Quantization of $\mathcal{N} = 1$ SYM theory

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Abstract

We identify the D-module structure on N=1 Superspace and classify the corresponding SUSY-invariant local functionals. Furthermore, we construct the Batalin–Vilkovisky complex of N=1 Super–Yang–Mills theory and carry out its quantization in the sense of homotopy renormalization. However, this note remains preliminary, and I'm still polishing it.

Contents

1 Introduction

In [1], Costello showed that the twisted N=1 gauge theory admits unique quantization on any Calabi-Yau manifold and analyzed its algebraic structure. With the same spirit, using the superfields construction and BV cohomology, we showed that the pure SYM theory has no quantization obstruction; thus, it also admits a unique quantization. Further, we try to calculate some observable structures. In Section 2, we identify the

D-module structure on N=1 Superspace and classify the corresponding SUSY-invariant local functionals into D-term and F-term. In Section 3, we give an introduction to SYM construction in BV formalism and review its twist. Section 4 is the quantization and calculation of the local BV cohomology, where we use the classification in section 2 and the Perturbative Non-renormalization property of N = 1 SYM.

2 SUSY D-module

2.1 SUSY algebra and Superfield

We work on **N=1 Superspace** $M = R^{4|4}$ consists of fourbonsonic coordinates x^μ and four complex Grassmann coordinates θ^α and $\bar{\theta}^{\dot{\alpha}}$. By analogy with the Poincaré group on R^4, the **SuperPoincaré group** is defined to be:

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu, \quad [M_{\mu\nu}, Q_\alpha] = i(\sigma_{\mu\nu})_\alpha^\beta Q_\beta, \quad [P_\mu, Q_\alpha] = 0,$$

$$\sigma_{\mu\nu} := -\frac{1}{4}(\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu), \qquad \bar{\sigma}^\mu := (1, -\sigma_1, -\sigma_2, -\sigma_3).$$

A **Suprefield** $Y(x^\mu, \theta, \bar{\theta})$ is a $\mathbb{C}$-valued function on the M. In Talor-extension, it decomposed to component fields:

$$\begin{aligned} Y(x^\mu, \theta, \bar{\theta}) = {} & \phi(x) + \theta^\alpha \psi_\alpha(x) + \bar{\theta}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}(x) + \theta^2 M(x) + \bar{\theta}^2 N(x) \\ & + \theta^\alpha \bar{\theta}^{\dot{\alpha}} V_{\alpha\dot{\alpha}}(x) + +\theta^2 \bar{\theta}_{\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}}(x) + \bar{\theta}^2 \theta^\alpha \rho_\alpha(x) + \theta^2 \bar{\theta}^2 D(x). \end{aligned}$$

It follows from the commutation relation:

$$[P_\mu, Y] = -i\partial Y, \qquad [Q_\alpha, Y] = \left(-i\frac{\partial}{\partial \theta^\alpha} - \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_\mu\right),$$

the differentials on M should be:

$$\begin{aligned} \mathcal{P}_\mu &= -i\partial_\mu, \\ \mathcal{Q}_\alpha &= -i\partial_\alpha - \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_\mu, \\ \bar{\mathcal{Q}}_{\dot{\alpha}} &= +i\bar{\partial}_{\dot{\alpha}} + \theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu. \end{aligned}$$

The N=1 SuperPoincaré group have two fundamental representation when it acts on Superfield:

$$\begin{cases} \Omega(x^\mu, \theta, \bar{\theta}) & \textbf{Chiral fields} & \bar{D}_{\dot{\alpha}} \Omega(x^\mu, \theta, \bar{\theta}) = 0, \\ \bar{\Omega}(x^\mu, \theta, \bar{\theta}) & \textbf{Anti-chiral fields} & D_\alpha \bar{\Omega}(x^\mu, \theta, \bar{\theta}) = 0, \\ V(x^\mu, \theta, \bar{\theta}) & \textbf{Vector fields} & V = \bar{V}. \end{cases}$$

where

$$D_\alpha := \partial_\alpha + i\sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_\mu, \qquad \bar{D}_{\dot{\alpha}} := -\bar{\partial}_{\dot{\alpha}} - i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu.$$

Also in Talor-extension:

$$\Omega(x,\theta,\bar{\theta}) = \phi(x) + \theta^{\alpha}\psi_{\alpha}(x) + \theta^2 F(x) + \mathrm{i}\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}\phi(x) - \frac{\mathrm{i}}{2}\theta^2\partial_{\mu}\psi(x)\sigma^{\mu}\bar{\theta} + \frac{1}{4}\theta^2\bar{\theta}^2\Box\phi(x) \quad (1)$$

$$\begin{aligned}\mathcal{V}(x,\theta,\bar{\theta}) &= C(x) + \theta\chi(x) + \bar{\theta}\bar{\chi}(x) + i\theta^2 M(x) - i\bar{\theta}^2 M^{\dagger}(x) + \theta\sigma^{\mu}\bar{\theta}A_{\mu}(x) \\ &+ \theta^2\bar{\theta}\left(\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^{\mu}\partial_{\mu}\chi(x)\right) + \bar{\theta}^2\theta\left(\lambda(x) + \frac{i}{2}\sigma^{\mu}\partial_{\mu}\bar{\chi}(x)\right) + \frac{1}{2}\theta^2\bar{\theta}^2\left(D(x) - \frac{1}{2}\Box C(x)\right) \quad (2) \\ &(C(x), D(x), A^{\mu}(x)\ are\ real\ fields)\end{aligned}$$

2.2 SUSY D-module

This part generalize algebraic D-module discovered in [2] to Superspace.

