Complex Analysis Quiz 2

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Pf. Let $w_1, \dots, w_n$ be the roots of P and $u_1, \dots, u_m$ be the roots of Q. With a_n the lead coefficient of P,

$$P(z) = a_n \prod_{k=1}^{n} (z - w_k) \implies P'(z) = a_n \sum_{k=1}^{n} \prod_{\substack{j=1 \\ j \neq k}}^{n} (z - w_j) = a_n \sum_{k=1}^{n} \frac{(z-w_1)(z-w_n)}{z - w_k} \implies \frac{P'(z)}{P(z)} = \sum_{k=1}^{n} \frac{1}{z - w_k}$$

For all $t \in \mathbb{R}$, set $z = it$, then since $\operatorname{Re} w_k < 0$,

$$\operatorname{Re} \frac{P'(z)}{P(z)} = \sum_{k=1}^{n} \operatorname{Re} \frac{1}{z - w_k} = \sum_{k=1}^{n} \operatorname{Re} \frac{\overline{z - w_k}}{|z - w_k|^2} = \sum_{k=1}^{n} \frac{\operatorname{Re}(z - w_k)}{|z - w_k|^2} > 0$$

and by similar arguments and the fact $\operatorname{Re} u_k > 0$,

$$\frac{Q'(z)}{Q(z)} = \sum_{k=1}^{m} \frac{1}{z - u_k} \implies \operatorname{Re} \frac{Q'(z)}{Q(z)} = \sum_{k=1}^{m} \frac{\operatorname{Re}(z - u_k)}{|z - u_k|^2} < 0$$

Putting the inequalities together,

$$\implies \operatorname{Re} \frac{P'(z)}{P(z)} \neq \operatorname{Re} \frac{Q'(z)}{Q(z)} \implies \frac{P'(z)}{P(z)} \neq \frac{Q'(z)}{Q(z)} \implies P'(z)Q(z) - P(z)Q'(z) \neq 0 \quad (1)$$

By the quotient rule,

$$R' = \frac{P'Q - PQ'}{Q^2}$$

which along with (1) means $R'(it) \neq 0$ for all $t \in \mathbb{R}$.