Quantization of $\mathcal{N} = 1$ SYM theory

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Abstract

We identify the D-module structure on N=1 and classify the corresponding SUSY-invariant local functionals. Furthermore, we construct the Batalin–Vilkovisky complex of N=1 Super–Yang–Mills theory and carry out its quantization in the sense of homotopy renormalization. However, this note remains preliminary; the new results are contained in Sections 4 and 5, while the earlier sections review well-known material.

Contents

1 Introduction

In [1], Costello showed that the twisted N=1 gauge theory admits unique quantization on any Calabi-Yau manifold and analyzed its algebraic structure. With the same spirit, using the superfields construction and BV cohomology, we showed that the pure SYM theory has no quantization obstruction; thus, it also admits a unique quantization. Further, we try to calculate some observables structure. In Section 2, we summarize the Super Symmetry Theory (SUSY) construction. In Section 3, we give an introduction to SYM construction in BV formalism and review its twist. In section 4, we give a general construction of a SUSY invariant functional on $R^{4|4}$. Section 5 is the quantization and calculation of the local BV cohomology. We highly recommend readers who are familiar with superspace construction go directly to section 4 to see the final construction.

2 SUSY algebra and representation

2.1 SUSY Algebra and superspace

For a $\mathbb{Z}_2$-graded super Lie algebra, $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$, impose $\mathcal{G}_0 = \mathcal{P} := \mathbb{C}[P_\mu, M_{\mu\nu}]$ with Lie algebra:

$$[P_\mu, P_\nu] = 0 \quad [M_{ab}, P_c] = i\eta_{ac}P_b - (a \leftrightarrow b) \quad [M_{ab}, M_{cd}] = i\eta_{ac}M_{bd} - i\eta_{ad}M_{bc} - (a \leftrightarrow b)$$

Since $[\mathcal{G}_,, \mathcal{G}_1] \subset \mathcal{G}_1$, $\mathcal{G}_1$ is a representation space of $\mathcal{G}_0$, so elements of $\mathcal{G}_1$ must be spinors. We thus take

$$\mathcal{G}_1 = \mathbb{C}[Q^a_\alpha] \ \ (\ or \ \mathbb{C}[Q^a_\alpha, \bar{Q}^a_{\dot\alpha}](after \ weyl \ \ decomposition \ when \ D = 2m),$$

where α is a spinor index and a counts the number of supercharges.

Survey on the structure constant [2] shows the Lie structure must be (As an example D=4):

$$\{Q^a_\alpha, \bar{Q}^b_{\dot\beta}\} = 2\delta^{ab}(\sigma^\mu)_{\alpha\dot\beta}P_\mu \quad [M_{\mu\nu}, Q^a_\alpha] = i(\sigma_{\mu\nu})^\beta_\alpha Q^a_\beta \quad [P_\mu, Q^a_\alpha] = 0 \quad (\sigma_{\mu\nu} := -\frac{1}{4}(\sigma_\mu\bar\sigma_\nu - \sigma_\nu\bar\sigma_\mu))$$

$$\bar\sigma^\mu := (1, -\sigma_1, -\sigma_2, -\sigma_3)$$

More generally, you can replace σ^μ with $\Gamma^\mu C$ in the Majorana case; with $P_+\Gamma^\mu C(P_+ \ is \ Weyl \ projection$ to the Weyl decomposition case, and add central charges...

Definition 2.1 *(Super Poincaré Group and Superspace)*

$$\mathcal{SP} := exp(i(-x^\mu P_\mu + \frac{1}{2}K^{ab}M_{ab} + \theta^\alpha_a Q^a_\alpha + \bar\theta^{\dot\alpha}_a \bar{Q}^a_{\dot\alpha})) \qquad (x^\mu, K^{ab} \in \mathbb{R}; \theta^\alpha, \bar\theta^{\dot\alpha} \in \mathbb{C}_a)$$

$$R^{n|s(n)} := \mathcal{SP}/SO(n-1,1) \qquad \text{s(n) is the number of spinors in n-dimension.}$$

The action of $\mathcal{SP}$ on $R^{n|s(n)}$ is given by:

$$\rho(A)(B) = e^A e^B \quad A \in \mathcal{SP}, B \in R^{n|s(n)}$$

Prop 2.1.1 *By BCH formula,(D=4)* $e^A \cdot e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots}$*, we have:*
Translation:

$$P_\mu : (x, \theta, \bar{\theta}) \mapsto (x^\mu + a^\mu, \theta, \bar{\theta})$$

Laurantz transformation:

$$\begin{aligned} x'^a &= (e^K)^a{}_b x^b \\ \theta'_\alpha &= \left(\exp\left(\frac{1}{2}K^{ab}\sigma_{ab}\right)\right)^\beta{}_\alpha \theta_\beta \\ \overline{\theta}'^{\dot{\alpha}} &= \left(\exp\left(\frac{1}{2}K^{ab}\tilde{\sigma}_{ab}\right)\right)^{\dot{\alpha}}{}_{\dot{\beta}} \overline{\theta}^{\dot{\beta}}. \end{aligned}$$

Supersymmetry transformation:

$$x'^a = x^a - i\epsilon\sigma^a\overline{\theta} + i\theta\sigma^a\bar{\epsilon} \tag{1}$$

$$\theta'^\alpha = \theta^\alpha + \epsilon^\alpha \quad \overline{\theta}'_{\dot{\alpha}} = \overline{\theta}_{\dot{\alpha}} + \bar{\epsilon}_{\dot{\alpha}}. \tag{2}$$

2.2 SUSY Representation and superfields on D=4

Since we primarily discuss the D=4 manifold in this paper, we provide the representation of the SUSY Algebra over it.

2.2.1 N=1 massive

In the rest frame, we have:$p^\mu = (m, 0, 0, 0)$. So the SUSY Algebra is:

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2m\delta_{\alpha\dot{\beta}} \quad \{Q_\alpha, Q_\alpha\} = \{\bar{Q}_{\dot{\beta}}, \bar{Q}_{\dot{\beta}}\} = 0 \quad [M_{12}, Q_\alpha] = \frac{1}{2}Q_\alpha$$

So it has two $SU(2)$ algebraic structures, the representation is:

$$\begin{cases} |-\frac{1}{2}\rangle \oplus 2|0\rangle \oplus |\frac{1}{2}\rangle & \text{chiral multiplet} \\ |-1\rangle \oplus 2|-\frac{1}{2}\rangle \oplus 2|0\rangle \oplus 2|\frac{1}{2}\rangle \oplus |1\rangle & \text{vector multiplet} \end{cases}$$

Remark: These two multiplets are irreducible representations of the SUSY algebra, although we haven't proven that. Differing from the Poincaré algebra, the spin J^2 is no longer a Casimir operator; instead, we construct C_2', called superspin.

$$B_\mu = W_\mu - \frac{1}{4}\bar{Q}_{\dot{\alpha}}(\bar{\sigma}_\mu)^{\dot{\alpha}\beta}Q_\beta \quad C_{\mu\nu} = B_\mu P_\nu - B_\nu P_\mu \quad C_2' = C_{\mu\nu}C^{\mu\nu}$$

2.2.2 N=1 massless

We will see later that this case corresponds to the gauge theory we discuss.
In the "rest frame" (E,0,0,E):

$$\{Q,\overline{Q}\} = \begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix}$$

So there is only one generate-annihilate operator pair, with CPT the representation is:

$$\begin{cases} |-\frac{1}{2}\rangle \oplus 2|0\rangle \oplus |\frac{1}{2}\rangle & \text{chiral multiplet} \\ |-1\rangle \oplus |-\frac{1}{2}\rangle \oplus |\frac{1}{2}\rangle \oplus |1\rangle & \text{vector multiplet} \end{cases}$$

The vector multiplet corresponds to a vector field; it consists of one gauge vector field and one fermion. This is the gauge theory we will discuss.

