Math 660

Quiz :

① Use residues to compute

$$h(\alpha) = \int_0^\infty \frac{1}{x^\alpha + 1} dx$$

for real $\alpha > 1.$

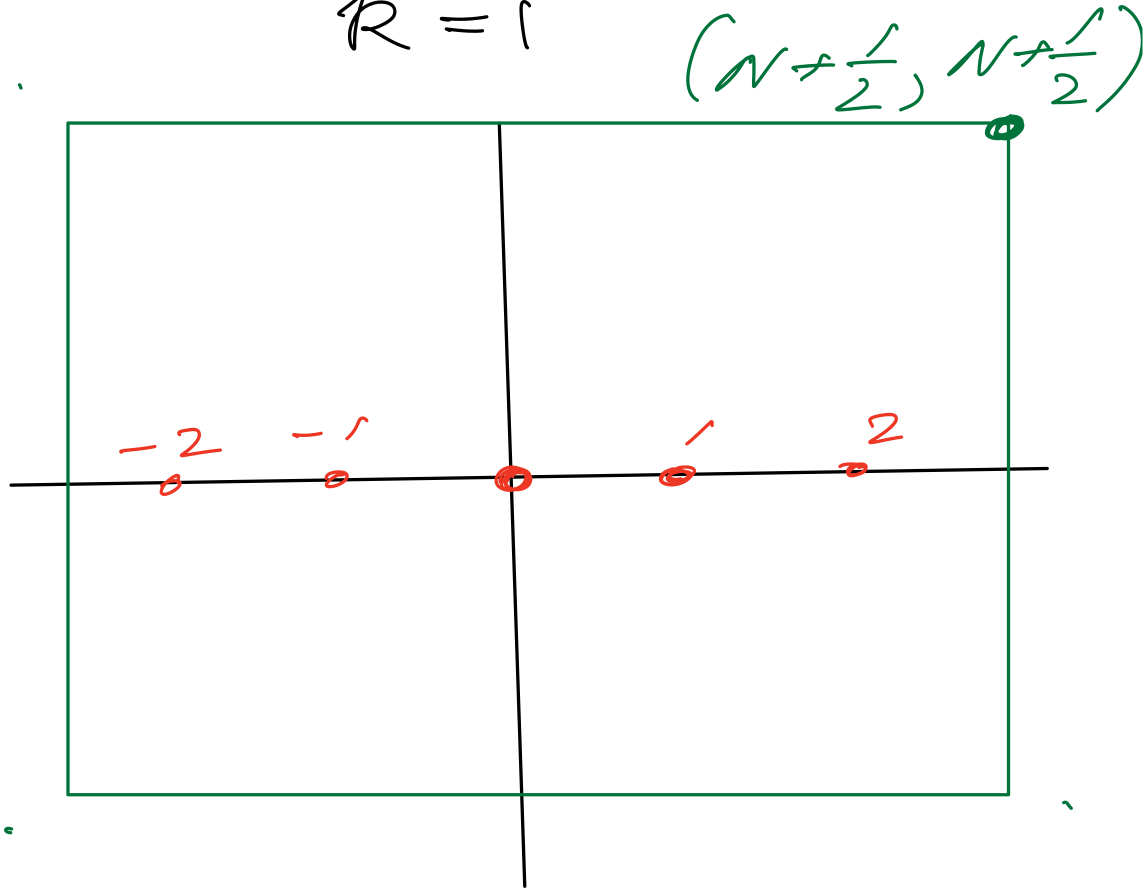
② For which $\alpha \in \mathbb{C}$ does the formula hold.

$h(\alpha)$ analytic function of α, provided

$\underline{\mathrm{Re}\,\alpha > 1}$

Basle Problem.

