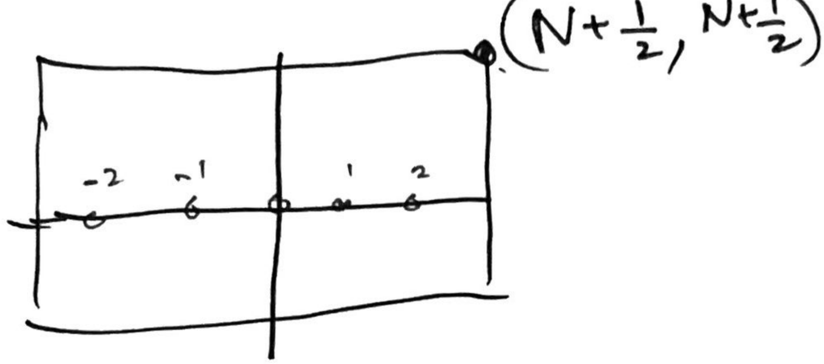
4/7/2022 ① Math 660

Basle Problem

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$



$$f(z) = \frac{\cot(\pi z)}{z^2}$$ has poles at the integers

Enclose in a big square so does not go through any of singularities.

$\cot(\pi z)$ has poles of order 1 at k with residue

$$\rho_k = \lim_{z \to k} (z-k) \frac{\cos(\pi z)}{\sin(\pi z)}$$

$$= (-1)^k \lim_{z \to k} \frac{z-k}{\sin(\pi z)}$$

$$= (-1)^k = \frac{1}{\cos(\pi k)} = \frac{1}{\pi}$$

(2)

$\therefore$ f has poles of order 1 at $\pm 1, \pm 2, \dots$

w. residue $\frac{1}{\pi k^2}$ at $z=0$!

$$\frac{1}{z^2} \frac{\cos(\pi z)}{\sin(\pi z)}$$

pole of order 3 expanded in Taylor Series:

$$= \frac{1}{z^2} \frac{\left(1 - \frac{z^2}{2!}\pi^2 + \dots\right)}{\left(\pi z - \frac{\pi^3}{\underset{6}{3!}} z^3 + \dots\right)}$$

$$= \frac{1}{\pi z^3} \ast \left(1 - \frac{\pi^2 z^2}{2} + \dots\right)\left(1 - \frac{\pi^2 z^2}{6} + \dots\right)^{-1}$$

$$\left(1 + \frac{\pi^2 z^2}{6} \dots\right)$$

need only terms in front

(there are no z terms)

$$= \frac{1}{\pi z^3} \ast \left(1 + \left(-\frac{1}{2} + \frac{1}{6}\right)\pi^2 z^2 + \dots\right)$$

$$= \frac{1}{\pi z^3}\left(1 - \frac{1}{3}\pi^2 z^2 \dots\right)$$

(3)

$$= \frac{1}{\pi z^3} - \frac{1}{3}\frac{\pi}{z} + \cdots$$

Residue of f at $z=0$ is $\boxed{\frac{-\pi}{3}}$

Γ is + orientated square with vertices $\pm (N+\frac{1}{2})$, $\pm i(N+\frac{1}{2})$

$N = 1, 2, \cdots$

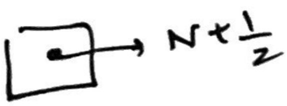
By residue Thm.

$$\int_\Gamma \frac{\cot(\pi z)}{z^2}\, dz = 2\pi i \left\{ -\frac{\pi}{3} + \sum_{k=1}^{N} \frac{1}{\pi k^2} \right\}$$

Lemma: $|\cot(\pi z)| < C$ on Γ

$$\left| \int_\Gamma \frac{\cot \pi z}{z^2}\, dz \right| \le C \int_\Gamma \frac{|\cot \pi z|}{|z|^2} \le$$

$$\le C \frac{4(2N+1)}{(N+\frac{1}{2})^2}$$



$$\le \frac{C}{N} \to 0 \text{ as } N \to \infty$$

4/7/23 (4)

In conclusion

$$0 = -\frac{\pi}{3} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{\pi^2}$$