Recall that the **ring of differential operators** on M, $\mathcal{D}_M$ is the ring of operators acting on the coordinate ring $\mathcal{O}_M$ as differential. Thus in our case:

$$\mathcal{O}_M = C^{\infty}(R^4) \otimes \Lambda[\theta^{\alpha}, \bar{\theta}^{\dot{\alpha}}] \qquad \mathcal{D}_M = \mathcal{O}_M[\partial_{\mu}, \partial_{\theta^{\alpha}}, \partial_{\bar{\theta}^{\dot{\alpha}}}] \cong \mathcal{O}_M[\partial_{\mu}, D_{\alpha}, \bar{D}_{\dot{\alpha}}] \text{ as } \mathcal{O}_M \text{ module.}$$

We define the category of **SUSY $\mathcal{D}$-modules** to be the category of graded modules over the ring $\mathcal{D}_M$, denoted by $\mathcal{M}od(\mathcal{D}_M)$.

Definition 2.1. *Let $F \in \mathcal{M}od(\mathcal{D}_M)$, we consider following bi-complex which we called* ***Super De-Rham complex*** $\mho^*(F)$*:*

$$\begin{tikzcd}
\cdots \arrow[r, "d_{DR}"] & \Omega^3_{\mathbb{R}^4} \otimes_{C^{\infty}(R^4)} \mathbb{C}[\theta^{\alpha}, \bar{\theta}^{\dot{\alpha}}] \otimes_{\mathcal{O}_M} F \arrow[r, "d_{DR}"] & \Omega^4_{\mathbb{R}^4} \otimes_{C^{\infty}(R^4)} \mathbb{C}[\theta^{\alpha}, \bar{\theta}^{\dot{\alpha}}] \otimes_{\mathcal{O}_M} F \\
\cdots \arrow[u, "\Phi"] \arrow[r, "d_{DR}"] & \Omega^3_{\mathbb{R}^4} \otimes_{C^{\infty}(R^4)} (\oplus_j(\mathbb{C}[\theta^i](i \neq j))\epsilon^j) \otimes_{\mathcal{O}_M} F \arrow[u, "\Phi"] \arrow[r, "d_{DR}"] & \Omega^4_{\mathbb{R}^4} \otimes_{C^{\infty}(R^4)} (\oplus_j(\mathbb{C}[\theta^i](i \neq j))\epsilon^j) \otimes_{\mathcal{O}_M} F \arrow[u, "\Phi"] \\
\cdots \arrow[u, "\Phi"] \arrow[r, "d_{DR}"] & \cdots \arrow[u, "\Phi"] \arrow[r, "d_{DR}"] & \Omega^4_{\mathbb{R}^4} \otimes_{C^{\infty}(R^4)} (\oplus_j(\mathbb{C}[\theta^i](i \neq j,k))\epsilon^j\epsilon^k) \otimes_{\mathcal{O}_M} F \arrow[u, "\Phi"] \\
\cdots \arrow[u, "\Phi"] \arrow[r, "d_{DR}"] & \cdots \arrow[u, "\Phi"] \arrow[r, "d_{DR}"] & \Omega^4_{\mathbb{R}^4} \otimes_{C^{\infty}(R^4)} (\oplus_j(\mathbb{C}[\theta^i](i \neq j,k,l))\epsilon^j\epsilon^k\epsilon^l) \otimes_{\mathcal{O}_M} F \arrow[u, "\Phi"] \\
\Omega^0_{\mathbb{R}^4} \otimes_{C^{\infty}(R^4)} \cdots \arrow[u, "\Phi"] \arrow[r, "d_{DR}"] & \cdots \arrow[u, "\Phi"] \arrow[r, "d_{DR}"] & \Omega^4_{\mathbb{R}^4} \otimes_{C^{\infty}(R^4)} \epsilon^1\epsilon^2\bar{\epsilon}^{\dot{1}}\bar{\epsilon}^{\dot{2}} \otimes_{\mathcal{O}_M} F \arrow[u, "\Phi"]
\end{tikzcd}$$

(1)θ^i goes over $\{\theta^1, \theta^2, \bar{\theta}^{\dot{1}}, \bar{\theta}^{\dot{2}}\}$, ϵ^j are corresponding formal odd variables.
(2)The horizontal differential d_{DR} is the usual De-Rham differentials: $f \otimes w \to f dx^i \otimes \partial_i w$
(3)The vertical differential Φ is odd map defined to be:

$$\Phi : \epsilon^{\alpha} \to 1 - \theta^{\alpha} \otimes_{\mathcal{O}_M} D_{\alpha},\ \bar{\epsilon}^{\dot{\alpha}} \to 1 - \bar{\theta}^{\dot{\alpha}} \otimes_{\mathcal{O}_M} \bar{D}_{\dot{\alpha}}.$$

(4)Our Super De Rham complex differs from that of Superform introduced in [3], in that we additionally impose the equivalence relations:

$$\int d^4x d\theta^1 Y \sim \int d^4x D_1 Y,$$

and similarly for the other odd variables.

Lemma 2.1. *The* ***Super de Rham functor***

$$\mho : \mathcal{M}od(\mathcal{D}_M) \longrightarrow \mathcal{CH}(\text{Sheaf}(M)),$$

extends naturally to a functor between bounded derived categories.

2.3 Classification of SUSY Local Functional

Let $\mathcal{E}$ be the Superfields space,

Definition 2.2. *(Jet bundle) The jet bundle assigned to sheaf $\mathcal{E}$ is defined as:*

$$J(\mathcal{E})(U) := D_M \otimes_{\mathcal{O}_M} \mathcal{E}(U).$$

Follows by definition, $J(\mathcal{E})$ is a $\mathcal{D}_M$-module. This also works for the completed symmetric algebra on its dual:

$$\check{J}(\mathcal{E}) \;\; := Hom(J(\mathcal{E}), \mathcal{O}_M) = D_M \otimes_{\mathbb{C}} \check{E}_0 \text{ ,where } E_0 \text{ is the trivial fiber.}$$

$$\mathcal{O}(J(\mathcal{E})) := \Pi_{n\geq 0} Sym^n \check{J}(E) \; .$$

Thus we can consider its Super De Rham complex $\mho^*(J(\mathcal{E}))$:

Lemma 2.2.

$$\mathcal{O}_{loc}(\mathcal{E}) = \mho^*(\mathcal{O}(J(\mathcal{E}))) \textit{ in derived sense.}$$

Definition of $\mathcal{O}_{loc}(\mathcal{E})$ can be found in [4] section 2.7.1, and the proof is straightforward.