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$



$$f(z) = \frac{\cot(\pi z)}{z^2}$$

$\cot(\pi z)$ has poles of order 1 at k with residue

$$\rho_k = \lim_{z \to k} (z-k) \frac{\cos(\pi z)}{\sin(\pi z)}$$

$$= (-1)^k \lim_{z \to k} \frac{(z-k)}{\sin(\pi z)}$$

$$= (-1)^k \frac{1}{\pi \cos(\pi k)}$$

$$= \frac{1}{\pi}$$

$\therefore f$ has poles
of order at $\pm 1, \pm 2 \cdots$
residue
$$\frac{1}{\pi k^2}$$

At $z = 0$:

$$\frac{1}{z^2} \frac{\cos(\pi z)}{\sin(\pi z)}$$

pole of order 3

$$= \frac{1}{z^2} \frac{\left(1 - \frac{\pi^2 z^2}{2!} + \cdots\right)}{\left(\pi z - \frac{\pi^3}{6} z^3 + \cdots\right)}$$

$$= \frac{1}{\pi z^3} * \left(1 - \frac{\pi^2 z^2}{2} + \cdots\right)\left(1 - \frac{\pi^2 z^2}{6} + \cdots\right)^{-1}$$

$$\left(1 + \frac{\pi^2}{6} z^2 \cdots\right)$$

$$= \frac{1}{\pi z^3} * \left(1 + \left(-\frac{1}{2} + \frac{1}{6}\right)\pi^2 z^2 + \cdots\right)$$

$$= \frac{1}{\pi z^3}\left(1 - \frac{1}{3}\pi^2 z^2 \ldots\right)$$

$$= \frac{1}{\pi z^3} - \frac{1}{3}\frac{\pi}{z} + \ldots$$

Residue of f

at $z = 0$ is

$$-\frac{\pi}{3}$$

Γ + orientated

square with vertices

$\pm\left(N+\frac{1}{2}\right) \pm i\left(N+\frac{1}{2}\right)$

$N = 1, 2, \ldots\ldots$

By Residue Th

$$\int_{\Gamma} \frac{\cot(\pi z)}{z^2}\, dz$$

$$= 2\pi i \left\{ -\frac{\pi}{3} + 2\sum_{k=1}^{N} \frac{1}{\pi k^2} \right\}$$

Lemma:
$|\cot(\pi z)| < C$
on Γ.

$$\left| \int_{\Gamma} \frac{\cot \pi z}{z^2} \, dz \right|$$

$$\leq \int_{\Gamma} \frac{|\cot \pi z|}{|z|^2} |dz|$$

$$\leq C \int_{\Gamma} \frac{|dz|}{|z|^2}$$

$$\leq C \frac{4(2N+1)}{\left(N+\frac{1}{2}\right)^2}$$

$$\leq \frac{C}{N} \to 0$$

as $N \to \infty$

In conclusion

$$0 = -\frac{\pi}{3} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{k^2}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

Method also works for expansions.

e.g.

$\cot(\pi z)$

poles of order 1

at $z = k \in \mathbb{Z}$,

residue $\frac{1}{\pi}$.

$$\cot(\pi z) \stackrel{?}{=} \sum_{k=-\infty}^{\infty} \frac{1}{\pi(z-k)}$$

$$= \lim_{N \to \infty} \sum_{-N}^{N} \frac{1}{\pi(z - k)}$$

allows cancelling

$$\frac{1}{z - k} + \frac{1}{z + k}$$

$$= \frac{2z}{z^2 - k^2}$$

$$\sum_{k=1}^{\infty} \left(\frac{1}{z-k} + \frac{1}{z+k} \right)$$

$$\leq 2|z| \sum_{k=r}^{\infty} \frac{1}{|k^2 - |z|^2|}$$

$$\sim C(z) \sum \frac{1}{k^2}$$

$$< \infty .$$

To prove this.

Fix $z \in \mathbb{C}$,

$z \notin \mathbb{Z}$

. . . . .

$$f(\xi) = \frac{\cot(\pi \xi)}{\xi - z}$$

poles of order 1

at $\xi = k \in \mathbb{Z}$

residue

$$\frac{1}{\pi} \frac{1}{(k - z)}$$

pole at $\xi = z$

residue: $\cot(\pi z)$

Integrating on Γ_N

$$\int_{\Gamma_N} \frac{\cot(\pi \xi)\, d\xi}{\xi - z}$$

$$= 2\pi i \left\{ \cot(\pi z) + \sum_{-N}^{N} \frac{1}{\pi(k-z)} \right\}$$

<u>problem</u>:

