Complex Analysis HW 4

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Note. Write $\operatorname{cis} t := \cos t + i \sin t$.

P81.3. Set

$$g(z) := \frac{z^3}{27} f(z)$$

so that

$$|g(z)| = \frac{|z|^3}{27} |f(z)|$$

Then on $|z| = 3$,

$$|g(z)| \leq \frac{3^3}{27} \cdot 1 = 1$$

and on $|z| = 1$,

$$|g(z)| \leq \frac{1^3}{27} \cdot 27 = 1$$

Thus by the maximum principle, on $1 \leq |z| \leq 3$,

$$|g(z)| \leq 1 \implies |f(z)| \leq \frac{27}{|z|^3}$$

P94.2 From $u(z) = z^2 + \overline{z}^2$, set

$$g(z) := \partial u(z) = 2z$$

$$G(z) := \int g(z) dz = z^2$$

$$f(z) := 2G(z) = 2z^2$$

We see f is entire and check that

$$\frac{1}{2} \left(f(z) + \overline{f(z)} \right) = \frac{1}{2} \left(2z^2 + 2\overline{z}^2 \right) = z^2 + \overline{z}^2 = u(z)$$

P94.4 Pf. With u harmonic, there exists an entire function f such that $\operatorname{Re} f = u$. Then e^f is entire and

$$|e^f| = e^u < e^c$$

so by Liouville's theorem e^f is constant, hence f is constant. In turn, u is constant.

P96.4 Set $u(x, y) := y$ and $v(x, y) := \cosh \pi x \sin \pi y$.

$$v_{xx} = \pi^2 \cosh \pi x \sin \pi y, \quad v_{yy} = -\pi^2 \cosh \pi x \sin \pi y \implies v_{xx} + v_{yy} = 0$$

Thus v is harmonic. Also, $v\big|_{y=0} = v\big|_{y=1} = 0$. Then the function $w := u + v$ is a sum of harmonic functions hence harmonic, and w satisfies the boundary conditions $w\big|_{y=0} = 0$ and $w\big|_{y=1} = 1$.

P107 Pf. Fix z in the unbounded component of α^c. The winding number is

$$n(z,\alpha) = \frac{1}{2\pi i}\int_\alpha \frac{dw}{w-z}$$

Since α is bounded and closed, $\frac{1}{|w-z|}$ is continuous in w on α, so it attains a maximum at some $w_0 \in \alpha$.

$$|n(z,\alpha)| \le \frac{1}{2\pi}\left|\int_\alpha \frac{dw}{w-z}\right| \le \frac{\text{length}(\alpha)}{2\pi|w_0 - z|} \xrightarrow{z\to\infty} 0$$

so that $\lim_{z\to\infty} |n(z,\alpha)| = 0$. Thus $|n(z,\alpha)| < 1$ for large enough $|z|$. Now $n(z,\alpha) \in \mathbb{Z}$, so $|n(z,\alpha)| = 0$ hence $n(z,\alpha) = 0$ for large enough $|z|$. Also $n(z,\alpha)$ must be constant on components of α^c, thus $n(z,\alpha) = 0$ on the unbounded component of α^c.

P114.1 The Laurent expansions are:

(i) For $0 < |z| < 2$,

$$\frac{1}{z(z^2+4)} = \frac{1}{4z(1-(-z^2/4))} = \frac{1}{4z}\sum_{k\ge 0}\frac{(-1)^k z^{2k}}{4^k} = \sum_{k\ge 0}\frac{(-1)^k z^{2k-1}}{4^{k+1}}$$

(ii) For $|z| > 2$,

$$\frac{1}{z(z^2+4)} = \frac{1}{z^3(1-(-4/z^2))} = \frac{1}{z^3}\sum_{k\ge 0}\frac{(-1)^k 4^k}{z^{2k}} = \sum_{k\ge 0}\frac{(-1)^k 4^k}{z^{2k+3}}$$

(iii) For $0 < |z+2i| < 2$,

$$\frac{1}{z(z^2+4)} = \frac{1}{z(z-2i)(z+2i)} = \frac{1}{z(z-2i)}\sum_{k\ge 0}(z+2i)^k$$

