Mirror Symmetry Report

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Abstract

In this report, I want to explain how to realize the D-brane in physics as a coherent sheaf and matrix factorization in Landau-Ginzburg model. And some applications of its mirror symmetry.

1 Derived category in physical point

Considering the open sigma model (Σ, M) and assigned Chan-Paton factors on the boundary, it naturally gives the coherent sheaf structure on each boundary:

$$V_1 \to \partial\Sigma_1 \xrightarrow{\phi^i|_{\partial\Sigma_1}} M$$

Since a string is connected with two D-branes, each D-brane can be equivalently described by the coherent sheaf structure. I will denote it as (E,F). In this report, I mainly use the following Landau–Ginzburg B Model:

$$\begin{aligned} S = \int d^2z \Big(& g_{i\bar{j}} h^{\mu\nu} \partial_\mu \phi^i \partial_\nu \overline{\phi}^{\bar{j}} \sqrt{h} - i g_{i\bar{j}} \psi^{\bar{j}} D_{\bar{z}} \rho^i_z + i g_{i\bar{j}} \overline{\psi^{\bar{j}}} D_z \rho^i_{\bar{z}} \\ & - \frac{1}{2} R_{i\bar{k}j\bar{l}} \rho^i_{\bar{z}} \rho^j_z \psi^{\bar{k}} \overline{\psi}^{\bar{l}} + \frac{1}{8} g^{j\bar{i}} \partial_{\bar{j}} \overline{W} \partial_i W + \frac{1}{4} (D_i \partial_j W) \rho^i_{\bar{z}} \rho^j_z + \frac{1}{4} (D_{\bar{i}} \partial_{\bar{j}} \overline{W}) \psi^{\bar{i}} \overline{\psi}^{\bar{j}} \Big) \\ & \psi^{\bar{\imath}} + \overline{\psi}^{\bar{\imath}} = -\eta^{\bar{\imath}}, \ \psi^{\bar{\imath}} - \overline{\psi}^{\bar{\imath}} = g^{\bar{\imath}j} \theta_j \end{aligned}$$

By simply making the correspondence:

$$\eta^{\bar{\imath}} \longleftrightarrow d\overline{z}^{\bar{\imath}}, \quad \theta_i \longleftrightarrow \frac{\partial}{\partial z^i}$$

As a well-known discussion in the B-sigma model, the Neumann condition brane imposes: $\theta = (*\rho) = 0 \quad Q = \bar{\partial}$ So the string spectrum, or as a morphism in $Brane(M)$ is:

$$H^p(X, E^* \otimes F) \cong \mathrm{Ext}^p(E, F)$$

Further, we usually impose: $E \sim E \oplus X \oplus \bar{X}$ when $X \oplus \bar{X}$ is brane-antibrane pair. Even more, when $E \sim T \oplus \bar{X}$, $0 \to E \to T \to X \to 0$. (We call T a bound state)

This says that: $E \sim T \oplus \bar{X}$ when $H^*_{Q_E}(E) \cong H^*(T \oplus \bar{X}, Q_E \oplus 1)$. Here 1 is the isomorphism $X \oplus \bar{X} \xrightarrow{1} Id$. That means we should consider the quasi-isomorphism. That's the origin of the derived category in physics.

2 D-brane in Landau–Ginzburg Model

In this section, I want to show how Landau–Ginzburg Model connects to matrix factorization. Now we have supper-potential: $W : M \to \mathbb{C}$ holomorphic. And by Morse lemma, it locally looks like:

$$W = z_1^2 + ... + z_n^2$$

For n=2k, it goes to:

$$W = z_1 z_2 + z_3 z_4 + \cdots + z_{2k-1} z_{2k}.$$

For n=2k+1, it goes to:

$$W = z_1 z_2 + z_3 z_4 + \cdots + z_{2k-1} z_{2k} + z_{2k+1}^2.$$

Now we only have to consider $W = xy$ and $W = xy + z^2$ two cases. And by localization, the correlation function only given by the zero set of W.

For $W = xy$, it only has 1 D_0 brane $x = 0, y = 0$ and two $x_i = 0$ D_2 brane. At D_0 brane, $\phi_1 = \phi_2 = \eta^{\bar{1}} = \eta^{\bar{2}} = 0$, which means the string spectrum are only composed by $\mathbb{C}[\theta^i]$ which isomorphic to $Cl(2, \mathbb{C})$.

For $W = xy + z^2$, the case is similar that D_0 brane gives $Cl(3, \mathbb{C})$ state and D_1 brane gives $Cl(1, \mathbb{C})$ state.

