Quantization of $\mathcal{N}=1$ Gauge theory and review on SUSY

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Contents

1 Introduction

In Section 2, we summarize the Super Symmetry Theory (SUSY) construction. In Section 3, we give an introduction to SYM construction in BV formalism and its twist. In section 4, we give

a general construction of SUSY invariant functional on $R^{4|4}$. And section 5 is the calculation of the local BV cohomology.

2 SUSY algebra and representation

2.1 Sigma Model

To simplify, we consider a general field theory in D space–time Σ whose Lagrangian functional description contains only scalar fields ϕ^i and spinors (fermion fields) ψ^a:

$$\mathcal{L} = \mathcal{L}(\phi, \psi)$$

It can be any functional $\mathcal{L} \in \mathcal{O}(\mathcal{E} \oplus \mathcal{S})$, here we suppose ϕ^i is a section of a vector bundle over Σ and denote the section set as $\mathcal{E}$ and the section set of the dual bundle as $\check{\mathcal{E}}$; and ψ^a is a section of the spinor bundle S, the set denoted as $\mathcal{S}$ dual sections as $\check{\mathcal{S}}$.

So $\mathcal{O}(\mathcal{E} \oplus \mathcal{S}) := Sym^*(\check{\mathcal{E}}) \otimes \wedge^*(\check{\mathcal{S}})$, but if we further impose the locality property—which means the functional is an integration pointwise, corresponding to causality in physics, after Taylor extension on ψ, we have:

$$S = \int_{R^D} \mathcal{L} \, d^D x$$

$$\begin{aligned}
\mathcal{L} = &-\frac{1}{2} g_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j + b_{ij}(\phi) \epsilon^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j + V(\phi) \\
&+ i h_{ab}(\phi) \bar{\psi}^a \gamma^\mu \partial_\mu \psi^b + i \tilde{h}_{ab}(\phi) \bar{\psi}^a \gamma_3 \gamma^\mu \partial_\mu \psi^b \\
&+ k_{abi}(\phi) \bar{\psi}^a \gamma^\mu \psi^b \partial_\mu \phi^i + \tilde{k}_{abi}(\phi) \bar{\psi}^a \gamma^\mu \gamma_3 \psi^b \partial_\mu \phi^i \\
&+ y_{ab}(\phi) \bar{\psi}^a \psi^b + \tilde{y}_{ab}(\phi) \bar{\psi}^a \gamma_3 \psi^b \\
&+ s_{abcd}(\phi) \bar{\psi}^a \psi^c \bar{\psi}^b \psi^d + \cdots (b_{ij}, \tilde{h}, \tilde{k} \neq 0, only \ when \ D = 2)
\end{aligned}$$

The Sigma model is a map : $\Sigma \to M$, $x \mapsto \phi^i(x)$, which gives a geometric explanation of this functional:

We regard $\{\phi^i\}$ as the coordinate functions on M, since g_{ij} is positive-definite and transfers as a metric under reparametrization, it defines a Riemannian metric on M.

For $\{\psi^a\}$, you know it has an index of the Spinor bundle, but its metric depends on ϕ, so it's a section of $S \otimes \phi^* V$, S is the Spinor bundle and $\phi^* V$ is the pull-back of a vector bundle V over M. When D=2, you can use the chiral decomposition to define $h_\pm$ [1]; the result is the same as in other dimensions to define $w_i{}^a{}_b := h^{ac} k_{cbi}$, and writing:

$$D_\mu \psi^a := \partial_\mu \psi^a + \partial_\mu \phi^a w_i{}^a{}_b \psi^b$$

as the covariant derivation. Since the symmetry part of k_{cbi} is arbitrary, you can always get $D_\mu(h_{ab}) = 0$, which means it's compatible with the metric.

So under the Sigma model, $\mathcal{L}$ is equivalent to geometric information, and symmetry is equivalent to constraints on the target manifold. One step more, since the coupling corresponds to the L^∞-structure over $(\mathcal{E} \otimes \mathcal{S})$, it gives a bridge from geometry to algebra and their deformations.

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2.2 Spinors in various dimensions

By definition, spinors are sections of the Spinor bundle, which is the associated bundle constructed from the principal Spin(n)-bundle and the Clifford algebra representation space. The Clifford algebra is:

$$Cl(V) := \oplus_k V^{\otimes k} \quad V = \mathbb{R}[\Gamma^\mu] \quad \{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$$

Its representation depends on the dimension. For D=even, its representation space $S = S^+ \oplus S^-$ is reducible as a representation of the Lorentz group $SO(d-1,1)$, and we call each part of the sections the Weyl spinors. For D=odd, it's irreducible, and for a physical purpose, we choose the real sections called Majorana spinors.

d	Majorana	Weyl	Majorana–Weyl
2	yes	self	yes
3	yes	–	–
4	yes	complex	–
5	–	–	–
6	–	self	–
7	–	–	–
8	yes	complex	–
9	yes	–	–
10	yes	self	yes
11	yes	–	–
12	yes	complex	–

Table 1: Dimensions in which various conditions are allowed for $SO(d-1,1)$ spinors. A dash indicates that the condition cannot be imposed. For the Weyl representation, it is indicated whether these are conjugate to themselves or to each other (complex).

