

# Scientific Computing HW 8

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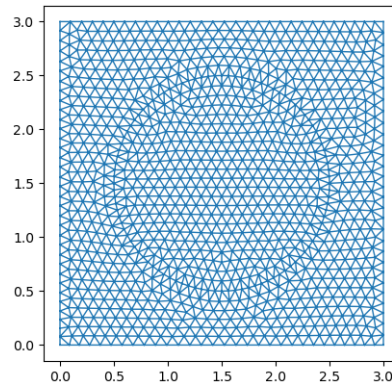
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**Problem 1.** Code: <https://github.com/RokettoJanpu/Scientific-Computing-2/blob/main/hw8.ipynb>

These functions were written:

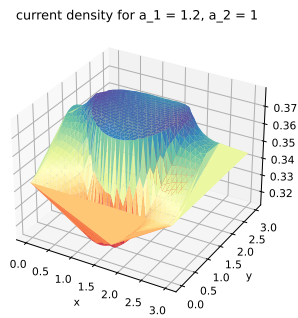
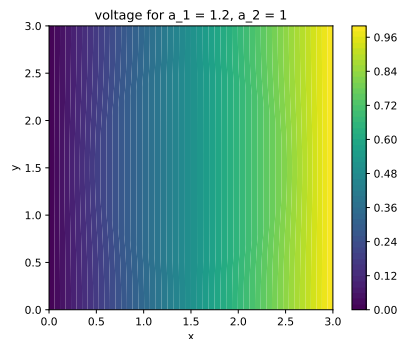
- `a_func`: Determine the conductivity at a point.
- `find_vert_line_pts`: Find the mesh points along a given vertical line
- `FEM_hw_solver`: FEM solver for the BVP in the HW.

We use the function `distmesh2D` to generate the mesh. For the fixed points, we pick the corners of the square and 100 evenly spaced points on the circle with center  $(1.5, 1.5)$  and radius 1.



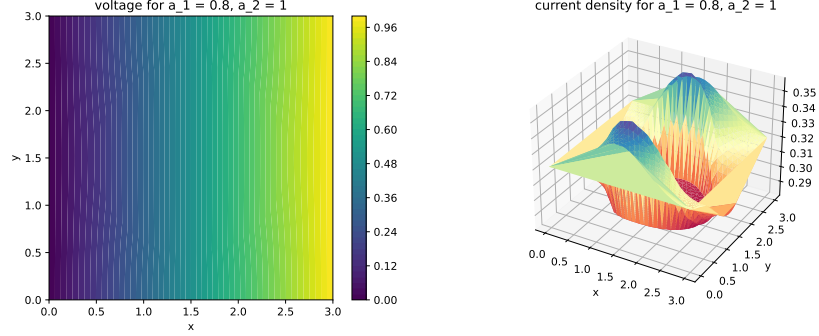
We solve for the voltage with the function `FEM_hw_solver`. We then plot the voltage and the current density.

- For  $a_1 = 1.2$  and  $a_2 = 1$ :



Note that in this case  $a_1 > a_2$ , which visually corresponds to the fact that the current density is greater on the inside of the circle than the outside.

- For  $a_1 = 0.8$  and  $a_2 = 1$ :



Note that in this case  $a_1 < a_2$ , which visually corresponds to the fact that the current density is greater on the outside of the circle than the inside.

These cases illustrate the principle of “the path of least resistance”, in the sense that current flow is higher in regions of greater electrical conductivity.

## Problem 2.

- (a) **Pf.** Write

$$\begin{aligned} u(x) &= \int_0^1 G(x, y) f(y) dy = \int_0^x G(x, y) f(y) dy + \int_x^1 G(x, y) f(y) dy \\ &= (1-x) \int_0^x y f(y) dy + x \int_x^1 (1-y) f(y) dy = (1-x) \int_0^x y f(y) dy - x \int_1^x (1-y) f(y) dy \end{aligned}$$

