Scientific Computing HW 4

Ryan Chen

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Problem 1. Pf. The characteristic polynomial is

$$\rho(z) = z^k + \sum_{j=0}^{k-1} \alpha_j z^{k-1-j}$$

and the recurrence is

$$u_{n+1} + \sum_{j=0}^{k-1} \alpha_j u_{n-j} = 0$$

Plug $u_n = n^l r^n$ into the LHS of the recurrence. Below, w depends on l and n, but for the sake of brevity we denote only the dependence on l.

$$w_l := u_{n+1} + \sum_{j=0}^{k-1} \alpha_j u_{n-j} = (n+1)^l r^{n+1} + \sum_{j=0}^{k-1} \alpha_j (n-j)^l r^{n-j} = r^{n-k+1} \left[(n+1)^l r^k + \sum_{j=0}^{k-1} \alpha_j (n-j)^l r^{k-1-j} \right]$$

Now induct on l for $0 \le l \le m-1$.

For the base case,

$$w_0 = r^{n-k+1} \left[r^k + \sum_{j=0}^{k-1} a_j r^{k-1-j} \right] = r^{n-k+1} \rho(r) = 0$$

For the inductive step, assume $w_{l-1} = 0$.

Problem 2. Pf. We first establish a lemma. For the $(k-1) \times k$ matrix

$$\begin{bmatrix} \lambda & -1 & & & \\ & \lambda & -1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & -1 \end{bmatrix}$$

The submatrix formed by deleting the jth column has determinant $(-1)^{k-j} \lambda^{j-1}$. To see this, write the submatrix in block form.

$$M := \begin{bmatrix} \lambda & -1 & & & & & & & \\ & \lambda & -1 & & & & & & \\ & & \ddots & \ddots & & & 0_{(j-1)\times(k-j)} & & \\ & & & \lambda & -1 & & & & \\ & & & & \lambda & & & & \\ & & & & & -1 & & & \\ & & & & & \lambda & -1 & & \\ & 0_{(k-j)\times(j-1)} & & & & & \ddots & \ddots & \\ & & & & & & & \lambda & -1 \end{bmatrix}$$

Using the fact that

$$\det \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} = \det(B)\det(C)$$

we have $\det M = \lambda^{j-1}(-1)^{k-j}$.

We now aim to show $\det(\lambda I - A) = \rho(\lambda)$.

$$\lambda I - A = \begin{bmatrix} \lambda & -1 & & & \\ & \lambda & -1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & -1 \\ \alpha_{k-1} & \alpha_{k-2} & \dots & \alpha_1 & \lambda + \alpha_0 \end{bmatrix}$$

Expanding over row k, starting at column k, and using the above lemma,

$$\det(\lambda I - A) = (\lambda+\alpha_0)(-1)^{k-k}\lambda^{k-1} + \sum_{j=1}^{k-1}(-1)^{k+k-j}\alpha_j(-1)^{k-k+j}\lambda^{k-j-1} = \lambda^k + \alpha_0\lambda^{k-1} + \sum_{j=1}^{k-1}(-1)^{2k}\alpha_j\lambda^{k-j-1}$$

$$= \lambda^k + \alpha_0\lambda^{k-1} + \sum_{j=1}^{k-1}\alpha_j\lambda^{k-j-1} = \lambda^k + \sum_{j=0}^{k-1}\alpha_j\lambda^{k-j-1} = \rho(\lambda)$$

Problem 4. Plug the supposed solution U_n into the LHS of the recurrence.

$$U_{n+1} = A^{n-k+2}U_{k-1} + h\sum_{j=0}^{n+1-k} A^j G_{n-j} = A^{n-k+2}U_{k-1} + hG_n + h\sum_{j=1}^{n+1-k} A^j G_{n-j}$$

$$= A^{n-k+2}U_{k-1} + hG_n + h\sum_{j=0}^{n-k} A^{j+1}G_{n-j-1}$$

Plug the solution into the RHS.

$$AU_n + hG_n = A^{n-k+2}U_{k-1} + h\sum_{j=0}^{n-k} A^{j+1}G_{n-j-1} + hG_n$$

The two expressions are equal, so U_n indeed solves the recurrence.

Problem 5.