2.3 Symmetries

As we mentioned, symmetries on Lagrangians impose constraints on the target space's geometric structure.

2.3.1 Poisson Symmetry

Suppose we have a symmetry group G that acts on scalar fields $\{\phi^i\}$. Preserving the Lagrangian is equivalent to preserving the metric $g_{ij}(\phi)$, so $G \subset Iso(M)$ and generators are Killing vector fields.

2.3.2 Fermion Symmetry

Suppose we have a symmetry group G that acts on fermion fields $\{\psi^a\}$. To match the metric h_{ab}, $G \subset U(m)(or\ SO(m)\ in\ Majorana)$, acting as:

$$\psi^a \mapsto M^a{}_b(\phi)\psi^b$$

It's a global symmetry iff:

$$D_\mu M^a{}_b = 0$$

Which means it's a flat section of $\mathcal{V} \otimes \check{\mathcal{V}}$, and for most cases, the Fermi number operator is the unique solution, which means most of the fermion symmetries are gauge theories. We refer to say under this gauge symmetry, the connection is twisted to:

$$D_\mu \mapsto D\mu - A_\mu{}^a{}_b(\phi)\psi^b$$

I will denote the fiber of the Spinor bundle as V. What we did was do the fiber product of $Spinor(n) \to M$ and an associated bundle $G \times_G End(V) \to M$. That means you twist the Spinor bundle by an associated bundle, and the new connection is that.

2.3.3 Super Symmetry

The next question is whether we could blend the two parts of symmetry. An observation is that the symmetry is split because, in our Sigma model, we separate it as an underlying manifold and fiber. What we need is to extend the spacetime manifold and give a map to the bundle. However, we must respect the anti-symmetry of fermion fields. That means we need fermion coordinates and the super-Poincaré group.

2.4 SUSY Algebra and superspace

Now for a $\mathbb{Z}_2$-graded super Lie algebra, $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$. If the Poincaré algebra $\mathcal{P} \subset \mathcal{G}_0$, then by S-matrix analysis [2], $\mathcal{G}_0 = \mathcal{P} \oplus \mathcal{T}$.

So we impose $\mathcal{G}_0 = \mathcal{P} := \mathbb{C}[P_\mu, M_{\mu\nu}]$ with Lie algebra:

$$[P_\mu, P_\nu] = 0 \quad [M_{ab}, P_c] = i\eta_{ac}P_b - (a \leftrightarrow b) \quad [M_{ab}, M_{cd}] = i\eta_{ac}M_{bd} - i\eta_{ad}M_{bc} - (a \leftrightarrow b)$$

Since $[\mathcal{G}_{,}\mathcal{G}_1] \subset \mathcal{G}_1$, $\mathcal{G}_1$ is a representation space of $\mathcal{G}_0$, so elements of $\mathcal{G}_1$ must be spinors. Impose $\mathcal{G}_1 = \mathbb{C}[Q_\alpha^a] = \mathbb{C}[Q_\alpha^a, \bar{Q}_{\dot\alpha}^a](decompose\ to\ weyl\ when\ D = 2m)$. α is the spinor index, and a is the number of spinors.

Survey on the structure constant [3] shows the Lie structure must be (As an example D=4):

$$\{Q_\alpha^a, \bar{Q}_{\dot\beta}^b\} = 2\delta^{ab}(\sigma^\mu)_{\alpha\dot\beta}P_\mu \quad [M_{\mu\nu}, Q_\alpha^a] = i(\sigma_{\mu\nu})_\alpha^\beta Q_\beta^a \quad [P_\mu, Q_\alpha^a] = 0 \quad (\sigma_{\mu\nu} := -\frac{1}{4}(\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu))$$

$$\bar{\sigma}^\mu := (1, -\sigma_1, -\sigma_2, -\sigma_3)$$

More generally, you can replace σ^μ with $\Gamma^\mu C$ in the Majorana case; with $P_+\Gamma^\mu C(P_+\ is\ Weyl\ projection)$ to the Weyl decomposition case, and add central charges...

