COMPLEX ANALYSIS

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Contents

Chapter 1

Introduction

A complex number is $z = x + \mathbf{i}y$, where x and y are real numbers, and $\mathbf{i}$ is the imaginary number $\sqrt{-1}$. This is just another way to write a point $(x,\ y)$ of the plane. With this notation the plane is the *complex plane* $\mathbb{C}$. Then with $z = x + \mathbf{i}y$ as above, $x = \Re(z)$ is called the real part of z, and $y = \Im(z)$ is called the imaginary part of z. The x axis is the real axis (which allows us to think of the real numbers $\mathbb{R}$ as a subset) and the y axis is the "imaginary axis." For example the real part of $3+\mathbf{i}2$ is 3 and the imaginary part 2. Often we write $y\mathbf{i}$ instead of $\mathbf{i}y$. There are other shortcut notations. For example, the complex number $3 + \mathbf{i}(-2)$ may be written as $3 - 2\mathbf{i}$. Also, every real number is a complex number; for example, $7 = 7 + \mathbf{i}(0)$. Furthermore, $z = 0$ means that the real part $x = 0$ and the imaginary part $y = 0$. However the complex plane is much more than being just different notation for the plane. This is because there is an addition and multiplication so we can do algebra like the ordinary numbers:

1.1 Algebra

The definitions of addition and multiplication of real numbers are extended to the complex numbers in the only reasonable way.

First, addition. Two complex numbers are added simply by adding together their real parts and imaginary parts: we define

$$(a + \mathbf{i}b) + (c + \mathbf{i}d) = (a + c) + \mathbf{i}(b + d).$$

For example, $(3+2\mathbf{i})+(4-6\mathbf{i})=(7-4\mathbf{i})$.

Next, multiplication. As we assume $\sqrt{-1}\sqrt{-1}=\mathbf{ii}=-1$

$$(2+3\mathbf{i})(4+5\mathbf{i})=2(4+5\mathbf{i})+3\mathbf{i}(4+5\mathbf{i})=8+10\mathbf{i}+12\mathbf{i}+15\mathbf{ii}=-7+22\mathbf{i}.$$

In general, the definition will be that

$$(a+\mathbf{i}b)(c+\mathbf{i}d)=(ac-bd)+\mathbf{i}(ad+bc)$$

With these definitions $\mathbb{C}$ enjoys all the usual arithmetical properties (e.g. addition and multiplication are commutative and associative; the distributive property holds; etc.).

Also we see that every $z=x+\mathbf{i}y\neq 0$ has a multiplicative inverse

$$z^{-1}=\frac{1}{x+\mathbf{i}y}=\frac{1}{x+\mathbf{i}y}\frac{x-\mathbf{i}y}{x-\mathbf{i}y}=\frac{x-\mathbf{i}y}{x^2+y^2},$$

i.e.

$$z^{-1}=\frac{x}{x^2+y^2}-\mathbf{i}\frac{y}{x^2+y^2}$$

For example

$$\frac{1}{2+\mathbf{i}}=\frac{2}{5}-\mathbf{i}\frac{1}{5}$$

In theory one complex equation for z can be converted into two real equations but this is not very effective, e.g. solve

$$(2+\mathbf{i})z=1-\mathbf{i}$$

which by sticking to complex notation has solution

$$z=\frac{1-\mathbf{i}}{2+\mathbf{i}}=(1-\mathbf{i})\left(\frac{2}{5}-\mathbf{i}\frac{1}{5}\right)=\frac{2}{5}-\frac{1}{5}-\mathbf{i}\frac{2}{5}-\mathbf{i}\frac{1}{5}=\frac{1}{5}-\mathbf{i}\frac{3}{5}$$

This is much easier than writing $z=x+\mathbf{i}y$ and converting the equation $(2+\mathbf{i})(x+\mathbf{i}y)=1-\mathbf{i}$ into two real equations

$$\begin{array}{rcrcr} 2x & - & y & = & 1 \\ x & + & 2y & = & -1 \end{array}$$

1.2 Geometry

Naturally the complex plane has a geometric side which is closely connected to its algebra. The “complex conjugate”:

$$\overline{z} = \overline{x + \mathbf{i}y} = x - \mathbf{i}y \ ,$$

gives the reflection $\rho(z) = \overline{z}$ in the real axis. One has the two fundamental properties:

$$\overline{z+w} = \overline{z} + \overline{w}, \ \ \overline{zw} = \overline{z}\ \overline{w} \ ,$$

furthermore $\Re(z) = (z + \overline{z})/2, \ \Im(z) = (z - \overline{z})/(2\mathbf{i})$.

Exercises

1. Compute $\overline{(1+2\mathbf{i})(1-2\mathbf{i})}$
2. Check the two fundamental properties of complex conjugate
3. Determine if it is true that

$$\overline{\left\{\frac{1}{z}\right\}} = \frac{1}{\overline{z}}$$

Modulus

To measure the distance between the points $(0,0), (x,y)$ we use Pythagorus:

$$|z| = |x + iy| = \sqrt{x^2 + y^2} \ .$$

This has such special properties for complex numbers we call it the modulus or “mod” rather then distance. For obvious reasons the modulus is also sometimes called absolute value. As $|z|^2 = z\overline{z}$ we use the complex conjugates to show

$$|zw| = |z|\,|w| \ .$$

which gives an easy proof of the Triangle inequality:

$$|z - w| \leq |z| + |w| \ .$$

For by previous

$$\begin{aligned}
|z-w|^2 &= (z-w)\overline{(z-w)} \\
&= z\overline{z} - z\overline{w} - w\overline{z} + w\overline{w} \\
&\leq |z|^2 + 2|z\overline{w}| + |w|^2 \\
&= |z|^2 + 2|z|\,|w| + |w|^2 \\
&= (|z|+|w|)^2 \ ,
\end{aligned}$$

where we used the fact that

$$z\overline{w} + w\overline{z} = 2\Re(z\overline{w}) \leq 2|z\overline{w}|$$

DEFINITION 1 *The open disk* $D(w,r) = \{z : |z-w| < r\}$.

Exercises

1. Compute

$$\left|\frac{(1-\mathbf{i})(1+2\mathbf{i})(5+5\mathbf{i})}{(1-2\mathbf{i})(1+\mathbf{i})(1+\mathbf{i})}\right|$$

2. Show that for any complex number z with $|z| = 1$

$$\overline{z} = \frac{1}{z}$$

3. Find the smallest $r > 0$ so that there is a disk $D(z,r)$ containing

$$D(0,1) \cap D(1+\mathbf{i},1) \ .$$

Arguments

This brings us to the geometric realization of a complex number z as a vector of length $r = |z|$ (its modulus) making an angle θ with the OX axis. θ is called the argument of z (or "arg"). We know that angles are real numbers regarded as equal if they differ by integer multiples of 2π. The argument of a nonzero complex number z is defined by considering $w = z/|z| = x + \mathbf{i}y$ which is a complex number of modulus 1. Now by considering the circle $x^2 + y^2 = 1$ we see there is a number θ, in the half open interval $(-\pi, \pi]$:

$$\cos(\theta) = x, \ \sin(\theta) = y.$$

This defines the Argument, $\theta = \mathrm{Arg}(z)$, of z. The argument, $\arg(z)$, is the class of $\theta + 2\pi n$, where n is an integer $0, \pm 1, \pm 2, ...$

THEOREM 1 *Suppose that z, w are nonzero complex numbers with moduli s, t, and with arguments θ, ϕ respectively. Then* $\arg(zw) = \theta + \phi$.

Now by definition $z = s(\cos(\theta) + \mathbf{i}\sin(\theta))$, $w = t(\cos(\phi) + \mathbf{i}\sin(\phi))$. Hence

$$\begin{aligned} zw &= s(\cos(\theta) + \mathbf{i}\sin(\theta))\, t(\cos(\phi) + \mathbf{i}\sin(\phi)) , \\ &= st\left\{(\cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)) + \mathbf{i}(\sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi))\right\} , \\ &= st(\cos(\theta + \phi) + \mathbf{i}\sin(\theta + \phi)) , \end{aligned}$$

by trig identities.

As a corollary we have the very useful "de Moivre's formula":

$$(\cos(\theta) + \mathbf{i}\sin(\theta))^n = \cos(n\theta) + \mathbf{i}\sin(n\theta) ,$$

for any integer n.

Exercises

1. Compute $\arg(-1 + \mathbf{i})^5$.

2. Find all complex numbers z, w so

$$\mathrm{Arg}(zw) = \mathrm{Arg}(z) + \mathrm{Arg}(w) .$$

3. Use De Moivre to show that for $n = 1, 2...$ $\cos(n\theta) = P_n(\cos(\theta))$ where P is a polynomial of degree n.

1.3 Roots of Polynomials

Polynomials over the complex numbers z of degree n may be defined as

$$p(z) = a_n\, z^n + ... + a_1\, z + a_0$$

where the coefficients $a_k \in \mathbb{C}$ and $a_n \neq 0$. Finding the roots of polynomials, i.e. ζ so that $p(\zeta) = 0$ has been of importance in science for hundreds of years. We have already discussed the linear case $a_1 z + a_0 = 0$ and know that it has a single solution $-a_0/a_1$. Of course we introduced the complex numbers to solve $z^2+1=0$ so we'd like to know what other equations can be solved by complex numbers. The simplest are the power equations

$$z^n = w$$

for given complex w and number n. Using polar form

$$w = R(\cos(\phi) + \mathbf{i}\sin(\phi))$$

with R, ϕ known and

$$z = r(\cos(\theta) + \mathbf{i}\sin(\theta))$$

for unknown r, θ. Then d'Moivre gives

$$r^n(\cos(n\theta) + \mathbf{i}\sin(n\theta)) = R(\cos(\phi) + \mathbf{i}\sin(\phi))$$

so hence for some integer k

$$r^n = R, \ n\theta = \phi + 2\pi k$$

Therefore we find

$$r = R^{1/n}, \ \theta = \frac{\phi}{n} + \frac{2\pi k}{n}$$

where $k = 0, 1, ...n-1$ (as other values of k just repeat). For example we can solve

$$z^3 = -\sqrt{2} + \mathbf{i}\sqrt{2}$$

Now $|-\sqrt{2} + \mathbf{i}\sqrt{2}| = 2, \arg(-\sqrt{2} + \mathbf{i}\sqrt{2}) = 3\pi/4 + 2\pi k$.

So if $z = r(\cos(\theta) + i\sin(\theta))$ we have

$$r^3 = 2, 3\theta = 3\pi/4 + 2\pi k$$

so $r = 2^{1/3}, \theta = \pi/4 + 2\pi k/3$. Therefore the three roots of $-\sqrt{2} + \mathbf{i}\sqrt{2}$ are

$$\begin{aligned}
z_0 &= 2^{1/3}(\cos(\pi/4) + \mathbf{i}\sin(\pi/4)) &&= 2^{-1/6}(1+\mathbf{i}) \\
z_1 &= 2^{1/3}(\cos(\pi/4 + 2\pi/3) + \mathbf{i}\sin(\pi/4 + 2\pi/3)) &&= 2^{-7/6}(-(1+\sqrt{3}) + \mathbf{i}(-1+\sqrt{3})) \\
z_2 &= 2^{1/3}(\cos(\pi/4 + 4\pi/3) + \mathbf{i}\sin(\pi/4 + 4\pi/3)) &&= 2^{-7/6}((-1+\sqrt{3}) - \mathbf{i}(1+\sqrt{3}))
\end{aligned}$$

In particular we see that we can find two complex square roots $\pm\sqrt{w}$ for any complex $w \neq 0$. Therefore by completing the square as usual we find that any *complex* quadratic equation $az^2 + bz + c = 0$ has two roots

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Back in the 16^{th} century similar formulas were discovered for 3^{rd} and 4^{th} degree polynomials. This lead to the conjecture that every n^{th} degree polynomial has n (counting multiplicity) roots. We discuss this history in the appendix. But the short answer is the Fundamental Theorem of Algebra:

THEOREM 2 (FTA)
Every polynomial p of degree n has n roots ζ_k in the complex plane, i.e.

$$p(z) = a(z - \zeta_1)(z - \zeta_2)..(z - \zeta_n)$$

Exercises

1. Find the roots of the equation $z^2 + 2z - \mathbf{i} = 0$.
2. Find the roots of the equation $(z - 1)^4 = z^4$.
3. Factorize $z^5 + 1$ into linear terms.

1.4 Sequences

Once we have the distance between points we can talk about limits and convergence of sequences etc. A sequence of complex numbers can be written $z_n = x_n + \mathbf{i}y_n$. So $\lim_{n\to\infty} z_n = w = u + \mathbf{i}v$ means $\lim_{n\to\infty} |z_n - w| = 0$ which is an ordinary real limit of calculus. Of course this is just the same as $\lim_{n\to\infty} x_n = u,\ \lim_{n\to\infty} y_n = v$, i.e two real limits.

For example with fixed $a + \mathbf{i}b \in \mathbb{C}$ we define the sequence

$$z_n = \left(1 + \frac{a + \mathbf{i}b}{n}\right)^n .$$

Let us write

$$1 + \frac{a + ib}{n} = r_n(\cos(\theta_n) + \mathbf{i}\sin(\theta_n))$$

where

$$r_n = \sqrt{(1 + \frac{a}{n})^2 + (\frac{b}{n})^2} = \sqrt{1 + 2\frac{a}{n} + \frac{a^2 + b^2}{n^2}}$$

and

$$\theta_n = \tan^{-1}\left(\frac{b}{n + a}\right)$$

Therefore if $z_n = R_n(\cos(\phi_n) + i\sin(\phi_n))$ by de Moivre we find

$$R_n = (r_n)^n = \left(1 + 2\frac{a}{n} + \frac{a^2+b^2}{n^2}\right)^{n/2} \to e^a$$

and

$$\phi_n = n\theta_n = n\tan^{-1}\left(\frac{b}{n+a}\right) \to b$$

Therefore

$$\left(1 + \frac{a+\mathbf{i}b}{n}\right)^n \to e^a\left(\cos(b) + \mathbf{i}\sin(b)\right)$$

Exercises

For which complex numbers z does the sequence $\omega_n = z^n$ converge?

Series

Series are just sequences of partial sums of $\sum_{n=0}^{\infty}\omega_n$. As with sequences the whole question of convergence comes down the convergence of the real and imaginary parts. So it is no surprise that complex series have exactly the same criteria for convergence that series of real (nonpositive) numbers have:

A complex series $\sum_{n=0}^{\infty}\omega_n$ converges if

1. $|\omega_n| < a_n$ for convergent positive series $\sum_{n=0}^{\infty} a_n$ "absolute convergence"
2. $|\omega_{n+1}|/|\omega_n| \le r < 1, n > N$ "ratio test"
3. $|\omega_n|^{1/n} \le r < 1, n > N$ "radical test"

Just like real series, we usually do not have an explicit formula for the limit. Again an exception is the geometric series:

$$\sum_{n=0}^{\infty}\omega^n ,$$

this time defined for fixed complex number $\omega = a + \mathbf{i}b$. Following exactly the same derivation as the real case

$$\sum_{k=0}^{n}\omega^k = \frac{1-\omega^{n+1}}{1-\omega}$$

provided $\omega \neq 1$. This converges to $1/(1-\omega)$ if and only if $|\omega| < 1$. Therefore

$$\sum_{n=0}^{\infty} \omega^n = \frac{1}{1-\omega}, \ |\omega| < 1 .$$

For real ω this is just the usual power series but we can have some fun by choosing some complex values.

Exercises

1. For what values of z does the series $\sum_{n=0}^{\infty} nz^n$ converge.

2. Prove that the infinite series, for $0 \leq r < 1$

$$1 + r\cos(\theta) + r^2\cos(2\theta) + r^3\cos(3\theta) + .. = \frac{1 - r\cos(\theta)}{1 + r^2 - 2r\cos(\theta)}$$

1.5 Complex Valued Functions

In calculus we have considered functions f defined on an interval $[a, b]$ of real numbers with real values. It is not much of a leap of the imagination to allow f to take complex values: $f(t) = x(t) + \mathbf{i}y(t)$ where $x(t), y(t)$ are ordinary real valued functions.

For example $f(t) = \cos(t) + \mathbf{i}\sin(t)$ which we already know gives a parametrization of the circle $x^2 + y^2 = 1$. In fact the whole theory of parametrized curves in the plane can be written in terms of complex variables. The derivative of such a function poses no problems:

$$z(t) = x(t) + \mathbf{i}y(t) \Rightarrow \frac{dz}{dt} = x'(t) + \mathbf{i}y'(t)$$

so for example the function $z(t) = 2\cos(t) + \mathbf{i}\sin(t)$ which parametrizes an elipse with major axis 4 and minor axis 2 has derivative

$$z'(t) = -2\sin(t) + \mathbf{i}\cos(t)$$

The usual rules of derivatives hold, and you can even integrate.

Many of the uses of complex variables in science and engineering involves complex valued functions of real variables, e.g. in electrical engineering or quantum theory one uses complex valued functions of (real) time t. However one also considers functions of a complex variable:

Functions of a Complex Variable

So now we are talking about functions $f(z)$ defined on a domain $D \subset \mathbb{C}$ taking complex values. Our polynomials $p(z)$ are examples of such functions. Other examples are rational functions like

$$f(z) = \frac{3}{z^2 + 1}$$

defined for all $z \neq \pm\mathbf{i}$. Complex valued functions can be added, multiplied and divided like ordinary functions. Also we define the complex derivative as

$$\frac{df}{dz} = \lim_{w \to z} \frac{f(w) - f(z)}{w - z} \ ,$$

where this means implicitly that that f is defined in some disk $D(z, r)$. As an exercise one can prove the usual properties:

THEOREM 3 *Suppose that f, g are differentiable at z then so is $f + g, fg$ with*

$$\frac{d(f+g)}{dz} = \frac{df}{dz} + \frac{dg}{dz}$$

and

$$\frac{d(fg)}{dz} = \frac{df}{dz} g + f \frac{dg}{dz}$$

While if $g(z) \neq 0$ also f/g is differentiable at z with

$$\frac{d}{dz} \frac{f}{g} = \{ \frac{df}{dz} g - f \frac{dg}{dz} \} \frac{1}{g^2} \ .$$

Finally, if f is differentiable at $w = g(z)$ and g differentiable at z then the composition $f \circ g$ is differentiable at z with

$$\frac{d}{dz} f \circ g = \frac{df}{dw} \frac{dg}{dz} \ .$$

Natural we often write

$$f'(z) = \frac{df}{dz} \ .$$

Exercises

These concern the problem: Suppose $E \subset \mathbb{C}$ is a connected set. If $f'(z) = 0$ on E is f is constant on E ?

1. Suppose $\gamma \subset \mathbb{C}$ is a smooth arc. If $f'(z) = 0$ on γ show that f is constant on γ.

2. Suppose $f'(z) = 0$ on an open connected set D show that f is constant on D.

3. Use a Koch Curve γ to construct a function f defined on γ (only) with $f' = 0$ on γ but f is nonconstant.

1.6 Analytic Functions

In the beginning there was polynomials over $\mathbb{C}$:

$$p(x) = a_0 + a_1 z + .. + a_n z^n ,$$

i.e. defined with coefficients $a_k \in \mathbb{C}$ and variable $z \in \mathbb{C}$. Power series generalize the notion of a polynomial as well as giving the promise of explicit computations. However there is now the question of convergence, i.e. which functions are equal to their Taylor Series. Consider a domain $D \subset \mathbb{C}$.

DEFINITION 2 *A function $f : D \to \mathbb{C}$ is analytic at z_0 if there is $r > 0$:*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \ |z - z_0| < r .$$

The prototypical example is a Geometric Series $1 + z + z^2 + ...$ which we know converges for $|z| < 1$ to

$$f(z) = \frac{1}{1 - z}$$

It is not obvious that even this function is analytic. To verify one needs to expand at any point $z_0 \neq 1$

$$\frac{1}{1-z} = \frac{1}{1 - z_0 - (z - z_0)} = \frac{1}{1 - z_0} \left\{ 1 - \frac{z - z_0}{1 - z_0} \right\}^{-1}$$

which may be expanded as another geometric series

$$\frac{1}{1-z_0}\sum_{n=0}^{\infty}\left\{\frac{z-z_0}{1-z_0}\right\}^n$$

which converges provided

$$\left|\frac{z-z_0}{1-z_0}\right| < 1,$$

i.e. in the disk $|z-z_0| < |1-z_0|$. Thus $f(z)$ is analytic.

Later we show that any convergent Power Series is analytic.

Although only convergence is required we need a stronger concept:

DEFINITION 3 *A series $\sum_{n=0}^{\infty}\alpha_n$ is absolutely convergent if the positive series $\sum_{n=0}^{\infty}|\alpha_n|$ is convergent.*

In practise we establish convergence of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ by finding sequence $b_n \geq |a_n|$ so that $\sum_{n=0}^{\infty} b_n|z-z_0|^n$ converges.

LEMMA 1 *An absolutely convergent series is convergent.*

Consider the partial sums $s_n = \sum_{j=0}^{n}\alpha_j$. Absolute convergence implies these are bounded. Thus by Bolzano-Weiertrass there is a subsequence n_k and a limit L so that $s_{n_k} \to L$. But then for $n_k > n$

$$|s_n - s_{n_k}| \leq \sum_{n+1}^{n_k}|\alpha_j| \ .$$

So as $n_k \to \infty$

$$|s_n - L| \leq \sum_{n+1}^{\infty}|\alpha_j| \ .$$

Therefore for $n \to \infty$ we have $s_n \to L$.

Absolute convergence is also important because these are the only series that can be arbitarily rearranged and retain convergence (to the same limit!):

$$\sum_{n=0}^{\infty} a_n \text{ absolutely convergent} \Rightarrow \sum_{n=0}^{\infty} a_{n(k)} \text{ absolutely convergent,}$$

for all permutations $n : \mathbb{N} \to \mathbb{N}$. The converse is true (Exercise).

Radius of Convergence

First we note there is a number $R \geq 0$ so that $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges absolutely for $|z-z_0| < R$ and diverges for $|z-z_0| > R$:
Suppose the series is convergent for $z = z_1 \neq z_0$, then the terms $a_n(z_1-z_0)^n$ must be bounded. Thus there is $C > 0$ so for all n

$$|a_n||z_1 - z_0|^n < C\ .$$

So for any z with $|z-z_0| < |z_0 - z_1|$ we have

$$|a_n||z-z_0|^n < Cr^n,\ r = \left|\frac{z-z_0}{z_1-z_0}\right| < 1$$

which gives absolute convergence as

$$\sum_{n=0}^{\infty} |a_n||z-z_0|^n \leq C\sum_{n=0}^{\infty} r^n < \infty\ .$$

Thus we may define $R = \sup|z_1 - z_0|$ so that the series is convergent. This R is called the *radius of convergence*, its precise value uses:

DEFINITION 4 *A sequence α_n has* $\limsup_{n\to\infty} \alpha_n = \lim_{n\to\infty} \sup_{k\geq n} \alpha_k$

Note that as $\beta_n = \sup_{k\geq n} \alpha_k$ is a decreasing sequence (or maybe always ∞) we always have a limit (maybe $\pm\infty$).

THEOREM 4 *(Hadamard) For a Power Series $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$, the radius of convergence (at z_0) is*

$$R = 1/\limsup_{n\to\infty} |a_n|^{1/n}\ ,$$

understanding:

$$(i)\quad \limsup_{n\to\infty} |a_n|^{1/n} = 0 \Rightarrow R = \infty$$

$$(ii)\quad \limsup_{n\to\infty} |a_n|^{1/n} = \infty \Rightarrow R = 0$$

For example the Power Series $\sum_{n=1}^{\infty} 2^n\, z^{2n}$ has radius of convergence

$$R = 1/\sqrt{2}\ .$$

However

$$\sum_{n=1}^{\infty} \frac{z^n}{n^n}$$

has radius of convergence ∞, i.e. always converges. The series

$$\sum_{n=1}^{\infty} n^n z^n$$

has radius of convergence $R = 0$ and thus actually only converges if $z = 0$.

Now we prove the theorem. We only deal with the case $0 < R < \infty$ and leave the others as an exercise. Without loss of generality assume $z_0 = 0$.

For fixed $z, |z| < R$, choose r so that $|z| < r < R$. Now

$$\limsup_{n\to\infty} |a_n|^{1/n} = \frac{1}{R} \text{ implies } |a_n| \leq \frac{1}{r^n} \text{ for } n > N\ .$$

Therefore we have absolute convergence:

$$\sum_N^{\infty} |a_n||z|^n \leq \sum_N^{\infty} \left\{ \frac{|z|}{r} \right\}^n$$

Conversely, if $|z| > R$, then for any $r, |z| > r > R$ there is a subsequence $k \to \infty$ so that

$$|a_{n_k}| > \frac{1}{r^{n_k}}$$

as $k \to \infty$. Therefore the terms

$$|a_{n_k} z^{n_k}| > \left\{ \frac{|z|}{r} \right\}^{n_k} > 1$$

so the series must diverge.

Remarks: The theorem gives the most general formula for the radius of convergence. However the essential formulation of the theorem is that convergence for say $z = z_1$ implies absolute convergence for $|z - z_0| < |z_0 - z_1|$. This means we may use whatever tests of convergence we find convenient. For example the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

is seen to converge for all $|z| < \infty$ by the ratio test.

Taylor coefficients

Next we show that analytic functions are differentiable:

THEOREM 5 *Suppose that $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$, converges for $|z-z_0| < R$. Then f is differentiable with*

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1},$$

converging absolutely for $|z-z_0| < R$.

First we note that the derived series converges with the same radius of convergence R as $n^{1/n} \to 1$. [1] Hence $\sum_{n=1}^{\infty} n|a_n|\, r^{n-1} < \infty$ for $r < R$.

Once again without loss of generality let $z_0 = 0$. Then the theorem is proved by considering

$$\frac{f(u)-f(v)}{u-v} = \sum_{n=1}^{\infty} a_n \frac{u^n - v^n}{u-v}$$

for any $u \neq v, |u|, |v| \leq r$ for fixed $r < R$. We make use of the identity

$$\frac{u^n - v^n}{u-v} = u^{n-1} + vu^{n-2} + ... + v^{n-1}$$

which implies

$$\left| \frac{u^n - v^n}{u-v} \right| \leq nr^{n-1}$$

Therefore given $\varepsilon > 0$ we may choose N so that

$$\sum_{n=N+1}^{\infty} |a_n| \left| \frac{u^n - v^n}{u-v} \right| \leq \sum_{n=N+1}^{\infty} n|a_n|\, r^{n-1} < \varepsilon/3$$

by the absolute convergence of the derivative series.

[1] To prove this let $n^{1/n} = 1 + a_n$ so by Bernoulli $n = (1+a_n)^n \geq n(n-1)a_n/2$, which gives $a_n < 2/(n-1) \to 0$.

Then once N is fixed, choose $\delta > 0$ so that $0 < |u - v| < \delta$ implies

$$\left| \sum_{n=1}^{N} a_n \frac{u^n - v^n}{u - v} - \sum_{n=1}^{N} n a_n u^{n-1} \right| < \varepsilon/3$$

since as $v \to u$

$$\lim_{v \to u} \frac{u^n - v^n}{u - v} = \lim_{v \to u} \left\{ u^{n-1} + v u^{n-2} + \ldots + v^{n-1} \right\} = n u^{n-1}$$

Thus for any $\varepsilon > 0$ there is $\delta > 0$ so that for any u, v with $|u|, |v| \leq r < R$ and $0 < |u - v| < \delta$ we have

$$\left| \frac{f(u) - f(v)}{u - v} - \sum_{n=1}^{\infty} n a_n u^{n-1} \right| \leq$$

$$\left| \sum_{n=1}^{N} a_n \frac{u^n - v^n}{u - v} - \sum_{n=1}^{N} n a_n u^{n-1} \right| + \sum_{n=N+1}^{\infty} |a_n| \left| \frac{u^n - v^n}{u - v} \right| + \sum_{n=N+1}^{\infty} n |a_n| |u|^{n-1}$$

which by previous is

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \ ,$$

which completes the proof.

In particular the theorem implies an analytic function is infinitely differentiable at z_0. Therefore we have the Taylor coefficients

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

Later, as a consequence of the integral theory, we show the converse:

THEOREM 6 *If f differentiable on a domain D then f is analytic on D. Furthermore the radius of convergence R at z_0 is the* $\sup\{r \geq 0 :$ *f has analytic extension to* $D(z_0, r)\}$.

1.7 Uniqueness and Isolated Zeros

An elementary famous result about analytic functions is:

THEOREM 7 (PRINCIPLE OF UNIQUENESS) *Let f be analytic on a region D. Suppose there is a point $a \in D$ so that*

$$\frac{d^k f(a)}{dz^k} = 0, \ k = 0, \ 1, \ 2...,$$

then f is identically zero on D.

Remarks: This vanishing of all the derivatives is equivalent to

$$f(z) = \sum_{k=0}^{\infty} a_k (z-a)^k \equiv 0 \ , \ |z-a| < r,$$

for some $r > 0$.

Now the Principle of Uniqueness is essentially equivalent to :

THEOREM 8 (PRINCIPLE OF ISOLATED ZEROS) *Let f be analytic at a point a. If there is a sequence of distinct points $z(n) \to a$ so that $f(z(n)) = 0$ then there is a $r > 0$ so that f is identically zero on $|z-a| < r$.*

We prove this first. Now if f is not zero then it has a first nonzero coefficient a_m. Taking out the common factor

$$f(z) = a_m (z-a)^m \left\{ 1 + \sum_{k=1}^{\infty} \frac{a_{k+m}}{a_m} (z-a)^k \right\}$$

Then as the sum is continuous for small enough r we see that f is nonzero if $0 < |z-a| < r$. This is a contradiction.

Next we prove the principle of uniqueness. Let A be the set of points b where $f^{(k)}(b) = 0$ for all k. Then A is open by the previous construction. Let B be the component of A containing a. Suppose $c \in \partial B \cap D$. Then there exists a sequence $z(n)$ in B converging to c. But $f(z(n)) = 0$ and hence by the principle of isolated zeros c must be in B too. Thus B is open and closed and must be a component of D. As D is connected we have $B = D$.

Here are two examples of the many applications of these principles:

1. An analytic identity true on a set with limit points inside a region is true throughout the domain of definition.

2. Principle of Analytic continuation

   A function f analytic on a region D has analytic continuation to a function F on a region R if $f = F$ on some ball B contained in both D and R. There can be only one function F equal to f on the ball B. If $R \supset D$ then we have a unique extension of f to a larger domain. For example the power series $\sum_{k=0}^{\infty} z^k$ extends to the unique function $1/(1-z)$.

These Principles are in complete contrast to real analysis where functions can be changed in a fairly arbitrary way. An analytic function is totally determined by its local behaviour, even just by the Taylor coefficients at a single point. This is in analogy to a biological organism which is determined by its DNA code taken from any cell. Thus complex analysts think of functions as single entities. This also means that analytic functions are "rigid objects" that cannot be locally modified without causing a global change. This is the source of a philosophical split in analysis, between those studying rigid objects and those looking at objects one can chop and patch.

Exercise

If f analytic at 0 with $f(1/n) = 1/(n+1)$ for $n = 1, 2, ...$ what is f?

1.8 Complex Exponential

This leads to the definition of complex exponential, using the familar power series

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

which we know converges for all z. Using the Power Series for e^x we see that $\exp(x) = e^x$ for real numbers x, i.e. $\exp(z)$ extends e^x to the complex plane.

Differentiating the power series gives

$$\frac{d}{dz}\exp(z) = \sum_{n=1}^{\infty} \frac{d}{dz}\frac{z^n}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = \exp(z) \ .$$

We use this to derive the addition formula for complex z, w:

$$\exp(z+w) = \exp(z)\exp(w)$$

First consider the function $g(z) = \exp(z)\exp(\omega - z)$ (for complex z, ω):

$$\frac{dg}{dz} = \exp(z)\exp(\omega - z) - \exp(z)\exp(\omega - z) = 0$$

by derivative rules. Therefore $g(z)$ is the constant $g(0)$, i.e

$$g(z) = \exp(z)\exp(\omega - z) = \exp(\omega)$$

for all z. Putting $\omega = z + w$ gives the addition formula.

Also we see some hitherto unexpected connections with the trig functions. For if we put $z = \mathbf{i}y$

$$\exp(\mathbf{i}y) = \sum_{n=0}^{\infty} \frac{\mathbf{i}^n y^n}{n!}$$

Now $\mathbf{i}^n$ takes the values $\mathbf{i}, -1, -\mathbf{i}, 1$ in turn so taking the real and imaginary parts of the the series gives

$$\exp(\mathbf{i}y) = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n)!} + \mathbf{i}\sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{(2n+1)!}$$

You may recognize that the first Power Series is $\cos(y)$ and the second one is $\sin(y)$.

Thus we have proved

$$\exp(\mathbf{i}y) = \cos(y) + \mathbf{i}\sin(y)$$

the famous "Euler's formula" which shows $\exp, \cos, \sin$ are different sides of the same thing. In particular putting $y = \pi$ we get the identity

$$e^{\mathbf{i}\pi} + 1 = 0$$

which Euler declared “proves God exists”.

The addition formula gives us an explicit formula for $\exp(z)$ for

$$\exp(x + \mathbf{i}y) = \exp(x)\exp(\mathbf{i}y) = e^x\left\{\cos(y) + \mathbf{i}\sin(y)\right\}$$

This gives De Moivre’s formula as well as all the trig identities as special cases of the addition formula for complex exponential. Indeed we now define complex sin/cos:

$$\cos(z) = \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!}\ ,\ \sin(z) = \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

and see that

$$\cos(z) = \frac{1}{2}\left\{\exp(\mathbf{i}z) + \exp(-\mathbf{i}z)\right\}\ ,\ \sin(z) = \frac{1}{2\mathbf{i}}\left\{\exp(\mathbf{i}z) - \exp(-\mathbf{i}z)\right\}$$

Exercises

1. Compute $\exp(1 - \mathbf{i}\frac{\pi}{4})$
2. From the addition formula for complex exp derive the trig addition formula for $\sin(a + b)$ and $\cos(a + b)$
3. Show that for all complex z, $\exp(z) \neq 0$.
4. Use the previous formula to show that for all complex z

   $$\cos^2(z) + \sin^2(z) = 1$$

5. Analytic continuation gives another way to prove trig identities, e.g.

   $$\sin(z + w) = \sin(z)\cos(w) + \cos(z)\sin(w)$$

   for all complex z, w from the known case for z, w real. Consider

   $$F(z, w) = \sin(z + w) - \sin(z)\cos(w) - \cos(z)\sin(w)$$

   which is analytic in z, w. For z, w real $F(z, w) = 0$. Hence by the principle of uniqueness for fixed w real $F(z, w) = 0$ for all complex z . Then fix complex z. By the principle of uniqueness $F(z, w) = 0$ for all complex w.

1.9 Fundamental Theorem of Algebra

The problem of finding roots of polynomials was very much tied up with the invention of the complex numbers themselves. Cardan (16^{th} century) discovered a formula for the roots of the equation $z^3 = 15z + 4$ which gave an answer involving $\sqrt{-121}$. Cardan knew that the equation had $z = 4$ as a solution and was able to manipulate 'complex numbers' to obtain the right answer without understanding what they were. Bombelli, in 1572, produced rules for manipulating these "complex numbers". Yet a 'proof' that the FTA was false was given by Leibniz in 1702 when he asserted that $z^4 + t^4$ could never be written as a product of two real quadratic factors. His mistake came in not realising that $\sqrt{\mathbf{i}}$ could be written in the form $a + b\mathbf{i}$.

D'Alembert in 1746 gives a sequence converging to a zero of the polynomial. His proof has the weakness that he did not have the necessary theory to prove convergence. In fact such a theory was not developed until the late 19^{th} century, never the less one sees his proof as being essentially correct. Indeed in France the FTA is called D'Alembert-Gauss.

Gauss in 1799 presented his first proof and also his objections to the other proofs. He is undoubtedly the first to spot the flaws in the earlier proofs. Actually Gauss does not claim to give the first proper proof. He merely calls his proof new and says of d'Alembert's proof, that despite his objections a rigorous proof could be constructed on the same basis. In fact Gauss's proof of 1799 also does not meet our present day standards.

In 1814 the Swiss accountant Jean Robert Argand published a proof based on d'Alembert's 1746 idea. Two years after Argand's proof appeared Gauss published a second proof of the FTA. This proof is complete and correct. Then he gave a third proof. Gauss introduced in 1831 the term 'complex number'. In 1849 (on the 50th anniversary of his first proof!) Gauss produced the first proof that a polynomial equation of degree n with complex coefficients has n complex roots. Despite the many proofs given by Gauss it seems fundamentally unfair to give to him the credit for FTA. A proof using calculus might be attributed to D'Alembert-Gauss, with important tidying up done by Argand.

A Proof in the style of D'Alembert-Argand

Consider a nonconstant polynomial $p = a_0 + a_1 z + .. + a_n z^n$, where $a_n \neq 0$. We argue by contradiction, i.e assume p does not have a root. Thus the function $u(z) = |p(z)|^2 > 0$. Now as $|p(z)| = |a_n||z|^n|1 + a_{n-1}/(a_n z) + ..|$ we have $\lim_{|z|\to\infty} u(z) = \infty$ so, by basic calculus, the function $u(z)$ achieves its minimum somewhere, say at z_0.

To make computations easier we make some simple transformations. So *without loss of generality* we may assume $z_0 = 0$ (otherwise just use translation $z - z_0$). Thus $|p(0)|^2 = |a_0|^2$ is the minimum of $u(z)$. Also since $p(0) \neq 0$, $p = a_0 + a_m z^m + .. + a_n z^n$ where $a_0 \neq 0$, and a_m is the next nonzero term. So without loss of generality (again) we may assume $a_0 = 1$ (by considering $p(z)/a_0$ instead), i.e. the minimum is now $u(0) = 1$. Finally we suppose that $a_m = \rho e^{i\phi}$ and let $\beta = \rho^{1/m} e^{i\phi/m}$ be an mth root. By making the substitution $p(z/\beta)$ we may assume $p(z) = 1 + z^m + ... + a_n z^n$.

So to obtain a contradiction we have only to show that $u = |p|^2$ cannot have its minimum at $z_0 = 0$. Expanding

$$u(z) = |1 + z^m + ...|^2 = \{1 + z^m + ...\}\overline{\{1 + z^m + ...\}}$$

So with $z = re^{i\theta}$,

$$u(z) = 1 + 2r^m \cos(m\theta) + \varepsilon r^m,$$

where the error $\varepsilon \to 0$ as $z \to 0$. But for $\theta = \pi/m$

$$u(z) = 1 - 2r^m + \varepsilon r^m,$$

so as $-2 < \varepsilon$ there is $z \to 0$ so that $u(z) < 1 = u(0)$. This means $z_0 = 0$ cannot give the minimum. Therefore p has no roots gives a contradiction. The only logical conclusion is that p has a root ζ.

Chapter 2

Holomorphic Functions

We now study complex valued functions $w = f(z)$ defined on disks D of the form $D = D(a,r) = \{z : |z-a| < r\}$ (which includes $r = +\infty$). We write $w = u + iv,\ z = x + iy$ (where x, y, u, v are always real) and set

$$u = u(x,y),\ v = v(x,y)$$

and consider pairs of real valued functions. Conversely any function $u(x,y) + iv(x,y)$ may be written purely in terms of complex variables $z, \overline{z}$, which can be thought of being "independent".