Since the quantum states are morphisms in $Brane(M)$ category, the $\mathbb{Z}_2$ structure imposes the complexes have $\mathbb{Z}_2$-decomposition. As a constraint condition of open Landau-Ginzburg Model, the variation of the bulk action contributes a non-zero boundary term-the Warner term $\frac{i}{2}\left(\psi_-^i + \psi_+^i\right)\partial_i W$. Usually, we fix the problem by deforming the boundary action to:

$$\mathcal{L}_b = \frac{i}{2}\left(\bar{\gamma} D_\tau \gamma + \psi^i \partial_i F \gamma + \psi^{\bar{i}} \partial_{\bar{i}} \bar{G} \gamma\right) - \frac{1}{4}\left(\bar{F}F + \bar{G}G\right) + \text{c.c.}$$

Here $F = F(\phi), G = G(\phi)$ are holomorphic sections of $Hom(E_1, E_2)$ and $Hom(E_2, E_1)$, respectively. They depend on the fields ϕ^i restricted to the boundary.

What we did was just glue two D-brane E_1, E_2 together through a pair tachyons F, G. Satisfied:

$$FG = i(W + const), \quad GF = i(W + const)$$

As a result, F,G is obvious the differential in complex with $\mathbb{Z}_2$-decomposition, the $Brane(M)$ category has the matrix factorization structure.

3 Correspondence in mirror symmetry

In this section, I want to discuss Homological mirror symmetry between non-linear sigma model and Landau–Ginzburg Model. Under the mirror map(T-duality on phase), $X = \mathbb{CP}^{N-1}$ goes to the Landau–Ginzburg model of $N-1$ periodic variables $Y_1, \ldots, Y_{N-1}$ of periodicity $2\pi i$ with superpotential

$$W = \mathrm{e}^{-Y_1} + \cdots + \mathrm{e}^{-Y_{N-1}} + \mathrm{e}^{-t+Y_1+\cdots+Y_{N-1}}.$$

Here $t = r - i\theta$ corresponds to the complexified Kähler class parameter of $\mathbb{CP}^{N-1}$. That means $\theta = -2\pi \int_{[\mathbb{CP}_1]} c_1$, c_1 is the Chern class of the U(1) bundle equipped in X.

First, we consider the trivial bundle $\mathcal{O}$. By our discussion in part 1, it corresponds to the total D_{N-1} brane, which means it satisfies the Neumann condition on every coordinates.

Naturally, T-duality exchanges Neumann and Dirichlet conditions, so the A-brane corresponds to B-brane that satisfies $ImY_i = 0$. And that gives the straight line emanating from $W(p_0) = N$ and extending in the positive real direction on W-image.

Then, we twist $\mathcal{O}$ by setting $\theta = -2n\pi$. For example, when $n = 1$, that corresponds to $\mathcal{O}(1)$. And that gives a shift on the phase of $Y_i \to Y_i + 2i\alpha$. And this is just a rotation of the straight line by angle $-\frac{2\pi}{N}$ on the W-plane. By analogy, $\mathcal{O}(-l)$ given by rotate $\frac{2l\pi}{N}$. After that, we have to rotate the line to be parallel to the real axis to make sure that the Lagrangian submanifold preserves the half super-charge that the A-twist needs. At last, this gives the corresponding spectrum from X-B brane to LG-A brane.

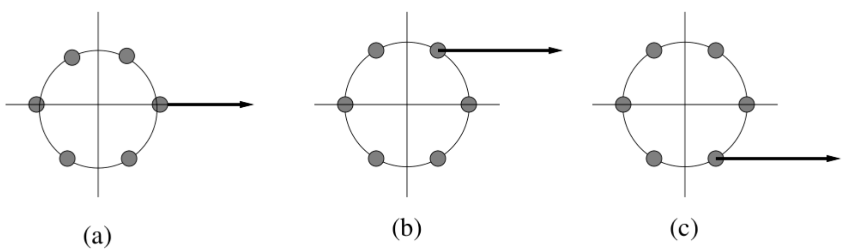

Figure 1: $a)\mathcal{O}$ $\quad b)\mathcal{O}-1)$ $\quad b)\mathcal{O}(1)$ $\quad$ cite from *Mirror Symmetry Hori, Vafa.*

For the inverse direction, there is no concise spectrum correspondence as before. However, when we focus on toric manifold X, defined by the charges Q_i^a with momentum map:$\sum_{i=1}^{N} Q_i^a s_i = t_a$. Considering the D-brane given by $|\phi_i|^2 = c_i$ equipped with connection

$$A_a = \sum_{i=1}^{N} \left[a_i \, d\varphi_i - \theta^a M_{ab} Q_i^b c_i \, d\varphi_i\right]$$

it corresponds to D_0 brane at $Y_i = c_i - ia_i$. We can calculate the Q-anomaly of A-brane in X by:

$$Q^2 \propto W\left(\mathrm{e}^{-Y^{(2)}}\right) - W\left(\mathrm{e}^{-Y^{(1)}}\right)$$

Therefore, the A-brane anomaly vanishes if and only if it corresponds to a "bound state" that cancels the Warner term as we discussed in Section 2.