D			
3	Supercharges		Majorana
	Reality		$(Q^a_\alpha)^\dagger = Q^a_\alpha \ (*)$
	Superalgebra		$\{Q^a_\alpha, Q^b_\beta\} = 2\delta^{ab}(\Gamma^\mu\Gamma_0)_{\alpha\beta}P_\mu$
	Automorphism(R-symmetry)		SO(N)
4	Supercharges		Weyl
	Reality		$(Q^a_\alpha)^\dagger = \overline{Q}_{\dot\alpha\, a}$
	Superalgebra		$\{Q^a_\alpha, \overline{Q}_{\dot\beta\, b}\} = 2\delta^a_b(\sigma^\mu)_{\alpha\dot\beta}P_\mu$
			$\{Q^a_\alpha, Q^b_\beta\} = \epsilon_{\alpha\beta}Z^{ab}$
	R-symmetry		U(N)
5	Supercharges		Symplectic–Majorana
	Reality		$(Q^a_\alpha)^\dagger = \Omega_{ab}B_\alpha{}^\beta Q^b_\beta$
	Superalgebra		$\{Q^a_\alpha, Q^b_\beta\} = 2\Omega^{ab}(\Gamma^\mu C)_{\alpha\beta}P_\mu$
	R-symmetry		Sp(N)
6	Supercharges		Symplectic–Majorana–Weyl
	Reality		$(Q^a_\alpha)^\dagger \equiv \bar{Q}_{\alpha\, a} = \Omega_{ab}B_\alpha{}^\beta Q^b_\beta$
	Superalgebra		$\{Q^a_\alpha, Q^b_\beta\} = 2\Omega^{ab}(\Sigma^\mu)_{\alpha\beta}P_\mu$
	R-symmetry		Sp(N_R)×Sp(N_L)
7	Supercharges		Symplectic–Majorana
	Reality		$(Q^a_\alpha)^\dagger = \Omega_{ab}B_\alpha{}^\beta Q^b_\beta$
	Superalgebra		$\{Q^a_\alpha, Q^b_\beta\} = 2\Omega^{ab}(\Gamma^\mu C)_{\alpha\beta}P_\mu$
	R-symmetry		Sp(N)
8	Supercharges		Weyl
	Reality		$(Q^a_\alpha)^\dagger = \overline{Q}_{\dot\alpha\, a}$
	Superalgebra		$\{Q^a_\alpha, \overline{Q}_{\dot\beta\, b}\} = 2\delta^a_b(\Sigma^\mu)_{\alpha\dot\beta}P_\mu$
			$\{Q^a_\alpha, Q^b_\beta\} = C_{\alpha\beta}Z^{ab}$
	R-symmetry		U(N)
9	Supercharges		Majorana
	Reality		$(Q^a)^\dagger = C(\bar{Q}^a)^t$
	Superalgebra		$\{Q^a_\alpha, Q^b_\beta\} = 2\delta^{ab}(\Gamma^\mu C)_{\alpha\beta}P_\mu$
	R-symmetry		SO(N)
10	Supercharges		Majorana–Weyl
	Reality		$(Q^a_\alpha)^\dagger = Q^a_\alpha$
	Superalgebra		$\{Q^a_\alpha, Q^b_\beta\} = 2\delta^{ab}(\Sigma^\mu)_{\alpha\beta}P_\mu$
			$\{Q^a_\alpha, \overline{Q}^b_\beta\} = CC$
	R-symmetry		SO(N_R)×SO(N_L)

Table 2: (1)CC stands for "central charges." (2) ($*$) holds in a Majorana rep. In a general rep. we have $Q^\dagger = BQ^T$, with $B = C\gamma^0$, where C is the charge-conjugation matrix. The same expression holds for the B matrices in $D = 5, 6, 7$; in the symplectic case we have $B^*B = -1$, in the Majorana one $B^*B = +1$. [1]

Remark: R-Symmetry means the freedom to choose the Supercharge(i.e., the Superalgebra still works after transformation). This symmetry is similar to the Fermion Symmetry, since the Supercharges are spinors. To this point, we usually equip the spacetime with an R-bundle to indicate the R-symmetry.

Definition 2.1 *(Super Poincaré Group and Superspace)*

$$\mathcal{SP} := exp(i(-x^\mu P_\mu + \frac{1}{2}K^{ab}M_{ab} + \theta^\alpha_a Q^a_\alpha + \bar{\theta}^{\dot\alpha}_a \bar{Q}^a_{\dot\alpha})) \qquad (x^\mu, K^{ab} \in \mathbb{R}; \theta^\alpha, \bar{\theta}^{\dot\alpha} \in \mathbb{C}_a)$$

$$R^{n|s(n)} := \mathcal{SP}/SO(n-1,1) \qquad s(n) \text{ is the number of spinors in n-dimension.}$$

