Scientific Computing HW 3

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Problem 1. Pf. From the Butcher array,

$$A = \begin{bmatrix} \gamma & 0 \\ 1-\gamma & \gamma \end{bmatrix}, \quad b = \begin{bmatrix} 1-\gamma \\ \gamma \end{bmatrix}, \quad c = \begin{bmatrix} \gamma \\ 1 \end{bmatrix}$$

Check the 1st order accuracy condition.

$$\sum_{l=1}^{2} b_l = (1-\gamma) + \gamma = 1$$

Check the 2nd order accuracy condition.

$$\sum_{l=1}^{2} b_l c_l = (1-\gamma)\gamma + \gamma \cdot 1 = \gamma - \gamma^2 + \gamma = 2\gamma - \gamma^2 = 2 - 2^{1/2} - 1 - 2^{-1} + 2^{1/2} = 1 - 2^{-1} = \frac{1}{2}$$

Thus the method is 2nd order accurate. To show it is A–stable, first find the stability function $R(z)$ and let $|z| \to \infty$.

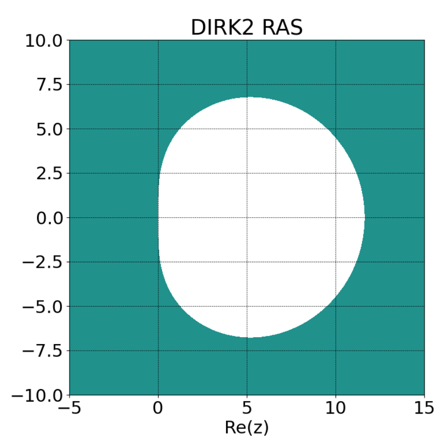
$$I - zA = \begin{bmatrix} 1-\gamma z & 0 \\ -(1-\gamma)z & 1-\gamma z \end{bmatrix} \implies D := \det(I - zA) = (1-\gamma z)^2 = \gamma^2 z^2 - 2\gamma z + 1$$

$$\implies (I - zA)^{-1} = \frac{1}{D}\begin{bmatrix} 1-\gamma z & 0 \\ (1-\gamma)z & 1-\gamma z \end{bmatrix} \implies (I-zA)^{-1}1_{s\times 1} = \frac{1}{D}\begin{bmatrix} 1-\gamma z \\ (1-\gamma)z + 1 - \gamma z \end{bmatrix} = \frac{1}{D}\begin{bmatrix} 1-\gamma z \\ (1-2\gamma)z + 1 \end{bmatrix}$$

$$R(z) - 1 = zb^T(I-zA)^{-1}1_{s\times 1} = \frac{z}{D}\left[(1-\gamma)(1-\gamma z) + \gamma((1-2\gamma)z+1)\right] = \frac{z}{D}\left[1 - \gamma z - \gamma + \gamma^2 z + (\gamma - 2\gamma^2)z + \gamma\right]$$

$$\implies R(z) - 1 = \frac{z}{D}\left[1 - \gamma^2 z\right] = \frac{-\gamma^2 z^2 + z}{\gamma^2 z^2 - 2\gamma z + 1} \implies R(z) = \frac{-\gamma^2 z^2 + z}{\gamma^2 z^2 - 2\gamma z + 1} + 1 \xrightarrow{|z|\to\infty} -1 + 1 = 0$$

To finish showing A–stability, we plot the RAS and see that it contains the left half plane. Code in 2nd cell of: `https://github.com/RokettoJanpu/Scientific-Computing-2/blob/main/hw3%20RAS.ipynb`



Problem 2. From the Butcher array,

$$A = \begin{bmatrix} \gamma & 0 \\ 1-2\gamma & \gamma \end{bmatrix}, b = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}, c = \begin{bmatrix} \gamma \\ 1-\gamma \end{bmatrix}$$

1. **Pf.** Check the 1st order accuracy condition.

$$\sum_{l=1}^{2} b_l = \frac{1}{2} + \frac{1}{2} = 1$$

Check the 2nd order accuracy condition.

$$\sum_{l=1}^{2} b_l c_l = \frac{1}{2}\gamma + \frac{1}{2}(1-\gamma) = \frac{1}{2}(\gamma + 1 - \gamma) = \frac{1}{2}$$

2. **Pf.** Check the 3rd order accuracy conditions.

$$\sum_{p,q,r} b_p a_{pq} a_{pr} = \frac{1}{2}[\gamma^2 + 2\gamma \cdot 0 + 0^2] + \frac{1}{2}[(1-2\gamma)^2 + 2(1-2\gamma)\gamma + \gamma^2]$$

