Scientific Computing HW 10

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Problem 1.

(a) Write $k_x = n\pi,\ ky = m\pi$ where $1 \le n, m \le J-1$, so that

$$v_{k_x,k_y}(x_r, y_s) = \sin n\pi x_r \sin m\pi y_s$$

Applying the discretized Laplace operator, we obtain

$$\frac{1}{h^2}\left[v|_{n+1,m} + v|_{n-1,m} + v|_{n,m+1} + v_{n,m-1} - 4v|_{n,m}\right] \quad (1.1)$$

The bracked expression is

$$[\sin((n+1)\pi x_r) + \sin((n-1)\pi x_r)] \sin m\pi y_s + \sin n\pi x_r \left[\sin((m+1)\pi y_s) + \sin((m-1)\pi y_s)\right] - 4 \sin n\pi x_r \sin m\pi y_s$$

Using the identity

$$\sin a + \sin b = 2 \sin \frac{a+b}{2} \cos \frac{a-b}{2}$$

the bracked expression is

$$2 \sin n\pi x_r \cos \pi x_r \mathrm{m}\pi y_s + 2 \sin m\pi y_s \cos \pi y_s \sin n\pi x_r - r \sin n\pi x_r \sin m\pi y_s$$

$$= \sin n\pi x_r \sin m\pi y_s \cdot 2\left[\cos \pi x_r + 2\cos \pi y_s - 2\right]$$

Hence the expression (1.1) equals

$$\sin n\pi x_r \sin m\pi y_s \cdot \frac{2}{h^2}\left[\cos \pi x_r + \cos \pi y_s - 2\right]$$

We then see that the eigenvalues are

$$\lambda = \frac{2}{h^2}\left[\cos \pi x_r + \cos \pi y_s - 2\right]$$

(b) For stability, we demand that the eigenvalues of $\Delta t A$,

$$\lambda \Delta t = \frac{2\Delta t}{h^2}[\cos \pi x_r + \cos \pi y_s - 2]$$

lie in the RAS of forward Euler, $[-2, 0]$. Note that $-4 \le \cos \pi x_r + \cos \pi y_s - 2 \le 0$, hence

$$-2 \le \lambda \Delta t \le 0 \iff \frac{2\Delta t}{h^2} \le \frac{1}{2} \iff \Delta t \le \frac{h^2}{4}$$

Problem 2.

(a) Fix a finite time T, integer N, and timestep $dt = \frac{T}{N}$. Using a scheme based on the trapezoidal rule,

$$u_{n+1} = u_n + \frac{1}{2}dt(\Delta u_{n+1} + \Delta u_n) + dt \implies u_{n+1} - \frac{1}{2}dt\Delta u_{n+1} = dt + u_n + \frac{1}{2}dt\Delta u_n$$

Fix a test function $v \in C^2(\Omega)$ with $v\big|_{\partial\Omega} = 0$. Multiply by v and integrate over Ω.

$$\int_\Omega u_{n+1} v dx - \frac{1}{2}dt \int_\Omega v\Delta u_{n+1} dx = dt \int_\Omega v dx + \int_\Omega u_n v dx + \frac{1}{2}dt \int_\Omega v\Delta u_n dx$$

Using Green's first identity and the fact $v|_{\partial\Omega} = 0$,

$$\int_\Omega v\Delta u_n dx = \int_{\partial\Omega} v\frac{\partial u_n}{\partial n} ds - \int_\Omega \boldsymbol{\nabla} v \cdot \boldsymbol{\nabla} u_n dx = -\int_\Omega \boldsymbol{\nabla} v \cdot \boldsymbol{\nabla} u_n dx$$

and similarly for the term involving Δu_{n+1}. We obtain the weak solution to the IBVP.

$$\int_\Omega u_{n+1} v dx + \frac{1}{2}dt \int_\Omega \boldsymbol{\nabla} v \cdot \boldsymbol{\nabla} u_{n+1} dx = dt \int_\Omega v dx + \int_\Omega u_n v dx + -\frac{1}{2}dt \int_\Omega \boldsymbol{\nabla} v \cdot \boldsymbol{\nabla} u_n dx$$

From this we write the FEM solution. Given a triangulation τ of Ω, let η_i be piecewise affine on each triangle in τ. Then for each timestep n, we solve

$$(B + \frac{1}{2}dtA)U_{n+1} = dtb + (B - \frac{1}{2}dtA)U_n$$

where

$$A_{ij} = \sum_{T\in\tau} \int_T \boldsymbol{\nabla}\eta_i \cdot \boldsymbol{\nabla}\eta_j dx, \quad b_j = \sum_{T\in\tau} \int_T \eta_j dx, \quad B_{ij} = \sum_{T\in\tau} \int_T \eta_i\eta_j dx$$