The action of $\mathcal{SP}$ on $R^{n|s(n)}$ is given by:

$$\rho(A)(B) = e^A e^B \quad A \in \mathcal{SP}, B \in R^{n|s(n)}$$

Prop 2.1.1 *By BCH formula,(D=4) $e^A \cdot e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots}$, we have:*
Translation:

$$P_\mu : (x, \theta, \bar{\theta}) \mapsto (x^\mu + a^\mu, \theta, \bar{\theta})$$

Laurantz transformation:

$$x'^a = (e^K)^a_{\ b} x^b$$

$$\theta'_\alpha = \left(\exp\left(\frac{1}{2}K^{ab}\sigma_{ab}\right)\right)^\beta_\alpha \theta_\beta$$

$$\bar{\theta}'^{\dot\alpha} = \left(\exp\left(\frac{1}{2}K^{ab}\tilde{\sigma}_{ab}\right)\right)^{\dot\alpha}_{\dot\beta} \bar{\theta}^{\dot\beta}.$$

Supersymmetry transformation:

$$x'^a = x^a - i\epsilon\sigma^a\bar{\theta} + i\theta\sigma^a\bar{\epsilon} \tag{1}$$

$$\theta'^\alpha = \theta^\alpha + \epsilon^\alpha \quad \bar{\theta}'_{\dot\alpha} = \bar{\theta}_{\dot\alpha} + \bar{\epsilon}_{\dot\alpha}. \tag{2}$$

2.5 SUSY Representation and superfields on D=4

Since we primarily discuss the D=4 manifold in this paper, we provide the representation of the SUSY Algebra over it.

2.5.1 N=1 massive

In the rest frame, we have:$p^\mu = (m, 0, 0, 0)$. So the SUSY Algebra is:

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2m\delta_{\alpha\dot\beta} \quad \{Q_\alpha, Q_\alpha\} = \{\bar{Q}_{\dot\beta}, \bar{Q}_{\dot\beta}\} = 0 \quad [M_{12}, Q_\alpha] = \frac{1}{2}Q_\alpha$$

So it has two $SU(2)$ algebraic structures, the representation is:

$$\begin{cases} |-\tfrac{1}{2}\rangle \oplus 2\,|0\rangle \oplus |\tfrac{1}{2}\rangle & \text{chiral multiplet} \\ |-1\rangle \oplus 2\,|-\tfrac{1}{2}\rangle \oplus 2\,|0\rangle \oplus 2\,|\tfrac{1}{2}\rangle \oplus |1\rangle & \text{vector multiplet} \end{cases}$$

Remark: These two multiplets are irreducible representations of the SUSY algebra, although we haven't proven that. Differing from the Poincaré algebra, the spin J^2 is no longer a Casimir operator; instead, we construct C_2', called superspin.

$$B_\mu = W_\mu - \frac{1}{4}\bar{Q}_{\dot{\alpha}}(\bar{\sigma}_\mu)^{\dot{\alpha}\beta}Q_\beta \quad C_{\mu\nu} = B_\mu P_\nu - B_\nu P_\mu \quad C_2' = C_{\mu\nu}C^{\mu\nu}$$

2.5.2 N=1 massless

We will see later that this case corresponds to the gauge theory we discuss.
In the "rest frame" (E,0,0,E):

$$\{Q, \overline{Q}\} = \begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix}$$

So there is only one generate-annihilate operator pair, with CPT the representation is:

$$\begin{cases} |-\tfrac{1}{2}\rangle \oplus 2\,|0\rangle \oplus |\tfrac{1}{2}\rangle & \text{chiral multiplet} \\ |-1\rangle \oplus |-\tfrac{1}{2}\rangle \oplus |\tfrac{1}{2}\rangle \oplus |1\rangle & \text{vector multiplet} \end{cases}$$

The vector multiplet corresponds to a vector field; it consists of one gauge vector field and one fermion. This is the gauge theory we will discuss.

2.5.3 N=2 massive and massless

In N=2, the crucial difference from N=1 is the central charge Z, which leads to the BPS boundary in the massive case: Consider $U = Q_1^1 + \lambda \bar{Q}_{\dot{2}}^1$, as it means the composition of SUSY transformations. We have:

$$\{U, U^\dagger\} \geq 0 \Rightarrow m \geq |Z|$$

$N = 2$		
Massive Z=0	SUSY Algebra	$\{Q_\alpha^A, \bar{Q}_{\dot\beta}^B\} = 2m\delta_{\alpha\dot\beta}\delta^{AB}$
	scalar multiplet	$5\phi + 4\psi + 1A_\mu$
Massive Z≠0	SUSY Algebra	$\{Q^A, \bar{Q}^B\} = 2\delta^{AB}\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$
		$\{Q^A, Q^B\} = \delta^{AB}\begin{pmatrix} 0 & -Z \\ Z & 0 \end{pmatrix}$
	m>Z scalar multiplet	same as Massive Z=0
	m=Z scalar multiplet	same as N=1 massive chiral
	m≥Z	prohibit
Massless	SUSY Algebra	$\left\{Q^A, \bar{Q}^B\right\} = \delta^{AB}\begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix}$
	scalar multiplet	$4\phi + 2\psi$
	vector multiplet	$2\phi + 2\psi + 1A_\mu$

Table 3: ϕ means the scalar representation under the Lorentz group, ψ stands for $spin - \frac{1}{2}$ representation, A_μ stands for $spin - 1$ representation.

