Quantization of $\mathcal{N} = 1$ SYM theory

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Abstract

We identify the D-module structure on N=1 Superspace and classify the corresponding SUSY-invariant local functionals. Furthermore, we construct the Batalin–Vilkovisky complex of N=1 Super–Yang–Mills theory and carry out its quantization in the sense of homotopy renormalization. However, this note remains preliminary, and I'm still polishing it.

Contents

1 Introduction

In [1], Costello showed that the twisted N=1 gauge theory admits unique quantization on any Calabi-Yau manifold and analyzed its algebraic structure. With the same spirit, using the superfields construction and BV cohomology, we showed that the pure SYM theory has no quantization obstruction; thus, it also admits a unique quantization. Further, we try to calculate some observable structures. In Section 2, we identify the

D-module structure on N=1 Superspace and classify the corresponding SUSY-invariant local functionals into D-term and F-term. In Section 3, we give an introduction to SYM construction in BV formalism and review its twist. Section 4 is the quantization and calculation of the local BV cohomology, where we use the classification in section 2 and the Perturbative Non-renormalization property of N = 1 SYM.

2 SUSY D-module

2.1 SUSY algebra and Superfield

We work on **N=1 Superspace** $M = R^{4|4}$ consists of fourbonsonic coordinates x^μ and four complex Grassmann coordinates θ^α and $\bar{\theta}^{\dot{\alpha}}$. By analogy with the Poincaré group on R^4, the **SuperPoincaré group** is defined to be:

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu, \quad [M_{\mu\nu}, Q_\alpha] = i(\sigma_{\mu\nu})_\alpha^\beta Q_\beta, \quad [P_\mu, Q_\alpha] = 0,$$

$$\sigma_{\mu\nu} := -\frac{1}{4}(\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu), \qquad \bar{\sigma}^\mu := (1, -\sigma_1, -\sigma_2, -\sigma_3).$$

A **Suprefield** $Y(x^\mu, \theta, \bar{\theta})$ is a $\mathbb{C}$-valued function on the M. In Talor-extension, it decomposed to component fields:

$$\begin{aligned} Y(x^\mu, \theta, \bar{\theta}) = \phi(x) + \theta^\alpha\psi_\alpha(x) + \bar{\theta}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}(x) + \theta^2 M(x) + \bar{\theta}^2 N(x) \\ + \theta^\alpha\bar{\theta}^{\dot{\alpha}}V_{\alpha\dot{\alpha}}(x) + +\theta^2\bar{\theta}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}(x) + \bar{\theta}^2\theta^\alpha\rho_\alpha(x) + \theta^2\bar{\theta}^2 D(x). \end{aligned}$$

It follows from the commutation relation:

$$[P_\mu, Y] = -i\partial Y, \qquad [Q_\alpha, Y] = \left(-i\frac{\partial}{\partial\theta^\alpha} - \sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu\right),$$

the differentials on M should be:

$$\begin{aligned} \mathcal{P}_\mu &= -i\partial_\mu, \\ \mathcal{Q}_\alpha &= -i\partial_\alpha - \sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu, \\ \bar{\mathcal{Q}}_{\dot{\alpha}} &= +i\bar{\partial}_{\dot{\alpha}} + \theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu. \end{aligned}$$

The N=1 SuperPoincaré group have two fundamental representation when it acts on Superfield:

$$\begin{cases} \Omega(x^\mu, \theta, \bar{\theta}) & \textbf{Chiral fields} & \bar{D}_{\dot{\alpha}}\Omega(x^\mu, \theta, \bar{\theta}) = 0, \\ \bar{\Omega}(x^\mu, \theta, \bar{\theta}) & \textbf{Anti-chiral fields} & D_\alpha\bar{\Omega}(x^\mu, \theta, \bar{\theta}) = 0, \\ V(x^\mu, \theta, \bar{\theta}) & \textbf{Vector fields} & V = \bar{V}. \end{cases}$$

where

$$D_\alpha := \partial_\alpha + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu, \qquad \bar{D}_{\dot{\alpha}} := -\bar{\partial}_{\dot{\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu.$$

Also in Talor-extension:

$$\Omega(x,\theta,\bar{\theta}) = \phi(x) + \theta^\alpha \psi_\alpha(x) + \theta^2 F(x) + \mathrm{i}\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) - \frac{\mathrm{i}}{2}\theta^2\partial_\mu\psi(x)\sigma^\mu\bar{\theta} + \frac{1}{4}\theta^2\bar{\theta}^2\Box\phi(x) \tag{1}$$

$$\begin{aligned}\mathcal{V}(x,\theta,\bar{\theta}) &= C(x) + \theta\chi(x) + \bar{\theta}\bar{\chi}(x) + i\theta^2 M(x) - i\bar{\theta}^2 M^\dagger(x) + \theta\sigma^\mu\bar{\theta}A_\mu(x)\\ &+ \theta^2\bar{\theta}\left(\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^\mu\partial_\mu\chi(x)\right) + \bar{\theta}^2\theta\left(\lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\bar{\chi}(x)\right) + \frac{1}{2}\theta^2\bar{\theta}^2\left(D(x) - \frac{1}{2}\Box C(x)\right)\\ &(C(x), D(x), A^\mu(x)\ are\ real\ fields)\end{aligned} \tag{2}$$

2.2 SUSY D-module

This part generalize algebraic D-module discovered in [2] to Superspace.

Recall that the **ring of differential operators** on M, $\mathcal{D}_M$ is the ring of operators acting on the coordinate ring $\mathcal{O}_M$ as differential. Thus in our case:

$$\mathcal{O}_M = C^\infty(R^4)\otimes\Lambda[\theta^\alpha,\bar{\theta}^{\dot{\alpha}}] \qquad \mathcal{D}_M = \mathcal{O}_M[\partial_\mu,\partial_{\theta^\alpha},\partial_{\bar{\theta}^{\dot{\alpha}}}] \cong \mathcal{O}_M[\partial_\mu, D_\alpha, \bar{D}_{\dot{\alpha}}] \text{ as } \mathcal{O}_M \text{ module.}$$

We define the category of **SUSY $\mathcal{D}$-modules** to be the category of graded modules over the ring $\mathcal{D}_M$, denoted by $\mathcal{M}od(\mathcal{D}_M)$.

