Notes on Seiberg–Witten Theory

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1 Introduction

It's a note for Vafa-Witten theory I studied in Amherst .

2 Cohomological Topology Field Theory

The basic construction and first example can refer[1]

2.1 Basic structure

Definition 2.1 (CTQFT)**.** *Cohomological Topology Field Theory is TQFT(Atiyah definite), with the generator of algebra* $\{P_\mu, Q, Q_\mu\}$*,satisfy*

$$Q^2 = [Q_\mu, Q_\nu] = [Q_\mu, P_\nu] = [Q, P_\nu] = 0, \quad [Q, Q_\mu] = iP_\mu. \quad (2.1)$$

and observable are Q-cohomology.

Here,Q usually comes from twist of Supersymmetry charge.

Definition 2.2. $Op_{\delta,x}$ *:=all operators supported in* $Ball_\delta(x)$

Prop 2.2.1. *Since the theory is topological,* $Op_{\delta,x}$ *is independent on* δ *and we denoted its cohomology class as* $\mathcal{A}_x$*.*

What's more for any operator supported on B_δ,we can use
$Operator \Rightarrow State$ and $Operator \Leftrightarrow State$ (details in [3]) to regard it as a local operator on x.

2.2 Topological Algebra(First product)

Definition 2.3. *The first product is a family of embedding defined on $C^2(M)$ given by:*

$$\begin{aligned} *_{x1,x2} : A \otimes A \to Op_1 \\ O_1, O_2 \mapsto O_1(x_1)O_2(x_2). \end{aligned}$$

The first order product in fact have the factorization structure:

Prop 2.3.1. *$*_{x1,x2}$ is equal in the same connected component of $C_2(M)$, in R^d case,since $C_2(R^d)$ is trivial when $d \geq 2$,we can define $* := *_{x1,x2}$*

$$\begin{aligned} Pf : O_1 *_{x1,x2+a} O_2 - O_1 *_{x1,x2} O_2 &= O_1(x^1)O_2(x^2+a) - O_1(x^1)O_2(x^2) \\ &= a \cdot O_1(x^1)\partial_\mu O_2(x^2) = a \cdot O_1(x^1)QQ_\mu O_2(x^2) \\ &= a \cdot (-1)^{F_1} Q(a \cdot O_1(x^1)Q_\mu O_2(x^2)) = 0 \end{aligned}$$

Prop 2.3.2. ** is associative*

Prop 2.3.3. *Obvious,* is graded-commutative.*

2.3 Secondary Product

Definition 2.4 (Descendant). $\mathcal{O}^{(k)}(x) = \frac{1}{k!}\left(Q_{\mu_1} \cdots Q_{\mu_k}\mathcal{O}\right)(x)\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}$
$\mathcal{O}^* = \sum_{k=0}^{\infty} \mathcal{O}^{(k)}$

Prop 2.4.1 (Descendant Equation). $Q\left(\mathcal{O}^{(k)}(x)\right) = d\mathcal{O}^{(k-1)}(x), \quad Q\mathcal{O}^* = d\mathcal{O}^*$

$$Pf : Q\mathcal{O}^{(k)}(x) = Q(\frac{1}{k!}Q_{\mu 1}...) = \frac{1}{k!} \cdot iP_{\mu 1}(...)dx^{\mu 1} = dO^{(k-1)}$$

Definition 2.5 (Homology Operator). *Denote $\mathcal{O}(\gamma) := \int_\gamma \mathcal{O}^*$,here $[\gamma]$ is homology class.*

Prop 2.5.1. *$O(\gamma)$ is operator supported on γ, and as Prop 2.2.1 $\mathcal{O}(\gamma) \in A$*

Definition 2.6 (Secondary Product). *The first product is a family of map defined on $C^1(M) = M$ given by:*

$$\begin{aligned} \{,\}_x : A \otimes A \to A_x \\ O_1, O_2 \mapsto \mathcal{O}_1\left(S_x^{d-1}\right)\mathcal{O}_2(x) \end{aligned}$$

Prop 2.6.1. *The same as the proof of* $*_{x1,x2}$ *is invariant under deformation,* $\{,\}_x$ *is invariant under deformation,so we can define* $\{,\}_x := \{,\}$ *in* R^d *so it's a product in* A.

