HYPERBOLIC EQUATIONS

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Contents

1. Linear advection equation

Please read Chapter 10 and Appendix E.3 in R. LeVeque "Finite difference methods for ordinary and partial differential equations". These notes are complimentary.

We will consider the linear advection equation on the interval $x \in [0,1]$ with the periodic boundary condition. the resulting initial and boundary value problem (IBVP) is

$$\begin{cases} u_t + au_x = 0, & x \in [0,1],\ t \geq 0, \\ u(0,t) = u(1,t), & t \geq 0 \\ u(x,0) = \eta(x). \end{cases} \tag{1}$$

The exact solution to (1) is $\eta(x - at)$ periodically extended, i.e., $\eta(x - at \mod 1)$.

1.1. Useful matrices and their spectra.

It is convenient to conduct the stability analysis of methods for linear advection equation by thinking of the *method of lines* and and the forward Euler time discretization. Below we will list matrices that arise the right-hand side of the method of lines and find their eigenvalues and eigenvectors.

Throughout this section, we will assume that the interval $[0,1]$ is partitioned to m subintervals of length $h = 1/m$. The periodic boundary conditions make the points $x_0 = 0$ and $x_m = 1$ identical. Therefore, we need to find the numerical solutions only at the points x_0, x_1, ..., x_{m-1} at each $t_n = kn$. Hence, the matrices in the right-hand side of the MOL will be $m \times m$.

- Matrix A_1 arises whenever the first derivative in space approximated using the central difference

$$\frac{U_{j+1} - U_{j-1}}{2h}$$

and the periodic boundary conditions are imposed:

$$A_1 = \begin{bmatrix} & 1 & & & -1 \\ -1 & & \ddots & & \\ & \ddots & & \ddots & \\ & & \ddots & & 1 \\ 1 & & & -1 & \end{bmatrix}. \tag{2}$$

In words, A_1 has 1s along its first superdiagonal, −1s along is first subdiagonal, −1 in the top right corner, and 1 in the bottom left corner. All other entries of A_1 are zero. We can guess the form of the eigenvector of A_1 taking into account that its jth entry must be proportional to the difference between its nearest neighbors. Moreover, the eigenvectors of A_1 should be extendable periodically. Hence, we try the vector v with entries $v_j = e^{ijb}$ as the candidate for the eigenvector. Periodicity requires

$$1 = v_0 = v_m = e^{ijbm}.$$

Hence $jbm = 2\pi p$ where $p \in \mathbb{Z}$. Therefore, $b = \frac{2\pi p}{m}$. For all $0 \le j \le m-1$ we have:

$$[Av]_j = v_{j+1} - v_{j-1} = e^{\frac{2\pi pi}{m}(j+1)} - e^{\frac{2\pi pi}{m}(j-1)} = e^{\frac{2\pi pi}{m}j} 2i \sin\left(\frac{2\pi p}{m}\right) = v_j 2i \sin\left(\frac{2\pi p}{m}\right).$$

Furthermore, the m distinct eigenpairs correspond to $p = 0, 1, \dots, m-1$. Therefore, the eigenvectors and eigenvalues of A_1 are

$$v^p = \begin{bmatrix} 1 \\ e^{\frac{2\pi pi}{m}} \\ e^{\frac{2\pi pi}{m}2} \\ \vdots \\ e^{\frac{2\pi pi}{m}(m-1)} \end{bmatrix}, \quad \lambda_p = 2i \sin\left(\frac{2\pi p}{m}\right), \quad p = 0, 1, \dots, m-1. \tag{3}$$

- Matrix A_2 arises when the artificial viscosity, i.e. a term with

$$\frac{U_{j+1} - 2U_j + U_{j-1}}{h^2},$$

is added to the finite difference scheme and the periodic boundary conditions are imposed. It is added in the Lax-Friedrichs scheme for stability purposes. In the Lax-Wendroff scheme, it compensates the error term proportional to u_{xx} resulting from the finite difference approximation of the first derivative and makes the method stable. The matrix A_2 is:

$$A_2 = \begin{bmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 1 & & & 1 & -2 \end{bmatrix}. \tag{4}$$