2.2.3 N=2 massive and massless

In N=2, the crucial difference from N=1 is the central charge Z, which leads to the BPS boundary in the massive case: Consider $U = Q_1^1 + \lambda \bar{Q}_2^1$, as it means the composition of SUSY transformations. We have:

$$\left\{U, U^\dagger\right\} \geq 0 \Rightarrow m \geq |Z|$$

$N=2$		
Massive Z=0	SUSY Algebra	$\{Q^A_\alpha, \bar{Q}^B_{\dot\beta}\} = 2m\delta_{\alpha\dot\beta}\delta^{AB}$
	scalar multiplet	$5\phi + 4\psi + 1A_\mu$
Massive Z≠0	SUSY Algebra	$\{Q^A, \bar{Q}^B\} = 2\delta^{AB}\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$
		$\{Q^A, Q^B\} = \delta^{AB}\begin{pmatrix} 0 & -Z \\ Z & 0 \end{pmatrix}$
	m>Z scalar multiplet	same as Massive Z=0
	m=Z scalar multiplet	same as N=1 massive chiral
	m≥Z	prohibit
Massless	SUSY Algebra	$\left\{Q^A, \bar{Q}^B\right\} = \delta^{AB}\begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix}$
	scalar multiplet	$4\phi + 2\psi$
	vector multiplet	$2\phi + 2\psi + 1A_\mu$

Table 1: ϕ means the scalar representation under the Lorentz group, ψ stands for $spin-\frac{1}{2}$ representation, A_μ stands for $spin-1$ representation.

2.3 Superfields

Just like fermion fields as $Spin-\frac{1}{2}$ representations of the Lorentz algebra, we now construct superfields as representations of the SUSY Algebra.

Following previous notes, we use Σ to denote the spacetime manifold and $\tilde{\Sigma}$ to denote the extended superspace in section 2.4. Now, consider the complex function bundle F over $\tilde{\Sigma}$, locally we have $F \cong R^{4|4} \times \mathbb{C}$ (we will focus on D=4 in the following sections). We name the section of this bundle "Superfield", and the set of the sections as $\mathcal{F}$.

In particular, a superfield can be written as $Y(x^\mu, \theta, \bar{\theta})$, we can expand it in terms of the odd elements:

$$\begin{aligned} Y(x^\mu, \theta, \bar{\theta}) = \phi(x) + \theta^\alpha \psi_\alpha(x) + \bar{\theta}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}(x) + \theta^2 M(x) + \bar{\theta}^2 N(x) \\ + \theta^\alpha \bar{\theta}^{\dot{\alpha}} V_{\alpha\dot{\alpha}}(x) + +\theta^2 \bar{\theta}_{\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}}(x) + \bar{\theta}^2 \theta^\alpha \rho_\alpha(x) + \theta^2 \bar{\theta}^2 D(x) \end{aligned}$$

Naturally, $\phi(x)$ is the map when you restrict Y to $\Sigma \subset \tilde{\Sigma}$, and ψ_α are the fermions of our original Sigma model. So Y is the construction we mentioned in section 2.3.3.

We can define the action of $\mathcal{SP}$ on Y by regarding Y as a "scalar field":

$$UY(x,\theta,\bar{\theta})U^\dagger = Y(x', \theta', \bar{\theta}') \quad U \in \mathcal{SP}, \quad (x', \theta', \bar{\theta}') \text{ is the corresponding coordinates transform}$$

Checking the infinitesimal transformation, we have:

$$\begin{aligned} [Q_\alpha, Y] &= \left(-i\frac{\partial}{\partial \theta^\alpha} - \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_\mu\right) Y \\ [\bar{Q}_{\dot{\alpha}}, Y] &= \left(+i\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + \theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu\right) Y \end{aligned}$$

So we define the following differential:

$$\begin{aligned} \mathcal{P}_\mu &= -i\partial_\mu \\ \mathcal{Q}_\alpha &= -i\partial_\alpha - \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_\mu \\ \bar{\mathcal{Q}}_{\dot{\alpha}} &= +i\bar{\partial}_{\dot{\alpha}} + \theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \end{aligned}$$

Their Lie algebra is the same as the SUSY Lie algebra.

Furthermore, we can define the following *covariant derivative*:

$$\begin{aligned} D_\alpha &= \partial_\alpha + i\sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_\mu \\ \bar{D}_{\dot{\alpha}} &= -\bar{\partial}_{\dot{\alpha}} - i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \end{aligned}$$

Definition 2.2 *(Chiral Field)* $\Omega \in \mathcal{F}$ *satisfies* $\bar{D}_{\dot{\alpha}} \Phi = 0$ *called the Chiral Field.*

Remark: We use $\mathcal{C}$ to denote the set of chiral fields, $\mathcal{C}$ is naturally a subsheaf of $\mathcal{F}$(Here I use the marks of sets to denote the sheaves).

Extend in fermion coordinates:

$$\Omega(x,\theta,\bar{\theta}) = \phi(x) + \theta^\alpha \psi_\alpha(x) + \theta^2 F(x) + \mathrm{i}\theta\sigma^\mu \bar{\theta} \partial_\mu \phi(x) - \frac{\mathrm{i}}{2}\theta^2 \partial_\mu \psi(x) \sigma^\mu \bar{\theta} + \frac{1}{4}\theta^2 \bar{\theta}^2 \Box \phi(x) \quad (3)$$

Definition 2.3 *(Vector Field)* $V \in \mathcal{F}$ *satisfies* $\bar{V} = V$ *called Vector Field*

The same, the set of vector fields $\mathcal{V}$ is a subsheaf, and in Taylor extension:

$$
\begin{aligned}
\mathcal{V}(x,\theta,\bar{\theta}) &= C(x) + \theta\chi(x) + \bar{\theta}\bar{\chi}(x) + i\theta^2 M(x) - i\bar{\theta}^2 M^\dagger(x) + \theta\sigma^\mu\bar{\theta}A_\mu(x) \\
&+ \theta^2\bar{\theta}\left(\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^\mu\partial_\mu\chi(x)\right) + \bar{\theta}^2\theta\left(\lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\bar{\chi}(x)\right) + \frac{1}{2}\theta^2\bar{\theta}^2\left(D(x) - \frac{1}{2}\Box C(x)\right) \\
&\qquad (C(x), D(x), A^\mu(x)\ are\ real\ fields)
\end{aligned}
\tag{4}
$$

The extension is complicated, but the component F(x) in Ω is an auxiliary field, and under the WZ gauge, $C = \chi = M = 0$. Therefore, the chiral and vector fields correspond to the chiral and the vector representations in section 2.5.2, just check the components.

