Claim: $F_n \to F$ converges with properties
① F 1:1 analytic on D_0
② $F(0) = 0$
③ $F(D_0) = U$

$u_n(z) = \log|F_n(z)|$ on $D_0 - \{0\}$
increasing seq of harmonic functions

$u_n(z)$ bounded above (by 0), and monotone increasing
so $u_n(z) \to u(z)$ harmonic on D_0 by Poisson formula
$u(z) \leq 0$.
$U_n + iV_n \longrightarrow$ locally analytic on D_0

$\Rightarrow F_n(z) \to F(z)$ analytic on D_0 by normal convergence

① F not constant. $0 = F(0)$.
By Schwarz lemma, $f_n'(0) > 1$
so by chain rule, $F_n'(0) > 1$
so $F'(0) \geq 1$, i.e. non constant.

② F conformal, i.e. 1:1.
F_n are 1:1, so fixed $z_1, z_2 \in D_0$ $0 \neq F_n(z_1) - F_n(z_2)$
By isolation of zeros (consequence of Rouché),
we get $F(z_1) - F(z_2) \neq 0$

③ $F(D_0) = D_\infty \subset U$. Show $D_\infty = U$
Check out $\omega_n \in U - D_n$

Normal Families

D domain

$H(D)$ vector space of f analytic on D.

Def. ① $\mathcal{F} \subset H(D)$ is normal if every sequence $f_n \in \mathcal{F}$ has a subsequence f_{n_k} which converges normally to $f \in H(D)$.

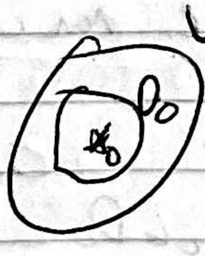
② $\mathcal{F}$ compact if every $f_n \in \mathcal{F}$ has a subsequence $f_{n_k} \to f \in \mathcal{F}$.

$\mathcal{F} \in H(D)$ bounded if for every compact $K \subset\subset D$ there is a $K \in \mathbb{R}^{\geq 0}$ so that $|f(z)| < k$, $f \in \mathcal{F}$, $z \in D$.

Montel's theorem "bullshit theorem"

$\mathcal{F} \in H(D)$ is compact $\iff$ $\mathcal{F}$ closed & bounded,

(Uses Ascoli-Arzela: $\mathcal{F}$ of cont. functions on compact $K \subset \mathbb{R}^n$ is normal $\iff$ $\mathcal{F}$ bounded & equicontinuous.)

Example:  U = unit disk. $H(D_0)$ holomorphic functions on D_0.

$\cup$

$S(D_0)$ conformal on D_0 s.t.

① $f(0)=0$, $f'(0)>0$.

② $f(D_0) \subseteq U$.

Claim: $S(D_0)$ is compact.

① $S(D_0)$ is closed: $f_n \in S(D_0)$, $f_n \xrightarrow{\text{normally}} f \in H(D_0)$
f holomorphic, $f_n(z_1) - f_n(z_2) \neq 0 \Rightarrow$
$f(z_1) - f(z_2) \neq 0$ so $f \in S(D_0)$.
(Same as earlier argument).

② choose sequence $z_k \in D_0$ so that
$\forall K \subset\subset D_0$, z_k is dense in K.
diagonalization style argument:
at any z_k, $f_n(z_k)$ is bounded.
so $\exists$ subsequence $f_{n_m}(z_k)$ convergent to $\lambda \in \mathbb{C}$.
Use diagonal process

z_0: $f_1(z_0)$, $f_2(z_0)$, ...
$\downarrow$
$f_1(z_0)$, $f_4(z_0)$, ... $\longrightarrow \lambda_0$
z_1: $f_1(z_0)$, $f_4(z_0)$, ...
$\downarrow$
$f_4(z_0)$, $f_{30}(z_0)$, ... $\longrightarrow \lambda_1$

take all diagonal terms (in this example, f_1, f_{30}, etc)
this converges for each z_K.

$\mathcal{F} \in S(D_0)$ is equicontinuous on D_0, i.e.
for every $K \subset\subset D_0$, and $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.
$|z_1 - z_2| \leq \delta \Rightarrow |f(z_1) - f(z_2)| < \varepsilon \quad \forall f \in \mathcal{F}$.

derivative is bounded $\Rightarrow$ equicontinuous, so $\{f_n\}$ is equicontinuous.

Define $f = \lim_{n \to \infty} f_n$ (taking ultimate subsequence)

claim: $f_n(z)$ close to $f_n(z_k) \to f(z)$ (by uniform continuity).
$\Rightarrow f_n \to f$ uniformly on K.

$S(D_0)$ compact,

$0 < f'(0) <$ something, i.e. $f'(0)$ bounded above

Let f_0 be the function in $S(D_0)$ that maximizes $f'(0)$.

Then $f_0(D) = U$

(use the old method to increase it otherwise)