P119.1

(i) The function

$$f(z) = \frac{1}{(z-4)(z^3-1)}$$

has singularities at $z = 4$ and $z = \text{cis}\,\frac{2\pi k}{3},\ 0 \le k \le 2$, i.e. $z = 4, 1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Considering the circle $\alpha : |z-2| = \frac{5}{2}$ has leftmost endpoint $-\frac{1}{2}$, the singularities that lie inside α are $z = 4, 1$. We now find

$$(z-4)f(z) = \frac{1}{z^3-1} \xrightarrow{z\to 4} \frac{1}{64-1} = \frac{1}{63}$$

so that $\text{Res}(f,4) = \frac{1}{63}$, and

$$(z-1)f(z) = \frac{1}{(z-4)(z^2+z+1)} \xrightarrow{z\to 1} \frac{1}{-3(1+1+1)} = -\frac{1}{9}$$

so that $\text{Res}(f,1) = -\frac{1}{9}$. Thus

$$\int_\alpha f(z)dz = 2\pi i[\text{Res}(f,4) + \text{Res}(f,1)] = 2\pi i\left[\frac{1}{63} - \frac{1}{9}\right] = 2\pi i\left[-\frac{6}{63}\right] = -\frac{4\pi i}{21}$$

(ii) The function

$$g(z) = \frac{3z^2+2}{(z-1)(z^2+9)}$$

has singularities at $z = 1, \pm 3i$ which all lie within the circle $\beta : |z| = 5$. We find

$$(z-1)g(z) = \frac{3z^2+2}{z^2+9} \xrightarrow{z\to 1} \frac{3+2}{1+9} = \frac{5}{10} = \frac{1}{2}$$

so that $\mathrm{Res}(g,1) = \frac{1}{2}$, and

$$(z \mp 3i)g(z) = \frac{3z^2+2}{(z-1)(z\pm 3i)} \xrightarrow{z\to\pm 3i} \frac{3(-9)+2}{(\pm 3i-1)(\pm 6i)} = \frac{-25}{-18\mp 6i}$$

$$= \frac{25}{18\pm 6i}\frac{18\mp 6i}{18\mp 6i} = \frac{450\mp 150i}{324+36} = \frac{450\mp 150i}{360} = \frac{15\mp 5i}{12}$$

so that $\mathrm{Res}(g, \pm 3i) = \frac{15\mp 5i}{12}$. Thus

$$\int_\beta g(z)dz = 2\pi i\left[\mathrm{Res}(g,1)+\mathrm{Res}(g,3i)+\mathrm{Res}(g,-3i)\right] = 2\pi i\left[\frac{1}{2}+\frac{15-5i}{12}+\frac{15+5i}{12}\right]$$

$$= 2\pi i\left[\frac{1}{2}+\frac{30}{12}\right] = 2\pi i\left[\frac{1}{2}+\frac{5}{2}\right] = 6\pi i$$

P119.2

(i) The function

$$f(z) = \frac{1}{\cosh 2z}$$

has singularities at (write $w = 2z = x + iy$)

$$\cosh 2z = 0 \implies \cosh x\cos y + i\sinh x\sin y = 0 \implies \cosh x\cos y = 0,\ \sinh x\sin y = 0$$

The first equation gives

$$\cos y = 0 \implies y = \frac{\pi}{2}+\pi n,\ n\in\mathbb{Z}$$

The second equation gives $x = 0$ or $y = \pi n$, so it must be the case $x = 0$ and $y = \frac{\pi}{2}+\pi n$.

$$w = i\left(\frac{\pi}{2}+\pi n\right) \implies z = \frac{w}{2} = \left(\frac{\pi}{4}+\frac{\pi}{2}n\right)i$$

The only singularities that lie within the circle $\gamma : |z| = 1$ are for $n = 0, -1$, i.e. $z = \pm\frac{\pi}{4}i$. Label the points as z_1 and z_2.

$$\mathrm{Res}(f, z_j) = \lim_{z\to z_j}(z-z_j)f(z) = \lim_{z\to z_j}\frac{z-z_j}{\cosh 2z}$$

Using L'Hoptital's rule,

$$\mathrm{Res}(f,z_j) = \lim_{z\to z_j}(z-z_j)f(z)\frac{1}{2\sinh 2z} = \frac{1}{2\sinh 2z_j} = \frac{1}{2\sinh\left(\pm\frac{\pi}{2}i\right)} = \frac{1}{2i\sin\left(\pm\frac{\pi}{2}\right)} = -\frac{i}{2(\pm 1)} = \mp\frac{i}{2}$$

Thus

$$\int_\gamma f(z)dz = 2\pi i\left[\mathrm{Res}\left(f,\frac{\pi}{4}i\right)+\mathrm{Res}\left(f,-\frac{\pi}{4}i\right)\right] = 2\pi i\left[-\frac{i}{2}+\frac{i}{2}\right] = 0$$

(ii) The function

$$g(z) = \cot z = \frac{\cos z}{\sin z}$$

has singularities at

$$\sin z = 0 \implies z = k\pi,\ k \in \mathbb{Z}$$

The only singularity that lies within the circle $\gamma : |z| = 1$ is for $k = 0$, i.e. $z = 0$.

$$\operatorname{Res}(g, 0) = \lim_{z\to 0} zg(z) = \lim_{z\to 0} \frac{z\cos z}{\sin z}$$

