Scientific Computing HW 11

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Problem 1.

(a) The form of solution is

$$u(x,t) = \frac{1}{2}[\varphi(x+at) + \varphi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(s)ds$$

First find u_{tt}.

$$\begin{aligned}
u_t &= \frac{1}{2}[\varphi'(x+at)\cdot a + \varphi'(x-at)\cdot(-a)] + \frac{1}{2a}[\psi(x+at)\cdot a - \psi(x-at)\cdot(-a)] \\
&= \frac{a}{2}[\varphi'(x+at) - \varphi'(x-at)] + \frac{1}{2}[\psi(x+at) + \psi(x-at)] \\
u_{tt} &= \frac{a^2}{2}[\varphi''(x+at) + \varphi''(x-at)] + \frac{a}{2}[\psi'(x+at) - \psi'(x-at)]
\end{aligned}$$

Then find $a^2 u_{xx}$ and see that it equals u_{tt}, hence u solves the PDE.

$$\begin{aligned}
u_x &= \frac{1}{2}[\varphi'(x+at) + \varphi'(x-at)] + \frac{1}{2a}[\psi(x+at) - \psi(x-at)] \\
u_{xx} &= \frac{1}{2}[\varphi''(x+at) + \varphi''(x-at)] + \frac{1}{2a}[\psi'(x+at) - \psi'(x-at)] \\
a^2 u_{xx} &= \frac{a^2}{2}[\varphi''(x+at) + \varphi''(x-at)] + \frac{a}{2}[\psi'(x+at) - \psi'(x-at)] = u_{tt}
\end{aligned}$$

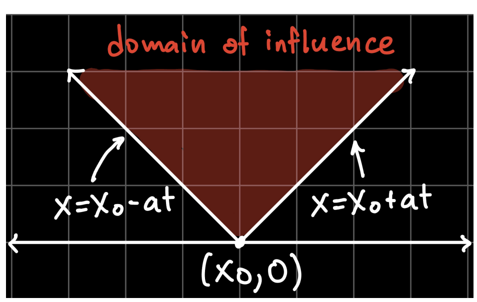
(b) Consider the form of solution

$$u(x,t) = \frac{1}{2}[\varphi(x+at) + \varphi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(s)ds$$

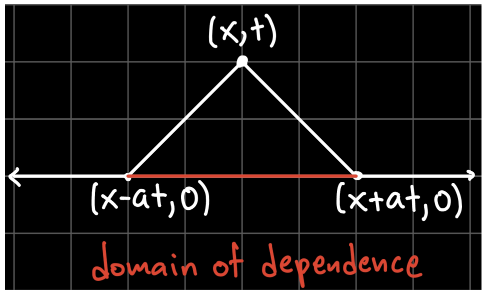
To find the domain of influence of a point x_0, note that $u(x,t)$ depends on the initial conditions at x_0 iff the integral in the second term depends on $\psi(x_0)$, which holds iff $x - at \leq x \leq x + at$. Then we have

$$x - at \leq x \leq x + at \iff -at \leq x_0 - x \leq at \iff at \geq x - x_0 \geq -at \iff x_0 + at \geq x \geq x_0 - at$$

Thus the domain of influence of x_0 is the region bounded by the lines $x = x_0 - at$ and $x = x_0 + at$.



Now to find the domain of dependence of a point (x,t), note that $u(x,t)$ depends precisely on the values of φ at $x \pm at$ and the values of ψ on $[x-at, x+at]$. Thus the domain of dependence of (x,t) is the line segment $\{(s,0) : x-at \le s \le x+at\}$.



(c) We see that

$$w := \begin{bmatrix} u_t \\ u_x \end{bmatrix} \implies w_x = \begin{bmatrix} u_{tx} \\ u_{xx} \end{bmatrix}$$

so that

$$w_t = \begin{bmatrix} u_{tt} \\ u_{xt} \end{bmatrix} = \begin{bmatrix} 0u_{tx} + a^2 u_{xx} \\ 1u_{tx} + 0u_{xx} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & a^2 \\ 1 & 0 \end{bmatrix}}_{=:A} \begin{bmatrix} u_{tx} \\ u_{xx} \end{bmatrix} = Aw_x$$