We now consider the complex of SUSY-invariant local functionals $\mathcal{O}_{loc}(\mathcal{E})^{SUSY}$, defined as the SUSY-invariant subcomplex of $\mathcal{O}_{loc}(\mathcal{E})$.

In the present case, the space of fields takes the form $\mathcal{E} = \mathcal{C} \oplus V \oplus \bar{\mathcal{C}}$, where $\mathcal{C}$ denotes the chiral superfield and V the vector superfield. This structure gives rise to the following classification of SUSY-invariant local functionals:

Theorem 2.1.
Let $M_1 := (\theta^1\theta^2/\mathcal{O}_M \cdot \theta^1\theta^2)$, $M_2 := (1/\mathcal{O}_M \cdot 1)$ be two trivial $\mathcal{O}_M$-modules, $D_{N=1} := \mathbb{C}[\partial_\mu, D_\alpha, \bar{D}_{\dot{\alpha}}]$.

Then we have following classification:

$$\mathcal{O}_{loc}(\mathcal{E})^{SUSY} = \begin{cases} \mho^*(M_1 \otimes_{\mathcal{O}_M} Sym^*(\mathbb{C}[\partial_\mu] \otimes_{\mathcal{O}_M} \check{\mathcal{E}})) & \textbf{\textit{F-term}} \quad \mathcal{E} = \mathcal{C},\ \textit{Pure Chiral}; \\ \mho^*(\bar{M}_1 \otimes_{\mathcal{O}_M} Sym^*(\mathbb{C}[\partial_\mu] \otimes_{\mathcal{O}_M} \check{\mathcal{E}})) & \textbf{\textit{F-term}} \quad \mathcal{E} = \bar{\mathcal{C}},\ \textit{Pure anti-Chiral}; \\ \mho^*(M_2 \otimes_{\mathcal{O}_M} Sym^*(D_{N=1} \otimes_{\mathcal{O}_M} \check{\mathcal{E}})) & \textbf{\textit{D-term}} \quad \textit{Othercases.} \end{cases}$$

As a D-module, D_M acts on Sym^* component as left-module.

Proof. The main idea is to exploit the $\mathcal{Q}_i$-closed property and the identity $[\mathcal{Q}_i, \theta^i] = i$ to deduce $\mathcal{Q}_i$-exactness. This requires introducing an explicit action of θ_i on the module and giving a separate discussion for the cases of purely chiral and anti-chiral fields.

To illustrate the definition of the action, we consider a representative example of a D-module of the form:

$$(D_i D_j \mathcal{E}_1)(D_k \mathcal{E}_2),\ D_i \text{ goes over } \{D_\alpha, \bar{D}_{\dot{\alpha}}\}.$$

The extension to the general case of Sym^* is straightforward.

Definition 2.3. *The **right action of** $\mathcal{Q}_i$ is defined by:*

$$(D_i D_j \mathcal{E}_1)(D_k \mathcal{E}_2) \cdot \mathcal{Q}_l = (D_i D_j Q_l \mathcal{E}_1)(D_k \mathcal{E}_2) + (D_i D_j \mathcal{E}_1)(D_k Q_l \mathcal{E}_2)$$

*Similarly, we define the **right action of** θ^l by:*

$$(D_i D_j \mathcal{E}_1)(D_k \mathcal{E}_2) \cdot \theta^l = (D_i D_j\ \theta^l \mathcal{E}_1)(D_k \mathcal{E}_2) + (D_i D_j \mathcal{E}_1)(D_k\ \theta^l \mathcal{E}_2).$$

Remark. The right action of θ^l is well defined when the usual product $\mathcal{E}$ consists of vector fields. However, if $\mathcal{E}$ is chiral, the induced quotient D-module structure

$$D^{\mathrm{Ch}}_{N=1} := D_{N=1}/D_{N=1}\,\bar{D}_{\dot{\alpha}},$$

is not compatible with this action, since $[\bar{\theta}^{\dot{\alpha}}, \bar{D}_{\dot{\alpha}}] \neq 0$. In this case, we instead define the action of θ^α on $\mathcal{C}$ to be the ordinary $\mathcal{O}_M$-module product, while the action of $\bar{\theta}^{\dot{\alpha}}$ is taken to be trivial. The anti-chiral case is defined in a completely analogous way.

Lemma 2.3. *When $\mathcal{E}$ is not purely chiral or anti-chiral, any $\mathcal{Q}_i$-closed element in $\mathcal{O}_{loc}(\mathcal{E})$ is $\mathcal{Q}_i$-exact.*

Proof. Since we have:

$$S \cdot \{\mathcal{Q}_l, \theta^l\} = iS, \quad S \cdot \mathcal{Q}_l = 0,$$

it follows that:

$$S = k(S \cdot \theta^1 \theta^2 \bar{\theta}^{\dot{1}} \bar{\theta}^{\dot{2}}) \cdot \mathcal{Q}_1 \mathcal{Q}_2 \bar{\mathcal{Q}}_{\dot{1}} \bar{\mathcal{Q}}_{\dot{2}}$$

□

This lemma shows that the $\mathcal{Q}_i$-invariant local functionals represented by:

$$\int_{\mathbb{R}^4} d^4x\, C^{\infty}(\mathbb{R}^4) \otimes_{\mathbb{R}[\partial_\mu]} D_1 D_2 \bar{D}_{\dot{1}} \bar{D}_{\dot{2}} \cdot Sym^*(D_{N=1} \otimes_{\mathcal{O}_M} \check{\mathcal{E}}).$$

Then use the translation-invariant property, this term is equivalent to:

$$\mho^*(M_2 \otimes_{\mathcal{O}_M} Sym^*(D_{N=1} \otimes_{\mathcal{O}_M} \check{\mathcal{E}})),$$

this finishes the proof for D-term.

