

HW

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Problem 1

(a) In this part we use the fact

$$\int_{-\infty}^{\infty} \exp(-ax^2 + bx) dx = \left(\frac{\pi}{a}\right)^{1/2} \exp\left[\frac{b^2}{4a}\right]$$

Take the Fourier transform of the PDE in x , using the fact $\partial_x^n \psi = (i\xi)^n \psi$.

$$\hat{\psi}_t = \frac{i}{2}(i\xi)^2 \hat{\psi} = -\frac{i}{2}\xi^2 \hat{\psi} \implies \hat{\psi}(\xi, t) = \hat{\psi}_0(\xi) \exp\left[-\frac{i}{2}\xi^2 t\right]$$

Take the inverse Fourier transform.

$$\psi(x, t) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \hat{\psi}_0(\xi) \exp\left[ix\xi - \frac{i}{2}t\xi^2\right] d\xi$$

Take the Fourier transform of the initial condition.

$$\begin{aligned} \hat{\psi}_0(\xi) &= \frac{1}{(2\pi)^{1/2}} \frac{1}{(2\pi\sigma_0^2)^{1/4}} \int_{-\infty}^{\infty} \exp\left[-\frac{x^2}{4\sigma_0^2} + ik_0x - i\xi x\right] dx \\ &= \frac{1}{(2\pi)^{3/4}\sigma_0^{1/2}} \int_{-\infty}^{\infty} \exp\left[-\frac{x^2}{4\sigma_0^2} + i(k_0 - \xi)x\right] dx \\ &= \frac{1}{(2\pi)^{3/4}\sigma_0^{1/2}} \pi^{1/2} 2\sigma_0 \exp[-(\xi - k_0)^2 \sigma_0^2] \\ &= \frac{2^{1/4}\sigma_0^{1/2}}{\pi^{1/4}} \exp[-(\xi - k_0)^2 \sigma_0^2] \end{aligned}$$

Then

$$\psi(x, t) = \frac{1}{(2\pi)^{1/2}} \frac{2^{1/4}\sigma_0^{1/2}}{\pi^{1/4}} \int_{-\infty}^{\infty} \exp\left[-\sigma_0^2(\xi - k_0)^2 - \frac{i}{2}t\xi^2 + ix\xi\right] d\xi$$

Rewrite the argument of exp as

$$-\sigma_0^2(\xi - k_0)^2 - \frac{i}{2}t\xi^2 + ix\xi = -\sigma_0^2(\xi^2 + k_0^2 - 2k_0\xi) - \frac{i}{2}t\xi^2 + ix\xi = -\left(\sigma_0^2 + \frac{i}{2}t\right)\xi^2 + (ix + 2\sigma_0^2 k_0)\xi - \sigma_0^2 k_0^2$$

so that

$$\begin{aligned}
\psi(x, t) &= \frac{\sigma_0^{1/2}}{2^{1/4}\pi^{3/4}} \int_{-\infty}^{\infty} \exp\left[-\left(\sigma_0^2 + \frac{i}{2}t\right)\xi^2 + (ix + 2\sigma_0^2 k_0)\xi - \sigma_0^2 k_0^2\right] d\xi \\
&= \frac{\sigma_0^{1/2}}{2^{1/4}\pi^{3/4}} \exp(-\sigma_0^2 k_0^2) \left(\frac{\pi}{\sigma_0^2 + \frac{i}{2}t}\right)^{1/2} \exp\left[\frac{-x^2 + 4\sigma_0^4 k_0^2 + 4i\sigma_0^2 k_0 x}{4(\sigma_0^2 + \frac{i}{2}t)}\right] \\
&= \frac{\sigma_0^{1/2}}{2^{1/4}\pi^{1/4}} \frac{1}{(\sigma_0^2 + \frac{i}{2}t)^{1/2}} \exp(-\sigma_0^2 k_0^2) \exp\left[\frac{-x^2 + 4\sigma_0^4 k_0^2 + 4i\sigma_0^2 k_0 x}{4(\sigma_0^2 + \frac{i}{2}t)}\right]
\end{aligned}$$

Problem 3

- (a) Fix a test function $v \in H_0^1(\Omega)$. Multiply the PDE by v and integrate over Ω .

$$\int_{\Omega} \epsilon v \Delta u dx = \int_{\Omega} (u^3 - u) v dx$$

Using Green's first identity and the fact $v = 0$ on $\partial\Omega$, the LHS is

$$\int_{\Omega} \epsilon v \Delta u dx = - \int_{\Omega} \epsilon \nabla u \cdot \nabla v dx + \int_{\partial\Omega} \epsilon v \frac{\partial u}{\partial n} ds = - \int_{\Omega} \epsilon \nabla u \cdot \nabla v dx$$

Then we obtain the weak formulation.

$$- \int_{\Omega} \epsilon \nabla u \cdot \nabla v dx = \int_{\Omega} (u^3 - u) v dx \implies \int_{\Omega} \epsilon \nabla u \cdot \nabla v dx - \int_{\Omega} (u - u^3) v dx = 0$$

- (b) e

- (c) Code for problem 3:

<https://github.com/RokettoJanpu/Scientific-Computing-2/blob/main/FINAL%20q3.ipynb>

Below is a mesh of Ω .