2.6 Superfields

Just like fermion fields as $Spin - \frac{1}{2}$ representations of the Lorentz algebra, we now construct superfields as representations of the SUSY Algebra.

Following previous notes, we use Σ to denote the spacetime manifold and $\tilde{\Sigma}$ to denote the extended superspace in section 2.4. Now, consider the complex function bundle F over $\tilde{\Sigma}$, locally we have $F \cong R^{4|4} \times \mathbb{C}$ (we will focus on D=4 in the following sections). We name the section of this bundle "Superfield", and the set of the sections as $\mathcal{F}$.

In particular, a superfield can be written as $Y(x^\mu, \theta, \bar{\theta})$, we can expand it in terms of the odd elements:

$$Y(x^\mu, \theta, \bar{\theta}) = \phi(x) + \theta^\alpha \psi_\alpha(x) + \bar{\theta}_{\dot\alpha} \bar{\chi}^{\dot\alpha}(x) + \theta^2 M(x) + \bar{\theta}^2 N(x)$$
$$+ \theta^\alpha \bar{\theta}^{\dot\alpha} V_{\alpha\dot\alpha}(x) + +\theta^2 \bar{\theta}_{\dot\alpha} \bar{\lambda}^{\dot\alpha}(x) + \bar{\theta}^2 \theta^\alpha \rho_\alpha(x) + \theta^2 \bar{\theta}^2 D(x)$$

Naturally, $\phi(x)$ is the map when you restrict Y to $\Sigma \subset \tilde{\Sigma}$, and ψ_α are the fermions of our original Sigma model. So Y is the construction we mentioned in section 2.3.3.

We can define the action of $\mathcal{SP}$ on Y by regarding Y as a "scalar field":

$$U Y(x, \theta, \bar{\theta}) U^\dagger = Y(x', \theta', \bar{\theta}') \quad U \in \mathcal{SP}, \quad (x', \theta', \bar{\theta}') \text{ is the corresponding coordinates transform}$$

Checking the infinitesimal transformation, we have:

$$[Q_\alpha, Y] = \left(-i\frac{\partial}{\partial\theta^\alpha} - \sigma_{\alpha\dot\alpha}^\mu \bar{\theta}^{\dot\alpha} \partial_\mu\right) Y$$

$$[\bar{Q}_{\dot\alpha}, Y] = \left(+i\frac{\partial}{\partial\bar{\theta}^{\dot\alpha}} + \theta^\alpha \sigma_{\alpha\dot\alpha}^\mu \partial_\mu\right) Y$$

So we define the following differential:

$$\mathcal{P}_\mu = -i\partial_\mu$$
$$\mathcal{Q}_\alpha = -i\partial_\alpha - \sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu$$
$$\bar{\mathcal{Q}}_{\dot\alpha} = +i\bar\partial_{\dot\alpha} + \theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$$

Their Lie algebra is the same as the SUSY Lie algebra.

Furthermore, we can define the following *covariant derivative*:

$$D_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu$$
$$\bar{D}_{\dot\alpha} = -\bar\partial_{\dot\alpha} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$$

Definition 2.2 *(Chiral Field)* $\Omega \in \mathcal{F}$ *satisfies* $\bar{D}_{\dot\alpha}\Phi = 0$ *called the Chiral Field.*

Remark: We use $\mathcal{C}$ to denote the set of chiral fields, $\mathcal{C}$ is naturally a subsheaf of $\mathcal{F}$ (Here I use the marks of sets to denote the sheaves).

