

# Scientific Computing HW 9

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**Problem 1.** The file Lagaris5.ipynb was modified:

<https://github.com/RokettoJanpu/Scientific-Computing-2/blob/main/hw9q1v2.ipynb>

- In the function ExactSolution, return  $y^2 \sin \pi x$ .
- In the function ActiveFun(x),

**Problem 2.** Using separation of variables, write

$$u(x, t) = X(x)T(t)$$

Plug into the PDE.

$$X(x)T'(t) = X''(x)T(t) \implies \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda \implies X'' + \lambda X = 0, T' + \lambda T = 0$$

The eigenvalue problem for  $X$ ,

$$X'' + \lambda X = 0, X(0) = X'(\pi) = 0$$

has eigenvalues and eigenfunctions

$$\lambda_n = (n + 1/2)^2, X_n(x) = \sin[(n + 1/2)x], n \geq 0$$

Solving the ODE for  $T$ ,

$$T' + \lambda_n T = 0 \implies T(t) = e^{-\lambda_n t}$$

Thus the solution to the BVP is, for some coefficients  $a_n$ ,

$$u(x, t) = \sum_{n \geq 0} a_n e^{-(n+1/2)^2 t} \sin[(n + 1/2)x]$$

Applying the IC,

$$x = u \Big|_{t=0} = \sum_{n \geq 0} a_n \sin[(n + 1/2)x]$$

Fix  $m \geq 0$ , multiply each side by  $X_m(x)$ , and integrate on  $[0, \pi]$ . Using the fact  $\int_0^\pi X_n(x)X_m(x)dx = \frac{\pi}{2} \delta_{nm}$ , the RHS becomes

$$\int_0^\pi \sum_{n \geq 0} a_n X_n(x)X_m(x)dx = \sum_{n \geq 0} a_n \int_0^\pi X_n(x)X_m(x)dx = \sum_{n \geq 0} a_n \frac{\pi}{2} \delta_{nm} = a_m \frac{\pi}{2}$$

Integrating by parts, the LHS becomes

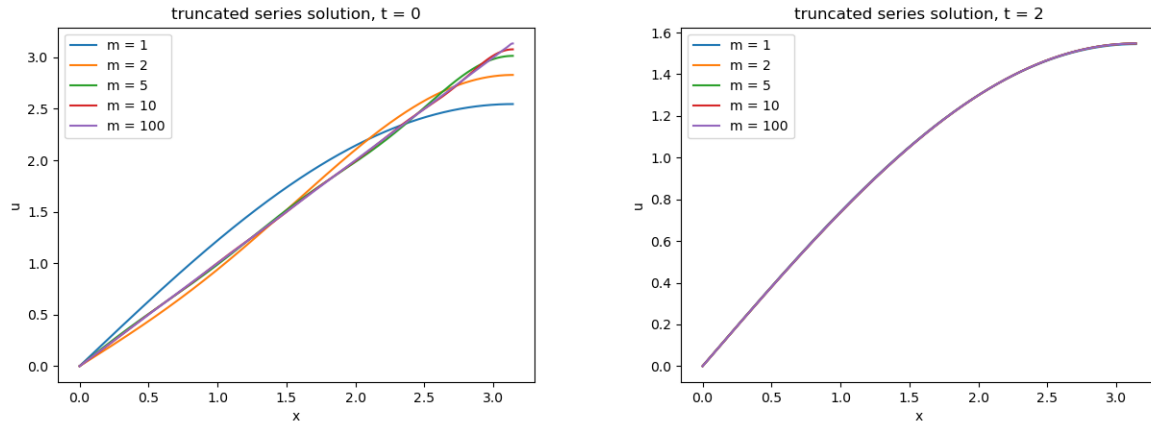
$$\int_0^\pi x \sin[(m + 1/2)x]dx = \left[ -\frac{1}{m + 1/2} x \cos[(m + 1/2)x] + \frac{1}{(m + 1/2)^2} \sin[(m + 1/2)x] \right] \Big|_0^\pi$$

$$= -0 + 0 + \frac{1}{(m + 1/2)^2}(-1)^m - 0 = \frac{(-1)^m}{(m + 1/2)^2}$$

Thus

$$a_m \frac{\pi}{2} = \frac{(-1)^m}{(m + 1/2)^2} \implies a_m = \frac{2(-1)^{m+1}}{\pi(m + 1/2)^2}$$

Code: <https://github.com/RokettoJanpu/Scientific-Computing-2/blob/main/hw9%20q2.ipynb>



In the same ipynb file, we find that the maximum value of  $|u(x, 0) - x|$  for  $m = 100$  is about  $6.37 \times 10^{-3}$ .