

# Scientific Computing HW 10

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## Problem 1.

- (a) Write  $k_x = n\pi$ ,  $k_y = m\pi$  where  $1 \leq n, m \leq J-1$ , so that

$$v_{k_x, k_y}(x_r, y_s) = \sin n\pi x_r \sin m\pi y_s$$

Applying the discretized Laplace operator, we obtain

$$\frac{1}{h^2} [v|_{n+1, m} + v|_{n-1, m} + v|_{n, m+1} + v|_{n, m-1} - 4v|_{n, m}] \quad (1.1)$$

The bracked expression is

$$[\sin((n+1)\pi x_r) + \sin((n-1)\pi x_r)] \sin m\pi y_s + \sin n\pi x_r [\sin((m+1)\pi y_s) + \sin((m-1)\pi y_s)] - 4 \sin n\pi x_r \sin m\pi y_s$$

Using the identity

$$\sin a + \sin b = 2 \sin \frac{a+b}{2} \cos \frac{a-b}{2}$$

the bracked expression is

$$\begin{aligned} & 2 \sin n\pi x_r \cos \pi x_r \sin m\pi y_s + 2 \sin m\pi y_s \cos \pi y_s \sin n\pi x_r - 4 \sin n\pi x_r \sin m\pi y_s \\ &= \sin n\pi x_r \sin m\pi y_s \cdot 2 [\cos \pi x_r + \cos \pi y_s - 2] \end{aligned}$$

Hence the expression (1.1) equals

$$\sin n\pi x_r \sin m\pi y_s \cdot \frac{2}{h^2} [\cos \pi x_r + \cos \pi y_s - 2]$$

We then see that the eigenvalues are

$$\lambda = \frac{2}{h^2} [\cos \pi x_r + \cos \pi y_s - 2]$$

- (b) For stability, we demand that the eigenvalues of  $\Delta t A$ ,

$$\lambda \Delta t = \frac{2\Delta t}{h^2} [\cos \pi x_r + \cos \pi y_s - 2]$$

lie in the RAS of forward Euler,  $[-2, 0]$ . Note that  $-4 \leq \cos \pi x_r + \cos \pi y_s - 2 \leq 0$ , hence

$$-2 \leq \lambda \Delta t \leq 0 \iff \frac{2\Delta t}{h^2} \leq \frac{1}{2} \iff \Delta t \leq \frac{h^2}{4}$$

**Problem 2.**

- (a) Fix a finite time  $T$ , integer  $N$ , and timestep  $dt = \frac{T}{N}$ . Using a scheme based on the trapezoidal rule,

$$u_{n+1} = u_n + \frac{1}{2}dt(\Delta u_{n+1} + \Delta u_n) + dt \implies u_{n+1} - \frac{1}{2}dt\Delta u_{n+1} = dt + u_n + \frac{1}{2}dt\Delta u_n$$

Fix a test function  $v \in C^2(\Omega)$  with  $v|_{\partial\Omega} = 0$ . Multiply by  $v$  and integrate over  $\Omega$ .

$$\int_{\Omega} u_{n+1}v dx - \frac{1}{2}dt \int_{\Omega} v\Delta u_{n+1} dx = dt \int_{\Omega} v dx + \int_{\Omega} u_n v dx + \frac{1}{2}dt \int_{\Omega} v\Delta u_n dx$$

Using Green's first identity and the fact  $v|_{\partial\Omega} = 0$ ,

$$\int_{\Omega} v\Delta u_n dx = \int_{\partial\Omega} v \frac{\partial u_n}{\partial n} ds - \int_{\Omega} \nabla v \cdot \nabla u_n dx = - \int_{\Omega} \nabla v \cdot \nabla u_n dx$$

and similarly for the term involving  $\Delta u_{n+1}$ . We obtain the weak solution to the IBVP.

$$\int_{\Omega} u_{n+1}v dx + \frac{1}{2}dt \int_{\Omega} \nabla v \cdot \nabla u_{n+1} dx = dt \int_{\Omega} v dx + \int_{\Omega} u_n v dx + -\frac{1}{2}dt \int_{\Omega} \nabla v \cdot \nabla u_n dx$$

From this we write the FEM solution. Given a triangulation  $\tau$  of  $\Omega$ , let  $\eta_i$  be the piecewise linear basis functions from prior FEM problems. Then for each timestep  $n$ , we solve

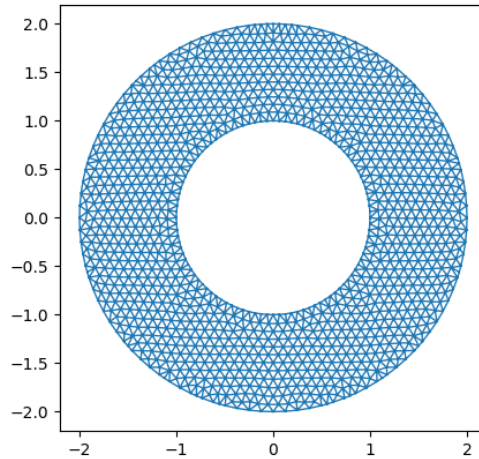
$$\left(B + \frac{1}{2}dtA\right)U_{n+1} = dtb + \left(B - \frac{1}{2}dtA\right)U_n$$

where

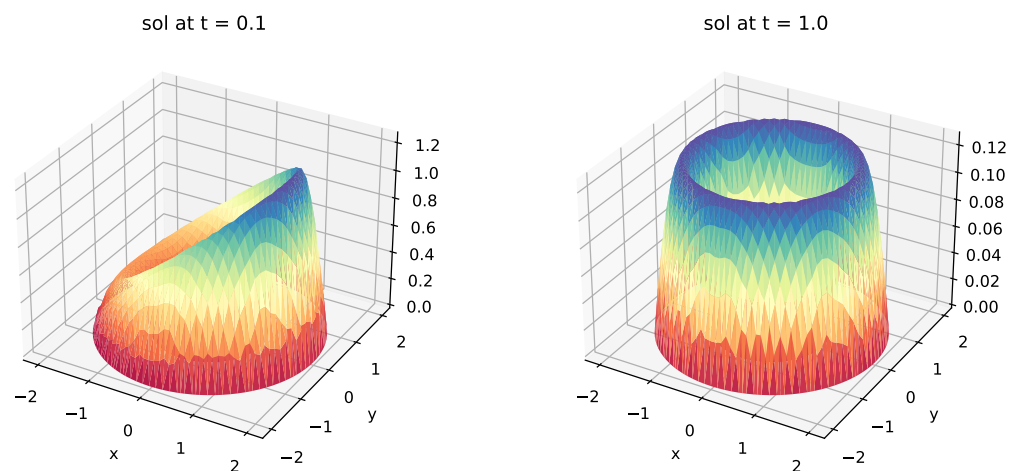
$$A_{ij} = \int_{\Omega} \nabla \eta_i \cdot \nabla \eta_j dx, \quad b_j = \int_{\Omega} \eta_j dx, \quad B_{ij} = \int_{\Omega} \eta_i \eta_j dx$$

- (b) Code: <https://github.com/RokettoJanpu/Scientific-Computing-2/blob/main/hw10.ipynb>

We set a mesh for the annulus with center  $(0,0)$ , inner radius 1, and outer radius 2.



Below is the numerical solution at  $t = 0.1$  and  $t = 1$ .



Below are the steady state numerical and exact solutions plotted as functions of  $r$ .