$I = \int$

$$\int_{\Gamma_N} \frac{\cot \pi\xi}{\xi - z}\, d\xi$$

$$\left| \frac{\cot \pi\xi}{\xi - z} \right| .$$

$$\leq \frac{C}{\big||\xi| - |z|\big|}$$

$$|I| \leq \frac{C}{N} N$$

$$\not\to 0 .$$

$$\frac{1}{\xi - z}$$

$$= \frac{1}{\xi} + \frac{z}{\xi(\xi - z)}$$

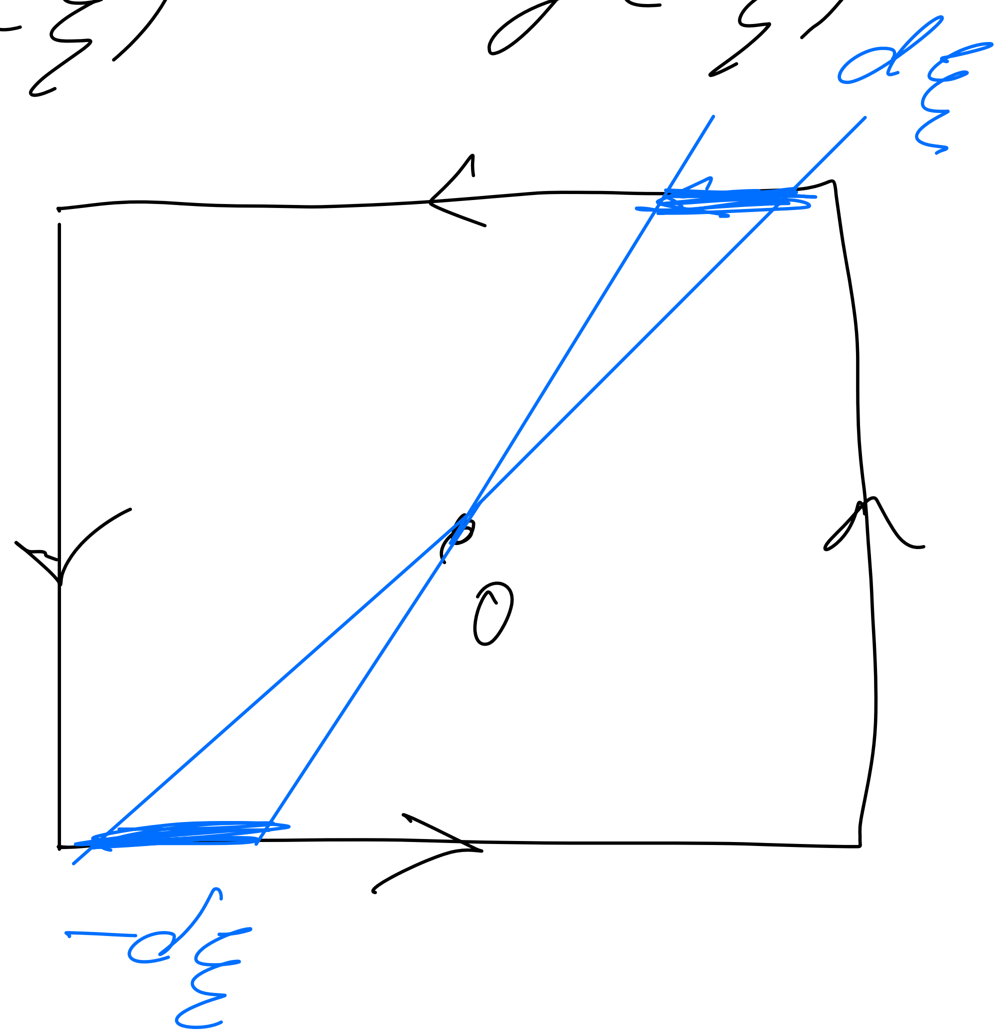
$$\int_{\Gamma_N} \frac{\cot \pi \xi}{\xi} \, d\xi$$