Example : The functions $u = x^2 - y^2, v = -3xy$ become

$$\begin{aligned} u + iv &= \{(\frac{z+\overline{z}}{2})^2 + (\frac{z-\overline{z}}{2i})^2\} - 3i(\frac{z+\overline{z}}{2})(\frac{z-\overline{z}}{2i}) \\ &= z\overline{z} - \frac{3}{4}z^2 + \frac{3}{4}\overline{z}^2. \end{aligned}$$

We can define limits by $u,\ v$ or by complex numbers:

$$\lim_{z \to \zeta} f(z) = w,$$

if $\forall \epsilon > 0,\ \exists \delta > 0 :\ 0 < |z - \zeta| < \delta \ \Rightarrow\ |f(z) - w| < \epsilon$. This is equivalent to familiar concepts from real calculus where (u,v) has limit $w = c + id$ at $\zeta = a + ib \in D$ if

$$\lim_{(x,y)\to(a,b)} u(x,y) = c, \text{ and } \lim_{(x,y)\to(a,b)} v(x,y) = d\ .$$

Whichever way one does it, limits may be used to define the continuous functions on D, i.e. $\lim_{z\to\zeta} f(z) = f(\zeta)$.

In this chapter we consider continuous functions $f = u + iv$ defined on a disk D so that f has first order partial devatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

on D. These are often grouped to form the Jacobian matrix

$$df = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} .$$

This is sometimes called the differential but we reserve that term for the symbolic form $df = f_x dx + f_y dy$ where we use complex valued partial derivatives

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} .$$

Actually for important technical reasons modern analysts prefer to use the complex partial derivatives defined by

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

often denoted ∂f, the del operator, and

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) ,$$

called the $\overline{\partial} f$, the del bar operator. The most trivial reason for using these is to ease computations when f is written in complex form.

Example: For the function $f = az^k + bz^n\overline{z}^m + c\overline{z}^q$ with $n, m, q \in \mathbb{N}$

$$\frac{\partial f}{\partial z} = kaz^{k-1} + bnz^{n-1}\overline{z}^m$$

$$\frac{\partial f}{\partial \overline{z}} = mbz^n\overline{z}^{m-1} + cq\overline{z}^{q-1} \ .$$

It is simple to recover real partials by

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \overline{z}} \ , \ \frac{\partial f}{\partial y} = i\left(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \overline{z}}\right) \ .$$

Thus we have

$$\frac{\partial f}{\partial x} = kaz^{k-1} + bnz^{n-1}\overline{z}^m + mbz^n\overline{z}^{m-1} + cq\overline{z}^{q-1}$$

$$\frac{\partial f}{\partial y} = ikaz^{k-1} + ibnz^{n-1}\overline{z}^m - imbz^n\overline{z}^{m-1} - icq\overline{z}^{q-1} \ ,$$

which would be next to impossible in do using real variables.

Exercises

Prove our assumptions of the previous example:

1. $\partial\, z^n = nz^{n-1}$ for numbers n,
2. $\partial\, \overline{z}^n = 0$,
3. $\overline{\partial\, f} = \overline{\partial}\, \overline{f}$, for any function f with partials.

2.1 Complex derivatives

Now suppose f is differentiable at z, i.e. f is defined in a disk $D(z, r),\ r > 0$ with

$$f'(z) = \lim_{\zeta \to 0} \frac{f(z + \zeta) - f(z)}{\zeta} \ .$$

By the previous section

$$\partial f(z) = f'(z),\ \overline{\partial} f(z) = 0.$$

This can also be seen by letting ζ approach z in different directions:

$$\frac{df(z)}{dz} = \lim_{x \to 0} \frac{f(z+x) - f(z)}{x} = \frac{df(z)}{dx}$$

and

$$\frac{df(z)}{dz} = \lim_{y \to 0} \frac{f(z+iy) - f(z)}{iy} = \frac{df(z)}{i\,dy} .$$

For $f = u + iv$ the condition that $\overline{\partial} f(z) = 0$ is equivalent to

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

the so called Cauchy Riemann equations. These are important not only because they characterise functions $u + iv$ which are complex differentiable but they have profound generalisations. Hence the Jacobian matrix has the form

$$df = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} ,$$

where $a,\ b$ are real numbers. Now one model (see exercise below) for $\mathbb{C}$ is the field of matrices of the form

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} , a, b \in \mathbb{R} .$$

In these terms CR equations express the idea that the Jacobian is itself a complex number.

DEFINITION 5 *We say that f is holomorphic at a point z if there is an open disk $D = D(z, r)$ where f has complex derivatives at each point of D .*

Naturally than holomorphic functions are defined on open sets only.

Exercises

1. Suppose that $f = u + \mathbf{i}v$ has continuous first partials prove CR equations $\Rightarrow$ holomorphic. [1]

2. Show that the previous model using 2×2 matrices is equivalent to $\mathbb{C}$.

2.2 Rational Functions

Although there are many similarities between calculus on the real line and in the complex plane, it is wise to remember there can be some big differences. For instance there is Rolle's Theorem:
Suppose that f is (real)differentiable on the interval $[a, b]$ and $f(a) = f(b) = 0$. Then there exists c, $a < c < b$, so that $f'(c) = 0$.

This is not even true for complex polynomials: e.g. $p(z) = z^3 - iz^2 - z + i$ has zeros 1, -1, i but $p'(z) = 3z^2 - 2iz - 1$ has zeros $(i \pm \sqrt{2})/3$ which do not lie on the lines joining the zeros. It is interesting application of the little theory we have so far developed that the following is true:

THEOREM 9 *(LUCAS'S THEOREM) Let K be the smallest convex polygon containing all the roots of a polynomial p of $deg(p) > 1$. Then all the roots of p' lie in K.*

Remark : A set K is convex iff.

$$\forall z, w \in K \Rightarrow tz + (1-t)w \in K,\ \forall t,\ 0 < t < 1.$$

This famous old result is proved by two preliminary results:
The first is geometrically obvious

LEMMA 2 *Any convex set K is equal to the intersection of all half planes H which contain K.*

[1] This corrects a problem of Lang who leaves out the "continuous first partials". Actually the theorem is true without this but it is a diffcult result of Melnikoff.

Let H any half plane containing all the zeros α_k of p. Without loss of generality H is defined for some a, $b \in \mathbf{C}$ as

$$\left\{ z : \Im\left(\frac{z-a}{b}\right) < 0 \right\} .$$

The second preliminary result is

LEMMA 3 *$p' \neq 0$ for $z \notin H$.*

For $z \notin H$

$$\Im\left(\frac{z-\alpha_k}{b}\right) = \Im\left(\frac{z-a}{b}\right) - \Im\left(\frac{\alpha_k - a}{b}\right) > 0$$

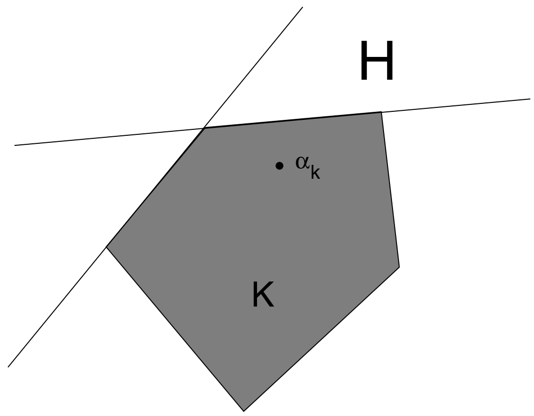
and therefore

$$\Im\left(\frac{b}{z-\alpha_k}\right) < 0 .$$

Hence

$$\Im\left(\frac{bp'(z)}{p(z)}\right) = \Im\sum_{k=1}^{n}\left(\frac{b}{z-\alpha_k}\right) < 0 .$$

Thus p' cannot be zero outside H. This proves the lemma and hence the theorem.



Convex sets

Suppose that P, Q are nonconstant polynomials with no common zeros (otherwise we cancel common factors) then

$$R(z) = \frac{P(z)}{Q(z)}$$

is a rational function of degree $l = \max\ \{deg(P),\ deg(Q)\}$. The zeros of Q are points where R is not defined as a complex valued function, called poles of R. However R is defined on the extended complex plane $\mathbf{S}$ and maps $\mathbf{S}$ onto $\mathbf{S}$. If we count the order of zeros then every value $w \in \mathbf{S}$ is taken l times. Consider what this means for $w = \infty$. If $deg(Q) = m$ then ∞ is taken m times (counting order) at the zeros of Q. If $deg(P) = n = l > m$ then as $z \to \infty$ we find

$$R(z) = az^{n-m} + o\{z^{n-m}\}\ .$$

Thus R has a pole of order $n - m$ at ∞.

One of the main results about rational functions is the factorisation into partial fractions. Namely

$$\frac{P(z)}{Q(z)} = P_0(z) + \sum_{k=1}^{k=m'} P_k((z-\beta_k)^{-1})$$

where Q has m' distinct zeros β_k of order n_k and P_k is a polynomial of order n_k with constant term equal to zero and if $n \geq m$ P_0 is a polynomial of degree $n-m$. The calculation by algebraic means of the R's is quite lengthy, we give an easy method by differentiation. For

$$P_k(z) = c_1 z + ... + c_{n_k} z^{n_k}$$

the problem is to determine the coefficents c_k. The easiest case is for $n_k = 1$

$$c_1 = \lim_{z \to \beta_k} (z - \beta_k) R(z)$$

which by L'Hopital's rule is

$$\frac{P(\beta_k)}{Q'(\beta_k)} .$$

Exercises

1. Find a formula for higher terms c_j.
2. Develop the following as partial fractions:

$$\frac{z^3}{1-z^3}, \frac{1}{z^3-1} .$$

3. Suppose f is rational and all its zeros are one side of a line L, all of its poles on the other side of L. Show that $f' \neq 0$ on L.
4. Suppose that R, S are rational functions of degree n. If there are $2n+1$ distinct points z_k so that $R(z_k) = S(z_k)$ show $R = S$.
5. Let R be a rational function so that $|R(z)| = 1$ when $|z| = 1$. Show that if ζ is a zero then $1/\overline{\zeta}$ is a pole. Hence show:

$$R(z) = c \prod_{k=1}^{n} \left(\frac{z - a_k}{1 - \overline{a_k} z} \right) ,$$

where a_k are the zeros and c is a constant of modulus 1.
(Hint : consider the function $1/\overline{R(1/\overline{z})}$).

2.3 Inverse functions

A function f has an inverse g if the compositions $f \circ g$ and $g \circ f$ are well defined and equal to the identity. Thus f must be 1:1 on a set D so we get g defined on $f(D)$. In the real case a sufficient condition for a function to be 1:1 on an interval [a,b] is that f' is nonzero on $[a, b]$. The same is not true in the plane -e.g. $\exp(z)$ has nonzero derivative but period $2\pi i$. (Only later we shall prove that if f is 1:1 on a disk D then f' is nonzero on D). However now we can prove

THEOREM 10 *Suppose that $f(z)$ is analytic on $|z-a| < R$ and $f(a) = b,\ f'(a) \neq 0$. Then there are positive numbers r, s so that f is 1:1 in $|z-a| < r$ and there is a function g analytic and 1:1 on $|z-b| < s$ so that $f \circ g = z$ for $|z-b| < s$. Furthermore $g'(z) = 1/f'(g(z))$.*

To simplify the computations, without loss of any generality, we assume that $a = 0,\ b = 0,\ f'(a) = 1$. (Otherwise just translate by a, subtract b and divide by $f'(0)$). Then if

$$f(z) = \sum_{n=1}^{\infty} a_n z^n ,$$

we obtain

$$\frac{f(z) - f(w)}{z - w} = 1 + \sum_{n=2}^{\infty} a_n \{z^{n-1} + ... + w^{n-1}\} \neq 0$$

provided z, w are inside $D(0, r)$ where $r > 0$ is any number so that

$$\sum_{n=2}^{\infty} n|a_n| r^{n-1} < 1 ,$$

which is certainly possible as the last series is continuous in r and zero when $r = 0$. Thus f is locally 1:1. Hence there is a function g on $f(|z-a| < r)$ so that $f \circ g = z$. By the chain rule g is holomorphic and $g'(z) = 1/f'(g(z))$. The last thing to do is show that g is analytic. Doing this with power series is not so easy, essentially we need an inverse function theorem. This is left as the following guided exercise. [2]

[2] An easier proof comes from our later result that any holomorphic function is analytic

Exercise

The idea is to show that the equation $z = f(g(z))$, has analytic solutions $g(z)$. Without loss of generality we assume $f(0) = 0,\ f'(0) = 1,\ g(0) = 0,\ g'(0) = 1$.

1. Writing

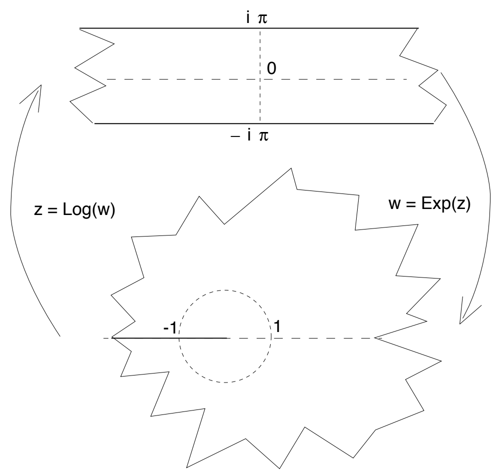
$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \ ,\ g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

one can solve for the unknown coefficients b_n recursively, i.e. show

$$b_n = a_2 P_{n,2}(b_{n-1}, ..., b_2) + ... + a_n P_{n,n-1}(b_{n-1}, ..., b_2)$$

where $P_{n,k}(b_{n-1}, ..., b_2)$ are polynomials in $(b_{n-1}, ..., b_2)$ with positive coefficients.

2. Having obtained a formal power series for $g(z)$ it remains to prove convergence. It suffices to prove $|b_n| \leq b^n$ for some positive constant b. As $f(z)$ has convergent power series we already know that $|a_n| \leq a^n$ for some postive constant a. Now by the recursive relationship the coefficients $|b_n| \leq c_n$ for any function $G(z) = z + \sum_{n=2}^{\infty} c_n z^n$ solving $z = F(G(z))$ for some function $F(z) = z - \sum_{n=2}^{\infty} d_n z^n$ where $|a_n| \leq d_n$. Choosing $d_n = a^n$ and show that $|b_n| \leq b^n$ for some positive constant b.



2.4 Logarithm

We solve $z = \exp w$ where $w = u + iv,\ z = r(\cos(\theta) + i\sin(\theta))$ and find $u = \log|z|,\ v = \arg(z)$. In particular there are an infinite number of solutions z. One is called the Principal branch of logarithm defined on $P = \mathbb{C} - (-\infty, 0]$ by

$$\mathrm{Log}(z) = \log|z| + i\mathrm{Arg}(z).$$

This is the image of the domain $D = \{w : |\Im(w)| < \pi\}$ where $\exp(w)$ is 1:1. Now $\exp(w)$ is holomorphic with nonzero derivative. As $\exp(w)$ is its own derivative

$$\frac{d\log(z)}{dz} = \frac{1}{z},\ z \in P.$$

Also by Abel's Theorem $\mathrm{Log}(z)$ is analytic. Now if $z \in P$,

$$\frac{1}{z} = \frac{1}{\zeta + (z - \zeta)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{\zeta^{k+1}}\,(z - \zeta)^k$$

which converges provided $|z-\zeta| < |\zeta|$. By antidifferentiation:

$$\mathrm{Log}(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k\,\zeta^k}\,(z-\zeta)^k\ ,$$

converging in the same disk.

"Other" logarithms may be found by choosing different sets D where $\exp(w)$ is 1:1. For instance $D = \{z : |\Im(w) - 2\pi n| < \pi\}$ gives a "branch" of logarithm defined on P with $\log(z) = \log|z| + i(\mathrm{Arg}(z) + 2\pi n)$. This does not exhaust the possibilities. Let $v = \phi(u)$ be any continuous function. Put $D = \{u + iv : \phi(u) < v < \phi(u) + 2\pi\}$, which looks like a wavy strip. The image of D is $H = \{r\exp(i\theta) : \phi(\log(r)) < \theta < \phi(\log(r)) + 2\pi\}$ which is the plane cut by the curve $\arg(z) = \phi(\log|z|)$. Then there is a well defined branch of logarithm mapping H onto D. The problem of understanding this behaviour will be dealt with in the next chapter with the introduction of Riemann surfaces.

Exercise

1. Find all complex numbers so that $e^{2z} + 1 = 0$.

2. Show
$$\log(2) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}.$$

3. For what values of complex z, w do we have
$$\mathrm{Log}(zw) = \mathrm{Log}(z) + \mathrm{Log}(w)$$

2.5 Arcfunctions

The three inverse trig functions can be given explicit form by use of complex log. We work out the details the complex arctan function. There are two ways to understand this. First by computation, $w = \arctan(z)$ so

$$z = \tan(w) = \frac{1}{i}\frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} = \frac{1}{i}\frac{e^{2iw} - 1}{e^{2iw} + 1}\ .$$

Solving for w gives

$$e^{2iw} = \frac{1+iz}{1-iz} \,.$$

To go further we need to choose a branch of logarithm. Taking the Principal branch we get

$$w = \frac{1}{2i}\mathrm{Log}\left\{\frac{1+iz}{1-iz}\right\} \,.$$

To finish up we observe where this branch of arctan is defined. Consider the bilinear map

$$T(z) = \frac{1+iz}{1-iz} \,.$$

This maps the plane minus the lines $\{iy : |y| \geq 1\}$ onto the region $\mathbb{C}-(-\infty, 0]$ where Log is defined. This inverse of arctan is defined on $\mathbb{C} - \{iy : |y| \geq 1\}$.

Another way of finding the inverse is thinking about where tan is 1:1. Now $\tan(w)$ has period π. From calculus we know that $\tan(w)$ maps $(-\pi/2, \pi/2)$ onto $(-\infty, +\infty)$. The line $\{w = -\pi/2 + it, t > 0\}$ is mapped onto the interval along the imaginary axis from $+i\infty$ to i. We see that $\tan(w)$ maps the positive imaginary axis onto the interval $[i, 0]$. Also $w = \tan(z)$ maps the strip $\{w : -\pi/2 < \Re(w) < 0, \Im(w) > 0\}$ 1:1 onto the quadrant $\{z : \Re(z) < 0, \Im(z) > 0\}$. The situation for other strips is symmetric, $\tan(w)$ maps $\{w : -\pi/2 < \Re(w) < 0, \Im(w) < 0\}$ onto $\{z : \Re(z) < 0, \Im(z) < 0\}$, $\{w : 0 < \Re(w) < \pi/2, \Im(w) > 0\}$ onto $\{z : \Re(z) > 0, \Im(z) > 0\}$, and $\{w : 0 < \Re(w) < \pi/2, \Im(w) < 0\}$ onto $\{z : \Re(z) > 0, \Im(z) < 0\}$. It follows that $\tan(w)$ maps $\{w : -\pi/2 < \Re(w) < \pi/2\}$ 1:1 onto $\mathbb{C} - \{it : t \geq 1\} - \{it : t \leq -1\}$. Going through the previous analysis we obtain

$$\mathrm{Arctan}(z) = \frac{1}{2i}\mathrm{Log}\left\{\frac{1+iz}{1-iz}\right\} \,,$$

the same as before.

Exercises

1. By direct computation show

$$\frac{d}{dw}\arctan(w) = \frac{1}{1+w^2}$$

2. Obtain the power series expansion at $z = 0$ for Arctan(z).

3. Find the range of $\sin(z)$ on the domain

$$\{z : -\frac{\pi}{2} < \Re(z) < \frac{\pi}{2},\ \Im(z) > 0\}$$

4. Give the explicit formula for $\arccos(w)$.

5. What is the domain for the Principal branch of $\arccos(w)$.

2.6 Roots and Powers

Now we define complex powers. When n is a positive integer we see (by de Moivre's Theorem) that the function $f(z) = z^n$ maps the domain $\mathbb{C} - \{0\}$ in a $n : 1$ way onto **C** - $\{0\}$. To obtain an inverse g we first find a domain where f is 1:1. There are many to choose from but the standard one is $D = \{z : -\pi/n < \text{Arg}(z) < \pi/n\}$ giving the Principal nth root of z as

$$g(z) = r^{1/n} \{\cos(\theta/n) + i \sin(\theta/n)\}\ ,$$

where $r = |z|$, $\theta =$ Arg(z). Now as $g(z) = \exp\{\text{ Log}(z)/n\}$ is the composition of analytic functions it too is analytic. This only defines roots on P, for other z we take other branches, altogether there are n roots. This approach suggests a consistent way of defining complex powers , i.e. for any complex a :

$$z^a = \exp\{a \text{ Log}(z)\}\ .$$

This is an analytic function on P, in general we obtain the multivalued function

$$z^a = \exp\{a \log(z)\}\ .$$

We have to be a little careful as the usual rules of exponents do not always hold (for the Principal value but always for the general function).

Example: Computing the values of i^{-2i}, we find $\exp[-2i \log(i)] = \exp[-2i(2n+1/2)\pi i] = \exp[(4n + 1)\pi],\ n \in \mathbb{Z}$.

When we compute the derivative of such a function we have to be careful to be consistent and differentiate the same branch:

$$\begin{aligned}\frac{dz^a}{dz} &= \frac{d}{dz}\exp[a\log(z)] = \frac{a}{z}\exp[a\log(z)]\\ &= a\exp[(a-1)\log(z)] = az^{a-1}\end{aligned}$$

where the same branch of log has been used at every step .

After all this we have defined $e^z = \exp[z \ \mathrm{Log}(e)]$. So only now we have

$$e^{i\pi} + 1 = 0\ .$$

Exercises

1. Show that $(-1+\sqrt{3}i)^{3/2} = \pm 2\sqrt{2}$.
2. For what values of z, a do we have $|z^a| = |z|^a$.
3. Show that for all complex a, b, z we have $z^a z^b = z^{a+b}$, where principal values are taken only.

2.7 Riemann sphere

We often work with "the point at infinity", ∞. Unfortunately it is different from the $\pm\infty$ used in real analysis, or the infinities used in the real projective plane, or in set theory for that matter. This special infinity is best understood by introducing another model for the complex plane.

First we consider a sphere in $\mathbb{R}^3 = \{(x_1, x_2, x_3)\ :\ x_j \in \mathbb{R}\}$ defined by

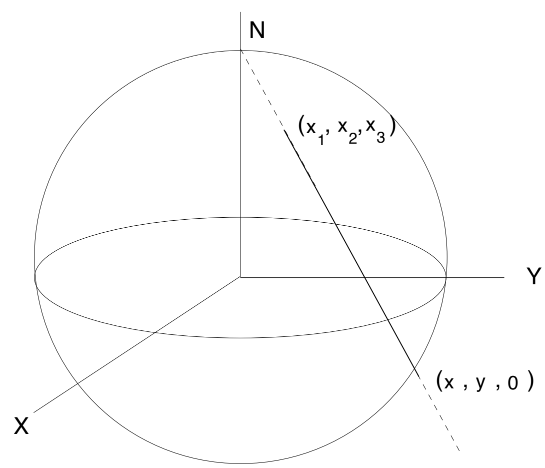
$$\mathbb{S} : x_1^2 + x_2^2 + x_3^2 = 1\ .$$

Also define a mapping $P : \mathbb{S} - \{(0,0,1)\} \to \mathbb{C}$ as

$$P(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3} = z = x + iy\ .$$

This $1:1$ onto map allows us to think of the complex plane as being a subset of the so called Riemann sphere $\mathbb{S}$, namely $\mathbb{S} - \{(0,0,1)\}$. The north pole $(0,0,1)$ is regarded as the point at infinity.

RIEMANN SPHERE



Consider the following computations:

$$x^2 + y^2 = |z|^2 = \frac{x_1^2 + x_2^2}{(1 - x_3)^2} = \frac{1 + x_3}{1 - x_3}$$

and thus

$$\begin{aligned} x_3 &= \tfrac{|z|^2 - 1}{|z|^2 + 1}, \\ 1 - x_3 &= \tfrac{2}{|z|^2 + 1}. \end{aligned}$$

Substituting into $x_1 = (1 - x_3)x,\ x_2 = (1 - x_3)y$ proves the formulae

$$x_1 = \frac{2x}{1 + x^2 + y^2}\ ,\ x_2 = \frac{2y}{1 + x^2 + y^2}\ .$$

Hence $(x, y, 0), (x_1, x_2, x_3),\ (0, 0, 1)$ are points on the same line, i.e.

THEOREM 11 *P is the stereographic projection of $\mathbb{S} - \{(0, 0, 1)\}$ onto $\mathbb{C}$.*

Next we prove

THEOREM 12 *Any circle C on $\mathbb{S}$ projects onto (and conversely) :*

1. *a line if* $(0,0,1) \in C$.

2. *a circle, otherwise.*

Any circle on $\mathbb{S}$ is defined by the intersection of $\mathbb{S}$ with a plane :

$$H: \; a_1x_1 + a_2x_2 + a_3x_3 = a_0 \; ,$$

for constants satisfying $a_1^2 + a_2^2 + a_3^2 = 1$. Using the previous equations we get

$$a_1 2x + a_2 2y + a_3(x^2 + y^2 - 1) = a_0(x^2 + y^2 + 1)$$

which may be written as

$$a_1 2x + a_2 2y + (a_3 - a_0)(x^2 + y^2) = (a_3 + a_0) \; .$$

This is the general equation of a circle unless $a_3 = a_0$, in which case we have a line and H intersects $(0,0,1)$.

Sometimes we refer to the chordal distance or metric. This uses the Euclidean distance between points on $\mathbb{S}$ to measure the distance between the corresponding points of the plane. In the exercises we prove the formula for complex numbers $z = P[(x_1, x_2, x_3)], \; w = P[(y_1, y_2, y_3)]$, the chordal metric is

$$\chi[z,w] = \frac{2|z-w|}{\sqrt{(|z|^2+1)(|w|^2+1)}} \; .$$

We found that circles are mapped by P to circle/lines. Other types of curves are not preserved. However if A and B are smooth curves on $\mathbb{S}$ which intersect at angle θ say then P maps A and B to smooth curves α and β which intersect at angle θ. This means that P is also a conformal mapping, i.e. preserves angles.

There is an algebraic model for the Riemann sphere. Recall that the rational numbers may be constructed from the integers by defining an equivalence class on ordered pairs (n, m) of integers where m is nonzero. If we apply the same construction to something that is already a field we essentially get the original field back. Motivated by this we define the complex projective plane as $\{(z,w) \in \mathbb{C}^2 - \{(0,0)\}\}$ with equivalence relation $(z,w) \equiv (a,b)$ if $zb = wa$. Notice that every ordinary complex number z is equivalent to the family $\{(zw, w) : w \in \mathbb{C} - \{0\}\}$. The point at infinity is the family $\{(w, 0) : w \in \mathbb{C} - \{0\}\}$. Essentially we are thinking of complex numbers as ratios

$$\frac{z}{w}\,, \quad \text{with } (z, w) \neq (0, 0)\,.$$

Geometrically the "points " are the complex lines through $(0, 0)$ which are written as $\{t(z, 1) : t \in \mathbb{C}\}$, except for the point at infinity which is the line $\{t(1, 0) : t \in \mathbb{C}\}$. Letting $z = x + iy$ and choosing $t = 2/(|z|^2 + 1)$ defines points $x_1 = tx,\ x_2 = ty,\ x_3 = t$ of a sphere in 3-space centered at $(0, 0, 1)$ and with radius 1. In this model the point at infinity now corresponds to the South Pole $(0, 0, 0)$. This realisation of the complex projective plane as a sphere shows the algebraic model is equivalent to the geometric model.

The ONLY arithmetical operations with ∞ consistently defined are:

1. $\infty + z = \infty$ for all finite z,
2. $\infty . z = \infty$ for all finite nonzero z,
3. $z/0 = \infty$ for all finite nonzero z.

Complex analysts happily switch between various models of the plane.

Exercises

1. Show that complex numbers z, w correspond to diametrically opposite points on the Riemann sphere iff $z\overline{w} = -1$.
2. Prove the formula for the chordal metric.
3. Show that the infinitesmal form of the chordal metric is
$$\sigma = \frac{2|dz|}{(1 + |z|^2)}\,,$$
i.e. the spherical length of any curve γ on $\mathbb{S}$ is given by
$$\sigma(\gamma) = \int_\gamma \frac{2|dz|}{(1 + |z|^2)}\,.$$
4. Prove that P is a conformal mapping.

2.8 Bilinear Transformations

Conformal mappings play a unique role in complex analysis. We have already seen that the mapping P is conformal. Other conformal self maps of the extended plane to itself may be found by considering conformal self maps of the Riemann sphere. For example any rotation of $\mathbb{S}$ about some axis between diametrically opposed points of $\mathbb{S}$. One can also consider anti-conformal mappings, i.e. those which reverse angles (such as reflections of $\mathbb{C}$). A simple example is the self map of the sphere:

$$F : (x_1, x_2, x_3) \to (x_1, x_2, -x_3) \ .$$

This is a reflection in the (x_1, x_2) plane and so reverses angles. Applying? the mapping P we see that F corresponds to the map of the extended plane $z \to 1/\overline{z}$. Next we apply the anti-conformal map $z \to \overline{z}$ we see that $z \to 1/z$ is a conformal map of the extended plane. The transformation $z \to 1/z$ converts any statement about the point at infinity to a statement about zero and is often used.

Of course there are some other conformal mappings of the plane:

$$z \to az + b \ , a \neq 0 \ ,$$

i.e. ordinary linear maps. These maps are the composition of

$$z \to az \ ,$$

a rotation of size $\arg(a)$ combined with dilatation of size $|a|$, with

$$z \to z + b \ ,$$

a pure translation by b. Composing these with the inversion $z \to 1/z$ gives more general conformal mappings:

$$z \to \frac{az + b}{cz + d} = \frac{bc - ad}{c(cz + d)} + \frac{a}{c} \ ,$$

which is nonconstant, even $1 : 1$, provided $ad - bc \neq 0$. Later we shall show that every conformal map of the extended complex plane has this form. These mappings are also called Möbius maps (after their discoverer). Möbius mappings are also called bilinear transformations because they are equivalent to 2×2 matrices with complex coefficents, as we now explain.

The bilinear transformation T defined by

$$T(z) = \frac{az+b}{cz+d} \ ,$$

may be written as

$$\begin{aligned} w_1 &= az_1 + bz_2 \ , \\ w_2 &= cz_1 + dz_2 \ , \end{aligned}$$

so that

$$w = \frac{w_1}{w_2} = \frac{az+b}{cz+d} \ , \text{ where } z = \frac{z_1}{z_2} \ .$$

This of course is using the realisation of the extended complex plane as ratios. We have to deal with the extended plane since if c is nonzero then $-d/c$ maps to ∞, and ∞ maps to a/c. The representation of T by matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a good representation because if S is another bilinear representation, say

$$S(z) = \frac{pz+q}{rz+s} \ ,$$

represented by matrix

$$B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

then the composition $T \text{o} S$, which is also bilinear, is represented by

$$C = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

This can be proved by 2 different ways. Firstly by just computing, actually as any bilinear transformation is the composition of basic transformations of the form:

$$T_1(z) = z + b, \ T_2(z) = az, T_3(z) = 1/z,$$

it suffices to consider only the 6 possible combinations of these. However if you understood the construction of the extended complex plane as ratios then the composition rule is pretty obvious.

In particular T has inverse given by

$$w = T^{-1}(z) = \frac{dz - b}{-cz + a} \, .$$

In any case we see that the bilinear transformations under the operation of function composition forms a noncommutative group with the function $\mathbf{I}(z) = z$ as the identity. It is equivalent to the class of 2×2 complex matrices with nonzero determinant. In modern parlance the set of all bilinear transformations is called the general linear group $\mathbf{P}(1, \mathbb{C})$.

To obtain further results we define the crossratio. Let $z_1, \ z_2, \ z_3, \ z_4$ be four distinct points of the extended complex plane. Then the crossratio is

$$[z_1, \ z_2, \ z_3, \ z_4] = \frac{z_1 - z_3}{z_1 - z_4} \frac{z_2 - z_4}{z_2 - z_3} \, .$$

For instance if $z_4 = \infty$ then the crossratio becomes

$$[z_1, \ z_2, \ z_3, \ z_4] = \frac{z_1 - z_3}{z_2 - z_3} \, ,$$

in particular $[z, \ 1, \ 0, \ \infty] = z$. Crossratios are invariant under bilinear transformations:

PROPOSITION 1 *For any bilinear transformation T*

$$[z_1, \ z_2, \ z_3, \ z_4] = [T(z_1), \ T(z_2), T(\ z_3), \ T(z_4)] \, .$$

This important result can be proved in a number of different ways. I don't suggest the obvious one, namely substitute any T, although after some calculations it will work. The proof can be simplified on realising that the general T is the composition of linear maps and inversions, and thus it suffices to prove the Proposition for these. Actually the Proposition is obvious for linear transformations so we can just check it for $T(z) = 1/z$. This result can be used to show

PROPOSITION 2 *Given any three distinct points $z_2, \ z_3, \ z_4$ of the extended plane and any other three distinct points $w_2, \ w_3, \ w_4$ there is a unique bilinear transformation T so that $T(z_j) = w_j, \ j = 2, 3, 4$.*

First we construct T. A map defined by

$$S_1(z) = [z,\ z_2,\ z_3,\ z_4] = \frac{z - z_3}{z - z_4}\,\frac{z_2 - z_4}{z_2 - z_3}\ ,$$

has the property that

$$S_1(z_2) = 1,\ S_1(z_3) = 0,\ S_1(z_4) = \infty\ .$$

On the other hand the map

$$S_2(w) = [w,\ w_2,\ w_3,\ w_4] = \frac{w - w_3}{w - w_4}\,\frac{w_2 - w_4}{w_2 - w_3}\ ,$$

satisfies

$$S_2(w_2) = 1,\ S_2(w_3) = 0,\ S_2(w_4) = \infty\ .$$

Consequently $T = S_2^{-1}\ \mathrm{o}\ S_1$ has the required properties. To show uniqueness we suppose that there is another map S so that $S(z_j) = w_j,\ j = 2,3,4.$. Then the map

$$R = S_1\ \mathrm{o}\, T^{-1}\ \mathrm{o}\, S\ \ \mathrm{o}\, S_1^{-1}$$

fixes the points $1, 0, \infty$. Now by the previous Proposition

$$R(z) = [R(z), 1, 0, \infty] = [z, 1, 0, \infty] = z,\ \forall z \notin \{1, 0, \infty\}\ .$$

Thus $T = S$, and this proves the lemma.

Although this gives an expression for T we should note that solving the equation for w in terms of z is usually slower then using properties of the transformation to find T. Finally we prove

PROPOSITION 3 *The crossratio* $[z_1,\ z_2,\ z_3,\ z_4]$ *is real iff* $z_1,\ z_2,\ z_3,\ z_4$ *lie on a circle or line.*

First we note that if T is a bilinear map (see exercises) and L is a circle/line then so is $T(L)$. We may use preliminary bilinear transformations, and so assume without loss of generality that

$$\{z_1,\ z_2,\ z_3,\ z_4\} = \{z,\ 1,\ 0,\ \infty\}\ .$$

That is we may assume that $z_2,\ z_3,\ z_4$ lie on the real line. But then $z = [z_1,\ z_2,\ z_3,\ z_4]$ is real iff. z is also on the real line. Thus z_1 lies on the same circle/line as the other three points.

Exercises

1. Find the bilinear transformation $w = T(z)$ so that
$$[z, 1, i, -1] = [w, i, -i, 1].$$

2. Prove:T bilinear and L is a circle/line $\Rightarrow T(L)$ is a circle/line.

3. Find the bilinear transformation T:
$$T(\{|z| < 1\}) = \{\Re(z) > 0\},\ T(0) = 1,\ T(-1) = 0$$

4. We say that w is a fixed point of T if $T(w) = w$.
Show that every bilinear map T, except for the identity, fixes at most two points of the extended complex plane.

5. If T fixes exactly one point of the extended complex plane show there is bilinear map S so that $S\ \mathrm{o}\ T\ \mathrm{o}\ S^{-1} = z + 1$.
(Then T is called parabolic).

6. If T fixes exactly two points of the extended complex plane show there is bilinear map S so that $S\ \mathrm{o}\ T\ \mathrm{o}\ S^{-1} = az$, where a is some fixed nonzero complex number. (We then call T hyperbolic if $|a| \neq 1$ or if a is real. If $|a| = 1$ then T is called elliptic , while it is loxodromic if $|a| \neq 1$ but not real.)

7. For bilinear T mapping $\mathbb{D} = \{|z| < 1\}$ onto itself show that
$$T(z) = e^{i\theta}\,\frac{z - a}{1 - \overline{a}z}\ ,$$
for some complex number a, $|a| < 1$, and angle θ, and conversely. (Hint: show $|T(z)| = 1$ for $|z| = 1$.)

8. Find a T a bilinear map of the disk $\mathbb{D}$ onto itself which has no fixed points inside the unit disk.

9. Let T a bilinear map of the disk $\mathbb{D}$ onto itself which has no fixed points in $\mathbb{D}$. Show there is a fixed point $\zeta \in \{|z| = 1\}$ so that
$$T^n(z) \to \zeta\ , \forall z \in \mathbb{D}\ .$$

10. Find all bilinear transformations that represent rotations of the Riemann sphere.

11. Show that T is a bilinear map of the halfplane $\mathbb{H} = \{\Im(z) > 0\}$ onto itself iff.
$$T(z) = \frac{az+b}{cz+d} ,$$
for some real numbers a, b, c, d so that $ad - bc = 1$.

12. Show that the following set of bilinear mappings
$$T(z) = \frac{az+b}{cz+d} , \text{ where } a, b, c, d \text{ are integers so that } ad - bc > 0 ,$$
is a group $\mathcal{M}$ generated by elements z, $-1/z$, $z+1$.

13. Different crossratios may be defined by using different permutations π of $\{z_1, z_2, z_3, z_4\}$. Show that for every such permutation π there is a bilinear mapping $T \in \mathcal{M}$ so that
$$T([z_1, z_2, z_3, z_4]) = [\pi(z_1), \pi(z_2), \pi(z_3), \pi(z_4)] .$$

2.9 Conformal Mapping

Conformal mapping has already been associated with bilinear mappings and the stereographic projection from the Riemann sphere. In general we have

DEFINITION 6 *(Conformal Mappings) A differentiable mapping f is conformal at a point ζ if for any smooth curves α, β intersecting at ζ with angle θ we have that $f(\alpha)$, $f(\beta)$ intersecting at $f(\zeta)$ with angle θ.*

As we are really only asking about the angles between tangents it suffices to check just for intersecting lines. Then from geometry we see that $f(x, y) = u + iv$ has nonsingular Jacobian matrix of the form
$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \rho \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} ,$$
i.e. a rotation and a scaling. But this is equivalent to $u_x = v_y$, $u_y = -v_x$, in other words the Cauchy Riemann equations hold. We call a mapping which is conformal at every point a locally conformal mapping, reserving the name conformal mapping for locally conformal maps which are $1:1$ on their domain of definition.

