

Scientific Computing HW 6

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Problem 1.

- (a) The system $Au = f$ is shown on the left, and the block structure of A is compactly written on the right.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
1	-4	2				1						u_1		-1		
2	1	-4	1				1					u_2		-1		
3		1	-4	1				1				u_3		-1		
4			1	-4	1				1			u_4		-1		
5				2	-4					1		u_5		-1		
6	1					-4	2			1		u_6		0		
7		1				1	-4	1			1	u_7		0		
8			1			1	-4	1			1	u_8		0		
9				1		1	-4	1			1	u_9		0		
10					1		1	-4	1			u_{10}		0		
11								-4	2			u_{11}		0		
12							1		-4	1		u_{12}		0		
13								1	-4	1		u_{13}		0		
14									1	-4	1	u_{14}		0		
15										2	-4	u_{15}		0		

$A \quad u \quad f$

$$A = \begin{bmatrix} & & & & & & & & & & & & & & & \\ & \textcolor{red}{T} & \textcolor{blue}{I} & & & & & & & & & & & & & \\ & \textcolor{blue}{I} & \textcolor{red}{T} & \textcolor{blue}{I} & & & & & & & & & & & & \\ & & \ddots & \ddots & \ddots & & & & & & & & & & & \\ & & & \ddots & \textcolor{red}{T} & \textcolor{blue}{I} & & & & & & & & & & \\ & & & & \textcolor{blue}{I} & \textcolor{red}{T} & & & & & & & & & & \end{bmatrix}$$

$$\textcolor{red}{T} = \begin{bmatrix} -4 & 2 & & & & & & & \\ 1 & -4 & 1 & & & & & & \\ & 1 & -4 & 1 & & & & & \\ & & 1 & -4 & 1 & & & & \\ & & & 1 & -4 & 1 & & & \\ & & & & 1 & -4 & 1 & & \\ & & & & & 1 & -4 & 1 & \\ & & & & & & 2 & -4 & \end{bmatrix}$$

- (b) The system $Au = f$ is shown on the left, and the block structure of A is compactly written on the right.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
1	-4	1	2									u_1		-1		
2	1	-4	1	2								u_2		0		
3		1	-4		2							u_3		0		
4	1		-4	1		1						u_4		-1		
5		1	1	-4	1		1					u_5		0		
6			1	1	-4	1		1				u_6		0		
7				1	-4	1	1		1			u_7		-1		
8					1	1	-4	1		1		u_8		0		
9						1	1	-4	1		1	u_9		0		
10							1	-4	1	1		u_{10}		-1		
11								1	-4	1		u_{11}		0		
12									1	-4	1	u_{12}		0		
13										2	-4	u_{13}		-1		
14										2	1	u_{14}		0		
15										1	-4	u_{15}		0		

$A \quad u \quad f$

$$A = \begin{bmatrix} & & & & & & & & & & & & & & & \\ & \textcolor{red}{T} & 2\textcolor{blue}{I} & & & & & & & & & & & & & \\ & \textcolor{blue}{I} & \textcolor{red}{T} & \textcolor{blue}{I} & & & & & & & & & & & & \\ & & \ddots & \ddots & \ddots & \ddots & & & & & & & & & & \\ & & & \ddots & \textcolor{red}{T} & \textcolor{blue}{I} & & & & & & & & & & \\ & & & & \textcolor{blue}{I} & \textcolor{red}{T} & \textcolor{blue}{I} & & & & & & & & & \\ & & & & & \ddots & \ddots & \ddots & & & & & & & & \\ & & & & & & \textcolor{red}{T} & \textcolor{blue}{I} & & & & & & & & \\ & & & & & & & \textcolor{blue}{I} & \textcolor{red}{T} & \textcolor{blue}{I} & & & & & & \\ & & & & & & & & \ddots & \ddots & \ddots & & & & & & \\ & & & & & & & & & \textcolor{red}{T} & \textcolor{blue}{I} & & & & & & \\ & & & & & & & & & & \textcolor{blue}{I} & \textcolor{red}{T} & \textcolor{blue}{I} & & & & \\ & & & & & & & & & & & 2\textcolor{red}{I} & \textcolor{blue}{T} & & & & \end{bmatrix}$$

$$\textcolor{red}{T} = \begin{bmatrix} -4 & 1 & & & & & & & \\ 1 & -4 & 1 & & & & & & \\ & 1 & -4 & 1 & & & & & \\ & & 1 & -4 & 1 & & & & \\ & & & 1 & -4 & 1 & & & \\ & & & & 1 & -4 & 1 & & \\ & & & & & 1 & -4 & 1 & \\ & & & & & & 2 & -4 & \end{bmatrix}$$

- (c) The system $Au = f$ is shown on the left, and the block structure of A is compactly written on the right.

(d) The system $Au = f$ is shown on the left, and the block structure of A is compactly written on the right.

(e) The system $Au = f$ is shown on the left. The block structure of A is highlighted.