Now we define a new product $\star$ defined on $C^n(M)$,and it captures the structure of Poisson structure of secondary product.

Definition 2.7 (Descendant on configuration space)**.** $(\mathcal{O}_1 \boxtimes \mathcal{O}_2 ... \boxtimes \mathcal{O}_n)^* = \mathcal{O}_1^* \wedge \sigma^{F_1}\mathcal{O}_2^* ... \wedge \sigma^{F_1 ... + F_{n-1}}\mathcal{O}_n^* .$, *where* σ *acts as* $(-1)^k$ *on the degree* k *part.*

Prop 2.7.1 (Descendant Equation)**.** $\mathcal{Q}\left(\mathcal{O}_1 ... \boxtimes \mathcal{O}_n\right)^* = d\left(\mathcal{O}_1 ... \boxtimes \mathcal{O}_n\right)^*.$

In particular,$\left(\mathcal{O}_1 \boxtimes \mathcal{O}_2\right)^{(k)}(x_1, x_2) = \sum_{n=0}^{k} (-1)^{(k-n)F_1} \mathcal{O}_1^{(n)}(x_1) \wedge \mathcal{O}_2^{(k-n)}(x_2)$
For $P \in H_\bullet(C^n(M), \mathbb{Z})$,we define

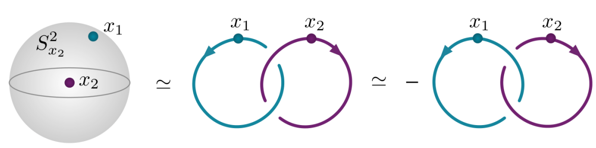
$$\star_P : A^{\otimes n} \to A$$
$$(O_1, ..., O_n) \mapsto \left(\mathcal{O}_1 \boxtimes \mathcal{O}_2 ... \boxtimes \mathcal{O}_n\right)(P) := \int_P \left(\mathcal{O}_1 \boxtimes \mathcal{O}_2 ... \boxtimes \mathcal{O}_n\right)^*$$

Then we prove some property of secondary product: .6.2

Prop 2.8.1 (Commutative property)**.** $\{,\}$ *is graded-commutative*

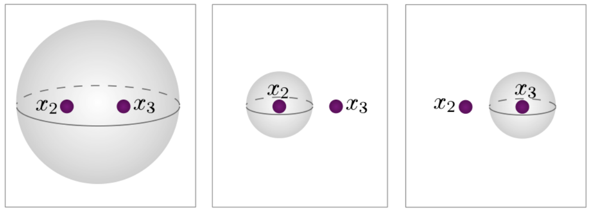
$$\begin{aligned} Pf : \{O_1, O_2\}_x &= O_1(S_x^{d-1})O_2(x) = (O_1 \boxtimes O_2)(S_x^{d-1}, \{x\}) = O_1 \star_{[S_x^{d-1} \times \{x\}]} O_2 \\ &\text{in } R^d \text{ case, } C^2(M) \cong S^{d-1}, \\ &[S_x^{d-1} \times \{x\}] \textit{ and } [\{x\} \times S_x^{d-1}] \text{ related by antipodal map} \\ &\therefore O_1 \star_{[S_x^{d-1} \times \{x\}]} O_2 = (-1)^d O_1 \star_{[\{x\} \times S_x^{d-1}]} O_2 = (-1)^d O_1(x) O_2(S^{d-1}) \\ &= (-1)^{F_1 F_2 + d}\{O_2, O_1\} \end{aligned}$$

In particular ,in d=3 case $\{,\}$ can be used to compute linking number. That is for Γ_1, Γ_2 two circles in R^3 , we have $[O_1 \boxtimes O_2]([\Gamma_1, \Gamma_2]) = l \ \cdot (O_1 \boxtimes O_2)(S_x^{d-1}, \{x\}) = l \ \cdot \{O_1, O_2\}$