PROPOSITION 4 *A mapping is locally conformal iff. it is holomorphic with nonzero derivative.*

Example: Consider the mapping $w = f(z) = z^2$ which, on the Riemann sphere is conformal except at $z = 0, \infty$. First we observe that circles $\{|z| = r\}$ are mapped to circles $\{|z| = r^2\}$ and rays $\{\arg(z) = \theta\}$ to rays $\{\arg(z) = 2\theta\}$. For other curves the transformation is more complicated and we use $z = x + iy,\ w = u + iv$ so $u = x^2 - y^2,\ v = 2xy$. Thus vertical lines $x = c$ are mapped into $u = c^2 - y^2,\ v = 2cy$ and eliminating y gives $u = c^2 - v^2/(4c^2)$, a parabola. Horizontal lines map into hyperbolae. The mapping is not 1:1 on the whole plane but restricting ourselves to $\{\Re(z) > 0\}$ gives a 1:1 mapping onto the plane cut along $(-\infty, 0)$.

Now analytic functions are conformal except at the zeros of the derivative. This means that conformal mappings are locally 1:1, but as exp shows, not necessarily globally 1:1. It is usual to define a mapping to be conformal on a region D if it is holomorphic and 1:1 on D. However the importance of conformal mappings means they have many names: univalent, simple or regular mappings, schlicht("nice"), biholomorphic (mainly used in several complex variables).

The importance of conformal mappings means that there are dictionaries of mappings between fundamental regions:

1. Any "disk/half plane" can be transformed into another by a bilinear mapping :
   (a). Disks to disks by linear maps $w = az + b$.
   (b). Disks to half planes, via $w = (1+z)/(1-z)$ which takes $\{|z| < 1\}$ onto the half plane $\{|\mathrm{Arg}(z)| < \pi/2\}$.

2. Wedges $\{|\mathrm{Arg}(z)| < a\pi/2\}$, $0 < a < 2$, to $\{|\mathrm{Arg}(z)| < \pi/2\}$ by the mapping $w = z^{1/a}$.

3. The function $w = \exp(z)$ maps the strip $\{|\Im(z)| < \pi/2\}$ 1:1 onto $\{|\mathrm{Arg}(z)| < \pi/2\}$.

4. The function $w = \sin(z)$ maps the half-strip $\{|\Re(z)| < \pi/2,\ \Im(z) > 0\}$ 1:1 onto $\{0 < \mathrm{Arg}(z)| < \pi\}$.

More complicated mappings can formed by compositions:

Example:

Find the conformal mapping of the region

$$R = \{z : \ |z-1| > 1, |z+1| > 1\}$$

onto the unit disk so that ∞ maps to 0.

First we map ∞ to 0 by the inversion $f_1 = 1/z$ which transforms R to the strip $\{|\Re(z)| < 1/2\}$. This is mapped to $\{|\Im(z)| < \pi/2\}$ by $f_2 = \pi i z$ which fixes 0. The strip $\{|\Im(z)| < \pi/2\}$ is mapped to $\{|\mathrm{Arg}(z)| < \pi/2\}$ by $f_3 = \exp(z)$ which takes 0 to 1. Finally $f_4 = (z-1)/(z+1)$ gives $\{|z| < 1\}$. Taking the composition we have:

$$f(z) = f_4 \circ f_3 \circ f_2 \circ f_1 = \frac{e^{i/z} - 1}{e^{i/z} + 1},$$

which is actually $i \tan(i/2z)$.

Exercises

1. For any rational mapping R of degree n so that $R(|z| < 1) = \mathbb{D}$ show

$$\int \int_{\mathbb{D}} |R'(z)|^2 \, dx \, dy = \pi n \ .$$

2. Let f be conformal on $\mathbb{D}$ and $Area(f(\mathbb{D})) < \infty$. Prove that

$$\int_0^1 |f'(re^{it})| \, dr < \infty,$$

for "almost all " $t \in [0, \, 2\pi]$.

3. Let f be conformal on $\mathbb{D}$. Prove that

$$\lim_{r \to 1} f(re^{it}),$$

exists for "almost all " $t \in [0, \, 2\pi]$.

Chapter 3

Complex Integration

The deeper results of complex analysis require Cauchy's theory of complex integration. Although this is superficially like calculus we soon find that its main aim is *not* the evaluation of definite integrals.

3.1 Connectivity and Curves

A subset E of a metric space is disconnected if there exist open sets A and B so that $E \cap A$ and $E \cap B$ are are nonempty and disjoint and

$$E = (E \cap A) \cup (E \cap B)$$

Otherwise a set E is connected. Furthermore any E is the union of connected subsets. These are called the components of E.

On the real line the situation for connected sets is very simple. The only connected open sets are the intervals I, i.e sets of the form $(a, b), [a, b), ..$ etc. In the plane connected sets can be very complicated indeed. However open connected sets are simpler. We call an open connected set a region (or a domain, from tradition).

Let the complex valued function f be continuous on some interval $[a, b]$. We observe that $f[a, b]$ is a connected set in $\mathbb{C}$. (Otherwise the inverse image of any disconnection is a disconnection of $[a, b]$). The function f is called a path curve $f(a)$ to $f(b)$. Notice there is nothing special about a and b, any points a and b with $a < b$ may be used. This definition would mean that the curves $f(t) = t,\ 0 \leq t \leq 1$ and $g(t) = t^2,\ 0 \leq t \leq 1$ would be different. Wishing them to be equivalent leads us to define:

$f(t),\ a \leq t \leq b$ and $g(t),\ c \leq t \leq d$ represent the same curve if there is a monotone increasing homeomorphism h from $[a, b]$ onto $[c, d]$ such that $f = g \circ h$ on $[a, b]$.

These are the different parametrisations for the same curve. We use Greek symbols α, β, $\gamma, ...$ for curves. It is a bit tedious to refer to the curve equivalence class a represented by f, so we save time and refer to the curve $\alpha :\ z = f(t),\ a \leq t \leq b$. We abuse language further by saying a curve is in a region R, when only its range is.

Examples:

(a) Let α be a curve represented by $f(t)$ continuous on $[0, 1]$ then the curve $-\alpha$ is represented by $f(1 - t),\ t \in [0, 1]$. Observe that these curves are not equivalent.

(b)Let α be a curve represented by $f(t) = \sin(t),\ t \in [0, \pi]$. This is equivalent to $g(t) = 4t(1 - t),\ t \in [0, 1]$.

A curve is polygonal (piecewise linear) if we can represent it as by a curve on $\alpha :\ z = f(t),\ 0 \leq t \leq n$:

$$f(t) = (t - k + 1)z_{k-1} + (k - t)z_k\ ,\ k - 1 \leq t \leq k, k = 1, ..n,$$

where $z_0, ..., z_n$ are points in the complex plane. This is called a polygon.

We now prove that if two points z, w belong to a region D there is a curve, in fact polygonal curve, in D from z to w. Let A be the class of all points z_1 so that there is a polygonal path from z to z_1, let $B = D - A$. Now A is nonempty as $z \in A$, furthermore if $z \in A$ then as D is open there is a $r > 0$ so that the ball $B(z_1, r)$ is in D. If $f[0, 1]$ is the path from z to z_1 for any $z_2 \in B(z_1, r)$ we define a polygonal curve from z to z_2 by:

$$F(t) = \begin{cases} f(t) & \text{if } 0 \leq t \leq 1 \\ (t - 1)z_2 + (2 - t)z_1 & \text{if } 1 \leq t \leq 2\ . \end{cases}$$

Thus $z_2 \in A$ too and so A is an open set. Supposing that B is nonempty we see, similarly, that B is open. Thus D is the union of two disjoint sets. As D is connected this is impossible unless B is empty. Thus $A = D$ and every point is connected by a polygonal curve.

DEFINITION 7 *(Jordan Curves) A curve $\alpha : z = f(t),\ t \in [0,1]$, is an open Jordan curve (or just a Jordan arc) if $f(s) \neq f(t)$ for all $s \neq t$.*
A curve $\alpha : z = f(t),\ t \in [0,1]$, is a closed Jordan curve (or just a Jordan curve) if $f(0) = f(1)$, and otherwise $f(s) \neq f(t)$ for $s \neq t$.

It is intuitively obvious that a Jordan curve α divides the complex plane into two regions, a bounded region called the interior of α , and an unbounded region called the exterior. This result is called the Jordan Curve Theorem after the Jordan who thought he proved it over 100 years ago. A correct proof was not done until topology had been properly developed around 1910. An accessible but not short proof is possible using the theory we prove in the next chapter. Part of the problem is that general Jordan curves can be extremely irregular. Not only may they have tangents at no points but they can have positive area. Examples of such curves were in the 19th century thought to be very pathological. However in the last 20 years it has been discovered that many natural processes produce such curves.

A curve $\alpha : z = f(t),\ t \in [0,1]$, is a smooth curve if $f(t) = u(t)+iv(t)$ has derivative which is nonzero and continuous. The tangent to α at $z = f(t)$ is $\tau = f'(t)/|f'(t)|$. In this course we usually only discuss curves which consists of the union of a finite number of smooth curves, called "piecewise smooth". These piecewise smooth curves $\alpha : z = f(t),\ t \in [a,b]$ have length given by $L(\alpha) = \int_a^b |f'(t)|\, dt$, which we note is independent of the particular parametrisation, see exercises. Thus gives a natural parametrisation, the so called arc length parametrisation, defined by $z = z(s)$ where $0 \leq s \leq L(a)$ is given by $s = \int_a^t |f'(t)|\, dt$.

To study more general curves with finite length we need a little measure theory. A curve $\alpha : z = f(t),\ t \in [0,1]$ is said to be rectifiable if there is a positive constant M:

$$\sum_{k=1}^{k=n} |z(t_k) - z(t_{k-1})| < M ,$$

for all partitions $a = t_0 \leq t_1 \leq ... \leq t_n = b$. The supremum over the set of bounds is called the length L of the curve, as it agrees with the definition for the smooth case. The bound on the differences is also the definition of the total variation of the function $z(t)$. From the Lebesgue theory of derivatives we deduce that $z = z(t)$ is differentiable on $[a, b]$, for almost all points t. However $z = z(t)$ is not necessarily absolutely continuous and need not give a good parametrisation. But we can define the length of subarcs of α and use arc length as a parametrisation. In this case, as $|z(s_1) - z(s_2)| \leq |s_1 - s_2|$, $z = f(s)$ is absolutely continuous and the derivative is bounded and satisfies $|f'(s)| = 1$ at all most all points. Thus we define the arc length

$$L(\alpha) = \int_\alpha |dz|$$

independently of any parametrisation.

Exercise

1. Prove that if E, G and H are nonempty sets with $E \cup G$, $G \cup H$ are connected sets then $E \cup G \cup H$ is connected.

2. We say that a curve α is approximated within ϵ by a curve β if there are parametrisations f and g of α, β both continuous on $[a, b]$ so that $|f(t) - g(t)| < \epsilon$ for all $t \in [a, b]$. For any a curve α contained inside region $R$and any $\epsilon > 0$ show that there is a polygonal curve Π contained inside R so that α is approximated within ϵ by Π.

3. Prove that arc lengths of equivalent curves are equal.

4. Let α be a curve in a region R with finite length L. Prove that α is approximated within ϵ by a polygonal curve Π with length no more then $2L$. (Actually the length of the approximating polygons may be arranged to tend towards the length of α.)

3.2 Line integrals

We could develop a theory of complex integrals from Riemann Sums using complex numbers, however it is simpler to assume the theory of line integrals from advanced calculus. It is first just a matter of using complex notation:

Let γ be a piecewise smooth curve parametrised by $z = z(t),\ 0 \le t \le 1$. If f and g are complex valued continuous functions defined on $\{z = z(t) :\ 0 \le t \le 1\}$ we define the line integral

$$\int_\gamma f\,dz + g\,\overline{dz} = \int_0^1 \{f(z(t))\,z'(t) + g(z(t))\,\overline{z'(t)}\}dt\ .$$

We make some remarks:

1. This has the form $\int_0^1 h(t)\,dt$ where $h(t) = u(t)+iv(t)$ is complex valued. Thus the integral may be evaluated by applying Riemann integration to the real and imaginary part separately.

2. The simplest way to evaluate this is to find a complex valued antiderivivative for h, i.e. if $H'(t) = h(t)$ then

$$\int_0^1 h(t)\,dt = H(1) - H(0)$$

   We find that our integrals can almost never be evaluated like this!

3. Now as $\int_\gamma g\,\overline{dz} = \overline{\int_\gamma \overline{g}\,dz}$ it suffices to study integrals in dz. Hence we assume $g = 0$.

We shall be considering all curves in a fixed region R and f continuous on R. If f is the derivative of F, a holomorphic function on R, the chain rule shows that

$$\int_\gamma f\,dz = F(b) - F(a)$$

for any curve γ in R beginning from a and ending at b. Unlike calculus the main question in complex analysis is not however to find antiderivatives.

Exercises

1. The line integral is defined in terms of the particular parametrisation. Show that it only depends only on the curve, i.e. the integral is the same for equivalent parametrisations.

2. Let γ be the straight line interval from 0 to i and $f = z^2,\ g = iz$. Evaluate the line integral.

3. Let K be a curve $z(t) = r\exp(it)$, fixed $r > 0,\ 0 \le t \le 2\pi$. Compute
$$\int_K \frac{dz}{z},\ \int_K |z-1|\frac{dz}{z}\ ,\ \int_K \frac{dz}{2z^2-1}\ .$$

4. Show that
$$\int_\gamma \frac{dz}{1+z^2} = 0$$
for any Jordan curve γ in the region $R = \mathbb{C} - \{it : t \ge 1,\ t \le -1\}$.

5. Let f be a power series convergent in $\mathbb{D} = \{|z| < r\}$. For fixed $a,\ b \in \mathbb{D}$ show that $\int_\gamma f\,dz$ does not depend on the curve $\gamma \subset D$ that goes from a to b.

6. If f is analytic and $|f(z) - 1| < 1$ on a region R show
$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = 0$$
for any closed curve γ in the region R .

7. Let Q_N be the square with vertices $\pm N\pi \pm N\pi i,\ N = 1,\ 2....$Show
$$\left|\int_{Q_N} \frac{dz}{z\ \cos(z)}\right| \le C,$$
for a constant C independent of N.

3.3 Integral Transforms

In analysis we commonly meet linear transforms defined as integrals over an interval $I = (a,\ b)$. Typically there is a "kernel" function $K(z, x)$ continuous in $z \in D$, integrable in $x \in I$. The integral transform $g = T(f)$ of f continuous on I is defined by
$$g(z) = \int_I K(z, x) f(x)\,dx\ .$$
It is easy to see that T is a linear transform of the space of functions continuous on I into the space of functions continuous on D.

In important cases the kernel function is actually analytic in $z \in D$, i.e. for each $a \in D$ there is a $r > 0$ so that for $|z - a| < r$

$$K(z, x) = \sum_{n=0}^{\infty} a_n(x)(z - a)^n \ ,$$

where $a_n(x)$ is continuous on I and satisfies

$$\int_I |a_n(x)| \, dx \leq C t^n \ ,$$

where $r < t$ (remembering Abel's Theorem for Power Series).

Under these circumstances we can interchange the order of integration and find that

$$g(z) = \sum_{n=0}^{\infty} \{ \int_I a_n(x) \, dx \} (z - a)^n \ ,$$

is analytic in $z \in D$.

Exercises

1. For any rectifiable curve γ find a domain D so that the "Cauchy Transform" of any f defined and continuous on γ:

   $$g(z) = \int_\gamma \frac{f(\zeta)}{z - \zeta} \, d\zeta$$

   is analytic on D.

2. Find domain D on which the "Laplace transform"

   $$\int_0^\infty f(t) e^{-zt} \, dt$$

   is analytic.

3. Find domain D on which the "finite" Fourier transform

   $$\int_a^b f(t) e^{izt} \, dt.$$

   is analytic.

For those knowing some measure theory: the assumption that our functions are continuous can be replaced by simply being integrable (and an interval replaced by integration over some measure).

3.4 Chains and Cycles

We introduce a standard notation. We wish to talk about the sum and differences of curves in our fixed region R. For α_1, ...α_n curves in R and m_1, ...m_n integers. We define a "formal" sum $m_1\,\alpha_1 + ... + m_n\,\alpha_n$. This is called a chain σ. Clearly chains can be added and subtracted formally. We use this formalism as notation to describe integrals over sums of curves :

$$\int_\sigma f(z)\,dz = m_1 \int_{\alpha_1} f(z)\,dz + ... + m_n \int_{\alpha_n} f(z)\,dz$$

Thus two chains σ, ρ will be regarded as equivalent iff. for every f continuous on R

$$\int_\sigma f(z)\,dz = \int_\rho f(z)\,dz.$$

Chains can be thought as generalisations of curves $z = z(t)$ to piecewise continuous $z(t)$. Recall that a closed curve $z = z(t),\ 0 \le t \le 1$, satisfies $z(0) = z(1)$. A chain is called a cycle if it is equivalent to $m_1\,\alpha_1 + ... + m_n\,\alpha_n$, where $\alpha_1, ..., \alpha_n$, are closed curves.

We want to decompose simple shapes into cycles . The simplest case is a triangle Δ which is a polygon with noncollinear vertices z_1, z_2, z_3. The triangle is positively orientated if $0 < \arg(z_3 - z_2)/(z_2 - z_1) < \pi$. We seek 4 congruent triangles Δ_j which are positively orientated and $\Delta = \Delta_1 + ... + \Delta_4$. Geometrically this is obvious. We bisect the sides of the original triangle and join these points. This gives 4 triangles Δ_j inside Δ, the sides of Δ_j on Δ determine the orientation of the subtriangles. Notice that as far as line integrals are concerned the interior sides of the Δ_j add up to zero. If we now apply the same process to each subtriangle... we can write Δ as larger and larger sum of smaller and smaller triangles.

Now a rectangle or more generally(by induction) any polygon is the sum of triangles. We say that the polygon is positively orientated if it is the sum of positively orientated triangles. We usually are studying polygons inside a region R. However this does not mean that if we decompose it into a cycle sum of triangles that these will also be inside the region. This will for instance be the case if the boundary of R is not connected (as a subset of the Riemann sphere). Now we have a choice of restricting ourselves to regions with connected boundaries or we have to be more specific about our polygons. In this chapter we restrict ourselves to regions with connected boundaries, usually just disks. In chapter 7 we consider polygonal cycles which are the boundaries of unions of squares contained inside the region.

3.5 Green's Theorem

This is one of a family of results (Gauss Theorem, Stokes Theorem). In some sense they are all just integration by parts. We give the version in complex notation. This is now rigorously proved.

THEOREM 13 *(Green's Theorem) Let f be differentiable on a region R such that $\overline{\partial} f$ is continuous on R. Then if Δ is a positively orientated triangle in R with interior D in R,*

$$\int_{\Delta} f(z)\, dz = 2i \int\int_{D} \overline{\partial} f \; dx\, dy.$$

We begin with a special case:

LEMMA 4 *Let Δ is a positively orientated triangle with interior D ,*

$$\int_{\Delta} \overline{z}\, dz = 2i Area(D).$$

Notice the LHS is invariant under rotations so we can assume the triangle has the form $0 < x < a, 0 < y < b$. This proves the result for right angle triangles. Hence as any triangle can be decomposed into the sum of two right angle triangles this proves the lemma.

To prove the theorem we use the decomposition of Δ into the sum of 4 congruent subtriangles. Each of these is the decomposition of 4 congruent subtriangles. Hence by induction after n decompositions we obtain 4^n triangles $\Delta_{n,k}$ with interior $D_{n,k},\ k = 1...4n$:

1. $\Delta_{n,k}$ is congruent to Δ, $dia(\Delta_{n,k}) = 4^{-n} dia(\Delta)$.

2. $\Delta = \Delta_{n,1} + ... + \Delta_{n,4^n}$.

Now we can write

$$\int_{\Delta_{n,j}} f(z)\,dz - 2i \int\int_{D_{n,j}} \overline{\partial} f \; dx\,dy$$

$$= \sum_{k=1}^{4} \left\{ \int_{\Delta_{n+1,k}} f(z)\,dz - 2i \int\int_{D_{n+1,k}} \overline{\partial} f \; dx\,dy \right\}$$

where the triangle $\Delta_{n,j}$ contains the four triangles $\Delta_{n+1,k}$. Assuming that for $n = 1$ the LHS is a nonzero number a then at least one of the subtriangles satisfy

$$\left| \int_{\Delta_{1,j}} f(z)\,dz - 2i \int\int_{D_{1,j}} \overline{\partial} f \; dx\,dy \right| \geq |a|/4.$$

Hence we obtain a sequence of nested triangles so that

$$\left| \int_{\Delta_{n,n(j)}} f(z)\,dz - 2i \int\int_{D_{n,n(j)}} \overline{\partial} f \; dx\,dy \right| \geq |a|/4^n.$$

Now by Cantor Intersection these triangles converge to point $w \in R$. But f is differentiable at w

$$f(z) = f(w) + \partial f(w)(z-w) + \overline{\partial} f \,\overline{(z-w)} + o(|z-w|).$$

Hence as $\overline{\partial} f$ is continuous

$$\int_{\Delta_{n,n(j)}} f(z)\,dz - 2i \int\int_{D_{n,n(j)}} \overline{\partial} f \; dx\,dy$$

$$= \int_{\Delta_{n,n(j)}} \left\{ f(w) + \partial f(w)(z-w) + \overline{\partial} f(w)\overline{(z-w)} + o(|z-w|) \right\} dz$$

$$-2i \int\int_{D_{n,n(j)}} \left\{ \overline{\partial} f(w) + o(1) \right\} \; dx\,dy$$

$$= \overline{\partial} f(w) \int_{\Delta_{n,n(j)}} \overline{(z-w)} dz - 2i \overline{\partial} f(w) \int\int_{D_{n,n(j)}} dx\,dy + o(4^{-n}) = o(4^{-n}).$$

This is a contradiction. Thus the theorem is proved.

From this we deduce

THEOREM 14 *Let f be differentiable on a region R and $\overline{\partial} f$ is continuous on R. Then if Π is a positively orientated closed Jordan polygon in R which is the sum of triangles all contained in R*

$$\int_{\Pi} f(z)\, dz = 2i \int \int_{P} \overline{\partial} f \; dx\, dy.$$

where P is the interior of the polygon, i.e. the sum of interiors of the triangles.

This is proved as polygon bounds a region which is the union of triangles in R. Applying the previous to each triangle and summing yields the result.

The reason for being so careful in this statement is that it is easy to find regions R and polygon P in R so that P is not the sum of triangles with interiors contained in R.

Exercise

Let f be holomorphic on a region $R - \{\zeta\}$ and $\lim_{z \to \zeta}(z - \zeta)f(z) = 0$. Then if Q is a square in $R - \{\zeta\}$ with interior in R show $\int_Q f(z)dz = 0$.

3.6 Cauchy's Theorem

We begin with a special case for simple regions and leave the general case to the next chapter.

THEOREM 15 *(Cauchy's Theorem) Let f be holomorphic on a disk D. Then if γ is any closed piecewise smooth curve in D*

$$\int_{\gamma} f(z)\, dz = 0.$$

Let $z(t),\ 0 \leq t \leq 1$ parametrise γ. Fix any $\delta > 0$. From our previous work for any $\epsilon > 0$ there is a closed polygon $\Pi \subset D$ where $w(t),\ 0 \leq t \leq 1$ parametrises Π :

1. $|z(t) - w(t)| < \epsilon,\ 0 \leq t \leq 1.$
2. $\int_0^1 |z'(t)) - w'(t)|\, dt < \epsilon.$

As f is continuous we choose ϵ so small that

$$|f(z(t)) - f(w(t))| < \delta \ ,$$

also there is a uniform bound M for $|f(z)|$ on all these curves.

Now f is complex differentiable and satisfies the C.R. equations. Thus for any triangle $\Delta \subset D$ by the previous section

$$\int_\Delta f(z)dz = 0.$$

Hence as the polygon can be written as the sum of triangles in D

$$\int_\Pi f(z)dz = 0.$$

But by the triangle inequality for integrals

$$\left|\int_\gamma f(z)dz - \int_\Pi f(z)dz\right| = \left|\int_0^1 \{f(z(t))z'(t) - f(w(t))w'(t)\}\, dt\right| \leq$$

$$\int_0^1 |f(z(t))z'(t) - f(w(t))z'(t)| + |f(w(t))z'(t) - f(w(t))w'(t)|\, dt \leq$$

$$\sup|f(z(t)) - f(w(t))| \int_0^1 |z'(t)|\, dt + \sup|f(w(t))| \int_0^1 |z'(t) - w'(t)|\, dt$$
$$\leq \delta \text{Length } (\gamma) + M\epsilon,$$

by (1),(2). Hence $\int_\gamma f(z)dz$ can be made arbitrarily small.

Exercise

1. Let f be holomorphic on a disk D. Prove that there is a function F holomorphic on D so that $F' = f$.

2. The following applies our method of proof to a previous exercise.

   LEMMA 5 *Let D be a disk, F be holomorphic on $D - \{\zeta\}$ and*

   $$\lim_{z\to\zeta}(z-\zeta)F(z) = 0$$

   Then if γ is any closed piecewise smooth closed curve in $D - \{\zeta\}$ we have $\int_\gamma F(z)dz = 0$.

3.7 Cauchy's Formula

Once again we prove a special case with the most general version in the next chapter.

THEOREM 16 *(Cauchy's Formula) Let f be holomorphic on a disk D. Then if γ is a any positively orientiated circle in D with z inside γ,*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)}\, d\zeta.$$

We apply the lemma proved in the previous exercise to the function

$$F(\zeta) = \frac{f(\zeta) - f(z)}{\zeta - z}$$

to obtain

$$\int_\gamma \frac{f(\zeta)}{(\zeta - z)}\, d\zeta = f(z) \int_\gamma \frac{d\zeta}{(\zeta - z)} = f(z) 2\pi i.$$

Now as the Cauchy kernel is analytic in z, we immediately deduce

COROLLARY 1 *: f be holomorphic on D implies f is analytic on D.*

Thus a function is holomorphic iff. it is analytic. In the next section we obtain further imformation.

Exercises

1. Suppose that $K : z(t) = \exp(it),\ 0 \le t \le 2\pi$. Then compute the following
$$\int_K \frac{e^{iz}}{z}\, dz \ , \int_K \frac{dz}{4z^2 + 1} \ , \int_K \frac{dz}{z(z^2 + 4)} \ .$$
2. Use the identities $|dz| = -i dz/z, z\overline{z} = 1$ to compute
$$\int_K \frac{|dz|}{|z - w|^2} \ .$$
3. Let f be analytic on $\mathbb{D} = \{|z| < 1\}$ and $\int\int_{\mathbb{D}} |f| dx\, dy < \infty$. Evaluate
$$\int\int_{\mathbb{D}} \frac{f(z)}{(1 - \overline{z}\zeta)^2}\, dx\, dy,\ z = x + iy \ ,$$
for arbitrary $\zeta \in \mathbb{D}$.

3.8 Taylor series

A more precise result than "holomorphic $\Rightarrow$ analytic" is:

THEOREM 17 *(Taylor's Series) Let f be holomorphic on a disk $D = \{z : |z-a| < R\}$. Then for any $r < R$ we have*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

is absolutely convergent for $|z-a| \leq r$ where $a_n = f^{(n)}(a)/n!$.

In other words the power series for f is convergent in the largest circle in which f is holomorphic. (e.g. this explains why although $(1+z^2)^{-1}$ is smooth on $\mathbb{R}$ its Taylor series has radius of converegnce 1).

Without loss of generality $a = 0$. For any $s,\ r < s < R$, Cauchy's formula gives for $|z| < s$

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)}\, d\zeta.$$

where γ is the circle $z(\theta) = s\exp(i\theta),\ 0 \leq \theta \leq 2\pi$. Also for fixed $z,\ |z| < s$

$$\frac{1}{(\zeta - z)} = \sum_{n=0}^{\infty} \frac{z^n}{\zeta^{n+1}},$$

uniformly in $z,\ |z| = s$. Thus we may integrate term by term to yield

$$f(z) = \frac{1}{2\pi i} \int_\gamma \left\{ \sum_{n=0}^{\infty} \frac{f(\zeta) z^n}{\zeta^{n+1}} \right\} d\zeta = \sum_{n=0}^{\infty} \left\{ \frac{1}{2\pi i} \int_\gamma \left(\frac{f(\zeta)}{\zeta^{n+1}} \right) d\zeta \right\} z^n,$$

which is $\sum_{n=0}^{\infty} a_n z^n$, where $a_n = f^n(a)/n!$.

Exercise

Obtain the Taylor series for : $z\cosh(z^2),\ z/(z^4+9)$.

3.9 Cauchy's inequalities

Suppose that $f(z)$ is holomorphic in $\{z : |z-a| < R\}$ then from Taylor's theorem for all nonnegative numbers n

$$f^{(n)}(a) = \frac{n!}{2\pi} \int_0^{2\pi} f(a + re^{i\theta})\, r^{-n} e^{-ni\theta}\, d\theta,$$

for any $r < R$. In particular we have the identity often called the "mean value theorem " :

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta})\, d\theta.$$

Also we have the so called "Cauchy inequalities":

$$|f^{(n)}(a)| \leq \frac{n!}{r^n} \sup_\theta |f(a + re^{i\theta})|,$$

which are also very useful. For instance if $f(z)$ is is analytic on $|z-a| \leq r$, bounded by M on $\{|z| = r\}$ then

$$|f'(a)| \leq \frac{M}{r}.$$

There are many functions bounded on the whole plane however if they are also analytic, the previous inequality with $r \to \infty$ gives:

THEOREM 18 *(Liouville's Theorem) Any function f analytic and bounded on the entire plane is constant.*

These inequalities have many other applications, including a nontopological proof of the fundamental theorem of algebra, which we include as exercises. Actually Liouville's Theorem is a big sledge hammer to prove the little old fundamental theorem of algebra. Liouville's Theorem was originally used to prove the existence of eigenvalues for o.d.e. For those of you who know some functional analysis Liouville's Theorem provides the method of showing every bounded operator has a nonempty spectrum. Here you consider operator valued functions analytic on the plane !

Exercises

1. Use Liouville's Theorem to prove the fundamental theorem of algebra.

2. Show from the mean value theorem that if f is holomorphic and non-constant on a region D then $|f(z)|$ does not have any local maximums in D. (i.e. the maximum principle can also be derived from Cauchy's formula).

3. Suppose that f is analytic on the entire plane and there is positive constants A, μ and R so that $|f(z)| < A|z|^\mu$ for all z, $|z| > R$. Prove that f is a polynomial of degree $n \leq \mu$.

4. Suppose that f is analytic on the entire plane and there is a ball $B(a, r)$ such that $f(z) \notin B(a, r)$ for all z. Show that f is constant.

5. Suppose that f is analytic and nonconstant on the entire plane. Show that $f(\mathbb{C})$ is dense in $\mathbb{C}$. This is Weierstrass - Casorti's Theorem.

6. Find all entire f such that $f(n\pi) = 0$ for $n \in \mathbb{Z}$ and there is a constant:
$$M \geq |f(z)|e^{-|\Im(z)|}\ ,\ z \in \mathbb{C}\ .$$

3.10 Normal convergence

In future chapters we shall be dealing with infinite expansions other then power series. We need general criteria for convergence, as well knowing that the limit of analytic functions is analytic with derivative obtained by differentiating the expansion term by term. In real analysis one usually has uniform convergence or convergence in some L^p norm. Although they have their place in complex analysis we usually don't have anything so strong. We found that power series converge uniformly on compact subdisks in the circle of convergence. This concept generalises to:

DEFINITION 8 *(Normal Convergence) Let $f_n(z)$ be a sequence of functions on a region R. Then we say that $f_n(z)$ converges normally to a function $f(z)$ defined on R if for every relatively compact set $A \subset R$ we have $f_n(z)$ converges uniformly to $f(z)$ on A, i.e.*

$$\forall \epsilon > 0,\ \exists N : n > N \Rightarrow |f_n(z) - f(z)| < \epsilon,\ \forall z \in A$$

This is often called "uniform convergence on compact subsets of R". It is convergence wrt the Euclidean metric. Normal convergence depends on which metric is being used.

Exercise

1. Show any power series converges normally inside its disk of convergence.
2. Show $f_n(z) = \exp(nz)$ converges normally to 0 on $\{\Re(z) < 0\}$.
3. Show that $f_n(z) = z + n$ does not converge normally on $\mathbb{C}$(wrt the Euclidean metric) but it does converge as a sequence on the Riemann sphere $\mathbb{S}$ using the spherical metric.

A set A is relatively compact in R if A is compact in the topology of R. Alternatively A is a compact subset of $\mathbb{C}$ contained inside R. This means that A is closed and bounded but a little more. For instance A must be disjoint from ∂R. Alternatively A can be covered by a finite number of closed balls of the form $\{z : |z-a_k| \leq r_k\} \subset R$. Consequently $f_n(z)$ converges normally to a function $f(z)$ defined on R iff for every closed balls of the form $B = \{z : |z-a| \leq r\} \subset R$, $f_n(z)$ converges uniformly to $f(z)$ on B.

In the previous chapter we observed that the uniform limit of continuous functions were also continuous. For analytic functions we have:

THEOREM 19 *Let $f_n(z)$ be a sequence of analytic functions on a region R which converges normally to a function $f(z)$ defined on R. Then $f(z)$ is analytic on R and $f_n'(z)$ converges normally to $f'(z)$.*

For any closed ball $B = \{z : |z-a| \leq r\} \subset R$ there exists $s > r$ so that $D = \{z : |z-a| \leq s\} \subset R$. Now for any point $z \in B$, by Cauchy formula

$$f_n(z) = \frac{1}{2\pi i} \int_K \frac{f_n(\zeta)}{(\zeta - z)}\, d\zeta,$$

where $K : z(t) = a + s\exp(it),\ 0 \leq t \leq 2\pi$. Now as f_n converges uniformly to f on D we know the integral of the limit is equal to the limit of the integral, thus

$$f(z) = \frac{1}{2\pi i} \int_K \frac{f(\zeta)}{(\zeta - z)}\, d\zeta.$$

Consequently f is analytic on B. Then by Cauchy formula we have that for $z \in B$

$$f'(z) = \frac{1}{2\pi i} \int_K \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta.$$

But by uniform convergence, for $z \in B$, RHS is the limit of

$$\frac{1}{2\pi i} \int_K \frac{f_n(\zeta)}{(\zeta - z)^2} \, d\zeta.$$

This, by Cauchy formula again, is $f_n'(z)$. Thus $f'(z)$ is the pointwise limit of $f_n'(z)$. The convergence is uniform on B as

$$\begin{aligned} |f'(z) - f_n'(z)| &\leq \sup_{z \in B} \left| \frac{1}{2\pi i} \int_K \frac{\{f(\zeta) - f_n(\zeta)\}}{(\zeta - z)^2} \, d\zeta \right|, \\ &\leq \frac{s}{(s-r)^2} \sup_{\zeta \in K} |f(\zeta) - f_n(\zeta)| \to 0 \end{aligned}$$

as $n \to \infty$.

The easiest (sufficient but not necessary) criteria for a series have (Euclidean) normal convergence on a region R is called the Weierstrass M-test. A series $\sum_{n=0}^{\infty} f_n(z)$ of functions analytic on a region R will converge normally provided that for any closed ball $B \subset R$ there is a bound $M_n = M_n(B) \ : \ |f_n(z)| \leq M_n(B), \ z \in B$ and

$$\sum_{n=0}^{\infty} M_n < \infty.$$

This is like the Abel test for power series but unlike it this test is not a necessary condition.The proof of the M-test is easy. On each closed ball in R we have uniform absolute convergence of the series. The triangle inequality shows that the sequence of partial sums $g_n = f_0 + ... + f_n$ is a Cauchy sequence in the uniform norm on B. Hence the series converges uniformly on B. Our earlier work tells us that we have normal convergence to an analytic function on R.

Exercises

1. Establish the normal convergence of the series :

$$\sum_{n=1}^{\infty} \frac{1}{n(n+z)}, \ \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+z)}$$

2. Let $a_n,\ b_n$ be two sequences of complex numbers so that

$$\overline{\lim}|a_n|^{1/n} = 0,\ \overline{\lim}|b_n|^{1/n} = 0.$$

Prove that the series

$$\sum_{n=0}^{\infty} a_n e^{inz} + \sum_{n=1}^{\infty} b_n e^{-inz}$$

converges normally on the complex plane to a function $F(z)$ with period 2π, i.e. $F(z+2\pi n) = F(z)$ for all integers n. (Later we shall prove the converse: i. e. every analytic function (of period 2π) can be represented by such a series.)

3. Suppose that $\omega_1, \omega_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$. Consider the double series defining the Weierstrass "Pe" function

$$\wp(z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(z-(m\omega_1+n\omega_2))^2}$$

This can be interpreted as

$$\lim_{N\to\infty} \sum_{m=-N}^{N} \sum_{n=-N}^{N} \frac{1}{(z-(m\omega_1+n\omega_2))^2}$$

Prove that the series converges to a doubly periodic meromorphic function by considering the equivalent

$$\sum_{(m,n)\in\mathbb{Z}^2-(0,0)} \frac{1}{(z-(m\omega_1+n\omega_2))^2} - \frac{1}{(m\omega_1+n\omega_2)^2}$$

4. Prove the following improved version of the M-test:
A series $\sum_{n=0}^{\infty} f_n(z)$ of functions analytic on a region R will converge absolutely and uniformily on every compact $K \subset R$ provided that for any closed ball $B \subset R$ there is a bound $M(B)$ so that

$$\sum_{n=0}^{\infty} |f_n(z)| \le M(B),\ z \in B.$$

(Hint: Although we have pointwise convergence we need to prove uniform convergence. One way is to apply monotone convergence to the Cauchy formula for circles).

Chapter 4

Geometric Properties of Analytic Functions

In the previous chapter we proved that holomorphic functions can be expanded in convergent power series. This strong analytic condition has surprising topological consequences.

4.1 Topological spaces

A topological space is a set W together with a subclass $\mathcal{T}$ of subsets U of W called the open sets of W. These are required to satisfy:

1. For any collection of $U_i \in \mathcal{T},\ i \in I$, we have $\cup U_i \in \mathcal{T}$.
2. For any finite collection $U_i \in \mathcal{T},\ i = 1,\ 2,\ ...n$, we have $\cap U_i \in \mathcal{T}$.
3. $W,\ \emptyset \in \mathcal{T}$.

A set A is said to be closed if $W - A \in \mathcal{T}$, i.e. is open.

There are obviously many different topologies on the same set W, e.g. two extreme examples of no use to us are $\mathcal{T} = \{W, \emptyset\}$ or $\mathcal{T}$ consists of all subsets of W. In analysis most topologies of interest are generated by some metric d, i.e. a generalisation of the Euclidean distance.

DEFINITION 9 *(Metric Spaces) A set W is a metric space if there is a distance function $d : W \times W \to [0, \infty)$:*

1. $\forall x,\, y \in W \; d(x, y) = 0 \Rightarrow x = y$,
2. $\forall x,\, y \in W \; d(x, y) = d(y, x)$.
3. $\forall x,\, y,\, z \in W \; d(x, z) \le d(x, y) + d(y, z)$.

