Complex Analysis Quiz 1

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Note. Define $\operatorname{cis}\theta := \cos\theta + i\sin\theta$.

Solve $(z-1)^6 = z^6$.

Ans. We know $z \neq 1$ since plugging in $z = 1$ gives $0 = 1$. So

$$\left(\frac{z}{z-1}\right)^6 = 1 = \operatorname{cis}(2\pi k),\ k \in \mathbb{Z} \implies \frac{z}{z-1} = \operatorname{cis}\left(\frac{\pi k}{3}\right),\ 1 \leq k \leq 5$$

Taking the 6th root, and the condition $z \neq 1$ means we take $1 \leq k \leq 5$. Then

$$z = z\operatorname{cis}\left(\frac{\pi k}{3}\right) - \operatorname{cis}\left(\frac{\pi k}{3}\right) \implies z\left[\operatorname{cis}\left(\frac{\pi k}{3}\right) - 1\right] = \operatorname{cis}\left(\frac{\pi k}{3}\right) \implies z = \frac{\operatorname{cis}\left(\frac{\pi k}{3}\right)}{\operatorname{cis}\left(\frac{\pi k}{3}\right) - 1}$$

Plugging in the values of k gives

$$z = \frac{1}{2},\ \frac{1}{2}\left(1 \pm \frac{i}{\sqrt{3}}\right),\ \frac{1}{2}(1 \pm i\sqrt{3})$$