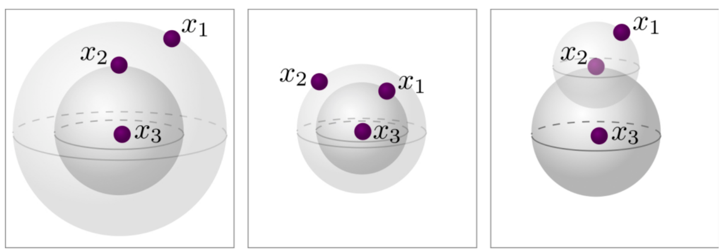
Prop 2.8.2 (Derivation property)**.** $\{[\mathcal{O}_1],\ [\mathcal{O}_2] * [\mathcal{O}_3]\} = \{[\mathcal{O}_1],\ [\mathcal{O}_2]\} * \{[\mathcal{O}_1],\ [\mathcal{O}_3]\} + (-1)^{(F_1+d-1)F_2}[\mathcal{O}_2] * \{[\mathcal{O}_1],\ [\mathcal{O}_3]\}$

The proof is directly,$(-1)^{F_1+d-1}$ from the order (d-1) form.And notice that two parts of the equation correspond to these graphs,and $[\Gamma_1] = [\Gamma_2] + [\Gamma_3]$



Prop 2.8.3 (Jacobi identity)**.**

$$\{[\![\mathcal{O}_1]\!], \{[\![\mathcal{O}_2]\!], [\![\mathcal{O}_3]\!]\}\} - (-1)^{(F_1+d-1)(F_2+d-1)}\{[\![\mathcal{O}_2]\!], \{[\![\mathcal{O}_1]\!], [\![\mathcal{O}_3]\!]\}\}$$
$$= (-1)^{(d-1)(F_1+d-1)}\{\{[\![\mathcal{O}_1]\!], [\![\mathcal{O}_2]\!]\}, [\![\mathcal{O}_3]\!]\}$$



2.4 Example:RW-twist of 3d N=4

Now,I give an example to illustrate what the component of A looks like and what the $\{,\}$ exactly is.

2.4.1 3d N=4 Super Algebra

Before this section,I highly commend reader skip to 3.1 part to study representation of Super-Poincare algebra.

Definition 2.9. *3d N=4 Super Algebra is given by* $\{P_\mu, Q^{a,b}_\alpha\}$*,satisfy:*

$$[Q^{ab}_\alpha, Q^{cd}_\beta] = \epsilon^{ac}\epsilon^{bd}\sigma^\mu_{\alpha\beta}P_\mu$$

the index α *means* Q *is Majorana spinor,index* a *is the spinor representation of* $SU(2)_H$ *(Higgs rotation),index* b *is spinor representation of* $SU(2)_C$ *(Coulomb rotation)*

Without proof,we claim 3d N=4 have a hyper-multiplet representation,consist of 4ϕ+2ψ ,we denote them $\phi^{a1},\phi^{a2},\psi^{b1}_\alpha,\psi^{b2}_\alpha$, the index "a" in ϕ^{a1} is the same as in Q, means it's $SU(2)_H$ representation,the indexes in ψ^b is similar.

2.4.2 Twist

Now,we will play twist on this super theory to make it be a CTQFT.We define:

$$Q := \delta^\alpha_b Q^{1b}_\alpha = Q^{11}_1 + Q^{12}_2 \quad Q_\mu := -\frac{i}{2}(\sigma^\mu)^\alpha_b \cdot Q^{2b}_\alpha$$

The new operator satisfy definition 2.1 ,so this is a twist CTQFT.

And now,we consider the pure hypermultiplets Lagrangian,as I have mentioned it's consist of $\phi^{aA},\psi^{bA}_\alpha \quad A\in[1,2N_f]$,(N_f) is the number of hypermultiplets.

