Scientific Computing HW 6

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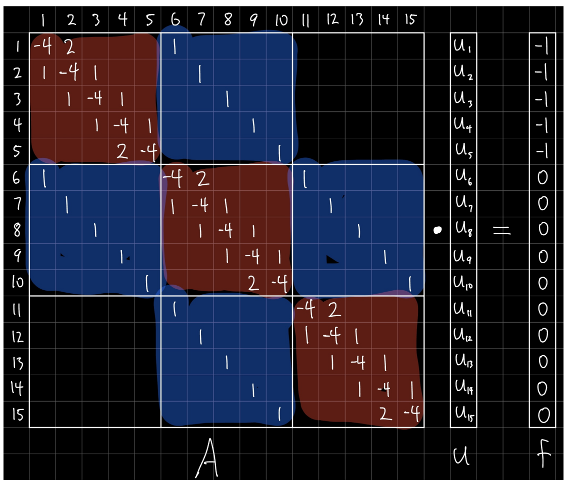
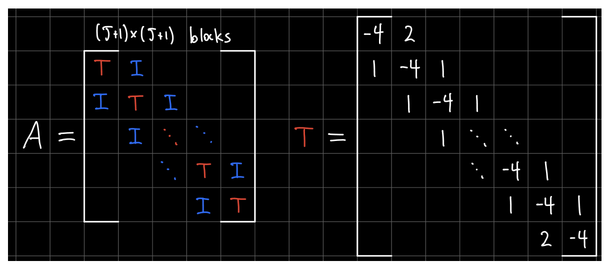
March 11, 2023

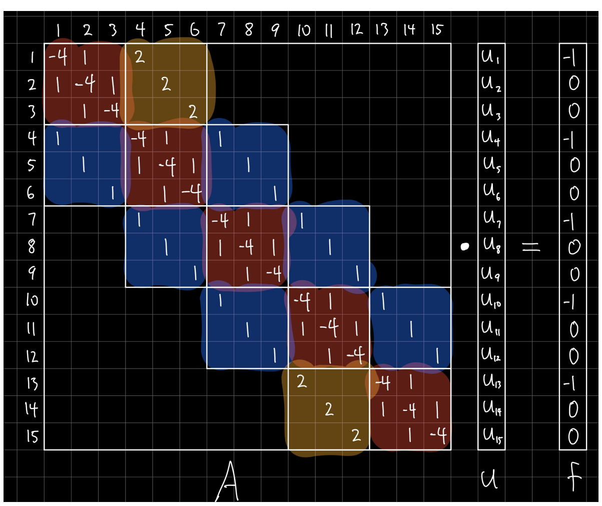
Problem 1.

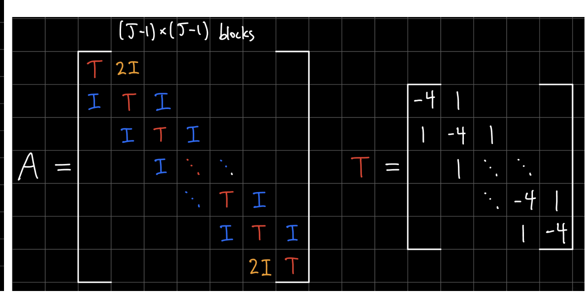
(a) The system $Au = f$ is shown on the left, and the block structure of A is compactly written on the right.



