Summary on Seiberg–Witten Theory

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Abstract

In this paper, I will give a summary on Seiberg-Witten theory, the main content I study in Umass Amherst.The key to Seiberg-Witten Theory is to calculate the Donaldson invariant using the TQFT method.

Keywords: Donaldson Invariant, Seiberg-Witten Theory, Super Quantum Filed Theory

1 Introduction

It's known that we have genus for 2D manifolds, knots for 3D manifolds to classify different topological manifolds. What's more, for 4D manifolds, we also have some invisible invariants, for instance Donaldson invariant.As will be shown in next section, it's well-defined but hard to calculate. But in 1988 [3], Witten gave the proof that this invariant can be realized as a correlation function of Twisted N=2 SQFT. It's a canonical example of how to use the field theory to discover mathematical objects. In the following section, I will give a short review about SQFT and then to its correlation function realization of Donaldson Invariant.

2 Review of Super Quantum Field Theory

2.1 Representation of Super-Pincare algebra

Poincare algebra is generated by $\{P_\mu, Q_a^A, \bar{Q}_{\dot{a}A}\}$, satisfied:

$$\left\{Q_\alpha^A, \overline{Q}_{\beta B}\right\} = 2\delta_B^A \sigma_{\alpha\beta}^\mu \mathbf{P}_\mu, \quad \left\{Q_\alpha^A, Q_\beta^B\right\} = \varepsilon_{\alpha\beta} Z^{AB}$$

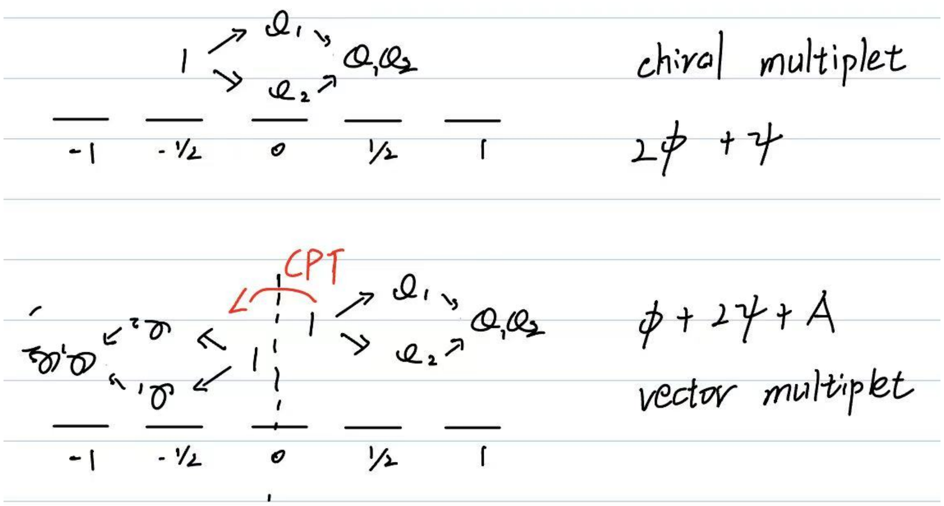
We claim it's the unique non-trivial extension of Pincare algebra.

2.1.1 N=1 massive

In the rest frame of a particle,$p_\mu = (m, 0, 0, 0)$ so

$$\{Q_\alpha, \overline{Q}_\beta\} = 2m\delta_{\alpha\beta}, \{Q, Q\} = \{\overline{Q}, \overline{Q}\} = 0$$

so we regard Q_a as creation operator and $\bar{Q}_{\dot{a}}$ as annihilation operator,begin with a state $|j\rangle$,we have $Q_1 |j\rangle, Q_2 |j\rangle, Q_1 Q_2 |j\rangle$,without gravity the superhelicity $J \leq 1$,at a result,we only have two representations called chiral multiplet and vector multiplet.