Definition 2.1. *Let $F \in \mathcal{M}od(\mathcal{D}_M)$, we consider following bi-complex which we called* ***Super De-Rham complex*** $\mho^*(F)$*:*

$$\begin{tikzcd}
 & \cdots \arrow[r, "d_{DR}"] & \Omega^3_{\mathbb{R}^4}\otimes_{C^\infty(R^4)} F \arrow[r, "d_{DR}"] & \Omega^4_{\mathbb{R}^4}\otimes_{C^\infty(R^4)} F \\
 & \cdots \arrow[r, "d_{DR}"] \arrow[u, "\Phi"] & \Omega^3_{\mathbb{R}^4}\otimes_{C^\infty(R^4)}(\oplus_j\epsilon^j)\otimes_{\mathcal{O}_M} F \arrow[r, "d_{DR}"] \arrow[u, "\Phi"] & \Omega^4_{\mathbb{R}^4}\otimes_{C^\infty(R^4)}(\oplus_j\epsilon^j)\otimes_{\mathcal{O}_M} F \arrow[u, "\Phi"] \\
 & \cdots \arrow[r, "d_{DR}"] \arrow[u, "\Phi"] & \cdots \arrow[r, "d_{DR}"] \arrow[u, "\Phi"] & \Omega^4_{\mathbb{R}^4}\otimes_{C^\infty(R^4)}(\oplus_{j,k}\epsilon^j\epsilon^k)\otimes_{\mathcal{O}_M} F \arrow[u, "\Phi"] \\
 & \cdots \arrow[r, "d_{DR}"] \arrow[u, "\Phi"] & \cdots \arrow[r, "d_{DR}"] \arrow[u, "\Phi"] & \Omega^4_{\mathbb{R}^4}\otimes_{C^\infty(R^4)}(\oplus_{j,k,l}\epsilon^j\epsilon^k\epsilon^l)\otimes_{\mathcal{O}_M} F \arrow[u, "\Phi"] \\
 & \Omega^0_{\mathbb{R}^4}\otimes_{C^\infty(R^4)}\cdots \arrow[r, "d_{DR}"] \arrow[u, "\Phi"] & \cdots \arrow[r, "d_{DR}"] \arrow[u, "\Phi"] & \Omega^4_{\mathbb{R}^4}\otimes_{C^\infty(R^4)}\epsilon^1\epsilon^2\bar{\epsilon}^{\dot{1}}\bar{\epsilon}^{\dot{2}}\otimes_{\mathcal{O}_M} F \arrow[u, "\Phi"]
\end{tikzcd}$$

(1)θ^i goes over $\{\theta^1,\theta^2,\bar{\theta}^{\dot{1}},\bar{\theta}^{\dot{2}}\}$, ϵ^j are corresponding formal odd variables.
(2)The horizontal differential d_{DR} is the usual De-Rham differentials: $f\otimes w \to f dx^i \otimes \partial_i w$
(3)The vertical differential Φ is odd map defined to be:

$$\Phi : \epsilon^\alpha \to 1 - \theta^\alpha \otimes_{\mathcal{O}_M} D_\alpha,\ \bar{\epsilon}^{\dot{\alpha}} \to 1 - \bar{\theta}^{\dot{\alpha}} \otimes_{\mathcal{O}_M} \bar{D}_{\dot{\alpha}}.$$

(4)Our Super De Rham complex differs from that of Superform introduced in [3], in that we additionally impose the equivalence relations, and write down its Koszul-type resolution:

$$\int d^4x d\theta^1 Y \sim \int d^4x D_1 Y,$$

and similarly for the other odd variables.

Lemma 2.1. *The **Super de Rham functor***

$$\mho : \mathcal{M}od(\mathcal{D}_M) \longrightarrow \mathcal{CH}(\mathrm{Sh}(M)),$$

extends naturally to a functor between bounded derived categories.

2.3 Classification of SUSY Local Functional

Let $\mathcal{E}$ be the Superfields space,

Definition 2.2. *(Jet bundle) The jet bundle assigned to sheaf $\mathcal{E}$ is defined as:*

$$J(\mathcal{E})(U) := D_M \otimes_{\mathcal{O}_M} \mathcal{E}(U).$$

Follows by definition, $J(\mathcal{E})$ is a $\mathcal{D}_M$-module. This also works for the completed symmetric algebra on its dual:

$$\check{J}(\mathcal{E}) \;\; := Hom(J(\mathcal{E}), \mathcal{O}_M) = D_M \otimes_{\mathbb{C}} \check{E}_0 \text{ ,where } E_0 \text{ is the trivial fiber.}$$

$$\mathcal{O}(J(\mathcal{E})) := \Pi_{n\geq 0} Sym^n \check{J}(E) \,.$$

Thus we can consider its Super De Rham complex $\mho^*(J(\mathcal{E}))$:

Lemma 2.2.

$$\mathcal{O}_{loc}(\mathcal{E}) = \mho^*(\mathcal{O}(J(\mathcal{E}))) \text{ in derived sense.}$$

Definition of $\mathcal{O}_{loc}(\mathcal{E})$ can be found in [4] section 2.7.1, and the proof is straightforward.

We now consider the complex of SUSY-invariant local functionals $\mathcal{O}_{loc}(\mathcal{E})^{SUSY}$, defined as the SUSY-invariant subcomplex of $\mathcal{O}_{loc}(\mathcal{E})$.

In the present case, the space of fields decomposes as

$$\mathcal{E} = \mathcal{C} \oplus V \oplus \bar{\mathcal{C}},$$

where $\mathcal{C}$ denotes the chiral superfield and V the vector superfield. This decomposition leads to the following classification of SUSY-invariant local functionals.