3 Super Yang-Mills Theory in BV formalism and its twist

Since we have a SUSY transformation $\mathcal{Q}$, we can ask whether it can be realized as something like a BRST charge. The usual answer is no, because they are not scalars under our structure group, the $Spin(n)$. So the solution is to twist our structure group by the R-bundle or to pick up a special structure to reduce the structure bundle, the resulting bundle we call $Spin(d)'$.

To simplify, we assume $[Q_i, Q_j] = \Gamma^\mu_{ij}P_\mu$ (now i include (α, A) $index$). Usually, the scalar $\mathcal{Q}$ under $Spin(n)'$ has: $\mathcal{Q} = q^i\mathcal{Q}_i$ ($q^i \in \mathbb{C}$). To assign a closed condition, we have $[\mathcal{Q},\mathcal{Q}] = \Gamma^\mu q_i q_j = 0\ \forall\mu$. The homogeneous coordinate $(q_1, ..., q_n)$ we call it *nilpotence variety*. There are two interesting cases in the nilpotence variety:

$$
\begin{cases}
[Q, Q_\mu] = P_\mu & \text{topological direction} \\
[Q, Q_{\bar{z}}] = P_{\bar{z}} & \text{anti-holomorphic deirection}
\end{cases}
$$

Just by the ward identity, we have that the correlation function is independent of $x^\mu (or\ \bar{z})$. Thus, after twisting, we realize a SUSY theory as a topological(holomorphic) theory.

The following table lists some twist theories in different dimensions and supercharge numbers. I will focus on the D=4 gauge theories in the following sections

D=2, $\mathcal{N}=(2,2)$	topological twist	R-symmetry $U(1)_A \times U(1)_V$	supercharge
	A-twist	$U(1)_R \xrightarrow{id} U(1)_V$	$Q_A = \bar{Q}_+ + Q_-$
	B-twist	$U(1)_R \xrightarrow{id} U(1)_A$	$Q_B = \bar{Q}_+ + \bar{Q}_-$
D=2, $\mathcal{N}=(0,2)$	holomorphic twist	R-symmetry $U(1)_A$	supercharge
		pick up complex structure	$Q = \bar{Q}_+$
D=3, $\mathcal{N}=2$	holo-topo twist	R-symmetry $SO(2)$	supercharge
		pick up complex structure	$Q = \bar{Q}_+$
D=4, $\mathcal{N}=1$	holomorphic twist	R-symmetry $U(1)$	supercharge
		pick up complex structure	$Q = Q_1$
D=4, $\mathcal{N}=2$	topological twist	R-symmetry $U(2)$	supercharge
		$SU(2)_r \xrightarrow{id} SU(2)_R$	$Q = \bar{Q}_{\dot{1}2} - \bar{Q}_{\dot{2}1}$
D=4, $\mathcal{N}=4$	topological twist	R-symmetry $U(4)$	supercharge
	GL twist	$\begin{pmatrix} SU(2)_l & 0 \\ 0 & SU(2)_r \end{pmatrix} \to SU(4)_R$	$Q = uQ_l + vQ_r$

Table 2: (1)Here, $SU(2)_I$ means the subgroup in $U(2)$, and we use $Spin(4) = SU(2)_l \times SU(2)_r$. (2)In D=4,$\mathcal{N}=2$, the second index from which we down the supercharge index A by ϵ_{ab}. (3)In D=4,$\mathcal{N}=4$,the Q_l, Q_r refer to [3].

3.1 D=4, $\mathcal{N}=1$ SYM

As a gauge theory with Lie algebra $\mathfrak{g}$ over 4D manifold M, its action functional is:

$$S_{SYM} = \frac{1}{16\pi} Im(\tau \int d^6z \; Tr(W^\alpha W_\alpha) + c.c) \quad W_a := -\frac{1}{8}\bar{D}^2(e^{-2V} D_\alpha e^{2V}) \quad \tau := \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$$

Here, d^6z is short for $d^4xd^2\theta$, and $c.c$ denotes the complex conjugate part. Tr stands for the Killing form of the Lie algebra $\mathfrak{g}$.

$$gauge\ transformation: \quad e^{2V} \to e^{i\bar{\Omega}} e^{2V} e^{-i\Omega}$$

Here, V is the vector field and Ω is the chiral field valued in $\mathfrak{g}$.

I will skip the introduction of the BV formalism(ref. [4]). Following our marks, $\mathcal{V}_g$ and $\mathcal{C}_g$ denote the field spaces of vector and chiral fields valued in $\mathfrak{g}$. In short, originally, we are considering $(\mathcal{V}_{\mathfrak{g}}, \mathcal{L}^{\mathfrak{g}})$ the gauge invariant functional over the field space. Instead of that, we consider the derived complex of the functor $Hom_{\mathfrak{g}}(\mathbb{C}, \_\_)$. The complex is the Chevalley-Eilenberg complex and its $H^0 = \{\mathcal{L}^{\mathfrak{g}}\}$. Based on this, we consider the Kaszul-Tate derived of the equation (df=0) over the CE-complex. As a result, we obtain something we call the BV complex and the corresponding BV derivation δ_{BV}.

Definition 3.1 *(local BV complex)*

$$\mathcal{BV}_{SYM} := (O_{loc}(T^*[-1](\mathcal{V}_g \oplus \mathcal{C}_g[1]), \delta_{SYM})$$

[1] denotes the left shift in a graded vector space; T^* means the cotangent bundle. Specifically, the graded vector space looks like:

$$\mathcal{C}_g[1] \oplus \mathcal{V}_g \oplus \mathcal{V}_g^*[-1] \oplus \mathcal{C}_g^*[-2] := \mathcal{E}$$

$O(\mathcal{E})$ means the functional over the function space $\mathcal{E}$;loc means it must take this form:

$$F(\mathcal{E}_1, ...\mathcal{E}_n) = \int_M \alpha(j_z(\mathcal{E}_1), ..., j_z(\mathcal{E}_n)) : \mathcal{E}^{\otimes n} \to R$$

Here, j_z is the jet bundle of a sheaf at point z; α is a density-valued function on the jet bundles.

Remark 3.1.1: Here I also use the $\mathcal{E}$ to denote the sheaf.
Remark 3.1.2: The odd graded fields are anti-commutative, and the even ones are commutative in the graded vector space. In physics, we call the components in $\mathcal{E}$ from left to right: ghost, field, anti-field, and anti-ghost.
Remark 3.1.3: There is a natural Poisson bracket on the Kaszul-Tate complex, which is the Schouten bracket, we also call it the BV bracket. In particular, we have $\delta_{BV} = \{\_\_, S_{SYM}\}$.

3.1.1 Wess-Zumino gauge and reduce to gauge theory

Extending the gauge transformation by the BCH formula, we have:

$$\delta V = \frac{i}{2}(\bar{\Omega} - \Omega) + O(V))$$

By comparing the formula (3)(4), the first six terms in (4) can be canceled out through the 0-order in gauge.