Using L'Hopital's rule,

$$\operatorname{Res}(g, 0) = \lim_{z\to 0} \frac{\cos z - z\sin z}{\cos z} = \frac{1-0}{1} = 1$$

Thus

$$\int_\gamma g(z)dz = 2\pi i \operatorname{Res}(g, 0) = 2\pi i$$

P127.4 Pf. Set

$$I := \int_0^\infty f(x)dx, \quad f(z) := e^{iz^2}$$

Fix large $R > 0$ and define the contours

$$\alpha := [0, R] + C_R - \gamma_R, \quad C_R : z = Re^{it},\ 0 \le t \le \frac{\pi}{4}, \quad \gamma_R : z = te^{i\pi/4},\ 0 \le t \le R$$

Since f is analytic on the region enclosed by α,

$$0 = \int_\alpha f(z)dz = \int_{[0,R]} f(z)dz + \int_{C_R} f(z)dz - \int_{\gamma_R} f(z)dz \quad (127.4.1)$$

We compute the third integral. On γ_R,

$$z = te^{i\pi/4},\ 0 \le t \le R \implies z^2 = t^2e^{i\pi/2} = it^2 \implies f(z) = e^{-it^2}$$

Using the Gaussian integral,

$$\int_{\gamma_R} f(z)dz = e^{i\pi/4}\int_0^R e^{-t^2}dt \xrightarrow{R\to\infty} e^{i\pi/4}\int_0^\infty e^{-t^2}dt = e^{i\pi/4}\frac{\sqrt{\pi}}{2}$$

We compute the second integral. On C_R,

$$z = Re^{it},\ 0 \le t \le \frac{\pi}{4}, \implies z^2 = R^2e^{i2t} = R^2\cos 2t + iR^2\sin 2t \implies iz^2 = iR^2\cos 2t - R^2\sin 2t \implies |f(z)| = e^{-R^2\sin 2t}$$

Now using the fact $\sin t \ge \frac{2t}{\pi}$ for $0 \le t \le \frac{\pi}{2}$, we have $\sin 2t \ge \frac{4t}{\pi}$ for $0 \le t \le \frac{\pi}{4}$, hence $|f(z)| \le e^{-R^2 4t/\pi}$.

$$\left|\int_{C_R} f(z)dz\right| \le \int_{C_R} |f(z)||dz| \le \int_0^{\pi/4} e^{-R^2 4t/\pi}Rdt = -\frac{R\pi}{4R^2}e^{-4R^2t/\pi}\Big|_0^{\pi/4} = -\frac{\pi}{4R}(e^{-R^2}-1) \xrightarrow{R\to\infty} 0(0-1) = 0$$

Putting together the calculations, as we take $R \to \infty$ in equation (127.4.1),

$$0 = I - e^{i\pi/4}\frac{\sqrt{\pi}}{2} \implies I = e^{i\pi/4}\frac{\sqrt{\pi}}{2} = \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\frac{\sqrt{\pi}}{2} \implies \int_0^\infty \cos(x^2)dx = \operatorname{im} I = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

P131.1 Set $D := \{|z| < 1\}$, so that $\partial D = \{|z| = 1\}$.

(i) Write

$$h(z) = f(z) + g(z), \quad f(z) = 5z^4, \quad g(z) = z^2 + 2z - i$$

On ∂D,

$$|f(z)| = 5|z|^4 = 5, \quad |g(z)| \leq |z|^2 + 2|z| + |i| = 4 \implies |f(z)| > |g(z)|$$

In D, f has a root at $z = 0$ of order 4, so by Rouche's theorem h has 4 roots in D.

(ii) Write

$$h(z) = g(z) + f(z), \quad g(z) = z^3, \quad f(z) = 8z^2 - 8z + 2$$

On ∂D, using the reverse triangle inequality,

$$|f(z)| \geq \left|8|z|^2 - 8|z| - |2|\right| = |8 - 8 - 2| = 2, \quad |g(z)| = |z|^3 = 1 \implies |f(z)| > |g(z)|$$

The roots of f are

$$0 = f(z) = 8z^2 - 8z + 2 = (4z - 2)(2z - 1) \implies z = \frac{1}{2}, \frac{1}{2} \in D$$

so by Rouche's theorem h has 2 roots in D.

(iii) Write

$$h(z) = g(z) + f(z), \quad g(z) = e^z, \quad f(z) = 3z$$

On ∂D, $|z| = 1$, so write $z = \operatorname{cis} t$.

$$|f(z)| = 3|z| = 3, \quad |g(z)| = |e^z| = e^{\cos t} \leq e^1 < 3 \implies |f(z)| > |g(z)|$$

In D, f has a root at $z = 0$ of order 1, so by Rouche's theorem h has 1 root in D.