Theorem 2.1.
Let

$$\tilde{\mathcal{O}}_M := \mathcal{O}_M/\mathbb{C},\ M_1 := (\theta^1\theta^2/\tilde{\mathcal{O}}_M \cdot \theta^1\theta^2),\ M_2 := (1/\tilde{\mathcal{O}}_M \cdot 1)$$

be two trivial $\tilde{\mathcal{O}}_M$-modules, and set

$$D_{N=1} := \mathbb{C}[\partial_\mu, D_\alpha, \bar{D}_{\dot{\alpha}}].$$

Then we have following classification:

$$\mathcal{O}_{loc}(\mathcal{E})^{SUSY} = \begin{cases} \mho^*(M_1 \otimes_{\mathcal{O}_M} Sym^*(\mathbb{C}[\partial_\mu] \otimes_{\mathcal{O}_M} \check{\mathcal{E}})) & \textbf{\textit{F-term}} \quad \mathcal{E} = \mathcal{C},\ \textit{Pure Chiral}; \\ \mho^*(\bar{M}_1 \otimes_{\mathcal{O}_M} Sym^*(\mathbb{C}[\partial_\mu] \otimes_{\mathcal{O}_M} \check{\mathcal{E}})) & \textbf{\textit{F-term}} \quad \mathcal{E} = \bar{\mathcal{C}},\ \textit{Pure anti-Chiral}; \\ \mho^*(M_2 \otimes_{\mathcal{O}_M} Sym^*(D_{N=1} \otimes_{\mathcal{O}_M} \check{\mathcal{E}})) & \textbf{\textit{D-term}} \quad for\ \forall \mathcal{E}. \end{cases}$$

Here the ring D_M acts on Sym^* component as a left-module.

Proof. The main idea is to exploit the $\mathcal{Q}_i$-closed conditions together with the identity

$$[\mathcal{Q}_i, \theta^i] = i$$

to deduce $\mathcal{Q}_i$-exactness. To make this argument precise, we introduce an explicit definition of the right action of θ^i on the module and treat separately the purely chiral and anti-chiral cases.

To illustrate the construction, consider a representative element of the form:

$$(D_i D_j \mathcal{E}_1)(D_k \mathcal{E}_2),\ D_i \in \{D_\alpha, \bar{D}_{\dot{\alpha}}\}.$$

The extension to the general case of Sym^* is straightforward.

Definition 2.3. *The* ***right action of*** $\mathcal{Q}_i$ *is defined by:*

$$(D_i D_j \mathcal{E}_1)(D_k \mathcal{E}_2) \cdot \mathcal{Q}_l = (D_i D_j Q_l \mathcal{E}_1)(D_k \mathcal{E}_2) + (D_i D_j \mathcal{E}_1)(D_k Q_l \mathcal{E}_2)$$

Similarly, we define the ***right action of*** θ^l *by:*

$$(D_i D_j \mathcal{E}_1)(D_k \mathcal{E}_2) \cdot \theta^l = (D_i D_j\ \theta^l \mathcal{E}_1)(D_k \mathcal{E}_2) + (D_i D_j \mathcal{E}_1)(D_k\ \theta^l \mathcal{E}_2).$$

Remark: The right action of θ^l is well defined as the usual product when $\mathcal{E}$ consists of vector fields. However, in the chiral case, the induced quotient D-module structure

$$D^{\mathrm{Ch}}_{N=1} := D_{N=1}/D_{N=1}\,\bar{D}_{\dot{\alpha}},$$

is not compatible with this action, since $[\bar{\theta}^{\dot{\alpha}}, \bar{D}_{\dot{\beta}}] \neq 0$. In this case, we instead define the action of θ^α on $\mathcal{C}$ to be the ordinary $\mathcal{O}_M$-module product, while the action of $\bar{\theta}^{\dot{\alpha}}$ is taken to be trivial. The anti-chiral case is defined in a completely analogous way.

Lemma 2.3. *When $\mathcal{E}$ is neither purely chiral nor purely anti-chiral, then every $\mathcal{Q}_i$-closed element in $\mathcal{O}_{loc}(\mathcal{E})$ is $\mathcal{Q}_i$-exact.*

Proof. Since we have:

$$S \cdot \{\mathcal{Q}_l, \theta^l\} = iS, \quad S \cdot \mathcal{Q}_l = 0,$$

it follows that:

$$S = k(S \cdot \theta^1 \theta^2 \bar{\theta}^{\dot{1}} \bar{\theta}^{\dot{2}}) \cdot \mathcal{Q}_1 \mathcal{Q}_2 \bar{\mathcal{Q}}_{\dot{1}} \bar{\mathcal{Q}}_{\dot{2}}$$

$\square$

This lemma shows that the $\mathcal{Q}_i$-invariant local functionals can be represented by:

$$\int_{\mathbb{R}^4} d^4x \; C^\infty(\mathbb{R}^4) \otimes_{\mathbb{R}[\partial_\mu]} D_1 D_2 \bar{D}_{\dot{1}} \bar{D}_{\dot{2}} \cdot Sym^*(D_{N=1} \otimes_{\mathcal{O}_M} \check{\mathcal{E}}).$$

Using translation invariance, this expression is equivalent to:

$$\mho^*(M_2 \otimes_{\mathcal{O}_M} Sym^*(D_{N=1} \otimes_{\mathcal{O}_M} \check{\mathcal{E}})),$$

this finishes the proof for D-term.

Then we focus on the purely chiral case, the anti-chiral case is completely analogous. In this situation, the ring of differential operators D_M is replaced by

$$\mathcal{D}_M^{Ch} := \mathcal{O}_M \otimes_{\mathbb{C}} D_{N=1}^{Ch},$$

we only have to consider resolution in $\mathcal{D}_M^{Ch}$-module category.

Lemma 2.4.
Let

$$\mathcal{D}_c := \mathbb{C}[\theta^\alpha] \otimes_{\mathbb{C}} Sym^*(D_{N=1}^{Ch} \otimes_{\mathcal{O}_M} \check{\mathrm{C}})$$

be another D-module. The following sequence provides another $\mathcal{D}_M^{Ch}$-module resolution of $H^0(\mathcal{O}_{loc}(\mathcal{C}))$:

$$\Omega^*(\mathcal{D}_c u^1 u^2)/\sim \xhookrightarrow{\Phi} \Omega^*(\mathcal{D}_c \otimes_{\mathbb{C}} \left(\mathbb{C}[\bar{\theta}^1] u^2 \oplus \mathbb{C}[\bar{\theta}^2] u^1\right))/\sim \xrightarrow{\Phi} \Omega^*(\mathcal{D}_c \otimes_{\mathbb{C}} \mathbb{C}[\bar{\theta}^{\dot{\alpha}}])/\sim \longrightarrow H^0(\mathcal{O}_{loc}(\mathcal{C}))$$

Here u_i are formal odd variables, and the differential Φ is defined similarly:

$$\Phi : u^\alpha \to 1 - \bar{\theta}^{\dot{\alpha}} \otimes_{\mathcal{O}_M} \bar{D}_{\dot{\alpha}}$$

The relation $\sim$ denotes the identification:

$$\mathcal{D}_c \sim \theta^\alpha \otimes D_\alpha \cdot \mathcal{D}_c$$

Remark:

- As a $D_M^{Ch} - module,\ \bar{\theta}^{\dot{\alpha}}$ acts on$\mathcal{D}_c$ as $\frac{\partial}{\partial \bar{D}_{\dot{\alpha}}}$.
- We define $\bar{\mathcal{Q}}_{\dot{\alpha}}$ acts on $\mathcal{D}_c$ as $\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - u^k \bar{D}_{\dot{\alpha}} \frac{\partial}{\partial u^k}$ on the complex.