Then we focus on the chiral case, the anti-chiral case is completely analogous. Since in chiral case, the ring of differential operators D_M is replaced by $\mathcal{D}_M^{Ch} := \mathcal{O}_M \otimes_{\mathbb{C}} D_{N=1}^{Ch}$, we only have to consider resolution in $\mathcal{D}_M^{Ch}$-module category.

Lemma 2.4. *Let $\mathcal{D}_c := \mathbb{C}[\theta^\alpha] \otimes_{\mathbb{C}} Sym^*(D_{N=1}^{Ch} \otimes_{\mathcal{O}_M} \check{C})$ be another D-module, we use it to construct a $\mathcal{D}_M^{Ch}$-module resolution of $Sym^*(D_M^{Ch} \otimes_{\mathcal{O}_M} \check{C})$:*

$$\mathcal{D}_c u^1 u^2 \xhookrightarrow{\Phi} \mathcal{D}_c \otimes \left(\mathbb{C}[\bar{\theta}^1]u^2 \oplus \mathbb{C}[\bar{\theta}^2]u^1\right) \xrightarrow{\Phi} \mathcal{D}_c \otimes \mathbb{C}[\bar{\theta}^{\dot{\alpha}}] \cong Sym^*(D_M^{Ch} \otimes_{\mathcal{O}_M} \check{C})$$

Here u_i are formal odd variables, and the differential Φ is defined similarly:

$$\Phi : u^\alpha \rightarrow 1 - \bar{\theta}^{\dot{\alpha}} \otimes_{\mathcal{O}_M} \bar{D}_{\dot{\alpha}}.$$

Remark: (1)As a $D_M^{Ch} - module$, $\bar{\theta}^{\dot{\alpha}}$ acts on$\mathcal{D}_c$ as $\frac{\partial}{\partial \bar{D}_{\dot{\alpha}}}$.
(2)We define $\bar{\mathcal{Q}}_{\dot{\alpha}}$ acts on $\mathcal{D}_c$ as $\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - u^k \bar{D}_{\dot{\alpha}} \frac{\partial}{\partial u^k}$ on u component.

Lemma 2.5.

$$H^*_{\bar{\mathcal{Q}}_i}(\mathcal{O}_{loc}(\mathcal{C})) \cong H^*_{\bar{\mathcal{Q}}_i}(\mho(\mathcal{D}_c))\bar{\theta}^2 u^1$$

Proof. Act the derived functor $\mho^*$ on the above resolution, and use spectrum sequence to calculate. □

Then we calculate $H^*_{\bar{\mathcal{Q}}_i}(\mho(\mathcal{D}_c))$.

Since $\{D_\alpha, \bar{D}_{\dot{\alpha}}\}$ is total derivative, we have the following quasi-isomorphism:

$$(\mho(\mathcal{D}_c), \bar{Q}_1) \cong \theta^1\theta^2\Omega^*(Sym^*(D_{N=1}^{Ch} \otimes \check{C}))$$

And notice that the spectral sequence of $(\theta^1\theta^2\Omega^*(Sym^*(D_{N=1}^{Ch} \otimes \check{C}), \bar{\mathcal{Q}}_1 + d_{dR})$ converges at E2, so we have:

$$H^*_{\bar{\mathcal{Q}}_1}(\mho(\mathcal{D}_c)) = (\theta^1\theta^2\Omega^*(Sym^*(D_{N=1}^{Ch} \otimes \check{\mathcal{C}}), \bar{\mathcal{Q}}_1 + d_{dR}) = \theta^1\theta^2 H^*_{dR}(Sym^*(H^*_{\bar{\mathcal{Q}}_1}(D_{N=1}^{Ch} \otimes \check{\mathcal{C}})))$$

Lemma 2.6.

$$H^*_{\bar{\mathcal{Q}}_1}(D^{Ch}_{N=1}\otimes\check{\mathcal{C}})=\mathbb{C}[\partial_{1\dot{2}},\partial_{2\dot{2}}]\otimes\check{\mathcal{C}}$$

Since the action of $\bar{\mathcal{Q}}_1$ is equivalent to the left action of $-2\partial_{\alpha 1}\frac{\partial}{\partial D_\alpha}$ on $\mathcal{O}^{Ch}_{loc}$, taking the dual, we have:

$$H^*(\mathbb{C}[x_{\alpha\dot{\alpha}},\frac{\partial}{\partial D_\alpha}]\otimes\mathcal{C},-2\partial_{\alpha 1}\frac{\partial}{\partial D_\alpha})=\mathbb{C}[x_{1\dot{2}},x_{2\dot{2}}]\otimes\mathcal{C}\quad\square$$

Summarize the above discussion, the $\mathcal{Q}_1$ cohomology of $\mathcal{O}_{loc}(\mathcal{C})$ is represented by :

$$\int d^2\theta d^2\bar{\theta} f(\theta)\bar{\theta}^1\bar{\theta}^2 W(\mathcal{C},\partial_{1\dot{2}}\mathcal{C},\partial_{2\dot{2}}\mathcal{C})$$

So for any S $\bar{\mathcal{Q}}_1$-invariant we write it as $S=S_0+\bar{\mathcal{Q}}_1\tilde{S}$.

And apply $\mathcal{Q}_1,\mathcal{Q}_2$-exact condition on S_0, it looks like:

$$\int d^2\theta d^2\bar{\theta}\bar{\theta}^1\bar{\theta}^2 W(\mathcal{C},\partial_{1\dot{2}}\mathcal{C},\partial_{2\dot{2}}\mathcal{C})$$

which is F-term.

Then consider: $\bar{\mathcal{Q}}_1\tilde{S}=\bar{\mathcal{Q}}_2 S'$, that means $[\bar{\mathcal{Q}}_1\tilde{S}]=0$ in $\mathcal{O}_{loc}/Im\mathcal{Q}_2$.

If $[\tilde{S}]=0$ in $H^*_{\bar{\mathcal{Q}}_1}(\mathcal{O}_{loc}(\mathcal{C})/Im\mathcal{Q}_2)$. Naturally, $\tilde{S}=\bar{\mathcal{Q}}_1 S_1+\bar{\mathcal{Q}}_2 S_2$, which means $S=\bar{\mathcal{Q}}_1\tilde{S}$ is $\bar{\mathcal{Q}}_1\bar{\mathcal{Q}}_2$-exact, applying $\mathcal{Q}_1\mathcal{Q}_2$-exact condition, it contributes D-term.