Problem 2. The BVP on the domain $\Omega := [-\pi, \pi] \times [0, 2]$ is

$$u_{xx} + u_{yy} = g(x) := \begin{cases} -\cos x, & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ 0, & \text{else} \end{cases} \quad (2.1)$$

with BCs

$$u \Big|_{x=\pi} = u \Big|_{x=-\pi}, \quad u_x \Big|_{x=\pi} = u_x \Big|_{x=-\pi}, \quad u \Big|_{y=0} = 0, \quad u_y \Big|_{y=2} = 0$$

Fix $J \in \mathbb{N}$. Set mesh steps in the x and y directions,

$$h_x := \frac{2\pi}{J}, \quad h_y := \frac{2}{J}$$

Then

$$\begin{aligned} u_{xx}(x, y) &= \frac{1}{h_x^2} [u(x + h_x, y) - 2u(x, y) + u(x - h_x, y)] + O(h_x^2) \\ u_{yy}(x, y) &= \frac{1}{h_y^2} [u(x, y + h_y) - 2u(x, y) + u(x, y - h_y)] + O(h_y^2) \end{aligned}$$

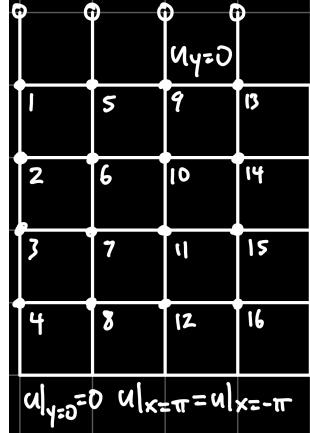
Plug these expressions into (2.1) and use the compass direction notation from lecture.

$$-2u_P \left[\frac{1}{h_x^2} + \frac{1}{h_y^2} \right] + \frac{1}{h_x^2}[u_E + u_W] + \frac{1}{h_y^2}[u_N + u_S] = g_P$$

Set $a := \frac{1}{h_x^2}$, $b := \frac{1}{h_y^2}$, $c := a + b$, so that

$$-2cu_P + a[u_E + u_W] + b[u_N + u_S] = g_P \quad (2.2)$$

To explore the appropriate numerical method, take $J = 4$. The mesh steps create a mesh from Ω with $(J+1)^2 = 25$ points. Using the BCs, it is enough to solve for the values of 16 points, labeled below.



Using the fact $u|_{y=0} = 0$, the values of the points on the bottom edge are all 0. Using the fact $u_y|_{y=2} = 0$, the values of the “ghost points” above the mesh from left to right are, respectively, u_2 , u_6 , u_{10} , u_{14} . Using the fact $u|_{x=-\pi} = u|_{x=\pi}$, the following pairs of points are treated as adjacent: 1 and 13, 2 and 14, 3 and 15, 4 and 16. Apply (2.2) to each point in the mesh to obtain a system $Au = f$, shown below on the left. The block structure of A is compactly written on the right.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	
1	-2c	2b		a					a				U ₁	g(-\pi + \frac{2\pi}{n})			
2	b	-2c	b		a					a			U ₂	g(-\pi - \frac{2\pi}{n})			
3		b	-2c	b		a					a		U ₃	g(\pi + \frac{2\pi}{n})			
4			b	-2c			a					a	U ₄	g(\pi - \frac{2\pi}{n})			
5	a			-2c	2b		a						U ₅	g(\pi - \frac{\pi}{n})			
6		a			b	-2c	b		a				U ₆	g(-\pi + \frac{3\pi}{n})			
7			a			b	-2c	b		a			U ₇	g(-\pi + \frac{4\pi}{n})			
8			a				b	-2c		a			U ₈	g(-\pi + \frac{5\pi}{n})	=		
9				a			-2c	2b		a			U ₉	g(-\pi + \frac{6\pi}{n})			
10					a			b	-2c	b		a	U ₁₀	g(-\pi + \frac{7\pi}{n})			
11						a			b	-2c	b		U ₁₁	g(-\pi + \frac{8\pi}{n})			
12							a			b	-2c		U ₁₂	g(-\pi + \frac{9\pi}{n})			
13	a						a				-2c	2b	U ₁₃	g(-\pi + \frac{10\pi}{n})			
14		a						a			b	-2c	b	U ₁₄	g(-\pi + \frac{11\pi}{n})		
15			a						a		b	-2c	b	U ₁₅	g(-\pi + \frac{12\pi}{n})		
16				a					a			b	-2c	U ₁₆	g(-\pi + \frac{13\pi}{n})		

$$A = \begin{array}{|ccc|} \hline & T & aI \\ aI & T & aI \\ & \ddots & \ddots \\ & \ddots & T & aI \\ aI & aI & T \\ \hline \end{array} \quad T = \begin{array}{|ccc|} \hline & -2c & 2b \\ b & -2c & b \\ b & -2c & b \\ b & \ddots & \ddots \\ \ddots & -2c & b \\ b & -2c & b \\ \hline \end{array}$$

The stationary heat distribution is solved for $J = 100$ and plotted below.

Code: <https://github.com/RokettoJanpu/Scientific-Computing-2/blob/main/hw6.ipynb>