Among the many examples there are several important to us:

1. $\mathbb{R}^N$ with the usual Euclidean metric.
2. The extended complex plane with the chordal metric χ (or the spherical metric σ).
3. The Poincaré disk which consists of $\mathbb{D} = \{z : |z| < 1\}$ with the metric

$$\rho(z,\, w) = \frac{1}{2} \log \left\{ \frac{1 + \left| \frac{z-w}{1-z\overline{w}} \right|}{1 - \left| \frac{z-w}{1-z\overline{w}} \right|} \right\} .$$

   Later we show that if T is any bilinear map of the unit disk onto itself then $\rho(Tz, Tw) = \rho(z, w)$.

   Not all our examples are finite dimensional:

4. Let $C[X]$ be the class of complex valued functions f defined on a metric space X such that f is bounded and continuous on X with the metric $d(f,\, g) = \sup_X |f(z) - g(z)|$.

In metric spaces we define an open ball with centre x and radius $r > 0$ by $D(x, r) = \{y : d(x,\, y) < r\}$. In examples (1) and (5) our balls really are round but in (2) they are square. Now every metric space generates a topology: a set U is open if for every $x \in U \; \exists\, r > 0 : \; D(x, r) \subset U$. It is easy to prove : *The open sets of a metric space form a topology .*

In Metric spaces we have the convergence of sequences: ie a sequence $a(n)$ converges to a point a if for every $\varepsilon > 0$ there exists an integer $N : \; n > N \;\Rightarrow\; d(a(n), a) < \epsilon$. We also define Cauchy sequences $a(n)$, i.e. for every $\varepsilon > 0$ there exists an integer $N : \; \forall n, \, m > N, \; d(a(n), a(m)) < \varepsilon$.

In particular a metric space W is complete if every Cauchy sequence in W converges to a limit (in W of course). However it is not so easy to prove from first principles that a given space is complete. The rationals were not complete but the real numbers were (actually by construction).

For example $\mathbb{R}^N$ (with the Euclidean metric) is complete. By the inequality

$$\sup_i |x_i - y_i| \leq \sqrt{\sum_{i=1}^{N} |x_i - y_i|^2} \leq \sqrt{N} \, \sup_i |x_i - y_i|$$

we see that a sequence $x(n)$ is Cauchy iff. each coordinate sequence $x_i(n)$ is Cauchy. As the reals are complete, for each i there is a $x_i : \; x_i(n) \to x_i$.

Actually even example (4) is complete. For each $x \in X$ the sequence $f_n(x)$ is Cauchy and thus converges to a value we call $f(x)$. In fact the convergence is uniform. It remains to show that f is bounded and continuous. Now as the f_n are bounded and Cauchy there exists N so that $n > N$ implies

$$\sup_X |f_n(x)| \leq \sup_X |f_N(x)| + 1 < \infty.$$

So f is bounded. Finally we show f is continuous. Fix x . For any $\varepsilon > 0$ there exists $n : \; |f_n(y) - f(y)| < \varepsilon/3$ for all $y \in X$. Fix n. As f_n is continuous there exists $\delta > 0$ so that $|f_n(x) - f_n(y)| < \varepsilon/3$ provided $d(x, \, y) < \delta$. Hence for $d(x, \, y) < \delta$ we have

$$|f(x) - f(y)| \leq |f_n(x) - f_n(y)| + |f(x) - f_n(x)| + |f_n(y) - f(y)|$$

$$\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \epsilon \; .$$

Thus f is continuous and a function of $C[X]$.

Unfortunately in many cases we cannot arrange for the sequence to be Cauchy. The next best thing is for some subsequence to converge, i.e. given a sequence $a(n)$ in $A \subset X$ when can we guarantee that there exists some subsequence $a(m(k))$ which is convergent to a point of A. We call this property "sequential compactness" of A. What does this tell us about A. Clearly we require that A is closed and bounded. In general metric spaces this is not enough. However in the case where A is a subset of $\mathbb{R}^N$, it is necessary and sufficient. This is often called the theorem of Bolzano (and) Weierstrass: *Every bounded sequence in $\mathbb{R}^N$ has a convergent subsequence* . This result is not true in normed vector space of infinite dimensions.

There is an equivalent property for metric spaces which is just as useful. This involves the concept of an open covering of a set A. Consider a collection of open sets $U_i,\ i \in I$. This is said to be an open covering of A if $A \subset \cup_I U_i$. A set A is said to be compact if for every open covering $U_i,\ i \in I$ has some finite subcovering, i.e. there are some $U_{i(1)}, ..., U_{i(n)}$, so that $U_{i(1)} \cup ... \cup U_{i(n)} \supset A$.

A famous result of Heine and Borel is *A set A in $\mathbb{R}^N$ is compact if and only if it is closed and bounded.* This is equivalent to Bolzano-Weierstrass. It is clear that compactness implies that A is closed. Otherwise there is a point $x \in cl(A) - A$ and a covering $B(x, 1/n)$ with no finite subcover. We show that compactness implies sequential compactness. Let $a(n)$ be a sequence in A. Cover A with balls of radius 1. There is a finite subcovering. At least one of these, U_1 say, contain an infinite number of the $a(n)$. Let $a(n(1))$ be the first one in U_1. Now cover A with balls of radius $1/2$. There is a finite subcovering. At least one, U_2 say, contains infinitely many of the $a(n)'s$ contained in U_1. The first one in U_2 after $a(n(1))$ we call $a(n(2))$ and so on to produce a sequence $a(n(k))$ where

$$|a(n(k)) - a(n(m))| < 1/(2N-1),\ \forall k, m > N.$$

Thus $a(n(k))$ is Cauchy and converges to a point $a \in A$ (as A is closed). We leave the converse as an exercise.

The use of these theorems in analysis is so widespread that we commonly just say "by compactness".

4.2 Mappings

We consider functions f with domain a metric space X and range a metric space Y. For any $A \subset X$ the set $f(A) = \{f(x) : x \in A\}$ is called the image of A. For any $B \subset Y$ the set $f^{-1}(B) = \{x \in X : f(x) \in B\}$ is called the inverse (or pullback) of B.

We have already defined continuous functions. There is an equivalent formulation in terms of open sets: A function f mapping X into Y is continuous if for every open set $U \subset Y$, $f^{-1}(U)$ is open in X. The inverse property is also defined:

DEFINITION 10 *(Open Mappings) A function f mapping a metric space X into a metric space Y is open if for every open set $U \subset X$, $f(U)$ is open in Y.*

Remarks: In both cases it is enough that we check only for open balls B instead of general open sets.

Finally we deal with homeomorphisms $f : X \to Y$, i.e. f is 1:1 with range Y, and f, f^{-1} are both continuous. Actually if f is 1:1 and continuous on an open set $U \subset \mathbf{R}^N$ then f is a homeomorphism of U to $f(U)$, this is a nontrivial result of topology which we do not use or prove. In calculus one "proves" the easier case for when smooth f has nonsingular Jacobian. This is called the implicit function theorem (actually proved rigorously by the contraction principle). Later we carefully prove this for analytic functions, and show the inverse was analytic too.

Examples:

(a) $f(z) = z^n$, n a positive integer, is an open mapping of $\mathbf{C}$. First we notice for any nonzero a there is a $r > 0$ so that f is 1:1 on $D = \{z : |z - a| < r\}$. For

$$\frac{f(z_1) - f(z_2)}{z_1 - z_2} = z_1^{n-1} + ... + z_2^{n-1} \neq 0 ,$$

provided $|z_j - a| < |a| \sin(\pi/2(n-1))$. Also f is continuous. Now the inverse of $f(D)$ consists of

$$\{z : |z - a| < r\},\ \{e^{2\pi i/n} z : |z - a| < r\},\ ...\{e^{2\pi i(n-1)/n} z : |z - a| < r\}.$$

Furthermore there are well defined analytic inverses f^{-1}, the nth roots, mapping $f(D)$ to each of these disks. These are continuous too. Thus $f(D)$ is open. Now f is not 1:1 on any disks $D = \{|z| < r\}$ but $f(D) = \{|z| < r^n\}$ is open. Since any open set U is the union of disks of the form D, $f(U)$ is the union of open sets $f(D)$. Thus $f(U)$ is open and so is f.

(b) In (a) we made good use of the fact that f was locally 1:1 away from points where the derivative is zero. We can say in general :
Suppose that f is analytic at a point a and $f'(a) \neq 0$. Then there is a $r > 0$ so that f is a homeomorphism on $\{|z - a| < r\}$. This was proved in the last chapter.

In example (b) we see that analytic functions are local homeomorphisms away from points where the derivative is zero. Finally we generalise (a) to the general case:
(c) Consider a nonconstant analytic function f with $f'(a) = 0$. We study the power series expansion around a. Clearly without loss of generality we may once again put $a = 0,\ f(a) = 0$. Expanding for $|z| < r$

$$f(z) = \sum_{n=m}^{\infty} a_n\, z^n \ ,$$

where we have a zero of order m. This may be rewritten as

$$\begin{aligned} f(z) &= a_m z^m \left\{ \sum_{n=0}^{\infty} \frac{a_{m+n}}{a_m}\, z^n \right\} \ , \\ &= a_m \{zh(z)\}^m \ , \end{aligned}$$

using the mth root

$$h(z) = \left\{ \sum_{n=0}^{\infty} \frac{a_{m+n}}{a_m}\, z^n \right\}^{1/m}$$

which is defined and analytic provided r is small. The function $zh(z)$ is analytic for $|z| < r$ and has derivative 1 at $z = 0$. Hence for small enough r this function is a homeomorphism. Consequently as f is the composition of the mth power with a homeomorphism we have that f is open near a.

Thus we have shown that in all cases a nonconstant analytic function is an open mapping.

Exercise

Without using the main result of 4.2. prove that $w = \exp(z)$ is an open mapping on the entire plane.

4.3 Maximum Principle

If u is real valued and continuous on a compact set K there is a point $x \in K : \ u(x) = \sup_K u$. To see this we take a sequence $x(n)$ in K such that $u(x(n)) \to \sup_K u$. As K is compact $x(n)$ has a convergent subsequence. Relabelling the sequence (if necessary) we may assume $x(n)$ converges to a point $x \in K$. Now as u is continuous $u(x(n)) \to u(x)$. Thus u achieves its supremum (which must therefore be finite).

Consider now a nonconstant analytic function f defined on a disk D. As f is open, every point in some small disk $\{|w - f(a)| < r\}$ is taken by points near a. Therefore $|f(z)|$ cannot have a local maximum at a point $a \in D$. Thus if f is nonconstant and analytic on a closed disk $\{|z - a| \leq r\}$ the maximum of $|f(z)|$ is achieved only on the boundary. This is called:
Maximum Principle: a nonconstant analytic function f does not achieve a local maximum of $|f(z)|$ at an interior point of its domain of definition.

Exercises

1. Find the maximum of the function $|2 + z - z^2|$ on $|z| \leq 1$.

2. If f is analytic on $1 \leq |z| \leq 3$ and $|f| \leq 1$ for $|z| = 3$, $|f| \leq 27$ for $|z| = 1$, show that $|f(z)| \leq 27/|z|^3, \ 1 \leq |z| \leq 3$.

4.4 Fundamental Theorem of Algebra, again!

The open mapping property gives an easy way of proving the Fundamental Theorem of Algebra. Let $p(z)$ be a polynomial of degree $n, \ n > 0$. Now p is an open mapping. The set $U(r) = p\{z : |z| < r\}$ is an open set with boundary curve which is a subset of $C(r) = p\{z : |z| = r\}$. As r increases $U(r)$ increases monotonely from the point $p(0)$. Now for r very large, $|z| = r$,

$p(z) = a_n z^n(1 + o(1))$ describes a curve $C(r)$ which winds n times around the origin. Thus everything inside $C(r)$ must be taken, i.e. there exists a z such that $p(z) = 0$. A variation on this argument is as follows: now for large r, $C(r)$ must always wind n times around the origin. If p never takes value 0 then for all r, $C(r)$ must always wind n times around the origin. However if $r \to 0$ this is impossible as $C(r)$ tends to nonzero complex number $p(0)$. Both arguments use simple topological ideas, so far unproved. Actually both of these "proofs" can be made rigorous, it just involves more topology then we want to see in an analysis course. These are two of the original proofs of Gauss. We now give a rigorous proof by use of the maximum principle:

Suppose that $p(z)$ has no zeros at all. Then the rational function $R(z) = 1/p(z)$ is analytic on the whole complex plane. (This is checked by giving Taylor expansions at every point). By the maximum principle

$$\sup\{|R(z)| : |z| = r\} = \sup\{|R(z)| : |z| < r\}$$

Now if r tends to infinity the LHS tends to zero as

$$|R(z)| \leq |1/p(z)| = |a_n|\, r^{-n}(1 + o(1)) \ .$$

Thus $R(z)$ is zero for all z which is impossible.

Exercise

Let f be analytic on the entire plane. Suppose there is a sequence $r(n) \to \infty$ so that $\inf_\theta |f(r(n)\exp(i\theta))| \to \infty$. Prove that f has a zero. (Actually polynomials are the only entire functions so that $\lim_{r\to\infty} \inf_\theta |f(r\exp(i\theta))| \to \infty$, nontrivial examples require infinite products like $\prod(1 - z/2^n)$.)

4.5 Schwarz Lemma

The maximum principle tells us that a nonconstant function $f(z)$ analytic on a disk $\mathbb{D} = \{z : |z - a| < R\}$ with $|f(z)| \leq M$ on $\mathbb{D}$ should have explicit bounds, less then M, on $|f(z)|$ for fixed z in $\mathbb{D}$. By way of normalisation we first assume that $R = M = 1$ and $a = 0 = f(0)$.

THEOREM 20 *(Schwarz Lemma) Suppose that $f(z)$ is analytic in $\{|z| < 1\}$ with $|f(z)| \leq 1$, $f(0) = 0$. Then $|f(z)| < |z|$ unless $f(z) = cz$ for some constant c, $|c| = 1$.*

As f is analytic and $f(0) = 0$ the function $g(z) = f(z)/z,\ 0 < |z| < 1$ with $g(0) = f'(0)$, is analytic for $|z| < 1$. The maximum principle proves that for positive $r < 1$, $\sup_{|z|\le r} |g(z)| \le 1/r$. Thus as $r \to 1$ we get $|g(z)| \le 1$. Hence by the maximum principle $|g(z)| < 1$ unless $g(z) = c$ for some constant.

We use bilinear maps to obtain a profound generalisation. For a function f bounded by M on $\{z : |z - a| < R\}$ with $f(a) = w$ we find that $h = S \circ f \circ T^{-1}$, where

$$S(z) = \frac{M(z - w)}{M^2 - z\overline{w}},\ T(z) = \frac{R(z - a)}{R^2 - z\overline{a}}\ ,$$

satisfies the hypothesis of Schwarz Lemma. Thus $|S \circ f(z)| \le |T(z)|$ or more explicitly

$$\left|\frac{M(f(z) - w)}{M^2 - f(z)\overline{w}}\right| \le \left|\frac{R(z - a)}{R^2 - z\overline{a}}\right|.$$

Exercises

1. Show that if f maps $\mathbb{D}$ into itself that

$$\frac{|f'(z)|}{1 - |f(z)|^2} \le \frac{1}{1 - |z|^2}.$$

   with equality only for a bilinear mapping of $\mathbb{D}$ onto itself.

2. Prove that if f is analytic on $\mathbb{D}$ mapping into itself with zeros of order m at $z = 0$ and also at $a_j \in \mathbb{D},\ j = 1, ...N$ that

$$|f(z)| \le |z|^m \prod_{j=1}^{N} \left|\frac{z - a_j}{1 - z\overline{a_j}}\right|\ .$$

3. Hence prove that if nonconstant f is analytic on $\mathbb{D}$ mapping into itself with zeros at $a_j \ne 0,\ j = 1, ...$ that

$$\prod_{j=1}^{\infty} |a_j| > 0\ .$$

4.6 Hyperbolic metric

Poincaré introduced a metric on the unit disk which has proved to have deep importance in Analysis, Geometry and Number Theory. We start with the infinitesmal form

$$\rho(z) = \frac{1}{1 - |z|^2}$$

which for any curve α (compactly contained in $\mathbb{D}$) defines a hyperbolic length

$$\rho(\alpha) = \int_\alpha \rho(z)\,|dz|\ .$$

We establish some of its properties in the following exercises. This approach is often called the invariant form of Schwarz lemma.

Exercise

1. Prove that
$$\rho([0,\, r]) = \frac{1}{2}\log\frac{1+r}{1-r},\ 0 < r < 1\ .$$

2. The hyperbolic distance $\rho(z,\, w)$ between two points z, w is defined to be the infimum of the hyperbolic lengths of all curves joining z to w. Show $\rho(0,\, r) = \rho([0,r])$. (Hint: consider the projection of the curve onto $[0,\, r]$.)

3. For any analytic function $f : \mathbb{D} \to \mathbb{D}$ show that
$$\rho(f(\alpha)) \leq \rho(\alpha)\ ,$$
with equality iff. f is a bilinear mapping of $\mathbb{D}$ onto itself.

4. Conclude that the example (5) of section 4.1 is the correct formula for the hyperbolic distance, i.e. the hyperbolic disk is a metric space with the bilinear transformations $T : \mathbb{D} \to \mathbb{D}$ being precisely the length preserving transformations.

4.7 Elementary Riemann Surfaces

The Principal value of $\sqrt{z}$ is defined as an analytic function $f(z)$ on the region $\{z : |\mathrm{Arg}(z)| < \pi\}$. This has the unfortunate result that $\sqrt{-1}$, our reason for introducing the complex numbers, is not defined. It is clear that $w = z^2$ has no analytic inverse on the whole plane. Let us to see what happens if we use analytic continuation. The function

$$g(z) = \exp\{1/2(\log|z| + i\arg(z))\}$$

where we take the values of $\arg(z)$ between $\pi/2$ and $2\pi+\pi/2$ is analytic on the region $\{z : \pi/2 < \arg(z) < 2\pi + \pi/2\}$. Furthermore $g = f$ on the subregion $\{z : \pi/2 < \arg(z) < \pi\}$. Thus g is an analytic continuation of f from the region $\{z : \pi/2 < \arg(z) < \pi\}$ to the region $\{z : \pi/2 < \arg(z) < 2\pi + \pi/2\}$. Note that f and g disagree on other parts of their domains of definition. Next we analytically continue g by $h(z) = \exp\{1/2(\log|z| + i\arg(z))\}$ where now we take the values of $\arg(z)$ between 2π and 4π. Now h is analytic on the region $\{z : 2\pi < \arg(z) < 4\pi\}$. Furthermore $g = h$ on the subregion $\{z : 2\pi < \arg(z) < 2\pi + \pi/2\}$ However what is really striking is that on the quadrant $\{z : -\pi < \arg(z) < 0\}$ we find $f = h$. It is as if by analytically continuing twice around the origin that we come back to f.

The solution to the problem of defining $\sqrt{z}$ is to be found not by multi-valued functions but by constructing a new type of object, a surface R which covers the plane. This surface is so constructed that each small piece is just a subregion of the plane. Let us construct the Riemann surface R for $\sqrt{z}$. We take two copies $R_1,\ R_3$ of $\{\Im(z) > 0\}$ and two copies $R_2,\ R_4$ of $\{\Im(z) < 0\}$. We join R_1 and R_2 along $(-\infty, 0)$, R_2 and R_3 along $(0, +\infty)$, R_3 and R_4 along $(-\infty, 0)$, R_4 and R_1 along $(0, +\infty)$. The easiest way to distinguish points of R is by using $\arg(z)$ where we no longer regard points as being equivalent if the arguments differ by $2\pi n$. Instead we regard points z as being equivalent if the arguments differ by $4\pi n$. The function $\exp\{\log(z)/2\}$ is now a well defined complex valued function on R.

4.8 The idea of a Riemann Surface

The previous discussion represents an example of the explicit "scissors and paste" construction used in the early days, starting from Riemann's first notions in the 1850's until the formal definition of Herman Weyl in 1913. Incidentally the naive approach posed no problems in the basic theory, up to and including the very deep Uniformisation theorem proved by Poincarè. The modern definition is:

DEFINITION 11 *(Riemann surfaces) A Riemann surface is a one dimensional complex manifold.*

However this formal definition is misleading in its own way, making the subject seem a branch of differential geometry which it is not, infact all Riemann surfaces are equivalent to surfaces consisting of pieces of the plane. They can be assumed to be coverings of planar regions. Furthermore the abstract approach does not give you much feel for these objects. So it is usual to introduce Riemann Surfaces by some explicit "scissors and paste" constructions.

We present a simplified construction, good enough for the elementary surfaces we are going to form. Consider a countable collection of regions of the plane denoted by R_k (not necessarily distinct, but if not distinct then tagged differently). We assume that these regions are bounded by a finite number of smooth Jordan arcs. These regions are pasted together according to a relationship called "pasting". If two domains R_n and R_m are to be pasted we require that these regions be disjoint and there is a set A of common boundary subarcs . We then join these two regions along the common border and say that they are pasted along A. If these domains are joined along A we require that no other domains are to be pasted to either domain along any part of A. Note that we do not necessarily paste together any two regions which happen to have a common border, only if specified. The Riemann surface is formed after all the specified pastings have been made. In order to ensure that R is connected we assume that our relationship is such that given any two regions $R_j,\ R_k$ there is a finite sequence $R_j,\ R_{j+1},\ \ldots R_{k-1},\ R_k$ so that consecutive regions are pasted along some common border.

Next we give a topology on R. All I have to do is tell you what the open balls are. Then any open set is defined to be the union of these balls. We simply take any balls contained in some R_k together with balls contained in any pasted pair $R_j \cup R_k \cup A$. We have to distinguish between balls which are on top of each other by using the tag that comes with the domains. Thus balls from R_k can only intersect other balls from R_k, or perhaps ones from $R_j \cup R_k \cup A$ if there is a pasting. Locally, a Riemann surface looks like subdomains of the plane. Hence it is a one dimensional complex manifold. Conversely, the Principle of analytic continuation can be used to prove that any one dimensional complex manifold can be so formed. Globally however the Riemann surface can be very complicated.

As a Riemann surface is locally just subdisks of the plane we can define continuous, even analytic functions f, which map one Riemann surface R into another one S say:

DEFINITION 12 *A function mapping R into S is analytic if for every point $a \in R$ there are disks $B(a,r) \subset R$ and $B(f(a),s) \subset S$ so that, when is restricted to $B(a,r)$, f is an analytic function mapping into $B(f(a),s)$.*

Our construction of the Riemann surface of $\sqrt{z}$ actually contains three ideas for the construction of Riemann surfaces. We now consider these:

1. (Riemann surfaces by pasting) Let n be a positive integer. We construct the Riemann surface for $\sqrt[n]{z}$. We are looking for a surface covering the plane so that $\exp\{\log(z)/n\}$ is well defined and analytic. So we expect this surface to cover the plane with n sheets. We take n copies $R_1,\ R_3,\ ...R_{2n-1}$ of $\{\Im(z) > 0\}$ and n copies $R_2,\ R_4,\ ...R_{2n}$ of $\{\Im(z) < 0\}$. We join R_1 and R_2 along $(-\infty, 0)$, R_2 and R_3 along $(0, +\infty)$, R_3 and R_4 along $(-\infty, 0)$, ... R_{2n} and R_1 along $(0, +\infty)$. This gives the n sheeted surface where $\sqrt[n]{z}$ is analytic. It is interesting to note that $g(z) = z^n$ may be considered as an analytic function mapping $\mathbf{C}\ \ -\{0\}$ 1:1 onto R. This done by defining $g(z)$ to be in R_j when $(j-1)\pi/n < \arg(z) < j\pi/n$.

   Our examples so far have a finite number of sheets. Next we construct the Riemann surface for $\log(z)$.

   We are looking for a surface covering the plane (- zero) so that $\log(z)$ is well defined and analytic. So we expect this surface R to cover the plane with an infinite number of sheets. We take infinitely copies $R_{-3},\ R_{-1},\ ...R_{2n-1}, ..$ of $\{\Im(z) > 0\}$ and $R_{-2},\ R_0,\ ...R_{2n}, ...$ of $\{\Im(z) <$

$0\}$. We join ...R_1 and R_2 along $(-\infty, 0)$, R_2 and R_3 along $(0, +\infty)$, R_3 and R_4 along $(-\infty, 0)$, ... R_{2n} and R_{2n+1} along $(0, +\infty)$, etc etc. in both directions. Now $g(z) = \exp(z)$ may be considered as an analytic function mapping **C** 1:1 onto R. This done by defining $g(z)$ to be in R_j when $(j-1)\pi < \Im(z) < j\pi$.

2. (Riemann surface of an analytic function) In the previous examples the Riemann surface was obtained by transforming the image of a function so that the inverse is well defined and 1:1. This is a general procedure.

   Suppose that f is a nonconstant analytic function on region D. We construct a Riemann surface R so that f may be considered to be a 1:1 analytic map onto R. Furthermore the inverse of f will be defined and analytic on R. We call R the Riemann Image of an analytic function. We know that f cannot be 1:1 in the neighborhood of the critical points $\zeta :\ f'(\zeta) = 0$, which we now know to be isolated and thus countable. Let the set of critical points be C. Then as f is locally 1:1 on $D - C$ we may partition up $D - C$ into a countable number of regions D_k upon which f is 1:1. These can be chosen so small that regions sharing a common border will be mapped to disjoint regions. To construct the Riemann surface of R we use the regions $R_k = f(D_k)$ pasted together along the curves $f(\partial D_k)$. From our previous work the inverse function are well defined and analytic on R.

   Example:
   We apply this construction to the inverse function of a rational function of degree $n > 0$. Except for at most n critical points every point w is taken by n values of z. Thus the Riemann Image is a n-sheeted covering of the Riemann sphere with singularities at $w = R(z)$ where $R'(z) = 0$. To be more specific consider $f(z) = z + 1/z$. This has critical points at $z = 1$ or -1. R maps these points to 2 and -2. The function f is 1:1 on $D_1 = \{|z| > 1\}$ also it is 1:1 on $D_2 = \{|z| < 1\}$. f maps both of these regions to $\mathbb{C} - [-2, 2]$. However as $f(z)$ is not 1:1 on ∂D we need to be a bit careful. It is better to choose $D_1 = \{|z| > 1, \Im(z) > 0\}$, $D_2 = \{|z| < 1, \Im(z) > 0\}$, $D_3 = \{|z| <$

$1, \Im(z) < 0\}$, $D_4 = \{|z| > 1, \Im(z) < 0\}$. The image of D_1, D_3 is $\{\Im(z) > 0\}$ which we call R_1, R_3. The image of D_2, D_4 is $\{\Im(z) < 0\}$ which we call R_2, R_4. Now R_1 is joined to R_2 across $(-2, 2)$, R_2 to R_3 across $(-\infty, -2)$ and $(2, +\infty)$. Then R_3 is joined to R_4 across $(-2, 2)$ and R_4 to R_1 across $(-\infty, -2)$ and $(2, +\infty)$. We get a 2-sheeted covering with singularities at -2 and $+2$.

(Weierstrass' Method) Next we consider a construction that includes the last as a special case. This works with the domain of definition instead of the image. The idea is to use analytic continuation as a method of constructing a Riemann surface. This time we start with a function f defined on a region D. This will produce a Riemann surface $R \supset D$ upon which there is an analytic function F so that $F|_D = f$. This we call the Riemann Domain of a function. This is in fact a generalisation of the previous process: if we consider analytic continuation of the inverse function we have the previous process. Our construction works purely with local power series expansions, beginning with any power series of f.

We say that one power series $g_1(z) = \sum_{k=0}^{\infty} a_k (z-a)^k$, convergent for $|z-a| < r$ is a direct continuation of another power series $g_2(z) = \sum_{k=0}^{\infty} b_k (z-b)^k$, convergent for $|z-b| < s$, if $r + s > |a-b|$, i.e. the disks of convergence intersect. Also, most importantly, $g_1(z) = g_2(z)$ on this intersection. Thus one power series may be considered as an extension of the other. In general power series g_1 and g_n are analytic continuations of each other if there is a sequence of pairs of direct analytic continuations $g_1,\ g_2,\ g_2,\ g_3,\ \ldots,\ g_{n-1},\ g_n$. We define the analytic continuation of a power series g_1 to be the class of all power series g obtained by analytic continuation. This process generates an enormous collection of disks. As a first step towards obtaining the surface we consider the collection of pairs

$$\{(z, g) : g \text{ is a power series converging on some disk } D \text{ with } z \in D\}.$$

We think of several (possibly countably many) points on top of each other tagged by different power series. The way to define the Riemann surface is to identify points (z, g_1) and (z, g_2) where $g_1(z) = g_2(z)$ in some disk $B(z,r)$. One finds a natural topology inherited from the disks used. We do not develop this at all except to say this is the method of "germs" and "sheafs" so important in several complex variables (but not necessary to use this language in one complex variable). Now F is defined at (z,g) as $g(z)$.

Example :
Begin with the power series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(z-1)^k}{k}$ which converges for $|z-1|<1$ where it is equal to log. Analytic continuation in all directions produces the usual function log defined on an infinite covering of $\mathbf{C}$ $-\{0\}$.

3. (Poincaré's Method) In some examples we made good use of equivalences to define a surface. The general idea is to have a group Γ of Möbius transformations acting on some region D, i.e. for all $g \in \Gamma$ we have $g(D) = D$. It is required that Γ is discontinuous with no fixed points, ie. for all $z \in D$ there is a disk $B(z,r)$ so that $g(B(z,r)) \cap B(z,r) = \emptyset$ unless $g = \mathrm{I}$. A set $F \subset D$ is a fundamental region for Γ if it is the largest domain set so that $g(F) \neq F$ unless $g = \mathrm{I}$. Then it follows that $\cup_\Gamma g(F) = D$. Any discontinuous group has a fundamental domain. The Riemann surface R is defined by the equivalence $z \equiv w$ iff $w = g(z)$ for some $g \in \Gamma$. Essentially this is the Fundamental domain with its boundary arcs equated in pairs.

   Example :
   The simplest is when Γ is a group of translations. For example if Γ_1 consists of the Mobiiüs transformations $g(z) = z + n,\ n \in \mathbb{Z}$ acting on the whole complex plane $\mathbb{C}$. It is easy to see that Γ_1 is discontinuous, one choice of fundamental region is $\{z:\ 0 < \Re(z) \leq 1\}$. We find that the surface R represents a cylinder. In another example Γ_2 consists of the Mobiiüs transformations $g(z) = z + n + mi,\ n, m \in \mathbb{Z}$. In this case Γ_2 is also discontinuous with no fixed points and R represents a torus. Now the rotation group $\Gamma_3 = \{g:\ g(z) = e^{i\theta}z\}$, fixed θ, acting on $\mathbb{C} - \{0\}$ need not be discontinuous. The condition for Γ_3 to be discontinuous is that θ/π be a rational number.

These examples represent elementary Riemann surfaces. More complicated ones can be formed if our group acts on the unit disk $\mathbb{D} = \{|z| < 1\}$ upon which it is supposed to be discontinuous with no fixed points on $\mathbb{D}$, (however there may be fixed points on $|z| = 1$). Such groups are called "Fuchsian". It is a fundamental result, "Hilbert's 22nd problem of 1900, that every Riemann surface is represented by such a group. This, the Uniformisation Theorem, was proved by Poincaré in 1905.

Exercises:

1. Sketch the Riemann surface upon which $\arctan(z)$ is a well defined analytic surface.
2. Sketch the Riemann image of the function $f(z) = z^3 - 3z$. .
3. Sketch a 2 sheeted covering over $\mathbb{C} - \{-1, 1\}$ upon which $\sqrt{1 - z^2}$ is a well defined analytic function.

4.9 Branches

We have previously defined branches for $\log(z)$ and powers. It is a general concept. Let $f(z)$ be analytic on a Riemann surface R covering the plane. We may consider an open connected subset $D \subset R$ so that no points of D cover the same point of the plane twice (or more). Hence D may be considered to be a region of the plane. The function f restricted to D is then a function F analytic on a planar region. The boundary of D is called a branch cut. The function F, restricted to D, is called a branch of f.

Example: $\mathrm{Log}(z)$ is a branch of $\log(z)$ with branch cut $(-\infty, 0)$.

Chapter 5

Harmonic functions

Harmonic functions arise in mathematical physics as the potentials of a heat(or electrical or fluid flow). In two (and only two) dimensions we find they are locally equivalent to analytic functions.

5.1 Laplacian

We now consider the function $u(z) = \Re\{f(z)\}$ where f is analytic on a region R. As $u(z) = \frac{1}{2}\{f(z) + \overline{f(z)}\}$ we get

$$0 = \frac{\partial^2 u}{\partial z\, \partial \overline{z}} = \frac{1}{4}\left\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right\}$$

We write

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \Delta u \ ,$$

the so called Laplacian of u. A function u on a region R with two partial derivatives and satisfying $\Delta u = 0$ on R is said to be harmonic on R. Thus every analytic function has harmonic real part. Is the converse true? That is, given a function u harmonic on a region R does there exist a function f analytic on R so that $\Re\{f\} = u$? Its easy to give a counterexample : check that the function $\log|z|$ is harmonic on $R = \{0 < |z| < \infty\}$. Now if $\log|z| = (f + \overline{f})/2$ for some function f analytic on R we differentiate with respect to z and find $2/z = f'$. There is no such f as the complex line integral of $2/z$ around the positively orientiated circle $\{|z| = 1\}$ is $4\pi i$. On the other hand the line integral of f' is 0.

However the problem is topological and locally the answer is yes. So we restrict ourselves to disks D and let u be harmonic on a disk D. To find f analytic on D with $u = \Re\{f\}$ we first observe that $g = \partial u$ is holomorphic on D as $\overline{\partial}\partial u = 0$. Thus there is an analytic function $G: \ G' = g$. We set $f = 2G$. To check that this is correct observe $\partial(u - f/2) = 0$ on D. Therefore $u - f/2$ is antiholomorphic, i.e. $u - f/2 = \overline{h}$ where h is analytic on D. Now as u is real valued on D, $\Im f = 2\Im h$ and thus $f = 2h + ic$ where c is a real constant. Hence $u(z) = \frac{1}{2}\{f(z) + \overline{f(z)}\}$.

We have shown u is harmonic on D iff $u + iv$ is analytic on D for some v. The same is true for the imaginary part, v, of f. However it is not true that any pair of harmonic functions u, v will give $u + iv$ analytic. We need the Cauchy-Riemann equations to hold. Suppose that u is harmonic then a harmonic function v is called a harmonic conjugate of u if $u + iv$ is analytic.

Exercises

1. Let $f(z)$ be analytic and nonzero on a region R. Show that $\log|f(z)|$ is harmonic on R.
2. Let $u(z) = z^2 - \overline{z}^2$. Find an analytic function f so that $\Re f = u$.
3. If u has harmonic conjugate v what is the harmonic conjugate of v?
4. If u is harmonic on $\mathbf{C}$ and $u(z) < c$ show u is constant.
5. Find all $u(re^{i\theta}) = u(r)$ which are harmonic on $\{s < |z| < t\}$.
6. If $u(re^{i\theta})$ harmonic on $\{|z| < 1\}$ show that there are real a_n, b_n :
$$u(re^{i\theta}) = \sum_{n=0}^{\infty}\{a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)\},$$
converges uniformly for $r < 1$.

5.2 Mean Value Theorem

Taking the real part of the mean value theorem for analytic functions immediately yields the mean value theorem for harmonic functions, i.e. if u is harmonic on $\{z : |z-a| \le r\}$ then

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a+re^{i\theta})\,d\theta.$$

Conversely one can use Green's theorem to show that this characterises harmonic functions which are sufficiently smooth. The mean value theorem also implies that harmonic functions satisfy the maximum principle (and a minimum principle).

THEOREM 21 *(Maximum Principle) Let u be harmonic on region R and $\limsup_{z\to\partial R} u(z) \le M$ then $u(z) \le M$ for all $z \in R$.*

Remark : $\limsup_{z\to\partial R}$ uses the boundary on the Riemann sphere $\mathbb{S}$.

Without loss of generality $\infty > m = \sup_R u > M$. Now as u is continuous there is point $a \in R$ so that $u(a_1) = m > M$. Then for small enough r the disk $\{z : |z-a| \le r\} \subset R$ and thus by the mean value property there is a point $b \in \{z : |z-a| = r\}$ so that $u(b) > m$ - contradiction.

In particular this gives us a boundary uniqueness theorem (where either R is bounded or we need to consider ∂R as a subset of the sphere):

If u, v are harmonic on R then

$$\lim_{z\to\partial R} |u(z) - v(z)| = 0 \Rightarrow u(z) = v(z),\ \forall z \in R\ .$$

One should note that any continuous function satisfying the mean value theorem satisfies the maximum principle. In the next section, as a consequence of the Poisson formula, we show that such a function is harmonic in the ordinary sense.

Brownian Motion

The usual applications of harmonic functions are to potential fields in physics. A less classical example comes from probability. For a region R we designate some subset $E \subset \partial R$. Consider a random walk beginning at a point z of R. Let $u(z)$ be the probability that we collide with E before we collide with $\partial R - E$. We can think of this being the chance that an virus will hit

E before escaping from the region. Now it can be seen that u satisfies the mean value theorem and hence (by a later section) is harmonic on R. Also u has boundary value 1 on E and 0 on $\partial R - E$. This is a typical example of the Dirichlet boundary value problem: Given a region R and a function h defined on ∂R find a function u which is harmonic on R and has boundary value h on ∂R.

Exercises

1. For nonconstant u harmonic on $\mathbb{C}$ let $\Omega = \{z : u(z) > c\}$. Show that every component of Ω is unbounded and simply connected.

2. Show $u(re^{i\theta}) = r\cos(\theta)$ solves the Dirichlet problem on $\{|z| < 1\}$ for $h(\theta) = \cos(\theta)$.

3. Show $u(re^{i\theta}) = \arg\{(1 + re^{i\theta})/(1 - re^{i\theta})\}$ solves the Dirichlet problem on $\{|z| < 1\}$ for $h(e^{i\theta}) = \pi/2,\ 0 < \theta < \pi;\ h(e^{i\theta}) = -\pi/2,\ -\pi < \theta < 0$ but not defined at $1,\ -1$.

4. Now $u(x, y) = y$ on the region $\{0 < y < 1\}$ has boundary value 0 on $\{y = 0\}$ and boundary value 1 on $\{y = 1\}$. Find another harmonic function with these boundary values.

