Complex Analysis HW 5

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P141.1. Pf. Recall the product expansion

$$\cos z = \prod_{k=1}^{\infty} \left[1 - \frac{4z^2}{\pi^2(2k-1)^2}\right]$$

hence

$$\cos \pi\sqrt{z} = \prod_{k=1}^{\infty} \left[1 - \frac{4\pi^2 z}{\pi^2(2(k-1/2))^2}\right] = \prod_{k=1}^{\infty} \left[1 - \frac{4z}{4(k-1/2)^2}\right] = \prod_{k=1}^{\infty} \left[1 - \frac{z}{(k-1/2)^2}\right]$$

P144.2. Pf. Form the Blaschke product

$$B(z) := z \cdot \frac{|-1/2|}{1/2} \frac{(z+1/2)}{1+z/2} \cdot \frac{|-3/4|}{3/4} \frac{z+3/4}{1+3z/4} = z \cdot \frac{z+1/2}{1+z/2} \cdot \frac{z+3/4}{1+3z/4}$$

Then

$$B\left(\frac{1}{2}\right) = \frac{1}{2} \cdot \frac{1}{5/4} \cdot \frac{5/4}{11/8} = \frac{1}{2} \cdot \frac{8}{11} = \frac{4}{11}$$

On the unit disk, $|f/B| \leq 1$, so that $|f| \leq |B|$, thus

$$\left|f\left(\frac{1}{2}\right)\right| \leq \left|B\left(\frac{1}{2}\right)\right| = \frac{4}{11}$$

P151.2. Pick the sequence $a_k := 2^k$.

$$\sum_{k=1}^{\infty} \frac{1}{|a_k|^{0+1}} = \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty$$

so its genus is $h = 0$. Its order is $\mu = 0$ since for all $\epsilon > 0$,

$$2^\epsilon > 1 \implies \frac{1}{2^\epsilon} < 1 \implies \sum_{k=1}^{\infty} \frac{1}{|a_k|^{0+\epsilon}} = \sum_{k=1}^{\infty} \frac{1}{2^{k\epsilon}} < \infty$$

Use the sequence a_k and its genus $h = 0$ to form the canonical product (a transcendental function)

$$P(z) := \prod_{k=1}^{\infty} E_0\left(\frac{z}{a_k}\right) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{2^k}\right)$$

Since the order of the sequence a_k is $\mu = 0$, by theorem 35 in the lecture notes the order of P is at most 0, i.e. the order of P is 0 itself.

P155.2. Pf. Recall that

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}$$

Taking logs,

$$\log \Gamma(z) = -\gamma z - \log z + \sum_{n=1}^{\infty} \left[-\log\left(1 + \frac{z}{n}\right) + \frac{z}{n} \right]$$

Differentiating,

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \left[-\frac{1/n}{1 + z/n} + \frac{1}{n} \right] = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \left[-\frac{1}{z+n} + \frac{1}{n} \right]$$

Differentiating again,

$$\frac{\mathrm{d}}{\mathrm{d}z} \frac{\Gamma'(z)}{\Gamma(z)} = \frac{1}{z^2} + \sum_{n=1}^{\infty} \frac{1}{(z+n)^2} = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}$$

P116.4. Pf. Write $E = \{z_1, \ldots, z_M\}$. For each $1 \le j \le M$, with f having a pole of finite order (say N_j) at z_j, write the principal part of f at z_j as

$$P_j(z) = \sum_{n=1}^{N_j} \frac{a_{j,-n}}{(z - z_j)^n}$$

Then

$$R(z) := \sum_{j=1}^{M} P_j(z)$$

is a sum of rational functions hence a rational function. Since each P_j is analytic on $D - E$, so is R, and in turn so is $f - R$. To see that $f - R$ is analytic on E, fix $1 \le j \le M$ and note that the principal part of R at z_j is P_j, so the principal part of $f - R$ at z_j is $P_j - P_j = 0$, hence $f - R$ is analytic at z_j. Thus $f - R$ is analytic on D.