$$= 0$$

$$g(\xi) = \frac{\cot \pi \xi}{\xi}$$

even

$$g(\xi) = -g(-\xi)$$



$$\int_{\Gamma} g(\xi)\, d\xi = 0.$$

$$\left| \int_{\Gamma_N} \frac{z \cot \pi \xi}{\xi (\xi - z)} \, d\xi \right|$$

$$\leq \frac{|z| \, C \, N}{N^2}$$

$$\leq \frac{C |z|}{N} \to 0.$$

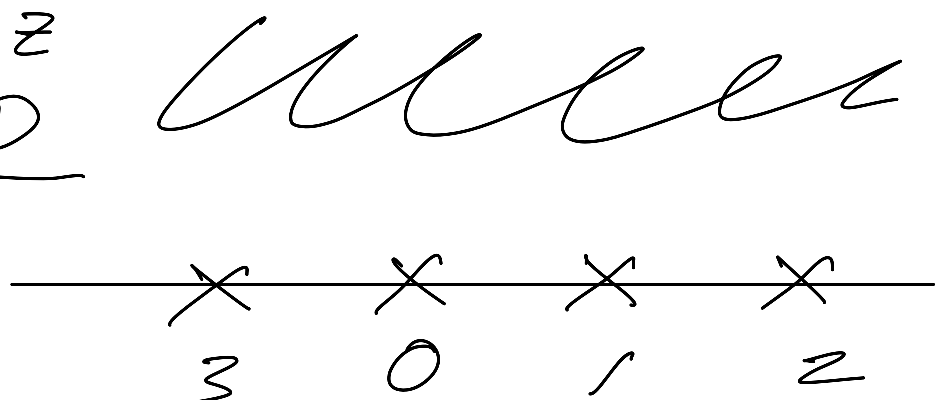
$$\cot(\pi z) =$$

$$\frac{1}{\pi}\sum_{-\infty}^{\infty}\frac{1}{z-k}$$

$$\cot(\pi z) = \frac{\cos \pi z}{\sin \pi z}$$

$$= \frac{1}{\pi}\frac{d}{dz}\log \sin(\pi z)$$

$\operatorname{Im} z$ > 0



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$$\frac{d}{dz} \log \sin(\pi z) = \sum_{-\infty}^{\infty} \frac{1}{z-k}$$

Integrate $\cot z$.

$$\log \sin(\pi z) = \sum_{-\infty}^{\infty} \log(z-k) + c_k + C$$

chosen so we have

convergence of series.

$$\log\left(1 - \frac{z}{k}\right)$$

$$= -\frac{z}{k} + \frac{1}{2}\frac{z^2}{k^2} + \cdots$$

suffices

$$\log(z - k)$$

$$= \log\left(\frac{z}{k} - 1\right) + \log k.$$

$$\log \sin(\pi z) =$$

$$c + \sum_{\substack{-\infty \\ k \neq 0}}^{\infty} \log\left(1 - \frac{z}{k}\right) + \log z$$

$$\sin(z\pi) =$$

$$c\, z \prod_{\substack{-\infty \\ k \neq 0}}^{\infty} \left(1 - \frac{z}{k}\right)$$