5. If u is harmonic on R and $f : D \to R$ is conformal prove $u \circ f$ is harmonic on D.

5.3 Poisson formula

We shall find that the Dirichlet problem is straight foreward for the unit disk $\mathbb{D} = \{z : |z| < 1\}$. First consider the case where u is harmonic on the closure of this disk, so that the boundary value of u is u itself. We use the fact that for any function u harmonic on the unit disk $\mathbb{D}$ and any conformal map $T : \mathbb{D} \to \mathbb{D}$ the composition $v = u \circ T$ is harmonic. The bilinear map T which maps 0 to a satisfies $v(0) = u(a)$. This bilinear map is given by $T(z) = (z + a)/(1 + \overline{a}z)$, inverse $T^{-1}(z) = (z - a)/(1 - \overline{a}z)$. Hence

$$\frac{dT^{-1}(e^{i\phi})}{d\phi} = \frac{1 - |a|^2}{(1 - \overline{a}e^{i\phi})^2}\ ,$$

which is calculated for later reference. So by the mean value theorem

$$u(a) = v(0) = \frac{1}{2\pi}\int_0^{2\pi} v(e^{i\psi})\,d\psi = \frac{1}{2\pi}\int_0^{2\pi} u(T(e^{i\psi}))\,d\psi\ .$$

Now if $T(e^{i\psi}) = e^{i\phi}$,

$$\frac{d\psi}{d\phi} = \frac{1-|a|^2}{|1-\overline{a}e^{i\phi}|^2}\ ,$$

and we find that if $a = re^{i\theta}$ we have

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\phi})P_r(\theta\ -\phi)\,d\phi\ ,$$

where the so called Poisson kernel is

$$P_r(\tau) = \frac{1-r^2}{1-2r\cos(\tau)+r^2} = \Re\left\{\frac{1+re^{i\tau}}{1-re^{i\tau}}\right\}\ .$$

We have only proved this for u harmonic on the closed disk. This is called the Poisson formula for u. There are various equivalent forms for the Poisson formula depending on the kernel used :

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \Re\left\{\frac{1+ze^{-i\phi}}{1-ze^{-i\phi}}\right\} u(e^{i\phi})\,d\phi\ .$$

There are several other ways of deriving the Poisson formula (it can also be derived from Cauchy's formula). Note that the Poisson formula solves the Dirichlet problem on the disk where we know that the given boundary value h is equal to u whenever u is harmonic on the closed disk. It is natural to think that the Poisson formula will solve the Dirichlet problem in all cases. Consider a function $h(e^{i\theta})$ integrable on the unit circle. Then we may use the Poisson integral as a means of defining the function

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \Re\left\{\frac{1+ze^{-i\phi}}{1-ze^{-i\phi}}\right\} h(e^{i\phi})\,d\phi\ .$$

Note that u will be well defined and harmonic on $\{|z|<1\}$. We hope that u has boundary values h. What precisely is meant by this. Naively we should like

$$\lim_{z\to e^{i\theta}} u(z) = h(e^{i\theta})$$

but this is impossible if h is not continuous at every point. Instead we ask that it hold at every point $e^{i\theta}$ where h is continuous. Thus we come to the first major result of this section:

THEOREM 22 *(Dirichlet Problem for the Disk) Consider a real valued function $h(e^{i\theta})$ integrable on $[0, 2\pi]$. Then*

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \Re\left\{\frac{1+ze^{-i\phi}}{1-ze^{-i\phi}}\right\} h(e^{i\phi})\, d\phi \ ,$$

is a harmonic function on $\{|z| < 1\}$ so that at every point $e^{i\theta}$ where h is continuous

$$\lim_{z\to e^{i\theta}} u(z) = h(e^{i\theta}).$$

First we observe some basic properties of Poisson integrals $u = P(h)$, which is an operator from L^1 into the harmonic functions. We shall need some properties of the Poisson kernel

$$p(r,\tau) = \frac{1}{2\pi}\Re\left\{\frac{1+re^{i\tau}}{1-re^{i\tau}}\right\}.$$

LEMMA 6 *The following are true :*

1. *$p(re^{i\tau}) = p(r,\tau)$ is harmonic on $\{|z| < 1\}$.*
2. *for all z, $|z| < 1$ we have $p(z) > 0$.*
3. *for fixed $\delta > 0$ and $U = \{|z| < 1\} - \{|z-1| > \delta\ \}$ we have*

$$\lim_{|z|\to 1,\, z\in U} p(z) = 0$$

Furthermore we have :

LEMMA 7 *(Properties of $u = P(h)$ considered as an operator)*

1. *$P(sh) = sP(h)$ for all constants s, and $h \in L^1$*
2. *$P(h+g) = P(h) + P(g)$, for all h, $g \in L^1$*
3. *if $h \geq 0$ then $P(h) \geq 0$*
4. *if $h \leq M$ then $P(h) \leq M$.*

5. $P(1) = 1$.

To prove the theorem all it remains to prove is the limit formula. Thus, as the Poisson kernel is rotationally invariant, without loss of generality h is continuous at $z = 1$ with value $h(1) = 0$. As h is continuous at 1 for any $\epsilon > 0$ there is a $\delta > 0$ so that $|h(e^{i\theta})| < \epsilon$ for $|\theta| < \delta$. Now

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \Re \left\{ \frac{1 + ze^{-i\phi}}{1 - ze^{-i\phi}} \right\} h(e^{i\phi})\, d\phi$$

is the sum of two integrals:

$$u_1(z) = \frac{1}{2\pi} \int_{-\delta}^{\delta} \Re \left\{ \frac{1 + ze^{-i\phi}}{1 - ze^{-i\phi}} \right\} h(e^{i\phi})\, d\phi$$

which by positivity is bounded above by

$$\frac{1}{2\pi} \int_0^{2\pi} \Re \left\{ \frac{1 + ze^{-i\phi}}{1 - ze^{-i\phi}} \right\} \epsilon\, d\phi = \epsilon.$$

Similary we obtain $u_1(z) \geq -\epsilon$. So $|u_1| \leq \epsilon$. The other integral is

$$u_2(z) = \frac{1}{2\pi} \int_{|e^{i\phi}-1|>\delta} \Re \left\{ \frac{1 + ze^{-i\phi}}{1 - ze^{-i\phi}} \right\} h(e^{i\phi})\, d\phi$$

However as the kernel converges uniformily to zero as $z \to 1$ the integral will converge to 0.

This result allows us to consider discontinuous boundary data. In fact the Poisson integral is well defined for a general integrable function, i. e. $h(e^{i\phi}) \in L^1[0,\, 2\pi]$, which may be nowhere continuous. In this case we cannot ask for an unrestricted limit but make use of the radial limit introduced for Abel's Second theorem. We restrict ourselves to bounded measureable function, and set

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \Re \left\{ \frac{1 + ze^{-i\phi}}{1 - ze^{-i\phi}} \right\} h(e^{i\phi})\, d(\phi).$$

We show that $u(z)$ has a radial boundary value at every point $e^{i\theta}$ where h has a so called Lebesgue points $e^{i\theta}$:

$$h(e^{i\theta}) = \lim_{t\to 0} \frac{1}{2t} \int_{\theta-t}^{\theta+t} h(e^{is})\, ds.$$

It is a basic result of Real Analysis that this is true for almost all θ.

THEOREM 23 *Let $u(z)$ be the Poisson integral of a $h \in L^\infty[0, 2\pi]$. Then at every Lebesgue point $e^{i\theta}$*

$$\lim_{r\to 1} u(re^{i\theta}) = h(e^{i\theta})\,.$$

To prove this we may assume $\theta = 0$ and $h(1) = 0$. Then integrating by parts

$$u(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} p(r,t)h(e^{it})dt$$

$$= p(r,\pi)h(-1) - p(r,-\pi)h(-1) - \int_{-\pi}^{\pi} \frac{\partial p(r,t)}{\partial t} H(e^{it})dt,$$

where $H'(e^{it})ie^{it} = h(e^{it})$, i.e. H is the absolutely continuous function with derivative $-ie^{-it}h(e^{it})$ (a.e.). The first term tends to zero as $r \to 1$. As before we consider the integral as the sum of two parts. We fix $\delta > 0$. The first integral is over the set $\delta < |t| < \pi$, which as

$$\left|\frac{\partial p(r,t)}{\partial t}\right| \le \frac{1}{2\pi}\frac{1-r^2}{(1-2r\cos\delta + r^2)^2}$$

tends to zero as $r \to 1$, uniformly. Hence $u(r)$ is dominated by

$$-\int_{-\delta}^{\delta} H(e^{it})\frac{\partial p(r,t)}{\partial t}dt = -2\int_0^{\delta} \frac{H(e^{it}) - H(e^{-it})}{2t} t\frac{\partial p(r,t)}{\partial t}dt.$$

Now as the 1 is a Lebesgue point, for every $\epsilon > 0$ we can choose small enough δ so that

$$|\frac{H(e^{it}) - H(e^{-it})}{2t}| < \epsilon$$

for $0 < t < \delta$. Hence the final integral is bounded by

$$-2\epsilon\int_0^{\delta} t\frac{\partial p(r,t)}{\partial t}dt$$

which in turn is bounded by

$$-\epsilon\int_{-\pi}^{\pi} t\frac{\partial p(r,t)}{\partial t}dt = \epsilon\int_{-\pi}^{\pi} p(r,t)dt = \epsilon.$$

Thus $u(r) \to 0$ as $r \to 1$ and the proof is complete.

Finally we tackle the problem of finding which harmonic functions are the Poisson integral of a $h \in L^\infty[0, 2\pi]$. We say that a function $u(z)$ harmonic on $|z| < 1$ belongs to the class h^∞ if $u(z)$ is uniformly bounded on $\mathbb{D}$. Now as $u(z)$ is continuous for $|z| < 1$ the functions $u_r(z) = u(rz)$ can be represented by Poisson integrals:

$$u_r(z) = \int_0^{2\pi} p(ze^{-it})u(re^{it})dt.$$

But $u_n = u((1-1/n)e^{it})$ is a bounded sequence in the space $L^\infty[0, 2\pi]$. From Real Analysis, the unit ball in $L^\infty[0, 2\pi]$ is "weak compact ". Thus there is a function $h \in L^\infty[0, 2\pi]$ and sequence $r_n \to 1$ so that u_n tends weakly to h. This means for any fixed continuous function $g(e^{it})$

$$\int_0^{2\pi} g(e^{it})u_n(e^{it})dt \to \int_0^{2\pi} g(e^{it})h(e^{it})dt\ .$$

Choosing $g(e^{it}) = p(ze^{-it})$ yields

$$u_n(z) \to \int_0^{2\pi} p(ze^{-it})h(e^{it})dt$$

for fixed $z \in \mathbb{D}$. Thus $u(z)$ is the Poisson integral of a bounded function if u is bounded. To see the converse observe that if $u = P(h)$ for $-M \leq h \leq M$ then

$$-P(M) \leq P(h) \leq P(M),\ \Rightarrow -M \leq u \leq M$$

as the Poisson integral of $h = 1$ is $u = 1$. Thus u is bounded. Furthermore by the previous theorem h is the radial boundary value of u (a.e.).

Now the previous discussion shows bounded harmonic functions have radial boundary values (a.e.). Furthermore two different bounded harmonic functions with the same radial boundary values (a.e.) are identical in the disk. This is called boundary uniqueness.

It can be shown that $u(z)$ is the Poisson integral of an L^p function if and only if

$$\sup_{r<1} \int_0^{2\pi} |u(re^{it})|^p dt < \infty,$$

i.e. the so called class the class h^p. This is true for $p > 1$. Somewhat surprisingly this does does NOT characterise the case $p = 1$ where it is found that there are bounded L^1 integrals if and only if u is the Poisson integral of a finite measure.

5.4 Mean Value implies Harmonic

As an important application of the Poisson integral we prove that continuous functions u which satisfy the mean value theorem on a region R are harmonic.

We first derive the Poisson formula for an arbitrary disk $D = D(a, R)$. If $u(a + Re^{i\theta})$ is continuous then for $|z| < R$

$$u(a + z) = \int_0^{2\pi} u(a + Re^{i\phi}) p(ze^{-i\phi}/R) d\phi$$

is harmonic with boundary value $u(a + Re^{i\theta})$. We denote this as $U(z, u, D)$.

Now we can prove that continuous functions satisfying the mean value theorem are harmonic. For any fixed disk D with $cl(D) \subset R$ consider

$$v = u - U(z, u, D)$$

which is a continuous function on $cl(D)$ satisfying the mean value theorem on D and thus the maximum/minimum principle. But $v = 0$ for $|z - a| = R$ so by the maximum/minimum principle $v = 0$ on D. So $u = U(z, u, D)$ is harmonic on D.

Exercises

1. Suppose that the Poisson integral of a function $h \in L^\infty$ is zero. Conclude that $h = 0$. Hence there is a unique representation of harmonic functions $u \in h^\infty$ by bounded functions(a.e).

2. Show that every bounded harmonic function u on $\mathbb{D}$ is the difference of positive harmonic functions, i.e. $u = v - w$ where harmonic v, $w \geq 0$ on $\mathbb{D}$.

3. Let $u(z)$ be harmonic and bounded on $\mathbb{D}$:
If $\lim_{r \to 1} u(re^{i\theta}) \leq M$, $\theta(a.e.)$ prove $u(z) \leq M$.

4. Let u be harmonic on $\mathbb{C}$ satisfy $\int |u(re^{i\phi})|\, d\phi \leq o(r)$, as $r \to \infty$. Show that u is constant.

5. Give an example of a function u harmonic on a domain D so that for every $\zeta \in \partial D$ we have $\limsup_{z \to \zeta} u(z) \leq M$ but u is unbounded.

5.5 The Reflection Principle

Symmetry principles are very important in complex analysis. Consider a symmetric region Ω, i.e. $z \in \Omega$ if and only if $\overline{z} \in \Omega$. In particular, as Ω is connected it must contain a subinterval I of the real line. Suppose that f is analytic on R and on I we have $f(x)$ is real valued. Then as the function $\overline{f(\overline{z})}$ is also analytic on Ω, equal to f on the interval I, by uniqueness we have $f(\overline{z}) = \overline{f(z)}$ on Ω. This is interesting but can be greatly improved. We now consider a region R which has an open interval I forming an accessible boundary arc. This means there is an open set O so that $\partial R \cap O = I$, and that $O \cap R$ is connected. Consider any function v harmonic on R, continuous on $R \cup I$ so that $v = 0$ on I. We show that v has a "harmonic" continuation through I. First we construct the reflected domain $R^* = \{z : \overline{z} \in R\}$. On R^* we define function $v^*(z) = -v(\overline{z})$ which is continuous on $R^* \cup I$ and harmonic on R^*.

We shall to show that the function v defined on a neighborhood O of I, equal to v on $O \cap R$, v^* on $O \cap R^*$ and $v = 0$ on I is harmonic. This means showing it is harmonic at each point of I. Now for any $x \in I$ and disk $B(x, r) \subset O$ the mean value of v around the circle $C(x, r)$ is zero. Thus v satisfies the mean value theorem on O and is therefore harmonic on all of O.

Therefore there is a function H analytic on a neighborhood O of I with $v = \Im(H)$. This is one method of proving the Schwarz reflection Principle:

THEOREM 24 *(Schwarz reflection Principle) Let $f(z)$ be analytic on a region R with accessible boundary interval I. Suppose that $v = \Im(f)$ is continuous on $R \cup I$ and $\lim_{z \to I} v = 0$. Then there is an analytic function H analytic on the symmetric domain $R \cup R^* \cup I$ so that $f(z) = H(z)$ on R.*

There is no need that this should just be restricted to intervals I, any subarc of a circle will do. Then we make use of reflection in the circle to achieve the same result.

Exercises

1. Let f, g be analytic on the upper half plane $\mathbb{H} = \{\Im(z) > 0\}$. Suppose that there is an interval $(a, b) \subset \mathbb{R}$ so that

$$\lim_{z \to I} |\Im(f - g)| = 0 \ .$$

   Show that $f = g + h$ where h is analytic on $\mathbb{C} - (-\infty, a] - [b, \infty)$.

2. Suppose that an entire function f is real on both the real and imaginary axes. Prove that

$$f(z) = g(z^2)$$

   for some entire function g.

Chapter 6

Winding Numbers

In Chapter V we obtained Cauchy's Theorem for closed curves in a disk. Our method easily generalises to such regions which are convex. And indeed one can convince oneself that Cauchy's Theorem is true for any simply connected domain. However we need an even more general result. In the process we are also led to determining the general conditions under which Cauchy's formula holds. The precise conditions are topological and are determined by counting how many times a curve winds around the boundary components of a region.

6.1 Variation in argument around an arc

Let $\alpha : \zeta = \zeta(t),\ 0 \le t \le 1$, be a piecewise smooth curve. In general the function $\arg(z(t)), 0 \le t \le 1$, is not continuously defined, e.g. $\zeta(t) = \exp(i2\pi t)$. Nevertheless we can define the variation in argument on α by decomposing the curve into a finite number of small subarcs $\alpha_j,\ j = 1, ..., k$, so that $\arg(z)$ has a well defined branch $\arg_j(z)$ on α_j . That is if α_j runs from ζ_{j-1} to ζ_j then

$$\Delta_{\alpha_j}(\arg(\zeta)) = \arg_j(\zeta_j) - \arg_j(\zeta_{j-1})$$

$$= \int_{\alpha_j} \frac{d\arg(\zeta)}{ds}\, ds = \int_{\alpha_j} \Im\left\{\frac{d\log(\zeta)}{ds}\right\} ds$$

$$= \Im\left\{\int_0^1 \frac{\zeta'(t)}{\zeta(t)}\, dt\right\} = \Im\left\{\int_{\alpha_j} \frac{d\zeta}{\zeta}\right\}.$$

Therefore we have

$$\Delta_\alpha(\arg(\zeta)) = \sum_{j=1}^{k} \Delta_{\alpha_j}(\arg(\zeta)) = \sum_{j=1}^{k} \Im \left\{ \int_{\alpha_j} \frac{d\zeta}{\zeta} \right\} = \Im \int_\alpha \frac{d\zeta}{\zeta}.$$

These integrals provide an alternative definition of $\Delta \arg(\zeta)$.

Example: Let $\alpha : \zeta = \zeta(t), 0 \leq t \leq 1$, be defined by $\zeta(t) = 2\exp(i2\pi t)$. The variation in argument is

$$\Delta_\alpha(\arg(\zeta)) = \Im \int_\alpha \frac{d\zeta}{\zeta} = \Im \int_0^1 i2\pi \frac{2e^{i2\pi t}}{2e^{i2\pi t}}\, dt = 2\pi.$$

This of course is intuitively obvious. In fact it is usually easier to see the answer then to calculate it.

Exercises

What is the variation in argument along the curves :

1. $\alpha : \zeta(t) = 2\exp(5i\pi t),\ 0 \leq t \leq 1.$
2. $\beta : \zeta(t) = \exp(3i\pi t)/\{1 + t(1-t)\},\ 0 \leq t \leq 1.$
3. $\gamma : \zeta(t) = 8\exp(6i\pi t) + \exp(2i\pi t),\ 0 \leq t \leq 1.$

6.2 Index of a curve

Let α be a cycle, i.e. the finite sum of piecewise smooth closed curves $\alpha_j,\ j = 1...k$. Let z be a point not on α. The index of z with respect to α, or the winding number with respect to α, is defined as

$$n(z, \alpha) = \Im \left\{ \frac{1}{2\pi} \int_\alpha \frac{d\zeta}{\zeta - z} \right\}.$$

First we prove

PROPOSITION 5 *For any cycle α and $z \notin \alpha$*

$$\int_\alpha \frac{d\zeta}{\zeta - z} = 2\pi n i,$$

for some integer n.

It suffices to assume that α is a closed curve $\alpha : \zeta = \zeta(t), 0 \leq t \leq 1$. Now consider the function defined by

$$h(t) = \int_0^t \frac{\zeta'(s)}{\zeta(s) - z}\, ds$$

which satisfies

$$\frac{d}{dt}\left\{e^{-h(t)}(\zeta(t) - z)\right\} = e^{-h(t)}\zeta'(t) - \frac{\zeta'(t)}{\zeta(t) - z}e^{-h(t)}(\zeta(t) - z) = 0$$

for all t. Therefore $(\zeta(t) - z)\exp(-h(t))$ is a constant c. But $h(0) = 0$ so

$$e^{h(t)} = \frac{\zeta(t) - z}{\zeta(0) - z}.$$

However as the curve is closed $\zeta(0) = \zeta(1)$ and thus $h(1) = 2\pi n i$ for some integer n.

As a corollary we deduce that

$$n(z, \alpha) = \frac{1}{2\pi i}\int_\alpha \frac{d\zeta}{\zeta - z}.$$

Therefore $n(z, \alpha)$ is an analytic function of z on each component of $\mathbb{C} - \alpha$. But an integer valued function analytic on each component of $\mathbb{C} - \alpha$ must be constant on each component of $\mathbb{C} - \alpha$.

What we have proved is intuitively obvious, that the winding number is constant on each component of the complement of a curve.

Exercise

Let α be a bounded closed curve. Compute the winding number with respect to points in the unbounded component of the complement.

6.3 Homology classes

Let R be a fixed region of the complex plane. We pick out the curves, and in general cycles, which do not wind about any exterior point.

DEFINITION 13 *: A cycle α contained in R is said to be homologous to zero if $n(z,\alpha) = 0$ for all z in the complement of R.*

Remarks: Being homologous to zero is a property of both the curve and the domain.

Examples:

1. The previous generalises to any simply connected region R: for the winding number $n(z,\alpha)$ is zero near ∞ and hence on the unbounded components of $\mathbb{C} - R$. But as R is simply connected $\mathbb{C} - R$ only has unbounded components. Any cycle in R is homologous to zero.

2. Suppose that R contains an annullus $\{z : s < |z-a| < r\}$ but not some point in $\{|z-a| < s\}$. Then the curve

$$z = a + (r+s)\exp(2\pi i t)/2,\ 0 \leq t \leq 1,$$

   is not homologous to zero in R.

Two cycles α, β contained in R is said to be homologous if

$$n(z, \alpha - \beta) = 0,\ z \notin R.$$

For instance if $R = \{z : |z| < 2\} - \{0, 1\}$ the cycle $\{\exp(it)/3 : 0 \leq t \leq 2\pi\} + \{1 + \exp(it)/3 : 0 \leq t \leq 2\pi\}$ is homologous to the cycle $\{1.5\exp(it) : 0 \leq t \leq 2\pi\}$ (with respect to R).

6.4 Cauchy's Theorem

In a strong sense this is the central result of our course.

THEOREM 25 *(Cauchy Theorem) Let $f(z)$ be analytic on a region R. Let α be a cycle in R which is homologous to zero. Then*

$$\int_\alpha f(z)\,dz = 0.$$

To prove this we need several preliminary results:

LEMMA 8 *Let Q be any positively orientated rectangle so that Q and its interior belong to R.*

$$\int_Q f(z)\,dz = 0.$$

This is Cauchy's formula for a rectangle. The proof is similar to Cauchy's formula for a circle (see chapter V).

The other result we need is topological. We first fix a compact set $K \subset R$. Consider squares of the form

$$Q(n,k,j) = \{k2^{-n} + j2^{-n} + x + iy : 0 < x, y < 2^{-n}\}$$

where n is a fixed positive number and k, j are any integers. For fixed n consider the set of these squares with closure in R, the union of these closed squares is a multiply connected polygonal domain $P(n)$.

LEMMA 9 *For large enough n there is a component D of $P(n)$ which contains K in its interior.*

This is a familar argument using connectivity which we sketch. Similar to the proof of polygonal connectivity of regions we prove that for any two fixed points $z,\ w \in R$ there is a number n and a finite sequence of adjacent squares $S(n,k,j) \subset R$ connecting w to z. As K is compact there is a finite n and finite number of squares $Q(n,k,j) \subset R$ which connect all points of K. (The details are an exercise for the reader).

Actually this result is intuitively obvious.

Let D be any polygonal region which is the connected union of a finite number, M, of squares $Q(n,k,j)$,(where n fixed). Let these squares have positively orientated boundaries Q_m say, $m = 1, ...M$. We now observe that the boundary of D may be written as a (orientated) cycle. This is seen by writing $\partial D = Q_1 + ... + Q_M$. Now all interior sides of squares cancel leaving those on the boundary only. To see that this gives a consistent orientation to the boundary of D we can use induction on M.

Next we set the cycle α to be our compact set K which is contained in the above polygonal D.

LEMMA 10 *For all $\zeta \notin D$ we have $n(\zeta, \alpha) = 0$.*

Follows as α is homologous to zero with respect to R, i.e. for all $\zeta \notin R$, $n(\zeta, \alpha) = 0$. But as $n(\zeta, \alpha)$ is constant on the components of $\mathbf{C} - \alpha$ we have that $n(\zeta, \alpha) = 0$ on ∂R. Now each side of ∂D bounds a square Q containing a point of ∂R and so $n(\zeta, \alpha) = 0$ on ∂D and hence on $\mathbf{C} - D$.

Proof of Cauchy Theorem:

Let α be our fixed cycle. As α is compact we apply the Lemma to find a polygonal region D which contains α. Let z be interior to the square Q_k of D. Then by the first Lemma

$$\frac{1}{2\pi i} \int_{Q_j} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \begin{cases} f(z) & j = k \\ 0 & j \neq k \end{cases}$$

Hence for all $z \in D - \cup_j Q_j$ by summing over all squares

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

However the RHS is continuous in the interior of D hence

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \, d\zeta,$$

for all z in the interior of D. Integrating both sides gives

$$\int_\alpha f(z) dz = \int_\alpha \left\{ \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \, d\zeta \right\} \, dz,$$

$$= \int_{\partial D} \left\{ \frac{1}{2\pi i} \int_\alpha \frac{f(\zeta)}{\zeta - z} \, dz \right\} \, d\zeta,$$

interchanging the order of integration. But $n(\zeta, \alpha) = 0$ for $\zeta \in \partial D$ so by the other Lemma 3 the inside term of the integral is zero thus completing the proof of Cauchy's Theorem.

6.5 Cauchy's formula

A general form of Cauchy's formula now follows easily from the general Cauchy Theorem :

THEOREM 26 *(Cauchy Formula) Let $f(z)$ be analytic on a region R. Let α be a cycle in R which is homologous to zero. Then for all z not on α*

$$n(z, \alpha)f(z) = \frac{1}{2\pi i}\int_\alpha \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

As before we apply Cauchy Theorem to the function

$$F(\zeta) = \frac{f(\zeta) - f(z)}{\zeta - z}$$

which is analytic on R. Hence

$$\frac{1}{2\pi i}\int_\alpha \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int_\alpha \frac{f(z)}{\zeta - z}\,d\zeta = n(z, \alpha)f(z).$$

Exercise

1. Let $f(z)$ be analytic on a region R which has finitely many boundary components b_1, ..., b_n. Show that $f(z) = f_1(z) + ... + f_n(z)$ where f_j is analytic on the component of $\mathbb{C} - b_j$ containing R.

2. Compute the possible values of

$$\int_\gamma \sqrt{z^2 - 1}\,dz$$

where γ is a closed curve on $\mathbb{C} - [-1, 1]$ and we take a branch of $\sqrt{z^2 - 1}$ which is analytic on $\mathbb{C} - [-1, 1]$. (The answer will depend only on the homology class of γ, i.e. the number of times it winds around [-1, 1]).

6.6 Laurent expansion

As a first application we obtain an expansion analogous to Taylor's theorem

THEOREM 27 *(Laurent expansion) Suppose that $f(z)$ is analytic on $A = \{r < |z-a| < R\}$. Then*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k \, (z-a)^k \ ,$$

where both the series in positive and negative powers are normally convergent on A. Furthermore

$$a_k \ = \frac{1}{2\pi i} \int_J \frac{f(\zeta)}{(\zeta-a)^{k+1}} \, d\zeta,$$

where J is any closed curve in A so that $n(a, J) = 1$.

First we fix numbers $r < s < p < |z-a| < q < t < R$. Next we define curves

$$\alpha = a + s\exp(i\theta), \ 0 \le \theta < 2\pi, \ \beta = a + t\exp(i\theta), \ 0 \le \theta < 2\pi$$

and cycle $\gamma = \beta - \alpha$. Now for $z, \ |z-a| < r, \ n(z, -\alpha) = -1$ and $n(z, \beta) = 1$ while for $z, \ |z-a| > R, \ n(z, -\alpha) = 0, \ n(z, \beta) = 0$. Thus γ is homologous to zero on A. On the other hand as $p < |z-a| < q, \ n(z, \alpha) = 1$. Thus by Cauchy Formula:

$$f(z) = \frac{1}{2\pi i} \int_\beta \frac{f(\zeta)}{\zeta - z} \, d\zeta - \frac{1}{2\pi i} \int_\alpha \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

The first term is expanded a la Taylor's formula to yield

$$\frac{1}{2\pi i} \int_\beta \frac{f(\zeta)}{\zeta - z} \, d\zeta = \sum_{k=0}^{\infty} a_k \, (z-a)^k \ ,$$

where

$$a_k \ = \frac{1}{2\pi i} \int_\beta \frac{f(\zeta)}{(\zeta-a)^{k+1}} \, d\zeta.$$

The series is uniformly convergent for $|z-a| < q$. For the second term we expand

$$\frac{1}{\zeta - z} = \frac{-1}{(z-a)(1-(\zeta-a)/(z-a))} = \sum_{k=0}^{\infty} -\frac{(\zeta-a)^k}{(z-a)^{k+1}} \ ,$$

uniformly convergent for $|\zeta - a| = s,\ |z-a| > p > s$. Substituting into the integral

$$\frac{-1}{2\pi i}\int_\alpha \frac{f(\zeta)}{\zeta - z}\, d\zeta = \sum_{k=1}^{\infty} b_k\,(z-a)^{-k}\ ,$$

where

$$b_k \;=\; \frac{1}{2\pi i}\int_\alpha \frac{f(\zeta)}{(\zeta-a)^{-k+1}}\, d\zeta.$$

Hence, uniformly for $p < |z-a| < q$,

$$f(z) = \sum_{k=0}^{\infty} a_k\,(z-a)^k \;+\; \sum_{k=1}^{\infty} a_{-k}\,(z-a)^{-k}\ ,$$

where if $k \geq 0$

$$a_k \;=\; \frac{1}{2\pi i}\int_\beta \frac{f(\zeta)}{(\zeta-a)^{k+1}}\, d\zeta,$$

or if $k < 0$

$$a_k \;=\; \frac{1}{2\pi i}\int_\alpha \frac{f(\zeta)}{(\zeta-a)^{k+1}}\, d\zeta.$$

Finally we note, by Cauchy's theorem, that as the curve J is homologous to both α and β we have that

$$\int_\alpha \frac{f(\zeta)}{(\zeta-a)^{k+1}}\, d\zeta = \int_\beta \frac{f(\zeta)}{(\zeta-a)^{k+1}}\, d\zeta$$

$$= \int_J \frac{f(\zeta)}{(\zeta-a)^{k+1}}\, d\zeta$$

which completes the proof of the Theorem.

It is usually easier to derive the Laurent expansion of a given function by means other then the evaluation of the Laurent expansion using the integral formulae, in fact it is often useful to reverse this and having "somehow" obtained the Laurent expansion then obtain the integral formula. The integral formula is also of great theoretical importance.

Examples

1. Find the Laurent expansion in powers of z for $f(z) = z\exp(1/z)$. Now f is analytic on $0 < |z| < \infty$ and we expand it on $0 < |z| < \infty$: from the power series for $\exp(z)$ we get

$$\exp(1/z) = \sum_{k=0}^{\infty} \frac{z^{-k}}{k!}$$

uniformly in $|z| \geq s$, where $s > 0$. Thus

$$z\exp(1/z) = 1 + z + \sum_{k=1}^{\infty} \frac{z^{-k}}{(k+1)!}$$

so the Laurent coefficents are $a_0 = a_1 = 1, a_2 = ... = 0$ and $a_{-k} = 1/(k+1)!,\ k > 0$.

2. If $c:\ z = \exp(it),\ 0 < t \leq 2\pi$, evaluate

$$\int_C \frac{dz}{z^2\ \sinh(z)}\ .$$

We find the first few terms of the Laurent expansion in $0 < |z| < \pi/2$ by

$$\frac{1}{\sinh(z)} = \left\{z + \frac{z^3}{6} + ...\right\}^{-1}$$
$$= \frac{1}{z}\left\{1 - \frac{z^2}{6} + ...\right\} = \frac{1}{z} - \frac{z}{6} + ...$$

and thus the Laurent formula gives

$$\int_C \frac{dz}{z^2\ \sinh(z)} = \frac{-2\pi i}{6}\ .$$

Exercises

1. Find Laurent expansions of $f(z) = 1/(z^3 + 4z)$ for :
(i) $0 < |z| < 2$ (in powers of z),
(ii) $2 < |z| < \infty$ (in powers of z),
(iii) $0 < |z + 2i| < 2$ (in powers of $z + 2i$)

2. Let f be analytic on $|z| = 1$ show

$$f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta},\ a_n = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}\,d\theta.$$

(This is the so called Fourier Expansion of a function defined on $|z| = 1$. Actually it holds (in some sense) for any continuous function however the general theory of how this converges is very difficult. It was only in 1966 that Carleson proved that the Fourier series converges at almost all points. There are continuous functions where the Fourier series diverges on a dense subset (of measure zero)).

3. Let f be analytic for $0 < |z| < \infty$. Prove $f = g + h$ where h is analytic for $|z| < \infty$ and g is analytic for $0 < |z|$ (including ∞).

4. If F is analytic on $\mathbb{C}$ and has period 2π prove that

$$F(z) = \sum_{n=-\infty}^{\infty} a_n e^{inz}.$$

6.7 Singularities

Understanding singularity is one of the main problems in modern mathematics, and is central to the application of complex analysis. The isolated singularities are the simplest, i.e. a is an isolated singularity of f if for some $R > 0$ we have that f is analytic in $\{z : 0 < |z - a| < R\}$ but not analytic on $\{z : |z - a| < R\}$. E.g. $f = 1/z$ $(a = 0)$ or $f = \tan(z)$ (singularities are $n\pi + \pi/2$).

The Laurent expansion may be used to classify singularities. Thus on $\{0 < |z - a| < R\}$ f has expansion

$$f(z) = \sum_{k=-\infty}^{\infty} a_k\,(z-a)^k\ ,$$

we define the Principal part to be

$$\sum_{k=1}^{\infty} a_{-k}\,(z-a)^{-k}\ ,$$

which is analytic in $0 < |z-a| < R$. If the Principal part is zero then f extends to a function analytic on $|z-a| < R$. There are a number of criteria which imply that f has such an analytic extension, see next section.

Otherwise we define the residue of f at a to be a_{-1}. This has the important property that

$$a_{-1} = \frac{1}{2\pi i}\int_C f(z)\,dz,\; C:\; z = a + re^{it},\; 0 < t \leq 2\pi,$$

for any $0 < r < R$. The case that the Principal part is a finite series is also important, i.e. $a_k = 0,\; k < -N$ but $a_{-N} \neq 0$. Then we say that f has a pole of order N at a. In this case f is said to have a removeable singularity at a since $(z-a)^N f(z)$ is analytic in $|z-a| < R$, e.g. $f = 1/\sin(z)$ has a pole of order 1 at $a = 0$. Otherwise f is said to have a transcendental, e.g. $f = \sin(1/z)$ has a transcendental singularity at $a = 0$. Every nonpolynomial function analytic on $\mathbb{C}$ has a transcendental singularity at ∞.

Exercises

1. Let f be analytic at a. Give a formula for the residue of $(z-a)^{-N} f(z),\; N$ a positive integer.

2. If a is a pole of order N for $f(z)$ prove that

$$a_{-1} = \frac{1}{(N-1)!}\frac{d^{N-1}}{dz^{N-1}}\{(z-a)^N f(z)\},\; z = a.$$

   (which can be evaluated by L'Hôpital's rule).

3. For each of the following determine singularities, the order and residue at each singularity:

$$(i) z/(z+3),\; (ii) \sin(z)/z^2,\; (iii) \cot(z),\; (iv) 1/(z^2\sin(z)).$$

4. Let D be a region and E a finite subset. Suppose that f is analytic on $D - E$ with a pole of finite order at each point of E. Prove there is a rational function $R(z)$ so that $f(z) - R(z)$ is analytic on D.

6.8 Removeable sets

Liouville's Theorem says that a bounded entire function is constant. An extension of this is

THEOREM 28 *(Riemann's Lemma) Suppose that f analytic and bounded on $\{0 < |z-a| < R\}$ then f has analytic extension to $z = a$.*

The proof shows that the Principal Part is zero: without loss of generality $a = 0$, so

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \; a_n = \frac{1}{2\pi}\int_0^{2\pi} r^{-n} f(re^{i\theta})e^{-in\theta}\,d\theta.$$

Hence if $|f| < M$ then for any negative integer n

$$|a_n| \le r^{-n}M \to 0 \text{ as } r \to 0.$$

Actually it is easy to see how this can be improved significantly. We end by very briefly considering nonisolated singularities. Let R be a region and E a compact totally disconnected subset of R. Suppose that f is analytic and bounded on $R - E$. When does f have analytic extension to R? Clearly from Riemann's Lemma provided E is finite. However there are even uncountable sets with this property. We measure the size of a set in terms of generalised length. A set E has zero length if for every $\epsilon > 0$ there exists a countable number of disks B_n of radius r_n so that

$$\cup_n B_n \supset E, \; \sum_{n=1}^{\infty} r_n < \epsilon.$$

For example any of the Cantor type sets (constructed with fixed ratio) will have zero length. Then we can show that f has analytic extension to E. It can be shown that for any compact set E of the real line with Lesbesgue measure $\lambda(E) > 0$ there is a function f bounded and analytic on $\mathbf{C} - E$ with no analytic extension to E. For more details refer to Garnett's book "Bounded analytic functions".

Exercises

1. Find all functions f analytic on $\mathbb{C} - \{0\}$ so that $f(2z) = f(z)$.

2. Suppose that analytic f is bounded by M on a domain R with boundary ∂R of zero length, i.e $\forall \varepsilon > 0$ the boundary ∂R is covered by open disks of radii r_n so that $\sum r_n < \varepsilon$. Show that there is g analytic on $\mathbb{C}$ so that

$$|f(z) - g(z)| \leq \frac{M\,\epsilon}{\delta}$$

where $\delta = dist(z, \partial R)$. hence f has analytic extension to $\mathbb{C}$.

6.9 Residue calculus

Cauchy proved that one only has to consider singularities to evaluate line integrals over closed curves. We say that a cycle α bounds a region D if there is a cycle $\alpha = \partial D$ and α consists of a finite number of piecewise smooth Jordan curves orientated so $n(z, \alpha) = 1$ for all $z \in D$ and for $z \notin D$ $n(z, \alpha) = 0$. Now let R be a region containing the closure of D. Then α is homologous to zero in R.

THEOREM 29 *(Residue Theorem) Suppose that a cycle α bounds a region D and f is analytic on the closure of D except for maybe some isolated singularities in D. Then*

$$\int_\alpha f(z)\,dz = 2\pi i \sum_{b \in D} Res(f, b)$$

where $Res(f, b) = a_{-1}$ is the residue at b.

Before proving this we consider an example. Let Q be the positively orientated square with vertices $\pm 3/2 \pm 3i/2$. Compute

$$\int_Q \frac{\exp(\sin^2(z))}{(z^2+1)(z-2)}\,dz.$$

This function is analytic on the plane except for poles at $\pm i, 2$. Only the first two lie inside Q and thus contribute. As these are poles of first order it is particularly easy to compute the residues. At i we have

$$a_{-1} = \frac{\exp(\sin^2(i))}{(2i)(i-2)} = \frac{\exp(\sinh^2(1))}{(2i)(i-2)}$$

while at $-i$

$$a_{-1} = \frac{\exp(\sin^2(-i))}{(-2i)(-i-2)} = \frac{\exp(\sinh^2(1))}{(2i)(i+2)}$$

Thus by the residue theorem

$$\int_Q \frac{\exp(\sin^2(z))}{(z^2+1)(z-2)}\, dz = 2\pi i \left\{ \frac{\exp(\sinh^2(1))}{(2i)(i-2)} + \frac{\exp(\sinh^2(1))}{(2i)(i+2)} \right\}$$

$$= -2\pi i \exp(\sinh^2(1))\,/5\ .$$

This example is obviously something which cannot be computed by finding the antiderivative.