Then

$$u_x \bigg|_{t=0} = \frac{1}{2}[\varphi'(x) + \varphi'(x)] + \frac{1}{2a}[\psi(x) - \psi(x)] = \varphi'(x)$$

so that the initial condition for w is

$$w \bigg|_{t=0} = \begin{bmatrix} u_t \big|_{t=0} \\ u_x \big|_{t=0} \end{bmatrix} = \begin{bmatrix} \psi(x) \\ \varphi'(x) \end{bmatrix}$$

(d) The eigenvalues of A are

$$0 = \det(A - \lambda I) = \begin{vmatrix} -\lambda & a^2 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - a^2 = (\lambda - a)(\lambda + a) \implies \lambda_1 = a,\ \lambda_2 = -a$$

Eigenvectors v_1, v_2 of A are

$$A - \lambda_1 I = \begin{bmatrix} -a & a^2 \\ 1 & -a \end{bmatrix} \implies v_1 = \begin{bmatrix} a \\ 1 \end{bmatrix}$$

$$A - \lambda_2 I = \begin{bmatrix} a & a^2 \\ 1 & a \end{bmatrix} \implies v_2 = \begin{bmatrix} -a \\ 1 \end{bmatrix}$$

Diagonalizing A,

$$A = C\Lambda C^{-1}, \quad C := [v_1, v_2] = \begin{bmatrix} a & -a \\ 1 & 1 \end{bmatrix}, \quad \Lambda := \operatorname{diag}(\lambda_1, \lambda_2) = \operatorname{diag}(a, -a)$$

Changing variable, we obtain independent PDEs.

$$y := C^{-1}w = \begin{bmatrix} \xi \\ \eta \end{bmatrix} \implies w = Cy \implies w_t = Cy_t,\ w_x = Cy_x$$

$$\implies 0 = w_t - Aw_x = Cy_t - C\Lambda C^{-1}Cy_x = C(y_t - \Lambda y_x) \implies y_t - \Lambda y_x = 0 \implies y_t = \Lambda y_x$$

$$\implies \xi_t = a\xi_x,\ \eta_t = -a\eta_x$$

First find C^{-1}.

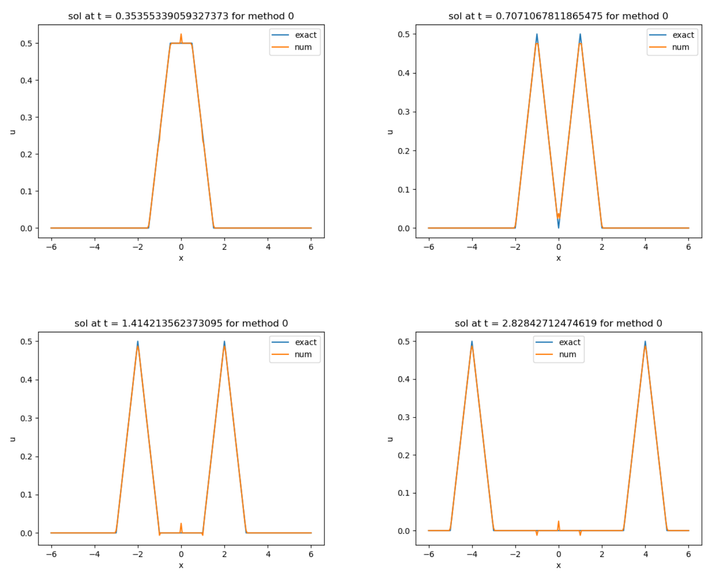
$$\det C = \begin{vmatrix} a & -a \\ 1 & 1 \end{vmatrix} = 2a \implies C^{-1} = \frac{1}{2a}\begin{bmatrix} 1 & a \\ -1 & a \end{bmatrix}$$

Then we find the initial condition for y, i.e. the initial conditions for ξ, η.