And these multiplets correspond to chiral and vector field.

$$\Phi(x, \theta, \bar{\theta}) = A(x) + \theta^\alpha \psi_\alpha(x) + \theta^2 F(x) + i\theta\sigma^a\bar{\theta}\partial_a A(x) + \tfrac{i}{2}\theta^2\bar{\theta}\tilde{\sigma}^a\partial_a\psi(x) + \tfrac{1}{4}\theta^2\bar{\theta}^2\Box A(x) \quad (1)$$

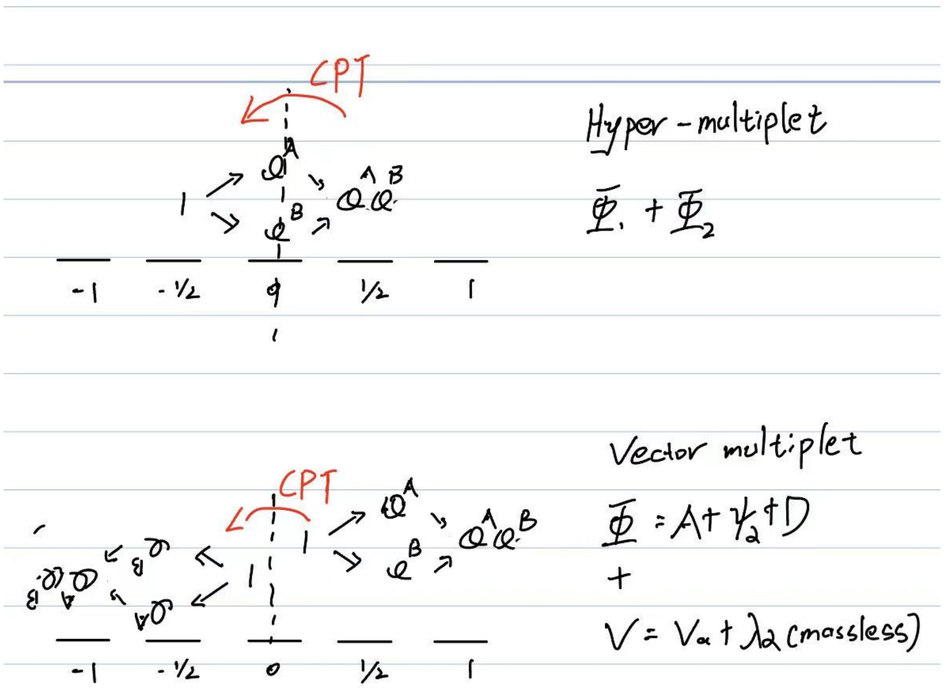
$$V = \theta\sigma^a\bar{\theta}V_a + \bar{\theta}^2\theta^\alpha\lambda_\alpha + \theta^2\bar{\theta}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} + \theta^2\bar{\theta}^2\mathcal{D} \quad (2)$$

2.1.2 N=1 massless

Also to the rest frame, $p_\mu = (E, 0, 0, E)$ .Now we only have one creation operator,as the same calculation,the representation component is the same as massive.

2.1.3 N=2 massless

Have "Hypermultiplet" and "Vector multiplet"



2.2 Pure N=2 massless Vector-mutiplet Theory

Since in this case,the Vector-mutiplet can be written as $V \oplus \Phi$. We consider N=1 massless theory with both these part,it is N=1 Super-Yang-Mills model:(this part following Witten's lecture https://youtu.be/9qZSqkn8-Qo, and for details about N=1 case,reference is[1][3.5.1])

$$S = \frac{1}{e^2}\int d^4x d\theta d\bar{\theta}\ tr(\bar{\Phi}e^{2V}\Phi) + \frac{1}{e^2}\int d^4x d\theta tr(W^{\alpha}W_{\alpha}) \quad (3)$$

$$Since \int dx d\theta \Phi = -\frac{1}{4}\int dx D^2\Phi|_{\theta=0,\bar{\theta}=0} \quad (4)$$

,what we need to do is using formulas (1)(2)(4) to extends (3) to euclidean case,and the result is :

$$S = \frac{1}{c^2}\int d^4x\ kinetic\ term\ of\ (A, \lambda, \bar{\lambda}, \phi) + \mathrm{Tr}\left[\phi, \phi^{\dagger}\right]^2 + (Yukawa\ potential)$$

The formula (3) have the same symmetry transformation as N=2 case after adding Yukawa potential,so it's the Lagrangian we are looking for.