Then we calculate $H^*_{\mathcal{Q}_1}(\mathcal{O}_{loc}/Im\mathcal{Q}_2)$ using the following resolution:

$$\begin{tikzcd}
...\mathcal{O}_{loc} \arrow[r, "\mathcal{Q}_2"] & \mathcal{O}_{loc} \arrow[r, "\mathcal{Q}_2"] & \mathcal{O}_{loc} \arrow[r] & \mathcal{O}_{loc}/Im\mathcal{Q}_2 \\
\oplus & & & \\
H^*_{\bar{\mathcal{Q}}_2}(\mathcal{O}_{loc}) \arrow[uur, "j"] & & &
\end{tikzcd}$$

We define deg_1 is the number of $\partial_{1\dot{2}}\ or\ \partial_{2\dot{2}}$; deg_1 is the number of $\partial_{1\dot{1}}\ or\ \partial_{2\dot{1}}$ in $\mathcal{O}_{loc}$. And j is an embedding by picking up a representation element.

Then E_1 page from applying $\mathcal{Q}_1$-cohomology on $\mathcal{O}_{loc}$, since we know that $\mathcal{Q}_2$ act on $H^*_{\mathcal{Q}_1}$) is 0, we have:

$$d_1=j:H^*_{\mathcal{Q}_2}\rightarrow H^*_{\mathcal{Q}_1}$$

is identity on the $deg_1=deg_2=0$ component, and 0 otherwise.

So E_2 is getting the mod degree-0 component of E_1 page. And trace the graph, d_n is replacing d $\partial_{1\dot{2}}\ or\ \partial_{2\dot{2}}$ component by $\partial_{1\dot{1}}\ or\ \partial_{2\dot{1}}$.

At last, the spectrum sequence converges to $\oplus_{k=2}^{\infty}H^{\geq k}_{\mathcal{Q}_2}(\mathcal{O}_{loc})$, and $H^{\geq k}_{\mathcal{Q}_2}(\mathcal{O}_{loc})$ corresponds to term with (k-1)-($\partial_{1\dot{2}}\ or\ \partial_{2\dot{2}}$) in $H^*_{\mathcal{Q}_1}(\mathcal{O}_{loc}/Im\mathcal{Q}_2)$. After applying $\mathcal{Q}_1$,$\mathcal{Q}_2$-exact condition, it contributes to F-term which has $Sym^*([\partial_\mu])$ structure.

$\square$

3 Super Yang-Mills Theory in classical BV formalism and its twist

3.1 D=4, $\mathcal{N}=1$ SYM

As a gauge theory with Lie algebra $\mathfrak{g}$ over M, its action functional is:

$$S_{SYM} = \frac{1}{16\pi} Im(\tau \int d^6z\, Tr(W^\alpha W_\alpha) + c.c) \quad W_a := -\frac{1}{8}\bar{D}^2(e^{-2V} D_\alpha e^{2V}) \quad \tau := \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$$

Here, d^6z is short for $d^4xd^2\theta$, and $c.c$ denotes the complex conjugate part. Tr stands for the Killing form of the Lie algebra $\mathfrak{g}$.

$$gauge\ transformation: \quad e^{2V} \to e^{i\bar{\Omega}} e^{2V} e^{-i\Omega}$$

Here, V is the vector field and Ω is the chiral field valued in $\mathfrak{g}$.

I will skip the introduction of the BV formalism(ref. [4]). Following our marks, $\mathcal{V}_g$ and $\mathcal{C}_g$ denote the field spaces of vector and chiral fields valued in $\mathfrak{g}$.

Definition 3.1. *(local BV complex)*

$$\mathcal{BV}_{SYM} := (O_{loc}(T^*[-1](\mathcal{V}_g \oplus \mathcal{C}_g[1]), \delta_{SYM})$$

[1] denotes the left shift in a graded vector space; T^* means the cotangent bundle. Specifically, the graded vector space looks like:

$$\mathcal{C}_g[1] \oplus \mathcal{V}_g \oplus \mathcal{V}_g^*[-1] \oplus \mathcal{C}_g^*[-2] := \mathcal{E}$$

$O(\mathcal{E})$ means the functional over the function space $\mathcal{E}$;loc means it must take this form:

$$F(\mathcal{E}_1, ...\mathcal{E}_n) = \int_M \alpha(j_z(\mathcal{E}_1), ..., j_z(\mathcal{E}_n)) : \mathcal{E}^{\otimes n} \to R$$

Here, j_z is the jet bundle of a sheaf at point z; α is a density-valued function on the jet bundles.

Remark 3.1.1: Here I also use the $\mathcal{E}$ to denote the sheaf.

Remark 3.1.2: The odd graded fields are anti-commutative, and the even ones are commutative in the graded vector space. In physics, we call the components in $\mathcal{E}$ from left to right: ghost, field, anti-field, and anti-ghost.

Remark 3.1.3: There is a natural Poisson bracket on the Kaszul-Tate complex, which is the Schouten bracket, we also call it the BV bracket. In particular, we have $\delta_{BV} = \{\_, S_{SYM}\}$.

3.1.1 Wess-Zumino gauge and reduce to gauge theory

Extending the gauge transformation by the BCH formula, we have:

$$\delta V = \frac{i}{2}(\bar{\Omega} - \Omega) + o(V)$$

By comparing the formula (1)(2), the first six terms in (2) can be canceled out through the 0-order in gauge.

Definition 3.2. *The Wess-Zumino gauge is to set: C, χ, M components in $\mathcal{V}$ equal to 0. Under this gauge:*

$$\mathcal{V} = \theta\sigma^{\mu}\bar{\theta}V_{\mu} + \bar{\theta}^2\theta^{\alpha}\lambda_{\alpha} + \theta^2\bar{\theta}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} + \theta^2\bar{\theta}^2 D$$

The higher order part looks like [V,...], you can't reach the WZ gauge just by transforming once, but the difference will reside in higher terms in (2), you can defer it to the later terms each time. Finally, we can reach the WZ gauge.