Definition 3.2 *The Wess-Zumino gauge is to set: C, χ, M components in $\mathcal{V}$ equal to 0. Under this gauge:*

$$\mathcal{V} = \theta\sigma^\mu\bar{\theta}V_\mu + \bar{\theta}^2\theta^\alpha\lambda_\alpha + \theta^2\bar{\theta}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} + \theta^2\bar{\theta}^2 D$$

The higher order part looks like [V,...], you can't reach the WZ gauge just by transforming once, but the difference will reside in higher terms in (4), you can defer it to the later terms each time. Finally, we can reach the WZ gauge.

So in WZ gauge,

$$W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V + \frac{1}{4}\bar{D}^2([V, D_\alpha V]) = \lambda_\alpha(x) + 2D\theta_\alpha + i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} - i\theta^2\sigma^\mu D_\mu\bar{\lambda}$$

Using the property:

$$\sigma_a\bar{\sigma}_b\sigma_c = (\eta_{ac}\sigma_b - \eta_{bc}\sigma_a - \eta_{ab}\sigma_c) + i\epsilon_{abcd}\sigma^d$$

Finally, we have:

$$S_{SYM} = \frac{1}{g^2}\int d^4x(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu D_\mu\bar{\lambda} + 2D^2) + \frac{\theta}{32\pi^2}\int d^4x F_{\mu\nu}{}^\star F^{\mu\nu}$$

which is the canonical N=1 gauge theory with θ term. Since in perturbative level, gauge fixing doesn't change the BV complex, we have:

Lemma 3.1 *Under Wess-Zumino gauge, the two BV complexes*

$$(\mathcal{O}_{loc}(\mathcal{E}^{SYM}), \delta_{SYM}) \overset{quasi-iso}{\cong} (\mathcal{O}_{loc}(\mathcal{E}^{gauge}), \delta_{gauge})$$

3.1.2 Twist theory

Induced from $\mathcal{V}_{\mathfrak{g}}$, we have the SUSY transform on component fields:

$$\begin{aligned}\delta A_\mu &= \epsilon\sigma_\mu\bar{\lambda} + \lambda\sigma_\mu\bar{\epsilon}\\ \delta\lambda &= \epsilon D + (\sigma^{\mu\nu}\epsilon)F_{\mu\nu}\\ \delta D &= i\epsilon\sigma^\mu\partial_\mu\bar{\lambda} - i\partial_\mu\lambda\bar{\sigma}^\mu\bar{\epsilon}\end{aligned}$$

Transferring to the first-order formalism, we have:

$$\begin{aligned}\delta_{Q_\alpha}A_\mu &= \sigma_{\mu\alpha\dot{\alpha}}\psi^{\dot{\alpha}}\\ \delta_{Q_\alpha}\psi_\beta &= \epsilon_\alpha^\gamma B_{\gamma\beta}\end{aligned}$$

We can choose any spinor as our twist charge, usually $Q := Q_1$. Since selecting a spinor is equivalent to choosing a complex structure, our structure group $Spin(4)$ now reduces to SU(2), which means that now Q is a U(1)-charge scalar but not a spinor as a derivation. This induces a second derivation in our BV complex:

$$0 \qquad 1 \qquad 2 \qquad 3$$

$$\begin{tikzcd}
 & \psi_- \arrow[r, "\not{D}"] \arrow[d, "Q"] & \psi'_+ & \\
\Omega^0 \arrow[r, "\mathrm{d}"] & \Omega^1 \arrow[r, "\mathrm{d}_+"] & \Omega^2_+ & \\
 & \Omega^2_+ \arrow[ur, "c\,\mathrm{Id}"] \arrow[r, "\mathrm{d}"] \arrow[d, "Q"] & \Omega^3 \arrow[r, "\mathrm{d}"] & \Omega^4 \\
 & \psi_+ \arrow[r, "\not{D}"] & \psi'_- &
\end{tikzcd}$$

The new derivation now has the wrong degree. We can add a BV degree 1, U(1) degree -1 element t to that. And we have:

$$(\mathcal{O}_{loc}(\mathcal{E}^{gauge})((t)), \delta_{gauge} + tQ)$$

$$0 \qquad 1 \qquad 2 \qquad 3$$

$$\begin{tikzcd}
\psi_- \arrow[r, "\not{D}"] \arrow[dr, "tQ"] & \psi'_+ \arrow[dr, "tQ"] & & \\
\Omega^0 \arrow[r, "\mathrm{d}"] & \Omega^1 \arrow[r, "\mathrm{d}_+"] & \Omega^2_+ & \\
 & \Omega^2_+ \arrow[ur, "c\,\mathrm{Id}"] \arrow[r, "\mathrm{d}"] \arrow[dr, "tQ"] & \Omega^3 \arrow[r, "\mathrm{d}"] \arrow[dr, "tQ"] & \Omega^4 \\
 & & \psi_+ \arrow[r, "\not{D}"] & \psi'_-
\end{tikzcd}$$

In short, what we did was construct a Q-cohomology by twisting and blending it with the BV-comology in the correct degree.

What's more, choosing Q not only gives us a complex structure, but also the following isomorphism:

$$\Gamma : \psi^- \xrightarrow{\cong} \Omega^{1,0} \tag{5}$$

Here, we substitute Q with the following left-spinor part:

$$V_\mu \rightarrow (\sigma^\mu V_\mu)_{\alpha\dot{\alpha}}$$

And,

$$Y : \Omega^0 \cdot w \oplus \Omega^{0,2} \xrightarrow{\cong} \psi^+ \tag{6}$$

Here, we use the isomorphism and pair one of ψ_+ with Q.

$$\Omega^2_+ \cong Sym^2(\psi^+)$$

With these isomorphisms, we have:

Theorem 3.1 *(Costello 4.0.2 [1]) The twisted N = 1 gauge theory on $\mathbb{C}^2$ is equivalent to the holomorphic BF theory.*

In the same paper, costello also proves that:

Theorem 3.2 *(Costello 4.1.1 [1]) The twisted N = 1 theory admits a unique quantization, compatible with certain natural symmetries, on any Calabi-Yau surface X.*

4 SUSY-invariant Local Functional

Our general super-functional space is denoted as $O_{loc}(\mathcal{E})$. We can further ask for SUSY invariant condition. In this case, we can write it down explicitly:

Lemma 4.1

$$\mathcal{O}_{loc}(\mathcal{E})^{SUSY} = \mathbb{C} \otimes_{\tilde{D}_{N=1}} Sym^*(\mathbb{C}[\partial_\mu, D_\alpha, \bar{D}_{\dot{\alpha}}] \otimes \check{\mathcal{E}})$$

Now we give some definitions of this lemma:

Definition 4.1 *(D-module)*

$$D_{N=1}(M) := \mathcal{O}(M)[\partial_\mu, \partial_{\theta^\alpha}, \partial_{\bar{\theta}^{\dot{\alpha}}}]$$

Here M is the Superspace $R^{4|4}$, $O(M)$ is the set of the superfields we mentioned in 2.6. It's just the polynomial ring of differential operators on $R^{4|4}$ as an $\mathcal{O}(M)$ module.