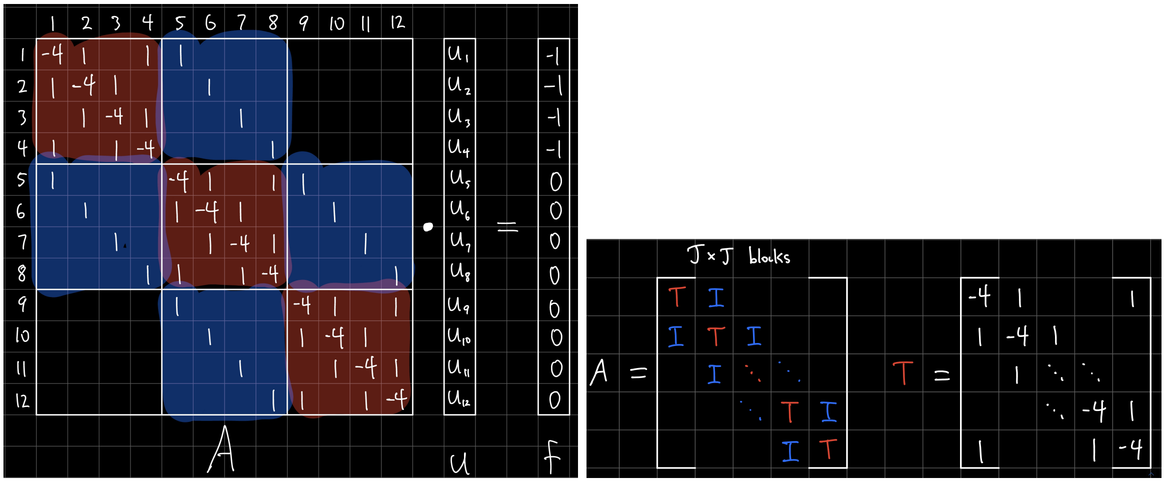


(b) The system $Au = f$ is shown on the left, and the block structure of A is compactly written on the right.

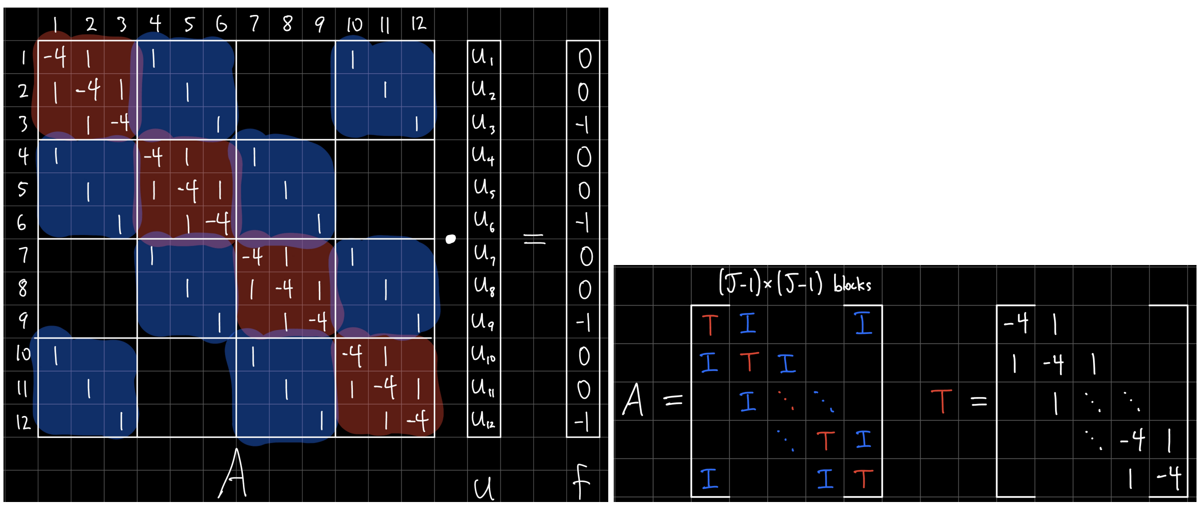




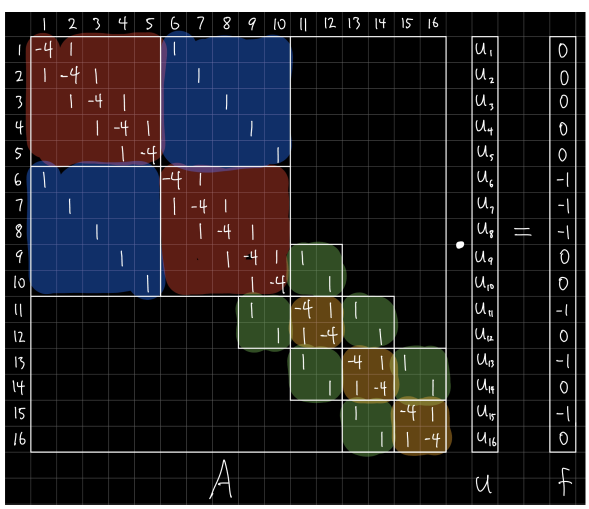
(c) The system $Au = f$ is shown on the left, and the block structure of A is compactly written on the right.



(d) The system $Au = f$ is shown on the left, and the block structure of A is compactly written on the right.



(e) The system $Au = f$ is shown on the left. The block structure of A is highlighted.