Now, as you seen the potential is :

$V(\phi) = Tr[\phi, \bar{\phi}]$,which means it's diagonal ,so we set $\phi = a\sigma^3$. Since vacua under $SU(2)$ are equivalent, we use the parameter $u := \frac{1}{2}a^2 =< \phi^2 >$ to parameterize modular space of vacua, as u is double covered by a.

2.3 Relation to QCD

When $a \neq 0$, $SU(2)$ symmetry is broken to $U(1)$ and generate massive boson $W^{\pm}$ from two color indexes of A^a_μ.So as we consider low-energy effective theory,the two massive bosons won't make effort.The residue field is A_μ, ψ with U(1) symmetry,which exactly is QCD.

3 Topological invariant on 4D manifold

Now we reach a subtle place which is the crossing of math and physics. We will use the information of bundles to construct N=2 SQFT I mention above.

Definition 3.1 (Basic setting)**.** *For a closed,oriented,compact,smooth 4D manifold X,with principle G- bundle P.Denote by $\mathcal{A}$ the connection on P compatible with G action,denote by $\mathcal{G}$ the gauge transformation.*

Definition 3.2. *ASD*

$$\mathcal{M}_k := \{A \in \mathcal{A} | F_A^+ = 0\}/\mathcal{G}$$

,this is called ***the moduli space of anti-self-dual instantons****. And here k is the topology number defined by A, since for S_{YM} to be finite, F_A must be 0 in infinite, which means $A_\mu = i\Omega\partial_\mu\Omega^{-1}$ at infinite,and Ω is a map from S^3 to G.Since $\pi_3(SU(n)) = \mathbb{Z}$ for $n \geq 2$, we can classify A by this number k.*

This is a mathematical object which is used to construct topological invariant on 4D manifold,but it has physics origin: As in pure Yang-Mills theory,

$$S_{YM} = \frac{1}{2g^2}\int d^4x \; tr(F_{\mu\nu}F^{\mu\nu})$$

since $tr(F_{\mu\nu}F^{\mu\nu}) = tr({}^\star F_{\mu\nu} \, {}^\star F^{\mu\nu})$ we have:

$$S_{\text{YM}} = \frac{1}{4g^2}\int d^4x \; \text{tr}(F_{\mu\nu} \mp {}^\star F_{\mu\nu})^2 \pm \frac{1}{2g^2}\int d^4x \; \text{tr}(F_{\mu\nu}{}^\star F^{\mu\nu}),$$

and we notice that the second term is $\theta - term$,which gives a k.So to make this action minimum we have to take $F_{\mu\nu} = -{}^\star F_{\mu\nu}$, that's the ASD.

Definition 3.3 (Donaldson map). *For bundle $\mathcal{P} : (P \times \mathcal{A})/\mathcal{G} \to X \times \mathcal{A}/\mathcal{G}$,we take its first poincare class $p_1(\mathcal{P}) \in H^4(X \times \mathcal{A}/\mathcal{G}, \mathbb{Z})$, and take slant product with a r-homology cycle Σ, we get $p_1(\mathcal{P})/\Sigma \in H^{4-r}(\mathcal{A}/\mathcal{G}, \mathbb{Z})$ and restrict this to cohomology to $\mathcal{M}_k$,this map $\mu_k : H_r(X, \mathbb{Z}) \to H^{4-r}(\mathcal{M}_k, \mathbb{Z})$ is called Donaldson map.*

Definition 3.4 (Donaldson invariants). $\{d_k(\Sigma_1, ..., \Sigma_n) := \int_{[M_k]} \mu_k(\Sigma_1) \cup ... \cup \mu_k(\Sigma_n)\}$ *is called Donaldson invariants*