Its eigenvectors are the same as those of A_1 (see (3)). The eigenvalues of A_2 are

$$\lambda_p = 2\cos\left(\frac{2\pi p}{m}\right) - 2, \quad p = 0, 1, \ldots, m-1. \tag{5}$$

- Matrices A_3 and A_4 arise in left and right upwind schemes, in which the first derivatives in x are approximated using

$$\frac{U_j - U_{j-1}}{h} \quad \text{and} \quad \frac{U_{j+1} - U_j}{h},$$

respectively:

$$A_3 = \begin{bmatrix} 1 & & & & -1 \\ -1 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & -1 \end{bmatrix}. \tag{6}$$

Their eigenvectors are the same as those of A_1 (see (3)). Their eigenvalues are

$$\lambda_p(A_3) = 1 - e^{-\frac{2\pi p i}{m}}, \quad \lambda_p(A_4) = e^{\frac{2\pi p i}{m}} - 1, \quad p = 0, 1, \ldots, m-1. \tag{7}$$

1.2. **Finite difference schemes and their spectra.**

1.2.1. *Lax-Friedrichs.* The scheme:

$$U_j^{n+1} = \frac{U_{j+1}^n + U_{j-1}^n}{2} - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right). \tag{8}$$

The local truncation error for Lax-Friedrichs is $O(h^2)+O(k)$ making it first-order accurate. The corresponding MOL discretization gives:

$$\frac{dU}{dt} = \frac{1}{2k}A_2 U - \frac{a}{2h}A_1 U. \tag{9}$$

The eigenvalues of the matrix $M := \frac{1}{2k}A_2 - \frac{a}{2h}A_1$ are:

$$\lambda_p = \frac{1}{k}\left(\cos\left(\frac{2\pi p}{m}\right) - 1\right) - i\frac{a}{h}\sin\left(\frac{2\pi p}{m}\right), \quad p = 0, 1, \ldots, m-1. \tag{10}$$

The Lax-Friedrichs method is obtained from the MOL equation (9) by using the forward Euler time stepping. Hence, the stability condition requires that $|k\lambda_p + 1| \le 1$, i.e.,

$$\left|\cos\left(\frac{2\pi p}{m}\right) - i\frac{ka}{h}\sin\left(\frac{2\pi p}{m}\right)\right| = \left[\cos^2\left(\frac{2\pi p}{m}\right) + \left(\frac{ka}{h}\right)^2 \sin^2\left(\frac{2\pi p}{m}\right)\right]^{1/2} < 1. \tag{11}$$

This condition holds iff

$$\left|\frac{ka}{h}\right| \le 1. \tag{12}$$

2. Modified equations

Consideration of modified equations is a convenient tool for the analysis of finite difference schemes for PDEs. A modified equation is an equation that is satisfied by the numerical solution more exactly than the original PDE. The extra terms in a modified equation explain the behavior of the numerical error.

2.1. A derivation of the modified equation for the Lax-Wendroff scheme.

We consider the linear advection equation

$$u_t + au_x = 0. \tag{13}$$

The Lax-Wendroff scheme for it is

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{a^2k^2}{2h^2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right). \tag{14}$$

Assume that a smooth function $v(t,x)$ satisfies the Lax-Wendroff scheme, i.e.,

$$\begin{aligned} v(t+k,x) = v(t,x) &- \frac{ak}{2h}\left(v(t,x+h) - v(t,x-h)\right) \\ &+ \frac{a^2k^2}{2h^2}\left(v(t,x+h) - v(t,x) + v(t,x-h)\right). \end{aligned} \tag{15}$$

To find a PDE for v we use Taylor expansions around (t,x):

$$\begin{aligned} &v + kv_t + \frac{k^2}{2}v_{tt} + \frac{k^3}{6}v_{ttt} + O(k^4) \\ =&v - \frac{ak}{2h}\left(2hv_x + 2\frac{h^3}{6}v_{xxx} + O(h^5)\right) + \frac{a^2k^2}{2h^2}\left(2\frac{h^2}{2}v_{xx} + 2\frac{h^4}{24}v_{xxxx} + O(h^6)\right) \end{aligned}$$

Canceling v and dividing by k we get:

$$v_t + av_x = \frac{k}{2}\left(a^2v_{xx} - v_{tt}\right) - \frac{k^2}{6}\left(v_{ttt} + \frac{ah^2}{k^2}v_{xxx}\right) + O(k^3). \tag{16}$$

Here we have taken into account that the Courant number $\nu := \frac{ak}{h}$ is constant.

