

Quantization of $\mathcal{N} = 1$ SYM theory

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Abstract

In this paper, we calculate the BV cohomology of $N=1$ Pure Super Yang-Mills field theory on $R^{4|4}$ space. The result shows that the $N = 1$ theory also admits a unique quantization in the semi-simple Lie algebra case, compatible with SUSY.

Contents

1	Introduction	2
2	SUSY algebra and representation	2
2.1	Sigma Model	2
2.2	Spinors in various dimensions	3
2.3	Symmetries	3
2.3.1	Poisson Symmetry	3
2.3.2	Fermion Symmetry	3
2.3.3	Super Symmetry	4
2.4	SUSY Algebra and superspace	5
2.5	SUSY Representation and superfields on D=4	7
2.5.1	$N=1$ massive	7
2.5.2	$N=1$ massless	8
2.5.3	$N=2$ massive and massless	8
2.6	Superfields	9
3	Super Yang-Mills Theory in BV formalism and its twist	11
3.1	$D=4, \mathcal{N} = 1$ SYM	12
3.1.1	Wess-Zumino gauge and reduce to gauge theory	13
3.1.2	Twist theory	14
4	SUSY-invariant Local Functional	16

1 Introduction

In [1], Costello has shown that the twisted $N=1$ gauge theory admits unique quantization on any Calabi-Yau manifold and calculated out its algebraic structure. With the same spirit, using the superfields construction and BV cohomology, we show in this paper that the pure SYM theory has no quantization obstruction; thus, it also admits a unique quantization. Further, we try to calculate some observables structure. In Section 2, we summarize the Super Symmetry Theory (SUSY) construction. In Section 3, we give an introduction to SYM construction in BV formalism and review its twist. In section 4, we give a general construction of a SUSY invariant functional on $R^{4|4}$. Section 5 is the quantization and calculation of the local BV cohomology. We highly recommend readers who are familiar with superspace construction go directly to section 4 to see the final construction.

2 SUSY algebra and representation

2.1 Sigma Model

To simplify, we consider a general field theory in D space-time Σ whose Lagrangian functional description contains only scalar fields ϕ^i and spinors (fermion fields) ψ^a :

$$\mathcal{L} = \mathcal{L}(\phi, \psi)$$

It can be any functional $\mathcal{L} \in \mathcal{O}(\mathcal{E} \oplus \mathcal{S})$, here we suppose ϕ^i is a section of a vector bundle over Σ and denote the section set as \mathcal{E} and the section set of the dual bundle as $\check{\mathcal{E}}$; and ψ^a is a section of the spinor bundle \mathcal{S} , the set denoted as \mathcal{S} dual sections as $\check{\mathcal{S}}$.

So $\mathcal{O}(\mathcal{E} \oplus \mathcal{S}) := \text{Sym}^*(\check{\mathcal{E}}) \otimes \wedge^*(\check{\mathcal{S}})$, but if we further impose the locality property—which means the functional is an integration pointwise, corresponding to causality in physics, after Taylor extension on ψ , we have:

$$\begin{aligned} S &= \int_{R^D} \mathcal{L} d^D x \\ \mathcal{L} &= -\frac{1}{2} g_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j + b_{ij}(\phi) \epsilon^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j + V(\phi) \\ &\quad + i h_{ab}(\phi) \bar{\psi}^a \gamma^\mu \partial_\mu \psi^b + i \tilde{h}_{ab}(\phi) \bar{\psi}^a \gamma_3 \gamma^\mu \partial_\mu \psi^b \\ &\quad + k_{abi}(\phi) \bar{\psi}^a \gamma^\mu \psi^b \partial_\mu \phi^i + \tilde{k}_{abi}(\phi) \bar{\psi}^a \gamma^\mu \gamma_3 \psi^b \partial_\mu \phi^i \\ &\quad + y_{ab}(\phi) \bar{\psi}^a \psi^b + \tilde{y}_{ab}(\phi) \bar{\psi}^a \gamma_3 \psi^b \\ &\quad + s_{abcd}(\phi) \bar{\psi}^a \psi^c \bar{\psi}^b \psi^d + \dots (b_{ij}, \tilde{h}, \tilde{k} \neq 0, \text{only when } D = 2) \end{aligned}$$

The Sigma model is a map : $\Sigma \rightarrow M$, $x \mapsto \phi^i(x)$, which gives a geometric explanation of this functional:

We regard $\{\phi^i\}$ as the coordinate functions on M , since g_{ij} is positive-definite and transfers as a metric under reparametrization, it defines a Riemannian metric on M .

For $\{\psi^a\}$, you know it has an index of the Spinor bundle, but its metric depends on ϕ , so it's a section of $S \otimes \phi^*V$, S is the Spinor bundle and ϕ^*V is the pull-back of a vector bundle V over M . When $D=2$, you can use the chiral decomposition to define h_\pm [2]; the result is the same as in other dimensions to define $w_i{}^a{}_b := h^{ac}k_{cbi}$, and writing:

$$D_\mu \psi^a := \partial_\mu \psi^a + \partial_\mu \phi^a w_i{}^a{}_b \psi^b$$

as the covariant derivation. Since the symmetry part of k_{cbi} is arbitrary, you can always get $D_\mu(h_{ab}) = 0$, which means it's compatible with the metric.

2.2 Spinors in various dimensions

By definition, spinors are sections of the Spinor bundle, which is the associated bundle constructed from the principal $\text{Spin}(n)$ -bundle and the Clifford algebra representation space. The Clifford algebra is:

$$Cl(V) := \bigoplus_k V^{\otimes k} \quad V = \mathbb{R}[\Gamma^\mu] \quad \{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$$

Its representation depends on the dimension. For $D=\text{even}$, its representation space $S = S^+ \oplus S^-$ is reducible as a representation of the Lorentz group $SO(d-1, 1)$, and we call each part of the sections the Weyl spinors. For $D=\text{odd}$, it's irreducible, and for a physical purpose, we choose the real sections called Majorana spinors.

2.3 Symmetries

As we mentioned, symmetries on Lagrangians impose constraints on the target space's geometric structure.

2.3.1 Poisson Symmetry

Suppose we have a symmetry group G that acts on scalar fields $\{\phi^i\}$. Preserving the Lagrangian is equivalent to preserving the metric $g_{ij}(\phi)$, so $G \subset Iso(M)$ and generators are Killing vector fields.

2.3.2 Fermion Symmetry

Suppose we have a symmetry group G that acts on fermion fields $\{\psi^a\}$. To match the metric h_{ab} , $G \subset U(m)$ (or $SO(m)$ in Majorana), acting as:

$$\psi^a \mapsto M^a{}_b(\phi) \psi^b$$

d	Majorana	Weyl	Majorana–Weyl
2	yes	self	yes
3	yes	–	–
4	yes	complex	–
5	–	–	–
6	–	self	–
7	–	–	–
8	yes	complex	–
9	yes	–	–
10	yes	self	yes
11	yes	–	–
12	yes	complex	–

Table 1: Dimensions in which various conditions are allowed for $SO(d - 1, 1)$ spinors. A dash indicates that the condition cannot be imposed. For the Weyl representation, it is indicated whether these are conjugate to themselves or to each other (complex).

It's a global symmetry iff:

$$D_\mu M^a{}_b = 0$$

Which means it's a flat section of $\mathcal{V} \otimes \check{\mathcal{V}}$, and for most cases, the Fermi number operator is the unique solution, which means most of the fermion symmetries are gauge theories. We refer to say under this gauge symmetry, the connection is twisted to:

$$D_\mu \mapsto D\mu - A_\mu{}^a{}_b(\phi)\psi^b$$

I will denote the fiber of the Spinor bundle as V . What we did was do the fiber product of $Spinor(n) \rightarrow M$ and an associated bundle $G \times_G End(V) \rightarrow M$. That means you twist the Spinor bundle by an associated bundle, and the new connection is that.

2.3.3 Super Symmetry

The next question is whether we could blend the two parts of symmetry. An observation is that the symmetry is split because, in our Sigma model, we separate it as an underlying manifold and fiber. What we need is to extend the spacetime manifold and give a map to the bundle. However, we must respect the anti-symmetry of fermion fields. That means we need fermion coordinates and the super-Poincaré group.

2.4 SUSY Algebra and superspace

Now for a \mathbb{Z}_2 -graded super Lie algebra, $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$. If the Poincaré algebra $\mathcal{P} \subset \mathcal{G}_0$, then by S-matrix analysis [3], $\mathcal{G}_0 = \mathcal{P} \oplus \mathcal{T}$.

