Scientific Computing HW 7

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Problem 1.

(a) Multiplying the BVP by -1 and integrating it, we find $k(x)u' = M$ for some constant M, i.e. $u' = \frac{M}{k(x)}$. The solution is then

$$u(x) = u_a + \int_a^x \frac{M}{k(s)} ds$$

If $x \leq c$ then

$$u(x) = u_a + \int_a^x \frac{M}{k_1} ds = u_a + \frac{M}{k_1}(x-a)$$

If $x > c$ then

$$u(x) = u_a + \int_a^c \frac{M}{k(s)} ds + \int_c^x \frac{M}{k(s)} ds = u_a + \int_a^c \frac{M}{k_1} ds + \int_c^x \frac{M}{k_2} ds = u_a + \frac{M}{k_1}(c-a) + \frac{M}{k_2}(x-c)$$

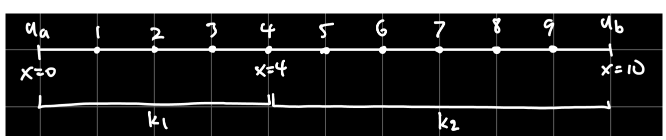
Apply BCs.

$$u_b = u(b) = u_a + \frac{M}{k_1}(c-a) + \frac{M}{k_2}(b-c) = u_a + M\left[\frac{c-a}{k_1} + \frac{b-c}{k_2}\right] \implies M = \frac{u_b - u_a}{\frac{c-a}{k_1} + \frac{b-c}{k_2}}$$

In summary, the solution is

$$u(x) = \begin{cases} u_a + \frac{M}{k_1}(x-a), & x \leq c \\ u_a + \frac{M}{k_1}(c-a) + \frac{M}{k_2}(x-c), & x > c \end{cases} \quad \text{where} \quad M = \frac{u_b - u_a}{\frac{c-a}{k_1} + \frac{b-c}{k_2}}$$

(b) Given the parameters, it is enough to solve for the values of the 9 mesh points shown below.



The finite difference scheme is

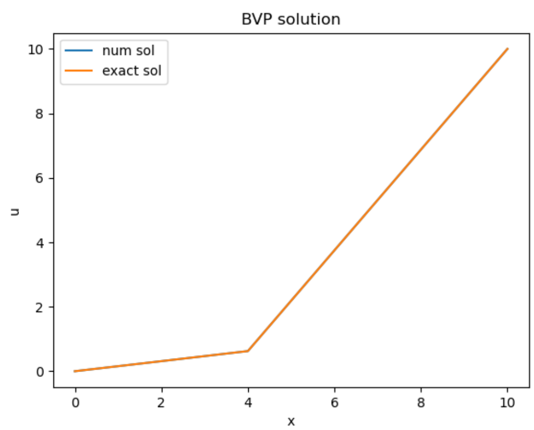
$$L_h u_P = -\frac{1}{h^2}\left[k_w u_W + k_e u_E - (k_e + k_w)u_P\right] = 0 \implies -(k_w + k_e)u_P + k_w u_W + k_e u_E = 0$$

Applying the scheme to each mesh point, we obtain a linear system.

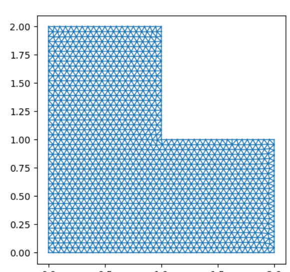
$$
\begin{array}{c|ccccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
1 & -2k_1 & k_1 & & & & & & & \\
2 & k_1 & -2k_1 & k_1 & & & & & & \\
3 & & k_1 & -2k_1 & k_1 & & & & & \\
4 & & & k_1 & -k_1-k_2 & k_2 & & & & \\
5 & & & & k_2 & -2k_2 & k_2 & & & \\
6 & & & & & k_2 & -2k_2 & k_2 & & \\
7 & & & & & & k_2 & -2k_2 & k_2 & \\
8 & & & & & & & k_2 & -2k_2 & k_2 \\
9 & & & & & & & & k_2 & -2k_2
\end{array}
\cdot
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{bmatrix}
=
\begin{bmatrix} -k_1u_a \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -k_2u_b \end{bmatrix}
$$

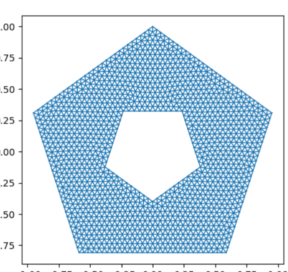
We solve it and plot the numerical solution u along with the exact solution from part (a). In this case the solutions agree exactly.