But now, what we did in twist is mix $SU(2)_C\times SU(2)_E$ diagonally to new Euclidean rotation group $SU(2)'_E$. So in twist theory, we don't have to consider ψ but the twisted one,so we define:

$$\eta_A := -\delta^\alpha_b\psi^{bA}_\alpha \quad \chi^A_\mu := \frac{i}{2}(\sigma^\mu)^\alpha_b\psi^{bA}_\alpha$$

is the $1\otimes 3$ representation of $SU(2)'$.

3 Physics describe

3.1 Representation of Super-Pincare algebra

Poincare algebra is generated by $\{P_\mu, Q^A_a, \bar{Q}_{\dot{a}A}\}$, satisfied:

$$\left\{Q^A_\alpha, \overline{Q}_{\beta B}\right\} = 2\delta^A_B\sigma^\mu_{\alpha\beta}\mathbf{P}_\mu, \quad \left\{Q^A_\alpha, Q^B_\beta\right\} = \varepsilon_{\alpha\beta}Z^{AB}$$

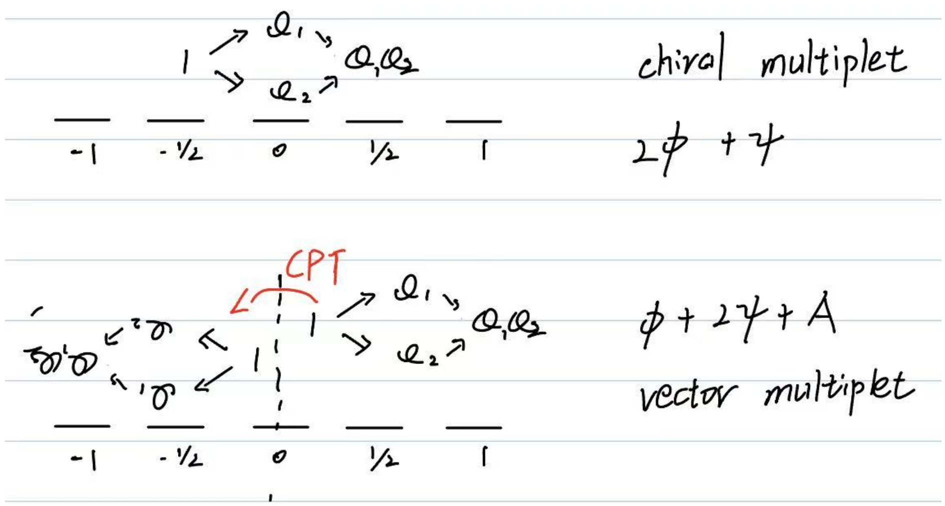
We claim it's the unique non-trivial extension of Pincare algebra.

3.1.1 N=1 massive

In the rest frame of a particle,$p_\mu=(m,0,0,0)$ so

$$\{Q_\alpha,\overline{Q}_\beta\} = 2m\delta_{\alpha\beta}, \{Q,Q\} = \{\overline{Q},\overline{Q}\} = 0$$

so we regard Q_a as creation operator and $\bar{Q}_{\dot{a}}$ as annihilation operator,begin with a state $|j\rangle$,we have $Q_1|j\rangle, Q_2|j\rangle, Q_1Q_2|j\rangle$,without gravity the superhelicity $J\leq 1$,at a result,we only have two representations called chiral multiplet and vector multiplet.



And these multiplets correspond to chiral and vector field.

$$\Phi(x,\theta,\bar{\theta}) = A(x) + \theta^{\alpha}\psi_{\alpha}(x) + \theta^2 F(x) + i\theta\sigma^a\bar{\theta}\partial_a A(x) + \tfrac{i}{2}\theta^2\bar{\theta}\tilde{\sigma}^a\partial_a\psi(x) + \tfrac{1}{4}\theta^2\bar{\theta}^2\Box A(x) \quad (1)$$

