Scientific Computing HW 12

Ryan Chen

May 3, 2023

(a) The PDE is

$$\rho_t + \partial_x f(\rho) = 0, \quad f(\rho) := -\rho \ln \rho$$

which becomes

$$\rho_t + f'(\rho)\rho_x = 0, \quad f'(\rho) = -\ln \rho - \rho \frac{1}{\rho} = -\ln \rho - 1$$

Consider the curve Γ given by $x(t)$ satisfying

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f'(\rho(x(t), t)), \quad x(0) = x_0$$

We see that Γ is a characteristic of the PDE since

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(x(t), t) = \rho_t + \rho_x \frac{\mathrm{d}x}{\mathrm{d}t} = \rho_t + \rho_x f'(\rho(x(t), t)) = 0$$

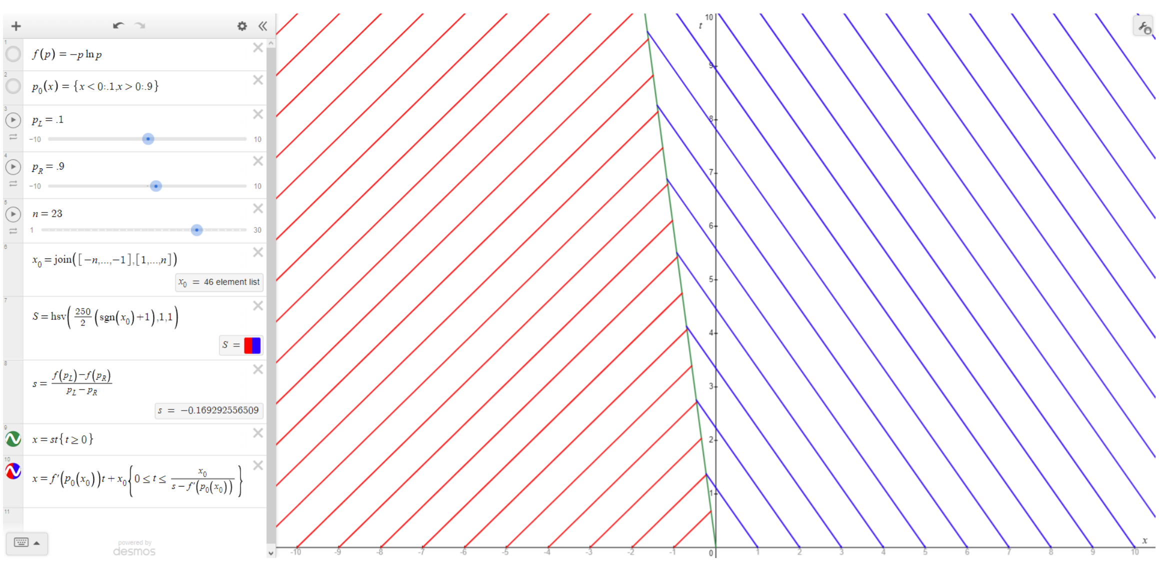
hence ρ is constant on Γ. In particular,

$$\rho(x(t), t) = \rho(x(0), 0) = \rho_0(x_0)$$

Thus Γ is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f'(\rho_0(x_0)) \implies x(t) = f'(\rho_0(x_0))t + x_0 = [-\ln \rho_0(x_0) - 1]t + x_0$$

(b) Some characteristics are plotted below. Those with $x_0 < 0$ are red, those with $x_0 > 0$ are blue, and the shock line is green.



(c) In this part the initial density is

$$\rho_0(x) = \frac{1}{2} + \frac{9}{10\pi}\arctan x$$

First compute

$$f''(\rho) = -\rho^{-1}, \quad \rho_0'(x) = \frac{9}{10\pi}(x^2+1)^{-1}$$

The equation of a characteristic starting at a point $(x_0, 0)$, considered as function of t and x_0, is

$$x = f'(\rho_0(x_0))t + x_0$$

Then the shock appears at time t_s when $\partial_{x_0} x = 0$, i.e.

$$f''(\rho_0(x_0))\rho_0'(x_0)t_s + 1 = 0$$

$$\implies t_s = -\left[f''(\rho_0(x_0))\rho_0'(x_0)\right]^{-1} = -\left[-\left(\frac{1}{2} + \frac{9}{10\pi}\arctan x_0\right)^{-1}\frac{9}{10\pi}(x_0^2+1)^{-1}\right]^{-1}$$

$$= \left(\frac{1}{2} + \frac{9}{10\pi}\arctan x_0\right)\frac{10\pi}{9}(x_0^2+1) = \left(\frac{5\pi}{9} + \arctan x_0\right)(x_0^2+1)$$

Now we find

$$\lim_{x\to\pm\infty}\rho_0(x) = \frac{1}{2} + \frac{9}{10\pi}\left(\pm\frac{\pi}{2}\right) = \frac{1}{2} \pm \frac{9}{20} \implies \rho_L = \frac{1}{20}, \quad \rho_R = \frac{19}{20}$$

Then compute

$$f(\rho_L) = -\frac{1}{20}\ln\frac{1}{20} = \frac{1}{20}\ln 20$$

$$f(\rho_R) = -\frac{19}{20}\ln\frac{19}{20} = \frac{19}{20}\ln\frac{20}{19}$$

Thus the eventual shock speed is

$$s = \frac{f(\rho_L) - f(\rho_R)}{\rho_L - \rho_R} = \frac{\frac{1}{20}\ln 20 - \frac{19}{20}\ln\frac{20}{19}}{\frac{1}{20} - \frac{19}{20}} = \frac{\ln 20 - 19\ln 20 + 19\ln 19}{1 - 19} = \ln 20 - \frac{19}{18}\ln 19 \approx -0.112$$