The quantity in the second bracket is

$$(1-2\gamma)^2 + 2(1-2\gamma)\gamma + \gamma^2 = 1 + 4\gamma^2 - 4\gamma + 2\gamma - 4\gamma^2 + \gamma^2 = \gamma^2 - 2\gamma + 1 = (\gamma-1)^2$$

giving

$$\sum_{p,q,r} b_p a_{pq} a_{pr} = \frac{1}{2}[\gamma^2 + \gamma^2 - 2\gamma + 1] = \gamma^2 - \gamma + \frac{1}{2}$$

We find

$$\gamma^2 = \frac{1}{2} + \frac{3}{36} + 2\frac{3^{1/2}}{12} = \frac{1}{12}[3 + 1 + 2 \cdot 3^{1/2}] = \frac{1}{12}[4 + 2 \cdot 3^{1/2}] = \frac{1}{6}[2 + 3^{1/2}]$$

so finally,

$$\sum_{p,q,r} b_p a_{pq} a_{pr} = \frac{1}{6}[2 + 3^{1/2} - 3 - 3^{1/2} + 3] = \frac{1}{3}$$

3. First find the stability function $R(z)$.

$$I - zA = \begin{bmatrix} 1-\gamma z & 0 \\ -(1-2\gamma)z & 1-\gamma z \end{bmatrix} \implies D := \det(I - zA) = (1-\gamma z)^2 = \gamma^2 z^2 - 2\gamma z + 1$$

$$\implies (I - zA)^{-1} = \frac{1}{D}\begin{bmatrix} 1-\gamma z & 0 \\ (1-2\gamma)z & 1-\gamma z \end{bmatrix} \implies (I - zA)^{-1}1_{s\times 1} = \frac{1}{D}\begin{bmatrix} 1-\gamma z \\ (1-2\gamma)z + 1 - \gamma z \end{bmatrix} = \frac{1}{D}\begin{bmatrix} 1-\gamma z \\ (1-3\gamma)z + 1 \end{bmatrix}$$

$$R(z) - 1 = zb^T(I - zA)^{-1}1_{s\times 1} = \frac{z}{2D}\left[1 - \gamma z + (1-3\gamma)z + 1)\right] = \frac{z}{2D}\left[(1-4\gamma)z + 2\right] = \frac{1}{2}\frac{(1-4\gamma)z^2 + 2z}{\gamma^2 z^2 - 2\gamma z + 1}$$

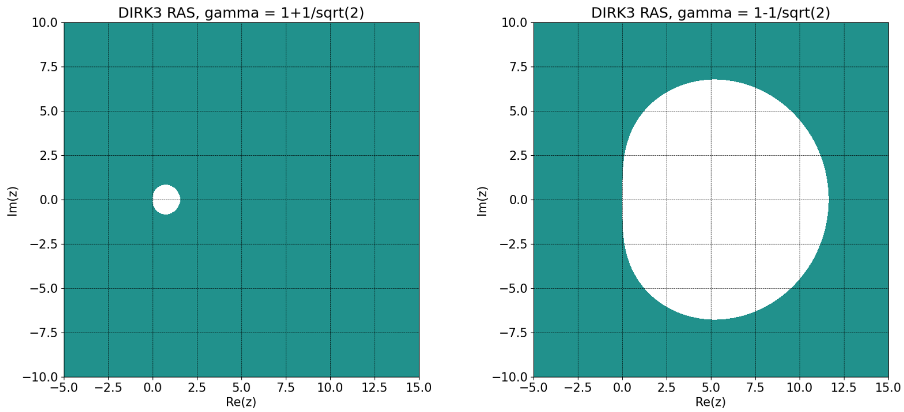
We find γ by imposing $\lim_{|z|\to\infty} R(z) = 0$.

$$\lim_{|z|\to\infty} R(z) = 0 \iff -1 = \frac{1}{2}\lim_{|z|\to\infty}\frac{(1-4\gamma)z^2 + 2z}{\gamma^2 z^2 - 2\gamma z + 1} \iff \lim_{|z|\to\infty}\frac{(1-4\gamma)z^2 + 2z}{\gamma^2 z^2 - 2\gamma z + 1} = -2 \iff \frac{1-4\gamma}{\gamma^2} = -2$$

$$\iff -2\gamma^2 = 1 - 4\gamma \iff 2\gamma^2 - 4\gamma + 1 = 0 \iff \gamma = \frac{4}{4} \pm \frac{(16-8)^{1/2}}{4} = 1 \pm \frac{2 \cdot 2^{1/2}}{4} = 1 \pm 2^{-1/2}$$

We check that the method for $\gamma = 1 \pm 2^{-1/2}$ is A–stable, hence L–stable, by plotting the RASes and seeing that they contain the left half plane. Code in 3rd cell of:

`https://github.com/RokettoJanpu/Scientific-Computing-2/blob/main/hw3%20RAS.ipynb`


DIRK3 RAS, gamma = 1+1/sqrt(2)
Im(z)
Re(z)
DIRK3 RAS, gamma = 1-1/sqrt(2)
Im(z)
Re(z)