Extend in fermion coordinates:

$$\Omega(x,\theta,\bar\theta) = \phi(x) + \theta^\alpha\psi_\alpha(x) + \theta^2 F(x) + i\theta\sigma^\mu\bar\theta\partial_\mu\phi(x) - \frac{i}{2}\theta^2\partial_\mu\psi(x)\sigma^\mu\bar\theta + \frac{1}{4}\theta^2\bar\theta^2\Box\phi(x) \qquad (3)$$

Definition 2.3 *(Vector Field)* $V \in \mathcal{F}$ *satisfies* $\bar{V} = V$ *called Vector Field*

The same, the set of vector fields $\mathcal{V}$ is a subsheaf, and in Taylor extension:

$$\mathcal{V}(x,\theta,\bar\theta) = C(x) + \theta\chi(x) + \bar\theta\bar\chi(x) + i\theta^2 M(x) - i\bar\theta^2 M^\dagger(x) + \theta\sigma^\mu\bar\theta A_\mu(x)$$
$$+ \theta^2\bar\theta\left(\bar\lambda(x) + \frac{i}{2}\bar\sigma^\mu\partial_\mu\chi(x)\right) + \bar\theta^2\theta\left(\lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\bar\chi(x)\right) + \frac{1}{2}\theta^2\bar\theta^2\left(D(x) - \frac{1}{2}\Box C(x)\right) \qquad (4)$$
$$(C(x), D(x), A^\mu(x) \ are \ real \ fields)$$

The extension is complicated, but the component F(x) in Ω is an auxiliary field, and under the WZ gauge, $C = \chi = M = 0$. Therefore, the chiral and vector fields correspond to the chiral and the vector representations in section 2.5.2, just check the components.

3   Super Yang-Mills Theory in BV formalism and its twist

Since we have a SUSY transformation $\mathcal{Q}$, we can ask whether it can be realized as something like a BRST charge. The usual answer is no, because they are not scalars under our structure group, the $Spin(n)$. So the solution is to twist our structure group by the R-bundle or to pick up a special structure to reduce the structure bundle, the resulting bundle we call $Spin(d)'$.

To simplify, we assume $[Q_i, Q_j] = \Gamma^\mu_{ij} P_\mu$ (now i include (α, A) *index*). Usually, the scalar $\mathcal{Q}$ under $Spin(n)'$ has: $\mathcal{Q} = q^i Q_i$ ($q^i \in \mathbb{C}$). To assign a closed condition, we have $[\mathcal{Q}, \mathcal{Q}] = \Gamma^\mu q_i q_j = 0 \ \forall \mu$. The homogeneous coordinate $(q_1, ..., q_n)$ we call it *nilpotence variety*. There are two interesting cases in the nilpotence variety:

$$\begin{cases} [Q, Q_\mu] = P_\mu & \text{topological direction} \\ [Q, Q_{\bar{z}}] = P_{\bar{z}} & \text{anti-holomorphic deirection} \end{cases}$$

Just by the ward identity, we have that the correlation function is independent of $x^\mu (or\ \bar{z})$. Thus, after twisting, we realize a SUSY theory as a topological(holomorphic) theory.

The following table lists some twist theories in different dimensions and supercharge numbers. I will focus on the D=4 gauge theories in the following sections

D=2, $\mathcal{N} = (2,2)$	topological twist	R-symmetry $U(1)_A \times U(1)_V$	supercharge
	A-twist	$U(1)_R \xrightarrow{id} U(1)_V$	$Q_A = \bar{Q}_+ + Q_-$
	B-twist	$U(1)_R \xrightarrow{id} U(1)_A$	$Q_B = \bar{Q}_+ + \bar{Q}_-$
D=2, $\mathcal{N} = (0,2)$	holomorphic twist	R-symmetry $U(1)_A$ pick up complex structure	supercharge $Q = \bar{Q}_+$
D=3, $\mathcal{N} = 2$	holo-topo twist	R-symmetry $SO(2)$ pick up complex structure	supercharge $Q = \bar{Q}_+$
D=4, $\mathcal{N} = 1$	holomorphic twist	R-symmetry $U(1)$ pick up complex structure	supercharge $Q = Q_1$
D=4, $\mathcal{N} = 2$	topological twist	R-symmetry $U(2)$ $SU(2)_r \xrightarrow{id} SU(2)_R$	supercharge $Q = \bar{Q}_{\dot{1}2} - \bar{Q}_{\dot{2}1}$
D=4, $\mathcal{N} = 4$	topological twist GL twist	R-symmetry $U(4)$ $\begin{pmatrix} SU(2)_l & 0 \\ 0 & SU(2)_r \end{pmatrix} \to SU(4)_R$	supercharge $Q = uQ_l + vQ_r$

Table 4: (1)Here, $SU(2)_I$ means the subgroup in $U(2)$, and we use $Spin(4) = SU(2)_l \times SU(2)_r$. (2)In D=4,$\mathcal{N} = 2$, the second index from which we down the supercharge index A by ϵ_{ab}. (3)In D=4,$\mathcal{N} = 4$,the Q_l, Q_r refer to [4].

