Scientific Computing HW 13

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Problem 1.

(a) Using $f(u) = au$,

$$u_j^* = u_j^n - \frac{ak}{h}(u_{j+1}^n - u_j^n)$$

In turn,

$$\begin{aligned}
u_j^{n+1} &= \frac{1}{2}\left[u_j^n + u_j^n - \frac{ak}{h}(u_{j+1}^n - u_j^n)\right] - \frac{ak}{2h}\left[u_j^n - \frac{ak}{h}(u_{j+1} - u_j^n) - u_{j-1}^n + \frac{ak}{h}(u_j^n - u_{j-1}^n)\right] \\
&= u_j^n - \frac{ak}{2h}(u_{j+1}^n - u_j^n + u_j^n - u_{j-1}^n) + \frac{a^2k^2}{2h^2}(u_{j+1}^n - u_j^n - u_j^n + u_{j-1}^n) \\
&= u_j^n - \frac{ak}{2h}(u_{j+1}^n - u_{j-1}^n) + \frac{a^2k^2}{2h^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n)
\end{aligned}$$

Which coincides with Lax–Wendroff.

Problem 4.

Write the PDE as

$$u_t = \underbrace{-u_{xxxx} - u_{xx}}_{=:Lu} \underbrace{-\frac{1}{2}(u^2)_x}_{=:N(u)}$$

First solving $u_t = Lu$, write

$$u(x,t) = \sum_{k=-\infty}^{\infty} u_k(t) e^{ikx/16}$$

so that

$$u_t = \sum_{k=-\infty}^{\infty} u_k'(t) e^{ikx/16}, \quad u_{xx} = \sum_{k=-\infty}^{\infty} u_k(t)\left(-\left(\frac{k}{16}\right)^2\right) e^{ikx/16}, \quad u_{xxxx} = \sum_{k=-\infty}^{\infty} u_k(t)\left(\frac{k}{16}\right)^4 e^{ikx/16}$$

Plugging in these derivatives and using the fact that the basis functions $e^{ikx/16}$ are linearly independent,

$$u_k'(t) = \left[\left(\frac{k}{16}\right)^2 - \left(\frac{k}{16}\right)^4\right] u_k(t) \implies u_k(t) = u_k(0) e^{[(k/16)^2 - (k/16)^4]t}$$

giving the solution

$$u(x,t) = \sum_{k=-\infty}^{\infty} u_k(0) e^{ikx/16} e^{[(k/16)^2 - (k/16)^4]t}, \quad u_k(0) = \frac{1}{32\pi}\int_0^{32\pi} u(x,0) e^{-ikx/16} dx$$

Define the solution operator e^{tL} by specifying its action on the basis functions $e^{ikx/16}$.

$$e^{tL}(e^{ikx/16}) := e^{ikx/16}e^{[(k/16)^2-(k/16)^4]t}$$

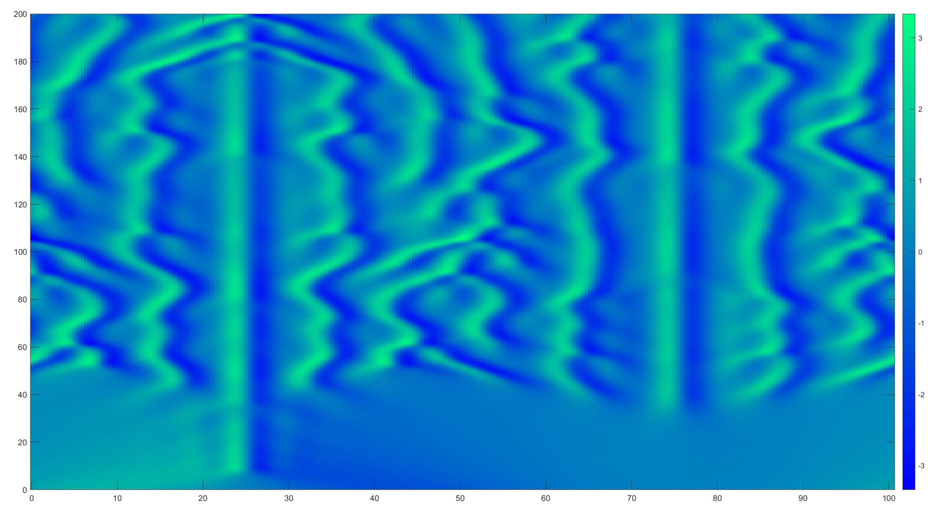
We check that we can rewrite the solution as

$$u(x,t) = \sum_{k=-\infty}^{\infty} u_k(0)e^{tL}(e^{ikx/16}) = e^{tL}\sum_{k=-\infty}^{\infty} u_k(0)e^{ikx/16} = e^{tL}u(x,0)$$

Let v satisfy $u = e^{tL}v$. Plugging into the equation $u_t = Lu + N(u)$, we obtain an equation for v.

$$Le^{tL}v + e^{tL}v_t = Le^{tL}v + N(e^{tL}v) \implies e^{tL}v_t = N(e^{tL}v) \implies v_t = e^{-tL}N(e^{tL}v)$$

The file KdVrkm.m was modified to solve this PDE.



The plot is very similar to the one in the linked article. The initial data does not vary significantly, but it splits into many high frequency waves, giving rise to a complex looking solution. On the other hand, characteristic lines appear, upon which solutions appear stationary, and moreover nearby solutions tend toward these lines.