Problem 2. The BVP on the domain $\Omega := [-\pi, \pi] \times [0, 2]$ is

$$u_{xx} + u_{yy} = g(x) := \begin{cases} -\cos x, & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ 0, & \text{else} \end{cases} \qquad (2.1)$$

with BCs

$$u\Big|_{x=\pi} = u\Big|_{x=-\pi}, \quad u_x\Big|_{x=\pi} = u_x\Big|_{x=-\pi}, \quad u\Big|_{y=0} = 0, \quad u_y\Big|_{y=2} = 0$$

Fix $J \in \mathbb{N}$. Set mesh steps in the x and y directions,

$$h_x := \frac{2\pi}{J}, \quad h_y := \frac{2}{J}$$

Then

$$u_{xx}(x, y) = \frac{1}{h_x^2}\left[u(x + h_x, y) - 2u(x, y) + u(x - h_x, y)\right] + O(h_x^2)$$

$$u_{yy}(x, y) = \frac{1}{h_y^2}\left[u(x, y + h_y) - 2u(x, y) + u(x, y - h_y)\right] + O(h_y^2)$$

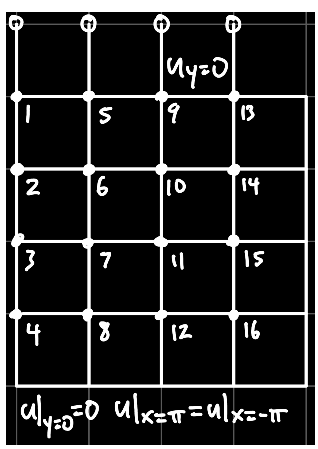
Plug these expressions into (2.1) and use the compass direction notation from lecture.

$$-2u_P\left[\frac{1}{h_x^2} + \frac{1}{h_y^2}\right] + \frac{1}{h_x^2}[u_E + u_W] + \frac{1}{h_y^2}[u_N + u_S] = g_P$$

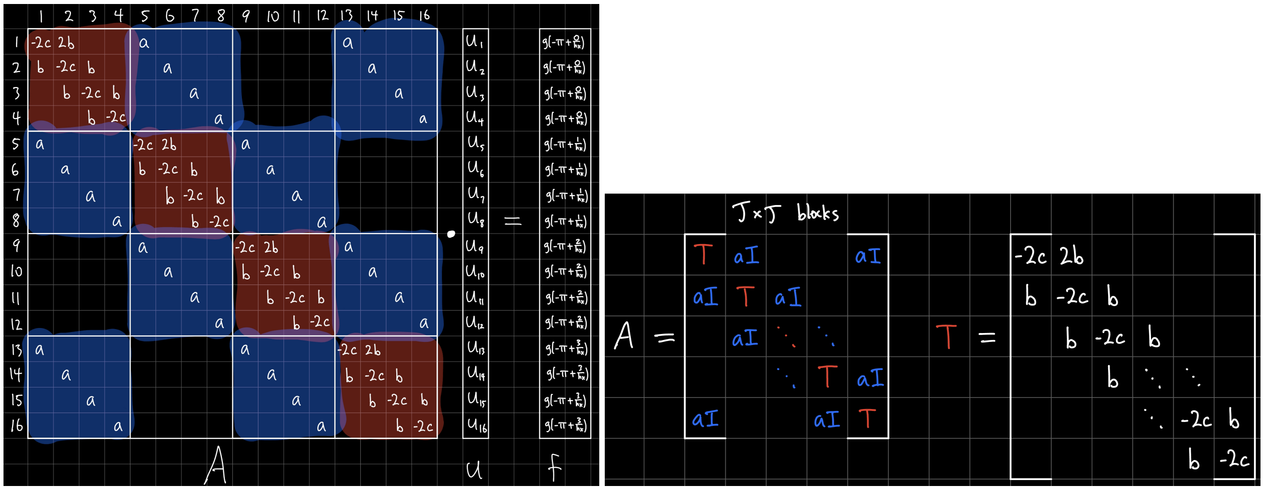
Set $a := \frac{1}{h_x^2}$, $b := \frac{1}{h_y^2}$, $c := a + b$, so that

$$-2cu_P + a[u_E + u_W] + b[u_N + u_S] = g_P \qquad (2.2)$$

To explore the appropriate numerical method, take $J = 4$. The mesh steps create a mesh from Ω with $(J + 1)^2 = 25$ points. Using the BCs, it is enough to solve for the values of 16 points, labeled below.



Using the fact $u_y\Big|_{y=2} = 0$, the values of the "ghost points" above the mesh from left to right are, respectively, u_2, u_6, u_{10}, u_{14}. Apply (2.2) to each point in the mesh to obtain a system $Au = f$, shown below on the left. The block structure of A is compactly written on the right.



The stationary heat distribution is solved for $J = 100$ and plotted below.
Code: `https://github.com/RokettoJanpu/Scientific-Computing-2/blob/main/hw6.ipynb`