$$= c\, z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)$$

$$\sin(\pi z) = \pi z - \frac{\pi^3 z^3}{3!} + \dots$$

$$c = \pi.$$

$$\sin(\pi z) =$$

$$\pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

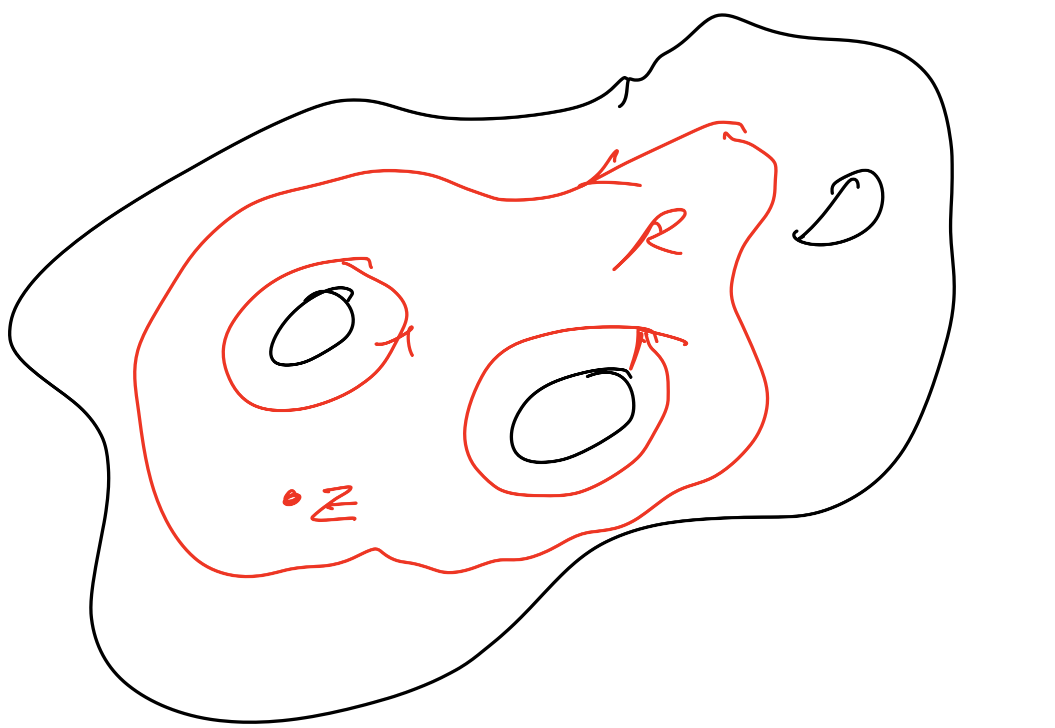
$$\sin(z) =$$

$$z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right)$$

Argument Principle

Suppose cycle $\alpha \subset D$

bounds a region $R \subset D$



* $z \in R$ winding #

$N(z, \alpha) = 1$

$\alpha \sim 0$ on D.

