Scientific Computing HW 8

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Problem 1. Code: `https://github.com/RokettoJanpu/Scientific-Computing-2/blob/main/hw8.ipynb`
Problem 2.

(a) **Pf.** Write

$$u(x) = \int_0^1 G(x,y)f(y)dy = \int_0^x G(x,y)f(y)dy + \int_x^1 G(x,y)f(y)dy$$

$$= (1-x)\int_0^x yf(y)dy + x\int_x^1 (1-y)f(y)dy = (1-x)\int_0^x yf(y)dy - x\int_1^x (1-y)f(y)dy$$

Then compute

$$u'(x) = -\int_0^x yf(y)dy + (1-x)xf(x) - \int_1^x (1-y)f(y)dy - x(1-x)f(x)$$

$$= -\int_0^x yf(y)dy - \int_1^x f(y)dy + \int_1^x yf(y)dy = -\int_1^x f(y)dy$$

$$\implies u''(x) = -f(x)$$

(b) **Pf.** Compute

$$G_y(x,y) = \begin{cases} 1-x, & y < x \\ -x, & y > x \end{cases}$$

Then

$$\int_0^1 v'(y)G_y(x,y)dy = \int_0^x v'(y)G_y(x,y)dy + \int_x^1 v'(y)G_y(x,y)dy$$

$$= (1-x)\int_0^x v'(y)dy - x\int_x^1 v'(y)dy = \int_0^x v'(y)dy - x\int_0^x v'(y)dy - x\int_x^1 v'(y)dy = \int_0^x v'(y)dy - x\int_0^1 v'(y)dy$$

$$= v(x) - v(0) - x[v(1) - v(0)] = v(x) - 0 - x[0-0] = v(x)$$

Since $u \in H_0^1([0,1])$, the identity in particular holds for u.

(c) The linear interpolant $I_h u$, as a linear combination of the basis functions φ_j, is

$$I_h u = \sum_{j=1}^n u(x_j)\varphi_j$$

(d) **Pf.** The function

$$\psi_j(x) := G(x_j, y) = \begin{cases} (1-x_j)y, & y \le x_j \\ x_j(1-y), & y > x_j \end{cases}$$

satisfies $\psi_j(0) = \psi_j(1) = 0$ and is linear on each interval $[x_i, x_{i+1}]$ for $0 \leq i \leq n$, so $\psi_j \in S_h$. Also note that ψ_j is not differentiable at x_j, but is differentiable at all other points.

Suppose the functions ψ_j are linearly dependent, i.e. there exist scalars c_j, not all 0, so that

$$\sum_{j=1}^{n} c_j \psi_j = 0$$

In particular the linear combination

$$\sum_{j=1}^{n} c_j \psi_j$$

is differentiable. From the assumption we can pick i so that $c_i \neq 0$. Since the functions ψ_j, for $j \neq i$, are differentiable at x_i, the linear combination

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} c_j \psi_j$$

is differentiable at x_i. Also, since ψ_i is not differentiable at x_i and $c_i \neq 0$, the function $c_i \psi_i$ is not differentiable at x_i. This means the linear combination $\sum_{j=1}^{n} c_j \psi_j$ is not differentiable at x_i, a contradiction. Thus the functions ψ_j are linearly independent.

Lastly, since $\dim(S_h) = n$, the n linearly independent functions ψ_j form a basis of S_h.

(e) **Pf.** Recall that

$$G_y(x, y) = \begin{cases} 1 - x, & y < x \\ -x, & y > x \end{cases}, \quad \varphi_i'(x) = \begin{cases} 0, & x < x_{i-1} \text{ or } x > x_{i+1} \\ \frac{1}{x_i - x_{i-1}}, & x_{i-1} < x < x_i \\ -\frac{1}{x_{i+1} - x_i}, & x_i < x < x_{i+1} \end{cases}$$

First write

$$\int_0^1 [I_h u]'(y) G_y(x_j, y) dy = \int_0^1 \left[\sum_{i=1}^{n} u(x_i) \varphi_i'(y) \right] G_y(x_j, y) dy$$

$$= \sum_{i=1}^{n} u(x_i) \int_0^1 \varphi_i'(y) G_y(x_j, y) dy = \sum_{i=1}^{n} u(x_i) \underbrace{\int_{x_{i-1}}^{x_{i+1}} \varphi_i'(y) G_y(x_j, y) dy}_{=: a_{ij}}$$

Now examine cases.