So $$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

This method also works for expansions (∞)

e.g. $\cot(\pi z)$

has poles of order 1 at $z = k \in \mathbb{Z}$

with residue $= \frac{1}{\pi}$

$$\cot(\pi z) \stackrel{?}{=}$$

$$\sum_{k=-\infty}^{\infty} \frac{1}{\pi(z-k)} = \lim_{N \to \infty} \sum_{-N}^{N} \frac{1}{\pi(z-k)}$$

allows cancelling

$$\frac{1}{z-k} + \frac{1}{z+k} = \frac{2z}{z^2-k^2}$$

So $$\sum_{k=1}^{\infty} \left| \frac{1}{z-k} + \frac{1}{z+k} \right| \le 2z \sum_{k=r}^{\infty} |k^2 - |z|^2| < \infty$$

To prove this,

Fix $\xi \in \mathbb{C}$

$\xi \notin \mathbb{Z}$

$\cancel{f(z)}$ $f(\xi) = \dfrac{\cot(\pi \xi)}{\xi - z}$

poles of order 1 at $\xi = k \in \mathbb{Z}$

residue $\frac{1}{k}$ $\frac{1}{k-z}$

pole at $\xi = z$ also of order 1

residue : $\cot(\pi z)$

So integrating on Γ_N

$$\frac{1}{2\pi i} \int_{\Gamma_N} \frac{\cot \pi \xi}{\xi - z} = 2\pi i \left\{ \cot(\pi z) \right.$$

$$+ \sum_{-N}^{N} \frac{1}{\pi(k - z)}$$

(6) <u>problem</u> Need to show integral goes to 0. Not that simple

b/c

$$I = \left| \frac{\cot \pi \xi}{\xi - z} \right| \le \frac{C}{\left(|\xi| - |z|\right)}$$

So $|I| \le \frac{CN}{N}$ does <u>not</u> converge to 0!

We need to work a little harder:

$$\frac{1}{\xi - z} = \frac{1}{\xi} + \frac{z}{\xi(\xi - z)}$$

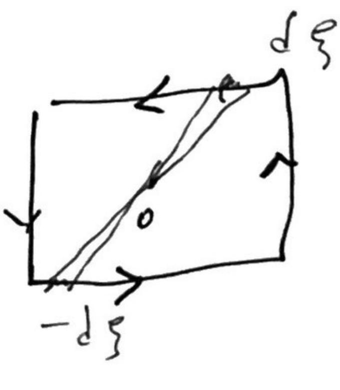
(looks hopeless)

$$\int_{\Gamma_N} \frac{\cot \pi \xi}{\xi} \, d\xi = 0$$

But

b/c $g(\xi) = \frac{\cot \pi \xi}{\xi}$ is even

So $g(\xi) = -g(-\xi)$



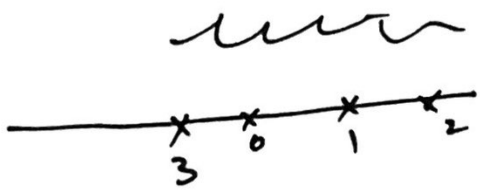
(7) So $\int_{\Gamma_N} g(\xi)\, d\xi = 0$

$$\left| \int_{\Gamma_N} \frac{z \cot \pi\xi}{\xi(\xi - z)}\, d\xi \right| \le \frac{|z|\, C N}{N^2}$$

$$\le \frac{C|z|}{N} \to 0 \quad \text{as } N \to \infty$$

$$\therefore \boxed{\cot \pi z = \frac{1}{\pi} \sum_{-\infty}^{\infty} \frac{1}{z-k}}$$

$$\cot(\pi z) = \frac{\cos \pi z}{\sin \pi z} = \frac{1}{\pi} \frac{d}{dz} \log \sin(\pi z)$$



$\text{Im } z > 0$

So $\frac{d}{dz} \log \sin(\pi z) = \sum_{-\infty}^{\infty} \frac{1}{z-k}$

integrate wrt z

(8) $\log \sin(\pi z)$

$$= \sum_{-\infty}^{\infty} \log(z-k) + c_k$$

$$+ c$$

chosen so we have ~~cgck~~ convergence of ~~sin~~ the series

?

$$\log\left(1-\frac{z}{k}\right) \approx -\frac{z}{k}$$

$$\log\left(1-\frac{z}{k}\right) = -\frac{z}{k} + \frac{1}{2}\frac{z^2}{k^2} + \cdots$$