Lemma 2.5. *There is a quasi-isomorphism:*

$$H^*_{\bar{\mathcal{Q}}_1}(\mathcal{O}_{loc}(\mathcal{C})) \cong \Omega^*(H^*_{\bar{D}_1}(\mathcal{D}_c))\bar{\theta}^2 u^1/(1-\bar{\theta}^2 \otimes \bar{D}_{\dot{2}}, \sim).$$

Proof. The statement follows by computing the spectral sequence associated to the double complex introduced in the previous lemma. □

Making the equivalence relation $\sim$ explicit, we obtain the following quasi-isomorphism:

$$\Omega^*((\mathcal{D}_c/\sim), \bar{D}_1 + d_{DR}) \cong (\theta^1\theta^2\Omega^*(Sym^*(D^{Ch}_{N=1} \otimes \check{\mathcal{C}})), \bar{D}_1 + d_{DR}).$$

Since the spectral sequence associated with the complex:

$$(\theta^1\theta^2\Omega^*(Sym^*(D^{Ch}_{N=1}) \otimes \check{\mathcal{C}}), \bar{D}_1 + d_{dR}),$$

degenerates at the E_2 page, we deduce that:

$$H^*_{\bar{D}_1}(\mathcal{D}_c/\sim) \cong \theta^1\theta^2 H^*_{DR}(Sym^*(H^*_{\bar{D}_1}(D^{Ch}_{N=1} \otimes \check{\mathcal{C}}))).$$

Consequently, the computation of the cohomology reduces to evaluating:

$$H^*_{\bar{D}_1}(D^{Ch}_{N=1} \otimes \check{\mathcal{C}}),$$

or equivalently, its dual description:

$$H^*(\mathbb{C}[x^\mu, \theta^\alpha], \bar{D}_1).$$

Recall that:

$$\bar{D}_1 = 2\partial_{1\dot{1}}\theta^1 + 2\partial_{2\dot{1}}\theta^2, \quad \partial_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu.$$

A direct computation then shows that:

$$H\Big(\mathbb{C}[x^\mu, \theta^\alpha], \bar{D}_1\Big) = \mathbb{C}[x_0 - x_3, x_1 - ix_2],$$

or equivalently

$$H\Big(D^{N=1}_{\text{chiral}}, \bar{D}_1\Big) = \mathbb{C}[\partial_{1\dot{2}}, \partial_{2\dot{2}}].$$

Tracing the spectral sequence back to the original complex, we find that the corresponding cohomology class in $\mathcal{O}_{loc}(\mathcal{C})$ is represented by:

$$\int d^4xd^2\theta d^2\bar{\theta} f(x,\theta)\bar{\theta}^1\bar{\theta}^2 W(\mathcal{C},\partial_{1\dot{2}}\mathcal{C},\partial_{2\dot{2}}\mathcal{C}).$$

Imposing further the conditions of $\mathcal{Q}^\alpha$-closed and translation invariance, this expression reduces to:

$$\int d^4xd^2\theta W(\mathcal{C},\partial_{1\dot{2}}\mathcal{C},\partial_{2\dot{2}}\mathcal{C}), \tag{3}$$

which contributes a part of the F-term functional form.

Remark: Directly from the structural lemma, we have

$$H^*_{\bar{\mathcal{Q}}_2}(H^*_{\bar{\mathcal{Q}}_1}(\mathcal{O}_{loc}(\mathcal{C}))) = 0,$$

which means the two cohomology group relate by $(\mathcal{O}_{loc}(\mathcal{C}), \bar{\mathcal{Q}}_{\dot{1}} + \bar{\mathcal{Q}}_{\dot{2}})$.

For a SUSY-invariant chiral local functional S, the $\bar{\mathcal{Q}}_1$-closed condition implies that it can be written as:

$$S = S_0 + \bar{\mathcal{Q}}_1 S_1$$

where S_0 is represented by (3).

Applying the same argument as above, we may further decompose S_1 into a $\bar{\mathcal{Q}}_2$-cohomology part and -exact part, where cohomology part is represented by:

$$\int d^4xd^2\theta W(\mathcal{C},\partial_{1\dot{1}}\mathcal{C},\partial_{2\dot{1}}\mathcal{C})$$

Finally, we discuss the remaining double-exact component. Imply

$$\bar{\mathcal{Q}}_1 S_1 = \bar{\mathcal{Q}}_2 S_2,$$

we have the following lemma, which will be used to finish the proof.

Lemma 2.6.

$$H^*_{\bar{\mathcal{Q}}_1}(\mathcal{O}_{loc}(\mathcal{C})/Im\bar{\mathcal{Q}}_2) = \mho^*(M_1 \otimes_{\mathcal{O}_M} Sym^*(\mathbb{C}[\partial_\mu] \otimes_{\mathcal{O}_M} \check{\mathcal{E}}))$$

Proof. To compute this cohomology, we consider the resolution:

$$\begin{tikzcd}
\ldots\mathcal{O}_{loc} \arrow[r, "\mathcal{Q}_2"] & \mathcal{O}_{loc} \arrow[r, "\mathcal{Q}_2"] & \mathcal{O}_{loc} \arrow[r] & \mathcal{O}_{loc}/Im\mathcal{Q}_2 \\
\oplus & & & \\
H^*_{\bar{\mathcal{Q}}_2}(\mathcal{O}_{loc}) \arrow[uur, "j"] & & &
\end{tikzcd}$$

where j is the inclusion map. The associated spectral sequence has E1-page:

$$\begin{tikzcd}
H^*_{\bar{\mathcal{Q}}_1}(\mathcal{O}_{loc}) \arrow[r, "\mathcal{Q}_2"] & H^*_{\bar{\mathcal{Q}}_1}(\mathcal{O}_{loc}) \arrow[r, "\mathcal{Q}_2"] & H^*_{\bar{\mathcal{Q}}_1}(\mathcal{O}_{loc}) \arrow[r] & H^*_{\bar{\mathcal{Q}}_1}(\mathcal{O}_{loc}/Im\mathcal{Q}_2) \\
\oplus & & & \\
H^*_{\bar{\mathcal{Q}}_2}(\mathcal{O}_{loc}) \arrow[uur, "j"] & & &
\end{tikzcd}$$

and the differential d_1 acts as the inclusion:

$$j : H^*_{\bar{\mathcal{Q}}_{\dot{2}}} \to H^*_{\bar{\mathcal{Q}}_{\dot{1}}}$$

identity on $sym^*(\mathbb{C})$ and vanishes on the remaining components.