Definition 4.2 *(Jet bundle)*

The jet bundle is a sheaf assigned to another sheaf $\mathcal{E}$ defined as:

$$J(\mathcal{E})(U) := C^\infty(U) \otimes_R \mathbb{R}[[\partial_1, ..., \partial_n]] \otimes_R \mathcal{E}(U)$$

Its elements look like $\partial_\mu \phi, \partial_\mu \partial_v \phi, ...$
To simplify the note, we use:

$$D_{N=1} := \mathbb{C}[\partial_\mu, \partial_{\theta^\alpha}, \partial_{\bar{\theta}^{\dot{\alpha}}}] \quad \tilde{D}_{N=1} := \mathbb{C}[\partial_\mu, D_\alpha, \bar{D}_{\dot{\alpha}}] \quad O_{N=1} := \mathbb{C}[\theta^\alpha, \bar{\theta}^{\dot{\alpha}}]$$

$$D_{N=1}(M) = \mathcal{O}(M) \otimes D_{N=1} \cong \mathcal{O}(M) \otimes \tilde{D}_{N=1}$$

Proof of Lemma 4.1:

(1) By definition:

$$\mathcal{O}_{loc}(\mathcal{E}) \cong Dens(M) \otimes_{D(M)} \mathcal{O}(J(\mathcal{E})) \cong Dens(M) \otimes_{D(M)} Sym^*(\check{J}(\mathcal{E})(M))$$

Here,$\check{J}(E)$ is the dual sheaf $\cong \mathcal{O} \otimes \tilde{D}_{N=1} \otimes \check{\mathcal{E}}$

(2) We perform the Taylor expansion on the fermion coordinate and move all the f(x) components to the left:

$$O_{loc}^{R^4} \cong O_{N=1} \otimes_{\tilde{D}(M)} Sym^*(\tilde{D}_{N=1} \otimes \check{\mathcal{E}}(M))$$

R^4 means the translation invariant functional, which imposes that f(x) is constant

(3) When variation occurs, it's a right action $\mathcal{Q}_\alpha$ on $\mathcal{O}_{loc}(\mathcal{E})$. We define the action of θ^α in the same way, using:

$$\{\mathcal{Q}_\alpha, \theta^\alpha\} = i$$

We have:

$$S \cdot \{\mathcal{Q}_\alpha, \theta^\alpha\} = iS \text{ and } \{\mathcal{Q}_\alpha, D_\alpha\} = 0 = \{\mathcal{Q}_\alpha, \tilde{D}_{\dot{\alpha}}\}$$

we move all θ^α to measurement part:

$$S = k(S \cdot \theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2) \mathcal{Q}_1 \mathcal{Q}_2 \bar{\mathcal{Q}}_1 \bar{\mathcal{Q}}_1$$

which imposes $O_{N=1}$ part in S = $\mathbb{C}$ $\square$

One step more, when we focus on the chiral fields $\mathcal{C}$, we have a similar discussion, and the conclusion is:

Theorem 4.1

$$\mathcal{O}_{loc}(\mathcal{C}, \bar{\mathcal{C}})^{SUSY} = \mathbb{C} \otimes_{\tilde{D}_{N=1}} (Sym^*(\tilde{D}_{N=1}^{Ch} \otimes \check{\mathcal{C}}) \otimes_{\mathbb{C}} Sym^*(\bar{\tilde{D}}_{N=1}^{Ch} \otimes \check{\bar{\mathcal{C}}}))$$

$$+ \mathbb{C}\bar{\theta}^1 \bar{\theta}^2 \otimes_{\mathbb{C}[\partial_\mu]} Sym^*(\mathbb{C}[\partial_\mu] \otimes \check{\mathcal{C}}) + c.c$$

As you can see, this is the canonical structure of the D-term + F-term. The proof of the D-term is similar to **Lemma 4.1**, but slightly delicate. Here

$$\tilde{D}^{Ch}_{N=1} := \tilde{D}_{N=1}/\tilde{D}_{N=1}\bar{D}_{\dot{\alpha}} \quad \bar{\tilde{D}}^{Ch}_{N=1} := \tilde{D}_{N=1}/\tilde{D}_{N=1}D_{\dot{\alpha}}$$

They are the differential operators acting separately on the chiral ring and the anti-chiral ring.

Now, following the previous proof, we have:

(here $\geq (1,1)$ means the number of Sym^*, this correspond to D-term)

$$O^{R^4}_{loc}(\mathcal{C},\bar{\mathcal{C}})_{\geq(1,1)} \cong O_{N=1} \otimes_{\tilde{D}_{N=1}} (Sym^*(\tilde{D}^{Ch}_{N=1} \otimes \check{\mathcal{C}}) \otimes_{\mathbb{C}} Sym^*(\bar{\tilde{D}}^{Ch}_{N=1} \otimes \check{\bar{\mathcal{C}}}))$$

different from **Lemma 4.1**, we can only define θ^α action on $Sym^*(\tilde{D}^{Ch}_{N=1})$, and $\bar{\theta}^{\dot{\alpha}}$ on $Sym^*(\bar{\tilde{D}}^{Ch}_{N=1})$, since $[D_\alpha, \theta_\alpha] \neq 0$, and that will break the quotient structure of $\bar{\tilde{D}}^{Ch}_{N=1}$. But the general $\{\mathcal{Q}_\alpha, \theta_\alpha\} = i$ still works, so we have:

$$O^{SUSY}_{loc}(\mathcal{C},\bar{\mathcal{C}})_{\geq(1,1)} \cong \mathbb{C} \otimes_{\tilde{D}_{N=1}} (Sym^*(\tilde{D}^{Ch}_{N=1} \otimes \check{\mathcal{C}}) \otimes_{\mathbb{C}} Sym^*(\bar{\tilde{D}}^{Ch}_{N=1} \otimes \check{\bar{\mathcal{C}}}))$$

The proof is the same as **Lemma 4.1**, and we call it the D-term.

Then we go to the F term, that means $\mathcal{O}_{loc}(\mathcal{C},\bar{\mathcal{C}})_{(*,0)} := \mathcal{O}_{loc}(\mathcal{C})$. It's still $\mathcal{Q}_\alpha$ exact, but $\{\bar{\mathcal{Q}}, \bar{\theta}\} = 0$ now, so we should talk about $H_{\bar{\mathcal{Q}}_1+\bar{\mathcal{Q}}_2}(\mathcal{O}_{loc}(\mathcal{C}))$

Lemma 4.2

$$H^*_{\bar{\mathcal{Q}}_1}(\mathcal{O}_{loc}(\mathcal{C})) \cong H^*(\mathcal{O}^{chiral}_{loc}(\mathcal{C}), \bar{D}_1)\bar{\theta}^2\bar{\epsilon}^1$$

Here $\mathcal{O}^{chiral}_{loc} := O^{chiral}_{N=1} \otimes Sym^*(\tilde{D}^{Ch}_{N=1} \otimes \check{\mathcal{C}}), \quad O^{chiral}_{N=1} := \mathbb{C}[\theta^\alpha]$