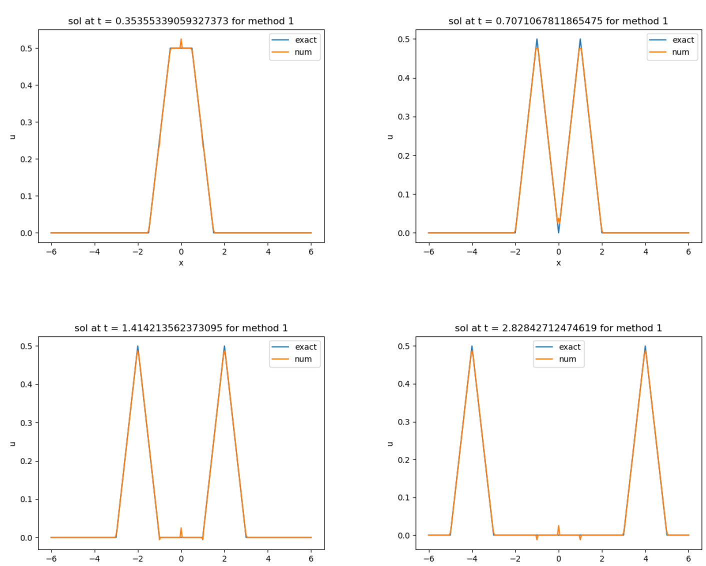
$$\begin{bmatrix} \psi(x) \\ \varphi'(x) \end{bmatrix} = w\Big|_{t=0} = Cy\Big|_{t=0} \implies y\Big|_{t=0} = C^{-1}\begin{bmatrix} \psi(x) \\ \varphi'(x) \end{bmatrix} = \frac{1}{2a}\begin{bmatrix} \psi(x) + a\varphi'(x) \\ -\psi(x) + a\varphi'(x) \end{bmatrix}$$

(e) Code: `https://github.com/RokettoJanpu/Scientific-Computing-2/blob/main/hw11.ipynb`

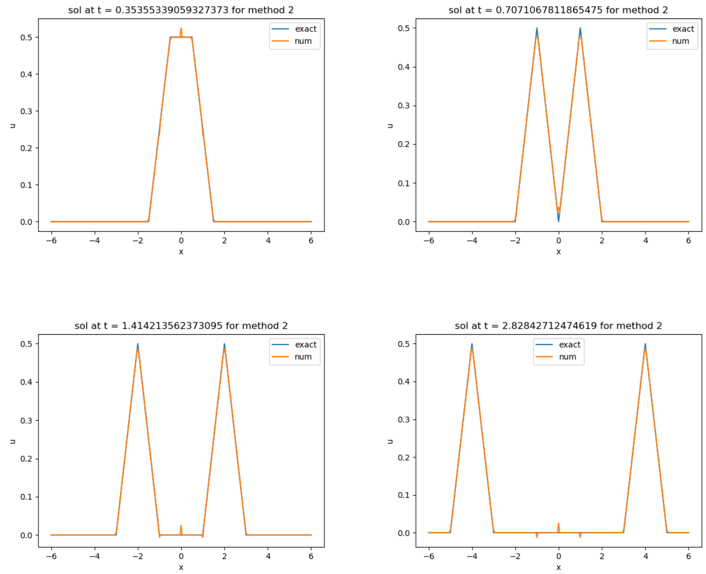
Lax–Friedrichs:



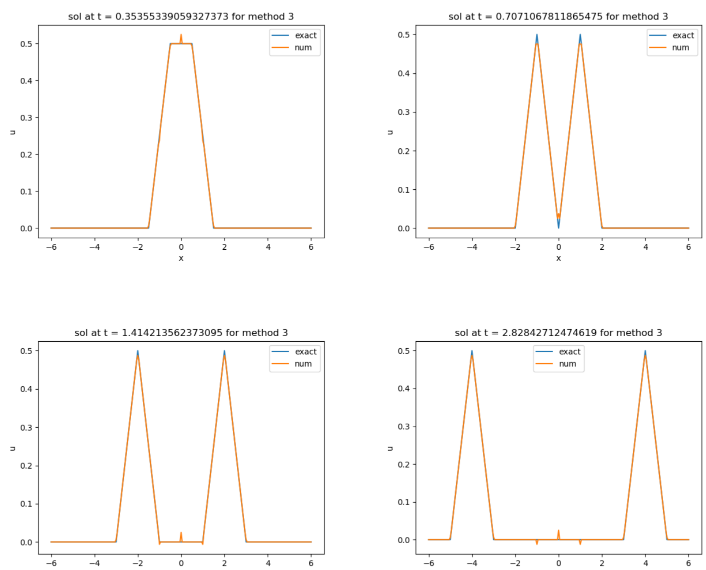
Upwind:



Lax–Wendroff:



Beam–Warming:



Each scheme is highly accurate in that it computes a solution that essentially coincides with the exact solution, except at finitely many cusps. Stationary cusps arise at points corresponding to cusps of the initial displacement $\varphi(x) = \max(1 - |x|, 0)$, i.e. at $x = 0, 1, -1$. Cusps that travel along with the solution arise from the form

$$u(x,t) = \frac{1}{2}[\varphi(x + at) + \varphi(x - at)]$$

meaning these cusps occur at $x \pm at = 0, 1, -1$, i.e. at $x = \mp at, 1 \mp at, -1 \mp at$.