(a) **Pf.** For BDF2, the interpolant is

$$p(t) = y_{n+1} + y[t_{n+1}, t_n](t - t_{n+1}) + y[t_{n+1}, t_n, t_{n-1}](t - t_{n+1})(t - t_n)$$

The derivative of the last term at t_{n+1} is

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_{n+1}}(t - t_{n+1})(t - t_n) = (t - t_n) + (t - t_{n+1})\Big|_{t=t_{n+1}} = 2t_{n+1} - t_n - t_{n+1} = t_{n+1} - t_n = h_n$$

and so

$$p'(t_{n+1}) = y[t_{n+1}, t_n] + y[t_{n+1}, t_n, t_{n-1}]h_n$$

Equating this with $f_{n+1} := f(t_{n+1}, u_{n+1})$ gives the BDF2 method.

$$\frac{u_{n+1}-u_n}{h_n} + \frac{\frac{u_{n+1}-u_n}{h_n} - \frac{u_n-u_{n-1}}{h_{n-1}}}{h_n+h_{n-1}} h_n = f_{n+1}$$

From $h_n = t_{n+1} - t_n$ and $\omega = \frac{h_n}{h_{n-1}}$,

$$1+\omega = 1 + \frac{h_n}{h_{n-1}} = \frac{h_{n-1}+h_n}{h_{n-1}} \implies (1+\omega)^2 = \frac{1}{h_{n-1}^2}\left[h_{n-1}^2 + h_n^2 + 2h_{n-1}h_n\right] \quad (5.1)$$

$$1+2\omega = 1 + 2\frac{h_n}{h_{n-1}} = \frac{h_{n-1}+2h_n}{h_{n-1}} \quad (5.2)$$

From (5.1) and (5.2),

$$\frac{(1+\omega)^2}{1+2\omega} = \frac{h_{n-1}}{h_{n-1}+2h_n}\frac{1}{h_{n-1}^2}\left[h_{n-1}^2 + h_n^2 + 2h_{n-1}h_n\right] = \frac{1}{h_{n-1}+2h_n}\left[h_{n-1} + \frac{h_n^2}{h_{n-1}} + 2h_n\right]$$

$$= h_{n-1}(1+2\omega)\left[h_{n-1} + \frac{h_n^2}{h_{n-1}} + 2h_n\right] \quad (5.3)$$

Now multiply the method by $h_n(h_n + h_{n-1})$.

$$(u_{n+1}-u_n)(h_n+h_{n-1}) + \left[\frac{u_{n+1}-u_n}{h_n} - \frac{u_n-u_{n-1}}{h_{n-1}}\right] h_n^2 = h_n(h_n+h_{n-1})f_{n+1}$$

Collect coefficients of the following terms. Rewrite them using (5.2) and (5.3).

$$\begin{aligned}
u_{n+1}: &\quad h_n + h_{n-1} + \frac{h_n^2}{h_n} = 2h_n + h_{n-1} \overset{(5.2)}{=} h_{n-1}(1+2\omega) \\
u_n: &\quad -(h_n+h_{n-1}) + \left[-\frac{1}{h_n} - \frac{1}{h_{n-1}}\right] h_n^2 = -\left[2h_n + h_{n-1} + \frac{h_n^2}{h_{n-1}}\right] \overset{(5.3)}{=} -h_{n-1}(1+2\omega)\frac{(1+\omega)^2}{1+2\omega} \\
u_{n-1}: &\quad \frac{h_n^2}{h_{n-1}} = h_{n-1}\omega^2
\end{aligned}$$

Putting these together, the method is

$$h_{n-1}(1+2\omega)u_{n+1} - h_{n-1}(1+2\omega)\frac{(1+\omega)^2}{1+2\omega}u_n + h_{n-1}\omega^2 = h_n(h_n+h_{n-1})f_{n+1}$$

Divide by $h_{n-1}(1+2\omega)$.

$$u_{n+1} - \frac{(1+\omega)^2}{1+2\omega}u_n + \frac{\omega^2}{1+2\omega}u_{n-1} = h_n\frac{\frac{h_n}{h_{n-1}}+1}{1+2\omega}f_{n+1} = h_n\frac{1+\omega}{1+2\omega}f_{n+1}$$

(b) From the LHS of the method, define

$$\rho(z) := z^2 - \frac{(1+\omega)^2}{1+2\omega}z + \frac{\omega^2}{1+2\omega}$$

By Vieta's formulas, the roots r, s of ρ satisfy

$$r+s = \frac{(1+\omega)^2}{1+2\omega} = \frac{1+\omega^2+2\omega}{1+2\omega} = \frac{\omega^2}{1+2\omega} + 1$$

$$rs = \frac{\omega^2}{1 + 2\omega}$$

From these we see

$$r = \frac{\omega^2}{1 + 2\omega}, \quad s = 1$$

Since $\omega < 1 + \sqrt{2}$ (I believe strict inequality is the right condition; non–strict would give 1 as a double root),

$$w^2 < (1 + \sqrt{2})^2 = 1 + 2 + 2 \cdot \sqrt{2} = 1 + 2(1 + \sqrt{2}) = 1 + 2\omega \implies 0 \le r = \frac{\omega^2}{1 + 2\omega} < 1 \implies |r| < 1$$

This leaves $s = 1$ as the only root with modulus 1, and it has multiplicity 1. Thus ρ satisfies the root condition, hence the method is stable.