So in WZ gauge,

$$W_{\alpha} = -\frac{1}{4}\bar{D}^2 D_{\alpha}V + \frac{1}{4}\bar{D}^2([V, D_{\alpha}V]) = \lambda_{\alpha}(x) + 2D\theta_{\alpha} + i(\sigma^{\mu\nu}\theta)_{\alpha}F_{\mu\nu} - i\theta^2\sigma^{\mu}D_{\mu}\bar{\lambda}$$

Using the property:

$$\sigma_a\bar{\sigma}_b\sigma_c = (\eta_{ac}\sigma_b - \eta_{bc}\sigma_a - \eta_{ab}\sigma_c) + i\epsilon_{abcd}\sigma^d$$

Finally, we have:

$$S_{SYM} = \frac{1}{g^2}\int d^4x(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^{\mu}D_{\mu}\bar{\lambda} + 2D^2) + \frac{\theta}{32\pi^2}\int d^4x F_{\mu\nu}{}^{\star}F^{\mu\nu}$$

which is the canonical N=1 gauge theory with θ term. Since in perturbative level, gauge fixing doesn't change the BV complex, we have:

Lemma 3.1. *Under Wess-Zumino gauge, the two BV complexes*

$$(\mathcal{O}_{loc}(\mathcal{E}^{SYM}), \delta_{SYM}) \overset{quasi-iso}{\cong} (\mathcal{O}_{loc}(\mathcal{E}^{gauge}), \delta_{gauge})$$

3.1.2 Twist theory

Induced from $\mathcal{V}_{\mathfrak{g}}$, we have the SUSY transform on component fields:

$$\begin{aligned}
\delta A_{\mu} &= \epsilon\sigma_{\mu}\bar{\lambda} + \lambda\sigma_{\mu}\bar{\epsilon} \\
\delta\lambda &= \epsilon D + (\sigma^{\mu\nu}\epsilon)F_{\mu\nu} \\
\delta D &= i\epsilon\sigma^{\mu}\partial_{\mu}\bar{\lambda} - i\partial_{\mu}\lambda\bar{\sigma}^{\mu}\bar{\epsilon}
\end{aligned}$$

Transferring to the first-order formalism, we have:

$$\delta_{Q_\alpha} A_\mu = \sigma_{\mu\alpha\dot{\alpha}} \psi^{\dot{\alpha}}$$
$$\delta_{Q_\alpha} \psi_\beta = \epsilon^\gamma_\alpha B_{\gamma\beta}$$

We can choose any spinor as our twist charge, usually $Q := Q_1$. Since selecting a spinor is equivalent to choosing a complex structure, our structure group $Spin(4)$ now reduces to SU(2), which means that now Q is a U(1)-charge scalar but not a spinor as a derivation. This induces a second derivation in our BV complex:

$$\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array}$$

$$\begin{tikzcd}
 & \psi_- \arrow[r, "\not{D}"] \arrow[d, "Q"] & \psi'_+ & \\
\Omega^0 \arrow[r, "\mathrm{d}"] & \Omega^1 \arrow[r, "\mathrm{d}_+"] & \Omega^2_+ & \\
 & \Omega^2_+ \arrow[r, "\mathrm{d}"] \arrow[ur, "c\,\mathrm{Id}"] \arrow[d, "Q"] & \Omega^3 \arrow[r, "\mathrm{d}"] & \Omega^4 \\
 & \psi_+ \arrow[r, "\not{D}"] & \psi'_- &
\end{tikzcd}$$

The new derivation now has the wrong degree. We can add a BV degree 1, U(1) degree -1 element t to that. And we have:

$$(\mathcal{O}_{loc}(\mathcal{E}^{gauge})((t)), \delta_{gauge} + tQ)$$

$$\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array}$$

$$\begin{tikzcd}
\psi_- \arrow[r, "\not{D}"] \arrow[dr, "tQ"] & \psi'_+ \arrow[dr, "tQ"] & & \\
\Omega^0 \arrow[r, "\mathrm{d}"] & \Omega^1 \arrow[r, "\mathrm{d}_+"] & \Omega^2_+ & \\
 & \Omega^2_+ \arrow[r, "\mathrm{d}"] \arrow[ur, "c\,\mathrm{Id}"] \arrow[dr, "tQ"] & \Omega^3 \arrow[r, "\mathrm{d}"] \arrow[dr, "tQ"] & \Omega^4 \\
 & & \psi_+ \arrow[r, "\not{D}"] & \psi'_-
\end{tikzcd}$$

In short, what we did was construct a Q-cohomology by twisting and blending it with the BV-comology in the correct degree.

What's more, choosing Q not only gives us a complex structure, but also the following isomorphism:

$$\Gamma : \psi^- \xrightarrow{\cong} \Omega^{1,0} \tag{3}$$

Here, we substitute Q with the following left-spinor part:

$$V_\mu \rightarrow (\sigma^\mu V_\mu)_{\alpha\dot{\alpha}}$$

And,

$$Y : \Omega^0 \cdot w \oplus \Omega^{0,2} \xrightarrow{\cong} \psi^+ \tag{4}$$

Here, we use the isomorphism and pair one of ψ_+ with Q.

$$\Omega^2_+ \cong Sym^2(\psi^+)$$

With these isomorphisms, we have:

Theorem 3.1. *(Costello 4.0.2 [1]) The twisted $N = 1$ gauge theory on $\mathbb{C}^2$ is equivalent to the holomorphic BF theory.*

In the same paper, costello also proves that:

Theorem 3.2. *(Costello 4.1.1 [1]) The twisted $N = 1$ theory admits a unique quantization, compatible with certain natural symmetries, on any Calabi-Yau surface X.*