When right acting on $\mathcal{O}_{loc}(\mathcal{C})$:

$$\bar{\mathcal{Q}}_{\dot{1}} = -2\partial_{1\dot{1}}\frac{\partial}{\partial D_1} - 2\partial_{2\dot{1}}\frac{\partial}{\partial D_2},\ \bar{\mathcal{Q}}_{\dot{2}} = -2\partial_{1\dot{2}}\frac{\partial}{\partial D_1} - 2\partial_{2\dot{2}}\frac{\partial}{\partial D_2}.$$

Tracing the higher differentials, one observes that the differential d_n acts by replacing (n-1) factors of $(\partial_{1\dot{1}}\ or\ \partial_{2\dot{1}})\ to\ (\partial_{1\dot{2}}\ or\ \partial_{2\dot{2}})$.
And the spectrum sequence converges to:

$$\begin{tikzcd}
\ldots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & H^*_{\bar{\mathcal{Q}}_1}(\mathcal{O}_{loc}(\mathcal{C})/\ Im\bar{\mathcal{Q}}_{\dot{2}}) \\
 & \oplus & \oplus & & & \\
 & H^*_{\bar{\mathcal{Q}}_2}(\mathcal{O}_{loc})_{\geq 3} & H^*_{\bar{\mathcal{Q}}_2}(\mathcal{O}_{loc})_{\geq 2} & & &
\end{tikzcd}$$

In $H^*_{\bar{\mathcal{Q}}_1}(\mathcal{O}_{loc}(\mathcal{C})/\ Im\bar{\mathcal{Q}}_2)$, these terms are represented by:

$$\mho^*(M_1 \otimes_{\mathcal{O}_M} Sym^*(\mathbb{C}[\partial_\mu] \otimes_{\mathcal{O}_M} \check{\mathcal{E}}))$$

□

Since $\bar{\mathcal{Q}}_{\dot{1}}S_1 = \bar{\mathcal{Q}}_{\dot{2}}S_2$, if $[S_1] \neq 0$, it is represented by the lemma. If $[S_1] = 0$, it is $\bar{\mathcal{Q}}_{\dot{2}}$-exact, then the double-closed term is double-exact, which means S contributes to the D-term.

This completes the proof of the theorem. □

3 Super Yang-Mills Theory in classical BV formalism and its twist

3.1 D=4, $\mathcal{N}=1$ SYM

As a gauge theory with Lie algebra $\mathfrak{g}$ over M, its action functional is:

$$S_{SYM} = \frac{1}{16\pi} Im(\tau \int d^6 z\, Tr(W^\alpha W_\alpha) + c.c) \quad W_a := -\frac{1}{8}\bar{D}^2(e^{-2V} D_\alpha e^{2V}) \quad \tau := \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$$

Here, $d^6 z$ is short for $d^4 x d^2\theta$, and $c.c$ denotes the complex conjugate part. Tr stands for the Killing form of the Lie algebra $\mathfrak{g}$.

$$gauge\ transformation: \quad e^{2V} \to e^{i\bar{\Omega}} e^{2V} e^{-i\Omega}$$

Here, V is the vector field and Ω is the chiral field valued in $\mathfrak{g}$.

I will skip the introduction of the BV formalism(ref. [4]). Following our marks, $\mathcal{V}_g$ and $\mathcal{C}_g$ denote the field spaces of vector and chiral fields valued in $\mathfrak{g}$.

Definition 3.1. *(local BV complex)*

$$\mathcal{BV}_{SYM} := (O_{loc}(T^*[-1](\mathcal{V}_g \oplus \mathcal{C}_g[1]), \delta_{SYM})$$

[1] denotes the left shift in a graded vector space; T^* means the cotangent bundle. Specifically, the graded vector space looks like:

$$\mathcal{C}_g[1] \oplus \mathcal{V}_g \oplus \mathcal{V}_g^*[-1] \oplus \mathcal{C}_g^*[-2] := \mathcal{E}$$

$O(\mathcal{E})$ means the functional over the function space $\mathcal{E}$;loc means it must take this form:

$$F(\mathcal{E}_1, ...\mathcal{E}_n) = \int_M \alpha(j_z(\mathcal{E}_1), ..., j_z(\mathcal{E}_n)) : \mathcal{E}^{\otimes n} \to R$$

Here, j_z is the jet bundle of a sheaf at point z; α is a density-valued function on the jet bundles.

Remark 3.1.1: Here I also use the $\mathcal{E}$ to denote the sheaf.
Remark 3.1.2: The odd graded fields are anti-commutative, and the even ones are commutative in the graded vector space. In physics, we call the components in $\mathcal{E}$ from left to right: ghost, field, anti-field, and anti-ghost.
Remark 3.1.3: There is a natural Poisson bracket on the Kaszul-Tate complex, which is the Schouten bracket, we also call it the BV bracket. In particular, we have $\delta_{BV} = \{\_, S_{SYM}\}$.

3.1.1 Wess-Zumino gauge and reduce to gauge theory

Extending the gauge transformation by the BCH formula, we have:

$$\delta V = \frac{i}{2}(\bar{\Omega} - \Omega) + o(V)$$

By comparing the formula (1)(2), the first six terms in (2) can be canceled out through the 0-order in gauge.