Suffices

$$\log(z-k) = \log\left(\frac{z}{k}-1\right) + \log k$$

So we get that $\log \sin(z) =$

$$\sum_{-\infty}^{\infty} \log\left(1-\frac{z}{k}\right)$$

$$\sin z + \log(z)$$

$$\sin z = c z \prod_{\substack{-\infty \\ k \neq 0}}^{\infty} \left(1-\frac{z}{k}\right)$$

9

$$= Cz \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)$$

We know that $\sin(\pi z) =$

$$\pi z - \frac{\pi^3 z^3}{3!} \cdots$$

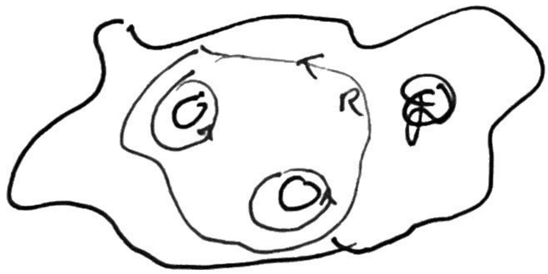
$$C = \pi$$

$$\sin(\pi z) = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)$$

$$\sin z = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 k^2}\right)$$

Argument Principle

Suppose a cycle $\alpha \in \mathcal{D}$ bounds the region $R \in \mathcal{D}$



$z \in R$ winding #

$N(z, \alpha) = 1$

$\alpha \sim 0$ on $\mathcal{D}$

(10) Theorem

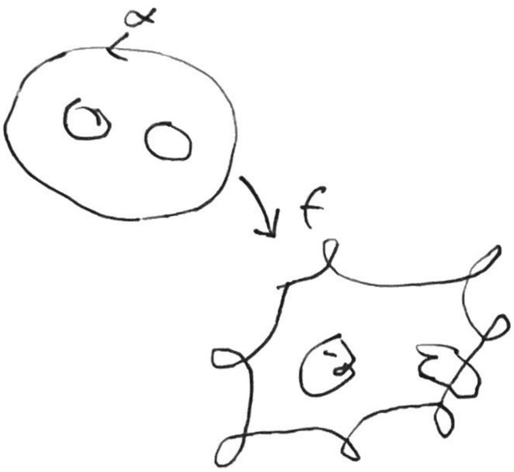
Suppose f is meromorphic on R
no ~~pole~~ poles or zeros on α

then if f has N zeros and M poles in R

then $N(0, f(\alpha)) = N - M$

Proof
Consider

$$g(z) = \frac{f'}{f}$$



So by the residue theorem

$$\int_\alpha \frac{f'}{f}\,dz = 2\pi i \sum \text{residues}$$

poles of $\frac{f'}{f}$ are ξ_k either poles or zeros

of f.
In latter case, $f = a_n (z - \xi_k)^{n-1} + \dots$

$$f' = n\, a_n (z - \xi_k)^{n-1}$$

$$\frac{f'}{f} = \frac{n}{z - \xi_k} + \dots$$ residue is n_k

(11) In the former case if

ξ is a pole of order m_k

$$f = \frac{a_{-m_k}}{(z - \xi_k)^{m_k}} + \cdots$$

$$f' = \frac{-m_k}{z - \xi_k}$$

$$\frac{f'}{f} = \frac{-m_k}{z - \xi_k}$$ residue is $-m_k$

thus

$$\frac{1}{2\pi i} \int_\alpha \frac{f'}{f}\, dz = \sum n_k - m_k$$

$$= N - M$$

Changing variables $w = f(z)$
$\beta = f(\alpha)$

then LHS

$$\frac{1}{2\pi} \int_\beta \frac{dw}{w} = N(0, \beta) = N(0, f(\alpha))$$

e.g. $$f(z) = z^m \prod_{k=1}^{n} \frac{z-a_k}{1-\bar{a}_k z}$$

$m, n > 0 \qquad 0 < |a_k| < 1$

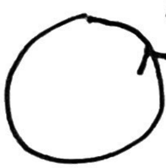
zeroes at 0, a_k

no poles in $|z|<1$

$\Gamma: \; z = e^{it} \qquad 0 \le t \le 2\pi$

So $N(0, f(\Gamma)) = \underline{m+n}$

$f(\Gamma)$



$\Rightarrow$ winds around every w, $|w|<1$ exactly $m+n$ times.

$\therefore$ For every w, $|w|<1$,

$f(z) = w$ has exactly $m+n$ roots.

<u>Finished the chapter</u>