Proof : Let f have singularities in D at $b_1, ...b_m \subset D$. Consider the cycle γ consisting of $\gamma_1 + ... + \gamma_m$ where $\gamma_j : \zeta(t) = b_j + s_j \exp(it),\ 0 < t \leq 2\pi$. We choose s_j small enough that the disks $\{|z - b_j| \leq s_j\}$ are disjoint and contained in D. Now α is homologous to γ on $R - \{b_1, ...b_m\}$. Thus by Cauchy theorem

$$\int_\alpha f(z)\, dz = \sum_{j=1}^{m} \int_{\gamma_j} f(z)\, dz.$$

While $\int_{\gamma_j} f(z)\, dz = 2\pi i Res(f, b_j)$.

Exercises

1. Evaluate the integrals of the functions

$$(i)\ \frac{1}{(z-4)(z^3-1)},\ (ii)\ \frac{3z^2+2}{(z-1)(z^2+9)}$$

on the positively orientated circles ;

$$\alpha :\ |z-2| = 5/2,\ \beta :\ |z| = 5.$$

2. Compute the integrals of the functions

$$(i)\ \frac{1}{\cosh(2z)},\ (ii)\ \cot(z)$$

on the positively orientated circle $\gamma :\ |z| = 1$.

3. Compute the integrals of the functions

$$(i)\ \frac{\sinh(\pi z)}{z^2(z-1)^2},\ (ii)\ \frac{\log(z+4)}{z(z-2)^2}$$

on the positively orientated circle $\gamma:\ |z| = 3/2$.

4. Show that $\sqrt{z^4+z}$ is a branch analytic on $D = \{|z| > 1\}$. Hence evaluate

$$\int_C f(z)\, dz,$$

where C is any curve in D winding once around 0.

5. Evaluate the integral

$$\int_C \frac{3z^{98}-4z^{49}}{4z^{100}-1}\, dz,$$

where $C : |z| = 2^{1/100}$.

6. Let P, Q be polynomials with $Deg(Q) - deg(P) > 1$. Prove that the sum of the residues of P/Q is zero.

7. Let P be a polynomial of deg > 1 with one zero outside $\Gamma : |z| = 1$, and all the other zeros inside Γ. Prove that

$$\int_\Gamma \frac{1}{P(z)}\, dz \neq 0\ .$$

6.10 Definite integrals

There is a vast literature concerning applications of residue calculus to both to pure and applied mathematics. We shall see some applications to pure mathematics, even number theory, later. In modern applied mathematics there is the need to obtain exact definite integrals, for example in scattering theory of fundamental particles. In general there is as much art as craft in being able to do this. However there are some standard cases where the application of the residue theorem is straight foreward:

(i) The simplest are integrals of the form:

$$\int_0^{2\pi} R(\cos(\theta), \sin(\theta))\, d\theta,$$

where $R(x, y)$ which is rational in x, y. A simple change of variables to $z = \exp(i\theta)$ and the interval $[0, 2\pi]$ becomes the positively orientated circle $C : |z| = 1$. The integral becomes

$$\int_C R((z + z^{-1})/2, (z - z^{-1})/2i)\, \frac{dz}{iz}.$$

which is computed via the residue theorem.

A simple example is

$$\int_0^{2\pi} \frac{d\theta}{a + \cos(\theta)}\ ,\ a > 1.$$

Now the transformed integral is

$$\int_C \frac{1}{a + (z + 1/z)/2}\, \frac{dz}{iz} = \int_C \frac{-2i\, dz}{2az + z^2 + 1}\ .$$

The integrand has poles at $-a \pm \sqrt{a^2 - 1}$, however only $-a + \sqrt{a^2 - 1}$ is inside C, where the residue is $-i(a^2 - 1)^{-1/2}$. Thus

$$\int_0^{2\pi} \frac{d\theta}{a + \cos(\theta)} = \frac{2\pi}{\sqrt{a^2 - 1}}.$$

It is remarkable that we can then use analytic continuation to assert that the formula is also true for all complex a provided $|a| > 1$ and the Principal Branch of $\sqrt{z}$ is used.

(ii) Integrals of the form

$$\int_{-\infty}^{\infty} R(x) e^{iax}\, dx$$

where R is a rational function which disappears at ∞, and a is a real constant. Actually, in theory at least, these can be evaluated by methods of calculus but this quickly becomes impractical. We show :

Let R be rational with no poles on the real axis and $zR(z) \to 0$ as $z \to \infty$ then

$$\int_{-\infty}^{\infty} R(x)\,dx = 2\pi i \sum_{\Im(b_j)>0} a_{-1,j}\ ,$$

where we sum residues over all poles in the upper half plane.

To prove this we use the curve $\alpha = [-r, r] + \{r\exp(it) : 0 < t \le \pi\}$. So if r is chosen so big that α bounds all the poles of R we have

$$\int_{\alpha} R(z)\,dz = 2\pi i \sum_{\Im(b_j)>0} a_{-1,j}\ ,$$

Now we have $\lim_{z\to\infty} zR(z) = 0$ so

$$\left| \int_0^{\pi} R(re^{it})\, rie^{it} dt \right| \le \pi r \sup_t |R(re^{it})| \to 0,$$

which proves the formula.

Example: Consider

$$\int_0^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx.$$

Now the integrand has poles at $i,\ 2i$ in the upper half plane with residues $-1/(2i.3),\ -4/(-3.4i)$ respectively. Hence

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx = 2\pi i \left\{ \frac{-1}{6i} + \frac{1}{3i} \right\} = \frac{\pi}{3}$$

As we have an even function for our integrand we obtain

$$\int_0^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \frac{\pi}{6}.$$

For these integrals it is obvious that

$$\int_{-\infty}^{\infty} |R(x)| dx < \infty$$

i.e. summable in the usual sense. For integrands involving $\exp(iax)$ we can relax the growth condition on $R(x)$ a little but need to extend the meaning of the improper integral. This is usually called the Principal Value of an integral. The simplest case is something like

$$\int_1^\infty \frac{\cos(x)}{x}\,dx$$

which is defined to be

$$\lim_{r\to\infty}\int_1^r \frac{\cos(x)}{x}\,dx = \lim_{r\to\infty}\left\{\frac{\sin(r)}{r} - \sin(1) + \int_1^r \frac{\sin(x)}{x^2}\,dx\right\}$$

which exists even though

$$\int_1^\infty \frac{|\cos(x)|}{x}\,dx = \infty.$$

In general we define the improper integrals when the limit exists. By the same integration by parts trick as before one can show that these limits do exist if $R(x)$ is rational with no poles on the real axis and $\lim_{z\to\infty} R(z) = 0$, i.e.

$$\int_0^\infty R(x)\left\{\begin{array}{c}\cos(ax)\\ \sin(ax)\end{array}\right\}dx, \int_{-\infty}^0 R(x)\left\{\begin{array}{c}\cos(ax)\\ \sin(ax)\end{array}\right\}dx$$

exist for any nonzero constant a. Having established the meaning of the improper integrals we compute them.

Assume $a > 0$ and R is rational with no poles on the real axis and $\lim_{z\to\infty} R(z) = 0$, then

$$\int_{-\infty}^\infty R(x)e^{iax}\,dx = 2\pi i \sum_{\Im(b_j)>0} a_{-1,j}\, e^{iab_j}\ ,$$

where we sum residues over all poles in the upper half plane.

To prove this we use the curve $\alpha = [-r, r] + \{r\exp(it) : 0 < t \leq \pi\}$. Thus if r is chosen so big that α bounds all the poles of R we have that

$$\int_\alpha R(z)e^{iaz}\,dz = 2\pi i \sum_{\Im(b_j)>0} a_{-1,j}e^{iab_j}\ .$$

To complete the computation we need

LEMMA 11 *(Jordan's Lemma) Now if we have* $\lim_{z\to\infty} R(z) = 0$ *then for some constants* c, s

$$\left|\int_0^\pi R(re^{it})\exp(iare^{it})\, rie^{it}\, dt\right| \le \frac{c}{r}\ ,\ r > s.$$

We are assuming that $|R(z)| < c/|z|,\ |z| > s$. In this case we get

$$\left|\int_0^\pi R(re^{it})\exp(iare^{it})\, rie^{it}\, dt\right| \le c\int_0^\pi \exp(-ar\sin(t))\, dt,$$

which is bounded by

$$c\int_0^\pi \exp(-art/\pi)\, dt \le \frac{c\pi}{ar}.$$

To get the integral of $R(x)\cos(ax)$ we take the real part of both sides. This requires that $R(x)$ be real on the real axis. If it is not then one needs to break it up into a real and imaginary part.
Example: Evaluate

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x-i}\, dx$$

We write $1/(x-i) = (x+i)/(x^2+1) = x/(x^2+1) + i/(x^2+1)$. Now

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+1}\, dx = 2\pi i\frac{ie^{i.i}}{2i} = \pi i e^{-1}.$$

Taking imaginary parts of both sides gives

$$\int_{-\infty}^{\infty} \frac{x\sin(x)}{x^2+1}\, dx = 2\pi i\frac{ie^{i.i}}{2i} = \frac{\pi}{e}.$$

On the other hand

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\, dx = 2\pi i\frac{1e^{i.i}}{2i} = \pi e^{-1}.$$

Hence

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x^2+1}\, dx = 0,$$

which in this case is obvious. Putting this altogether

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x-i}\, dx = \frac{\pi}{e}.$$

By the way this cannot be directly obtained by applying "Cauchy Formula" which if we could would yield

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x-i}\, dx = -\pi(e-e^{-1}).$$

The same method can be extended somewhat to the cases where $R(x)$ has a pole on one of the zeros of $\sin(x)$ (or $\cos(x)$). Rather then write the general formula we shall compute an important example

$$\int_{0}^{\infty} \frac{\sin(x)}{x}\, dx.$$

The integrand is even and has Principal value, however we have to deal wth a pole at 0. We use the curve

$$\alpha = [-r,-s] + \{se^{it} : \pi < t \leq 2\pi\} + [s,r] + \{re^{it} : 0 < t \leq \pi\},$$

where $r, 1/s$ are large. Now

$$\int_{\alpha} \frac{e^{iz}}{z}\, dz = 2\pi i.$$

by the Residue Theorem. By Jordan's Lemma the integral over $\{re^{it} : 0 < t \leq \pi\}$ tends to zero as $r \to \infty$. We compute the integral over $\{se^{it} : \pi < t \leq 2\pi\}$ to be

$$\int_{\pi}^{2\pi} \frac{1+se^{it}+O(s^2)}{se^{it}}\, ise^{it}\, dt = \pi i + O(s),$$

as $s \to 0$. Putting all of these terms together gives

$$2\pi i = \int_{\alpha} \frac{e^{iz}}{z}\, dz = \pi i + \int_{-r}^{-s} \frac{e^{ix}}{x}\, dx + \int_{s}^{r} \frac{e^{ix}}{x}\, dx + O(s) + O(1/r)\ ,$$

as $s, 1/r \to 0$. Now at this point we take imaginary parts of both sides so that taking limits yields

$$\int_{0}^{\infty} \frac{\sin(x)}{x}\, dx = \frac{\pi}{2}.$$

This formula is important to harmonic analysis and its applications to optics.

(iii) The next class of integrals have the form

$$\int_0^\infty R(x)\left\{\begin{array}{c} x^a \\ \log(x) \end{array}\right\} dx$$

where a is a real constant and once again where zR is a rational function which disappears at ∞. Consider the example

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2}dx$$

We use the branch of $\sqrt{z}$ defined on $\{0 < \arg(z) < 2\pi\}$. We use the curve $\alpha = -a+b+c-d$ where $a = \{se^{it} : q < t \le 2\pi - q\},\ b = \{te^{iq} : s < t < r\},\ c = \{re^{it} : q < t \le 2\pi - q\},\ d = \{te^{-iq} : s < t < r\}$, where r, $1/s$, $1/q$ are large. The residue theorem shows

$$\int_\alpha \frac{\sqrt{z}}{1+z^2}dz = 2\pi i\left\{\frac{\sqrt{i}}{2i} + \frac{\sqrt{-i}}{-2i}\right\}$$

$$= \pi\left\{e^{i\pi/4} - e^{3\pi i/4}\right\} = \pi\sqrt{2}.$$

As before the integrals on the circles tend to zero as s, q tend to zero and r tends to ∞. On the other hand the integral on b tends to

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2}dx$$

while that on $-d$ tends to

$$\int_0^\infty \frac{\sqrt{e^{2\pi i}}\sqrt{x}}{1+x^2}(-dx) = \int_0^\infty \frac{\sqrt{x}}{1+x^2}dx.$$

Thus in the limit

$$2\int_0^\infty \frac{\sqrt{x}}{1+x^2}dx = \pi\sqrt{2}$$

so that

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2}dx = \frac{\pi}{\sqrt{2}}.$$

Exercises

1. Compute the following integrals:

$$(i)\ \int_{-\infty}^{\infty} \frac{dx}{x^4+1},\ (ii)\ \int_0^{\infty} \frac{\log(x)}{(x^2+1)^2}dx,\ (iii)\ \int_0^{\pi} \frac{dx}{(2+\cos(x))^2}.$$

2. Evaluate

$$(i)\ \int_0^{\infty} \frac{dx}{x^3+8},\ (ii)\ \int_0^{\infty} \frac{x^{a-1}}{1+x}dx,\ a<1 (iii)\ \int_0^{\infty} \frac{\cos(ax)}{(b^2+x^2)^2}dx.$$

3. Show that

$$\int_0^{\pi} \sin^{2n}(x)\,dx = \frac{(2n)!}{2^{2n}\,(n!)^2}\pi.$$

4. By considering the wedge $\{0 < \arg(z) < \pi/4\}$ prove

$$\int_0^{\infty} \cos(x^2)\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}}\ .$$

5. Given that for $a > 0$

$$\int_0^{\infty} \frac{\cos(x)}{(a^2+x^2)^2}\,dx = \frac{\pi}{4a^3}(1+a)e^{-a}$$

evaluate the integral for all complex a, without computing the integral.

6.11 Applications to Series

We shall use Residue Calculus to obtain the exact answer for a variety of series of the sort seen in calculus, e.g.

$$\sum_{n=1}^{\infty} \frac{1}{n^k}$$

which converges for any number $k > 1$. (Actually we obtain explicit answers for even k only).

The idea is to make use of the function $\cot(\pi z)$ which has simple poles with residue $1/\pi$ at the integers n. We consider the positively orientated square Q_N with vertices $\pm(N+1/2) \pm (N+1/2)i$. Thus for any rational function $R(z)$, with poles at b_j $(j = 1, .., M)$ contained inside Q_N, R having residue a_j at b_j,

$$\int_{Q_N} R(z)\, \cot(\pi z)\, dz = 2\pi i \left\{ \sum_{-N}^{N}{}^{*} \frac{R(n)}{\pi} + \sum_{j=1}^{M}{}^{*} a_j\, \cot(\pi b_j) \right\}.$$

where the $\sum{}^{*}$ indicates we have to take special account if any of the poles of R is an integer. If as $N \to \infty$ the integral tends to zero then we would obtain

$$\sum_{-\infty}^{\infty}{}^{*} R(n) = -\pi \sum_{j=1}^{M}{}^{*} a_j\, \cot(\pi b_j).$$

which now has to be justified.

First consider the integral, as

$$\cot(\pi z) = i \frac{e^{2\pi i z} + 1}{e^{2\pi i z} - 1}$$

if $z = x + iy, |y| = N + 1/2$, then

$$|\cot(\pi z)| = \left| \frac{e^{2\pi i x \pm 2\pi(N+1/2)} + 1}{e^{2\pi i x \pm 2\pi(N+1/2)} - 1} \right| \leq \frac{2}{1 - e^{-2\pi(N+1/2)}},\ N \to \infty$$

and if $|x| = N + 1/2$

$$|\cot(\pi z)| = \left| \frac{e^{\pm 2\pi i(N+1/2) - 2\pi y} + 1}{e^{\pm 2\pi i(N+1/2) - 2\pi y} - 1} \right| \leq \frac{1 - e^{-2\pi y}}{1 + e^{-2\pi y}} \leq 1,$$

so that

$$\left| \int_{Q_N} R(z)\, \cot(\pi z)\, dz \right| \leq 16(N + 1/2) \sup_{Q_N} |R(z)|.$$

Thus the formula holds provided $\lim_{|z| \to \infty} |zR(z)| = 0$.

As an example we prove the formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

We use the function $R(z) = z^{-2}$. This has one pole at 0 which is also a pole of $\cot(\pi z)$. Therefore we work out the residue of $\cot(\pi z)\, z^{-2}$ at 0. We need the Laurent expansion of $\cot(\pi z)$:

$$\cot(z) = \frac{\cos(z)}{\sin(z)} = \left(1 - \frac{z^2}{2} + \frac{z^4}{24}\right)\frac{1}{z}\left(1 - \frac{z^2}{6} + \frac{z^4}{120}\right)^{-1}$$
$$= \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - ...$$

Thus $z^{-2}\cot(\pi z)$ has residue $\frac{-\pi}{3}$ at $z = 0$ and the previous theory gives

$$2\sum_{n=1}^{\infty} \frac{1}{\pi n^2} = \frac{\pi}{3}.$$

We can compute $\sum_{n=\infty}^{\infty}(-1)^n R(n)$ using $\operatorname{cosec}(\pi z)$ instead of $\cot(\pi z)$ in the integrand. These methods also yield some interesting series expansions. We know that every rational function is the sum of its Principal Parts, is the same true for meromorphic functions such as $\cot(\pi z)$? We might guess

$$\cot(\pi z) = \frac{1}{\pi}\sum_{n=-\infty}^{\infty} \frac{1}{z-n} .$$

Using the same notation as before

$$\int_{Q_N} \frac{\cot(\pi\zeta)}{\zeta - z}\, d\zeta = 2\pi i\left\{\sum_{-N}^{N} \frac{1}{\pi(n-z)} + \cot(\pi z)\right\}.$$

Unfortunately the earlier bound used to prove that the integral tends to zero does not suffice. Instead we write

$$\frac{1}{\zeta - z} = \frac{1}{\zeta} + \frac{z}{\zeta(\zeta - z)}$$

Now as cot is odd

$$\int_{Q_N} \frac{\cot(\pi\zeta)}{\zeta}\, d\zeta = 0$$

while as before

$$\left|\int_{Q_N} \frac{\cot(\pi\zeta)}{\zeta(\zeta - z)}\, d\zeta\right| \leq \frac{C(N+1/2)}{(N+1/2)(N+1/2-|z|)} \to 0 ,$$

as $N \to \infty$. Hence the series converges for every z not an integer. If we try a little harder we can show that the convergence is uniform on every compact subset of $\mathbb{C} - \mathbb{Z}$.

Exercises

1. Compute the following

$$(i)\ \sum_{n=1}^{\infty}\frac{1}{n^6},\ (ii)\ \sum_{-\infty}^{\infty}\frac{1}{8n^3-1},\ (iii)\ \sum_{n=1}^{\infty}\frac{(-1)^n}{n^4},$$

2. Prove the expansion

$$\sec(z)=\sum_{-\infty}^{\infty}\frac{(-1)^n}{z-\pi(n-1/2)}$$

converges normally on $\mathbb{C}-\{\pi(n-\frac{1}{2}):\ n\in\mathbb{Z}\}$.

3. Prove the infinite product expansion

$$\sin(\pi z)=\pi z\lim_{N\to\infty}\Pi_{n=1}^{N}\left\{1-\frac{z^2}{n^2}\right\}.$$

6.12 Argument Principle

In this section we bring togther some of the concepts of this chapter for:

THEOREM 30 *(Argument Principle) Let α be a cycle which bounds a region R. Suppose that $f(z)$ is meromorphic on the closure of R with no zeros or poles on α. Suppose that f has N zeros in R and M poles in R, (counting multiplicities) then*

$$n(0,\ f(\alpha))=N-M.$$

The left hand side is the winding number about 0 of the image of α under f. This can be equivalently written as

$$\frac{1}{2\pi i}\int_{\alpha}\frac{f'(z)}{f(z)}\,dz=N-M.$$

The proof is easy : f'/f is meromorphic wth a pole with residue $+1$ at each zero of f and a pole with residue -1 at each pole of f.

This has many applications. Let us start with a simple one:
Suppose that f is analytic on and inside α which bounds R and $f(\alpha)$ bounds a region D. We show that f is 1:1 on R. Now $n(w, f(\alpha)) = 1$ for any w in D. Hence by the argument principle for each $w \in D$ there is exactly one point $z \in R$ so that $f(z) - w$ has a zero. On the other hand for w outside $f(\alpha)$ there is no point $z \in R$ with $f(z) = w$. Thus there is no points $w \in f(\alpha)$ itself taken by some point in R.

Yet another application is usually called Rouché's Theorem:

THEOREM 31 *(Rouché's Theorem) Let f, g be analytic in and on a bounding cycle α. Suppose that on α we have that $|f| > |g|$. Then f and $f+g$ have the same number of zeros inside α.*

Just consider the variation in argument around α of $1 + g/f$.

Exercises

1. Find the number of zeros in $|z| < 1$ of the following functions:

$$(i)\ 5z^4 + z^2 + 2z - i,\ (ii)\ z^3 + 8z^2 - 8z + 2\ ,\ (iii)\ e^z + 3z.$$

2. How many zeros of

$$z^4 + 8z^3 + 3z^2 + 8z + 3$$

lie in $\{z : \Re(z) > 0\}$. (Hint : sketch the image of the curve $\{r\exp(it) : -\pi < t < \pi\} + \{it : -r < t < r\}$ for large r).

3. Suppose that f is analytic in and inside a bounding cycle α. Interprete

$$\frac{1}{2\pi i}\int \frac{zf'(z)\,dz}{f(z) - w}.$$

4. A meromorphic function f is elliptic if it is has periods ω_1, ω_2 (linearly independent over $\mathbb{R}$). Its fundamental region is

$$\mathcal{F} = \{x\omega_1 + y\omega_2 : 0 < x,\ y \leq 1\}.$$

Show that nonconstant f has at least one pole inside $\mathcal{F}$. Show that

(a) The sum of the residues in $\mathcal{F}$ is zero.

(b) Let M be the number of poles inside $\mathcal{F}$. Show that for any $w \in \mathbb{S}$ $f(z) - w$ has exactly M zeros inside $\mathcal{F}$.

5. Let f be analytic on the closed disk $B = \{|z| \leq 1\}$ and

$$f(B) \subset \{|z| < 1\} .$$

Prove that f has a fixed point ζ, i.e. $f(\zeta) = \zeta$.

6. Let f be analytic on the unit disk $\mathbb{D}$ and

$$f(\mathbb{D}) \subset \mathbb{D} .$$

Prove that either

(a) f has a fixed point $\zeta \in \mathbb{D}$, i.e. $f(\zeta) = \zeta$.
(Hint: apply the previous result to $f((1 - 1/n)z)$.)

(b) or, $\exists \zeta \in \partial\mathbb{D}$ so that the angular limit:

$$\lim_{z \to \zeta} f(z) = \zeta$$

7. Let f be analytic on the unit disk $\mathbb{D}$ and

$$f(B) \subset \mathbb{D} .$$

Prove that there exists $\zeta \in \{|z| \leq 1\}$ so that the compositions

$$f^n(z) = f \circ ... \circ f(z) \to \zeta$$

normally on $\mathbb{D}$. (Hint: Use previous results and Schwarz Lemma).

Chapter 7

Infinite Products

Two of our subthemes have been the factorisation of functions and the convergence of infinite expansions. These will now be brought together. An infinite product of complex numbers p_k

$$\prod_{k=1}^{\infty} p_k$$

converges if there exists an integer m so that $p_k \neq 0,\ k \geq m$ and the sequence of partial products

$$\prod_{k=m}^{n} p_k \to p_0 \neq 0,\ n \to \infty\ .$$

Then we set

$$\prod_{k=1}^{\infty} p_k = p_0 \prod_{k=1}^{m-1} p_k.$$

For example

$$(i)\ \prod_{k=2}^{\infty} \left\{1 - \frac{1}{k^2}\right\} = \frac{1}{2}$$

and

$$(ii)\ \prod_{k=1}^{\infty} \left\{1 - \frac{1}{k^2}\right\} = 0$$

but

$$(iii)\ \prod_{k=1}^{\infty} \{1 - (-1)^k\}$$

is not defined.

It is not necessary to develop a theory of infinite products from scratch:

LEMMA 12 *An infinite product*

$$\prod_{k=1}^{\infty} p_k \to p$$

if and only if for some m *the the terms* $\log(p_k)$ *are defined for* $k \geq m$ *and*

$$\sum_{k=m}^{\infty} \log(p_k) \to s\ .$$

Then $p = p_1 \ldots p_{m-1} \exp(s)$.

As this is for complex numbers this implicitly assumes that the Principal branch of logarithm is being used.

Proof: If for any branch of logarithm $\sum_{k=m}^{\infty} \log(p_k) \to s$ then as the function exponential is continuous

$$\exp\left\{\sum_{k=m}^{n} \log(p_k)\right\} = \prod_{k=m}^{n} p_k \to e^s \neq 0.$$

Conversely if $\prod_{k=1}^{\infty} p_k \to p \neq 0$ then $p_k \to 1$. Thus for $m,\ k > m$ we have $|\arg(p_k)| < \pi$. Also by convergence the partial products $p_{m+1} \ldots p_n$ may be assumed to lie in the region $\{z : |\arg(z)| < \pi\}$. Thus we may apply the continuous branch $\log(z)$ to the this chopped off sequence and use the product rule. Therefore

$$\sum_{k=m}^{n} \log(p_k) \to \log(p_0)\ ,$$

and the result follows.

For example the infinite product

$$\prod_{k=1}^{\infty} \left\{1 + \frac{z}{k^2}\right\}$$

we have that

$$\log\left\{1+\frac{z}{k^2}\right\} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}(\frac{z}{k^2})^n$$

and thus for fixed z there is N so that if $k > N$

$$\left|\log\left\{1+\frac{z}{k^2}\right\}\right| \leq \frac{c}{k^2}.$$

Thus the log series is absolutely convergent. In general if we have $\sum |\log(p_k)| < \infty$ we say the product is absolutely convergent. This has the important property that we can rearrange the order of terms in the product. The same expansion of $\log(1+z)$ applied in the general situation gives:

LEMMA 13 *The product*

$$\prod_{k=1}^{\infty}(1+a_k)$$

is absolutely convergent if and only if $\sum |a_k| < \infty$.

BEWARE: The above without "absolute" fails to be true:

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$$

is convergent but

$$\prod_{n=1}^{\infty}\left\{1+\frac{(-1)^n}{\sqrt{n}}\right\}$$

is not.

Also we may use the terms "uniformly convergent infinite product" or "infinite product normally convergent on a region R" by considering the equivalent infinite series. Furthermore it is easy to show that

LEMMA 14 *Let $p_k(z) \neq 0$ be analytic on a region R . Then*

$$\prod_{k=1}^{\infty} p_k(z) \to p(z) \neq 0$$

uniformly on a compact set $K \subset R$ if and only if for some m the following infinite sum is defined on K and

$$\sum_{k=m}^{\infty} \log(p_k(z)) \to s(z)$$

uniformly on K. Then $p(z) = p_1(z)..p_{m-1}(z)\exp(s(z))$. Furthermore we have absolute normal convergence of the product on R provided we have the same for the sum $\sum\{p_k - 1\}$.

Thus in a converging infinite product of analytic functions we can have only finitely many terms with zeros on any given compact set. This implies that the only zeros of the limit $p(z)$ are points where one of the factors p_k have a zero.

1. First consider $\prod_{k=1}^{\infty}(1 - kz^k)$. This is seen to be uniformly convergent on $|z| < r$ for any fixed $r < 1$. Let the limit function be $f(z)$ which is analytic on $|z| < 1$. Then $f(z)$ has zeros at the points $\exp(2\pi ji/k)/\sqrt[k]{k}$ where $k = 1, 2, ...,$ and $j = 1, ..., k$. These cluster on the unit circle so it follows that $f(z)$ has no analytic continuation to any point of $|z| = 1$.

2. Next we consider
$$\prod_{k=1}^{\infty}\left(1 - (-1)^k \frac{z}{k}\right).$$
This is not absolutely convergent for any nonzero z. Nevertheless we show uniform convergence on the disk $|z| < R$, any $R > 0$ by pairing consecutive terms.

3. The infinite product
$$\prod_{k=1}^{\infty}\left(1 - \frac{z}{k}\right)$$

does not converge at all (except at $z = 0$). However multiplying by an extra nonzero factor will ensure convergence of an infinte product that has zeros at $z = k$. We consider then

$$\prod_{k=1}^{\infty} \left(1 - \frac{z}{k}\right) e^{z/k},$$

which we show is absolutely convergent. Fix $r > 0$ and consider $|z| < r/2$. Now for $k > r$ from the Taylor series

$$\begin{aligned} \left|\log(1 - \frac{z}{k}) + \frac{z}{k}\right| &\leq \frac{r^2}{k^2}(1 - r/k)^{-1} \\ &\leq 2\frac{r^2}{k^2}. \end{aligned}$$

Thus for fixed $r > 0$ the product is uniformly convergent for $|z| < r/2$.

Exercises

1. Discuss the convergence of the following

$$(i) \prod_{k=1}^{\infty} \left(1 - e^{kz}\right), \ (ii)\ z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 k^2}\right) \ (iii) \prod_{k=1}^{\infty} \left(1 - \frac{i^k z}{\sqrt{k}}\right) .$$

2. Suppose that we have a sequence of complex numbers $a_k \to \infty$, and

$$\prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right)$$

converges uniformly on compact sets to an entire function $f(z)$. Let H be any closed convex set containing all the a's. Prove that all the zeros of f' are also contained in H.

3. Suppose that we have a sequence of real numbers $a_k \to \infty$, and

$$\prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right)$$

converges uniformly on compact sets to an entire function $f(z)$. Prove that all the zeros of f' are real.

4. Find an entire function $f(z)$ with real zeros so f' has no zeros at all. (Thus Lucas Theorem does not hold in general).

7.1 Canonical Products

The aim of this chapter is to show that every entire function has a factorisation similar to the factorisation of polynomials. Unfortunately our previous examples show that this is not just a matter of multiplying terms like $(1-z/a)$. Also there are entire functions which have no zeros at all. We deal with them first :

PROPOSITION 6 *For any entire function f, nonzero on $\mathbb{C}$, there is an entire function g so that $f(z) = \exp(g(z))$*

Now f'/f is also an entire function equal to g' for some entire function g. Thus on differentiating we find that $f \exp(-g) = c$ a constant. So $f = c \exp(g)$. (The constant can be included in the exponential too.)

This result is not necessarily true for nonsimply connected regions. Next we give a canonical method of constructing infinite products. We first introduce the canonical factors of order $h = 1,\ 2, ...$

$$E_h(z) = (1-z)\exp(z + \frac{z^2}{2} + \ ...\ + \frac{z^h}{h})$$

and call $E_0(z) = (1-z)$. Let $f(z)$ be an nonconstant entire function with nonzero zeros $a_k \to \infty$. We shall prove that

$$f(z) = z^m\ e^{g(z)} \prod_{k=1}^{\infty} E_{h_k}\left(\frac{z}{a_k}\right)$$

where f has a zero of order m at $z = 0$, g is an entire function and the sequence of nonnegative integers h_k will be assigned. In order to take this any further we need an estimate:

LEMMA 15 *With the above notation*

$$\log|E_h(z)| \leq \begin{cases} (2h+1)|z|^{h+1} & ,0<|z|<1 \\ (h+1)|z|^h & ,\ |z| \geq 1. \end{cases} .$$

Now by definition

$$(i)\ \log|E_h(z)| \leq \log|E_{h-1}(z)| + |z|^h$$

for all z. As $\log|E_0(z)| \leq \log(1+|z|) \leq |z|$ for all z we have

$$\log|E_h(z)| \leq (h+1)|z|^h,\ |z| \geq 1$$

by induction. For $|z| < 1$ we use the power series expansion to give

$$\log|E_h(z)| \leq \frac{|z|^{h+1}}{h+1} + \frac{|z|^{h+2}}{h+2} + ... \leq \frac{|z|^{h+1}}{1-|z|}$$

and hence

$$\log|E_h(z)| \leq |z|\log|E_h(z)| + |z|^{h+1} \leq |z|\log|E_{h-1}(z)| + 2|z|^{h+1}$$

by (i). Thus by induction for $|z| < 1$

$$\log|E_h(z)| \leq (2h+1)\,|z|^{h+1}.$$

Having proved the lemma we are ready to state the famous Weierstrass Factorisation Theorem:

THEOREM 32 *(Weierstrass) Consider a sequence of nonzero points $a_k \to \infty$. Let sequence h_k of nonnegative integers satisfy*

$$\sum_{k=1}^{\infty} \frac{1}{h_{k+1}} \left|\frac{r}{a_k}\right|^{h_k+1} < \infty,$$

for all $r > 0$. Then the infinite product

$$\prod_{k=1}^{\infty} E_{h_k}\left(\frac{z}{a_k}\right)$$

converges normally on $\mathbf{C}$.

For instance choosing $h_k = k$ will ensure convergence in every case. This follows from Abel's First Theorem. However we don't want to be as extreme as this. In an example such as $a_k = \{1, -1, 2, -2...\}$ we see that $h_k = 1$ suffices. Thus

$$\prod_{k=-\infty}^{\infty} \left(1 - \frac{z}{k}\right) \exp\left(\frac{z}{k}\right)$$

converges to an entire function, which we already know to be $\sin(\pi z)/\pi z$.

In general we say that a sequence a_k has genus h if

$$\sum_{k=1}^{\infty} \frac{1}{|a_k|^{h+1}} < \infty.$$

Evidently the canonical product

$$\prod_{k=1}^{\infty} E_h\left(\frac{z}{a_k}\right)$$

will converge to an entire function.
Proof of the Theorem: Fix any $r > 0$. It suffices to show

$$\prod_{|a_k|>2r}^{\infty} E_{h_k}\left(\frac{z}{a_k}\right)$$

is uniformly convergent on $|z| < r$. From the Lemma

$$\left|\log\left|E_{h_k}\left(\frac{z}{a_k}\right)\right|\right| \le \frac{2}{h_k+1}\left|\frac{r}{a_k}\right|^{h_k+1}$$

for $|a_k| > 2r,\ |z| < r$. Our requirements tell us that this product is absolutely and uniformly convergent on $|z| < r$.

This Theorem has a number of implications:

COROLLARY 2 *Any entire function f may be written*

$$f(z) = z^m\, e^{g(z)} \prod_{k=1}^{\infty} E_{h_k}\left(\frac{z}{a_k}\right)$$

where f has a zero of order m at $z = 0$, g is an entire function and for some sequence of nonnegative integers h_k.

It is evident that there is no uniqueness in this factorisation. This and the question of the best choice of the canonical factors will be considered in later sections.

It is well known that every rational function is the ratio of polynomials. A generalisation of the rational functions is the class of meromorphic functions: we say that $h(z)$ is meromorphic on the plane if h is analytic except at isolated singularities where h has poles of finite order. e.g. $\cot(z)$, $\mathcal{P}(z, \omega_1, \omega_2)$ are meromorphic, $\exp(1/z)$ is not .

COROLLARY 3 *:Any meromorphic function h may be written $h = f/g$ where f, g are entire functions with no common zeros.*

Let $h(z)$ have poles of order N_k at a_k. Construct an entire function $g(z)$ with zeros of order N_k at a_k. Then hg has no poles and must therefore be equal to an entire function f.

Exercises

1. Show that
$$\cos(\pi\sqrt{z}) = \prod_{k=1}^{\infty}\left(1 - \frac{z}{(k-\frac{1}{2})^2}\right).$$

2. Let f, g be entire functions. Show that there is an entire functions h, F, G so that $f = hF, g = hG$ and F, G have no zeros in common. Hence prove there are entire functions P, Q so that $h = Pf + Qg$.

3. Prove that
$$\frac{\pi}{2} = \prod_{k=1}^{\infty}\left(1 - \frac{1}{4k^2}\right)^{-1} .$$

7.2 Bounded functions in the disk

Before going onto develop more precise results about entire functions we develop the analogous theory for the disk. We consider functions f analytic and bounded on the unit disk $\mathbb{D} = \{|z| < 1\}$, say $|f(z)| \leq 1$ for $|z| < 1$. Suppose that f is nonconstant and has zeros of order m at the origin and nonzero zeros a_k , $k = 1, 2....$ Instead of canonical factors we consider terms of the form
$$\frac{z-a}{1-\overline{a}z},$$

i.e. bilinear maps of the disk onto itself. From these we define the so called Blaschke product:

$$B_n(z) = \prod_{k=1}^{n} \frac{|a_k|}{-a_k} \frac{z - a_k}{1 - \overline{a_k} z} \ ,$$

where it is convenient to assume no roots at $z = 0$, i. e. no factors z^m.

THEOREM 33 *B_n converges normally on $\mathbb{D}$ if and only if*

$$(1) \ \sum_{k=1}^{\infty} (1 - |a_k|) \ < \infty \ .$$

Condition (1) is necessary as

$$B_n(0) = \prod_{k=1}^{\infty} |a_k|$$

is convergent iff. (1) satisfied. Conversely

$$\left| 1 - \frac{|a|}{-a} \left(\frac{z - a}{1 - \overline{a}\, z} \right) \right| = \left| \frac{a - a|a| + z(|a| - |a|^2)}{a(1 - \overline{a}\, z)} \right|$$

$$\leq (1 - |a|) \left| \frac{1 + |z|}{1 - \overline{a}\, z} \right| \leq 2\, \frac{1 - |a|}{1 - |z|},$$

for $|z| < 1$. Thus by the theory of infinite products, as $n \to \infty$,

$$B_n(z) \to z^m \prod_{k=1}^{\infty} \frac{|a_k|}{-a_k} \left(\frac{z - a_k}{1 - \overline{a_k}\, z} \right) ,$$

on compact subsets of $\mathbb{D}$. The infinite Blaschke product (including z^m) will be denoted as $B(z)$.

Conversely we shall see that if f is bounded and nonconstant with zeros a_k then

$$\sum_{k=1}^{\infty} (1 - |a_k|) < \infty \ .$$

We first prove

LEMMA 16 *The function f/B_n is analytic on $\mathbb{D}$ and bounded by* 1.

This is a simple application of the maximum principle. The function B_n, being the finite product of linear transformations which map the unit disk onto itself, satisfies $|B_n| = 1$ for $|z| = 1$. Thus as B_n is continuous, for every $\epsilon > 0$ there is a $r < 1$ so that $|B_n| > 1 - \epsilon$ for $1 > |z| > r$. Thus $|f/B_n| < 1/(1-\epsilon)$ for $|z| > r$. Now as the zeros of B_n are also zeros of f (up to order) f/B_n is analytic on $\{|z| < 1\}$ and hence by the maximum principle $|f/B_n| < 1/(1-\epsilon)$ on $\{|z| < 1\}$.

