

Scientific Computing HW 5

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March 7, 2023

Problem 1.

1. From $H(p, q) = T(p) + U(q)$,

$$\partial_p H(p, q) = T'(p), \quad \partial_q H(p, q) = U'(q)$$

Plug into the Stoermer-Verlet method.

$$p_{n+1/2} = p_n - \frac{1}{2}hU'(q_n)$$

$$q_{n+1} = q_n + \frac{1}{2}h[T'(p_{n+1/2}) + T'(p_{n+1/2})] = q_n + hT' \left(p_n - \frac{1}{2}hU'(q_n) \right)$$

$$p_{n+1} = p_n - \frac{1}{2}hU'(q_n) - \frac{1}{2}hU'(q_{n+1}) = p_n - \frac{1}{2}h \left[U'(q_n) + U' \left(q_n + hT' \left(p_n - \frac{1}{2}hU'(q_n) \right) \right) \right]$$

The RHS quantities are independent of p_{n+1}, q_{n+1} , so the method is explicit.

The Hamiltonian for the 1D simple harmonic oscillator is

$$H(p, q) = T(p) + U(q), \quad T(p) := \frac{p^2}{2m}, \quad U(q) := \frac{m\omega^2 q^2}{2}$$

First compute

$$T'(p) = \frac{p}{m}, \quad U'(q) = m\omega^2 q$$

Plug into the method.

$$q_{n+1} = q_n + hT' \left(p_n - \frac{1}{2}hm\omega^2 q_n \right) = q_n + \frac{h}{m} \left[p_n - \frac{1}{2}hm\omega^2 q_n \right] = \frac{h}{m}p_n + \left(1 - \frac{1}{2}h^2\omega^2 \right) q_n$$

$$p_{n+1} = p_n - \frac{1}{2}h \left[m\omega^2 q_n + m\omega^2 \left(q_n + \frac{h}{m} \left(p_n - \frac{1}{2}hm\omega^2 q_n \right) \right) \right]$$

In the above expression, collect coefficients of the following terms.

$$p_n : \quad 1 - \frac{1}{2}hm\omega^2 \frac{h}{m} = 1 - \frac{1}{2}h^2\omega^2$$

$$q_n : \quad -\frac{1}{2}h \left[m\omega^2 + m\omega^2 \left(1 + \frac{h}{m} \left(-\frac{1}{2}hm\omega^2 \right) \right) \right] = -\frac{1}{2}hm\omega^2 \left(2 - \frac{1}{2}h^2\omega^2 \right) = hm\omega^2 \left(\frac{1}{4}h^2\omega^2 - 1 \right)$$

Therefore

$$\begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = A \begin{bmatrix} p_n \\ q_n \end{bmatrix}, \quad A := \begin{bmatrix} a & b \\ c & a \end{bmatrix}, \quad a := 1 - \frac{1}{2}h^2\omega^2, \quad b := hm\omega^2 \left(\frac{1}{4}h^2\omega^2 - 1 \right), \quad c := \frac{h}{m}$$

2. **Pf.** We compute

$$\begin{aligned}
JA &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & a \end{bmatrix} = \begin{bmatrix} c & a \\ -a & -b \end{bmatrix} \\
\implies A^T JA &= \begin{bmatrix} a & c \\ b & a \end{bmatrix} \begin{bmatrix} c & a \\ -a & -b \end{bmatrix} = \begin{bmatrix} ac - ca & a^2 - bc \\ bc - a^2 & ba - ab \end{bmatrix} = \begin{bmatrix} 0 & a^2 - bc \\ -(a^2 - bc) & 0 \end{bmatrix} \\
a^2 - bc &= 1 + \frac{1}{4}h^4\omega^4 - h^2\omega^2 - h^2\omega^2 \left(\frac{1}{4}h^2\omega^2 - 1 \right) = 1 + \frac{1}{4}h^4\omega^4 - h^2\omega^2 - \frac{1}{4}h^4\omega^4 + h^2\omega^2 = 1 \\
\implies A^T JA &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = J
\end{aligned}$$

3. **Pf.** The shadow Hamiltonian is

$$H^*(p_n, q_n) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 \left[1 - \frac{1}{4}h^2\omega^2 \right] = \begin{bmatrix} p_n \\ q_n \end{bmatrix}^T S \begin{bmatrix} p_n \\ q_n \end{bmatrix}, \quad S := \begin{bmatrix} d & 0 \\ 0 & e \end{bmatrix}, \quad d := \frac{1}{2m}, \quad e := \frac{1}{2}m\omega^2 \left[1 - \frac{1}{4}h^2\omega^2 \right]$$