So we impose $\mathcal{G}_0 = \mathcal{P} := \mathbb{C}[P_\mu, M_{\mu\nu}]$ with Lie algebra:

$$[P_\mu, P_\nu] = 0 \quad [M_{ab}, P_c] = i\eta_{ac}P_b - (a \leftrightarrow b) \quad [M_{ab}, M_{cd}] = i\eta_{ac}M_{bd} - i\eta_{ad}M_{bc} - (a \leftrightarrow b)$$

Since $[\mathcal{G}_0, \mathcal{G}_1] \subset \mathcal{G}_1$, \mathcal{G}_1 is a representation space of \mathcal{G}_0 , so elements of \mathcal{G}_1 must be spinors. Impose $\mathcal{G}_1 = \mathbb{C}[Q_\alpha^a] = \mathbb{C}[Q_\alpha^a, \bar{Q}_\dot{\alpha}^a]$ (*decompose to weyl when D = 2m*). α is the spinor index, and a is the number of spinors.

Survey on the structure constant [4] shows the Lie structure must be (As an example D=4):

$$\{Q_\alpha^a, \bar{Q}_\dot{\beta}^b\} = 2\delta^{ab}(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu \quad [M_{\mu\nu}, Q_\alpha^a] = i(\sigma_{\mu\nu})_\alpha^\beta Q_\beta^a \quad [P_\mu, Q_\alpha^a] = 0 \quad (\sigma_{\mu\nu} := -\frac{1}{4}(\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu))$$

$$\bar{\sigma}^\mu := (1, -\sigma_1, -\sigma_2, -\sigma_3)$$

More generally, you can replace σ^μ with $\Gamma^\mu C$ in the Majorana case; with $P_+ \Gamma^\mu C$ (P_+ is Weyl projection to the Weyl decomposition case, and add central charges...).

D		
3	Supercharges Reality Superalgebra Automorphism(R-symmetry)	Majorana $(Q_\alpha^a)^\dagger = Q_\alpha^a$ (*) $\{Q_\alpha^a, Q_\beta^b\} = 2\delta^{ab}(\Gamma^\mu \Gamma_0)_{\alpha\beta} P_\mu$ $\text{SO}(N)$
4	Supercharges Reality Superalgebra R-symmetry	Weyl $(Q_\alpha^a)^\dagger = \bar{Q}_{\dot{\alpha}}{}_a$ $\{Q_\alpha^a, \bar{Q}_{\dot{\beta}}{}_b\} = 2\delta^a_b(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu$ $\{Q_\alpha^a, Q_\beta^b\} = \epsilon_{\alpha\beta} Z^{ab}$ $\text{U}(N)$
5	Supercharges Reality Superalgebra R-symmetry	Symplectic–Majorana $(Q_\alpha^a)^\dagger = \Omega_{ab} B_\alpha{}^\beta Q_\beta^b$ $\{Q_\alpha^a, Q_\beta^b\} = 2\Omega^{ab}(\Gamma^\mu C)_{\alpha\beta} P_\mu$ $\text{Sp}(N)$
6	Supercharges Reality Superalgebra R-symmetry	Symplectic–Majorana–Weyl $(Q_\alpha^a)^\dagger \equiv \bar{Q}_{\dot{\alpha}}{}_a = \Omega_{ab} B_\alpha{}^\beta Q_\beta^b$ $\{Q_\alpha^a, Q_\beta^b\} = 2\Omega^{ab}(\Sigma^\mu)_{\alpha\beta} P_\mu$ $\text{Sp}(N_R) \times \text{Sp}(N_L)$
7	Supercharges Reality Superalgebra R-symmetry	Symplectic–Majorana $(Q_\alpha^a)^\dagger = \Omega_{ab} B_\alpha{}^\beta Q_\beta^b$ $\{Q_\alpha^a, Q_\beta^b\} = 2\Omega^{ab}(\Gamma^\mu C)_{\alpha\beta} P_\mu$ $\text{Sp}(N)$
8	Supercharges Reality Superalgebra R-symmetry	Weyl $(Q_\alpha^a)^\dagger = \bar{Q}_{\dot{\alpha}}{}_a$ $\{Q_\alpha^a, \bar{Q}_{\dot{\beta}}{}_b\} = 2\delta^a_b(\Sigma^\mu)_{\alpha\dot{\beta}} P_\mu$ $\{Q_\alpha^a, Q_\beta^b\} = C_{\alpha\beta} Z^{ab}$ $\text{U}(N)$
9	Supercharges Reality Superalgebra R-symmetry	Majorana $(Q^a)^\dagger = C(\bar{Q}^a)^t$ $\{Q_\alpha^a, Q_\beta^b\} = 2\delta^{ab}(\Gamma^\mu C)_{\alpha\beta} P_\mu$ $\text{SO}(N)$
10	Supercharges Reality Superalgebra R-symmetry	Majorana–Weyl $(Q_\alpha^a)^\dagger = Q_\alpha^a$ $\{Q_\alpha^a, Q_\beta^b\} = 2\delta^{ab}(\Sigma^\mu)_{\alpha\beta} P_\mu$ $\{Q_\alpha^a, \bar{Q}_\beta{}^b\} = CC$ $\text{SO}(N_R) \times \text{SO}(N_L)$

Table 2: (1)CC stands for "central charges." (2) (*) holds in a Majorana rep. In a general rep. we have $Q^\dagger = BQ^T$, with $B = C\gamma^0$, where C is the charge-conjugation matrix. The same expression holds for the B matrices in $D = 5, 6, 7$; in the symplectic case we have $B^*B = -1$, in the Majorana one $B^*B = +1$. [2]

Remark: R-Symmetry means the freedom to choose the Supercharge(i.e., the Super-algebra still works after transformation). This symmetry is similar to the Fermion Symmetry, since the Supercharges are spinors. To this point, we usually equip the spacetime with an R-bundle to indicate the R-symmetry.

Definition 2.1 (*Super Poincaré Group and Superspace*)

$$\mathcal{SP} := \exp(i(-x^\mu P_\mu + \frac{1}{2} K^{ab} M_{ab} + \theta_a^\alpha Q_\alpha^a + \bar{\theta}_a^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}}^a)) \quad (x^\mu, K^{ab} \in \mathbb{R}; \theta^\alpha, \bar{\theta}^{\dot{\alpha}} \in \mathbb{C}_a)$$

$R^{n|s(n)} := \mathcal{SP}/SO(n-1, 1) \quad s(n) \text{ is the number of spinors in } n\text{-dimension.}$

The action of \mathcal{SP} on $R^{n|s(n)}$ is given by:

$$\rho(A)(B) = e^A e^B \quad A \in \mathcal{SP}, B \in R^{n|s(n)}$$

Prop 2.1.1 By BCH formula,(D=4) $e^A \cdot e^B = e^{A+B+\frac{1}{2}[A,B]+\dots}$, we have:

Translation:

$$P_\mu : (x, \theta, \bar{\theta}) \mapsto (x^\mu + a^\mu, \theta, \bar{\theta})$$

Laurantz transformation:

$$\begin{aligned} x'^a &= (e^K)_b^a x^b \\ \theta'_\alpha &= \left(\exp\left(\frac{1}{2} K^{ab} \sigma_{ab}\right) \right)_\alpha^\beta \theta_\beta \\ \bar{\theta}'^{\dot{\alpha}} &= \left(\exp\left(\frac{1}{2} K^{ab} \tilde{\sigma}_{ab}\right) \right)_{\dot{\beta}}^{\dot{\alpha}} \bar{\theta}^{\dot{\beta}}. \end{aligned}$$

Supersymmetry transformation:

$$x'^a = x^a - i\epsilon \sigma^a \bar{\theta} + i\bar{\theta} \sigma^a \bar{\epsilon} \quad (1)$$

$$\theta'^\alpha = \theta^\alpha + \epsilon^\alpha \quad \bar{\theta}'_{\dot{\alpha}} = \bar{\theta}_{\dot{\alpha}} + \bar{\epsilon}_{\dot{\alpha}}. \quad (2)$$

2.5 SUSY Representation and superfields on D=4

Since we primarily discuss the D=4 manifold in this paper, we provide the representation of the SUSY Algebra over it.