Proof of Lemma 4.2: We consider the super-De Rham complex of O^{chiral}_{loc}

$$\mathcal{O}^{ch}_{loc}\bar{\epsilon}^1\bar{\epsilon}^2 \overset{\Phi}{\hookrightarrow} \mathcal{O}^{ch}_{loc} \otimes \left(\mathbb{C}[\bar{\theta}^1]\bar{\epsilon}^2 \oplus \mathbb{C}[\bar{\theta}^2]\bar{\epsilon}^1\right) \xrightarrow{\Phi} \mathcal{O}^{ch}_{loc} \otimes \mathbb{C}[\bar{\theta}^1, \bar{\theta}^2] \cong \mathcal{O}_{loc}(\mathcal{C})$$

Here $\bar{\epsilon}^{\dot{\alpha}}$ is just a formally odd parameter, and

$$\Phi : \bar{\epsilon}^{\dot{\alpha}} \to 1 - \bar{\theta}^{\dot{\alpha}} \otimes_D \bar{D}_{\dot{\alpha}}$$

$$\bar{\mathcal{Q}}_1 := \frac{\partial}{\partial \bar{\theta}^1} - \bar{\epsilon}^{\dot{\alpha}}\bar{D}_1\frac{\partial}{\partial \bar{\epsilon}^{\dot{\alpha}}} \text{ left act on } \epsilon \text{ component for first two complex}$$

In this way, Φ is a well-defined chain map, as an example:

$$\begin{array}{ccc}
\theta_1\theta_2 \otimes D_1D_2\mathcal{C} \otimes \bar{\theta}^1\bar{\epsilon}^2 & \xrightarrow{\quad\Phi\quad} & -\theta^1\theta^2\bar{\theta}_1 \otimes D_1D_2\mathcal{C} + \theta^1\theta^2\bar{\theta}^1\bar{\theta}^2 \otimes \bar{D}_2D_1D_2\mathcal{C} \\
\downarrow{\scriptstyle \mathcal{Q}_1} & & \downarrow{\scriptstyle \mathcal{Q}_1} \\
\theta_1\theta_2 \otimes D_1D_2\mathcal{C} \otimes (\bar{\epsilon}^2 + \bar{\epsilon}^2\bar{D}_1\bar{\theta}^1) & \xrightarrow{\quad\Phi\quad} & -\theta^1\theta^2\bar{\theta}_1 \otimes \bar{D}_1D_1D_2\mathcal{C} - \theta^1\theta^2\bar{\theta}^1\bar{\theta}^2 \otimes \bar{D}_1\bar{D}_2D_1D_2\mathcal{C}
\end{array}$$

So the $H^*_{\bar{\mathcal{Q}}_1}(\mathcal{O}_{loc}(\mathcal{C}))$ we need is the De-Rham degree 2 cohomology of the double complex; calculating by spectral sequence, we have:

$$H^*_{\bar{\mathcal{Q}}_1}(\mathcal{O}_{loc}(\mathcal{C})) \cong H^*(\mathcal{O}^{chiral}_{loc}(\mathcal{C}), \bar{D}_1)\bar{\theta}^2\bar{\epsilon}^1 \square$$

Since $\{D_\alpha, \bar{D}_{\dot{\alpha}}\}$ is total derivative, we have the following quasi-isomorphism:

$$(\mathcal{O}^{Ch}_{loc}, \bar{D}_1) \cong \theta^1\theta^2\Omega^*(Sym^*(\tilde{D}^{Ch}_{N=1}))$$

And notice that the spectral sequence of $(\theta^1\theta^2\Omega^*(Sym^*(\tilde{D}^{Ch}_{N=1}), \bar{\mathcal{Q}}_1 + d_{dR})$ converges at E2, so we have:

$$H^*_{\bar{\mathcal{Q}}_1}(\mathcal{O}^{Ch}_{loc}) = (\theta^1\theta^2\Omega^*(Sym^*(\tilde{D}^{Ch}_{N=1}\otimes\check{\mathcal{C}}), \bar{\mathcal{Q}}_1 + d_{dR}) = \theta^1\theta^2 H^*_{dR}(Sym^*(H^*_{\bar{\mathcal{Q}}_1}(\tilde{D}^{Ch}_{N=1}\otimes\check{\mathcal{C}})))$$

Lemma 4.3

$$H^*_{\bar{\mathcal{Q}}_1}(\tilde{D}^{Ch}_{N=1}\otimes\check{\mathcal{C}}) = \mathbb{C}[\partial_{1\dot{2}}, \partial_{2\dot{2}}]\otimes\check{\mathcal{C}}$$

Since the action of $\bar{\mathcal{Q}}_1$ is equivalent to the left action of $-2\partial_{\alpha 1}\frac{\partial}{\partial D_\alpha}$ on $\mathcal{O}^{Ch}_{loc}$, taking the dual, we have:

$$H^*(\mathbb{C}[x_{\alpha\dot{\alpha}}, \frac{\partial}{\partial D_\alpha}]\otimes\mathcal{C}, -2\partial_{\alpha 1}\frac{\partial}{\partial D_\alpha}) = \mathbb{C}[x_{1\dot{2}}, x_{2\dot{2}}]\otimes\mathcal{C} \quad \square$$

Summarize the above discussion, the $\mathcal{Q}_1$ cohomology of $\mathcal{O}_{loc}(\mathcal{C})$ is represented by :

$$\int d^2\theta d^2\bar{\theta} f(\theta)\bar{\theta}^1\bar{\theta}^2 W(\mathcal{C}, \partial_{1\dot{2}}\mathcal{C}, \partial_{2\dot{2}}\mathcal{C})$$

So for any S $\bar{\mathcal{Q}}_1$-invariant we write it as $S = S_0 + \bar{\mathcal{Q}}_1\tilde{S}$.
And apply $\mathcal{Q}_1, \mathcal{Q}_2$-exact condition on S_0, it looks like:

$$\int d^2\theta d^2\bar{\theta}\bar{\theta}^1\bar{\theta}^2 W(\mathcal{C}, \partial_{1\dot{2}}\mathcal{C}, \partial_{2\dot{2}}\mathcal{C})$$

which is F-term.

Then consider: $\bar{\mathcal{Q}}_1\tilde{S} = \bar{\mathcal{Q}}_2 S'$, that means $[\bar{\mathcal{Q}}_1\tilde{S}] = 0$ in $\mathcal{O}_{loc}/Im\mathcal{Q}_2$.

If $[\tilde{S}] = 0$ in $H^*_{\mathcal{Q}_1}(\mathcal{O}_{loc}/Im\mathcal{Q}_2)$. Naturally, $\tilde{S} = \bar{\mathcal{Q}}_1 S_1 + \bar{\mathcal{Q}}_2 S_2$, which means $S = \bar{\mathcal{Q}}_1\tilde{S}$ is $\bar{\mathcal{Q}}_1\bar{\mathcal{Q}}_2$-exact, applying $\mathcal{Q}_1\mathcal{Q}_2$-exact condition, it contributes D-term.

