Complex Analysis HW 2

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Note. Let f_x denote $\frac{\partial f}{\partial x}$.

P31.3. Pf. Write $f = u + iv$.

$$\overline{f_x} = \overline{u_x + iv_x} = u_x - iv_x = \overline{f}_x$$

and similarly $\overline{f_y} = \overline{f}_y$. Using these facts,

$$\overline{\partial f} = \overline{\frac{1}{2}(f_x - if_y)} = \frac{1}{2}(\overline{f_x} + i\overline{f_y}) = \frac{1}{2}(\overline{f}_x + i\overline{f}_y) = \overline{\partial}\ \overline{f}$$

P33.1. Pf. Fix a point z_0 and fix $r > 0$. To show f is holomorphic at z_0, we fix $z = x + iy \in D(z_0, r)$ and aim to show $f'(z)$ exists. The CR equations let us set

$$A := u_x(x, y) = v_y(x, y), \quad B := v_x(x, y) = -u_y(x, y)$$

Since u has continuous first partials, u is differentiable at (x, y), so that as $(h, k) \to (0, 0)$,

$$u(x+h, y+k) = u(x, y) + hu_x(x, y) + ku_y(x, y) + o(\|(h, k)\|) \implies u(x+h, y+k) - u(x, y) = hA - kB + o(\|(h, k)\|)$$

and by similar arguments,

$$v(x+h, y+k) = v(x, y) + hv_x(x, y) + kv_y(x, y) + o(\|(h, k)\|) \implies v(x+h, y+k) - v(x, y) = hB + kA + o(\|(h, k)\|)$$

Using these expressions,

$$\frac{f(x + iy + h + ik) - f(x + iy)}{h + ik} = \frac{1}{h + ik}\left[u(x + h, y + k) - u(x, y) + iv(x + h, y + k) - iv(x, y)\right]$$

$$= \frac{1}{h + ik}\left[hA - kB + ihB + ikA + o(\|(h, k)\|\right] = \frac{1}{h + ik}\left[hA - kB + ihB + ikA\right] + \frac{o\|(h, k)\|}{h + ik}$$

The second term vanishes as $(h, k) \to (0, 0)$ since

$$\frac{o\|(h, k)\|}{h + ik} \leq \left|\frac{o\|(h, k)\|}{h + ik}\right| = \frac{|o\|(h, k)\||}{\|(h, k)\|} \xrightarrow{(h,k)\to(0,0)} 0$$

The first term becomes

$$\frac{h - ik}{h^2 + k^2}\left[hA - kB + ihB + ikA\right] = \frac{1}{h^2 + k^2}\left[h^2A + ih^2B - hkB + ihkA - ihkA + hkB + ik^2B + k^2A\right]$$

$$= \frac{1}{h^2 + k^2}\left[A(h^2 + k^2) + i(h^2 + k^2)B\right] = A + iB$$

From our calculations, we finally get

$$f'(z) = \lim_{w \to 0} \frac{f(z+w) - f(z)}{w} = \lim_{(h,k)\to(0,0)} \frac{f(x+iy+h+ik) - f(x+iy)}{h+ik} = A + iB$$

P36.3. Pf. Write $f = \frac{P}{Q}$ for polynomials P and Q with no common roots. Let $w_1, \dots, w_n$ be the roots of P and $u_1, \dots, u_m$ be the roots of Q. With a_n the lead coefficient of P,

$$P(z) = a_n \prod_{k=1}^{n} (z - w_k) \implies P'(z) = a_n \sum_{k=1}^{n} \prod_{\substack{j=1 \\ j \neq k}}^{n} (z - w_j) = a_n \sum_{k=1}^{n} \frac{(z-w_1)(z-w_n)}{z - w_k} \implies \frac{P'(z)}{P(z)} = \sum_{k=1}^{n} \frac{1}{z - w_k}$$

For all $t \in \mathbb{R}$, set $z = it$, then since $\operatorname{Re} w_k < 0$,

$$\operatorname{Re} \frac{P'(z)}{P(z)} = \sum_{k=1}^{n} \operatorname{Re} \frac{1}{z - w_k} = \sum_{k=1}^{n} \operatorname{Re} \frac{\overline{z - w_k}}{|z - w_k|^2} = \sum_{k=1}^{n} \frac{\operatorname{Re}(z - w_k)}{|z - w_k|^2} > 0$$

and by similar arguments and the fact $\operatorname{Re} u_k > 0$,

$$\frac{Q'(z)}{Q(z)} = \sum_{k=1}^{m} \frac{1}{z - u_k} \implies \operatorname{Re} \frac{Q'(z)}{Q(z)} = \sum_{k=1}^{m} \frac{\operatorname{Re}(z - u_k)}{|z - u_k|^2} < 0$$