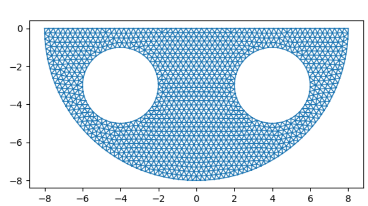
Code: `https://github.com/RokettoJanpu/Scientific-Computing-2/blob/main/hw7q1.ipynb`



Problem 2. Code:
`https://github.com/RokettoJanpu/Scientific-Computing-2/blob/main/hw7q2.ipynb`
Below are meshes for the L–shape, the pentagon with a smaller pentagon removed, and the semicircle with two smaller circles removed.







Problem 3.

(a) Multiply the BVP by -1 and multiply by w.

$$w(x)u''(x) = -w(x)f(x)$$

Integrate on $[0,1]$. Integrating the LHS by parts, we repeatedly differentiate w and integrate u.

$$\begin{aligned}\int_0^1 w(x)u''(x)dx &= w(x)u'(x)\Big|_0^1 - w'(x)u(x)\Big|_0^1 + \int_0^1 w''(x)u(x)dx \\ &= w(1)u'(1) - w(0)u'(0) - w'(1)u(1) + w'(0)u(0) + \int_0^1 w''(x)u(x)dx \\ &= -w'(1) + w'(0) + \int_0^1 w''(x)u(x)dx\end{aligned}$$

We obtain an integral equation for u.

$$-w'(1)+w'(0)+\int_0^1 w''(x)u(x)dx = -\int_0^1 w(x)f(x)dx \implies \int_0^1 w''(x)u(x)dx = -\int_0^1 w(x)f(x)dx+w'(1)-w'(0)$$

(b) First define other basis functions

$$\varphi_0(x) := \begin{cases} 0, & x \geq x_1 \\ \frac{x_1-x}{x_1}, & x < x_1 \end{cases}, \quad \varphi_{N+1}(x) := \begin{cases} 0, & x \leq x_N \\ \frac{x-x_N}{x_{N+1}-x_N}, & x > x_N \end{cases}$$

When computing the stiffness matrix (with rows and columns starting at 0 instead of 1)

$$A_{ij} = \int_0^1 \varphi_i'(x)\varphi_j'(x)dx$$

we will use the fact it is symmetric. First compute

$$\varphi_i'(x) = \begin{cases} 0, & x < x_{i-1} \text{ or } x > x_{i+1} \\ \frac{1}{x_i-x_{i-1}}, & x_{i-1} < x < x_i \\ -\frac{1}{x_{i+1}-x_i}, & x_i < x < x_{i+1} \end{cases}, \quad \varphi_0'(x) = \begin{cases} 0, & x > x_1 \\ -\frac{1}{x_1}, & x < x_1 \end{cases}, \quad \varphi_{N+1}'(x) = \begin{cases} 0, & x < x_N \\ \frac{1}{x_{N+1}-x_N}, & x > x_N \end{cases}$$

Fix $1 \leq i \leq N$ and examine cases of the value of j.

- If $j = i$ then

  $$\varphi_i'(x)\varphi_j'(x) = \begin{cases} 0, & x < x_{i-1} \text{ or } x > x_{i+1} \\ \frac{1}{(x_i-x_{i-1})^2}, & x_{i-1} < x < x_i \\ \frac{1}{(x_{i+1}-x_i)^2}, & x_i < x < x_{i+1} \end{cases}$$

  hence

  $$A_{ij} = \frac{x_i - x_{i-1}}{(x_i - x_{i-1})^2} + \frac{x_{i+1} - x_i}{(x_{i+1} - x_i)^2} = \frac{1}{x_i - x_{i-1}} + \frac{1}{x_{i+1} - x_i}$$

- If $j = i+1$ then

  $$\varphi_i'(x)\varphi_j'(x) = \begin{cases} 0, & x < x_i \text{ or } x > x_{i+1} \\ -\frac{1}{(x_{i+1}-x_i)^2}, & x_i < x < x_{i+1} \end{cases}$$

  hence

  $$A_{ij} = -\frac{x_{i+1} - x_i}{(x_{i+1} - x_i)^2} = -\frac{1}{x_{i+1} - x_i}$$

  In particular, if $j = i-1$ then $i = j+1$, so that by symmetry

  $$A_{ij} = A_{ji} = -\frac{1}{x_{j+1} - x_j} = -\frac{1}{x_i - x_{i-1}}$$

- $j \geq i+2$ then $\varphi_i'\varphi_j' = 0$ hence $A_{ij} = 0$. In particular, if $j \leq i-2$ then $i \geq j+2$, so that by symmetry $A_{ij} = A_{ji} = 0$.