2.5.1 N=1 massive

In the rest frame, we have: $p^\mu = (m, 0, 0, 0)$. So the SUSY Algebra is:

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2m\delta_{\alpha\dot{\beta}} \quad \{Q_\alpha, Q_\alpha\} = \{\bar{Q}_{\dot{\beta}}, \bar{Q}_{\dot{\beta}}\} = 0 \quad [M_{12}, Q_\alpha] = \frac{1}{2} Q_\alpha$$

So it has two $SU(2)$ algebraic structures, the representation is:

$$\begin{cases} |-\frac{1}{2}\rangle \oplus 2|0\rangle \oplus |\frac{1}{2}\rangle & \text{chiral multiplet} \\ | -1\rangle \oplus 2|-\frac{1}{2}\rangle \oplus 2|0\rangle \oplus 2|\frac{1}{2}\rangle \oplus |1\rangle & \text{vector multiplet} \end{cases}$$

Remark: These two multiplets are irreducible representations of the SUSY algebra, although we haven't proven that. Differing from the Poincaré algebra, the spin J^2 is no longer a Casimir operator; instead, we construct C'_2 , called superspin.

$$B_\mu = W_\mu - \frac{1}{4} \bar{Q}_{\dot{\alpha}} (\bar{\sigma}_\mu)^{\dot{\alpha}\beta} Q_\beta \quad C_{\mu\nu} = B_\mu P_\nu - B_\nu P_\mu \quad C'_2 = C_{\mu\nu} C^{\mu\nu}$$

2.5.2 N=1 massless

We will see later that this case corresponds to the gauge theory we discuss.

In the "rest frame" ($E, 0, 0, E$):

$$\{Q, \bar{Q}\} = \begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix}$$

So there is only one generate-annihilate operator pair, with CPT the representation is:

$$\begin{cases} |-\frac{1}{2}\rangle \oplus 2|0\rangle \oplus |\frac{1}{2}\rangle & \text{chiral multiplet} \\ | -1\rangle \oplus |-\frac{1}{2}\rangle \oplus |\frac{1}{2}\rangle \oplus |1\rangle & \text{vector multiplet} \end{cases}$$

The vector multiplet corresponds to a vector field; it consists of one gauge vector field and one fermion. This is the gauge theory we will discuss.

2.5.3 N=2 massive and massless

In $N=2$, the crucial difference from $N=1$ is the central charge Z , which leads to the BPS boundary in the massive case: Consider $U = Q_1^1 + \lambda \bar{Q}_2^1$, as it means the composition of SUSY transformations. We have:

$$\{U, U^\dagger\} \geq 0 \Rightarrow m \geq |Z|$$

$N = 2$			
Massive $Z=0$	SUSY Algebra scalar multiplet	$\{Q_\alpha^A, \bar{Q}_\beta^B\} = 2m\delta_{\alpha\beta}\delta^{AB}$ $5\phi + 4\psi + 1A_\mu$	
Massive $Z \neq 0$	SUSY Algebra m>Z scalar multiplet m=Z scalar multiplet m≥Z	$\{Q^A, \bar{Q}^B\} = 2\delta^{AB} \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$ $\{Q^A, Q^B\} = \delta^{AB} \begin{pmatrix} 0 & -Z \\ Z & 0 \end{pmatrix}$ same as Massive $Z=0$	
Massless	SUSY Algebra scalar multiplet vector multiplet	$\{Q^A, \bar{Q}^B\} = \delta^{AB} \begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix}$ $4\phi + 2\psi$ $2\phi + 2\psi + 1A_\mu$	same as $N=1$ massive chiral prohibit

Table 3: ϕ means the scalar representation under the Lorentz group, ψ stands for $spin - \frac{1}{2}$ representation, A_μ stands for $spin - 1$ representation.

2.6 Superfields

Just like fermion fields as $Spin - \frac{1}{2}$ representations of the Lorentz algebra, we now construct superfields as representations of the SUSY Algebra.

Following previous notes, we use Σ to denote the spacetime manifold and $\tilde{\Sigma}$ to denote the extended superspace in section 2.4. Now, consider the complex function bundle F over $\tilde{\Sigma}$, locally we have $F \cong \mathbb{R}^{4|4} \times \mathbb{C}$ (we will focus on D=4 in the following sections). We name the section of this bundle "Superfield", and the set of the sections as \mathcal{F} .

In particular, a superfield can be written as $Y(x^\mu, \theta, \bar{\theta})$, we can expand it in terms of the odd elements:

$$Y(x^\mu, \theta, \bar{\theta}) = \phi(x) + \theta^\alpha \psi_\alpha(x) + \bar{\theta}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}(x) + \theta^2 M(x) + \bar{\theta}^2 N(x) + \theta^\alpha \bar{\theta}^{\dot{\alpha}} V_{\alpha\dot{\alpha}}(x) + \theta^2 \bar{\theta}_{\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}}(x) + \bar{\theta}^2 \theta^\alpha \rho_\alpha(x) + \theta^2 \bar{\theta}^2 D(x)$$

Naturally, $\phi(x)$ is the map when you restrict Y to $\Sigma \subset \tilde{\Sigma}$, and ψ_α are the fermions of our original Sigma model. So Y is the construction we mentioned in section 2.3.3.

We can define the action of \mathcal{SP} on Y by regarding Y as a "scalar field":

$$UY(x, \theta, \bar{\theta})U^\dagger = Y(x', \theta', \bar{\theta}') \quad U \in \mathcal{SP}, \quad (x', \theta', \bar{\theta}') \text{ is the corresponding coordinates transform}$$

Checking the infinitesimal transformation, we have:

$$[Q_\alpha, Y] = \left(-i \frac{\partial}{\partial \theta^\alpha} - \sigma_{\alpha\dot{\alpha}}^\mu \bar{\theta}^{\dot{\alpha}} \partial_\mu \right) Y$$

$$[\bar{Q}_{\dot{\alpha}}, Y] = \left(+i \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + \theta^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu \right) Y$$

So we define the following differential:

$$\mathcal{P}_\mu = -i \partial_\mu$$

$$\mathcal{Q}_\alpha = -i \partial_\alpha - \sigma_{\alpha\dot{\alpha}}^\mu \bar{\theta}^{\dot{\alpha}} \partial_\mu$$

$$\bar{\mathcal{Q}}_{\dot{\alpha}} = +i \bar{\partial}_{\dot{\alpha}} + \theta^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu$$

Their Lie algebra is the same as the SUSY Lie algebra.

Furthermore, we can define the following *covariant derivative*:

$$D_\alpha = \partial_\alpha + i \sigma_{\alpha\dot{\alpha}}^\mu \bar{\theta}^{\dot{\alpha}} \partial_\mu$$

$$\bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - i \theta^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu$$

Definition 2.2 (Chiral Field) $\Omega \in \mathcal{F}$ satisfies $\bar{D}_{\dot{\alpha}} \Phi = 0$ called the Chiral Field.

Remark: We use \mathcal{C} to denote the set of chiral fields, \mathcal{C} is naturally a subsheaf of \mathcal{F} (Here I use the marks of sets to denote the sheaves).

Extend in fermion coordinates:

$$\Omega(x, \theta, \bar{\theta}) = \phi(x) + \theta^\alpha \psi_\alpha(x) + \theta^2 F(x) + i \theta \sigma^\mu \bar{\theta} \partial_\mu \phi(x) - \frac{i}{2} \theta^2 \partial_\mu \psi(x) \sigma^\mu \bar{\theta} + \frac{1}{4} \theta^2 \bar{\theta}^2 \square \phi(x) \quad (3)$$

Definition 2.3 (Vector Field) $V \in \mathcal{F}$ satisfies $\bar{V} = V$ called Vector Field

The same, the set of vector fields \mathcal{V} is a subsheaf, and in Taylor extension:

$$\begin{aligned} \mathcal{V}(x, \theta, \bar{\theta}) &= C(x) + \theta \chi(x) + \bar{\theta} \bar{\chi}(x) + i \theta^2 M(x) - i \bar{\theta}^2 M^\dagger(x) + \theta \sigma^\mu \bar{\theta} A_\mu(x) \\ &+ \theta^2 \bar{\theta} \left(\bar{\lambda}(x) + \frac{i}{2} \bar{\sigma}^\mu \partial_\mu \chi(x) \right) + \bar{\theta}^2 \theta \left(\lambda(x) + \frac{i}{2} \sigma^\mu \partial_\mu \bar{\chi}(x) \right) + \frac{1}{2} \theta^2 \bar{\theta}^2 \left(D(x) - \frac{1}{2} \square C(x) \right) \end{aligned} \quad (4)$$

$(C(x), D(x), A^\mu(x) \text{ are real fields})$

The extension is complicated, but the component $F(x)$ in Ω is an auxiliary field, and under the WZ gauge, $C = \chi = M = 0$. Therefore, the chiral and vector fields correspond to the chiral and the vector representations in section 2.5.2, just check the components.