Equation (16) implies that $v_t + av_x$ is at most $O(k)$. Let us show that it is actually $O(k^2)$ and that $v_{tt} - a^2v_{xx}$ is also $O(k^2)$. The next two equations are obtained by differentiating (16) with respect to t and x, respectively:

$$v_{tt} + av_{xt} = \frac{k}{2}\left(a^2v_{xxt} - v_{ttt}\right) + O(k^2), \tag{17}$$

$$v_{tx} + av_{xx} = \frac{k}{2}\left(a^2v_{xxx} - v_{ttx}\right) + O(k^2). \tag{18}$$

Multiplying (18) by a and subtracting it from (17) we obtain:

$$v_{tt} - a^2v_{xx} = \frac{k}{2}\left[\left(a^2v_{xx} - v_{tt}\right)_t - a\left(a^2v_{xx} - v_{tt}\right)_x\right] + O(k^2). \tag{19}$$

Equation (19) implies that $v_{tt} - a^2 v_{xx}$ is at most $O(k)$. But this means that

$$(20)\qquad v_{tt} - a^2 v_{xx} = \frac{k}{2}\left([O(k)]_t - a[O(k)]_x\right) + O(k^2) = O(k^2).$$

Therefore, $v_{tt} - a^2 v_{xx} = O(k^2)$ which, together with (16) implied that $v_t + av_x = O(k^2)$.

Plugging the relationship $v_{tt} - a^2 v_{xx} = O(k^2)$ into (16) we get

$$(21)\qquad v_t + av_x = -\frac{k^2}{6}\left(v_{ttt} + \frac{ah^2}{k^2} v_{xxx}\right) + O(k^3).$$

Since $v_t + av_x = O(k^2)$, we have

$$(22)\qquad \frac{\partial}{\partial t} = -a\frac{\partial}{\partial x} + O(k^2).$$

Hence

$$(23)\qquad \frac{\partial^3}{\partial t^3} = -a^3\frac{\partial^3}{\partial x^3} + O(k^2).$$

This means that, in (21), the term in the large parensethes on the right-hand side is:

$$v_{ttt} + \frac{ah^2}{k^2} v_{xxx} = a\left(-a^2 + \frac{h^2}{k^2}\right) v_{xxx}.$$

From the fact that $\nu = \frac{ak}{h}$ is constant we get $k = \frac{\nu h}{a}$ and $\frac{h}{k} = \frac{a}{\nu}$. Therefore, (21) can be rewritten as

$$(24)\qquad v_t + av_x = -\frac{\nu^2 h^2}{6a^2} a\left(-a^2 + \frac{a^2}{\nu^2}\right) v_{xxx} + O(h^3) = -\frac{ah^2}{6}\left(-\nu^2 + 1\right) v_{xxx} + O(h^3).$$

Equation (24) is the modified equation whose solution satisfies the Lax-Wendroff scheme exactly. If we are willing to truncate the higher-order terms on the right-hand side of (24), we obtain a simpler modified equation that the numerical solution satisfies approximately, but more exactly than the original PDE (13):

$$(25)\qquad \boxed{v_t + av_x + \frac{ah^2}{6}\left(1 - \nu^2\right) v_{xxx} = 0.}$$

The modified equations (24) and (25) have a leading-order extra extra term in comparison with the original advection equation (13) which is of the order $O(h^2)$ and dispersive, i.e., proportional to v_{xxx}. This indicates that the Lax-Wendroff scheme is second-order accurate and the numerical error in its numerical solution will be oscillatory.