AMSC661, Spring 2023 Maria Cameron

Homework 5. Due March 8

Please upload a single pdf file on ELMS. Link your codes to your pdf (i.e., put your codes to dropbox, Github, google drive, etc. and place links to them in your pdf file with your solutions.

1. **(10 pts)** Solve exercise 2 in my lecture notes `SymplecticMethods.pdf`.

2. **(10 pts)** Consider the motion in the gravitational field with the Hamiltonian

$$H(u,v,x,y) = \frac{1}{2}u^2 + \frac{1}{2}v^2 - \frac{1}{\sqrt{x^2+y^2}}, \tag{1}$$

where $x,\ y$ are the coordinates and $u,\ v$ are the momenta in the reduced units.

(a) Write the Hamiltonian equations of motion. Set the initial conditions to $u(0) = 0,\ v(0) = \frac{1}{2},\ x(0) = 2,\ y(0) = 0$. Check that the total energy is negative, hence the motion will follow an elliptic trajectory.
The exact motion according to these Hamiltonian equations with the initial conditions $u(0) = 0,\ v(0) = \frac{1}{2},\ x(0) = 2,\ y(0) = 0$ occurs by the elliptical orbit with one focus at the origin, the major semiaxis $a = 4/3$, eccentricity $e = 1/2$, and the exact period of revolution $T = 2\pi a^{3/2} = 9.673596609249161$. Hence, for the exact orbit, $x_{\max} = a(1+e) = 2$, and $x_{\min} = -a(1-e) = -2/3$.

(b) Integrate the system for 10 revolutions using the implicit midpoint rule. Proceed as follows. Define

$$z := \begin{bmatrix} u \\ v \\ x \\ y \end{bmatrix}, \quad f(z) := \frac{dz}{dt};$$

then

$$k = f(z_n + \tfrac{h}{2}k), \quad z_{n+1} = z_n + hk.$$

At each time step, you will need to solve the 4D nonlinear system

$$F(k) := k - f(z_n + \tfrac{h}{2}k) = 0. \tag{2}$$

Find the initial approximation for k by linearizing f and solving the linear system:

$$k = f(z_n) + \tfrac{h}{2}Df(z_n)k,$$

where $Df(z_n)$ is the Jacobian matrix of f evaluated at z_n. Then find the solution of Eq. (2) using Newton's iteration. Plot x and y components of your numerical solutions on the same xy-plane. Plot the Hamiltonian versus time for your numerical solution. Do this task with time steps such that there are 100, 1000, and 10000 steps per period. You should generate a total of 6 figures.

(c) Integrate the same system using the Stoermer-Verlet method described in the previous problem using the same time steps and generate the same plots.

(d) Conclude which method is more accurate, while both of them are 2nd order.

3. **(3 pts)** Find the region of absolute stability of the implicit midpoint rule described in the previous problem.