Putting the inequalities together,

$$\implies \operatorname{Re} \frac{P'(z)}{P(z)} \neq \operatorname{Re} \frac{Q'(z)}{Q(z)} \implies \frac{P'(z)}{P(z)} \neq \frac{Q'(z)}{Q(z)} \implies P'(z)Q(z) - P(z)Q'(z) \neq 0 \quad (36.3.1)$$

By the quotient rule,

$$f' = \frac{P'Q - PQ'}{Q^2}$$

which along with (1) means $f'(it) \neq 0$ for all $t \in \mathbb{R}$.

P40.2. Pf. Using the power series for $|z - w| < |w|$,

$$\operatorname{Log} z = \sum_{k \geq 1} \frac{(-1)^{k+1}}{kw^k} (z - w)^k$$

set $z = 2$ and $w = 1$. Since $\operatorname{Arg} 2 = 0$, we have $\operatorname{Log} 2 = \ln 2$.

$$\ln 2 = \sum_{k \geq 1} \frac{(-1)^{k+1}}{k \cdot 1^k} 1^k = \sum_{k \geq 1} \frac{(-1)^{k+1}}{k}$$

P41.6. Pf. Let $P := (x_1, x_2, x_3)$ and $Q := (y_1, y_2, y_3)$ be the points on the Riemann sphere whose stereographic projections are z and w, respectively, i.e.

$$z = \frac{x_1 + ix_2}{1 - x_3}, \quad w = \frac{y_1 + iy_2}{1 - y_3}$$

($\implies$) If P and Q are diametrically opposite then $y_j = -x_j$ for $j = 1, 2, 3$, so

$$z\overline{w} = \frac{(x_1 + ix_2)(y_1 - iy_2)}{(1 - x_3)(1 - y_3)} = \frac{(x_1 + ix_2)(-x_1 + iy_2)}{(1 - x_3)(1 + x_3)} = -\frac{x_1^2 + x_2^2}{1 - x_3^2} = -\frac{1 - x_3^2}{1 - x_3^2} = -1$$

($\Longleftarrow$) We first find

$$\overline{w}w = |w|^2 = \frac{y_1^2+y_2^2}{(1-y_3)^2} = \frac{1-y_3^2}{(1-y_3)^2} = \frac{1+y_3}{1-y_3} \implies \frac{\overline{w}w}{w} = \frac{(1+y_3)(1-y_3)}{(1-y_3)(y_1+iy_2)} = \frac{1+y_3}{y_1+iy_2}$$

Under the assumption $z\overline{w} = -1$,

$$-1 = z\overline{w} = \frac{z\overline{w}w}{w} = \frac{(x_1+ix_2)(1+y_3)}{(1-x_3)(y_1+iy_2)} \implies (x_1+ix_2)(1+y_3) = -(1-x_3)(y_1+iy_2)$$

Equating real and imaginary parts,

$$x_1(1+y_3) = -(1-x_3)y_1,\ x_2(1+y_3) = -(1-x_3)y_2 \implies \frac{x_1}{y_1} = \frac{x_2}{y_2} = \frac{x_3-1}{y_3+1}$$

For convenience later, set $c := \frac{x_1}{y_1}$, so that

$$x_1 = cy_1,\ x_2 = cy_2,\ x_3 - 1 = c(y_3+1)$$

Now consider the chord from $(0,0,1)$ to P. It has a direction vector $a := (x_1, x_2, x_3 - 1)$. Similarly the chord from $(0,0,1)$ to Q has a direction vector $b := (y_1, y_2, y_3 - 1)$. We find

$$a \cdot b = x_1y_1 + x_2y_2 + (x_3-1)(y_3-1) = c(y_1^2+y_2^2+y_3^2-1) = c \cdot 0 = 0$$

Using the fact $|a \cdot b| = \|a\|\|b\|\cos\theta$, where θ is the angle between the aforementioned chords, gives $\cos\theta = 0$ hence $\theta = \frac{\pi}{2}$. From Euclidean geometry, the angle that P and Q subtend at the origin is $2\theta = \pi$, i.e. P and Q are diametrically opposite.