f meromorphic on D

no poles or zeros on α

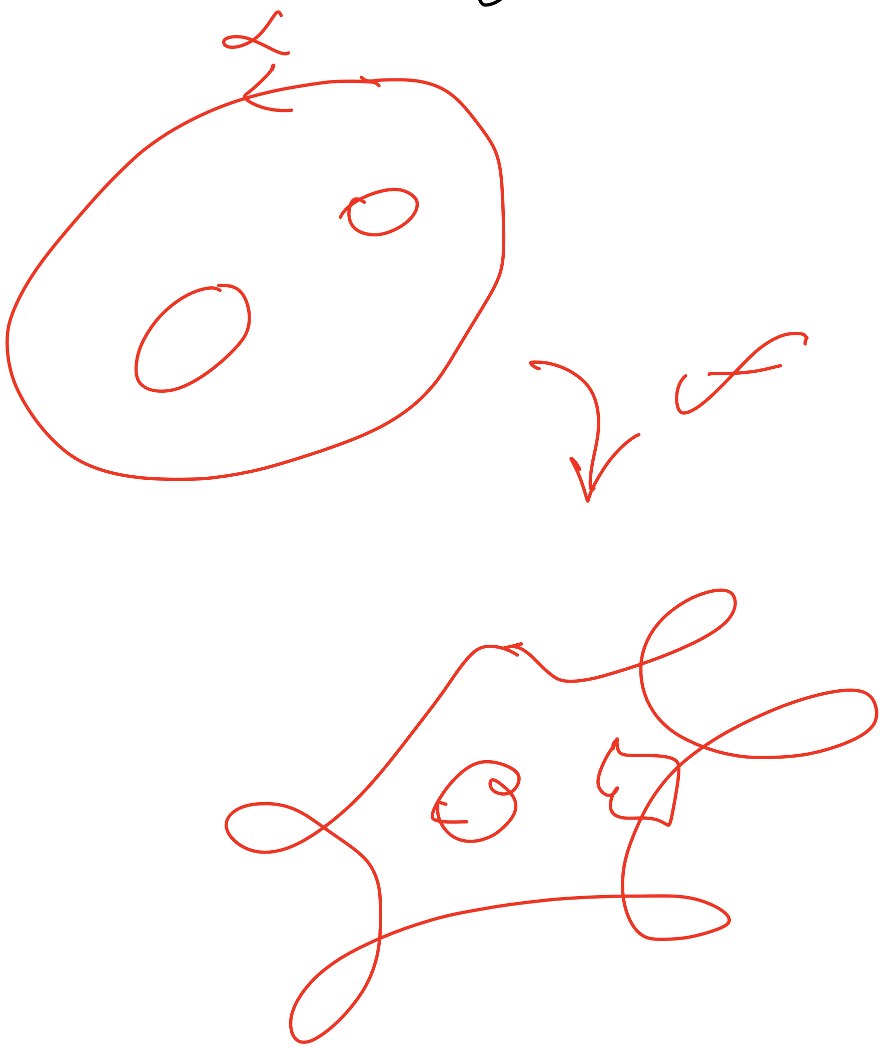
f N zeros in R

M poles in R

$N(0, f(\alpha)) = N - M$.

Consider

$$g(z) = \frac{f'}{f} \text{ on } R.$$



So by residue th.

$$\int_\alpha \frac{f'}{f}\, dz =$$

$$2\pi i \sum \underline{\text{residues}}$$

poles of $\frac{f'}{f}$ are ξ_k either poles or zeros of f.

In latter case.

$$f = a_{n_k}(z-\xi_k)^{n_k} + \dots$$

$$f' = n_k a_n (z-\xi_k)^{n_k - 1}$$

$$\frac{f'}{f} = \frac{n_k}{z - \xi_k} + \cdots$$

residue is n_k.

In former case

if ξ_k is a pole

of order m_k

$$f = \frac{a_{-m_k}}{(z - \xi_k)^{m_k}} + \cdots$$

$$f' = \frac{-m_k \, a_{-m_k}}{} + \cdots$$

$$(z - \xi_k)^{m_k + 1}$$

$$\frac{f'}{f} = \frac{-m_k}{(z - \xi_k)}$$

residue is $-m_k$.

Thus

$$\frac{1}{2\pi i} \int_\alpha \frac{f'}{f} \, dz$$

$$= \sum n_k - m_k$$

$$= N - M$$

Changing variable

$$w = f(z)$$

$$\beta = f(\alpha)$$

LHS

$$\frac{1}{2\pi i} \int_{\beta} \frac{dw}{w}$$

$$= N(0, \beta)$$

$$= N(0, f(\alpha)).$$

$$= N - M.$$

e.g.

$$f(z) = z^m \prod_{k=1}^{n} \frac{z - a_k}{1 - \bar{a}_k z}$$

$m, n > 0$ $\quad 0 < |a_k| < 1$

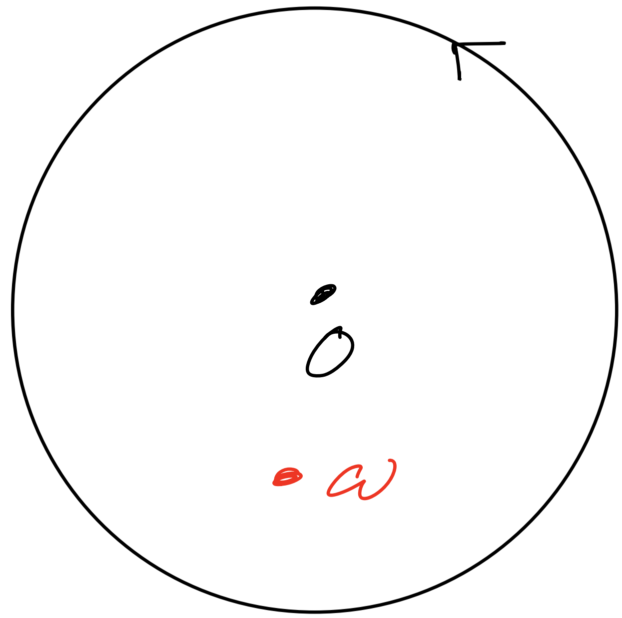
zeros at 0, a_k.

no poles in $|z| < 1$

$\Gamma : z = e^{it}, 0 \leq t \leq 2\pi,$

$$N(0, f(\Gamma)) = m + n$$

$f(\Gamma)$



$\Rightarrow$ winds around every w, $|w|<1$ exactly $m+n$.

$\therefore$ For every w, $|w|<1$,

$$f(z) = w$$

has $m+n$ roots.