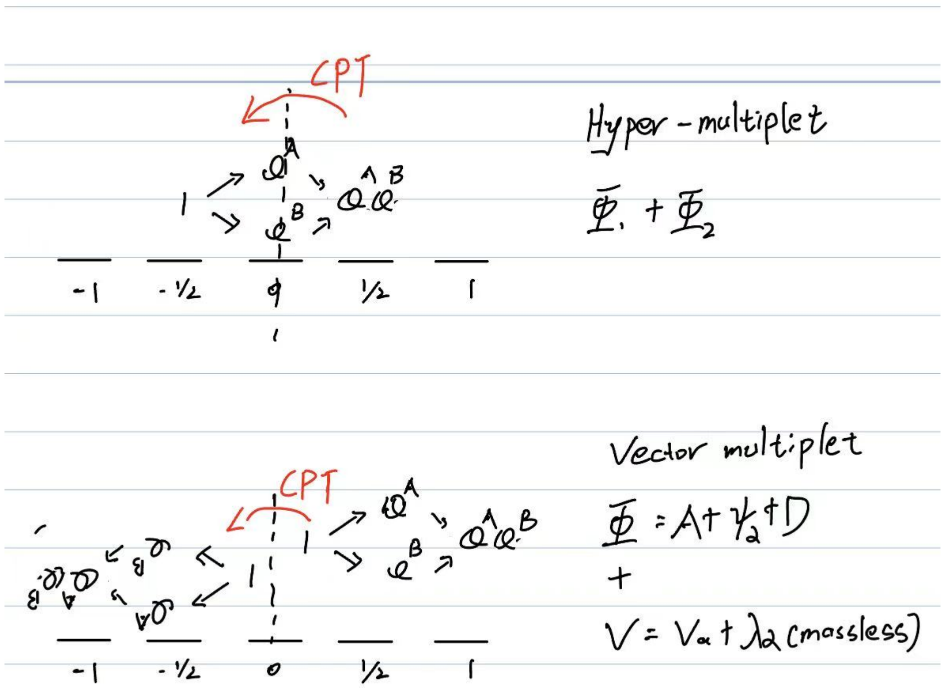
$$V = \theta\sigma^a\bar{\theta}V_a + \bar{\theta}^2\theta^{\alpha}\lambda_{\alpha} + \theta^2\bar{\theta}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} + \theta^2\bar{\theta}^2\mathcal{D} \quad (2)$$

3.1.2 N=1 massless

Also to the rest frame, $p_{\mu} = (E,0,0,E)$ .Now we only have one creation operator,as the same calculation,the representation component is the same as massive.

3.1.3 N=2 massless

Have "Hypermultiplet" and "Vector multiplet"



3.2 Pure N=2 massless Vector-mutiplet Theory

Since in this case,the Vector-mutiplet can be written as $V \oplus \Phi$. We consider N=1 massless theory with both these part,it is N=1 Super-Yang-Mills model:(this part following Witten's lecture https://youtu.be/9qZSqkn8-Qo, and for details about N=1 case,reference is[2][3.5.1])

$$S = \frac{1}{e^2}\int d^4x d\theta d\bar{\theta}\ tr(\bar{\Phi}e^{2V}\Phi) + \frac{1}{e^2}\int d^4x d\theta tr(W^\alpha W_\alpha) \quad (3)$$

$$Since \int dx d\theta \Phi = -\frac{1}{4}\int dx D^2\Phi|_{\theta=0,\bar{\theta}=0} \quad (4)$$

,what we need to do is using formulas (1)(2)(4) to extends (3) to euclidean case,and the result is :

$$S = \frac{1}{c^2}\int d^4x\ kinetic\ term\ of\ (A, \lambda, \bar{\lambda}, \phi) + \mathrm{Tr}\left[\phi, \phi^\dagger\right]^2 + (Yukawa\ potential)$$

The formula (3) have the same symmetry transformation as N=2 case after adding Yukawa potential,so it's the Lagrangian we are looking for.

Now, as you seen the potential is :

$V(\phi) = Tr[\phi, \bar{\phi}]$,which means it's diagonal ,so we set $\phi = a\sigma^3$. Since vacua under $SU(2)$ are equivalent, we use the parameter $u := \frac{1}{2}a^2 =< \phi^2 >$ to parameterize modular space of vacua, as u is double covered by a.