Now examine cases for $i = 0$.

- If $j = 0$,
$$\varphi_0'(x)^2 = \begin{cases} 0, & x > x_1 \\ \frac{1}{x_1^2}, & x < x_1 \end{cases}$$
hence $A_{00} = \frac{1}{x_1}$.
- If $j = 1$, by symmetry $A_{01} = A_{10} = -\frac{1}{x_1 - x_0} = -\frac{1}{x_1}$.
- If $2 \leq j \leq N+1$ then $\varphi_0'\varphi_j' = 0$, hence $A_{0j} = 0$.

Examine cases for $i = N+1$.

- If $0 \leq j \leq N-1$ then $\varphi_{N+1}'\varphi_j' = 0$, hence $A_{N+1,j} = 0$.
- If $j = N$, by symmetry $A_{N+1,N} = A_{N,N+1} = -\frac{1}{x_{N+1} - x_N}$.
- If $j = N+1$,
$$\varphi_{N+1}'(x)^2 = \begin{cases} 0, & x < x_N \\ \frac{1}{(x_{N+1} - x_N)^2}, & x > x_N \end{cases}$$
hence $A_{N+1,N+1} = \frac{1}{x_{N+1} - x_N}$.

To summarize, for $1 \leq i \leq N$,

$$A_{ij} = \begin{cases} \frac{1}{x_i - x_{i-1}} + \frac{1}{x_{i+1} - x_i}, & j = i \\ -\frac{1}{x_{i+1} - x_i}, & j = i+1 \\ -\frac{1}{x_i - x_{i-1}}, & j = i-1 \\ 0, & j \leq i-2 \text{ or } j \geq i+2 \end{cases}$$

For $i = 0$,

$$A_{0j} = \begin{cases} \frac{1}{x_1}, & j = 0 \\ -\frac{1}{x_1}, & j = 1 \\ 0, & 2 \leq j \leq N+1 \end{cases}$$

For $i = N+1$,

$$A_{N+1,j} = \begin{cases} 0, & 0 \leq j \leq N-1 \\ -\frac{1}{x_{N+1} - x_N}, & j = N \\ \frac{1}{x_{N+1} - x_N}, & j = N+1 \end{cases}$$

We now compute the load vector b. For $1 \leq i \leq N$, we approximate

$$\int_0^1 \varphi_i(x) f(x) dx = \int_{x_{i-1}}^{x_i} \varphi_i(x) f(x) dx + \int_{x_i}^{x_{i+1}} \varphi_i(x) f(x) dx \approx f\left(\frac{x_{i-1} + x_i}{2}\right) \frac{x_i - x_{i-1}}{2} + f\left(\frac{x_i + x_{i+1}}{2}\right) \frac{x_{i+1} - x_i}{2}$$

From the BCs, set the vector u_D with components

$$(u_D)_i := \begin{cases} 1, & i = 0 \text{ or } i = N+1 \\ 0, & 1 \leq i \leq N \end{cases}$$

so that

$$Au_D = \begin{bmatrix} \frac{1}{x_1} \\ -\frac{1}{x_1} \\ 0 \\ \vdots \\ 0 \\ -\frac{1}{x_{N+1}-x_N} \\ \frac{1}{x_{N+1}-x_N} \end{bmatrix}$$

Thus for $1 \leq i \leq N$,

$$b_i = -(Au_D)_i + \int_0^1 \varphi_i(x) f(x) dx \approx \begin{cases} \frac{1}{x_1} + f\left(\frac{x_1}{2}\right)\frac{x_1}{2} + f\left(\frac{x_1+x_2}{2}\right)\frac{x_2-x_1}{2}, & i = 1 \\ f\left(\frac{x_{i-1}+x_i}{2}\right)\frac{x_i - x_{i-1}}{2} + f\left(\frac{x_i+x_{i+1}}{2}\right)\frac{x_{i+1}-x_i}{2}, & 2 \leq i \leq N-1 \\ \frac{1}{x_{N+1}-x_N} + f\left(\frac{x_{N-1}+x_N}{2}\right)\frac{x_N - x_{N-1}}{2} + f\left(\frac{x_N+x_{N+1}}{2}\right)\frac{x_{N+1}-x_N}{2}, & i = N \end{cases}$$

(c) In what case the FEM solution would coincide with the finite difference solution using the central difference scheme?