Definition 3.2. *The Wess-Zumino gauge is to set: C, χ, M components in $\mathcal{V}$ equal to 0. Under this gauge:*

$$\mathcal{V} = \theta\sigma^{\mu}\bar{\theta}V_{\mu} + \bar{\theta}^2\theta^{\alpha}\lambda_{\alpha} + \theta^2\bar{\theta}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} + \theta^2\bar{\theta}^2 D$$

The higher order part looks like [V,...], you can't reach the WZ gauge just by transforming once, but the difference will reside in higher terms in (2), you can defer it to the later terms each time. Finally, we can reach the WZ gauge.

So in WZ gauge,

$$W_{\alpha} = -\frac{1}{4}\bar{D}^2 D_{\alpha}V + \frac{1}{4}\bar{D}^2([V, D_{\alpha}V]) = \lambda_{\alpha}(x) + 2D\theta_{\alpha} + i(\sigma^{\mu\nu}\theta)_{\alpha}F_{\mu\nu} - i\theta^2\sigma^{\mu}D_{\mu}\bar{\lambda}$$

Using the property:

$$\sigma_a\bar{\sigma}_b\sigma_c = (\eta_{ac}\sigma_b - \eta_{bc}\sigma_a - \eta_{ab}\sigma_c) + i\epsilon_{abcd}\sigma^d$$

Finally, we have:

$$S_{SYM} = \frac{1}{g^2}\int d^4x(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^{\mu}D_{\mu}\bar{\lambda} + 2D^2) + \frac{\theta}{32\pi^2}\int d^4x F_{\mu\nu}{}^{\star}F^{\mu\nu}$$

which is the canonical N=1 gauge theory with θ term. Since in perturbative level, gauge fixing doesn't change the BV complex, we have:

Lemma 3.1. *Under Wess-Zumino gauge, the two BV complexes*

$$(\mathcal{O}_{loc}(\mathcal{E}^{SYM}), \delta_{SYM}) \overset{quasi-iso}{\cong} (\mathcal{O}_{loc}(\mathcal{E}^{gauge}), \delta_{gauge})$$

3.1.2 Twist theory

Induced from $\mathcal{V}_{\mathfrak{g}}$, we have the SUSY transform on component fields:

$$\begin{aligned} \delta A_{\mu} &= \epsilon\sigma_{\mu}\bar{\lambda} + \lambda\sigma_{\mu}\bar{\epsilon} \\ \delta\lambda &= \epsilon D + (\sigma^{\mu\nu}\epsilon)F_{\mu\nu} \\ \delta D &= i\epsilon\sigma^{\mu}\partial_{\mu}\bar{\lambda} - i\partial_{\mu}\lambda\bar{\sigma}^{\mu}\bar{\epsilon} \end{aligned}$$

Transferring to the first-order formalism, we have:

$$\delta_{Q_\alpha} A_\mu = \sigma_{\mu\alpha\dot{\alpha}} \psi^{\dot{\alpha}}$$
$$\delta_{Q_\alpha} \psi_\beta = \epsilon_\alpha^\gamma B_{\gamma\beta}$$

We can choose any spinor as our twist charge, usually $Q := Q_1$. Since selecting a spinor is equivalent to choosing a complex structure, our structure group $Spin(4)$ now reduces to SU(2), which means that now Q is a U(1)-charge scalar but not a spinor as a derivation. This induces a second derivation in our BV complex:

$$\begin{tikzcd}
0 & 1 & 2 & 3 \\
 & \psi_- \arrow[r, "\not{D}"] \arrow[d, "Q"] & \psi'_+ & \\
\Omega^0 \arrow[r, "\mathrm{d}"] & \Omega^1 \arrow[r, "\mathrm{d}_+"] & \Omega^2_+ & \\
 & \Omega^2_+ \arrow[ur, "c\,\mathrm{Id}"] \arrow[r, "\mathrm{d}"] \arrow[d, "Q"] & \Omega^3 \arrow[r, "\mathrm{d}"] & \Omega^4 \\
 & \psi_+ \arrow[r, "\not{D}"] & \psi'_- &
\end{tikzcd}$$

The new derivation now has the wrong degree. We can add a BV degree 1, U(1) degree -1 element t to that. And we have:

$$(\mathcal{O}_{loc}(\mathcal{E}^{gauge})((t)), \delta_{gauge} + tQ)$$

$$\begin{tikzcd}
0 & 1 & 2 & 3 \\
\psi_- \arrow[r, "\not{D}"] \arrow[dr, "tQ"] & \psi'_+ \arrow[dr, "tQ"] & & \\
\Omega^0 \arrow[r, "\mathrm{d}"] & \Omega^1 \arrow[r, "\mathrm{d}_+"] & \Omega^2_+ & \\
 & \Omega^2_+ \arrow[ur, "c\,\mathrm{Id}"] \arrow[r, "\mathrm{d}"] \arrow[dr, "tQ"] & \Omega^3 \arrow[r, "\mathrm{d}"] \arrow[dr, "tQ"] & \Omega^4 \\
 & & \psi_+ \arrow[r, "\not{D}"] & \psi'_-
\end{tikzcd}$$

In short, what we did was construct a Q-cohomology by twisting and blending it with the BV-comology in the correct degree.

What's more, choosing Q not only gives us a complex structure, but also the following isomorphism:

$$\Gamma : \psi^{-} \xrightarrow{\cong} \Omega^{1,0} \tag{4}$$

Here, we substitute Q with the following left-spinor part:

$$V_{\mu} \rightarrow (\sigma^{\mu} V_{\mu})_{\alpha\dot{\alpha}}$$

And,

$$Y : \Omega^{0} \cdot w \oplus \Omega^{0,2} \xrightarrow{\cong} \psi^{+} \tag{5}$$

Here, we use the isomorphism and pair one of ψ_{+} with Q.

$$\Omega^{2}_{+} \cong Sym^{2}(\psi^{+})$$

With these isomorphisms, we have:

Theorem 3.1. *(Costello 4.0.2 [1]) The twisted N = 1 gauge theory on $\mathbb{C}^2$ is equivalent to the holomorphic BF theory.*

In the same paper, costello also proves that:

Theorem 3.2. *(Costello 4.1.1 [1]) The twisted N = 1 theory admits a unique quantization, compatible with certain natural symmetries, on any Calabi-Yau surface X.*

