \{Q, Q\} = \{\bar{Q}, \bar{Q}\} = 0$$

so we regard Q_a as creation operator and $\bar{Q}_{\dot{a}}$ as annihilation operator, begin with a state $|j\rangle$, we have $Q_1|j\rangle, Q_2|j\rangle, Q_1Q_2|j\rangle$, without gravity the superhelicity $J \leq 1$, at a result, we only have two representations called chiral multiplet and vector multiplet.



And these multiplets correspond to chiral and vector field.

$$\Phi(x, \theta, \bar{\theta}) = A(x) + \theta^\alpha \psi_\alpha(x) + \theta^2 F(x) + i\theta\sigma^a\bar{\theta}\partial_a A(x) + \frac{i}{2}\theta^2\bar{\theta}\tilde{\sigma}^a\partial_a\psi(x) + \frac{1}{4}\theta^2\bar{\theta}^2\Box A(x) \quad (1)$$

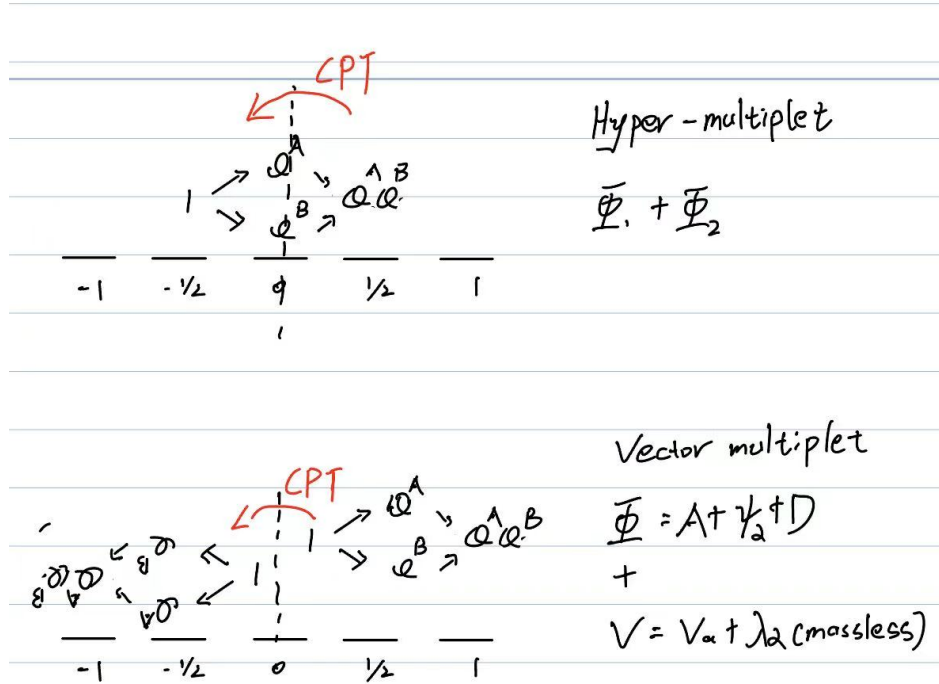
$$V = \theta\sigma^a\bar{\theta}V_a + \bar{\theta}^2\theta^\alpha\lambda_\alpha + \theta^2\bar{\theta}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} + \theta^2\bar{\theta}^2\mathcal{D} \quad (2)$$

3.1.2 N=1 massless

Also to the rest frame, $p_\mu = (E, 0, 0, E)$. Now we only have one creation operator, as the same calculation, the representation component is the same as massive.

3.1.3 N=2 massless

Have "Hypermultiplet" and "Vector multiplet"



3.2 Pure N=2 massless Vector-multiplet Theory

Since in this case, the Vector-multiplet can be written as $V \oplus \Phi$. We consider N=1 massless theory with both these parts, it is N=1 Super-Yang-Mills model: (this part following Witten's lecture <https://youtu.be/9qZSqkn8-Qo>, and for details about N=1 case, reference is [2][3.5.1])

$$S = \frac{1}{e^2} \int d^4x d\theta d\bar{\theta} \text{tr}(\bar{\Phi} e^{2V} \Phi) + \frac{1}{e^2} \int d^4x d\theta \text{tr}(W^\alpha W_\alpha) \quad (3)$$

Since $\int dx d\theta \Phi = -\frac{1}{4} \int dx D^2 \Phi|_{\theta=0, \bar{\theta}=0}$ (4), what we need to do is using formulas (1)(2)(4) to extend (3) to the euclidean case, and the result is:

$$S = \frac{1}{e^2} \int d^4x \text{kinetic term of } (A, \lambda, \bar{\lambda}, \phi) + \text{Tr} [\phi, \phi^\dagger]^2 + (\text{Yukawa potential})$$

The formula (3) has the same symmetry transformation as N=2 case after adding Yukawa potential, so it's the Lagrangian we are looking for.