3 Super Yang-Mills Theory in BV formalism and its twist

Since we have a SUSY transformation \mathcal{Q} , we can ask whether it can be realized as something like a BRST charge. The usual answer is no, because they are not scalars under our structure group, the $Spin(n)$. So the solution is to twist our structure group by the R-bundle or to pick up a special structure to reduce the structure bundle, the resulting bundle we call $Spin(d)'$.

To simplify, we assume $[Q_i, Q_j] = \Gamma_{ij}^\mu P_\mu$ (now i include (α, A) index). Usually, the scalar \mathcal{Q} under $Spin(n)'$ has: $\mathcal{Q} = q^i \mathcal{Q}_i$ ($q^i \in \mathbb{C}$). To assign a closed condition, we have $[\mathcal{Q}, \mathcal{Q}] = \Gamma^\mu q_i q_j = 0 \forall \mu$. The homogeneous coordinate (q_1, \dots, q_n) we call it *nilpotence variety*. There are two interesting cases in the nilpotence variety:

$$\begin{cases} [Q, Q_\mu] = P_\mu & \text{topological direction} \\ [Q, Q_{\bar{z}}] = P_{\bar{z}} & \text{anti-holomorphic direction} \end{cases}$$

Just by the ward identity, we have that the correlation function is independent of x^μ (or \bar{z}). Thus, after twisting, we realize a SUSY theory as a topological(holomorphic) theory.

The following table lists some twist theories in different dimensions and supercharge numbers. I will focus on the D=4 gauge theories in the following sections

D=2, $\mathcal{N} = (2, 2)$	topological twist	R-symmetry $U(1)_A \times U(1)_V$	supercharge
	A-twist	$U(1)_R \xrightarrow{id} U(1)_V$	$Q_A = \bar{Q}_+ + Q_-$
	B-twist	$U(1)_R \xrightarrow{id} U(1)_A$	$Q_B = \bar{Q}_+ + \bar{Q}_-$
D=2, $\mathcal{N} = (0, 2)$	holomorphic twist	R-symmetry $U(1)_A$ pick up complex structure	supercharge $Q = \bar{Q}_+$
D=3, $\mathcal{N} = 2$	holo-topo twist	R-symmetry $SO(2)$ pick up complex structure	supercharge $Q = \bar{Q}_+$
D=4, $\mathcal{N} = 1$	holomorphic twist	R-symmetry $U(1)$ pick up complex structure	supercharge $Q = Q_1$
D=4, $\mathcal{N} = 2$	topological twist	R-symmetry $U(2)$ $SU(2)_r \xrightarrow{id} SU(2)_R$	supercharge $Q = \bar{Q}_{i2} - \bar{Q}_{\dot{2}1}$
D=4, $\mathcal{N} = 4$	topological twist	R-symmetry $U(4)$	supercharge
	GL twist	$\begin{pmatrix} SU(2)_I & 0 \\ 0 & SU(2)_r \end{pmatrix} \rightarrow SU(4)_R$	$Q = u Q_l + v Q_r$

Table 4: (1)Here, $SU(2)_I$ means the subgroup in $U(2)$, and we use $Spin(4) = SU(2)_l \times SU(2)_r$. (2)In D=4, $\mathcal{N} = 2$, the second index from which we down the supercharge index A by ϵ_{ab} . (3)In D=4, $\mathcal{N} = 4$, the Q_l, Q_r refer to [5].

3.1 D=4, $\mathcal{N} = 1$ SYM

As a gauge theory with Lie algebra \mathfrak{g} over 4D manifold M , its action functional is:

$$S_{SYM} = \frac{1}{16\pi} Im(\tau \int d^6z Tr(W^\alpha W_\alpha) + c.c) \quad W_a := -\frac{1}{8} \bar{D}^2(e^{-2V} D_a e^{2V}) \quad \tau := \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$$

Here, d^6z is short for $d^4x d^2\theta$, and *c.c* denotes the complex conjugate part. *Tr* stands for the Killing form of the Lie algebra \mathfrak{g} .

$$\text{gauge transformation: } e^{2V} \rightarrow e^{i\bar{\Omega}} e^{2V} e^{-i\Omega}$$

Here, V is the vector field and Ω is the chiral field valued in \mathfrak{g} .

I will skip the introduction of the BV formalism(ref. [6]). Following our marks, \mathcal{V}_g and \mathcal{C}_g denote the field spaces of vector and chiral fields valued in \mathfrak{g} . In short, originally,

we are considering $(\mathcal{V}_g, \mathcal{L}^g)$ the gauge invariant functional over the field space. Instead of that, we consider the derived complex of the functor $\text{Hom}_g(\mathbb{C}, \underline{})$. The complex is the Chevalley-Eilenberg complex and its $H^0 = \{\mathcal{L}^g\}$. Based on this, we consider the Kaszul-Tate derived of the equation ($df=0$) over the CE-complex. As a result, we obtain something we call the BV complex and the corresponding BV derivation δ_{BV} .

Definition 3.1 (*local BV complex*)

$$\mathcal{BV}_{SYM} := (O_{loc}(T^*[-1](\mathcal{V}_g \oplus \mathcal{C}_g[1]), \delta_{SYM})$$

[1] denotes the left shift in a graded vector space; T^* means the cotangent bundle. Specifically, the graded vector space looks like:

$$\mathcal{C}_g[1] \oplus \mathcal{V}_g \oplus \mathcal{V}_g^*[-1] \oplus \mathcal{C}_g^*[-2] := \mathcal{E}$$

$O(\mathcal{E})$ means the functional over the function space \mathcal{E} ; loc means it must take this form:

$$F(\mathcal{E}_1, \dots, \mathcal{E}_n) = \int_M \alpha(j_z(\mathcal{E}_1), \dots, j_z(\mathcal{E}_n)) : \mathcal{E}^{\otimes n} \rightarrow R$$

Here, j_z is the jet bundle of a sheaf at point z ; α is a density-valued function on the jet bundles.

Remark 3.1.1: Here I also use the \mathcal{E} to denote the sheaf.

Remark 3.1.2: The odd graded fields are anti-commutative, and the even ones are commutative in the graded vector space. In physics, we call the components in \mathcal{E} from left to right: ghost, field, anti-field, and anti-ghost.

Remark 3.1.3: There is a natural Poisson bracket on the Kaszul-Tate complex, which is the Schouten bracket, we also call it the BV bracket. In particular, we have $\delta_{BV} = \{\underline{}, S_{SYM}\}$.

3.1.1 Wess-Zumino gauge and reduce to gauge theory

Extending the gauge transformation by the BCH formula, we have:

$$\delta V = \frac{i}{2}(\bar{\Omega} - \Omega) + O(V)$$

By comparing the formula (3)(4), the first six terms in (4) can be canceled out through the 0-order in gauge.

Definition 3.2 The Wess-Zumino gauge is to set: C, χ, M components in \mathcal{V} equal to 0. Under this gauge:

$$\mathcal{V} = \theta \sigma^\mu \bar{\theta} V_\mu + \bar{\theta}^2 \theta^\alpha \lambda_\alpha + \theta^2 \bar{\theta}_{\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}} + \theta^2 \bar{\theta}^2 D$$

The higher order part looks like $[V, \dots]$, you can't reach the WZ gauge just by transforming once, but the difference will reside in higher terms in (4), you can defer it to the later terms each time. Finally, we can reach the WZ gauge.