Then we calculate $H^*_{\mathcal{Q}_1}(\mathcal{O}_{loc}/Im\mathcal{Q}_2)$ using the following resolution:

$$\begin{tikzcd}
...\mathcal{O}_{loc} \arrow[r, "\mathcal{Q}_2"] & \mathcal{O}_{loc} \arrow[r, "\mathcal{Q}_2"] & \mathcal{O}_{loc} \arrow[r] & \mathcal{O}_{loc}/Im\mathcal{Q}_2 \\
\oplus & & & \\
H^*_{\bar{\mathcal{Q}}_2}(\mathcal{O}_{loc}) \arrow[uur, "j"] & & &
\end{tikzcd}$$

We define deg_1 is the number of $\partial_{1\dot{2}}\ or\ \partial_{2\dot{2}}$; deg_1 is the number of $\partial_{1\dot{1}}\ or\ \partial_{2\dot{1}}$ in $\mathcal{O}_{loc}$. And j is an embedding by picking up a representation element.

Then E_1 page from applying $\mathcal{Q}_1$-cohomology on $\mathcal{O}_{loc}$, since we know that $\mathcal{Q}_2$ act on $H^*_{\mathcal{Q}_1}$) is 0, we have:

$$d_1 = j : H^*_{\mathcal{Q}_2} \to H^*_{\mathcal{Q}_1}$$

is identity on the $deg_1 = deg_2 = 0$ component, and 0 otherwise.

So E_2 is getting the mod degree-0 component of E_1 page. And trace the graph, d_n is replacing d $\partial_{1\dot{2}}\ or\ \partial_{2\dot{2}}$ component, by $\partial_{1\dot{1}}\ or\ \partial_{2\dot{1}}$.

At last, the spectrum sequence converges to $\oplus_{k=2}^{\infty} H^{\geq k}_{\mathcal{Q}_2}(\mathcal{O}_{loc})$, and $H^{\geq k}_{\mathcal{Q}_2}(\mathcal{O}_{loc})$ corresponds to term with (k-1)-($\partial_{1\dot{2}}\ or\ \partial_{2\dot{2}}$) in $H^*_{\mathcal{Q}_1}(\mathcal{O}_{loc}/Im\mathcal{Q}_2)$. After applying $\mathcal{Q}_1$,$\mathcal{Q}_2$-exact condition, it contributes to F-term which has $Sym^*([\partial_\mu])$ structure. $\square$ *Thm* 4.1

5 Quantization of $\mathcal{N} = 1$ SYM on superspace

As a *pre-theory*([4] section 13) on R^4 , $\mathcal{N} = 1$ SYM given by following second-order BV complex:

$$\begin{tikzcd}
-1 & 0 & 1 & 2 \\
\mathcal{C}_g \arrow[r, "i"] & \mathcal{V}_g \arrow[ddr, red, "\bar{D}^2 D_\alpha"] & \mathcal{V}_g^* \arrow[r, "\bar{D}^2"] \arrow[ddr, "D^2"] & \mathcal{C}_g^* \\
\oplus & \oplus & \oplus & \oplus \\
\bar{\mathcal{C}}_g \arrow[uur, "-i"] & \mathcal{B}_\alpha \arrow[uur, "D^\alpha"] \arrow[r, "="] & \mathcal{B}_\alpha^* & \bar{\mathcal{C}}_g^*
\end{tikzcd}$$

And Q^{GF} given by:

$$\begin{tikzcd}
-1 & 0 & 1 & 2 \\
\mathcal{C}_g & \mathcal{V}_g \arrow[l, "\bar{D}^2 D^2"'] \arrow[ddl, "D^2\bar{D}^2"] & \mathcal{V}_g^* \arrow[ddl, red, "\bar{D}^2 D_\alpha"] & \mathcal{C}_g^* \arrow[l, "D^2"] \\
\oplus & \oplus & \oplus & \oplus \\
\bar{\mathcal{C}}_g & \mathcal{B}_\alpha & \mathcal{B}_\alpha^* \arrow[uul, "D^\alpha"] & \bar{\mathcal{C}}_g^* \arrow[uul, "\bar{D}^2"]
\end{tikzcd}$$

By simple calculation, we have the property $[Q, Q^{GF}] = \square$.

The second-order SYM action is:

$$S_{SYM} = \frac{1}{16\pi} Im(\tau \int d^6z\ Tr(W^\alpha B_\alpha + B^\alpha B_\alpha) + c.c)$$

Quantization means we can assign R^4 a factorization algebra Obs^q, which locally(for U a open disk) looks like following DGLA at level Φ:

$$Obs^q_\Phi(U) \cong (\oplus_n Sym^n(T^*[-1](\mathcal{V}_g(U) \oplus \mathcal{C}_g(U)[1]))^*[[\hbar]], Q + \hbar\triangle_\Phi + \{I[\Phi], \_\_\_\})$$

Here we define parametrix following [5] section 7.2.4 as an element:

$$\Phi \in \bar{\mathcal{E}}(U)\widehat{\otimes}_\pi \bar{\mathcal{E}}(U)$$

Satisfying several properties from elliptic regularity. $\bar{\mathcal{E}}(U)$ means the distributional section of $\mathcal{E}(U)$. Define:

$$P(\Phi) = \frac{1}{2} Q^{GF}\Phi \in \overline{\mathcal{E}}\widehat{\otimes}_\pi \overline{\mathcal{E}}.$$

This is the propagator associated with Φ. We let

$$K_\Phi = K_{\mathrm{Id}} - QP(\Phi).$$

K_{Id} is the kernel of $Id : \mathcal{E} \rightarrow \bar{\mathcal{E}}$

$$\triangle_\Phi := \partial_{K_\Phi} : \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E})$$

$$\{I, J\}_\Phi := \triangle_\Phi(IJ) - (\triangle_\Phi I)J - (-1)^{|I|} I \triangle_\Phi J$$

on the space $\mathcal{O}(\mathcal{E})$.

Also, we demand the quantum field theory $I[\Phi]$ corresponds to our classical field theory $I_0 = S_{SYM}$, that means it satisfies:

- $I[\Phi] = W(P(\Phi) - P(\Psi), I[\Psi])$ (1)
- $\lim_{\Phi \longrightarrow 0} I[\Phi] - I_0 = 0 \ mod\ \hbar$ (2)
- $I[\Phi]$ satisfies *quantum master equation* for $\forall\ \Phi$ (3)

Condition (1)(2)(3) are satisfied at the same time if we can find a lift $\tilde{I}_0$ of I_0 at scale 0, that satisfies quantum master equation. Since we can define:

$$I[\Phi] := W(P(\Phi), \tilde{I}_0)$$

And we have the following lemma:

Lemma 9.2.2 [4]

If $I[\Phi]$ satisfies the Φ-QME, then $I[\Psi] = W(P(\Psi) - P(\Phi), I[\Phi])$ satisfies the Ψ-QME.