3.1 Twist of pure N=2 massless theory

In our setting, the theory I state in 3.2 now has concrete components:

$$A \in \mathcal{A} \quad \lambda \in \Gamma(X, ad\ P \otimes S_+ \otimes C^2) \quad \phi \in \Gamma(X, ad\ P)$$

To get a topological field theory,we need Q cohomology, so we choose A-twist:

$$Q = Q_1^1 + \epsilon_{\dot{\alpha}i}\bar{Q}^{\dot{\alpha}i} \quad Q_\mu = \sigma_\mu^{\alpha\dot{\alpha}} \left(Q_\alpha^1 \epsilon_{\dot{\alpha}1} + \bar{Q}_{\dot{\alpha}2}\right)$$

you can check this satisfying property (2.1)

Under this twist, we mix $SU(2)_R$ and $SU(2)_-$ diagonally, so we have to replace C^2 by S_+. The result is $\lambda, \bar{\lambda}$ *divieded to* ψ, η, χ where $\psi \in \Omega^1(X, ad\ P), \eta \in \Omega^0(X, ad\ P), \chi \in \Omega^2_+(X, ad\ P)$.

What's more, under this twist, we have the commutative relation:

$$Q^2 = 0 \quad [Q, Q_\mu] = iP_\mu$$

This is the key point, since to preserve symmetry, every observable in the twisted theory should satisfy Q-cohomology. So we have:

$$\begin{aligned}\partial_{\mu,x_2} \langle O(x_1)O(x_2)...\rangle &= i \langle O(x_1)P_\mu(O(x_2))...\rangle = \langle O(x_1)[Q, Q_\mu]O(x_2)...\rangle \\ &= \langle Q(...)\rangle - \langle O(x_2)Q(...)\rangle = [0] \in H^\bullet(Obs)\end{aligned}$$

The correlation functions of physical observable are topological invariants. And since $\delta\phi = 0$ in the twisted transformation, we define:

$$W_0(P) = \frac{1}{2}\operatorname{Tr}\phi^2(P)$$
$$0 = i\{Q, W_0\}, \quad dW_0 = i\{Q, W_1\}$$
$$dW_1 = i\{Q, W_2\}, \quad dW_2 = i\{Q, W_3\}$$
$$dW_3 = i\{Q, W_4\}, \quad dW_4 = 0$$

This is standard descendant, and write it down concretely, we have:

$$W_0 = -\frac{1}{16\pi^2}\langle\phi,\phi\rangle, \ldots, W_2 = \frac{1}{8\pi^2}\left(\frac{1}{2}\langle\psi\wedge\psi\rangle + i\langle\phi, F_A\rangle\right), \ldots,$$
$$W_4 = \frac{1}{16\pi^2}\langle F_A \wedge F_A\rangle$$

And then pack it up:

$$\sum_{i=1}^{4} W_i = \frac{1}{8\pi}\langle\mathcal{F}\wedge\mathcal{F}\rangle \quad \mathcal{F} = F_A - \psi + i\phi$$

So $\mathcal{F}$ is the curvature of bundle $\mathcal{P}$ in Definition 3.3, so $\langle\mathcal{F}\wedge\mathcal{F}\rangle \in H^4(X \times \mathcal{A}/\mathcal{G}, \mathbb{Z})$ Just following the definition, as a result, we have:

$$d_k(\Sigma_1, \ldots, \Sigma_n) = |Z(G)| \left\langle \prod_{i=1}^{n} \int_{\Sigma_i} W_{r_i} \right\rangle$$

□

4 AG describe

As mentioned before, we use the differential geometry language to describe invariants. Also, there is the algebraic geometry version which is easier to calculate.