So in WZ gauge,

$$W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V + \frac{1}{4}\bar{D}^2([V, D_\alpha V]) = \lambda_\alpha(x) + 2D\theta_\alpha + i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} - i\theta^2\sigma^\mu D_\mu \bar{\lambda}$$

Using the property:

$$\sigma_a \bar{\sigma}_b \sigma_c = (\eta_{ac}\sigma_b - \eta_{bc}\sigma_a - \eta_{ab}\sigma_c) + i\epsilon_{abcd}\sigma^d$$

Finally, we have:

$$S_{SYM} = \frac{1}{g^2} \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu D_\mu \bar{\lambda} + 2D^2 \right) + \frac{\theta}{32\pi^2} \int d^4x F_{\mu\nu} \star F^{\mu\nu}$$

which is the canonical N=1 gauge theory with θ term. Since in perturbative level, gauge fixing doesn't change the BV complex, we have:

Lemma 3.1 *Under Wess-Zumino gauge, the two BV complexes*

$$(\mathcal{O}_{loc}(\mathcal{E}^{SYM}), \delta_{SYM}) \xrightarrow{quasi-iso} (\mathcal{O}_{loc}(\mathcal{E}^{gauge}), \delta_{gauge})$$

3.1.2 Twist theory

Induced from \mathcal{V}_g , we have the SUSY transform on component fields:

$$\begin{aligned} \delta A_\mu &= \epsilon \sigma_\mu \bar{\lambda} + \lambda \sigma_\mu \bar{\epsilon} \\ \delta \lambda &= \epsilon D + (\sigma^{\mu\nu}\epsilon) F_{\mu\nu} \\ \delta D &= i\epsilon \sigma^\mu \partial_\mu \bar{\lambda} - i\partial_\mu \lambda \bar{\sigma}^\mu \bar{\epsilon} \end{aligned}$$

Transferring to the first-order formalism, we have:

$$\begin{aligned} \delta_{Q_\alpha} A_\mu &= \sigma_{\mu\alpha\dot{\alpha}} \psi^{\dot{\alpha}} \\ \delta_{Q_\alpha} \psi_\beta &= \epsilon_\alpha^\gamma B_{\gamma\beta} \end{aligned}$$

We can choose any spinor as our twist charge, usually $Q := Q_1$. Since selecting a spinor is equivalent to choosing a complex structure, our structure group $Spin(4)$ now reduces to $SU(2)$, which means that now Q is a U(1)-charge scalar but not a spinor as a derivation. This induces a second derivation in our BV complex:

0 1 2 3

$$\begin{array}{ccccc}
 & \psi_- & \xrightarrow{\mathcal{D}} & \psi'_+ & \\
 & \downarrow Q & & & \\
 \Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{d_+} & \Omega^2_+ \\
 & & & \nearrow c\text{Id} & \\
 & & \Omega^2_+ & \xrightarrow{d} & \Omega^3 \xrightarrow{d} \Omega^4 \\
 & & \downarrow Q & & \\
 & \psi_+ & \xrightarrow{\mathcal{D}} & \psi'_- &
 \end{array}$$

The new derivation now has the wrong degree. We can add a BV degree 1, U(1) degree -1 element t to that. And we have:

$$(\mathcal{O}_{loc}(\mathcal{E}^{gauge})(t), \delta_{gauge} + tQ)$$

0 1 2 3

$$\begin{array}{ccccc}
 & \psi_- & \xrightarrow{\mathcal{D}} & \psi'_+ & \\
 & \searrow tQ & & \swarrow tQ & \\
 & \Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{d_+} \Omega^2_+ \\
 & & & \nearrow c\text{Id} & \\
 & & \Omega^2_+ & \xrightarrow{d} & \Omega^3 \xrightarrow{d} \Omega^4 \\
 & & \searrow tQ & & \swarrow tQ \\
 & \psi_+ & \xrightarrow{\mathcal{D}} & \psi'_- &
 \end{array}$$

In short, what we did was construct a Q -cohomology by twisting and blending it with the BV-comology in the correct degree.

What's more, choosing Q not only gives us a complex structure, but also the following isomorphism:

$$\Gamma : \psi^- \xrightarrow{\cong} \Omega^{1,0} \tag{5}$$

Here, we substitute Q with the following left-spinor part:

$$V_\mu \rightarrow (\sigma^\mu V_\mu)_{\alpha\dot{\alpha}}$$

And,

$$Y : \Omega^0 \cdot w \oplus \Omega^{0,2} \xrightarrow{\cong} \psi^+ \quad (6)$$

Here, we use the isomorphism and pair one of ψ_+ with Q.

$$\Omega_+^2 \cong \text{Sym}^2(\psi^+)$$

With these isomorphisms, we have:

Theorem 3.1 (Costello 4.0.2 [1]) *The twisted $N = 1$ gauge theory on \mathbb{C}^2 is equivalent to the holomorphic BF theory.*

In the same paper, costello also proves that:

Theorem 3.2 (Costello 4.1.1 [1]) *The twisted $N = 1$ theory admits a unique quantization, compatible with certain natural symmetries, on any Calabi-Yau surface X.*

4 SUSY-invariant Local Functional

Our general super-functional space is denoted as $\mathcal{O}_{loc}(\mathcal{E})$. We can further ask for SUSY invariant condition. In this case, we can write it down explicitly:

Lemma 4.1

$$\mathcal{O}_{loc}(\mathcal{E})^{SUSY} = \mathbb{C} \otimes_{\tilde{D}_{N=1}} \text{Sym}^*(\mathbb{C}[\partial_\mu, D_\alpha, \bar{D}_{\dot{\alpha}}] \otimes \check{\mathcal{E}})$$

Now we give some definitions of this lemma:

Definition 4.1 (*D-module*)

$$D_{N=1}(M) := \mathcal{O}(M)[\partial_\mu, \partial_{\theta^\alpha}, \partial_{\bar{\theta}^{\dot{\alpha}}}]$$

Here M is the Superspace $R^{4|4}$, $\mathcal{O}(M)$ is the set of the superfields we mentioned in 2.6. It's just the polynomial ring of differential operators on $R^{4|4}$ as an $\mathcal{O}(M)$ module.

Definition 4.2 (*Jet bundle*)

The jet bundle is a sheaf assigned to another sheaf \mathcal{E} defined as:

$$J(\mathcal{E})(U) := C^\infty(U) \otimes_R \mathbb{R}[[\partial_1, \dots, \partial_n]] \otimes_R \mathcal{E}(U)$$

Its elements look like $\partial_\mu \phi, \partial_\mu \partial_\nu \phi, \dots$

To simplify the note, we use:

$$D_{N=1} := \mathbb{C}[\partial_\mu, \partial_{\theta^\alpha}, \partial_{\bar{\theta}^{\dot{\alpha}}}] \quad \tilde{D}_{N=1} := \mathbb{C}[\partial_\mu, D_\alpha, \bar{D}_{\dot{\alpha}}] \quad O_{N=1} := \mathbb{C}[\theta^\alpha, \bar{\theta}^{\dot{\alpha}}]$$

$$D_{N=1}(M) = \mathcal{O}(M) \otimes D_{N=1} \cong \mathcal{O}(M) \otimes \tilde{D}_{N=1}$$

Proof of Lemma 4.1:

(1) By definition:

$$\mathcal{O}_{loc}(\mathcal{E}) \cong Dens(M) \otimes_{D(M)} \mathcal{O}(J(\mathcal{E})) \cong Dens(M) \otimes_{D(M)} Sym^*(\check{J}(\mathcal{E})(M))$$

Here, $\check{J}(E)$ is the dual sheaf $\cong \mathcal{O} \otimes \tilde{D}_{N=1} \otimes \check{\mathcal{E}}$

(2) We perform the Taylor expansion on the fermion coordinate and move all the $f(x)$ components to the left:

$$O_{loc}^{R^4} \cong O_{N=1} \otimes_{\tilde{D}(M)} Sym^*(\tilde{D}_{N=1} \otimes \check{\mathcal{E}}(M))$$

R^4 means the translation invariant functional, which imposes that $f(x)$ is constant

(3) When variation occurs, it's a right action \mathcal{Q}_α on $\mathcal{O}_{loc}(\mathcal{E})$. We define the action of θ^α in the same way, using:

$$\{\mathcal{Q}_\alpha, \theta^\alpha\} = i$$

We have:

$$S \cdot \{\mathcal{Q}_\alpha, \theta^\alpha\} = iS \text{ and } \{\mathcal{Q}_\alpha, D_\alpha\} = 0 = \{\mathcal{Q}_\alpha, \bar{D}_{\dot{\alpha}}\}$$

we move all θ^α to measurement part:

$$S = k(S \cdot \theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2) \mathcal{Q}_1 \mathcal{Q}_2 \bar{\mathcal{Q}}_1 \bar{\mathcal{Q}}_2$$

which imposes $O_{N=1}$ part in $S = \mathbb{C}$ \square

One step more, when we focus on the chiral fields \mathcal{C} , we have a similar discussion, and the conclusion is:

Theorem 4.1

$$\begin{aligned} \mathcal{O}_{loc}(\mathcal{C}, \bar{\mathcal{C}})^{SUSY} &= \mathbb{C} \otimes_{\tilde{D}_{N=1}} (Sym^*(\tilde{D}_{N=1}^{Ch} \otimes \check{\mathcal{C}}) \otimes_{\mathbb{C}} Sym^*(\tilde{D}_{N=1}^{Ch} \otimes \check{\mathcal{C}})) \\ &+ \mathbb{C} \bar{\theta}^1 \bar{\theta}^2 \otimes_{\mathbb{C}[\partial_\mu]} Sym^*(\mathbb{C}[\partial_\mu] \otimes \check{\mathcal{C}}) + c.c \end{aligned}$$

As you can see, this is the canonical structure of the D-term + F-term. The proof of the D-term is similar to **Lemma 4.1**, but slightly delicate. Here

$$\tilde{D}_{N=1}^{Ch} := \tilde{D}_{N=1}/\tilde{D}_{N=1}\bar{D}_{\dot{\alpha}} \quad \tilde{\bar{D}}_{N=1}^{Ch} := \tilde{D}_{N=1}/\tilde{D}_{N=1}D_{\dot{\alpha}}$$

They are the differential operators acting separately on the chiral ring and the anti-chiral ring.