3.1 D=4, $\mathcal{N} = 1$ SYM

As a gauge theory with Lie algebra $\mathfrak{g}$ over 4D manifold M, its action functional is:

$$S_{SYM} = \frac{1}{16\pi} Im(\tau \int d^6z\ Tr(W^\alpha W_\alpha) + c.c)\quad W_a := -\frac{1}{8}\bar{D}^2(e^{-2V}D_\alpha e^{2V})\quad \tau := \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$$

Here, d^6z is short for $d^4xd^2\theta$, and $c.c$ denotes the complex conjugate part. Tr stands for the Killing form of the Lie algebra $\mathfrak{g}$.

$$gauge\ transformation: \quad e^{2V} \to e^{i\bar{\Omega}}e^{2V}e^{-i\Omega}$$

Here, V is the vector field and Ω is the chiral field valued in $\mathfrak{g}$.

I will skip the introduction of the BV formalism(ref. [5]). Following our marks, $\mathcal{V}_g$ and $\mathcal{C}_g$ denote the field spaces of vector and chiral fields valued in $\mathfrak{g}$. In short, originally, we are considering $(\mathcal{V}_{\mathfrak{g}}, \mathcal{L}^{\mathfrak{g}})$ the gauge invariant functional over the field space. Instead of that, we consider the derived complex of the functor $Hom_{\mathfrak{g}}(\mathbb{C}, \_\_)$. The complex is the Chevalley-Eilenberg complex and its $H^0 = \{\mathcal{L}^{\mathfrak{g}}\}$. Based on this, we consider the Kaszul-Tate derived of the equation (df=0) over the CE-complex. As a result, we obtain something we call the BV complex and the corresponding BV derivation δ_{BV}.

Definition 3.1 *(local BV complex)*

$$\mathcal{BV}_{SYM} := (O_{loc}(T^*[-1](\mathcal{V}_g \oplus \mathcal{C}_g[1]), \delta_{SYM})$$

[1] denotes the left shift in a graded vector space; T^* means the cotangent bundle. Specifically, the graded vector space looks like:

$$\mathcal{C}_g[1] \oplus \mathcal{V}_g \oplus \mathcal{V}_g^*[-1] \oplus \mathcal{C}_g^*[-2] := \mathcal{E}$$

$O(\mathcal{E})$ means the functional over the function space $\mathcal{E}$;loc means it must take this form:

$$F(\mathcal{E}_1, ... \mathcal{E}_n) = \int_M \alpha(j_z(\mathcal{E}_1), ..., j_z(\mathcal{E}_n)) : \mathcal{E}^{\otimes n} \to R$$

Here, j_z is the jet bundle of a sheaf at point z; α is a density-valued function on the jet bundles.

Remark 3.1.1: Here I also use the $\mathcal{E}$ to denote the sheaf.

Remark 3.1.2: The odd graded fields are anti-commutative, and the even ones are commutative in the graded vector space. In physics, we call the components in $\mathcal{E}$ from left to right: ghost, field, anti-field, and anti-ghost.

Remark 3.1.3: There is a natural Poisson bracket on the Kaszul-Tate complex, which is the Schouten bracket, we also call it the BV bracket. In particular, we have $\delta_{BV} = \{\_\_, S_{SYM}\}$.

3.1.1 Wess-Zumino gauge and reduce to gauge theory

Extending the gauge transformation by the BCH formula, we have:

$$\delta V = \frac{i}{2}(\bar{\Omega} - \Omega) + O(V))$$

By comparing the formula (3)(4), the first six terms in (4) can be canceled out through the 0-order in gauge.

Definition 3.2 *The Wess-Zumino gauge is to set: C, χ, M components in $\mathcal{V}$ equal to 0. Under this gauge:*

$$\mathcal{V} = \theta \sigma^\mu \bar{\theta} V_\mu + \bar{\theta}^2 \theta^\alpha \lambda_\alpha + \theta^2 \bar{\theta}_{\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}} + \theta^2 \bar{\theta}^2 D$$

The higher order part looks like [V,...], you can't reach the WZ gauge just by transforming once, but the difference will reside in higher terms in (4), you can defer it to the later terms each time. Finally, we can reach the WZ gauge.

So in WZ gauge,

$$W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V + \frac{1}{4}\bar{D}^2([V, D_\alpha V]) = \lambda_\alpha(x) + 2D\theta_\alpha + i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} - i\theta^2 \sigma^\mu D_\mu \bar{\lambda}$$

Using the property:

$$\sigma_a \bar{\sigma}_b \sigma_c = (\eta_{ac}\sigma_b - \eta_{bc}\sigma_a - \eta_{ab}\sigma_c) + i\epsilon_{abcd}\sigma^d$$

Finally, we have:

$$S_{SYM} = \frac{1}{g^2}\int d^4x(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu D_\mu\bar{\lambda} + 2D^2) + \frac{\theta}{32\pi^2}\int d^4x F_{\mu\nu}{}^\star F^{\mu\nu}$$

which is the canonical N=1 gauge theory with θ term.