4.1 Module Space of sheaves

Definition 4.1. *For X smooth,projective,dimension d variety,with ample sheaf H. Its modular space of sheaves with fix rank r,and Chern Class c_i,denoted as $M_X^H(r, c_i)$*

Theorem 4.1. *The Zariski tangent space at [E]:=*$T_M|_{[E]} \cong Ext^1_X(E,E)$ﬁ$T_{M_L}|_{[E]} \cong Ext^1_X(E,E)_0$*,here* $Ext^1_X(E,E)_0$ *denote the kernel of trace map:* $tr^i : Ext^i(E,E) \cong H^i(End(E)) \to H^i(O_X)$

4.2 Hilbert scheme and generating series

Definition 4.2 (Hilbert Scheme)**.** $X^{[n]} := \{Z \subseteq X : Z \text{ is 0-dimensional with } \dim H^0(X, \mathcal{O}_Z) = n\}$

it should be regarded as n-point construction space with degree.

$$W_0 = -\frac{1}{16\pi^2}\langle\phi,\phi\rangle,\ldots,W_2 = \frac{1}{8\pi^2}\left(\frac{1}{2}\langle\psi\wedge\psi\rangle + i\langle\phi,F_A\rangle\right),\ldots,$$
$$W_4 = \frac{1}{16\pi^2}\langle F_A \wedge F_A\rangle$$

Now,we define generating series:
$Z_{S,E_1,\ldots,E_\ell}(q) = 1 + \sum_{n=1}^{\infty} q^n \int_{S^{[n]}} P_n$
P_n is polynomial in Chern class of $E_i^{[n]}, T_{S^{[n]}}, \mathcal{Z}$ is universal subscheme and $E_i^{[n]} = p_* q^* E_i$.

Definition 4.3 (Hilbert scheme of divisor)**.** *For a effective divisor class* $\beta \in H_{2d-2}(M)$*,we define:* $\mathrm{Hilb}_\beta(X) = \{D \subseteq X : D \text{ effective divisor such that } [D] = \beta\}$

We pass the proof of the set is a scheme,it's equivalent to prove the modular functor is representable.

Prop 4.3.1. *When* $h_1(X) = 0$*(especially K3 surface),*$Hilb_\beta(X)$ *is linear system,and we denote it as* $|\beta|$

Pf :Consider $H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \cong Pic(X) \xrightarrow{c_1} H^2(X, \mathbb{Z})$

since $h^1 = 0$, c_1 is injective ,the equivalent class of divisor is determined by its Chern class.

4.3 Seiberg–Witten invariants

Definition 4.4 (Virtual Dimension)**.** *For modular space* $M_S^H(r, L, c_2)$ *(S is projective surface),*

$$vd(r,L,c_2) := dim_C(Ext^1_S(E,E)_0) - dim_C(Ext^2_S(E,E)_0) =$$
$$2rc_2 - (r-1)\,c_1\,(L)^2 - (r^2-1)\,\chi\,(\mathcal{O}_S)$$

as $Ext^1_S(E,E)_0$ is the deformation as obstruction in $Ext^2_S(E,E)_0$

Definition 4.5 (Virtual fundamental Class)**.** *For M a* $\mathbb{C}-$*Scheme of finite type,the virtual fundamental class* $[M]^{vir} \in H_{2vd}(M,\mathbb{Z})$ *is well-defined ,if there exists a perfect obstruction theory over M*

Prop 4.5.1. *For projective surface S, fix $\beta \in H_2(S, \mathbb{Z})$, the Hilbert space $Hilb_\beta(S) := |\beta|$ have perfect obstruction theory. And $[|\beta|]^{vir} \neq 0$ only when $vd(|\beta|) = 0$*

Definition 4.6 (Seiberg–Witten invariants)**.** $SW(\beta) := \int_{[|\beta|]^{vir}} 1$

And finally, I will end with the formula to calculate the Donaldson Invariant using Seiberg-Witten Invariant without proof.([2])

$$\Phi(p, \Sigma) = 2^{1+\frac{1}{4}(7\chi+11\sigma)} \left(e^{\frac{1}{2}\Sigma\cdot\Sigma+2p} \sum_x \mathrm{SW}(x)\, e^{\Sigma\cdot x} \right. \\ \left. + i^{\Delta}\, e^{-\frac{1}{2}\Sigma\cdot\Sigma-2p} \sum_x \mathrm{SW}(x)\, e^{-i\Sigma\cdot x} \right)$$

References

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[3] E Witten. Topological quantum field theory. *Commun.Math. Phys*, 1988.