Now, following the previous proof, we have:

(here $\geq (1, 1)$ means the number of Sym^* , this correspond to D-term)

$$O_{loc}^{R^4}(\mathcal{C}, \check{\mathcal{C}})_{\geq(1,1)} \cong O_{N=1} \otimes_{\tilde{D}_{N=1}} (Sym^*(\tilde{D}_{N=1}^{Ch} \otimes \check{\mathcal{C}}) \otimes_{\mathbb{C}} Sym^*(\tilde{\bar{D}}_{N=1}^{Ch} \otimes \check{\mathcal{C}}))$$

different from **Lemma 4.1**, we can only define θ^α action on $Sym^*(\tilde{D}_{N=1}^{Ch})$, and $\bar{\theta}^{\dot{\alpha}}$ on $Sym^*(\tilde{\bar{D}}_{N=1}^{Ch})$, since $[D_\alpha, \theta_\alpha] \neq 0$, and that will break the quotient structure of $\tilde{D}_{N=1}^{Ch}$. But the general $\{\mathcal{Q}_\alpha, \theta_\alpha\} = i$ still works, so we have:

$$O_{loc}^{SUSY}(\mathcal{C}, \check{\mathcal{C}})_{\geq(1,1)} \cong \mathbb{C} \otimes_{\tilde{D}_{N=1}} (Sym^*(\tilde{D}_{N=1}^{Ch} \otimes \check{\mathcal{C}}) \otimes_{\mathbb{C}} Sym^*(\tilde{\bar{D}}_{N=1}^{Ch} \otimes \check{\mathcal{C}}))$$

The proof is the same as **Lemma 4.1**, and we call it the D-term.

Then we go to the F term, that means $\mathcal{O}_{loc}(\mathcal{C}, \check{\mathcal{C}})_{(*,0)} := \mathcal{O}_{loc}(\mathcal{C})$. It's still \mathcal{Q}_α exact, but $\{\bar{\mathcal{Q}}, \bar{\theta}\} = 0$ now, so we should talk about $H_{\bar{\mathcal{Q}}_1 + \bar{\mathcal{Q}}_2}(\mathcal{O}_{loc}(\mathcal{C}))$

Lemma 4.2

$$H_{\bar{\mathcal{Q}}_1}^*(\mathcal{O}_{loc}(\mathcal{C})) \cong H^*(\mathcal{O}_{loc}^{chiral}(\mathcal{C}), \bar{D}_1)\bar{\theta}^2\bar{\epsilon}^1$$

$$\text{Here } \mathcal{O}_{loc}^{chiral} := O_{N=1}^{chiral} \otimes Sym^*(\tilde{D}_{N=1}^{Ch} \otimes \check{\mathcal{C}}), \quad O_{N=1}^{chiral} := \mathbb{C}[\theta^\alpha]$$

Proof of Lemma 4.2: We consider the super-De Rham complex of O_{loc}^{chiral}

$$\mathcal{O}_{loc}^{ch}\bar{\epsilon}^1\bar{\epsilon}^2 \xrightarrow{\Phi} \mathcal{O}_{loc}^{ch} \otimes (\mathbb{C}[\bar{\theta}^1]\bar{\epsilon}^2 \oplus \mathbb{C}[\bar{\theta}^2]\bar{\epsilon}^1) \xrightarrow{\Phi} \mathcal{O}_{loc}^{ch} \otimes \mathbb{C}[\bar{\theta}^1, \bar{\theta}^2] \cong \mathcal{O}_{loc}(\mathcal{C})$$

Here $\bar{\epsilon}^{\dot{\alpha}}$ is just a formally odd parameter, and

$$\Phi : \bar{\epsilon}^{\dot{\alpha}} \rightarrow 1 - \bar{\theta}^{\dot{\alpha}} \otimes_D \bar{D}_{\dot{\alpha}}$$

$$\bar{\mathcal{Q}}_1 := \frac{\partial}{\partial \bar{\theta}^1} - \bar{\epsilon}^{\dot{\alpha}} \bar{D}_1 \frac{\partial}{\partial \bar{\epsilon}^{\dot{\alpha}}} \text{ left act on } \epsilon \text{ component for first two complex}$$

In this way, Φ is a well-defined chain map, as an example:

$$\begin{array}{ccc} \theta_1 \theta_2 \otimes D_1 D_2 \mathcal{C} \otimes \bar{\theta}^1 \bar{\epsilon}^2 & \xrightarrow{\Phi} & -\theta^1 \theta^2 \bar{\theta}_1 \otimes D_1 D_2 \mathcal{C} + \theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2 \otimes \bar{D}_2 D_1 D_2 \mathcal{C} \\ \downarrow \mathcal{Q}_1 & & \downarrow \mathcal{Q}_1 \\ \theta_1 \theta_2 \otimes D_1 D_2 \mathcal{C} \otimes (\bar{\epsilon}^2 + \bar{\epsilon}^2 \bar{D}_1 \bar{\theta}^1) & \xrightarrow{\Phi} & -\theta^1 \theta^2 \bar{\theta}_1 \otimes \bar{D}_1 D_1 D_2 \mathcal{C} - \theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2 \otimes \bar{D}_1 \bar{D}_2 D_1 D_2 \mathcal{C} \end{array}$$

So the $H_{\bar{\mathcal{Q}}_1}^*(\mathcal{O}_{loc}(\mathcal{C}))$ we need is the De-Rham degree 2 cohomology of the double complex; calculating by spectral sequence, we have:

$$H_{\bar{\mathcal{Q}}_1}^*(\mathcal{O}_{loc}(\mathcal{C})) \cong H^*(\mathcal{O}_{loc}^{chiral}(\mathcal{C}), \bar{D}_1) \bar{\theta}^2 \bar{\epsilon}^1 \square$$

Since $\{D_\alpha, \bar{D}_{\dot{\alpha}}\}$ is total derivative, we have the following quasi-isomorphism:

$$(\mathcal{O}_{loc}^{Ch}, \bar{D}_1) \cong \theta^1 \theta^2 \Omega^*(Sym^*(\tilde{D}_{N=1}^{Ch}))$$

And notice that the spectral sequence of $(\theta^1 \theta^2 \Omega^*(Sym^*(\tilde{D}_{N=1}^{Ch})), \bar{\mathcal{Q}}_1 + d_{dR})$ converges at E2, so we have:

$$H_{\bar{\mathcal{Q}}_1}^*(\mathcal{O}_{loc}^{Ch}) = (\theta^1 \theta^2 \Omega^*(Sym^*(\tilde{D}_{N=1}^{Ch} \otimes \check{\mathcal{C}})), \bar{\mathcal{Q}}_1 + d_{dR}) = \theta^1 \theta^2 H_{dR}^*(Sym^*(H_{\bar{\mathcal{Q}}_1}^*(\tilde{D}_{N=1}^{Ch} \otimes \check{\mathcal{C}})))$$

Lemma 4.3

$$H_{\bar{\mathcal{Q}}_1}^*(\tilde{D}_{N=1}^{Ch} \otimes \check{\mathcal{C}}) = \mathbb{C}[\partial_{1\dot{2}}, \partial_{2\dot{2}}] \otimes \check{\mathcal{C}}$$

Since the action of $\bar{\mathcal{Q}}_1$ is equivalent to the left action of $-2\partial_{\alpha 1} \frac{\partial}{\partial D_\alpha}$ on \mathcal{O}_{loc}^{Ch} , taking the dual, we have:

$$H^*(\mathbb{C}[x_{\alpha\dot{\alpha}}], \frac{\partial}{\partial D_\alpha}) \otimes \mathcal{C}, -2\partial_{\alpha 1} \frac{\partial}{\partial D_\alpha} = \mathbb{C}[x_{1\dot{2}}, x_{2\dot{2}}] \otimes \mathcal{C} \quad \square$$

Summarize the above discussion, the \mathcal{Q}_1 cohomology of $\mathcal{O}_{loc}(\mathcal{C})$ is represented by :

$$\int d^2\theta d^2\bar{\theta} f(\theta) \bar{\theta}^1 \bar{\theta}^2 W(\mathcal{C}, \partial_{1\dot{2}} \mathcal{C}, \partial_{2\dot{2}} \mathcal{C})$$

So for any $S \bar{\mathcal{Q}}_1$ -invariant we write it as $S = S_0 + \bar{\mathcal{Q}}_1 \tilde{S}$.