Lemma 3.1 *The gauge fixing induces a quasi-isomorphism on BV complex.*

$$
\begin{array}{ccc}
\mathcal{L} \in \mathcal{O}_{loc}(\mathcal{E}) & \xrightarrow{e^{\{L_{gf},--\}}} & e^{\{L_{gf},--\}}(\mathcal{L}) \in \mathcal{O}_{loc}(\mathcal{E}^{gf}) \\
\downarrow{\scriptstyle \delta_{BV}} & & \downarrow{\scriptstyle e^{(L_{gf'}-1)}} \\
\{S, \mathcal{L}\} & \xrightarrow{e^{\{L_{gf},--\}}} & \{e^{\{L_{gf},--\}}(S), e^{\{L_{gf},--\}})(\mathcal{L})\}
\end{array}
$$

Corollary 3.1.1 Under Wess-Zumino gauge, the two BV complexes

$$(\mathcal{O}_{loc}(\mathcal{E}^{SYM}), \delta_{SYM}) \overset{quasi-iso}{\cong} (\mathcal{O}_{loc}(\mathcal{E}^{gauge}), \delta_{gauge})$$

3.1.2 Twist theory

Induced from $\mathcal{V}_{\mathfrak{g}}$, we have the SUSY transform on component fields:

$$\delta A_\mu = \epsilon\sigma_\mu\bar{\lambda} + \lambda\sigma_\mu\bar{\epsilon}$$
$$\delta\lambda = \epsilon D + (\sigma^{\mu\nu}\epsilon)F_{\mu\nu}$$
$$\delta D = i\epsilon\sigma^\mu\partial_\mu\bar{\lambda} - i\partial_\mu\lambda\bar{\sigma}^\mu\bar{\epsilon}$$

Transferring to the first-order formalism, we have:

$$\delta_{Q_\alpha} A_\mu = \sigma_{\mu\alpha\dot{\alpha}}\psi^{\dot{\alpha}}$$
$$\delta_{Q_\alpha}\psi_\beta = \epsilon_\alpha^\gamma B_{\gamma\beta}$$

We can choose any spinor as our twist charge, usually $Q := Q_1$. Since selecting a spinor is equivalent to choosing a complex structure, our structure group $Spin(4)$ now reduces to SU(2), which means that now Q is a U(1)-charge scalar but not a spinor as a derivation. This induces a second derivation in our BV complex:

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$$
\begin{array}{cccc}
0 & 1 & 2 & 3
\end{array}
$$

$$
\begin{array}{ccc}
 & \psi_- \xrightarrow{\ \not{D}\ } \psi'_+ & \\
 & \downarrow Q & \\
\Omega^0 \xrightarrow{\ d\ } \Omega^1 \xrightarrow{\ d_+\ } \Omega^2_+
\end{array}
$$

$$
\Omega^2_+ \xrightarrow{\ d\ } \Omega^3 \xrightarrow{\ d\ } \Omega^4
$$

with $c\,\mathrm{Id}$ arrow and

$$
\psi_+ \xrightarrow{\ \not{D}\ } \psi'_-
$$

The new derivation now has the wrong degree. We can add a BV degree 1, U(1) degree -1 element t to that. And we have:

$$
(\mathcal{O}_{loc}(\mathcal{E}^{gauge})((t)), \delta_{gauge} + tQ)
$$

$$
\begin{array}{cccc}
0 & 1 & 2 & 3
\end{array}
$$

In short, what we did was construct a Q-cohomology by twisting and blending it with the BV-comology in the correct degree.

What's more, choosing Q not only gives us a complex structure, but also the following isomorphism:

$$
\Gamma: \psi^- \xrightarrow{\ \cong\ } \Omega^{1,0} \tag{5}
$$

Here, we substitute Q with the following left-spinor part:

$$
V_\mu \to (\sigma^\mu V_\mu)_{\alpha\dot\alpha}
$$

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And,

$$Y : \Omega^0 \cdot w \oplus \Omega^{0,2} \xrightarrow{\cong} \psi^+ \tag{6}$$

Here, we use the isomorphism and pair one of ψ_+ with Q.

$$\Omega^2_+ \cong Sym^2(\psi^+)$$

With these isomorphisms, we have:

Theorem 3.1 *(Costello 4.0.2 [6]) The twisted N = 1 gauge theory on $\mathbb{C}^2$ is equivalent to the holomorphic BF theory.*

In the same paper, costello also proves that:

Theorem 3.2 *(Costello 4.1.1 [6]) The twisted N = 1 theory admits a unique quantization, compatible with certain natural symmetries, on any Calabi-Yau surface X.*

3.2   D=4, $\mathcal{N} = 2$ Seiberg-Witten theory

3.3   D=4, $\mathcal{N} = 4$ GL twist

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