4 Quantization of $\mathcal{N} = 1$ SYM on Superspace

As a *pre-theory*([4] section 13) on R^4 , $\mathcal{N} = 1$ SYM given by following second-order BV complex:

$$\begin{tikzcd}
-1 & 0 & 1 & 2 \\
\mathcal{C}_g \arrow[r, "i"] & \mathcal{V}_g \arrow[ddr, red, "\bar{D}^2 D_\alpha"] & \mathcal{V}_g^* \arrow[r, "\bar{D}^2"] \arrow[ddr, "D^2"] & \mathcal{C}_g^* \\
\oplus & \oplus & \oplus & \oplus \\
\bar{\mathcal{C}}_g \arrow[uur, "-i"] & \mathcal{B}_\alpha \arrow[r, "="] \arrow[uur, "D^\alpha"] & \mathcal{B}_\alpha^* & \bar{\mathcal{C}}_g^*
\end{tikzcd}$$

And Q^{GF} given by:

$$
\begin{tikzcd}
-1 & 0 & 1 & 2 \\
\mathcal{C}_g & \mathcal{V}_g \arrow[l, "\bar{D}^2 D^2"'] \arrow[ddl, "D^2\bar{D}^2"] & \mathcal{V}_g^* \arrow[ddl, red, "\bar{D}^2 D_\alpha"] & \mathcal{C}_g^* \arrow[l, "D^2"'] \\
\oplus & \oplus & \oplus & \oplus \\
\bar{\mathcal{C}}_g & \mathcal{B}_\alpha & \mathcal{B}_\alpha^* \arrow[uul, "D^\alpha"] & \bar{\mathcal{C}}_g^* \arrow[uul, "\bar{D}^2"]
\end{tikzcd}
$$

By simple calculation, we have the property $[Q, Q^{GF}] = \Box$.

The second-order SYM action is:

$$S_{SYM} = \frac{1}{16\pi} Im(\tau \int d^6 z\, Tr(W^\alpha B_\alpha + B^\alpha B_\alpha) + c.c)$$

Quantization means we can assign R^4 a factorization algebra Obs^q, which locally(for U a open disk) looks like following DGLA at level Φ:

$$Obs^q_\Phi(U) \cong (\oplus_n Sym^n(T^*[-1](\mathcal{V}_g(U) \oplus \mathcal{C}_g(U)[1]))^*[[\hbar]], Q + \hbar\triangle_\Phi + \{I[\Phi], \underline{\quad}\})$$

Here we define parametrix following [5] section 7.2.4 as an element:

$$\Phi \in \bar{\mathcal{E}}(U)\widehat{\otimes}_\pi \bar{\mathcal{E}}(U)$$

Satisfying several properties from elliptic regularity. $\bar{\mathcal{E}}(U)$ means the distributional section of $\mathcal{E}(U)$. Define:

$$P(\Phi) = \frac{1}{2} Q^{GF}\Phi \in \overline{\mathcal{E}}\widehat{\otimes}_\pi \overline{\mathcal{E}}.$$

This is the propagator associated with Φ. We let

$$K_\Phi = K_{\mathrm{Id}} - QP(\Phi).$$

K_{Id} is the kernel of $Id : \mathcal{E} \to \bar{\mathcal{E}}$

$$\triangle_\Phi := \partial_{K_\Phi} : \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E})$$

$$\{I, J\}_\Phi := \triangle_\Phi(IJ) - (\triangle_\Phi I)J - (-1)^{|I|} I \triangle_\Phi J$$

on the space $\mathcal{O}(\mathcal{E})$.

Also, we demand the quantum field theory $I[\Phi]$ corresponds to our classical field theory $I_0 = S_{SYM}$, that means it satisfies:

- $I[\Phi] = W(P(\Phi) - P(\Psi), I[\Psi])$ (1)
- $\lim_{\Phi \to 0} I[\Phi] - I_0 = 0 \ mod\ \hbar$ (2)

- $I[\Phi]$ satisfies *quantum master equation* for $\forall\ \Phi$ (3)

Condition (1)(2)(3) are satisfied at the same time if we can find a lift $\tilde{I}_0$ of I_0 at scale 0, that satisfies quantum master equation. Since we can define:

$$I[\Phi] := W(P(\Phi), \tilde{I}_0)$$

And we have the following lemma:

Lemma 9.2.2 [4]

If $I[\Phi]$ satisfies the Φ-QME, then $I[\Psi] = W(P(\Psi) - P(\Phi), I[\Phi])$ satisfies the Ψ-QME.

This follows from the identity

$$\left[Q, \partial_{P(\Phi)} - \partial_{P(\Psi)}\right] = \triangle_\Psi - \triangle_\Phi$$

Then

$$(Q + \hbar\triangle_\Psi)e^{I[\Psi]/\hbar} = (Q + \hbar\triangle_\Psi)e^{\hbar\partial_{P(\Psi)-P(\Phi)}}e^{I[\Phi]/\hbar} = e^{\hbar\partial_{P(\Psi)-P(\Phi)}}(Q + \hbar\triangle_\Phi)e^{I[\Phi]/\hbar}$$

Naturally, $I[\Phi] := W(P(\Phi), I_0)$ satisfies mod $\hbar$ *quantum master equation*. To lift it to a theory satisfies mod $\hbar^2$ *quantum master equation*, or more generally, to lift a $\hbar^n$ theory $I[L]$ to a $\hbar^{n+1}$, we choose any lift $\tilde{I}[L]$ and define its obstruction at scale L as:

$$O_{n+1}[L] = \hbar^{-n-1}\left(Q\tilde{I}[L] + \frac{1}{2}\{\tilde{I}[L], \tilde{I}[L]\}_L + \hbar\Delta_L\tilde{I}[L]\right)$$

And we have the following property:

- The obstruction of the lift $\tilde{I}[L]$: $O_{n+1} \in H^1_{Q+\{I_0,-\}}(\mathcal{O}_{loc}(\mathcal{E}))$
- When the obstruction vanishes, the classification set of the lift equals $H^0_{Q+\{I_0,-\}}(\mathcal{O}_{loc}(\mathcal{E}))$

In this paper, we want to show that when $I_0 = S_{SYM}$, $H^1_{Q+\{I_0,-\}}(\mathcal{O}_{loc}(\mathcal{E})) = 0$, then our quantization is well-defined.