This follows from the identity

$$\left[Q, \partial_{P(\Phi)} - \partial_{P(\Psi)}\right] = \triangle_\Psi - \triangle_\Phi$$

Then

$$(Q + \hbar\triangle_\Psi)e^{I[\Psi]/\hbar} = (Q + \hbar\triangle_\Psi)e^{\hbar\partial_{P(\Psi)-P(\Phi)}}e^{I[\Phi]/\hbar} = e^{\hbar\partial_{P(\Psi)-P(\Phi)}}(Q + \hbar\triangle_\Phi)e^{I[\Phi]/\hbar}$$

Naturally, $I[\Phi] := W(P(\Phi), I_0)$ satisfies mod $\hbar$ *quantum master equation*. To lift it to a theory satisfies mod $\hbar^2$ *quantum master equation*, or more generally, to lift a $\hbar^n$ theory $I[L]$ to a $\hbar^{n+1}$, we choose any lift $\tilde{I}[L]$ and define its obstruction at scale L as:

$$O_{n+1}[L] = \hbar^{-n-1}\left(Q\tilde{I}[L] + \frac{1}{2}\{\tilde{I}[L], \tilde{I}[L]\}_L + \hbar\Delta_L \tilde{I}[L]\right)$$

And we have the following property:

- The obstruction of the lift $\tilde{I}[L]$: $O_{n+1} \in H^1_{Q+\{I_0,\_\}}(\mathcal{O}_{loc}(\mathcal{E}))$
- When the obstruction vanishes, the classification set of the lift equals $H^0_{Q+\{I_0,\_\}}(\mathcal{O}_{loc}(\mathcal{E}))$

In this paper, we want to show that when $I_0 = S_{SYM}$, $H^1_{Q+\{I_0,\_\}}(\mathcal{O}_{loc}(\mathcal{E})) = 0$, then our quantization is well-defined.

The proof of these properties is from *Lemma 11.1.1* [4].

In short, the first property is obtained easily from DGLA:

Let $\mathfrak{g}$ be a differential graded Lie algebra, and let $X \in \mathfrak{g}$ be an odd element. Let

$$O(X) = \mathrm{d}_{\mathfrak{g}} X + \frac{1}{2}[X, X].$$

Then,

$$\mathrm{d}_{\mathfrak{g}} O(X) + [X, O(X)] = 0.$$

That means

$$QO_{n+1} + \{I_0, O_{n+1}\} = 0$$

For $\tilde{I}[L]$ $\forall$lift, then any other lift is given by $\tilde{I}[L] + \hbar^{n+1}J[L]$. The obstruction vanishes if only:

$$QJ[L] + \{I_0[L], J[L]\} = O_{n+1}[L] \quad \square$$

Now we calculate $H^*_{Q+\{I_0,\_\}}(\mathcal{O}^{SUSY}_{loc}(\mathcal{E}))$. Recall in *Lemma 4.1*, we have proved that

$$\mathcal{O}_{loc}(\mathcal{E})^{SUSY} = \mathbb{C} \otimes_{\tilde{D}_{N=1}} Sym^*(\mathbb{C}[\partial_\mu, D_\alpha, \bar{D}_{\dot{\alpha}}] \otimes \check{\mathcal{E}})$$

And in SYM,

$$\mathcal{E} := \mathcal{C}_g[1] \oplus \mathcal{V}_g \oplus \mathcal{V}^*_g[-1] \oplus \mathcal{C}^*_g[-2]$$

as a Q-complex. And the $Q{+}[I_o, \_\_\,]$ cohomology is quasi-iso to the $Q{+}[I_o, \_\_\,]$ cohomology of the following bi-complex as a resolution:

$$\begin{tikzcd}
\dots \arrow[r] & \wedge^1\mathbb{R}^4\otimes\mathbb{C}[\theta^i\otimes D^i]\otimes Sym^*(\dots) \arrow[r, "\partial_i"] & \mathbb{C}[\theta^i\otimes D^i]\otimes Sym^*(\mathbb{C}[\partial_\mu, D_\alpha, \bar{D}_{\dot\alpha}]\otimes\check{\mathcal{E}}) \\
& \dots \arrow[r, "\partial_i"] & (\oplus_j(\mathbb{C}[\theta^i\otimes D^i](i\neq j))\epsilon^j)\otimes_{\tilde{D}_{N=1}} Sym^*(\mathbb{C}[\partial_\mu, D_\alpha, \bar{D}_{\dot\alpha}]\otimes\check{\mathcal{E}}) \arrow[u, "\Phi"] \\
& & \dots \arrow[u, "\Phi"]
\end{tikzcd}$$

The row is the canonical De Rham complex, and the column is the super-De Rham we defined in *Lemma 4.2*, θ^i go over $\theta^1, \theta^2, \bar{\theta}^1, \bar{\theta}^2$. The $Q+[I_0, \_\_\ ]$ only acts on Sym^* component, so we calculate $H_{Q+[I_0,\_\_\ ]}(Sym^*(\dots))$ as the E_1-page.

Now we filter the Sym^* component by its $*$-index, under this filtration, the E_1-page given by $Sym^i(\mathbb{C}[\partial_\mu, D_\alpha, \bar{D}_{\dot\alpha}]\otimes H^j_Q(\check{\mathcal{E}}))\cong Sym^i(\mathbb{C}[\partial_\mu, D_\alpha, \bar{D}_{\dot\alpha}]\otimes \check{H}^j_Q(\mathcal{E}))$.

Recall as in [4]*Lemma 6.7.1*, as a L^∞ CE cohomology, $H_{Q+\{I_0,-\}}(\mathcal{O}^{SUSY}_{loc})$ can be calculated localized to its fiber $(\mathcal{O}_{loc})_0 := Sym^*(\mathbb{C}[\partial_\mu, D_\alpha, \bar{D}_{\dot\alpha}]\otimes\check{\mathcal{E}}_0)$, because of its SUSY invariant. Then, just by Talor extending $\mathcal{C}$ and $\mathcal{V}$. For example, using the extension (3)(4) in section 2.6, we have:

$$e_1 = C(x)-\frac{1}{4}\theta^2\bar{\theta}^2\Box C(x) = Q\Big(\phi(x)+\mathrm{i}\theta^\mu\theta\partial_\mu\phi(x)+\frac{1}{4}\theta^2\bar{\theta}^2\Box\phi(x)\Big) = Qs_1 \text{ when } \phi \text{ is pure imagine}$$

$$e_2 = \theta\sigma^\mu\bar{\theta}A_\mu = Q\Big(\phi(x)+\mathrm{i}\theta^\mu\theta\partial_\mu\phi(x)+\frac{1}{4}\theta^2\bar{\theta}^2\Box\phi(x)\Big) = Qs_2 \text{ when } \phi \text{ is pure real}$$

Here a bit different from (3)(4) $\phi, \psi, F, \chi, \bar{\chi}, \lambda, \bar{\lambda}, M \in \mathbb{C}[[x_1, \dots, x_4]] \quad C, D, A_\mu \in \mathbb{R}[[x_1, \dots, x_4]]$

Finally, we have $H^*_Q(\mathcal{E}) = \mathfrak{g}[1]$. As a result, $H_{Q+[I_0,\_\_\ ]}(Sym^*(\dots))\cong H^*_{CE}(\mathfrak{g})$. When considering semi-simple $\mathfrak{g}$, the degree 1 component vanishes, and the degree 0 component is just $\mathbb{C}$, which means there is no obstruction and quantization is unique up to a constant. $\Box$

Acknowledgements

This paper benefited greatly from Si Li's notes about the D-module construction and Kevin Costello's work on homotopic renormalization. Besides this, I am deeply grateful to my supervisor, Prof. Si Li, for his invaluable guidance and constructive feedback throughout the research.

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