3.3 Relation to QCD

When $a \neq 0$, $SU(2)$ symmetry is broken to $U(1)$ and generate massive boson $W^{\pm}$ from two color indexes of A_μ^a.So as we consider low-energy effective theory,the two massive bosons won't make effort.The residue field is A_μ, ψ with U(1) symmetry,which exactly is QCD.

4 AG describe

4.1 Module Space of sheaves

Definition 4.1. *For X smooth,projective,dimension d variety,with ample sheaf H. Its modular space of sheaves with fix rank r,and Chern Class c_i,denoted as $M_X^H(r, c_i)$*

Theorem 4.1. *The Zariski tangent space at [E]:=$T_M|_{[E]} \cong Ext_X^1(E,E)fiT_{M_L}|_{[E]} \cong Ext_X^1(E,E)_0$,here $Ext_X^1(E,E)_0$ denote the kernel of trace map: $tr^i : Ext^i(E,E) \cong H^i(End(E)) \to H^i(O_X)$*

4.2 Hilbert scheme and generating series

Definition 4.2 (Hilbert Scheme)**.** $X^{[n]} := \{Z \subseteq X : Z \textit{ is 0-dimensional with } \dim H^0(X, \mathcal{O}_Z) = n\}$

it should be regarded as n-point construction space with degree.

$$\begin{tikzcd} & \mathcal{Z} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ X & & X^{[n]} \end{tikzcd}$$

Now,we define generating series:
$Z_{S,E_1,\ldots,E_\ell}(q) = 1 + \sum_{n=1}^{\infty} q^n \int_{S^{[n]}} P_n$
P_n is polynomial in Chern class of $E_i^{[n]}, T_{S^{[n]}}$,$\mathcal{Z}$ is universal subscheme and $E_i^{[n]} = p_* q^* E_i$.

Definition 4.3 (Hilbert scheme of divisor)**.** *For a effective divisor class $\beta \in H_{2d-2}(M)$,we define:* $\mathrm{Hilb}_\beta(X) = \{D \subseteq X : D \textit{ effective divisor such that } [D] = \beta\}$

We pass the proof of the set is a scheme,it's equivalent to prove the modular functor is representable.

Prop 4.3.1. *When* $h_1(X) = 0$*(especially K3 surface),*$Hilb_\beta(X)$ *is linear system,and we denote it as* $|\beta|$

Pf :*Consider* $H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \cong Pic(X) \xrightarrow{c_1} H^2(X, \mathbb{Z})$

since $h^1 = 0,\ c_1$ is injective ,the equivalent class of divisor is determined by its Chern class.

4.3 Seiberg–Witten invariants

Definition 4.4 (Virtual Dimension)**.** *For modular space* $M_S^H(r, L, c_2)$ *(S is projective surface),*

$$vd(r, L, c_2) := dim_C(Ext_S^1(E, E)_0) - dim_C(Ext_S^2(E, E)_0) =$$
$$2rc_2 - (r-1)\, c_1 (L)^2 - (r^2 - 1)\, \chi\, (\mathcal{O}_S)$$

as $Ext_S^1(E, E)_0$ is the deformation as obstruction in $Ext_S^2(E, E)_0$

Definition 4.5 (Virtual fundamental Class)**.** *For M a* $\mathbb{C} - Scheme$ *of finite type,the virtual fundamental class* $[M]^{vir} \in H_{2vd}(M, \mathbb{Z})$ *is well-defined ,if there exists a perfect obstruction theory over M*

Prop 4.5.1. *For projective surface S, fix* $\beta \in H_2(S, \mathbb{Z})$*, the Hilbert space* $Hilb_\beta(S) := |\beta|$ *have perfect obstruction theory. And* $[|\beta|]^{vir} \neq 0$ *only when* $vd(|\beta|) = 0$

Definition 4.6 (Seiberg–Witten invariants)**.** $SW(\beta) := \int_{[|\beta|]^{vir}} 1$

References

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[2] Sergio M Kuzenko. Ideas and methods of supersymmetry and supergravity or a walk through superspace, 1998.

[3] David Simmons-Duffin. Tasi lectures on the conformal bootstrap, 2016.