Complex Analysis HW 3

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P60.6. Pf. Fix $z \in R$ and write $f(z) = u + iv$.

$$|f(z) - 1| < 1 \implies (u-1)^2 \leq (u-1)^2 + v^2 < 1 \implies |u-1| < 1 \implies -1 < u - 1 < 1 \implies u > 0$$

so $f(R)$ is contained in the right half plane, on which Ln is analytic. With f analytic on R, this means $\operatorname{Ln} f$ is analytic on R, and by the chain rule, $(\operatorname{Ln} f)' = \frac{f'}{f}$. Then γ being closed gives us $\int_\gamma \frac{f'}{f} = 0$.

P60.7. Since Q_N is a square with side length $2N\pi$, the length of Q_N is

$$L_N := \int_{Q_N} |dz| = 4 \cdot 2N\pi = 8N\pi$$

Set $f(z) := \frac{1}{z \cos z}$. We seek a bound on f along Q_N. First examining the left and right sides of Q_N, write $z = \pm N\pi + ti, \ -N\pi \leq t \leq N\pi$.

$$\cos z = \cos(\pm N\pi)\cos(it) - \sin(\pm N\pi)\sin(it) = (-1)^N \cosh t \implies |\cos z| = \cosh t \geq 1 \implies \frac{1}{|\cos z|} \leq 1$$

$$|z|^2 = (N\pi)^2 + t^2 \geq (N\pi)^2 \implies \frac{1}{|z|} \leq \frac{1}{N\pi}$$

$$|f(z)| = \frac{1}{|z||\cos z|} \leq \frac{1}{N\pi}$$

Then examining the top and bottom sides of Q_N, write $z = t \pm N\pi i, \ -N\pi \leq t \leq N\pi$.

$$\cos z = \cos(t)\cos(\pm N\pi i) - \sin(t)\sin(\pm N\pi i) = \cosh N\pi \cos t \mp i \sinh N\pi \sin t$$

$$\implies |\cos z| = \cosh^2 N\pi \cos^2 t + \sinh^2 N\pi \sin^2 t = (\sinh^2 N\pi + 1)\cos^2 t + \sinh^2 N\pi \sin^2 t$$

$$= \sinh^2 N\pi(\cos^2 t + \sin^2 t) + \cos^2 t = \sinh^2 N\pi + \cos^2 t \geq \sinh^2 N\pi \geq \sinh^2 \pi \geq 1$$

$$\implies \frac{1}{|\cos z|} \leq 1$$

$$|z|^2 = t^2 + (N\pi)^2 \geq (N\pi)^2 \implies \frac{1}{|z|} \leq \frac{1}{N\pi}$$

$$|f(z)| = \frac{1}{|z||\cos z|} \leq \frac{1}{N\pi}$$

Thus $|f| \leq \frac{1}{N\pi}$ on Q_N. We then have a constant bound on the integral below.

$$\left| \int_{Q_N} f(z) dz \right| \leq \frac{1}{N\pi} L_N = \frac{8N\pi}{N\pi} = 8$$

P66.1. Pf. With f analytic hence holomorphic on D, for $z_0 \in D$ there exists $r > 0$ such that for $|z - z_0| < r$,

$$f(z) = \sum_{j \geq 0} a_j z^j$$

Now set

$$F(z) := \sum_{j \geq 0} \frac{1}{j+1} a_j z^{j+1}$$

so that $F'(z) = f(z)$ for $|z - z_0| < r$ (in particular $F'(z_0) = f(z_0)$), giving F as holomorphic at z_0.

P68. First we find the Maclaurin series for $z \cosh z^2$. Using the Maclaurin series for exp,

$$e^z = \sum_{n \geq 0} \frac{z^n}{n!} \implies e^{-z} = \sum n \geq 0 \frac{(-1)^n z^n}{n!}$$

Due to the $(-1)^n$ term, adding the two series will cancel the odd terms and double the even terms.

$$\cosh z = \frac{1}{2}(e^z - e^{-z}) = \frac{1}{2} \sum_{n \geq 0} 2 \frac{z^{2n}}{(2n)!} = \sum_{n \geq 0} \frac{z^{2n}}{(2n)!}$$

We then get the Maclaurin series for $z \cosh z^2$.

$$\implies \cosh z^2 = \sum_{n \geq 0} \frac{z^{4n}}{(2n)!} \implies \boxed{z \cosh z^2 = \sum_{n \geq 0} \frac{z^{4n+1}}{(2n)!}}$$

Perform the ratio test for convergence.

$$\left| \frac{z^{4n+5}}{(2n+2)!} \frac{(2n)!}{z^{4n+1}} \right| = \frac{|z|^4}{(2n+2)(2n+1)} \xrightarrow{n \to \infty} 0$$

The series converges absolutely for all z.

Using the formula for the geometric series,

$$\frac{1}{1-z} = \sum_{n \geq 0} z^n \implies \frac{1}{z^4 + 9} = \frac{1}{9} \frac{1}{1 - (-z^4/9)} = \frac{1}{9} \sum_{n \geq 0} \left(-\frac{z^4}{9} \right)^n = \frac{1}{9} \sum_{n \geq 0} \frac{(-1)^n z^{4n}}{9^n}$$

$$\implies \boxed{\frac{z}{z^4 + 9} = \frac{1}{9} \sum_{n \geq 0} \frac{(-1)^n z^{4n+1}}{9^n}}$$

Perform the ratio test. The series converges absolutely iff

$$\left| \frac{(-1)^{n+1} z^{4n+5}}{9^{n+1}} \frac{9^n}{(-1)^n z^{4n+1}} \right| = \frac{|z|^4}{9} < 1 \iff |z|^4 < 9 \iff |z| < \sqrt{3}$$

P71.3. Pf. For all z, the sequence $z + n$ diverges to ∞ as $n \to \infty$, so f_n has no piecewise limit, hence no uniform limit, on any (nonempty) subset of $\mathbb{C}$. Thus f_n does not converge normally to any function $\mathbb{C} \to \mathbb{C}$.

Now fix $z = x + yi \in \mathbb{C}$ and consider the sequence $z + n$ on the Riemann sphere with the spherical metric χ. It converges to ∞ since

$$\chi(z+n, \infty) = \frac{2}{[1 + |z+n|^2]^{1/2}} = \frac{2}{[1 + (x+n)^2 + y^2]^{1/2}} = \frac{2}{[n^2 + 2xn + x^2 + y^2 + 1]^{1/2}} \leq \frac{2}{[n^2 + 2xn]^{1/2}} \cdot \frac{1/n}{[1/n^2]^{1/2}}$$

$$= \frac{2/n}{[1 + 2x/n]^{1/2}} \xrightarrow{n \to \infty} \frac{0}{[1+0]^{1/2}} = 0$$