4 Quantization of $\mathcal{N} = 1$ SYM on Superspace

As a *pre-theory*([4] section 13) on R^4 , $\mathcal{N} = 1$ SYM given by following second-order BV complex:

$$\begin{tikzcd}
-1 & 0 & 1 & 2 \\
\mathcal{C}_g \arrow[r, "i"] & \mathcal{V}_g \arrow[dr, red, "\bar{D}^2 D_\alpha"] & \mathcal{V}^*_g \arrow[r, "\bar{D}^2"] \arrow[dr, "D^2"] & \mathcal{C}^*_g \\
\oplus & \oplus & \oplus & \oplus \\
\bar{\mathcal{C}}_g \arrow[uur, "-i"] & \mathcal{B}_\alpha \arrow[r, "="] \arrow[uur, "D^\alpha"] & \mathcal{B}^*_\alpha & \bar{\mathcal{C}}^*_g
\end{tikzcd}$$

And Q^{GF} given by:

$$\begin{tikzcd}
-1 & 0 & 1 & 2 \\
\mathcal{C}_g & \mathcal{V}_g \arrow[l, "\bar{D}^2 D^2"'] \arrow[ddl, "D^2\bar{D}^2"] & \mathcal{V}_g^* \arrow[ddl, red, "\bar{D}^2 D_\alpha"] & \mathcal{C}_g^* \arrow[l, "D^2"'] \\
\oplus & \oplus & \oplus & \oplus \\
\bar{\mathcal{C}}_g & \mathcal{B}_\alpha & \mathcal{B}_\alpha^* \arrow[uul, "D^\alpha"] & \bar{\mathcal{C}}_g^* \arrow[uul, "\bar{D}^2"]
\end{tikzcd}$$

By simple calculation, we have the property $[Q, Q^{GF}] = \Box$.

The second-order SYM action is:

$$S_{SYM} = \frac{1}{16\pi} Im(\tau \int d^6 z\, Tr(W^\alpha B_\alpha + B^\alpha B_\alpha) + c.c)$$

Quantization means we can assign R^4 a factorization algebra Obs^q, which locally(for U a open disk) looks like following DGLA at level Φ:

$$Obs^q_\Phi(U) \cong (\oplus_n Sym^n(T^*[-1](\mathcal{V}_g(U) \oplus \mathcal{C}_g(U)[1]))^*[[\hbar]], Q + \hbar\triangle_\Phi + \{I[\Phi], \underline{}\})$$

Here we define parametrix following [5] section 7.2.4 as an element:

$$\Phi \in \bar{\mathcal{E}}(U) \widehat{\otimes}_\pi \bar{\mathcal{E}}(U)$$

Satisfying several properties from elliptic regularity. $\bar{\mathcal{E}}(U)$ means the distributional section of $\mathcal{E}(U)$. Define:

$$P(\Phi) = \frac{1}{2} Q^{GF} \Phi \in \overline{\mathcal{E}} \widehat{\otimes}_\pi \overline{\mathcal{E}}.$$

This is the propagator associated with Φ. We let

$$K_\Phi = K_{\mathrm{Id}} - QP(\Phi).$$

K_{Id} is the kernel of $Id : \mathcal{E} \to \bar{\mathcal{E}}$

$$\triangle_\Phi := \partial_{K_\Phi} : \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E})$$

$$\{I, J\}_\Phi := \triangle_\Phi(IJ) - (\triangle_\Phi I)J - (-1)^{|I|} I \triangle_\Phi J$$

on the space $\mathcal{O}(\mathcal{E})$.

Also, we demand the quantum field theory $I[\Phi]$ corresponds to our classical field theory $I_0 = S_{SYM}$, that means it satisfies:

- $I[\Phi] = W(P(\Phi) - P(\Psi), I[\Psi])$ (1)
- $\lim_{\Phi \to 0} I[\Phi] - I_0 = 0 \; mod \; \hbar$ (2)

- $I[\Phi]$ satisfies *quantum master equation* for $\forall\ \Phi$ (3)

Condition (1)(2)(3) are satisfied at the same time if we can find a lift $\tilde{I}_0$ of I_0 at scale 0, that satisfies quantum master equation. Since we can define:

$$I[\Phi] := W(P(\Phi), \tilde{I}_0)$$

And we have the following lemma:

Lemma 9.2.2 [4]

If $I[\Phi]$ satisfies the Φ-QME, then $I[\Psi] = W(P(\Psi) - P(\Phi), I[\Phi])$ satisfies the Ψ-QME.

This follows from the identity

$$\left[Q, \partial_{P(\Phi)} - \partial_{P(\Psi)}\right] = \triangle_\Psi - \triangle_\Phi$$

Then

$$(Q + \hbar\triangle_\Psi)e^{I[\Psi]/\hbar} = (Q + \hbar\triangle_\Psi)e^{\hbar\partial_{P(\Psi)-P(\Phi)}}e^{I[\Phi]/\hbar} = e^{\hbar\partial_{P(\Psi)-P(\Phi)}}(Q + \hbar\triangle_\Phi)e^{I[\Phi]/\hbar}$$

Naturally, $I[\Phi] := W(P(\Phi), I_0)$ satisfies mod $\hbar$ *quantum master equation*. To lift it to a theory satisfies mod $\hbar^2$ *quantum master equation*, or more generally, to lift a $\hbar^n$ theory $I[L]$ to a $\hbar^{n+1}$, we choose any lift $\tilde{I}[L]$ and define its obstruction at scale L as:

$$O_{n+1}[L] = \hbar^{-n-1}\left(Q\tilde{I}[L] + \frac{1}{2}\{\tilde{I}[L], \tilde{I}[L]\}_L + \hbar\Delta_L\tilde{I}[L]\right)$$

And we have the following property:

- The obstruction of the lift $\tilde{I}[L]$: $O_{n+1} \in H^1_{Q+\{I_0,-\}}(\mathcal{O}_{loc}(\mathcal{E}))$
- When the obstruction vanishes, the classification set of the lift equals $H^0_{Q+\{I_0,-\}}(\mathcal{O}_{loc}(\mathcal{E}))$

In this paper, we want to show that when $I_0 = S_{SYM}$, $H^1_{Q+\{I_0,-\}}(\mathcal{O}_{loc}(\mathcal{E})) = 0$, then our quantization is well-defined.

The proof of these properties is from *Lemma 11.1.1* [4].