Now, as you see, the potential is:

$V(\phi) = \text{Tr}[\phi, \bar{\phi}]$, which means it's diagonal, so we set $\phi = a\sigma^3$. Since vacua under $SU(2)$ are equivalent, we use the parameter $u := \frac{1}{2}a^2 = \langle \phi^2 \rangle$ to parameterize the modular space of vacua, as u is double covered by a .

3.3 Relation to QCD

When $a \neq 0$, $SU(2)$ symmetry is broken to $U(1)$ and generate massive boson W^\pm from two color indexes of A_μ^a . So as we consider low-energy effective theory, the two massive bosons won't make effort. The residue field is A_μ, ψ with $U(1)$ symmetry, which exactly is QCD.

4 AG describe

4.1 Module Space of sheaves

Definition 4.1. For X smooth, projective, dimension d variety, with ample sheaf H . Its modular space of sheaves with fix rank r , and Chern Class c_i , denoted as $M_X^H(r, c_i)$

Theorem 4.1. The Zariski tangent space at $[E] := T_M|_{[E]} \cong Ext_X^1(E, E) / \text{fit}_{T_{M_L}|_{[E]}} \cong Ext_X^1(E, E)_0$, here $Ext_X^1(E, E)_0$ denote the kernel of trace map: $tr^i : Ext^i(E, E) \cong H^i(End(E)) \rightarrow H^i(O_X)$

4.2 Hilbert scheme and generating series

Definition 4.2 (Hilbert Scheme). $X^{[n]} := \{Z \subseteq X : Z \text{ is } 0\text{-dimensional with } \dim H^0(X, \mathcal{O}_Z) = n\}$

it should be regarded as n -point construction space with degree.

$$\begin{array}{ccc} & \mathcal{Z} & \\ q \swarrow & & \searrow p \\ X & & X^{[n]} \end{array}$$

Now, we define generating series:

$$Z_{S, E_1, \dots, E_\ell}(q) = 1 + \sum_{n=1}^{\infty} q^n \int_{S^{[n]}} P_n$$

P_n is polynomial in Chern class of $E_i^{[n]}$, $T_{S^{[n]}}$, \mathcal{Z} is universal subscheme and $E_i^{[n]} = p_* q^* E_i$.

Definition 4.3 (Hilbert scheme of divisor). For a effective divisor class $\beta \in H_{2d-2}(M)$, we define: $\text{Hilb}_\beta(X) = \{D \subseteq X : D \text{ effective divisor such that } [D] = \beta\}$

We pass the proof of the set is a scheme, it's equivalent to prove the modular functor is representable.

Prop 4.3.1. When $h_1(X) = 0$ (especially $K3$ surface), $\text{Hilb}_\beta(X)$ is linear system, and we denote it as $|\beta|$

Pf : Consider $H^1(X, \mathcal{O}_X) \rightarrow H^1(X, \mathcal{O}_X^*) \cong \text{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z})$

since $h^1 = 0$, c_1 is injective, the equivalent class of divisor is determined by its Chern class.

4.3 Seiberg–Witten invariants

Definition 4.4 (Virtual Dimension). *For modular space $M_S^H(r, L, c_2)$ (S is projective surface),*

$$vd(r, L, c_2) := \dim_C(\text{Ext}_S^1(E, E)_0) - \dim_C(\text{Ext}_S^2(E, E)_0) = \\ 2rc_2 - (r-1)c_1(L)^2 - (r^2-1)\chi(\mathcal{O}_S)$$

as $\text{Ext}_S^1(E, E)_0$ is the deformation as obstruction in $\text{Ext}_S^2(E, E)_0$

Definition 4.5 (Virtual fundamental Class). *For M a \mathbb{C} –Scheme of finite type, the virtual fundamental class $[M]^{vir} \in H_{2vd}(M, \mathbb{Z})$ is well-defined, if there exists a perfect obstruction theory over M*

Prop 4.5.1. *For projective surface S , fix $\beta \in H_2(S, \mathbb{Z})$, the Hilbert space $\text{Hilb}_\beta(S) := |\beta|$ have perfect obstruction theory. And $[|\beta|]^{vir} \neq 0$ only when $vd(|\beta|) = 0$*

Definition 4.6 (Seiberg–Witten invariants). $SW(\beta) := \int_{[|\beta|]^{vir}} 1$

References

- [1] Christopher Beem, David Ben-Zvi, Mathew Bullimore, Tudor Dimofte, and Andrew Neitzke. Secondary products in supersymmetric field theory. *Annales Henri Poincaré*, 21(4):1235–1310, February 2020.
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- [3] David Simmons-Duffin. Tasi lectures on the conformal bootstrap, 2016.