Homework 10. Due April 19

1. **(5 pts)** Consider the heat equation in the unit square $[0,1]^2$

$$u_t = \Delta u, \quad (x,y) \in [0,1]^2. \tag{1}$$

Impose the homogeneous Dirichlet boundary conditions: $u = 0$ on $\partial\Omega$. Consider the uniform square mesh $(J+1)\times(J+1)$ in the unit square $[0,1]^2$ and discretize Laplace's operator to this mesh using central differences and taking the boundary conditions into account. The resulting "method of lines" equation will be of the form

$$\frac{d}{dt}U = AU, \tag{2}$$

where A is the discretized operator Δ, an $(J-1)^2 \times (J-1)^2$ matrix.

(a) Verify that the eigenvectors of A are

$$v_{k_x,k_y}(x_r, y_s) = \sin(k_x x_r)\sin(k_y y_s), \quad k_x, k_y = \pi, 2\pi, \ldots, (J-1)\pi, \tag{3}$$

where (x_r, y_s) are the mesh points. Find the corresponding eigenvalues.

(b) Suppose we are applying the forward Euler time discretization to (2). Find the relationship between the time step Δt and the mesh step h such that this method is stable for all J.

2. **(10 pts)** Consider the following Initial and Boundary Value Problem (IBVP) in 2D:

$$u_t = \Delta u + 1, \quad (x,y) \in \Omega = \{(x,y) \in \mathbb{R}^2 \mid 1 < r < 2\}, \tag{4}$$
$$u|_{t=0} = r + \cos(\phi), \tag{5}$$
$$u|_{r=1} = u|_{r=2} = 0, \tag{6}$$

where r and ϕ are the polar coordinates. Solve this problem using the finite element method and a scheme based on the trapezoidal rule:

$$u_{n+1} = u_n + \tfrac{1}{2}\Delta t\,(\Delta u_{n+1} + \Delta u_n) + \Delta t.$$

(a) Derive equations for the weak and the FEM solutions of the IBVP (4)-(6) analogous to Eq. (13) and the two unnumbered equations right below it in Section 9 on page 127 in Remarks around 50 lines of Matlab: short finite element implementation. Use time step `dt = 0.01`.

(b) Make your program plot the following figures:

- with the computed solution at $t = 0.1$ (use `trisurf`);

- with the computed solution at $t = 1$ (use `trisurf`);
- with the computed solution at time $t = 1$ as a function of r. You can do it e.g., as follows:

```
u = U(:,N+1); % N+1 corresponds to t=1.
r = sqrt(coordinates(:,1).^2 + coordinates(:,2).^2);
[rsort,isort] = sort(r,'ascend');
usort = u(isort);
plot(rsort,usort,'Linewidth',2);
```

At $t = 1$, the function u will virtually reach the stationary solution $\Delta u + 1 = 0$ satisfying the BC (6). This stationary solution can be found exactly:

$$u(r) = \frac{1 - r^2}{4} + \frac{3\log(r)}{4\log 2}. \tag{7}$$

Plot the graph of the exact stationary solution (7) in the same figure.

Hint: You might find helpful my code MyFEMheat.m implementing the Backward Euler time integrator described in "Remarks around 50 lines of Matlab: ...". This code is found on ELMS in Files/Codes/Elliptic/MyFEMcat.