

Complex Analysis HW 1

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Note. Define $\text{cis } \theta := \cos \theta + i \sin \theta$.

9.2. Pf. Write $z = x + yi$ and $w = u + vi$.

1.

$$\overline{z+w} = \overline{(x+u) + (y+v)i} = (x+u) - (y+v)i = (x-yi) + (u-vi) = \bar{z} + \bar{w}$$

2.

$$\overline{zw} = \overline{(xu-yv) + (xv+yu)i} = (xu-yv) - (xv+yu)i = xu - xvi - yui - yv = (x-yi)(u-vi) = \bar{z}\bar{w}$$

11.2. Ans. Set $r := |z|$, $s := |w|$, $\theta := \text{Arg } z$, $\varphi := \text{Arg } w$. Then $zw = rs \text{cis}(\theta + \varphi)$, so that

$$\text{Arg}(zw) = \theta + \varphi \iff -\pi < \theta + \varphi \leq \pi \iff -\pi - \varphi < \theta \leq \pi - \varphi$$

In other words, $\text{Arg}(zw) = \text{Arg } z + \text{Arg } w$ iff $-\pi - \text{Arg } w < \text{Arg } z \leq \pi - \text{Arg } w$.

13.2. Ans. We know $z \neq 1$ since plugging $z = 1$ gives $0 = 1$. Now

$$\left(\frac{z}{z-1}\right)^4 = 1 = \text{cis}(2\pi k), \quad k \in \mathbb{Z} \implies \frac{z}{z-1} = \text{cis}\left(\frac{\pi k}{2}\right), \quad 1 \leq k \leq 3$$

Taking the 4th root and the condition $z \neq 1$ means we take $1 \leq k \leq 3$. Then

$$z = z \text{cis}\left(\frac{\pi k}{2}\right) - \text{cis}\left(\frac{\pi k}{2}\right) \implies z \left[\text{cis}\left(\frac{\pi k}{2}\right) - 1 \right] = \text{cis}\left(\frac{\pi k}{2}\right) \implies z = \frac{\text{cis}\left(\frac{\pi k}{2}\right)}{\text{cis}\left(\frac{\pi k}{2}\right) - 1}$$

Plugging in the values of k gives

$$\frac{\text{cis}\left(\frac{\pi(1)}{2}\right)}{\text{cis}\left(\frac{\pi(1)}{2}\right) - 1} = \frac{i}{i-1} \cdot \frac{-i-1}{-i-1} = \frac{1-i}{2}$$

$$\frac{\text{cis}\left(\frac{\pi(2)}{2}\right)}{\text{cis}\left(\frac{\pi(2)}{2}\right) - 1} = \frac{-1}{-1-1} = \frac{1}{2}$$

$$\frac{\text{cis}\left(\frac{\pi(3)}{2}\right)}{\text{cis}\left(\frac{\pi(3)}{2}\right) - 1} = \frac{-i}{-i-1} \cdot \frac{i-1}{i-1} = \frac{1+i}{2}$$

Thus the solutions are $z = \frac{1}{2}, \frac{1}{2}(1 \pm i)$.

15.2. Pf. Since $|r \text{cis } \theta| = |r| |\text{cis } \theta| = |r| < 1$, the following geometric series converges.

$$\sum_{n \geq 0} (r \text{cis } \theta)^n = \frac{1}{1 - r \text{cis } \theta} = \frac{1 - r \cos \theta + ir \sin \theta}{(1 - r \cos \theta - ir \sin \theta)(1 - r \cos \theta + ir \sin \theta)} = \frac{1 - r \cos \theta + ir \sin \theta}{(1 - r \cos \theta)^2 + r^2 \sin^2 \theta}$$

$$= \frac{1 - r \cos \theta + ir \sin \theta}{1 - 2r \cos \theta + r^2 \cos^2 \theta + r^2 \sin^2 \theta} = \frac{1 - r \cos \theta + ir \sin \theta}{1 + r^2 - 2r \cos \theta}$$

Now by de Moivre's formula, for all $n \geq 0$,

$$(r \operatorname{cis} \theta)^n = r^n \operatorname{cis}^n \theta = r^n \operatorname{cis} n\theta = r^n \cos n\theta + ir^n \sin n\theta$$

so that

$$\sum_{n \geq 0} (r \operatorname{cis} \theta)^n = \sum_{n \geq 0} r^n \cos n\theta + i \sum_{n \geq 0} r^n \sin n\theta$$

Using the above calculation,

$$\sum_{n \geq 0} r^n \cos n\theta + i \sum_{n \geq 0} r^n \sin n\theta = \frac{1 - r \cos \theta + ir \sin \theta}{1 + r^2 - 2r \cos \theta}$$

Equating real parts,

$$\sum_{n \geq 0} r^n \cos n\theta = \frac{1 - r \cos \theta}{1 + r^2 - 2r \cos \theta}$$

26.2. Pf. Euler's formula gives

$$\exp[i(a + b)] = \cos(a + b) + i \sin(a + b)$$

The addition formula for exp gives

$$\exp[i(a + b)] = \exp[ia + ib] = \exp(ia) \exp(ib)$$

$$= (\cos a + i \sin a)(\cos b + i \sin b) = [\cos a \cos b - \sin a \sin b] + i[\cos a \sin b + \sin a \cos b]$$

Equating real and imaginary parts,

$$\cos(a + b) = \cos a \cos b - \sin a \sin b$$

$$\sin(a + b) = \cos a \sin b + \sin a \cos b$$