Problem 4.

(a) **Pf.** The matrix G is given by

$$G = A^{-1}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A := \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

We find A^{-1} by its adjugate. By cofactor expansion over the first row,

$$D := \det(A) = \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = x_2y_3 - x_3y_2 - x_1y_3 + x_3y_1 + x_1y_2 - x_2y_1$$

The matrix of cofactors is

$$\operatorname{cof}(A) = \begin{bmatrix} x_2y_3 - x_3y_2 & -x_1y_3 + x_3y_1 & x_1y_2 - x_2y_1 \\ -y_3 + y_2 & y_3 - y_1 & -y_2 + y_1 \\ x_3 - x_2 & -x_3 + x_1 & x_2 - x_1 \end{bmatrix}$$

The adjugate of A is

$$\operatorname{adj}(A) = \operatorname{cof}(A)^T = \begin{bmatrix} x_2y_3 - x_3y_2 & -y_3 + y_2 & x_3 - x_2 \\ -x_1y_3 + x_3y_1 & y_3 - y_1 & -x_3 + x_1 \\ x_1y_2 - x_2y_1 & -y_2 + y_1 & x_2 - x_1 \end{bmatrix}$$

Thus

$$G = \frac{1}{D}\operatorname{adj}(A)\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{D}\begin{bmatrix} y_2 - y_3 & x_3 - x_2 \\ y_3 - y_1 & x_1 - x_3 \\ y_1 - y_2 & x_2 - x_1 \end{bmatrix}$$

Fix an even permutation (i, j, k) of $1, 2, 3$ (i.e. one of $(1,2,3),\ (2,3,1),\ (3,1,2)$).

$$\eta_i(x, y) = \frac{\begin{vmatrix} 1 & x & y \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix}}{\begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix}}$$

Since the permutation (i,j,k) is even, the denominator is $\det\left(A^T\right) = \det(A) = D$. By cofactor expansion over the first row, the numerator is $x_j y_k - x_k y_j - (y_k - y_j)x + (x_k - x_j)y$, so that

$$\partial_x \eta_i(x,y) = \frac{1}{D}(y_j - y_k), \quad \partial_y \eta_i(x,y) = \frac{1}{D}(x_k - x_j)$$

Thus

$$\begin{bmatrix} \partial_x \eta_1(x,y) & \partial_y \eta_1(x,y) \\ \partial_x \eta_2(x,y) & \partial_y \eta_2(x,y) \\ \partial_x \eta_3(x,y) & \partial_y \eta_3(x,y) \end{bmatrix} = \frac{1}{D} \begin{bmatrix} y_2 - y_3 & x_3 - x_2 \\ y_3 - y_1 & x_1 - x_3 \\ y_1 - y_2 & x_2 - x_1 \end{bmatrix} = G$$

(b) Using the functions η_j from (a) and the fact $\boldsymbol{\nabla}\eta_j$ is the jth row of G,

$$u(x,y) = \sum_{j=1}^{3} u_j \eta_j(x,y) \implies \boldsymbol{\nabla} u(x,y) = \sum_{j=1}^{3} u_j \boldsymbol{\nabla} \eta_j(x,y) = \frac{1}{D}\left(u_1 \begin{bmatrix} y_2 - y_3 \\ x_3 - x_2 \end{bmatrix} + u_2 \begin{bmatrix} y_3 - y_1 \\ x_1 - x_3 \end{bmatrix} + u_3 \begin{bmatrix} y_1 - y_2 \\ x_2 - x_1 \end{bmatrix} \right)$$

$$\implies \boldsymbol{\nabla} u(x,y) = (x_2 y_3 - x_3 y_2 - x_1 y_3 + x_3 y_1 + x_1 y_2 - x_2 y_1)^{-1} \begin{bmatrix} u_1(y_2 - y_3) + u_2(y_3 - y_1) + u_3(y_1 - y_2) \\ u_1(x_3 - x_2) + u_2(x_1 - x_3) + u_3(x_2 - x_1) \end{bmatrix}$$