In short, the first property is obtained easily from DGLA:

Let $\mathfrak{g}$ be a differential graded Lie algebra, and let $X \in \mathfrak{g}$ be an odd element. Let

$$O(X) = \mathrm{d}_\mathfrak{g}X + \frac{1}{2}[X, X].$$

Then,

$$\mathrm{d}_\mathfrak{g}O(X) + [X, O(X)] = 0.$$

That means

$$QO_{n+1} + \{I_0, O_{n+1}\} = 0$$

For $\tilde{I}[L]$ $\forall$lift, then any other lift is given by $\tilde{I}[L]+\hbar^{n+1}J[L]$. The obstruction vanishes if only:

$$QJ[L]+\{I_0[L],J[L]\}=O_{n+1}[L] \quad \square$$

Now we calculate $H^*_{Q+\{I_0,\_\}}(\mathcal{O}^{SUSY}_{loc}(\mathcal{E}))$.

Lemma 4.1. *(Perturbative Non-renormalization of $\mathcal{N}=1$ SYM)*

In perturbation SYM theory, quantum corrections can only generate counterterms of D-term type, i.e. integrals over the full superspace of the form

$$\int d^4\theta\, \mathcal{K}(\Phi,\bar{\Phi},V),$$

Non-perturbative effects, such as instantons, may generate effective F-terms, but these are absent in perturbation theory.

Using this lemma, we only have to calculate D-term, and recall in *Lemma 4.1*, we have proved that

$$\mathcal{O}_{loc}(\mathcal{E})^{SUSY}=\mho^*(\mathbb{C}\otimes Sym^*(\mathbb{C}[\partial_\mu,D_\alpha,\bar{D}_{\dot\alpha}]\otimes\check{\mathcal{E}}))$$

And in SYM,

$$\mathcal{E}:=\mathcal{C}_g[1]\oplus\mathcal{V}_g\oplus\mathcal{V}^*_g[-1]\oplus\mathcal{C}^*_g[-2]$$

as a Q-complex. So the $Q+[I_o,\_\_]$ cohomology is quasi-iso to the $Q+[I_o,\_\_]$ cohomology of the following Super De Rham-complex:

$$\begin{array}{ccc}
\ldots\to\wedge^1\mathbb{R}^4\otimes\mathbb{C}[\theta^i\otimes D^i]\otimes Sym^*(\ldots) & \xrightarrow{\partial_i} & \mathbb{C}[\theta^i\otimes D^i]\otimes Sym^*(\mathbb{C}[\partial_\mu,D_\alpha,\bar{D}_{\dot\alpha}]\otimes\check{\mathcal{E}}) \\
 & & \Phi\uparrow \\
\ldots & \xrightarrow{\partial_i} & (\oplus_j(\mathbb{C}[\theta^i\otimes D^i](i\neq j))\epsilon^j)\otimes_{\tilde{D}_{N=1}} Sym^*(\mathbb{C}[\partial_\mu,D_\alpha,\bar{D}_{\dot\alpha}]\otimes\check{\mathcal{E}}) \\
 & & \Phi\uparrow \\
 & & \ldots
\end{array}$$

As the $Q+[I_0,\_\_]$ only acts on Sym^* component, so we calculate $H_{Q+[I_0,\_\_]}(Sym^*(\ldots))$ as the E_1-page.

Now we filter the Sym^* component by its $*$-index, under this filtration, the E_1-page given by $Sym^i(\mathbb{C}[\partial_\mu,D_\alpha,\bar{D}_{\dot\alpha}]\otimes H^j_Q(\check{\mathcal{E}}))\cong Sym^i(\mathbb{C}[\partial_\mu,D_\alpha,\bar{D}_{\dot\alpha}]\otimes\check{H}^j_Q(\mathcal{E}))$.

Recall as in [4]*Lemma 6.7.1*, as a L^∞ CE cohomology, $H_{Q+\{I_0,\_\}}(\mathcal{O}^{SUSY}_{loc})$ can be calculated localized to its fiber $(\mathcal{O}_{loc})_0:=Sym^*(\mathbb{C}[\partial_\mu,D_\alpha,\bar{D}_{\dot\alpha}]\otimes\check{\mathcal{E}}_0)$, because of its SUSY invariant.

Then, just by Talor extending $\mathcal{C}$ and $\mathcal{V}$. For example, using the extension (1)(2) in section 2.1, we have:

$$e_1 = C(x) - \frac{1}{4}\theta^2\bar{\theta}^2\Box C(x) = Q\Big(\phi(x) + \mathrm{i}\theta^\mu\theta\partial_\mu\phi(x) + \frac{1}{4}\theta^2\bar{\theta}^2\Box\phi(x)\Big) = Qs_1 \text{ when } \phi \text{ is pure imagine}$$

$$e_2 = \theta\sigma^\mu\bar{\theta}A_\mu = Q\Big(\phi(x) + \mathrm{i}\theta^\mu\theta\partial_\mu\phi(x) + \frac{1}{4}\theta^2\bar{\theta}^2\Box\phi(x)\Big) = Qs_2 \text{ when } \phi \text{ is pure real}$$

$Here$ a bit different from (1)(2), $\phi, \psi, F, \chi, \bar{\chi}, \lambda, \bar{\lambda}, M \in \mathbb{C}[[x_1, \ldots, x_4]] \quad C, D, A_\mu \in \mathbb{R}[[x_1, \ldots, x_4]]$

Finally, we have $H_Q^*(\mathcal{E}) = \mathfrak{g}[1]$. As a result, $H_{Q+[I_0,\_\_]}(Sym^*(...)) \cong H_{CE}^*(\mathfrak{g})$. When considering semi-simple $\mathfrak{g}$, the degree 1 component vanishes, and the degree 0 component is just $\mathbb{C}$, which means there is no obstruction and quantization is unique up to a constant. $\Box$

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References

[1] Kevin Costello. Supersymmetric gauge theory and the yangian, 2013.

[2] A. Borel. *Algebraic D-modules*. Perspectives in mathematics. Academic Press, 1987.

[3] R. Catenacci, P. A. Grassi, and S. Noja. Superstring field theory, superforms and supergeometry, 2018.

[4] Kevin Costello. *Renormalization and Effective Field Theory*. American Mathematical Society, 2010.

[5] Kevin Costello and Owen Gwilliam. *FIRST EXAMPLES OF FIELD THEORIES AND THEIR OBSERVABLES*, page 87–88. New Mathematical Monographs. Cambridge University Press, 2016.