- If $i = j$,
$$a_{ij} = \int_{x_{j-1}}^{x_j} \frac{1 - x_j}{x_j - x_{j-1}} dy + \int_{x_j}^{x_{j+1}} \frac{-x_j}{-(x_{j+1} - x_j)} dy = 1 - x_j + x_j = 1$$
- If $i \geq j + 1$,
$$a_{ij} = \int_{x_{i-1}}^{x_i} \frac{-x_j}{x_i - x_{i-1}} dy + \int_{x_i}^{x_{i+1}} \frac{-x_j}{-(x_{i+1} - x_i)} dy = -x_j + x_j = 0$$
- If $i \leq j - 1$,
$$a_{ij} = \int_{x_{i-1}}^{x_i} \frac{1 - x_j}{x_i - x_{i-1}} dy + \int_{x_i}^{x_{i+1}} \frac{1 - x_j}{-(x_{i+1} - x_i)} dy = 1 - x_j - (1 - x_j) = 0$$

This means $a_{ij} = \delta_{ij}$, thus

$$\int_0^1 [I_h u]'(y) G_y(x_j, y) dy = \sum_{i=1}^n u(x_i)\delta_{ij} = u(x_j)$$

(f) **Pf.** e

Problem 3.

(a) **Pf.** Using the identity $\boldsymbol{\nabla} \cdot (fv) = \boldsymbol{\nabla} f \cdot v + f \boldsymbol{\nabla} \cdot v$ for any scalar function f and vector field v,

$$\boldsymbol{\nabla} \cdot \left(e^{-\beta V(x)} \boldsymbol{\nabla} u\right) + \beta e^{-\beta V(x)} f(x) \cdot \boldsymbol{\nabla} u = e^{-\beta V(x)} \Delta u - \beta e^{-\beta V(x)} \boldsymbol{\nabla} V(x) \cdot \boldsymbol{\nabla} u + \beta e^{-\beta V(x)} f(x) \cdot \boldsymbol{\nabla} u$$

$$= \beta e^{-\beta V(x)} \left[\beta^{-1} \Delta u - \boldsymbol{\nabla} V(x) \cdot \boldsymbol{\nabla} u + f(x) \cdot \boldsymbol{\nabla} u\right] = \beta e^{-\beta V(x)} \left[\beta^{-1} \Delta u + b(x) \cdot \boldsymbol{\nabla} u\right]$$

and since $\beta e^{-\beta V(x)} \neq 0$ for all x,

$$\beta^{-1} \Delta u + b(x) \cdot \boldsymbol{\nabla} u = 0 \iff \boldsymbol{\nabla} \cdot \left(e^{-\beta V(x)} \boldsymbol{\nabla} u\right) + \beta e^{-\beta V(x)} f(x) \cdot \boldsymbol{\nabla} u = 0$$

hence the two BVPs are equivalent.

(b) Decompose

$$b(x, y) = \begin{bmatrix} x - x^3 - 10xy^2 \\ -y - x^2 y \end{bmatrix} = \underbrace{\begin{bmatrix} x - x^3 - xy^2 \\ -y - 3y^3 - x^2 y \end{bmatrix}}_{=:F(x,y)} + \underbrace{\begin{bmatrix} -9xy^2 \\ 3y^3 \end{bmatrix}}_{=:f(x,y)}$$

We check

$$\boldsymbol{\nabla} \times F = \partial_x(-y - 3y^3 - x^2 y) + \partial_y(x - x^3 - xy^2) = -2xy + 2xy = 0$$
$$\boldsymbol{\nabla} \cdot f = \partial_x(-9xy^2) + \partial_y(3y^3) = -9y^2 + 9y^2 = 0$$

so the decomposition is as desired. For the vector field

$$-F(x, y) = \begin{bmatrix} -x + x^3 + xy^2 \\ y + 3y^3 + x^2 y \end{bmatrix}$$

a potential V is found by

$$V(x, y) = \int (-x + x^3 + xy^2) dx = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{2}x^2 y^2 + g(y)$$

for some function g, and

$$V(x, y) = \int (y + 3y^3 + x^2 y) dy = \frac{1}{2}y^2 + \frac{3}{4}y^4 + \frac{1}{2}x^2 y^2 + h(x)$$

for some function h. Putting the calculations together,

$$V(x, y) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{2}x^2 y^2 + \frac{1}{2}y^2 + \frac{3}{4}y^4$$