HW

Ryan Chen

May 16, 2023

Problem 1

(a) In this part we use the fact

$$\int_{-\infty}^{\infty} \exp\bigl(-ax^2+bx\bigr)dx = \left(\frac{\pi}{a}\right)^{1/2} \exp\left[\frac{b^2}{4a}\right]$$

Take the Fourier transform of the PDE in x, using the fact $\hat{\partial_x^n \psi} = (i\xi)^n \psi$.

$$\hat{\psi}_t = \frac{i}{2}(i\xi)^2\hat{\psi} = -\frac{i}{2}\xi^2\hat{\psi} \implies \hat{\psi}(\xi,t) = \hat{\psi}_0(\xi)\exp\left[-\frac{i}{2}\xi^2 t\right]$$

Take the inverse Fourier transform.

$$\psi(x,t) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\hat{\psi}_0(\xi)\exp\left[ix\xi - \frac{i}{2}t\xi^2\right]d\xi$$

Take the Fourier transform of the initial condition.

$$\begin{aligned}
\hat{\psi}_0(\xi) &= \frac{1}{(2\pi)^{1/2}}\frac{1}{(2\pi\sigma_0^2)^{1/4}}\int_{-\infty}^{\infty}\exp\left[-\frac{x^2}{4\sigma_0^2}+ik_0x - i\xi x\right]dx \\
&= \frac{1}{(2\pi)^{3/4}\sigma_0^{1/2}}\int_{-\infty}^{\infty}\exp\left[-\frac{x^2}{4\sigma_0^2}+i(k_0-\xi)x\right]dx \\
&= \frac{1}{(2\pi)^{3/4}\sigma_0^{1/2}}\pi^{1/2}2\sigma_0\exp\bigl[-(\xi-k_0)^2\sigma_0^2\bigr] \\
&= \frac{2^{1/4}\sigma_0^{1/2}}{\pi^{1/4}}\exp\bigl[-(\xi-k_0)^2\sigma_0^2\bigr]
\end{aligned}$$

Then

$$\psi(x,t) = \frac{1}{(2\pi)^{1/2}}\frac{2^{1/4}\sigma_0^{1/2}}{\pi^{1/4}}\int_{-\infty}^{\infty}\exp\left[-\sigma_0^2(\xi-k_0)^2 - \frac{i}{2}t\xi^2 + ix\xi\right]d\xi$$

Rewrite the argument of exp as

$$-\sigma_0^2(\xi-k_0)^2 - \frac{i}{2}t\xi^2 + ix\xi = -\sigma_0^2(\xi^2+k_0^2-2k_0\xi) - \frac{i}{2}t\xi^2 + ix\xi = -\left(\sigma_0^2+\frac{i}{2}t\right)\xi^2 + (ix+2\sigma_0^2k_0)\xi - \sigma_0^2k_0^2$$

so that

$$\begin{aligned}
\psi(x,t) &= \frac{\sigma_0^{1/2}}{2^{1/4}\pi^{3/4}} \int_{-\infty}^{\infty} \exp\left[-\left(\sigma_0^2 + \frac{i}{2}t\right)\xi^2 + (ix + 2\sigma_0^2 k_0)\xi - \sigma_0^2 k_0^2\right] d\xi \\
&= \frac{\sigma_0^{1/2}}{2^{1/4}\pi^{3/4}} \exp\left(-\sigma_0^2 k_0^2\right) \left(\frac{\pi}{\sigma_0^2 + \frac{i}{2}t}\right)^{1/2} \exp\left[\frac{-x^2 + 4\sigma_0^4 k_0^2 + 4i\sigma_0^2 k_0 x}{4\left(\sigma_0^2 + \frac{i}{2}t\right)}\right] \\
&= \frac{\sigma_0^{1/2}}{2^{1/4}\pi^{1/4}} \frac{1}{(\sigma_0^2 + \frac{i}{2}t)^{1/2}} \exp\left(-\sigma_0^2 k_0^2\right) \exp\left[\frac{-x^2 + 4\sigma_0^4 k_0^2 + 4i\sigma_0^2 k_0 x}{4\left(\sigma_0^2 + \frac{i}{2}t\right)}\right]
\end{aligned}$$

Problem 3

(a) Fix a test function $v \in H_0^1(\Omega)$. Multiply the PDE by v and integrate over Ω.

$$\int_\Omega \epsilon v \Delta u dx = \int_\Omega (u^3 - u) v dx$$

Using Green's first identity and the fact $v = 0$ on $\partial\Omega$, the LHS is

$$\int_\Omega \epsilon v \Delta u dx = -\int_\Omega \epsilon \nabla u \cdot \nabla v dx + \int_{\partial\Omega} \epsilon v \frac{\partial u}{\partial n} ds = -\int_\Omega \epsilon \nabla u \cdot \nabla v dx$$

Then we obtain the weak formulation.

$$-\int_\Omega \epsilon \nabla u \cdot \nabla v dx = \int_\Omega (u^3 - u) v dx \implies \int_\Omega \epsilon \nabla u \cdot \nabla v dx - \int_\Omega (u - u^3) v dx = 0$$

(b) e

(c)