And apply $\mathcal{Q}_1, \mathcal{Q}_2$ -exact condition on S_0 , it looks like:

$$\int d^2\theta d^2\bar{\theta} \bar{\theta}^1 \bar{\theta}^2 W(\mathcal{C}, \partial_{1\dot{2}} \mathcal{C}, \partial_{2\dot{2}} \mathcal{C})$$

which is F-term.

Then consider: $\bar{\mathcal{Q}}_1 \tilde{S} = \bar{\mathcal{Q}}_2 S'$, that means $[\bar{\mathcal{Q}}_1 \tilde{S}] = 0$ in $\mathcal{O}_{loc}/Im\mathcal{Q}_2$.

If $[\tilde{S}] = 0$ in $H_{\bar{\mathcal{Q}}_1}^*(\mathcal{O}_{loc}/Im\mathcal{Q}_2)$. Naturally, $\tilde{S} = \bar{\mathcal{Q}}_1 S_1 + \bar{\mathcal{Q}}_2 S_2$, which means $S = \bar{\mathcal{Q}}_1 \tilde{S}$ is $\bar{\mathcal{Q}}_1 \bar{\mathcal{Q}}_2$ -exact, applying $\mathcal{Q}_1 \mathcal{Q}_2$ -exact condition, it contributes D-term.

Then we calculate $H_{\bar{\mathcal{Q}}_1}^*(\mathcal{O}_{loc}/Im\mathcal{Q}_2)$ using the following resolution:

$$\begin{array}{ccccccc} \dots & \mathcal{O}_{loc} & \xrightarrow{\bar{\mathcal{Q}}_2} & \mathcal{O}_{loc} & \xrightarrow{\bar{\mathcal{Q}}_2} & \mathcal{O}_{loc} & \longrightarrow \mathcal{O}_{loc}/Im\mathcal{Q}_2 \\ & j & \nearrow & & & & \\ & & \oplus & & & & \\ & & & & & & \\ & & H_{\bar{\mathcal{Q}}_2}^*(\mathcal{O}_{loc}) & & & & \end{array}$$

We define \deg_1 is the number of $\partial_{1\dot{2}}$ or $\partial_{2\dot{2}}$; \deg_2 is the number of $\partial_{1\dot{1}}$ or $\partial_{2\dot{1}}$ in \mathcal{O}_{loc} . And j is an embedding by picking up a representation element.

Then E_1 page from applying \mathcal{Q}_1 -cohomology on \mathcal{O}_{loc} , since we know that \mathcal{Q}_2 act on $H_{\mathcal{Q}_1}^*$ is 0, we have:

$$d_1 = j : H_{\mathcal{Q}_2}^* \rightarrow H_{\mathcal{Q}_1}^*$$

is identity on the $\deg_1 = \deg_2 = 0$ component, and 0 otherwise.

So E_2 is getting the mod degree-0 component of E_1 page. And trace the graph, d_n is replacing d $\partial_{1\dot{2}}$ or $\partial_{2\dot{2}}$ component, by $\partial_{1\dot{1}}$ or $\partial_{2\dot{1}}$.

At last, the spectrum sequence converges to $\bigoplus_{k=2}^{\infty} H_{\mathcal{Q}_2}^{\geq k}(\mathcal{O}_{loc})$, and $H_{\mathcal{Q}_2}^{\geq k}(\mathcal{O}_{loc})$ corresponds to term with $(k-1)$ - $(\partial_{1\dot{2}}$ or $\partial_{2\dot{2}})$ in $H_{\mathcal{Q}_1}^*(\mathcal{O}_{loc}/Im\mathcal{Q}_2)$. After applying $\mathcal{Q}_1, \mathcal{Q}_2$ -exact condition, it contributes to F-term which has $Sym^*([\partial_\mu])$ structure. \square Thm 4.1

5 Quantization of $\mathcal{N} = 1$ SYM on superspace

As a pre-theory([6] section 13) on R^4 , $\mathcal{N} = 1$ SYM given by following second-order BV complex:

$$\begin{array}{ccccccc} & & -1 & & 0 & & 1 & & 2 \\ & & \mathcal{C}_g & \xrightarrow{i} & \mathcal{V}_g & & \mathcal{V}_g^* & \xrightarrow{\bar{D}^2} & \mathcal{C}_g^* \\ & & \oplus & \nearrow -i & \oplus & & \oplus & & \oplus \\ & & \bar{\mathcal{C}}_g & & \mathcal{B}_\alpha & \xrightarrow{D^\alpha} & \mathcal{B}_\alpha^* & \xrightarrow{=\bar{D}^2 D_\alpha} & \bar{\mathcal{C}}_g^* \\ & & & & & & & & \end{array}$$

And Q^{GF} given by:

$$\begin{array}{ccccccc} & & -1 & & 0 & & 1 & & 2 \\ & & \mathcal{C}_g & \xleftarrow{\bar{D}^2 D^2} & \mathcal{V}_g & \xleftarrow{D^\alpha} & \mathcal{V}_g^* & \xleftarrow{D^2} & \mathcal{C}_g^* \\ & & \oplus & \nearrow D^2 \bar{D}^2 & \oplus & \nearrow D^2 \bar{D}_\alpha & \nearrow D^2 & \nearrow \bar{D}^2 & \oplus \\ & & \bar{\mathcal{C}}_g & & \mathcal{B}_\alpha & & \mathcal{B}_\alpha^* & & \bar{\mathcal{C}}_g^* \\ & & & & & & & & \end{array}$$

By simple calculation, we have the property $[Q, Q^{GF}] = \square$.

The second-order SYM action is:

$$S_{SYM} = \frac{1}{16\pi} Im(\tau \int d^6 z Tr(W^\alpha B_\alpha + B^\alpha B_\alpha) + c.c)$$

Quantization means we can assign R^4 a factorization algebra Obs^q , which locally(for U a open disk) looks like following DGLA at level Φ :

$$\text{Obs}_\Phi^q(U) \cong (\oplus_n \text{Sym}^n(T^*[-1](\mathcal{V}_g(U) \oplus \mathcal{C}_g(U)[1]))^*[[\hbar]], Q + \hbar \Delta_\Phi + \{I[\Phi], __\}\})$$

Here we define parametrix following [7] section 7.2.4 as an element:

$$\Phi \in \bar{\mathcal{E}}(U) \widehat{\otimes}_\pi \bar{\mathcal{E}}(U)$$

Satisfying several properties from elliptic regularity. $\bar{\mathcal{E}}(U)$ means the distributional section of $\mathcal{E}(U)$. Define:

$$P(\Phi) = \frac{1}{2} Q^{GF} \Phi \in \bar{\mathcal{E}} \widehat{\otimes}_\pi \bar{\mathcal{E}}.$$

This is the propagator associated with Φ . We let

$$K_\Phi = K_{\text{Id}} - QP(\Phi).$$

K_{Id} is the kernel of $\text{Id} : \mathcal{E} \rightarrow \bar{\mathcal{E}}$

$$\Delta_\Phi := \partial_{K_\Phi} : \mathcal{O}(\mathcal{E}) \rightarrow \mathcal{O}(\mathcal{E})$$

$$\{I, J\}_\Phi := \Delta_\Phi(IJ) - (\Delta_\Phi I)J - (-1)^{|I|} I \Delta_\Phi J$$

on the space $\mathcal{O}(\mathcal{E})$.

