Complex Analysis HW 1

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Note. Define $\operatorname{cis}\theta := \cos\theta + i\sin\theta$.

P9.2. Pf. Write $z = x + yi$ and $w = u + vi$.

1.
$$\overline{z+w} = \overline{(x+u)+(y+v)i} = (x+u)-(y+v)i = (x-yi)+(u-vi) = \overline{z}+\overline{w}$$

2.
$$\overline{zw} = \overline{(xu-yv)+(xv+yu)i} = (xu-yv)-(xv+yu)i = xu - xvi - yui - yv = (x-yi)(u-vi) = \overline{z}\,\overline{w}$$

P11.2. Ans. Set $r := |z|,\ s := |w|,\ \theta := \operatorname{Arg} z,\ \varphi := \operatorname{Arg} w$. Then $zw = rs\operatorname{cis}(\theta+\varphi)$, so that

$$\operatorname{Arg}(zw) = \theta + \varphi \iff -\pi < \theta + \varphi \le \pi \iff -\pi - \varphi < \theta \le \pi - \varphi$$

In other words, $\operatorname{Arg}(zw) = \operatorname{Arg} z + \operatorname{Arg} w$ iff $-\pi - \operatorname{Arg} w < \operatorname{Arg} z \le \pi - \operatorname{Arg} w$.

P13.2. Ans. We know $z \neq 1$ since plugging $z = 1$ gives $0 = 1$. Now

$$\left(\frac{z}{z-1}\right)^4 = 1 = \operatorname{cis}(2\pi k),\ k \in \mathbb{Z} \implies \frac{z}{z-1} = \operatorname{cis}\left(\frac{\pi k}{2}\right),\ 1 \le k \le 3$$

Taking the 4th root and the condition $z \neq 1$ means we take $1 \le k \le 3$. Then

$$z = z\operatorname{cis}\left(\frac{\pi k}{2}\right) - \operatorname{cis}\left(\frac{\pi k}{2}\right) \implies z\left[\operatorname{cis}\left(\frac{\pi k}{2}\right) - 1\right] = \operatorname{cis}\left(\frac{\pi k}{2}\right) \implies z = \frac{\operatorname{cis}\left(\frac{\pi k}{2}\right)}{\operatorname{cis}\left(\frac{\pi k}{2}\right) - 1}$$

Plugging in the values of k gives

$$\frac{\operatorname{cis}\left(\frac{\pi(1)}{2}\right)}{\operatorname{cis}\left(\frac{\pi(1)}{2}\right) - 1} = \frac{i}{i-1} \cdot \frac{-i-1}{-i-1} = \frac{1-i}{2}$$

$$\frac{\operatorname{cis}\left(\frac{\pi 2}{2}\right)}{\operatorname{cis}\left(\frac{\pi 2}{2}\right) - 1} = \frac{-1}{-1-1} = \frac{1}{2}$$

$$\frac{\operatorname{cis}\left(\frac{\pi 3}{2}\right)}{\operatorname{cis}\left(\frac{\pi 3}{2}\right) - 1} = \frac{-i}{-i-1} \cdot \frac{i-1}{i-1} = \frac{1+i}{2}$$

Thus the solutions are $z = \frac{1}{2}, \frac{1}{2}(1 \pm i)$.

P15.2. Pf. Since $|r\operatorname{cis}\theta| = |r||\operatorname{cis}\theta| = |r| < 1$, the following geometric series converges.

$$\sum_{n\ge 0}(r\operatorname{cis}\theta)^n = \frac{1}{1 - r\operatorname{cis}\theta} = \frac{1 - r\cos\theta + ir\sin\theta}{(1 - r\cos\theta - ir\sin\theta)(1 - r\cos\theta + ir\sin\theta)} = \frac{1 - r\cos\theta + ir\sin\theta}{(1 - r\cos\theta)^2 + r^2\sin^2\theta}$$

$$= \frac{1 - r\cos\theta + ir\sin\theta}{1 - 2r\cos\theta + r^2\cos^2\theta + r^2\sin^2\theta} = \frac{1 - r\cos\theta + ir\sin\theta}{1 + r^2 - 2r\cos\theta}$$

Now by de Moivre's formula, for all $n \geq 0$,

$$(r \operatorname{cis}\theta)^n = r^n \operatorname{cis}^n\theta = r^n \operatorname{cis} n\theta = r^n \cos n\theta + ir^n \sin n\theta$$

so that

$$\sum_{n\geq 0} (r \operatorname{cis}\theta)^n = \sum_{n\geq 0} r^n \cos n\theta + i \sum_{n\geq 0} r^n \sin n\theta$$

Using the above calculation,

$$\sum_{n\geq 0} r^n \cos n\theta + i \sum_{n\geq 0} r^n \sin n\theta = \frac{1 - r\cos\theta + ir\sin\theta}{1 + r^2 - 2r\cos\theta}$$

Equating real parts,

$$\sum_{n\geq 0} r^n \cos n\theta = \frac{1 - r\cos\theta}{1 + r^2 - 2r\cos\theta}$$

P26.2. Pf. Euler's formula gives

$$\exp[i(a+b)] = \cos(a+b) + i\sin(a+b)$$

The addition formula for exp gives

$$\exp[i(a+b)] = \exp[ia + ib] = \exp(ia)\exp(ib)$$

$$= (\cos a + i\sin a)(\cos b + i\sin b) = [\cos a\cos b - \sin a\sin b] + i[\cos a\sin b + \sin a\cos b]$$

Equating real and imaginary parts,

$$\cos(a+b) = \cos a\cos b - \sin a\sin b$$

$$\sin(a+b) = \cos a\sin b + \sin a\cos b$$