Complex Analysis Quiz 3

Ryan Chen

April 7, 2023

1. Write

$$h(\alpha) = \int_0^\infty f(x)dx, \quad f(z) := \frac{1}{z^\alpha + 1}$$

The simple poles z_k of f are

$$z^\alpha = -1 = e^{i(2k+1)\pi} \implies z_k = e^{i(2k+1)\pi/\alpha}$$

Consider the contour

$$\Gamma := [0, R] + C_R - \gamma_R, \quad C_R : z = Re^{it},\ 0 \le t \le \frac{2\pi}{\alpha}, \quad \gamma_R : z = te^{i2\pi/\alpha},\ 0 \le t \le R$$

so that

$$\int_\Gamma f(z)dz = \int_{[0,R]} f(z)dz + \int_{C_R} f(z)dz - \int_{\gamma_R} f(z)dz$$

and the only pole of f that lies inside Γ is $z_0 = e^{i\pi/\alpha}$. We find

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} \frac{z - z_0}{z^\alpha + 1}$$

Using L'Hopital's rule,

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{\alpha z^{\alpha - 1}} = \frac{1}{\alpha z_0^{\alpha-1}} = \frac{z_0}{\alpha z_0^\alpha} = -\frac{z_0}{\alpha}$$

Thus by the residue theorem,

$$\int_\Gamma f(z)dz = 2\pi i \operatorname{Res}(f, z_0) = -2\pi i \frac{z_0}{\alpha} = -2\pi i \frac{e^{i\pi/\alpha}}{\alpha}$$

On C_R,

$$|z| = R \implies |f(z)| \le \frac{1}{||z|^\alpha - 1|} = \frac{1}{R^\alpha - 1} = O\left(\frac{1}{R^\alpha}\right)$$

so that, using the fact $\alpha > 1$,

$$\left|\int_{C_R} f(z)dz\right| = O\left(\frac{1}{R^{\alpha-1}}\right) \xrightarrow{R \to \infty} 0$$

Then we compute

$$\int_{\gamma_R} f(z)dz = \int_0^R \frac{1}{t^\alpha e^{i2\pi} + 1} e^{i2\pi/\alpha} dt = e^{i2\pi/\alpha} \int_0^R f(t)dt$$

Putting together the computations, as $R \to \infty$,

$$-2\pi i \frac{e^{i\pi/\alpha}}{\alpha} = \int_0^\infty f(x)dx - e^{i2\pi/\alpha} \int_0^R f(x)dx = \left[1 - e^{i2\pi/\alpha}\right] \int_0^\infty f(x)dx$$

Multiply by $e^{-i\pi/\alpha}$.

$$\frac{-2\pi i}{\alpha} = \left[e^{-i\pi/\alpha} - e^{i\pi/\alpha}\right] \int_0^\infty f(x)dx = -2i \sin\frac{\pi}{\alpha} \int_0^\infty f(x)dx$$

$$\implies h(\alpha) = \int_0^\infty f(x)dx = \frac{\pi/\alpha}{\sin \pi/\alpha}$$

2. From the last part, the expression for $h(\alpha)$ is valid for

$$\alpha \in \mathbb{C} - \left[\left\{\frac{1}{k} : k \in \mathbb{Z}\right\} \cup \{0\}\right]$$