Also, we demand the quantum field theory $I[\Phi]$ corresponds to our classical field theory $I_0 = S_{\text{SYM}}$, that means it satisfies:

- $I[\Phi] = W(P(\Phi) - P(\Psi), I[\Psi])$ (1)
- $\lim_{\Phi \rightarrow 0} I[\Phi] - I_0 = 0 \text{ mod } \hbar$ (2)
- $I[\Phi]$ satisfies *quantum master equation* for $\forall \Phi$ (3)

Condition (1)(2)(3) are satisfied at the same time if we can find a lift \tilde{I}_0 of I_0 at scale 0, that satisfies quantum master equation. Since we can define:

$$I[\Phi] := W(P(\Phi), \tilde{I}_0)$$

And we have the following lemma:

Lemma 9.2.2 [6]

If $I[\Phi]$ satisfies the Φ -QME, then $I[\Psi] = W(P(\Psi) - P(\Phi), I[\Phi])$ satisfies the Ψ -QME. This follows from the identity

$$[Q, \partial_{P(\Phi)} - \partial_{P(\Psi)}] = \Delta_\Psi - \Delta_\Phi$$

Then

$$(Q + \hbar \Delta_\Psi) e^{I[\Psi]/\hbar} = (Q + \hbar \Delta_\Psi) e^{\hbar \partial_{P(\Psi)} - P(\Phi)} e^{I[\Phi]/\hbar} = e^{\hbar \partial_{P(\Psi)} - P(\Phi)} (Q + \hbar \Delta_\Phi) e^{I[\Phi]/\hbar}$$

Naturally, $I[\Phi] := W(P(\Phi), I_0)$ satisfies mod \hbar quantum master equation. To lift it to a theory satisfies mod \hbar^2 quantum master equation, or more generally, to lift a \hbar^n theory $I[L]$ to a \hbar^{n+1} , we choose any lift $\tilde{I}[L]$ and define its obstruction at scale L as:

$$O_{n+1}[L] = \hbar^{-n-1} \left(Q\tilde{I}[L] + \frac{1}{2} \{\tilde{I}[L], \tilde{I}[L]\}_L + \hbar \Delta_L \tilde{I}[L] \right)$$

And we have the following property:

- The obstruction of the lift $\tilde{I}[L]$: $O_{n+1} \in H_{Q+\{I_0,\cdot\}}^1(\mathcal{O}_{loc}(\mathcal{E}))$
- When the obstruction vanishes, the classification set of the lift equals $H_{Q+\{I_0,\cdot\}}^0(\mathcal{O}_{loc}(\mathcal{E}))$

In this paper, we want to show that when $I_0 = S_{SYM}$, $H_{Q+\{I_0,\cdot\}}^1(\mathcal{O}_{loc}(\mathcal{E})) = 0$, then our quantization is well-defined.

The proof of these properties is from *Lemma 11.1.1* [6].

In short, the first property is obtained easily from DGLA:

Let \mathfrak{g} be a differential graded Lie algebra, and let $X \in \mathfrak{g}$ be an odd element. Let

$$O(X) = d_{\mathfrak{g}} X + \frac{1}{2} [X, X].$$

Then,

$$d_{\mathfrak{g}} O(X) + [X, O(X)] = 0.$$

That means

$$QO_{n+1} + \{I_0, O_{n+1}\} = 0$$

For $\tilde{I}[L]$ lift, then any other lift is given by $\tilde{I}[L] + \hbar^{n+1}J[L]$. The obstruction vanishes if only:

$$QJ[L] + \{I_0[L], J[L]\} = O_{n+1}[L] \quad \square$$

Now we calculate $H_{Q+\{I_0,\cdot\}}^*(\mathcal{O}_{loc}^{SUSY}(\mathcal{E}))$. Recall in *Lemma 4.1*, we have proved that

$$\mathcal{O}_{loc}(\mathcal{E})^{SUSY} = \mathbb{C} \otimes_{\tilde{D}_{N=1}} Sym^*(\mathbb{C}[\partial_\mu, D_\alpha, \bar{D}_{\dot{\alpha}}] \otimes \check{\mathcal{E}})$$

And in SYM,

$$\mathcal{E} := \mathcal{C}_g[1] \oplus \mathcal{V}_g \oplus \mathcal{V}_g^*[-1] \oplus \mathcal{C}_g^*[-2]$$

as a Q-complex. And the $Q+\{I_0, \cdot\}$ cohomology is quasi-iso to the $Q+\{I_0, \cdot\}$ cohomology of the following bi-complex as a resolution:

$$\begin{array}{ccc}
\cdots \rightarrow \wedge^1 \mathbb{R}^4 \otimes \mathbb{C}[\theta^i \otimes D^i] \otimes Sym^*(\cdots) & \xrightarrow{\partial_i} & \mathbb{C}[\theta^i \otimes D^i] \otimes Sym^*(\mathbb{C}[\partial_\mu, D_\alpha, \bar{D}_{\dot{\alpha}}] \otimes \check{\mathcal{E}}) \\
& \Phi \uparrow & \\
\cdots & \xrightarrow{\partial_i} & (\oplus_j (\mathbb{C}[\theta^i \otimes D^i](i \neq j))\epsilon^j) \otimes_{\tilde{D}_{N=1}} Sym^*(\mathbb{C}[\partial_\mu, D_\alpha, \bar{D}_{\dot{\alpha}}] \otimes \check{\mathcal{E}}) \\
& \Phi \uparrow & \\
& \cdots &
\end{array}$$

The row is the canonical De Rham complex, and the column is the super-De Rham we defined in *Lemma 4.2*, θ^i go over $\theta^1, \theta^2, \bar{\theta}^1, \bar{\theta}^2$. The $Q + [I_0, \dots]$ only acts on Sym^* component, so we calculate $H_{Q+[I_0, \dots]}(Sym^*(\cdots))$ as the E_1 -page.

Now we filter the Sym^* component by its * -index, under this filtration, the E_1 -page given by $Sym^i(\mathbb{C}[\partial_\mu, D_\alpha, \bar{D}_{\dot{\alpha}}] \otimes H_Q^j(\check{\mathcal{E}})) \cong Sym^i(\mathbb{C}[\partial_\mu, D_\alpha, \bar{D}_{\dot{\alpha}}] \otimes \check{H}_Q^j(\mathcal{E}))$.

Recall as in [6]*Lemma 6.7.1*, as a L^∞ CE cohomology, $H_{Q+[I_0, \dots]}(\mathcal{O}_{loc}^{SUSY})$ can be calculated localized to its fiber $(\mathcal{O}_{loc})_0 := Sym^*(\mathbb{C}[\partial_\mu, D_\alpha, \bar{D}_{\dot{\alpha}}] \otimes \check{\mathcal{E}}_0)$, because of its SUSY invariant. Then, just by Talor extending \mathcal{C} and \mathcal{V} . For example, using the extension (3)(4) in section 2.6, we have:

$$\begin{aligned}
e_1 &= C(x) - \frac{1}{4}\theta^2\bar{\theta}^2\square C(x) = Q\left(\phi(x) + i\theta^\mu\theta\partial_\mu\phi(x) + \frac{1}{4}\theta^2\bar{\theta}^2\square\phi(x)\right) = Qs_1 \text{ when } \phi \text{ is pure imagine} \\
e_2 &= \theta\sigma^\mu\bar{\theta}A_\mu = Q\left(\phi(x) + i\theta^\mu\theta\partial_\mu\phi(x) + \frac{1}{4}\theta^2\bar{\theta}^2\square\phi(x)\right) = Qs_2 \text{ when } \phi \text{ is pure real}
\end{aligned}$$

Here a bit different from (3)(4) $\phi, \psi, F, \chi, \bar{\chi}, \lambda, \bar{\lambda}, M \in \mathbb{C}[[x_1, \dots, x_4]] \quad C, D, A_\mu \in \mathbb{R}[[x_1, \dots, x_4]]$

Finally, we have $H_Q^*(\mathcal{E}) = \mathfrak{g}[1]$. As a result, $H_{Q+[I_0, \dots]}(Sym^*(\cdots)) \cong H_{CE}^*(\mathfrak{g})$. When considering semi-simple \mathfrak{g} , the degree 1 component vanishes, and the degree 0 component is just \mathbb{C} , which means there is no obstruction and quantization is unique up to a constant. \square

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