

# Scientific Computing HW 1

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**P1.** Pick  $T < t^*$  where  $t^* := t_0 + \frac{1}{y_0}$ . Fix  $r \in \mathbb{R}$ . Then  $f(t, y) := y^2$  is continuous on the cylinder  $Q := \{t_0 \leq t \leq T, |y - y_0| \leq r\}$ . Now for all  $(t, y) \in Q$ ,

$$|y - y_0| \leq r \implies |y| = |y - y_0 + y_0| \leq |y - y_0| + |y_0| \leq r + |y_0| \implies |f(t, y)| = |y^2| = |y|^2 \leq (r + |y_0|)^2$$

i.e.  $|f| \leq M$  on  $Q$  where  $M := (r + |y_0|)^2$ . Then by theorem 1, the IVP has a solution for  $0 \leq t - t_0 \leq \min(\frac{r}{M}, T - t_0)$ .

**P2.** Pick  $0 \leq T < \infty$ . Fix  $r \in \mathbb{R}$ . Then  $f(t, y) := 2y^{1/2}$  is continuous on the cylinder  $Q := \{t_0 \leq t \leq T, |y - y_0| \leq r\}$ . Now for all  $(t, y) \in Q$  with  $y \geq 0$ , using similar arguments as in P1,

$$|y - y_0| \leq r \implies y = |y| \leq r + |y_0| \implies |f(t, y)| = |2y^{1/2}| = 2y^{1/2} \leq 2(r + |y_0|)^{1/2}$$

i.e.  $|f| \leq M$  on  $Q$  where  $M := 2(r + |y_0|)^{1/2}$ . Then by theorem 1, the IVP has a solution for  $0 \leq t - t_0 \leq \min(\frac{r}{M}, T - t_0)$ .

To see that  $f(y) := 2y^{1/2}$  is not Lipschitz at  $y = 0$ , suppose there exists  $C > 0$  such that  $|f(x) - f(0)| \leq C|x - 0|$  for all  $x \geq 0$ , i.e.  $2x^{1/2} \leq Cx$ . But if we set  $x = \frac{1}{C^2}$ , we get  $2 \leq Cx^{1/2} = C\frac{1}{C} = 1$ , a contradiction.

**P3.** We compute the Picard iterates for  $f(t, y) := y^2$ ,  $t_0 = 0$ ,  $y_0 = 1$ , starting at  $y_1(t) = 1$ .

$$y_2(t) = 1 + \int_0^t f(s, y_1(s))ds = 1 + \int_0^t 1ds = 1 + t$$

$$y_3(t) = 1 + \int_0^t f(s, y_2(s))ds = 1 + \int_0^t (1 + s)^2 ds = 1 + \int_0^t (1 + 2s + s^2)ds = 1 + t + t^2 + \frac{1}{3}t^3$$

$$y_4(t) = 1 + \int_0^t \left(1 + 2s + 3s^2 + \frac{8}{3}s^3 + \frac{5}{3}s^4 + \frac{2}{3}s^5 + \frac{1}{9}s^6\right) ds = 1 + t + t^2 + t^3 + \frac{2}{3}t^4 + \frac{1}{3}t^5 + \frac{1}{9}t^6 + \frac{1}{63}t^7$$

The Picard iterates seem to converge to  $\frac{1}{1-t}$  for  $t < 1$ . Plugging values into the time interval given in P1, we get

$$M = (r + |y_0|)^2 = (r + 1)^2$$

$$\frac{r}{M} = \frac{r}{(r + 1)^2} \leq 1$$

$$T - t_0 = T < t^* = t_0 + \frac{1}{y_0} = 1$$

$$0 \leq t \leq \min\left(\frac{r}{M}, T - t_0\right) < 1$$

which is a smaller interval.

**P4. Pf.** Using the Taylor expansion of  $y$  at  $t_n$ ,

$$\begin{aligned}\tau_{n+1} = & y + hy' + \frac{1}{2}h^2y'' + \frac{1}{6}h^3y''' + O(h^4) - y \\ & - h \left[ \frac{23}{12}y' - \frac{4}{3} \left( y' - hy'' + \frac{1}{2}h^2y''' - \frac{1}{6}h^3y'''' + O(h^4) \right) + \frac{5}{12} \left( y' - 2hy'' + 2h^2y''' - \frac{4}{3}h^3y'''' + O(h^4) \right) \right]\end{aligned}$$

Collect coefficients of the following terms:

$$\begin{aligned}y : & 1 - 1 = 0 \\ hy' : & 1 - \frac{23}{12} + \frac{4}{3} - \frac{5}{12} = \frac{1}{12}(12 - 23 + 16 - 5) = 0 \\ h^2y'' : & \frac{1}{2} - \frac{4}{3} + \frac{5}{6} = \frac{1}{6}(3 - 8 + 5) = 0 \\ h^3y''' : & \frac{1}{6} + \frac{2}{3} - \frac{5}{6} = \frac{1}{6}(1 + 4 - 5) = 0 \\ h^4y'''' : & \frac{1}{24} - \frac{2}{9} + \frac{5}{9} \neq 0\end{aligned}$$

This gives  $\tau_{n+1} = O(h^4)$ . Thus the method is consistent of order 3.

**P6.**

- (a) Since we have four undetermined coefficients, we take a third order Taylor expansion. Any higher order expansion would give an overconstrained system.

$$\begin{aligned}\tau_{n+1} = & y + hy' + \frac{1}{2}h^2y'' + \frac{1}{6}h^3y''' + O(h^4) - a_0y - a_1 \left[ y' - hy' + \frac{1}{2}h^2y'' - \frac{1}{6}h^3y''' + O(h^4) \right] \\ & - h \left[ b_0y' + b_1(y' - hy'' + \frac{1}{2}h^2y''' - \frac{1}{6}h^3y'''' + O(h^4)) \right]\end{aligned}$$

Collect coefficients of the following terms. To seek consistency of the highest possible order (in this case order 3), set each sum equal to 0:

$$\begin{aligned}y : & 1 - a_0 - a_1 = 0 \\ hy' : & 1 + a_1 - b_0 - b_1 = 0 \\ h^2y'' : & \frac{1}{2} - \frac{1}{2}a_1 + b_1 = 0 \\ h^3y''' : & \frac{1}{6} + \frac{1}{6}a_1 - \frac{1}{2}b_1 = 0\end{aligned}$$

These equations form a linear system whose solution is  $a_0 = -4$ ,  $a_1 = 5$ ,  $b_0 = 4$ ,  $b_1 = 2$ .

- (b) Applying the method (with coefficients found in the last part) to  $y' = 0 =: f(t, y)$  with initial condition  $y(0) = a$ .

$$u_{n+1} + 4u_n - 5u_{n-1} = 0$$

With ansatz  $u_n = r^n$ , we get

$$0 = r^2 + 4r - 5 = (r + 5)(r - 1) \implies r = 1, -5 \implies u_n = A + B(-5)^n$$

The exact solution of the IVP is  $y(t) = a$ . Perturb the values of the first two iterates, say  $u_0 = a + \delta_0$  and  $u_1 = a + \delta_1$ . Then

$$a + \delta_0 = A + B, \quad a + \delta_1 = A - 5B \implies \delta_0 - \delta_1 = 6B$$

If  $\delta_0 \neq \delta_1$  then  $B \neq 0$ , in which case the solution blows up. Thus the method is unstable.