We compute

$$\begin{aligned}
SA &= \begin{bmatrix} d & 0 \\ 0 & e \end{bmatrix} \begin{bmatrix} a & b \\ c & a \end{bmatrix} = \begin{bmatrix} da & db \\ ec & ea \end{bmatrix} \\
\implies A^T SA &= \begin{bmatrix} a & c \\ b & a \end{bmatrix} \begin{bmatrix} da & db \\ ec & ea \end{bmatrix} = \begin{bmatrix} da^2 + ec^2 & dba + eac \\ bda + aec & db^2 + ea^2 \end{bmatrix} = \begin{bmatrix} da^2 + ec^2 & a(bd + ec) \\ a(bd + ec) & db^2 + ea^2 \end{bmatrix} \\
bd + ec &= \frac{1}{2}h\omega^2 \left[\frac{1}{4}h^2\omega^2 - 1 \right] + \frac{1}{2}h\omega^2 \left[1 - \frac{1}{4}h^2\omega^2 \right] = 0 \\
da^2 + ec^2 &= \frac{1}{2m} \left[1 + \frac{1}{4}h^4\omega^4 - h^2\omega^2 \right] + \frac{1}{2}m\omega^2 \left[1 - \frac{1}{4}h^2\omega^2 \right] \frac{h^2}{m^2} \\
&= \frac{1}{2m} \left[1 + \frac{1}{4}h^4\omega^4 - h^2\omega^2 + h^2\omega^2 - \frac{1}{4}h^4\omega^4 \right] \\
&= \frac{1}{2m} \\
&= d \\
db^2 + ea^2 &= \frac{1}{2m}h^2m^2\omega^4 \left[\frac{1}{4}h^2\omega^2 - 1 \right]^2 + \frac{1}{2}m\omega^2 \left[1 - \frac{1}{4}h^2\omega^2 \right] \left[1 + \frac{1}{4}h^4\omega^4 - h^2\omega^2 \right] \\
&= \frac{1}{2}m\omega^2 \left[1 - \frac{1}{4}h^2\omega^2 \right] \left[h^2\omega^2 \left(1 - \frac{1}{4}h^2\omega^2 \right) + 1 + \frac{1}{4}h^4\omega^4 - h^2\omega^2 \right] \\
&= \frac{1}{2}m\omega^2 \left[1 - \frac{1}{4}h^2\omega^2 \right] \left[h^2\omega^2 - \frac{1}{4}h^4\omega^4 + 1 + \frac{1}{4}h^4\omega^4 - h^2\omega^2 \right] \\
&= \frac{1}{2}m\omega^2 \left[1 - \frac{1}{4}h^2\omega^2 \right] \\
&= e
\end{aligned}$$

Put together,

$$\begin{aligned}
A^T SA &= \begin{bmatrix} d & a \cdot 0 \\ a \cdot 0 & e \end{bmatrix} = \begin{bmatrix} d & 0 \\ 0 & e \end{bmatrix} = S \\
\implies H^*(p_{n+1}, q_{n+1}) &= \begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix}^T S \begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} p_n \\ q_n \end{bmatrix}^T A^T SA \begin{bmatrix} p_n \\ q_n \end{bmatrix} = \begin{bmatrix} p_n \\ q_n \end{bmatrix}^T S \begin{bmatrix} p_n \\ q_n \end{bmatrix} = H^*(p_n, q_n)
\end{aligned}$$

Thus H^* is conserved.

Problem 2.

(a) The Hamiltonian equations of motion are

$$\dot{u} = -\partial_x H = -\frac{x}{(x^2 + y^2)^{3/2}}$$

$$\dot{v} = -\partial_y H = -\frac{y}{(x^2 + y^2)^{3/2}}$$

$$\dot{x} = -\partial_u H = u$$

$$\dot{y} = -\partial_v H = v$$

From the initial conditions, the total energy is

$$H \Big|_{t=0} = \frac{1}{2}0^2 + \frac{1}{2} \left(\frac{1}{2} \right)^2 - \frac{1}{(2^2 + 0^2)^{1/2}} = \frac{1}{8} - \frac{1}{2} = -\frac{3}{8} < 0$$