Complex Analysis HW 6

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May 8, 2023

P166.2.

We set $R_n := n$ so that $R_n \to \infty$, and claim that p_n has no roots in $|z| < n$.

Write

$$g(z) := z^n p_n\left(\frac{n}{z}\right) = z^n \sum_{k=0}^{n} \frac{n^k z^{-k}}{k!} = \sum_{k=0}^{n} \frac{n^k z^{n-k}}{k!} = z^n + \frac{n}{1!} z^{n-1} + \frac{n^2}{2!} z^{n-2} + \cdots + \frac{n^{n-1}}{(n-1)!} z + \frac{n^n}{n!} = \sum_{k=0}^{n} a_k z^k$$

so that

$$a_0 = a_1 > a_2 > \cdots > a_n$$

Then

$$(1-z)g(z) = (a_0 + a_1 z + \dots) - (a_0 z + a_1 z^2 + \dots) = a_0 + (a_1 - a_0)z + (a_2 - a_1)z^2 + \cdots + (a_{n-1} - a_n)z^n - a_n z^{n+1}$$
$$= a_0 - \left[(a_0 - a_1)z + (a_1 - a_2)z^2 + \cdots + (a_{n-1} - a_n)z^n + a_n z^{n+1}\right]$$

For $|z| \le 1$, we bound the bracketed expression.

$$|\dots| \le (a_0 - a_1)|z| + (a_1 - a_2)|z|^2 + \cdots + (a_{n-1} - a_n)|z|^n + a_n|z|^{n+1} \le (a_0 - a_1) + (a_1 - a_2) + \cdots + (a_{n-1} - a_n) + a_n = a_0$$

Thus for $|z| \le 1,\ z \ne 1$,

$$|(1-z)g(z)| \ge |a_0 - |\dots|| \ge 0 \implies |g(z)| \ge 0$$

The above expression for $(1-z)g(z)$ equals 0 only if $z = 1$. Thus $|g(z)| > 0$ for $|z| \le 1$, i.e. the roots of g lie in $|z| > 1$. Using this, we can bound the roots z of p_n by setting $w := \frac{n}{z}$ and observing

$$p_n\left(\frac{n}{w}\right) = 0 \implies g(w) = w^n p_n\left(\frac{n}{w}\right) = 0 \implies |w| > 1 \implies |z| < n$$

P168.2.

Pf. Suppose the negation of the result holds, i.e. there exists a compact set $K \subset R$ and a subsequence P_{n_k} such that $P_{n_k} \cap K \ne \varnothing$. From this we can pick a pole $z_{n_k} \in K$ of f_{n_k}.

$$\sigma(f_{n_k}(z_{n_k}), f(z_{n_k})) = \sigma(\infty, f(z_{n_k})) = \frac{2}{[|f(z_{n_k})|^2 + 1]^{1/2}} \implies \sup_{z \in K} \sigma(f_{n_k}(z), f(z)) \ge \frac{2}{[|f(z_{n_k})|^2 + 1]^{1/2}} \ge 0$$

Since $f_n \to f$ normally, $\sigma(f_n(z), f(z)) \to 0$ uniformly on K, hence

$$\sup_{z \in K} \sigma(f_n(z), f(z)) \xrightarrow{n \to \infty} 0 \implies \sup_{z \in K} \sigma(f_{n_k}(z), f(z)) \xrightarrow{k \to \infty} 0$$

The above facts, along with the squeeze theorem, give

$$\frac{2}{[|f(z_{n_k})|^2 + 1]^{1/2}} \xrightarrow{k \to \infty} 0 \implies f(z_{n_k}) \xrightarrow{k \to \infty} \infty$$

But f is analytic hence continuous on the compact set K, so the sequence $f(z_{n_k})$ is bounded, a contradiction.

P172.1.

Pf. ($\Longrightarrow$) Fix $\epsilon > 0$. Since Φ is equicontinuous, there exists $\delta > 0$ such that for all $t \in \mathbb{R}$, $|x-y| < \delta$ implies $|\varphi(x+t) - \varphi(y+t)| < \epsilon$. It holds for $t = 0$ in particular, so $|x-y| < \delta$ implies $|\varphi(x) - \varphi(y)| < \epsilon$. Thus φ is uniformly continuous.
($\Longleftarrow$) Fix $\epsilon > 0$. Since φ is uniformly continuous, there exists $\delta > 0$ such that $|x-y| < \delta$ implies $|\varphi(x) - \varphi(y)| < \epsilon$. For all $t \in \mathbb{R}$, if $|x-y| < \delta$ then $|(x+t) - (y+t)| = |x-y| < \delta$, and in turn $|\varphi(x+t) - \varphi(y+t)| < \epsilon$. Thus Φ is equicontinuous.

P174.1.

First some preliminary computations. Writing $z = x + iy$, the function

$$g_n(z) := \frac{|f_n'(z)|}{1+|f_n(z)|^2} = \frac{|ne^{nz}|}{1+|e^{nz}|^2} = \frac{ne^{nx}}{1+e^{2nx}} = \frac{n}{e^{-nx}+e^{nx}} = \frac{n}{2\cosh nx}$$

attains a maximum value of $\frac{n}{2}$ at $x = 0$.

We claim that the f_n's are normal precisely for domains D where $0 \notin D$.

First consider the case $0 \in D$. This assumption along with D being open means we can pick a compact set E with $0 \in E \subset D$. By the preliminary computations, the sequence $\sup_{z\in E} g_n(z) = \frac{n}{2}$ is unbounded, hence the f_n's are not normal.

Now consider the case $0 \notin D$. Fix a compact set $E \subset D$. From the properties of D and E, we have $0 \notin E$ and that 0 is an isolated point of E, giving some $r > 0$ such that the disk $B(0,r)$ and E are disjoint. Then for all $z = x + iy \in E$,

$$|x| \geq r \implies \cosh nx \geq \cosh nr \implies g_n(z) \leq g_n(r) =: K_n$$

This gives $\sup_{z\in E} g_n(z) \leq K_n$. The sequence K_n converges to 0, hence it is bounded by some $K(E)$ independent of n. Thus the f_n's are normal.

P187.7.

Pf. We can take $a = 0,\ b = 1,\ c = \infty$ since, otherwise, we can pick d such that $ad - bd = 1$, so that

$$F(z) := \frac{af(z)+b}{cf(z)+d}$$

is meromorphic and does not attain 0 or 1, and we can proceed replacing f by F.

Let μ be the elliptic modular function and set

$$g(z) := \mu^{-1}(f(z)), \quad h(z) := i\frac{1+z}{1-z}$$

Then $h^{-1} \circ g^{-1} \circ f$ is entire and bounded, so by Liouville's theorem it is constant. In turn f is constant.