Then compute

$$\begin{aligned} u'(x) &= - \int_0^x y f(y) dy + (1-x) x f(x) - \int_1^x (1-y) f(y) dy - x(1-x) f(x) \\ &= - \int_0^x y f(y) dy - \int_1^x f(y) dy + \int_1^x y f(y) dy = - \int_1^x f(y) dy \\ &\implies u''(x) = -f(x) \end{aligned}$$

- (b) **Pf.** Compute

$$G_y(x, y) = \begin{cases} 1-x, & y < x \\ -x, & y > x \end{cases}$$

Then

$$\begin{aligned} \int_0^1 v'(y) G_y(x, y) dy &= \int_0^x v'(y) G_y(x, y) dy + \int_x^1 v'(y) G_y(x, y) dy \\ &= (1-x) \int_0^x v'(y) dy - x \int_x^1 v'(y) dy = \int_0^x v'(y) dy - x \int_0^x v'(y) dy - x \int_x^1 v'(y) dy = \int_0^x v'(y) dy - x \int_0^1 v'(y) dy \\ &= v(x) - v(0) - x[v(1) - v(0)] = v(x) - 0 - x[0 - 0] = v(x) \end{aligned}$$

Since  $u \in H_0^1([0, 1])$ , the identity in particular holds for  $u$ .

(c) The linear interpolant  $I_h u$ , as a linear combination of the basis functions  $\varphi_j$ , is

$$I_h u = \sum_{j=1}^n u(x_j) \varphi_j$$

(d) **Pf.** The function

$$\psi_j(x) := G(x_j, y) = \begin{cases} (1 - x_j)y, & y \leq x_j \\ x_j(1 - y), & y > x_j \end{cases}$$

satisfies  $\psi_j(0) = \psi_j(1) = 0$  and is linear on each interval  $[x_i, x_{i+1}]$  for  $0 \leq i \leq n$ , so  $\psi_j \in S_h$ . Also note that  $\psi_j$  is not differentiable at  $x_j$ , but is differentiable at all other points.

Suppose the functions  $\psi_j$  are linearly dependent, i.e. there exist scalars  $c_j$ , not all 0, so that

$$\sum_{j=1}^n c_j \psi_j = 0$$

In particular the linear combination

$$\sum_{j=1}^n c_j \psi_j$$

is differentiable. From the assumption we can pick  $i$  so that  $c_i \neq 0$ . Since the functions  $\psi_j$ , for  $j \neq i$ , are differentiable at  $x_i$ , the linear combination

$$\sum_{\substack{j=1 \\ j \neq i}}^n c_j \psi_j$$

is differentiable at  $x_i$ . Also, since  $\psi_i$  is not differentiable at  $x_i$  and  $c_i \neq 0$ , the function  $c_i \psi_i$  is not differentiable at  $x_i$ . This means the linear combination  $\sum_{j=1}^n c_j \psi_j$  is not differentiable at  $x_i$ , a contradiction. Thus the functions  $\psi_j$  are linearly independent.

Lastly, since  $\dim(S_h) = n$ , the  $n$  linearly independent functions  $\psi_j$  form a basis of  $S_h$ .

(e) **Pf.** Recall that

$$G_y(x, y) = \begin{cases} 1 - x, & y < x \\ -x, & y > x \end{cases}, \quad \varphi'_i(x) = \begin{cases} 0, & x < x_{i-1} \text{ or } x > x_{i+1} \\ \frac{1}{x_i - x_{i-1}}, & x_{i-1} < x < x_i \\ -\frac{1}{x_{i+1} - x_i}, & x_i < x < x_{i+1} \end{cases}$$

First write

$$\begin{aligned} \int_0^1 [I_h u]'(y) G_y(x_j, y) dy &= \int_0^1 \left[ \sum_{i=1}^n u(x_i) \varphi'_i(y) \right] G_y(x_j, y) dy \\ &= \sum_{i=1}^n u(x_i) \int_0^1 \varphi'_i(y) G_y(x_j, y) dy = \sum_{i=1}^n u(x_i) \underbrace{\int_{x_{i-1}}^{x_{i+1}} \varphi'_i(y) G_y(x_j, y) dy}_{=: a_{ij}} \end{aligned}$$

Now examine cases.