This shows the interesting bound for $f(z) = c_m z^m + ...,\ c_m \neq 0$

$$0 < |c_m| \leq \prod_{k=1}^{n} |a_k|$$

for all n, which implies for any analytic f bounded on $|z| < 1$ that

$$\sum_{k=1}^{\infty} (1 - |a_k|) < \infty \ .$$

Hence by the previous theorem B_n converges normally to B. From the lemma we have that $|f/B| \leq 1$ on $\{|z| < 1\}$. Thus $F = f/B$ has no zeros and satisfies $|F| \leq 1$ on $\{|z| < 1\}$. Finally we have that $f = BF$ which provides a unique factorisation for bounded functions. In the exercises we show that the nonzero term F may be represented by a Poisson type integral representation.

EXAMPLE: Consider the infinite product

$$\prod_{k=1}^{\infty} \frac{(1-n^{-2}) - z^n}{1 - (1-n^{-2}) z^n} \ .$$

Now we see that the Blaschke condition is clearly satisfied. The function B is a bounded function with zeros clustering at every point $z, |z| = 1$. (Thus B has no analytic continuation from $\{|z| < 1\}$.) However we shall show that for all most every point $\zeta \in \partial\mathbb{D}$, $B(z)$ has radial boundary value ω with $|\omega| = 1$.

Exercises

1. Let f, g be bounded functions on the unit disk so that
$$f(1-1/k) = g(1-1/k)\ ,\ k = 1, 2.....$$
Show that $f = g$.

2. Let f be analytic and bounded by 1 on the unit disk with zeros at $z = 0, -1/2, -3/4$. Prove
$$|f(1/2)| \leq 4/11\ .$$

3. Suppose F is analytic and bounded by 1 with no zeros. Show there exists a positive measure $d\mu$ on $\partial\mathbb{D}$ so that
$$F(z) = c \exp\left\{-\int \frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\, d\mu\right\}$$
for some constant c, $|c| = 1$.

We apply these results to obtain the

7.3 Poisson-Jensen Formula

Suppose that f is analytic and nonconstant on $|z| \leq R$, with zeros a_k, $k = 1, ...n$, satisfying $0 < |a_k| < R$. Then for any $z = re^{it}\,, r < R$

$$\log|f(z)| = \sum_{k=1}^{n} \log\left|\frac{R(z-a_k)}{R^2 - \overline{a_k}z}\right| + \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\log|f(Re^{is})|}{R^2 - 2Rr\cos(s-t) + r^2}\, ds$$

We prove this in several simple steps beginning by assuming $R = 1$. Then if $f(z)$ has no zeros on $|z| = 1$ the formula follows by applying the Poisson formula applied to the harmonic function $u(z) = \log|f(z)/B(z)|$, where $B(z)$ is the Blaschke product

$$B(z) = \prod_{k=1}^{n} \frac{z - a_k}{1 - \overline{a_k}z}\ .$$

Note that $|f| = |f/B|$ on $|z| = 1$ as $|B| = 1$ on $|z| = 1$.

It remains only to prove the case where f has zeros on $|z| = R$. Consider a sequence R_m tending to R from below. For R_m close enough to R there are no zeros on the circle $|z| = R_m$. Thus we have the formula:

$$\log|f(z)| = \sum_{k=1}^{n} \log\left|\frac{R_m(z-a_k)}{R_m^2 - \overline{a_k}z}\right| + \frac{1}{2\pi}\int_0^{2\pi} \frac{(R_m^2 - r^2)\log|f(R_m e^{is})|}{R_m^2 - 2R_m r\cos(s-t) + r^2}\, ds$$

As m tends to infinity the sum term on the right has the required limit. In the integral the kernel tends uniformly to the limit: The term $\log|f(Re^{is})|$ has integrable singularities at the zeros and is the limit of $\log|f(R_m e^{is})|$ in the L^1 sense. Thus the formula holds in all cases.

Our first deduction is the so called Jensen inequality:
Suppose that f is analytic and nonconstant on $|z| \leq R$, then for any $z = re^{it}\,, r < R$

$$\log|f(z)| \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\log|f(Re^{is})|}{R^2 - 2Rr\cos(s-t) + r^2}\, ds\ .$$

In particular if f is analytic and nonconstant on $|z| \leq R$, then

$$\log|f(0)| \leq \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{is})|\, ds\ .$$

Exercises

1. Suppose that f is analytic and nonconstant for $z = re^{it}, r \leq R$. Prove that

   $$\frac{1}{2\pi}\int_0^{2\pi} -\log^-|f(Re^{is})|\ ds \leq \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(Re^{is})|\ ds\ -\ \log|f(0)|\ ,$$

   where $\log^+(x) = \max\{\log x,\, 0\},\ \log^-(x) = -\max\{-\log x,\, 0\}$.

2. Let $f(z)$ be analytic on $|z| < 1$. Prove that

   $$\sup_{0<r<1}\int_0^{2\pi} |\log|f(re^{is})||\ ds < \infty$$

   if and only if

   $$\sup_{0<r<1}\int_0^{2\pi} \log^+|f(re^{is})|\ ds < \infty.$$

3. Let $f(z)$ be analytic and nonconstant on $|z| < 1$. Suppose that

$$\sup_{0<r<1} \int_0^{2\pi} \log^+ |f(re^{is})|\, ds < \infty.$$

Show that if f has (nonzero) roots $a_1, ...$ in $|z| < 1$ then

$$\sum_{k=1}^{\infty} \log |a_k^{-1}| < \infty$$

and hence the Blaschke condition holds for the zeros.

4. Let $f(z)$ be analytic and nonconstant on $|z| < 1$. Suppose that

$$\sup_{0<r<1} \int_0^{2\pi} \log^+ |f(re^{is})|\, ds < \infty$$

show there is a Blaschke product B and a nonzero function F satisfying

$$\sup_{0<r<1} \int_0^{2\pi} \log^+ |F(re^{is})|\, ds < \infty$$

so that $f = BF$.

5. Let $F(z)$ be analytic and nonzero on $|z| < 1$. Suppose that

$$\sup_{0<r<1} \int_0^{2\pi} \log^+ |F(re^{is})|\, ds < \infty$$

show there is a real measure $d\mu$ on $\{|\zeta| = 1\}$ so that

$$F(z) = c \exp\left\{ - \int \frac{1 + z\overline{\zeta}}{1 - z\overline{\zeta}}\, d\mu \right\}$$

for some constant c, $|c| = 1$.

6. Show that functions $f(z)$ analytic and nonconstant on $|z| < 1$ satisfying

$$\sup_{0<r<1} \int_0^{2\pi} \log^+ |f(re^{is})|\, ds < \infty$$

have radial boundary values (a.e.). (Hint use previous results and chapter on harmonic functions).

7.4 Entire functions of finite order

Let f be an entire function. We define its characteristic function by

$$T(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{is})|\; ds\;.$$

where $\log^+(x) = \log(x)$ if $x > 1$ and 0 otherwise. Now we say that f has order λ(of growth) if for any $\epsilon > 0$ there is a constant C so that

$$T(r,f) < Cr^{\lambda+\epsilon}\;,\; r > R(f,\epsilon)\;,$$

and λ is the least nonnegative number with this property.

For example $\exp(z^n)$ has order of growth n. Polynomials (and some others too) have order of growth 0.

By the Poisson -Jensen inequality, assuming (without loss of generality) that $f(0)$ is nonzero

$$\frac{1}{2\pi}\int_0^{2\pi} -\log^- |f(re^{is})|\; ds \le \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{is})|\; ds - \log|f(0)|\;,$$

and thus having finite order λ is equivalent to

$$\frac{1}{2\pi}\int_0^{2\pi} \big|\log|f(re^{is})|\big|\; ds \le Cr^{\lambda+\epsilon}\;, r > R.$$

Sometimes the order of growth is defined by the apparently stronger:

$$\log|f(re^{is})| \;\le\; Cr^{\lambda+\epsilon}\;, r > R.$$

For if this is satisfied so surely does $T(r,f)$ satisfy the required bound. Conversely, by Poisson-Jensen again,

$$\begin{aligned}\log|f(\frac{r}{2}\,e^{it})| \;\le\; &\frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{is})|\,\frac{r^2 - r^2/4}{(r-r/2)^2}\; ds\\ &\le 3T(r,f)\;.\end{aligned}$$

This shows the two definitions are in fact equivalent. The characteristic function is prefered as it is the right object for Nevanlinna's deep theory of meromorphic functions, a theory that not only generalises to several complex variables but which recently has led to progress in number theory.

Exercises

1. Compute the characteristic functions for :

$$\exp(z),\ \sin(z),\ \cos(\sqrt{z})\ .$$

2. What is the order of growth of the functions in the previous exercise.

3. If f, g are entire with order ν, μ respectively show that fg has order $\lambda \leq \max\{\nu, \mu\}$.

4. If f, g are entire with order $\nu < \mu$ respectively show that fg has order $\lambda = \mu$.

5. let f be an entire function of order λ with no zeros. Prove that $1/f$ is an entire function of order λ.

7.5 Order of a Sequence

Without loss of generality we order a sequence so that $1 < |a_1| \leq |a_2| \leq .. \to \infty$. We say that the sequence has order of growth μ (not necessarily an integer) if for any ϵ we have

$$\sum_{k=1}^{\infty} \frac{1}{|a_k|^{\mu+\epsilon}} < \infty\ .$$

and μ is the least nonnegative real number with this property.
e.g. $a_k = k^{\alpha}$ has order of growth $\mu = 1/\alpha$.

Also we use the counting function $N(r) = \#\{|a_k| < r\}$. Now μ is also the least number so that for any $\epsilon > 0$ there is a constant $C\ :\ N(r) \leq Cr^{\mu+\epsilon}$. To see this we use a Stieltjes integral to write

$$\sum_{k=1}^{\infty} \frac{1}{|a_k|^{\mu+\epsilon}} \geq \int_1^r \frac{dN(x)}{x^{\mu+\epsilon}} = |_1^r \frac{N(x)}{x^{\mu+\epsilon}} + (\mu+\epsilon)\int_1^r \frac{dN(x)}{x^{\mu+1+\epsilon}}$$

$$\geq \frac{N(r)}{r^{\mu+\epsilon}}\ .$$

for any $r > 1$ as $N(1) = 0$. Simple comparisions give what we want.

Finally we define the genus of the sequence is the least nonnegative integer h so that

$$\sum_{k=1}^{\infty} \frac{1}{|a_k|^{h+1}} < \infty ,$$

clearly $h \leq m \leq h+1$.
Example: The sequence $a_k = k^2$ has order $1/2$ and genus 0, while $a_k = k$ has order 1 and genus 1. Also $a_k = k(\log k)^2$ has order 1 and genus 0.

We finish this section by estimating the order of growth of the zeros a_k of an entire function of order λ.

THEOREM 34 *Let μ be the order of growth of the zeros a_k of an entire function of order λ. Then $\mu \leq \lambda$.*

This is another application of Poisson Jensen. For if $a_1, a_2, ..., a_N$ are the zeros contained in $|z| < r$

$$\sum_{k=1}^{N} \log \frac{r}{|a_k|} \leq T(r,f) - \log |f(0)| ,$$

and hence

$$\log(2) N(\frac{r}{2}) \leq C r^{\lambda+\epsilon} ,$$

and thus $\mu \leq \lambda$.

7.6 Order of the canonical product

Let a_k be a nonzero sequence of order μ and genus h. We construct the canonical product:

$$P(z) = \prod_{k=1}^{\infty} E_h(\frac{z}{a_k}) \text{ , where } E_h(z) = (1-z)\exp(z + ... + z^h/h) .$$

As the sequence has genus h the canonical product converges. We prove

THEOREM 35 *$P(z)$ has order at most μ.*

From the lemma on Weierstrass Factorisation if $\epsilon > 0$ satisfies $m = \mu + \epsilon < h + 1$ there is a constant C so that

$$|E_h(z)| \leq C^{|z|^m}$$

and hence

$$\log|P(z)| \leq \sum \log(C) \left|\frac{z}{a_k}\right|^m = K|z|^m ,$$

which proves the theorem.

As a corollary we obtain that for an entire function $f(z)$ of order λ the canonical product $P(z)$, formed from the zeros, has order at most λ.

7.7 Hadamard factorisation

Let f be an entire function of finite order λ, with zeros a_k. In previous sections we showed that the sequence a_k has order $\mu \leq \lambda$, and genus h, with $\mu \leq h + 1$. Also we proved that canonical product

$$P(z) = \prod_{k=1}^{\infty} E_h(\frac{z}{a_k}) \ , \text{ where } E_h(z) = (1 - z)\exp(z + ... + z^h/h) \ .$$

converges absolutely and is an entire function of order μ. Thus we may write

$$f(z) = z^n \, P(z) e^{g(z)}$$

where n is the order of zeros of f at 0 and g is an entire function. By previous bounds $\exp(g)$ has order less then or equal to λ, i.e. $\Re\{g(z)\} \leq C|z|^{\lambda+\epsilon}$, $|z| > R$. We have only to prove

LEMMA 17 *$g(z)$ is a polynomial of degree $q \leq \lambda$.*

From the complex form of the Poisson formula

$$g(z) = ic + \frac{1}{2\pi}\int_0^{2\pi} \frac{r + ze^{-is}}{r - ze^{-is}} \Re\{g(re^{is})\} \, ds \ ,$$

which holds for $|z| < r$, and some real constant $C = \Im g(0)$. In particular

$$|g(\frac{re^{it}}{2})| \leq 3Cr^{\lambda+\epsilon} \ , \ r > R \ .$$

Hence by Cauchy's Inequality $g(z)$ is a polynomial of degree at most $\lambda + \epsilon$.

We have proved the famous Hadamard Factorisation Theorem:

THEOREM 36 *Let f be a nonconstant entire function of finite order λ, with zeros a_k. Then the sequence a_k has order $\mu \leq \lambda$, and genus h, $h \leq \mu \leq h+1$. The canonical product*

$$P(z) = \prod_{a_k \neq 0} E_h(\frac{z}{a_k})$$

converges absolutely and is an entire function of order μ. Also we may write

$$f(z) = z^n \, P(z) \, e^{g(z)}$$

where g is a polynomial of order $q \leq \lambda$ and n is the order of zeros of f at 0. The factorisation is unique.

There is nothing left to do but

Exercises

1. Prove that
$$\sin z = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 k^2} \right) \, .$$

2. Construct a transcendental function of order 0.

3. Prove that every entire function of finite but noninteger order has infinitely many zeros.

4. Show that $e^z - q(z)$ has infinitely many zeros, provided $q(z)$ is a nonconstant entire function of order $\mu < 1$.

5. Prove that Lucas' Theorem holds for entire functions of order $\lambda < 1$.

6. Find a representation for all functions $f(z)$ of order 1 so that
$$f(z) = g(z^2)$$
for some entire function $g(z)$.

Chapter 8

Special functions

The two most famous nonelementary functions of all of mathematics are the Gamma function and the Riemann Zeta function, both introduced by Euler. The Gamma function extends the function $(n-1)!$ to the plane. It is used throughout mathematics.

8.1 Euler's Gamma Function

In order to obtain the Gamma function we first define Euler's constant:

$$\gamma = \lim_{n\to\infty} \left\{ 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \log(n+1) \right\} = .57722..$$

which exists by

LEMMA 18 *: The sequence*

$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{n} - \log(n+1)$$

is monotone increasing and bounded above by 1*.*

Now

$$\gamma_n - \gamma_{n+1} = \log(1 + \frac{1}{n+1}) - \frac{1}{n+1} \leq 0$$

since $\log(1+x) \leq x\,,\ x > 0$. The upper bound comes from

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{n} - \log(n+1) \leq 1 + \int_1^n \frac{dx}{x} - \log(n+1)$$

$$= 1 + \log(n) - \log(n+1) \leq 1$$

Next we define the infinite product

$$J(z) = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n} .$$

Its main property is defined in

LEMMA 19 *J(z) is an entire function.*

The sequence $a_n = -n,\ n \in \mathbb{N}$ has order 1 and genus 1. Thus the infinite product for J converges to an entire function.

Remarks: By the Hadamard Theorem J has order 1.

Finally we come to the Gamma function

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n} .$$

Thus $\Gamma = 1/J$ is meromorphic with simple poles at 0, -1, -2..., and no zeros. We now deduce its main properties:

PROPOSITION 7 *The Gamma function* Γ *satisfies*
(i) $\Gamma(z+1) = z\Gamma(z)$
(ii) $\Gamma(n) = (n-1)!, n \in \mathbb{N}$
(iii) $\Gamma(z)\,\Gamma(1-z) = \pi/\sin(\pi z)$

First we observe (iv)

$$e^{-\gamma} \lim_{N\to\infty} \prod_{n=1}^{N} \frac{ne^{1/n}}{n+1} = 1 .$$

and hence $\Gamma(1) = 1$.

To prove (i) we use the infinite product representation:

$$\Gamma(z+1) = \frac{e^{-\gamma\,(z+1)}}{z+1} \prod_{n=1}^{\infty} \left(1 + \frac{z+1}{n}\right)^{-1} e^{(z+1)/n}$$

$$= \frac{e^{-\gamma z}}{z+1} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n+1}\right)^{-1} e^{z/n} \, e^{-\gamma} \lim_{N\to\infty} \prod_{n=1}^{N} \frac{n e^{1/n}}{n+1} \,,$$

by (iv)

$$= \frac{e^{-\gamma z}}{z+1} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n+1}\right)^{-1} e^{z/n} \,,$$

$$= e^{-\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n} \,,$$

$$= z \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n} \,,$$

which is $z\Gamma(z)$.

To prove (ii) we apply induction to (i) and use $\Gamma(1) = 1$.

Finally we prove (iii):
By the product representation for Γ and $\sin(z)$

$$\Gamma(z)\Gamma(-z) = \frac{1}{z^2} \prod_{n=-\infty, n\neq 0}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} = \frac{\pi}{\sin(z)} \,,$$

and thus (iii) is proved as $-z\Gamma(-z) = \Gamma(1-z)$ by (i).

Exercises

1. Prove that

$$\Gamma\{\frac{1}{2}\} = \sqrt{\pi}$$

2. Show

$$\frac{d}{dz} \frac{\Gamma'}{\Gamma} = \sum_{-\infty}^{\infty} \frac{1}{(z+n)^2}$$

3. By computation show

$$\frac{d}{dz} \frac{\Gamma'(z)}{\Gamma(z)} + \frac{d}{dz} \frac{\Gamma'(z+1/2)}{\Gamma(z+1/2)} = 2 \frac{d}{dz} \frac{\Gamma'(2z)}{\Gamma(2z)} \,.$$

4. Prove that there are constants a, b so that

$$\Gamma(z)\Gamma(z+1/2) = e^{az+b}\Gamma(2z) \ .$$

5. Hence prove

$$2^{2z-1}\,\Gamma(z)\Gamma(z+1/2) = \sqrt{\pi}\Gamma(2z) \ .$$

8.2 Representation Theorem

We shall prove:

THEOREM 37 *For* $\Re(z) > 0$

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,dt \ ,$$

where $t^{z-1} = \exp((z-1)\log(t))$ *is the Principal branch.*

Define

$$F(z) = \int_0^\infty e^{-t}t^{z-1}\,dt \ ,$$

which is analtyic for $\Re(z) > 0$. First we find

LEMMA 20 $F(z+1) = zF(z),\ \Re(z) > 0.$

The proof used integration by parts to see that

$$F(z+1) = \int_0^\infty e^{-t}t^z\,dt = e^{-t}t^z|_0^\infty + z\int_0^\infty e^{-t}t^{z-1}\,dt$$

which is $zF(z)$

Also we see that $F(1) = 1$.

Consider the function $H = F/G$, defined and analytic for $Re(z) > 0$.

LEMMA 21 *$H(z)$ is analytic on* $\mathbf{C}$ *with period 1.*

Now as

$$H(z+1) = \frac{F(z+1)}{\Gamma(z+1)} = \frac{zF(z)}{z\Gamma(z)} = H(z)$$

for $\Re(z) > 0$ we can extend H to each strip $-n < \Re(z) < -n+1$ and hence to an entire function with period 1. $H(z)$ can transformed onto $h(z) = H(Log(\zeta)/2\pi)$ which is analytic for $0 < |\zeta| < \infty$, and hence by Laurents' Theorem

$$h(\zeta) = \sum_{n=-\infty}^{\infty} a_n z^n, \text{ where } a_n = \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{H(\frac{\log(\zeta)}{2\pi i})}{\zeta^{n+1}}\, d\zeta$$

or changing back to z variables

$$H(z) = \sum_{-\infty}^{\infty} a_n e^{2\pi i n}$$

We intend to apply Riemann's Lemma to show that H is constant i.e. if

$$\lim_{z\to\infty} |H(z)| \exp(-|2\pi\Im(z)|) = 0$$

then H is constant. To estimate the size of H we begin with F, restricting ourself to $1 < \Re(z) < 2$, and observing that $|F(z)| \le F(\Re(z)) \le C < \infty$. For J we have by (iii)

$$\left|J(\frac{1}{2}+iy)J(\frac{1}{2}-iy)\right| = \frac{\left|\sin(\frac{\pi}{2}+iy)\right|}{\pi} ,$$

for $z = x+iy$, and as (by the product formula) $J(x-iy) = \overline{J(x+iy)}$,

$$\left|J(\frac{1}{2}+iy)\right| = \sqrt{\frac{\cosh(\pi y)}{\pi}} \le \frac{e^{(\pi|y|/2)}+1}{\sqrt{2\pi}}] .$$

Thus we have proved

LEMMA 22 *There is a constant C*

$$\log\left|J(\frac{1}{2}+iy)\right| \le C + \frac{\pi}{2}|y| \text{ , for real } y$$

To estimate the difference between $\log |J(1/2+iy)|$ and $\log |J(x+iy)|$, for $1 < x < 2$ we bound the derivative of $\log J$. Thus as

$$\left|\frac{J'}{J}\right| = \left|-\gamma + \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{n+z} - \frac{1}{n}\right)\right| \leq C + \sum_{n=1}^{\infty}\frac{|z|}{n^2}, \quad \text{for } \Re(z) > 0 \ .$$

$$\leq C + \frac{\pi^2}{6}(1+o(1))|\Im(z)| \ ,$$

for $1/2 < \Re(z) < 2$. By integrating the above inequality we have that

$$\log |J(x+iy)| \leq \log |J(1/2+iy)| + \int_{1/2}^{x}\left|\frac{d\log J|}{dt}\right| dt$$

$$\leq C + \frac{\pi|y|}{2} + \frac{3\pi^2}{12}|y| < C + 4.5|y|$$

for $1 < x < 2$. Thus $\log |H(z)| < C + 4.5..|\Im(z)|$. As $4.5.. < 2\pi$ we have that H is constant. Hence $\Gamma(z) = F(z)$.

8.3 Riemann Zeta Function

Riemann found the fundamental connection of the Zeta function with the most important questions of number theory which is why the Zeta function bears his name rather than Euler's. The Zeta function is

$$\zeta(z) = \sum_{n=1}^{\infty}\frac{1}{n^z} \ .$$

LEMMA 23 *$\zeta(z)$ is defined and analytic for $\Re(z) > 1$.*

ζ is defined and analytic for $\Re(z) > 1$ by the Weierstrass M-test. Actually this can be improved to

LEMMA 24 *$\zeta(z) - \frac{1}{z-1}$ is extends analytically to $\Re(z) > 0$.*

Remarks: It can shown that $\zeta(z) - 1/(z-1)$ is entire (see Ahlfors).

For $\Re(z) > 1$,

$$\zeta(z) - \frac{1}{z-1} = \sum_{n=1}^{\infty} \frac{1}{n^z} - \int_1^{\infty} \frac{dx}{x^z} = \sum_{n=1}^{\infty} \left\{ \frac{1}{n^z} - \int_n^{n+1} \frac{dx}{x^z} \right\} .$$

Thus as

$$\left| \int_n^{n+1} \left\{ \frac{1}{n^z} - \frac{1}{x^z} \right\} dx \right| \leq \max_{n \leq x \leq n+1} \left| \frac{1}{n^z} - \frac{1}{x^z} \right| \leq \max_{n \leq x \leq n+1} \left| \frac{z}{x^{z+1}} \right| = \frac{|z|}{n^{\Re(z)+1}} .$$

So the M-test again shows the series is analytic for $\Re(z) > 0$.

The next result is due to Euler who proved the product formula:

LEMMA 25 *For $\Re(z) > 1$,*

$$\frac{1}{\zeta(z)} = \prod_{n=1}^{\infty} \left(1 - \frac{1}{p_n^z} \right)$$

where p_n is the nth prime.

Remarks: We set $p_1 = 2, \ldots$.

To prove this we use the unique factorisation of numbers into the product of primes (done in Euclid 2200 years ago). Thus $n = p_1^{m_1} \ldots p_k^{m_k}$, for nonnegative integers $m_1, \ldots$ unique up to order.

$$\zeta(z) = \sum_{0 \leq m_1 \leq m_2} (p_1^{m_1} p_2^{m_2})^{-z} = \prod_{n=1}^{\infty} \left(\sum_{k=0}^{\infty} p_n^{-kz} \right) = \prod_{n=1}^{\infty} \left(\frac{1}{1 - p_n^{-z}} \right)$$

for $\Re(z) > 1$.

We observe that the product formula shows that ζ has no zeros for $\Re(z) > 1$. Next we show that $\zeta(z)$ has no zeros for $\Re(z) \geq 1$. This uses a clever trick of Hadamard. There are many functions associated with the zeta function. We make special use of

$$\Theta(z) = \sum_{n=1}^{\infty} \frac{\log(p_n)}{p_n^z} .$$

Now by the product formula

$$-\frac{\zeta'(z)}{\zeta(z)} = \sum_{n=1}^{\infty} \frac{\log(p_n)}{p_n^z - 1} = \sum_{n=1}^{\infty} \left\{ \frac{1}{p_n^z} + \frac{1}{p_n^z(p_n^z - 1)} \right\} \log(p_n)$$

$$= \Theta(z) + \sum_{n=1}^{\infty} \frac{\log(p_n)}{p_n^z(p_n^z - 1)} \ .$$

Now a simple estimate which shows the second term is bounded for $\Re(z) > 1/2$ we see that $\Theta(z)$ is meromorphic for $\Re(z) \geq 1$, analytic except for a pole of order 1 with residue 1 at $z = 1$, and poles of order 1 at the zeros of $\zeta(z)$.

LEMMA 26 $\Theta(z)$ *has exactly one pole (at* $z = 1$*) in* $\Re(z) \geq 1$*.*

Suppose that $\zeta(z)$ has a zero of order N at $z = 1 + iy$, and order M at $1 + 2iy$. Then $\Theta(z)$ has a simple pole with residue - N at $z = 1 + iy$, and residue - M at $1 + 2iy$. Therefore

$$\lim_{t \to 0+} t\Theta(1 + t) = 1, \lim_{t \to 0+} t\Theta(1 \pm iy + t) = -N, \lim_{t \to 0+} t\Theta(1 \pm 2iy + t) = -M$$

Now by the binomial formula

$$\Theta(1 - 2iy + t) + 4\Theta(1 - iy + t) + 6\Theta(1 + t) + 4\Theta(1 + iy + t) + \Theta(1 + 2iy + t) =$$

$$\sum_{n=1}^{\infty} \frac{\log(p_n)}{p_n^{1+t}} \left\{ p_n^{-iy/2} + p_n^{iy/2} \right\}^4 > 0 \ .$$

Multiplying by $t \to 0$, we get - M - 4N + 6 - 4N - M ≥ 0. Thus

$$8N + 2M \leq 6 \ .$$

If say N is 1, 2,.. this is impossible. Therefore N = 0.

Exercises

1. Prove that
$$\sum_{n=1}^{\infty} \frac{1}{p_n} = \infty$$

2. What is the value of $\zeta(2)$

8.4 Prime Number Theorem

It is in number theory that the Zeta function achieves its true significance. Gauss on the evidence of thousands of computations guessed the prime number conjecture, that $\pi(x) = \#\{primes \le x\}$ is asymptotic to $x/\log(x)$, i.e.

$$\lim_{x\to\infty} \frac{\pi(x)\log(x)}{x} = 1 \; .$$

Riemann introduced the use of the Zeta function and his famous conjecture that the nontrivial zeros lie on the axis $\Re(z) = 1/2$. This conjecture is still open. In 1896 Hadamard and de la Vallee -Poussin, separately, proved the prime number conjecture. The proofs were regarded as the most difficult ever constructed. The proofs used complex analysis of the Zeta Function and Number Theory, the essential difficulty was the use of a "Tauberian" theorem. (These are related to Abel's second theorem). This was simplified by Weiner in the 30's. However in the 80's D. C. Newman came up with an easy proof which has now been simplified the number theory to the point that it is trivial. Thus we have now a truly quite elementary proof.

We use the functions

$$\Theta(z) = \sum_{n=1}^{\infty} \frac{\log(p_n)}{p_n^z} \; , \text{ and } \phi(x) = \sum_{p_n \le x} \log(p_n)$$

For $\Re(z) > 1$ we have

$$\Theta(z) = \int_1^\infty \frac{d\phi(x)}{x^z} = z\int_1^\infty \frac{\phi(x)}{x^{z+1}}\,dx = z\int_0^\infty e^{-zt}\phi(e^t)\,dt$$

and we may apply the following Tauberian result (a sort of converse to Abel's second theorem). This will be proved in the next section.

THEOREM 38 *Let $f(t)$ be bounded on $\{t > 0\}$ and suppose that the function*

$$g(z) = \int_0^\infty f(t)e^{-zt}\,dt, \Re(z) > 0 \; ,$$

extends analytically to $\Re(z) \ge 0$. Then

$$\lim_{T\to\infty} \int_0^T f(t)\,dt = g(0) \; .$$

This is applied to the functions

$$f(t) = \phi(e^t)e^{-t} - 1, \ g(z) = \frac{\Theta(z+1)}{z+1} - \frac{1}{z}$$

provided we can show that $f(t)$ is bounded which comes from some simple number theory:

LEMMA 27 *(Tchebyshev)* $\phi(x) \leq Cx$

For $n = 1, \ 2, ...$

$$2^{2n} = (1+1)^{2n} = \binom{2n}{0} + \ ... \ + \binom{2n}{2n} \geq \binom{2n}{n} \geq \prod_{n \leq p_k \leq 2n} p_k = e^{\phi(2n)-\phi(n)}$$

Hence $\phi(2n) - \phi(n) \leq 2n \log(2)$ for all n. Summing over n, n/2, n/4,... gives

$$\phi(n) < 2n \log(2), \ n = 2^k \ .$$

Now for any x not a power of 2 there is a $n = 2^k$ so that $n < x < 2n$, and $\phi(x) < \phi(2n)$. Hence $f(x) < 2^{k+1} \log 2 < Cx$ where $C = 4 \log(2)$.

Thus we get that

$$\int_1^\infty \frac{\phi(x) - x}{x^2} \, dx$$

is a convergent integral which will show that $\lim_{x \to \infty} \phi(x)/x = 1$.

Assume on the contrary that there is some $q > 1$ so that for some large values of $x, \ \phi(x)/x > q$. Now

$$\int_x^{qx} \frac{\phi(t) - t}{t^2} \, dt \geq \int_x^{qx} \frac{qx - t}{t^2} \, dt = \int_1^q \frac{q-t}{t^2} \, dt = c > 0$$

contradicting the convergence of the integral. Similarly if there is some $0 < q < 1$ so that for some large values of $x, \ \phi(x)/x < q$. Then

$$\int_{qx}^{x} \frac{\phi(t) - t}{t^2} \, dt \leq \int_{qx}^{x} \frac{qx - t}{t^2} \, dt = \int_q^1 \frac{q-t}{t^2} \, dt = c < 0$$

again a contradiction.

This easily yields the prime number theorem as

$$\phi(x) = \sum_{p_n \le x} \log(p_n) \le \sum_{p_n \le x} \log(x) = \pi(x)\log(x)$$

and

$$\phi(x) \ge \sum_{x^{1-\epsilon} \le p_n \le x} \log(p_n) \ge \sum_{x^{1-\epsilon} \le p_n \le x} (1-\epsilon)\log(x) \ge (1-\epsilon)\log(x)\{\pi(x) - Cx^{1-\epsilon}\}$$

It remains to prove the Tauberian Theorem:

8.5 A Tauberian Theorem

We define $g_T = \int_0^T f(t)e^{-zt}\,dt$ and show $\lim_{T\to\infty} g_T(0) = g(0)$.

Let K be the positively oriented boundary of

$$\{z : |z| < R, \Re(z) > -r\},$$

R is large positive and r small and positive so that g analytic in and on K. Then by Cauchy's theorem

$$g_T(0) - g(0) = \frac{1}{2\pi i}\int_K (g_T(z) - g(z))e^{Tz}\left(1 + \frac{z^2}{R^2}\right)\frac{dz}{z}$$

The contribution from the part $K+$ of K in $\{\Re(z) > 0\}$ is bounded by

$$\begin{aligned}
&\frac{1}{2\pi}\int_{K+} |g_T(z) - g(z)|e^{T\Re(z)}\left|\frac{1}{z} + \frac{z}{R^2}\right||dz| \\
&\le \frac{1}{2\pi}\int_{K+} |\int_T^\infty f(t)e^{-zt}dt|e^{T\Re(z)}\left|\frac{2\Re(z)}{R^2}\right||dz| \\
&\le \frac{1}{2\pi}\int_{K+} \left|\int_T^\infty C|e^{-zt}|dt\right| e^{T\Re(z)}\left|\frac{2\Re(z)}{R^2}\right||dz| \\
&\le \frac{1}{2\pi}\int_{K+} \frac{C}{\Re(z)}e^{-T\Re(z)}e^{T\Re(z)}\left|\frac{2\Re(z)}{R^2}\right||dz| \\
&\le \frac{C}{R}
\end{aligned}$$

We now evaluate the contribution from $K- = \{z \in K, \Re(z) < 0\}$. Now as $g_T(z)$ is entire we have

$$\frac{1}{2\pi}\int_{K-} g_T(z)e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z} = \frac{1}{2\pi}\int_{K*} g_T(z)e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z}$$

where $K* = \{Re^{i\theta} : \pi/2 < \theta < 3\pi/2\}$. Similarly to above

$$\left|\frac{1}{2\pi i}\int_{K*} g_T(z)e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z}\right|$$

$$\leq \frac{1}{2\pi}\int_{K*} |\int_0^T f(t)e^{-zt}dt| e^{T\Re(z)}\left|\frac{2\Re(z)}{R^2}\right||dz|$$

$$\leq \frac{1}{2\pi}\int_{K*}\int_{-\infty}^T Ce^{-\Re(z)t}dte^{T\Re(z)}\left|\frac{2\Re(z)}{R^2}\right||dz| \leq \frac{C}{R}$$

Thus for any $\epsilon > 0$ we choose R so that $\epsilon < 2C/R$ and find $r > 0$ so that

$$|g_T(0) - g(0)| \leq \epsilon + \left|\frac{1}{2\pi i}\int_{K-} g(z)e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z}\right|,$$

independently of T. Letting $T \to \infty$ we see that the integral tends to zero as $e^{Tz} \to 0$ in $L^1(K^*)$ as $K^- \subset \{\Re(z) < 0\}$.

Chapter 9

Normal Families

We now introduce a very important tool of analysis: the notion of a compact family of functions. French mathematicians such as Montel, Frechet formalised the idea, sometime around 1905. It may be regarded as the first tool of functional analysis, useful in real analysis as well as complex analysis. The original presentation predates the definition of a topological space, Banach or Hilbert space by several decades.

9.1 Normal Convergence

Recall our definition of normal convergence *in the Euclidean metric*(i.e. uniform convergence on compact subsets):

A sequence f_n of functions on a domain $W \subset \mathbb{C}$ converges normally to f, a function on W, if for every compact $E \subset W$, we have

$$\lim_{n\to\infty} \sup_E |f_n(z) - f(z)| = 0 \ .$$

We have already seen that normal convergence arises naturally in the representation of analytic functions by infinite series. Also we proved

If a sequence f_n of analytic functions on a domain $W \subset \mathbb{C}$ converges normally to f, a function on W, then f is analytic.

Our proof used Cauchy's formula. Not only does f_n converge but so do all its derivatives. A result we haven't proved before is due to Hurwitz:

LEMMA 28 *If a sequence f_n of functions on a domain $W \subset \mathbb{C}$ converges normally to f, a nonconstant function on W, then every zero ζ of f is the limit of the zeros of f_n, i.e. if ζ has multiplicity m then for every (small) $r > 0$ there is a N such that for $n > N$, f_n has exactly m zeros(counting multiplicity) in the disk $\{z : |z - \zeta| < r\}$.*

As f is not identically zero the zeros of f are isolated. For any zero ζ there is $r > 0$ so that f is nonzero for $\{z : 0 < |z - \zeta| < r\}$. "It follows" that $1/f_n$ converges uniformly to $1/f$ on $\gamma : z = \zeta + r\exp(it), 0 < t < 2\pi$. Also $f_n'(z)$ converges uniformly to $f'(z)$ in γ. Thus

$$\frac{1}{2\pi}\int_\gamma \frac{f_n'(z)}{f_n(z)}\,dz \to \frac{1}{2\pi}\int_\gamma \frac{f'(z)}{f(z)}\,dz\ ,$$

i.e. by the argument principle, # zeros of f_n inside γ converges to # zeros of f inside γ.

For example, if a sequence of functions conformal on a domain D is normally convergent then the limit is constant or a conformal mapping of D : for fixed $w \in D$, $f_n(z) - f_n(w)$ has no zeros on $D - \{w\}$ as f_n is 1:1. So if the limit is nonconstant $f(z) - f(w)$ has no zeros on $D - \{w\}$. Thus the limit is 1:1.

Exercises

1. Prove that a convergent power series

$$\sum_{k=0}^{\infty} a_k z^k$$

converges normally inside its circle of convergence.

2. Find a disk R_n so that the polynomial

$$p_n(z) = 1 + z + .. + \frac{z^n}{n!}$$

has no zeros inside $\{|z| < R_n\}$ with $\lim_{n\to\infty} R_n = \infty$.

3. Prove that
$$\lim_{n\to\infty} \sum_{-N}^{N} \frac{1}{z-\pi n} = \cot(z)$$
normally on $\mathbb{C}$ in the spherical metric.

9.2 Normal Families

Let R be a fixed region in the complex plane. There are two commonly used approaches to normal families. We begin with the classical approach which makes no distinction for ∞, interestingly enough the older approach is most useful in the most modern theory of Riemann surfaces as well as Complex Dynamics. The other approach is from functional analysis and uses the vector space $H(R)$ which consists of all functions analytic on R. For fixed R we abbreviate this to H. The corresponding space of meromorphic functions is denoted by M. The classical definition is:

DEFINITION 14 *A sequence $f_n \in M(R)$ is normally convergent if there exists a function $f \in M(R) \cup \{\infty\}$ so that in the spherical metric*
$$\sigma(f_n(z),\, f(z)) \to 0$$
uniformly on every compact subset of R.