The proof of these properties is from *Lemma 11.1.1* [4].

In short, the first property is obtained easily from DGLA:

Let $\mathfrak{g}$ be a differential graded Lie algebra, and let $X \in \mathfrak{g}$ be an odd element. Let

$$O(X) = \mathrm{d}_\mathfrak{g}X + \frac{1}{2}[X, X].$$

Then,

$$\mathrm{d}_\mathfrak{g}O(X) + [X, O(X)] = 0.$$

That means

$$QO_{n+1} + \{I_0, O_{n+1}\} = 0$$

For $\tilde{I}[L]$ $\forall$lift, then any other lift is given by $\tilde{I}[L]+\hbar^{n+1}J[L]$. The obstruction vanishes if only:

$$QJ[L]+\{I_0[L],J[L]\}=O_{n+1}[L] \quad \square$$

Now we calculate $H^*_{Q+\{I_0,\_\}}(\mathcal{O}^{SUSY}_{loc}(\mathcal{E}))$.

Lemma 4.1. *(Perturbative Non-renormalization of $\mathcal{N}=1$ SYM)*
In perturbation SYM theory, quantum corrections can only generate counterterms of D-term type, i.e. integrals over the full superspace of the form

$$\int d^4\theta\, \mathcal{K}(\Phi,\bar{\Phi},V),$$

Non-perturbative effects, such as instantons, may generate effective F-terms, but these are absent in perturbation theory.

Using this lemma, we only have to calculate D-term, and recall in *Lemma 4.1*, we have proved that

$$\mathcal{O}_{loc}(\mathcal{E})^{SUSY}=\mho^*(\mathbb{C}\otimes Sym^*(\mathbb{C}[\partial_\mu,D_\alpha,\bar{D}_{\dot{\alpha}}]\otimes\check{\mathcal{E}}))$$

And in SYM,

$$\mathcal{E}:=\mathcal{C}_g[1]\oplus\mathcal{V}_g\oplus\mathcal{V}^*_g[-1]\oplus\mathcal{C}^*_g[-2]$$

as a Q-complex. So the $Q+[I_o,\_\_]$ cohomology is quasi-iso to the $Q+[I_o,\_\_]$ cohomology of the following Super De Rham-complex:

$$\begin{array}{ccc}
\ldots \to \wedge^1\mathbb{R}^4\otimes\mathbb{C}[\theta^i\otimes D^i]\otimes Sym^*(\ldots) & \xrightarrow{\partial_i} & \mathbb{C}[\theta^i\otimes D^i]\otimes Sym^*(\mathbb{C}[\partial_\mu,D_\alpha,\bar{D}_{\dot{\alpha}}]\otimes\check{\mathcal{E}}) \\
 & & \Phi\uparrow \\
\ldots & \xrightarrow{\partial_i} & (\oplus_j(\mathbb{C}[\theta^i\otimes D^i](i\neq j))\epsilon^j)\otimes_{\tilde{D}_{N=1}} Sym^*(\mathbb{C}[\partial_\mu,D_\alpha,\bar{D}_{\dot{\alpha}}]\otimes\check{\mathcal{E}}) \\
 & & \Phi\uparrow \\
 & & \ldots
\end{array}$$

As the $Q+[I_0,\_\_]$ only acts on Sym^* component, so we calculate $H_{Q+[I_0,\_\_]}(Sym^*(\ldots))$ as the E_1-page.

Now we filter the Sym^* component by its *-index, under this filtration, the E_1-page given by $Sym^i(\mathbb{C}[\partial_\mu,D_\alpha,\bar{D}_{\dot{\alpha}}]\otimes H^j_Q(\check{\mathcal{E}}))\cong Sym^i(\mathbb{C}[\partial_\mu,D_\alpha,\bar{D}_{\dot{\alpha}}]\otimes\check{H}^j_Q(\mathcal{E}))$.

Recall as in [4]*Lemma 6.7.1*, as a L^∞ CE cohomology, $H_{Q+\{I_0,\_\}}(\mathcal{O}^{SUSY}_{loc})$ can be calculated localized to its fiber $(\mathcal{O}_{loc})_0:=Sym^*(\mathbb{C}[\partial_\mu,D_\alpha,\bar{D}_{\dot{\alpha}}]\otimes\check{\mathcal{E}}_0)$, because of its SUSY invariant.

Then, just by Talor extending $\mathcal{C}$ and $\mathcal{V}$. For example, using the extension (1)(2) in section 2.1, we have:

$$e_1 = C(x) - \frac{1}{4}\theta^2\bar{\theta}^2\Box C(x) = Q\Big(\phi(x) + \mathrm{i}\theta^\mu\theta\partial_\mu\phi(x) + \frac{1}{4}\theta^2\bar{\theta}^2\Box\phi(x)\Big) = Qs_1 \text{ when } \phi \text{ is pure imagine}$$

$$e_2 = \theta\sigma^\mu\bar{\theta}A_\mu = Q\Big(\phi(x) + \mathrm{i}\theta^\mu\theta\partial_\mu\phi(x) + \frac{1}{4}\theta^2\bar{\theta}^2\Box\phi(x)\Big) = Qs_2 \text{ when } \phi \text{ is pure real}$$

$Here$ a bit different from (1)(2), $\phi, \psi, F, \chi, \bar{\chi}, \lambda, \bar{\lambda}, M \in \mathbb{C}[[x_1, \ldots, x_4]] \quad C, D, A_\mu \in \mathbb{R}[[x_1, \ldots, x_4]]$

Finally, we have $H_Q^*(\mathcal{E}) = \mathfrak{g}[1]$. As a result, $H_{Q+[I_0,\_\_]}(Sym^*(...)) \cong H_{CE}^*(\mathfrak{g})$. When considering semi-simple $\mathfrak{g}$, the degree 1 component vanishes, and the degree 0 component is just $\mathbb{C}$, which means there is no obstruction and quantization is unique up to a constant. □

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