- If  $i = j$ ,

$$a_{ij} = \int_{x_{j-1}}^{x_j} \frac{1 - x_j}{x_j - x_{j-1}} dy + \int_{x_j}^{x_{j+1}} \frac{-x_j}{-(x_{j+1} - x_j)} dy = 1 - x_j + x_j = 1$$

- If  $i \geq j + 1$ ,

$$a_{ij} = \int_{x_{i-1}}^{x_i} \frac{-x_j}{x_i - x_{i-1}} dy + \int_{x_i}^{x_{i+1}} \frac{-x_j}{-(x_{i+1} - x_i)} dy = -x_j + x_j = 0$$

- If  $i \leq j - 1$ ,

$$a_{ij} = \int_{x_{i-1}}^{x_i} \frac{1 - x_j}{x_i - x_{i-1}} dy + \int_{x_i}^{x_{i+1}} \frac{1 - x_j}{-(x_{i+1} - x_i)} dy = 1 - x_j - (1 - x_j) = 0$$

This means  $a_{ij} = \delta_{ij}$ , thus

$$\int_0^1 [I_h u]'(y) G_y(x_j, y) dy = \sum_{i=1}^n u(x_i) \delta_{ij} = u(x_j)$$

(f) **Pf.** e

**Problem 3.**

- (a) **Pf.** Using the identity  $\nabla \cdot (fv) = \nabla f \cdot v + f \nabla \cdot v$  for any scalar function  $f$  and vector field  $v$ ,

$$\begin{aligned} \nabla \cdot (e^{-\beta V(x)} \nabla u) + \beta e^{-\beta V(x)} f(x) \cdot \nabla u &= e^{-\beta V(x)} \Delta u - \beta e^{-\beta V(x)} \nabla V(x) \cdot \nabla u + \beta e^{-\beta V(x)} f(x) \cdot \nabla u \\ &= \beta e^{-\beta V(x)} [\beta^{-1} \Delta u - \nabla V(x) \cdot \nabla u + f(x) \cdot \nabla u] = \beta e^{-\beta V(x)} [\beta^{-1} \Delta u + b(x) \cdot \nabla u] \end{aligned}$$

and since  $\beta e^{-\beta V(x)} \neq 0$  for all  $x$ ,

$$\beta^{-1} \Delta u + b(x) \cdot \nabla u = 0 \iff \nabla \cdot (e^{-\beta V(x)} \nabla u) + \beta e^{-\beta V(x)} f(x) \cdot \nabla u = 0$$

hence the two BVPs are equivalent.

- (b) Decompose

$$b(x, y) = \begin{bmatrix} x - x^3 - 10xy^2 \\ -y - x^2y \end{bmatrix} = \underbrace{\begin{bmatrix} x - x^3 - xy^2 \\ -y - 3y^3 - x^2y \end{bmatrix}}_{=: F(x, y)} + \underbrace{\begin{bmatrix} -9xy^2 \\ 3y^3 \end{bmatrix}}_{=: f(x, y)}$$

We check

$$\begin{aligned} \nabla \times F &= \partial_x(-y - 3y^3 - x^2y) + \partial_y(x - x^3 - xy^2) = -2xy + 2xy = 0 \\ \nabla \cdot f &= \partial_x(-9xy^2) + \partial_y(3y^3) = -9y^2 + 9y^2 = 0 \end{aligned}$$

so the decomposition is as desired. For the vector field

$$-F(x, y) = \begin{bmatrix} -x + x^3 + xy^2 \\ y + 3y^3 + x^2y \end{bmatrix}$$

a potential  $V$  is found by

$$V(x, y) = \int (-x + x^3 + xy^2) dx = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{2}x^2y^2 + g(y)$$

for some function  $g$ , and

$$V(x, y) = \int (y + 3y^3 + x^2y) dy = \frac{1}{2}y^2 + \frac{3}{4}y^4 + \frac{1}{2}x^2y^2 + h(x)$$

for some function  $h$ . Putting the calculations together,

$$V(x, y) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{2}x^2y^2 + \frac{1}{2}y^2 + \frac{3}{4}y^4$$