Complex Analysis HW 1

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Note. Define $\operatorname{cis}\theta := \cos\theta + i\sin\theta$.

9.2. Pf. Write $z = x + yi$ and $w = u + vi$.

1. $$\overline{z+w} = \overline{(x+u)+(y+v)i} = (x+u)-(y+v)i = (x-yi)+(u-vi) = \overline{z}+\overline{w}$$

2. $$\overline{zw} = \overline{(xu-yv)+(xv+yu)i} = (xu-yv)-(xv+yu)i = xu-xvi-yui-yv = (x-yi)(u-vi) = \overline{z}\,\overline{w}$$

11.2. Ans. Set $r := |z|,\ s := |w|,\ \theta := \operatorname{Arg} z,\ \varphi := \operatorname{Arg} w$. Then $zw = rs\operatorname{cis}(\theta+\varphi)$, so that

$$\operatorname{Arg}(zw) = \theta+\varphi \iff -\pi < \theta+\varphi \le \pi \iff -\pi-\varphi < \theta \le \pi-\varphi$$

13.2. Ans.

$$(z-1)^4 = z^4 \iff z^4-(z-1)^4 = 0 \iff [z^2-(z-1)^2][z^2+(z-1)^2] = 0 \iff z^2-(z-1)^2 = 0 \text{ or } z^2+(z-1)^2 = 0$$

In the first above case,

$$0 = z^2-(z^2-2z+1) = 2z-1 \iff z = \frac{1}{2}$$

In the second above case,

$$0 = z^2+z^2-2z+1 = 2z^2-2z+1 \iff z = \frac{1}{4}\left[2\pm\sqrt{4-4(2)(1)}\right] = \frac{1}{4}\left[2\pm\sqrt{-4}\right] = \frac{1}{4}\left[2\pm 2i\right] = \frac{1}{2}\left[1\pm i\right]$$

Thus the solutions are $z = \frac{1}{2}, \frac{1}{2}[1\pm i]$.

15.2. Pf. Since $|r\operatorname{cis}\theta| = |r||\operatorname{cis}\theta| = |r| < 1$, the following geometric series converges.

$$\sum_{n\ge 0}(r\operatorname{cis}\theta)^n = \frac{1}{1-r\operatorname{cis}\theta} = \frac{1-r\cos\theta+ir\sin\theta}{(1-r\cos\theta-ir\sin\theta)(1-r\cos\theta+ir\sin\theta)} = \frac{1-r\cos\theta+ir\sin\theta}{(1-r\cos\theta)^2+r^2\sin^2\theta}$$

$$= \frac{1-r\cos\theta+ir\sin\theta}{1-2r\cos\theta+r^2\cos^2\theta+r^2\sin^2\theta} = \frac{1-r\cos\theta+ir\sin\theta}{1+r^2-2r\cos\theta}$$

Now by de Moivre's formula, for all $n \ge 0$,

$$(r\operatorname{cis}\theta)^n = r^n\operatorname{cis}^n\theta = r^n\operatorname{cis}n\theta = r^n\cos n\theta + ir^n\sin n\theta$$

so that

$$\sum_{n\geq 0}(r\operatorname{cis}\theta)^n = \sum_{n\geq 0} r^n\cos n\theta + i\sum_{n\geq 0} r^n\sin n\theta$$

Using the above calculation,

$$\sum_{n\geq 0} r^n\cos n\theta + i\sum_{n\geq 0} r^n\sin n\theta = \frac{1 - r\cos\theta + ir\sin\theta}{1 + r^2 - 2r\cos\theta}$$

Equating real parts,

$$\sum_{n\geq 0} r^n\cos n\theta = \frac{1 - r\cos\theta}{1 + r^2 - 2r\cos\theta}$$

26.2. Pf. Euler's formula gives

$$\exp[i(a+b)] = \cos(a+b) + i\sin(a+b)$$

The addition formula for exp gives

$$\exp[i(a+b)] = \exp[ia+ib] = \exp(ia)\exp(ib)$$

$$= (\cos a + i\sin a)(\cos b + i\sin b) = [\cos a\cos b - \sin a\sin b] + i[\cos a\sin b + \sin a\cos b]$$

Equating real and imaginary parts,

$$\cos(a+b) = \cos a\cos b - \sin a\sin b$$

$$\sin(a+b) = \cos a\sin b + \sin a\cos b$$