In analysis the following is sometimes called precompact:

DEFINITION 15 *A family $\mathcal{F} \subset M(R)$ is normal if for every sequence $f_n \in \mathcal{F}$ there is a normally convergent subsequence.*

Note that we do not demand that the limit $f \in \mathcal{F}$. However the case where the limit belongs to $\mathcal{F}$ is important:

DEFINITION 16 *A family $\mathcal{F} \subset M(R)$ is closed if every sequence $f_n \in \mathcal{F}$ which converges normally has a limit $f \in \mathcal{F}$.*

For example the family $\{f(z) = z + \tau : \tau \in \mathbb{C}\}$ is closed in $H(\mathbb{C})$ but not normal. However it is normal as a subset of $M(\mathbb{C})$ but not closed. Finally we say

DEFINITION 17 *A family $\mathcal{F} \subset M(R)$ is compact if it is normal and closed.*

Exercise

1. Prove that a family $\mathcal{F} \in H(R)$ (i.e. analytic but not meromorphic) is normal iff every sequence $f_n \in \mathcal{F}$ has a subsequence $f_{n(k)}$ which converges (in the euclidean metric) to some analytic function (uniformly on compact subsets) OR $1/f_{n(k)} \to 0$ converges uniformly (in the euclidean metric) on compact subsets.

2. Suppose that $f_n \in M(R)$ converges normally to $f \in H(R)$. Prove that the set of poles P_n (of f_n) tend to the boundary, i.e. given any compact subset $K \subset R$ there exists N so that $P_n \cap K = \emptyset$ for $n > N$.

3. Some "famous" families of analytic functions, all defined on the unit disk $\mathbb{D} = \{z : |z| < 1\}$, we shall meet are:

   (a) $\mathcal{B}_0 = \{f \in H : \ f(0) = 0 \text{ and } |f(z)| < 1, \ \forall z \in \mathbb{D}\}$,

   (b) $\mathcal{B} = \{f \in H : \ |f(z)| < 1, \ \forall z \in \mathbb{D}\}$,

   (c) $\mathcal{P} = \{g \in H : \ g(0) = 1 \text{ and } \Re\{g(z)\} > 0, \ \forall z \in \mathbb{D}\}$,

   (d) $\mathcal{S} = \{f \in H : \ f(0) = 0, \ f'(0) = 1 \text{ and } f \text{ is } 1:1 \text{ on } \mathbb{D}\}$,

The conditions $f(0) = 0$ etc are normalisations which ensure that the family will be closed in the topology to be defined. Some of these families are related, e.g families (a) and (c) by $g = (1 + f)/(1 - f)$. Discuss whether the examples are closed in the relevant H or M.

9.3 Frechet space H(R)

A different approach is to use ideas of functional analysis. We first consider $H(R)(= H)$ as an object in itself. Evidently H is a infinite dimensional vector space over the complex scalars. The definition of closed sets in terms of normal convergence defines some sort of topology on H. One next asks if there is a norm on H which gives this topology. In fact there are many different norms on H. For example let D_k be closed disks in R with union equal to R. One norm would be $||f||_k = \sup\{|f(z)| : \ z \in D_k\}$. These do not even give a complete normed space. In fact no norm can generate the normal topology on H. This is because the closed unit ball in a normed space is compact iff. the space is finite dimensional. However it turns out that H has a metric. For we define the metric

$$d(f,\, g) = \sum_{k=1}^{\infty} \frac{||f-g||_k\, 2^{-k}}{1+||f-g||_k}.$$

In the exercises it is shown that this metric generates the normal topology. Curiously such Frechet metrics predate the abstract idea of a Banach or Hilbert Space. There is nothing unique about this metric, there are other ways which are equivalent to the normal topology. Observe that these are "linear" spaces. (However in many important examples we need to include functions taking the value ∞. For these cases the classical definition is best.)

Exercises

1. Prove that d is a metric on H .

2. Prove that (H, d) is a complete metric space.

3. Prove that a set $\mathcal{F} \subset H$ is closed in the normal sense iff. $\mathcal{F}$ is closed in (H, d).

4. Let A_p be the space of functions f analytic on a domain R with finite norm :

$$||f||_p = \left(\int\int_R |f(x+iy)|^p\, dx\, dy \right)^{1/p},$$

   where $1 \geq p < \infty$ (and if $p = \infty$ we use the uniform norm). Prove that A_p is a Banach space for $p \geq 1$ and a Hilbert space for $p = 2$.

5. Suppose that $\#\{\mathbb{C} - R\} < \infty$ Prove that $A_p = \{0\}$.

6. Prove $\mathcal{U} = \{f \in A_p : ||f||_p \leq 1\}$ is a normal family.

9.4 Equicontinuous families

Let R be a region of the plane. We give necessary and sufficient conditions that a family $\mathcal{F}$ of complex valued continuous functions on R to be "normal". First we have:

DEFINITION 18 *Let E be a compact subset of the plane. A family $\mathcal{F}$ is equicontinuous if for every $\varepsilon > 0$ there exists a $\delta > 0$ so that for any $z, w \in E$, $|z - w| < \delta \Rightarrow |f(z) - f(w)| < \varepsilon$, $\forall f \in \mathcal{F}$.*

Examples

1. Let $E = [0, 1]$ and let $\mathcal{F}$ be the class of (real) functions f on E with $|f'(x)| \leq 1$, $\forall x \in E$. Then $\mathcal{F}$ is equicontinuous.

2. Let $K(x, y)$ be continuous in $[0, 1] \times [0, 1]$ and let $\mathcal{F}$ be the class of

$$f(x) = \int_0^1 K(x, y) g(y) dy ,$$

for g continuous on $[0, 1]$ with $\sup_{[0,1]} |g(y)| \leq 1$. Then $\mathcal{F}$ is also equicontinuous.

DEFINITION 19 *The family $\mathcal{F}$ is bounded if there exists a $M > 0$ so that $\sup_E |f(z)| < M$, $\forall f \in \mathcal{F}$.*

Observe that the second example is bounded but the first was not.

As a preliminary result we prove the theorem of Arzela and Ascoli's ("Arzela-Ascoli") :

THEOREM 39 *A family $\mathcal{F}$ of complex valued continuous functions defined on a compact set E is relatively sequentially compact in the uniform norm if and only if:*

1. *$\mathcal{F}$ is equicontinuous,*

2. *$\mathcal{F}$ is bounded.*

First we show that these conditions are necessary:

(i) If $\mathcal{F}$ is not equicontinuous then there exists $\epsilon > 0$ and a sequence of points $z_n, w_n \in E$ and functions $f_n \in \mathcal{F}$ so that $|z_n - w_n| \to 0$ but :
(1) $|f_n(z_n) - f_n(w_n)| > \varepsilon$. Without loss of generality, and avoiding multiple subindices by relabelling if necessary, as E is compact and $\mathcal{F}$ is normal,

we assume that there is a point $z \in E$ and a function f continuous on E so that $z_n,\ w_n \to z$ and $f_n \to f$ uniformly on E. As f is uniformly continuous, there is N so that $n > N\ :\ $ (2) $|f(z_n) - f(w_n)| < \varepsilon/3$. Then for large enough n, by uniform convergence (3) $|f_n(z_n) - f(z_n)| < \epsilon/3$, and (4) $|f_n(w_n) - f(w_n)| < \varepsilon/3$. Thus by applying the triangle inequality to (2), (3), (4)

$$|f_n(z_n) - f_n(w_n)| \leq |f_n(z_n) - f(z_n)| + |f(z_n) - f(w_n)| + |f_n(w_n) - f(w_n)| < \varepsilon\ .$$

This contradicts inequality (1).

(ii) If $\mathcal{F}$ is not bounded then there exists a sequence of points $z_n \in E$ and functions $f_n \in \mathcal{F}$ so that (5) $|f_n(z_n)| \to \infty$. Without loss of generality, as E is compact and $\mathcal{F}$ is normal, we assume that there is a point $z \in E$ and a function f continuous on E so that $z_n \to z$ and $f_n \to f$ uniformly on E. As f is uniformly continuous, there is $N :$ (6) $|f(z_n)| < N$. Then for large enough n, by uniform convergence (7) $|f_n(z_n) - f(z_n)| < N$. Thus by the triangle inequality and (6), (7)

$$|f_n(z_n)| \leq |f_n(z_n) - f(z_n)| + |f(z_n)| < 2N$$

This contradicts inequality (5).

Next we prove that conditions (i) and (ii) are sufficient. Given a sequence $f_n \in \mathcal{F}$ we produce a subsequence uniformly convergent on E. Let $z_k \in E$ be an everywhere dense countable set we call G. We use $n(k, j)$ to denote an array of positive integers , $j = 1, 2..$ and $k = 1, 2..$, constructed so that (8) $n(k, j) < n(k, m)$ for $j < m$ and $k = 1, 2....$ For fixed k we construct (9) $n(k+1, j)$ as a subsequence of $n(k, j)$. This is formed by letting $n(1, j)$ be a subsequence of n so that the sequence $f_{n(1,j)}(z_1)$ converges as $j \to \infty$. This is possible as f_n is bounded. Suppose that $n(k, j)$ has been chosen to satisfy (8), (9) as well as (10) $f_{n(k,j)}(z_k)$ converges to say $f(z_k)$, as $j \to \infty$. Then there exists a subsequence $n(k + 1, j)$ of $n(k, j)$ so $f_{n(k,j)}(z_{k+1})$ converges as $j \to \infty$. This defines the array by induction. Now a subsequence $n(j)$ is chosen to be the diagonal sequence of the array: (11) $n(j) = n(j, j)$. So $n(j)$ is eventually a subsequence of each row $n(k, j)$. Thus for each k we have that $f_{n(j)}(z_k) \to f(z_k),\ j \to \infty$. Now we observe that f is uniformly continuous on the set G. Next we define a function f on $E - G$. Now for any $z \in E$, we obtain a sequence $z_k \in G$ converging to z. As f is uniformly continuous on G we see that $f(z_k)$ has a unique limit which we call $f(z)$. Hence f is uniformly continuous on E. Finally we show uniform convergence of $f_{n(j)}(z)$ to $f(z)$. Fix $\varepsilon > 0$. Then there is $\delta > 0$ so that if $|z - w| < \delta$ then

$$|f_n(z) - f_n(w)| < \varepsilon/3, |f(z) - f(w)| < \varepsilon/3$$

for all n. As E is compact it has a finite covering by disks $\{z : |z - z_k| < \delta\}$. As there are only a finite number of k, there is N so that $j > N$ imply that for $n = n_j$

$$|f_n(z_k) - f(z_k)| < \varepsilon/3 \ .$$

Now for any $z \in E$ there is a z_k so that $|z - z_k| < \delta$. Thus for $j > N, \ n = n(j)$

$$|f_n(z) - f(z)| \leq |f_n(z_k) - f(z_k)| + |f(z_k) - f(z)| + |f_n(z) - f_n(z_k)| < \varepsilon \ .$$

This proves the theorem.

Exercises

1. Let $\phi : \mathbb{R} \to \mathbb{R}$ be continuous. Define the family

$$\Phi = \{\phi(x + t) : t \in \mathbb{R}\} \ .$$

Show that Φ is equicontinous if and only if ϕ is uniformly continuous.

2. Let $\phi : \mathbb{R} \to \mathbb{R}$ be continuous increasing (homeomorphism). It is said to be quasisymmetric if there is a constant k:

$$\frac{\phi(x + t) - \phi(x)}{\phi(x) - \phi(x - t)} \leq k \ , \forall x \in \mathbb{R}, \ t > 0$$

What does this mean.

3. For real constants $a, \ b$ define $\phi(x) = e^{iax} + e^{ibx}$, and the family

$$\Phi = \{\phi(x + t) : t \in \mathbb{R}\} \ .$$

Show Φ is closed if and only if $a, \ b$ are commensurate.

9.5 Montel's Theorem

We now apply the previous to the case of $\mathcal{F} \subset H(R)$.

DEFINITION 20 *A family $\mathcal{F} \subset H(R)$ is (locally) bounded if for every relatively compact $E \subset R$ there is a positive number $M = M(E)$:*

$$\sup_E |f(z)| < M, \ \forall f \in \mathcal{F}.$$

Obviously our example $\mathcal{B}_0$ is bounded (uniformly). In fact by Schwarz lemma $|f(z)| \leq |z|$, for any z in the unit disk. Now for $g \in \mathcal{P}$, using the representation $g = (1+f)/(1-f),\ f \in \mathcal{B}_0$ yields

$$|g(z)| \leq \frac{1+|z|}{1-|z|},\ |z| \leq 1 \ .$$

Thus $\mathcal{P}$ is locally bounded.

Our main result is a result by Montel.

THEOREM 40 *A family $\mathcal{F} \subset H(R)$ is normal iff it is (locally) bounded.*

Remarks: i.e. precompactness for uniform convergence on compact subsets. For classical normality this theorem is only a sufficient condition.

Let D_k be a countable collection of disks $\{z : |z - z_k| \leq r_k\} \subset R$ so that $\cup_k \{z : |z - z_k| \leq r_k/2\} = R$. Then for points $z \in \{z : |z - z_k| \leq r_k\}$, by Cauchy's integral theorem for f analytic on R

$$f'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(z-w)^2}\, dw \ ,$$

with $\gamma = z_k + r_k e^{it}$, $0 < t \leq 2\pi$. So $\mathcal{F}$ is bounded by M_k on D_k gives

$$|f'(z)| \leq \frac{4M_k}{r_k} \ , \forall z : |z - z_k| \leq r_k/2$$

i.e. the derivative is locally bounded too. Thus $\mathcal{F}$ is equicontinuous on $\{z : |z - z_k| \leq r_k/2\}$. So by Arzela-Ascoli any $f_n \in \mathcal{F}$ has a convergent subsequence. Using the same diagonal trick as before we then obtain a subsequence

$f_{n(j)}$of the f_n which is uniformly convergent on every $\{z : |z - z_k| \le r_k/2\}$. Since any compact $E \subset R$ is covered by finitely many $\{z : |z - z_k| \le r_k/2\}$ this implies $f_{n(j)}$ converges uniformily on every compact subset, i.e. we have normal convergence. The converse immediately follows from Arzela-Ascoli, i.e. "normal $\Rightarrow$ locally bounded".

COROLLARY 4 *$\mathcal{F} \subset H(R)$ is compact iff it is (locally) bounded and closed.*

In many applications one wishes to to allow "uniform convergence to infinity". There are two approaches to this. One is to use the Riemann sphere$\mathbb{S}$ with the spherical metric ρ. One can repeat all the previous arguments using ρ instead of the euclidean metric and show that equicontinuity with repect to ρ is necessary and sufficient. However one is really interested in considering meromorphic functions f on R, i.e. f is analytic except for isolated singularities of finite order. As above one can prove:

THEOREM 41 *(Marty) A family $\mathcal{F} \subset M(R)$ is normal with repect to the spherical metric if and only if for every compact $E \subset R$*

$$\sup_E \frac{|f'(z)|}{1 + |f(z)|^2} < K(E) < \infty,$$

uniformly over $\mathcal{F}$.

Notice that boundedness in the spherical metric certainly doesn't give compactness!

Exercises

1. Find all domains D where the family $f_n = e^{nz}$ is normal (in $H(D)$ or $M(D)$).

2. Discuss whether the examples of 8.2 are normal in the relevant H or M.

3. Let f be analytic on the unit disk. Consider the family

$$\mathcal{F} = \{f \circ g \text{ , for any bilinear } g : \mathbb{D} \to \mathbb{D}\} \text{ .}$$

Show that $\mathcal{F}$ is normal in $H(\mathbb{D})$ if and only if there is a constant K :

$$\frac{|f'(z)|}{1-|z|^2} \leq K, \ \forall z \in \mathbb{D} \ .$$

4. Prove that a normal family $\mathcal{F}$ of rational functions has degree $deg(p) < N$ for some N. Is the converse true? (Hint: by Hurwitz the functional $f \to deg(f)$ is continuous).

9.6 Normal region

Instead of considering if some family is normal on a given region we consider which region a given family will be normal on. For example in Newton-Raphson we seek to solve $f(\zeta) = 0$ for a given rational function f by first defining an auxilary function

$$F(z) = z - \frac{f(z)}{f'(z)} \ .$$

We guess a value z_0 then iterate by setting $z_{n+1} = F(z_n)$. Equivalently we consider the family of nth iterates $F^n(z) = F \circ ... \circ F(z)$. The sequence $z_n = F^n(z_0)$ will converge to ζ at least when z_0 is close enough to ζ. The problem is to ascertain for which values of z_0 will there be convergence. Once we require stability in convergence with respect to the initial choice we are led to the region of normality of F^n, i.e. the largest open set where the iterates F^n form a normal family (in the spherical metric). This set is named after its discoverer:

DEFINITION 21 *(Fatou and Julia Sets) Let f be rational. The Fatou set $\mathcal{F}$ is the largest open set upon which the iterates f^n form a normal family. Its complement, necessarily closed, is the Julia set $\mathcal{J}$.*

Examples:

It is easy to see that the Julia set of z^n is $|z| = 1$. In fact any rational f , of $degf > 1$, has nonempty Julia set. (Otherwise the family f^n is normal on the sphere, thus the $degf$ being continuous is bounded whereas $degf^n =$

$N^n \to \infty$.) For polynomials $\mathcal{J}$ is compact (as $f^n(z) \to \infty$, for $|z| > R$). On the other hand there are rational functions such that $\mathcal{J} = \mathbb{S}$. It is a famous conjecture of Fatou as to whether there exist f with $\mathcal{J}$ of positive area but not equal to the sphere (Sullivan, Shishikura constructed examples with dimension 2).

Exercises

1. Let $f(z) = \lambda z + ...$ be analytic in $|z| < R$. If $|\lambda| < 1$ show there is $r > 0$ so that $g_n = f^n/\lambda^n \to g$ uniformly on $|z| < r$.

2. Let $f(z) = \lambda z + ...$ be analytic in $|z| < R$. If $|\lambda| < 1$ show there is $r > 0$ and $g = z + ..$ conformal near 0 with

$$g \circ f \circ g^{-1} = \lambda z, \quad |z| < r \; .$$

3. Let $f(z) = \lambda z + ...$ be analytic in $|z| < R$. If $|\lambda| > 1$ show there is no $r > 0$ so that $g_n = f^n \to g$ uniformly on $|z| < r$.

4. Suppose that $f = a_n z^n + ... + a_0$ has real distinct roots, a_n real. Let $F = z - f/f'$. Show that the Julia set of F is a totally disconnected subset of $\mathbb{C}$

Chapter 10

Conformal Mappings

We now study functions which are analytic and 1:1 on a domain D. These are called conformal mappings because of the property of preserving angles between curves. Conformal mappings were used by Gauss in his work on orthogonal coordinate systems on arbitrary surfaces. Riemann discovered the connection with boundary value problems. For instance we teach undergraduate engineers how to solve problems in fluid flow by using conformal mapping. Today numerical analysts use conformal maps to set up orthogonal co-ordinate systems. A final application which lies very deep is the use of the space of all conformal maps, so called Teichmüller Space, in Grand Unified Theory for fundamental physics (Witten, Field's Medal, 1990).

Notation: Conformal maps are also referred to as being: simple, schlicht, univalent, biholomorphic. The unit disk $\mathbb{D} = \{|z| < 1\}$.

10.1 Riemann Mapping Theorem

The climax of this course is the famous mapping theorem:

THEOREM 42 *(Riemann) Let R be a simply connected domain in $\mathbb{C}$, $R \neq \mathbb{C}$. Let $\zeta \in R$. Then there is a unique conformal map f from $\mathbb{D}$ onto R such that $f(0) = \zeta$, $f'(0) > 0$.*

Riemann thought he proved the existence of f. Hilbert (1900) gave this as one of the 23 major problems of mathematics. Proof by and Koebe in 1907. (Actually Koebe and Poincaré simultaneously proved the generalisation to simply connected Riemann surfaces called the Uniformisation Theorem). In fact the American mathematician Osgood essentially proved the Riemann Mapping Theorem a few years earlier but did not realise it. The main feature of the theorem is that the restriction on R is almost purely topological (plus the requirement that $\mathbb{C} - R$ be nonempty). In higher dimensions there are extra geometric requirements on the boundary to make something like the RMT true.

The simplest conformal maps are the bilinear transformations which map the Riemann sphere onto itself. Of particular importance are the transformations

$$T(z) = c\,\frac{z-a}{1-\overline{a}z},\ |a| < 1,\ |c| = 1,$$

which map the unit disk onto itself. We note that these are the only conformal maps of $\mathbb{D}$ onto itself. For if $f(0) = \zeta$ then let $g : \mathbb{D} \to \mathbb{D}$ be the bilinear mapping with $g(\zeta) = 0$. So if $F = g \circ f$ we have $F(0) = 0$. Then by Schwarz lemma $|F'(0)| \leq 1$. But the same applies to F^{-1} so $1/|F'(0)| = 1$. Thus $|F'(0)| = 1$ and so $F(z) = cz$, for some $c, |c| = 1$. Finally : $f = g^{-1} \circ (cz)$ is bilinear too.

We consider elementary conformal maps which will be used later:

1. Let R be a simply connected domain with point $0 \notin R$. Then

$$f(z) = \sqrt{z},$$

   has an analytic branch on R. Furthermore f maps R onto a domain R^* omitting an open disk Δ. (For if $z \in R^*$ then $-z \notin R^*$.)

2. Let R be a simply connected domain omitting a disk with centre w and radius $r > 0$. Then there is a conformal map f which maps R onto a domain $R^* \subset \mathbb{D}$. This is achieved by

$$f(z) = \frac{r}{z-w}.$$

By (1) and (2) we have

LEMMA 29 *For any simply connected domain R with some point $a \notin R$, there is a conformal map f which maps R onto a domain $R^* \subset \mathbb{D}$.*

Thus we shall restrict our attention to a subdomain R of the unit disk. Without loss of generality we assume that $0 \in R$.

Now we define $\mathcal{S}(R)$ to be the subclass of $H(R)$ consisting of functions f univalent on R such that:

1. $f(R) \subset \mathbb{D}$,
2. $f(0) = 0$, and
3. $f'(0) > 0$.

The following result is clear:

LEMMA 30 *$\mathcal{S}(R) \cup \{0\}$ is a compact subclass of $H(R)$.*

We may generate nontrivial members of $\mathcal{S}(R)$ by the following operation. Suppose that some point $a \in \mathbb{D} - R$ then

$$g_1(z) = \sqrt{\frac{z-a}{1-\overline{a}z}}$$

maps R onto a domain $R_1 \subset \mathbb{D}$, with $0 \notin R_1$. Thus

$$g_2(z) = \frac{g_1(z) - g_1(0)}{1 - \overline{g_1(0)}g_1(z)}$$

conformally maps R_1 onto a domain $R_2 \subset \mathbb{D}$, with $g_2(0) = 0$. Finally to ensure that the derivative at 0 is positive we define

$$F_a(z) = \frac{g_2(z)}{g_2'(0)}$$

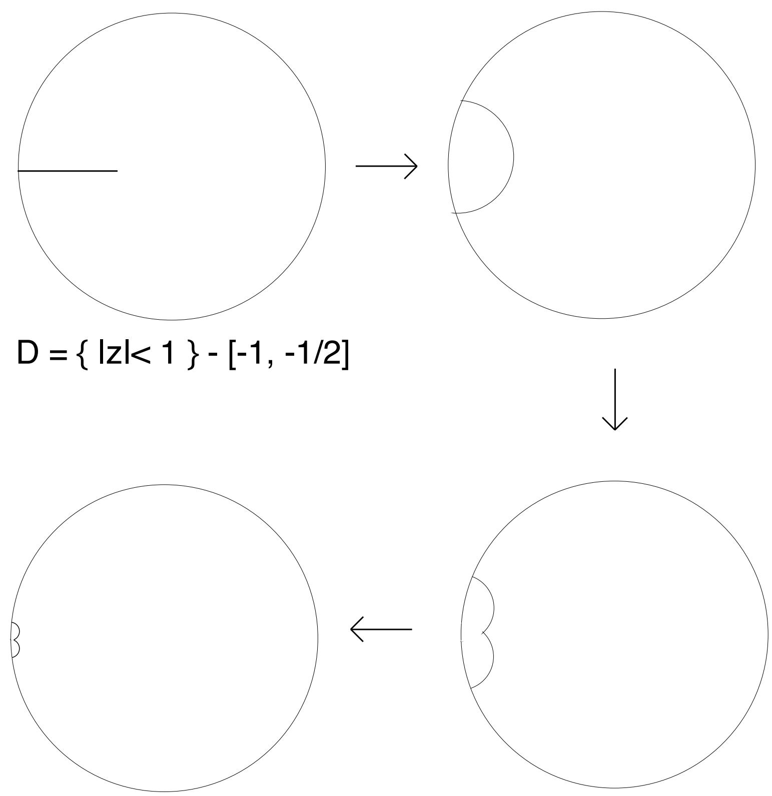
which belongs to $\mathcal{S}(R)$.

It is important to note:

LEMMA 31 *For any point $a \notin \mathbb{D} - R$ we have that $F_a'(0) > 1$.*

Now if $w = F_a(z)$ we observe directly that $z = F_a^{-1}(w)$ is defined and analytic on the unit disk $\mathbb{D}$, also $F_a^{-1}(\mathbb{D}) \subset \mathbb{D}$. Thus by Schwarz Lemma

$$1 > \frac{d}{dw}F_a^{-1}(0) = \left\{\frac{dF_a(0)}{dz}\right\}^{-1} .$$



Iterated Square Root/Bilinear Transforms

We are now ready to prove the Riemann Mapping Theorem. We could used iterated squre root transforms and obtain a sequence of domains $R_n \to \mathbb{D}$. However it is quicker to use a little functional analysis. We consider an extremal problem on $\mathcal{S}(R)$. Define the functional $\Lambda(f) = f'(0)$, and let h be an extremal function maximising $\Re\{\Lambda(f)\}$ over $\mathcal{S}(R)$. Suppose that h does not map R onto $\mathbb{D}$. Then there is some point $a \in \mathbb{D} - h(W)$ and function $F_a \in \mathcal{S}(h(R))$ with $F_a'(0) > 1$. But the function $f(z) = F_a(h(z)) \in$

$\mathcal{S}(R)$ satisfies $f'(0) = F_a'(0)h'(0) > h'(0)$ contradicting the extremality of h. Actually we proved the existence of some conformal map of R onto $\mathbb{D}$. The inverse function g is the function we are looking for.

Finally we prove uniqueness. Suppose that there are functions f and g conformally mapping $\mathbb{D}$ onto R such that $f(0) = \zeta = g(0)$ and $f'(0) > 0, g'(0) > 0$. The function $T = f^{-1} \circ g$ maps $\mathbb{D}$ onto $\mathbb{D}$, and $T(0) = 0,\ T'(0) > 0$. Thus $T = z$

Exercises:

First we consider some issues connected with the Mapping Theorem

1. Let $0 \in R \subset \mathbb{D}$ and $a \in \mathbb{D} - R$. Prove that
$$\min\{|z|\ :\ z \in \mathbb{D} - R\} < \min\{|z|\ :\ z \in \mathbb{D} - F_a(R)\}\ .$$

2. Show for domain $0 \in R \subset \mathbb{D}$ there is a sequence of Square Root transforms G_n so that
$$R \xrightarrow{G_1} R_1 \xrightarrow{G_2} R_2 \to ..R_n \to \mathbb{D}$$

   Next we have some simple consequences of the RMT

3. Any two simply connected domains R and W, both not equal to $\mathbb{C}$, are conformally equivalent.

4. Prove that all conformal maps of the sphere are bilinear.

5. Let g be the Riemann mapping onto a domain $R \neq \mathbb{C}$ be symmetric about the OX axis, and $g(0) = \zeta \in \mathbb{R}$ and $g'(0) > 0$. Prove that all the Taylor coefficients (at 0) of g are real.

   Any symmetry in the image R gives symmetry in the map f:

6. Let R be symmetric about the OX axis, i.e. $z \in R \Rightarrow \overline{z} \in R$, and $\zeta \in \mathbb{R} \cap R$. Then the map $g(z) = \overline{f(\overline{z})}$ is conformal and maps $\mathbb{D}$ onto R. Also $g(0) = \zeta,\ g'(0) > 0$ and so by uniqueness $f = g$.

7. A domain R has "n-fold "symmetry if $w \in R \Rightarrow e^{2\pi i/n} w \in R$.
Let $R \neq \mathbb{C}$ be have "n-fold "symmetry and contain 0 and let f be the conformal mapping of the disk $\mathbb{D}$ onto R so that $f(0) = 0,\ f'(0) > 0$. Show there is a conformal mapping of $\mathbb{D}$ so that

$$f(z) = \{f(z^n)\}^{1/n} \ .$$

10.2 Analytic Boundary Values

Caratheódory proved the general result that a conformal mapping between domains bordered by Jordan curves must extend to a homeomorphism of the boundary. This is by no means obvious: it is easy to find homeomorphisms of the unit disk onto itself which do not extend to a homeomorphism of the closed disk.

However the most important cases are when we have a conformal mapping $f : \Omega \to \mathbb{D}$ where $\partial\Omega$ consists of a finite number $\gamma_1,\ \gamma_n$ of OPEN analytic arcs : i.e. for each j there is a conformal mapping f_j of a nieghborhood U_j of $(-1,\, 1)$ so that $f_j((-1,\, 1)) = \gamma_j$. In fact in most examples the arcs are line intervals or circular subarcs. We use the simple symmetry principles of the previous section to show that f has extension through each γ_j so that f is conformal on some nieghborhood of γ_j.

The simplest case is when γ_j is $(-1,\, 1)$ and $\Omega \subset \{\Im(z) > 0\}$. Consider the domain R obtained by joining Ω to its reflection $\overline{\Omega}$ across $(0,\, 1)$. This is evidently a simply connected domain, symmetric about $(-1,\, 1)$. Let F be the conformal mapping of the unit disk onto R so that $F(0) = 0$ and $F'(0) > 0$. By symmetry (or uniqueness) $F(\overline{z}) = \overline{F(z)}$. In particular, for $H = \{z : |z| < 1, \Im(z) > 0\}$, we have $F(H) = \Omega$. On the other hand there is a conformal mapping of $\mathbb{D}$ onto H, analytic on $|z| = 1$ except at two boundary points which may be chosen to be -1, 1 and map onto -1, 1. (What is it?). Thus we get a map of $\mathbb{D}$ onto Ω whose inverse is analytic on γ_j.

Obviously we require $\Omega \subset \{\Im(z) > 0\}$. This assumption may be removed by a preliminary conformal mapping g which fixes maps $(-1,\, 1)$ to a line interval and Ω to the upper half plane. Choose a simple open arc α in

$\mathbb{C} - cl(\Omega)$ joining -1 to ∞. Then there is a branch g of $(z+1)^{1/n}$, for large n, conformal on $\mathbb{C} - \alpha$ which maps $(-1, \infty)$ to $(0, \infty)$. Thus the extension result holds for $g(\Omega)$ and hence for Ω itself.

Clearly the above result may be generalised to if the γ_j are arbitary line intervals or circular subarcs. To get the final result that the γ_j need only be analytic arcs we use the previous trick of composing with a conformal mapping, this time the f_j itself does the trick.

Although the Riemann Mapping Theorem does not hold in $\mathbb{C}^N$ if f is a biholomorphic mapping between smooth domains in $\mathbb{C}^N$ then f does extend to a smooth homeomorphism of the boundary. (Fefferman, Field's Medal 1978). However a map between the interior of topological spheres need not extend to the boundary !

Exercises

1. Let $\alpha,\ \beta$ be lines meeting at the point $z = 0$ with angle θ. Suppose $\alpha \cup \beta$ is an accessible boundary arc of a simply connected domain R. Prove that the conformal mapping f from R onto the upper half plane is such that $f^{\pi/\theta}$ is analytic at 0.

2. Let Q be a square with sides 0, 1, $1+i$, i. Prove there is a conformal mapping f of the upper half plane $\Im(z) > 0$ onto Q so that -1, 0, 1, ∞ correspond to 0, 1, $1+i$, i. Hence prove that f^{-1} is the restriction of a function $G(z)$ meromorphic on $\mathbb{C}$ with poles of order 2 at $i + 2n + 2mi,\ n, m \in \mathbb{Z}$.

3. Let A be a triangle with (noncollinear)vertices a, b, c ordered positively. Let f be the conformal mapping of the upper half plane so that $0,\ 1,\ \infty$ corresponds to a, b, c. Under what circumstances does the inverse function f^{-1} have analytic continuation to $\mathbb{C}$.

10.3 The Triangle Functions

To finish Part I we sketch the functions called Schwarz triangle functions. We begin with a triangle Δ with vertices p, q, r and angles θ, ϕ, ψ respectively. For the moment we assume that the edges of Δ are straight lines, i.e. we have a Euclidean triangle. Consider now the conformal mapping of $H = \{\Im(z) > 0\}$ onto Δ so that 0, 1, ∞ corresponds to p, q, r respectively. By the reflection principle ennunciated above we reflect Δ in its edges. Each time we reflect in an edge of a triangle this corresponds to reflecting H to $\overline{H}$ (or viceversa) through one of the intervals (0, 1), (1, ∞), (∞, 0). Consider the set of all triangles obtained from Δ by reflection. In general this forms an infinite sheeted covering of the plane winding around all of the vertices. If however there are numbers n, m, k so that $n\theta = \pi, m\phi = \pi$, $k\psi = \pi$ the collection of triangles form a one sheeted covering, i.e. we obtain a tessalation of the plane by triangles. On the other hand we must have $\theta + \phi + \psi = \pi$, so

$$\frac{1}{n} + \frac{1}{m} + \frac{1}{k} = 1 .$$

Thus the only possible sets of angles are

$$\{\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}\}, \{\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}\}, \{\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\} .$$

These three configurations determine Δ uniquely up to a Euclidean transformation.

We now consider the inverse function F of f. Evidently it is well defined on the collection of triangles and the edges, mapping onto $\mathbb{C} - \{0, 1, \infty\}$. In the previous section exercises we show that at each vertice the function is algebraic, i.e. at angle θ the function $F = h^{\pi/\theta}$ where h is conformal at the vertex.In particular in the case that we have a complete tesselation the orders of F at the vertices are

$$\{2, 4, 4\}, \{2, 3, 6\}, \{3, 3, 3\} .$$

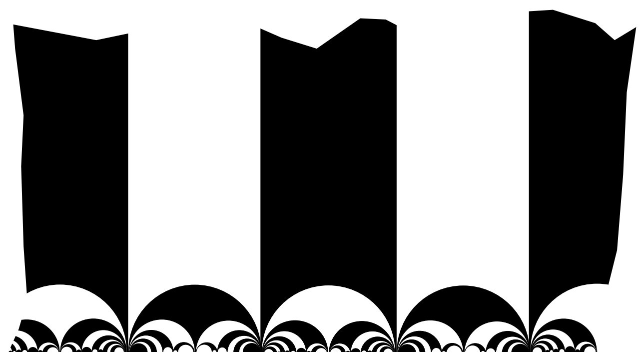
So that in each of these cases F is meromorphic with poles at the points of the lattice of vertices which correspond to the reflections of the point r. Furthermore F will be doubly periodic.

10.4 The elliptic modular function

One could obtain similar constructions for the spherical case, where one obtains the regular polyhedrons inscribed on a sphere or the hyperbolic case, i.e. p, q, r are points of the unit disk $\mathbb{D}$ joined by circular arcs othogonal to

$|z| = 1$. In hyperbolic geometry the sum of the angles in a triangle is less than π so the calculations are more intricate. However to finish the course we consider a special case, when all the angles are zero. In this case the vertices are on $|z| = 1$. Actually it is standard to use $\mathbb{H} = \{\Im(z) > 0\}$ with the hyperbolic metric, instead of the unit disk. Let us define a nonelementary function, called $\lambda(\tau)$, the modular function. We begin with the curvilinear triangle

$$\Delta = \{\tau : \Im(\tau) > 0, 0 < \Re(\tau) < 1, |\tau - 1/2| > 1/2\}.$$



Circle construction for Modular Function

First we recall the discussion of groups of bilinear mappings. Let $\rho_1,\ \rho_2$ be relections in the "disks" $C_1 = \{\Re(\tau) < 0\},\ C_2 = \{\Re(\tau) > 1\}$ respectively. Let ρ_3 be the reflection in the circle $C_3 = \{|\tau - 1/2| < 1/2\}$. These reflections on disjoint disks generate a group Γ of anticonformal/conformal mappings which permutes the images of the original circles. Let the collection of all circles be $\mathcal{C}$. This group has a Möbius Subgroup G generated equal to elements of the form $r_1 \circ r_2$ where $r_j \in \Gamma$. This group is in fact generated by elements $g_1 = z + 2$ and $g_2 = z/(2z - 1)$ and has fundamental region

$$F = \{\tau : \Im(\tau) > 0, 0 < \Re(\tau) < 1, |\tau - 1/4| > 1/4, \, |\tau - 3/4| > 1/4\}$$

As an exercise(below) the circles cluster on the whole real line, i.e. there is a sequence of distinct circles $C_n \to \zeta$ if and only if ζ is real (including ∞). Thus the set of discontinuouity is $\mathbb{C} - \mathbb{R}$. We shall consider the action of G on a region R, choosing $R = \mathbb{H}$. We shall see that the Riemann surface $\mathbb{H}/G$ is conformally equivalent $\mathbb{C} - \{0, \, 1, \, \infty\}$.

By assumption of the Riemann Mapping Theorem there is a conformal mapping $\lambda(\tau)$ which maps Δ onto $\mathbb{H}$ so that 0, 1, ∞ correspond to 0, 1, ∞. We now apply Schwarz reflection to analytically continue through the circles that form the boundaries of Δ. For each such circle C the image of $\mathbb{H} \cap C$ will be one of the complementary subarcs of $\mathbb{R} - \{0, \, 1, \, \infty\}$. Each curvilinear triangle in $\mathbb{H}$ bounded by circles of $\mathcal{C}$ will map to either the upper half or lower half plane, the sides mapping to one of the subintervals of $\mathbb{R} - \{0, 1, \infty\}$. Hence we obtain the analytic function $\lambda = \lambda(\tau)$ defined on $\mathbb{H}$ with image $\mathbb{C} - \{0, 1, \infty\}$. The mapping is locally 1:1.

The group G of bilinear transformations of the form

$$T(z) = \frac{az + b}{cz + d}, \; a, b, c, d \in \mathbb{Z}, \; ad - dc > 0$$

is called the Modular group. Thus

$$\lambda(T(z)) = \lambda(z), \; \forall T \in G,$$

i.e. λ is a modular function.

The proof of Fermat's Last Theorem is to show that certain analytic functions are in fact modular !

Exercises

1. Prove that the circles cluster on the whole real line, i.e. there is a sequence of distinct circles $C_n \to \zeta$ if and only if ζ is real (including ∞).

2. Show that λ has no analytic continuation over the real line.

3. By definition $\lambda(g(\tau)) = \lambda(\tau)$ for all $g \in G$. Other relationships may be found by using symmetry using the uniqueness of the Riemann mapping. Show that
$$\lambda(g(-1/\tau)) = 1/\lambda(\tau) \; .$$

4. Find $\lambda(i)$.

5. Construct the analogous function μ defined from the unit disk $\mathbb{D}$ to $\mathbb{C} - \{0, 1\}$.

6. Let f be entire and $f(z) \neq 0,\ 1$ (for all z). Show that $\mu^{-1} \circ f$ is entire and bounded.

7. (Picard's theorem) Let f be meromorphic on $\mathbb{C}$. Suppose there are three distinct values $a, b, c \in \mathbb{S}$ so that
$$f(\mathbb{C}) \subset \mathbb{S} - \{a, b, c\} \; .$$
Show that f is constant.

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