Problem 2.

(a) The LaxFR scheme is

$$u_j^{n+1} = \frac{1}{2}[u_{j+1}^n + u_{j-1}^n] - \frac{1}{2}\nu[u_{j+1}^n - u_{j-1}^n]$$

Let v satisfy $v(x_j, t_n) = u_j^n$.

$$v(x, t+k) = \frac{1}{2}[v(x+h,t) + v(x-h,t)] - \frac{1}{2}\nu[v(x+h,t) - v(x-h,t)]$$

Taylor expand at (x, t).

$$\begin{aligned} v(x,t+k) &= v + kv_t + \frac{1}{2}k^2 v_{tt} + O(k^3) \\ v(x+h,t) &= v + hv_x + \frac{1}{2}h^2 v_{xx} + \frac{1}{6}h^3 v_{xxx} + O(h^4) \\ v(x-h,t) &= v - hv_x + \frac{1}{2}h^2 v_{xx} - \frac{1}{6}h^3 v_{xxx} + O(h^4) \end{aligned}$$

Plug in the expansions.

$$\begin{aligned} v + kv_t + \frac{1}{2}k^2 v_{tt} + O(k^3) &= \frac{1}{2}[2v + h^2 v_{xx} + O(h^4)] - \frac{1}{2}\nu\left[2hv_x + \frac{2}{6}h^3 v_{xxx} + O(h^5)\right] \\ &= v + \frac{1}{2}h^2 v_{xx} - \nu h v_x + \frac{1}{6}\nu h^3 v_{xxx} + O(h^4) \end{aligned}$$

Cancel v then divide by k.

$$v_t + \frac{1}{2}kv_{tt} + O(k^2) = \frac{1}{2}\frac{h^2}{k}v_{xx} - av_x + \frac{1}{6}\nu\frac{h^3}{k}v_{xxx} + O(h^3)$$

Truncate the $O(h^2)$ and smaller terms.

$$\begin{aligned} v_t + \frac{1}{2}kv_{tt} &= \frac{1}{2}\frac{h^2}{k}v_{xx} - av_x + O(h^2) \\ v_t + av_x &= \frac{1}{2}\frac{h^2}{k}v_{xx} - \frac{1}{2}kv_{tt} + O(h^2) \qquad (1) \end{aligned}$$

To express v_{tt} in terms of v_{xx}, take ∂_t and ∂_x separately.

$$\begin{aligned} v_{tt} + av_{xt} &= \frac{1}{2}\frac{h^2}{k}v_{xxt} - \frac{1}{2}kv_{ttt} + O(h) = O(h) \qquad (2) \\ v_{tx} + av_{xx} &= \frac{1}{2}\frac{h^2}{k}v_{xxx} - \frac{1}{2}kv_{ttx} + O(h) = O(h) \qquad (3) \end{aligned}$$

Take (2) $- a$(3).

$$v_{tt} - a^2 v_{xx} = O(h) \implies v_{tt} = a^2 v_{xx} + O(h)$$

Plug into (1).

$$\begin{aligned} v_t + av_x &= \frac{1}{2}\frac{h^2}{k}v_{xx} - \frac{1}{2}k[a^2 v_{xx} + O(h)] + O(h^2) \\ &= \frac{1}{2}\frac{h^2}{k}\left[v_{xx} - \frac{1}{2}\frac{k^2}{h^2}a^2 v_{xx}\right] + O(h^2) \\ &= \frac{1}{2}\frac{h^2}{k}[1 - \nu^2]v_{xx} + O(h^2) \\ &= \frac{1}{2}\frac{ah}{\nu}[1 - \nu^2]v_{xx} + O(h^2) \end{aligned}$$

Thus the modified equation for LaxFR (first order) is

$$\boxed{v_t + av_x = \frac{1}{2}\frac{ah}{\nu}[1 - \nu^2]v_{xx}}$$

The LF scheme is

$$\frac{1}{2k}[u_j^{n+1} - u_j^{n-1}] = -\frac{a}{2h}[u_{j+1}^n - u_{j-1}^n]$$

Let v satisfy $v(x_j, t_n) = u_j^n$.

$$\frac{1}{2k}[v(x, t+k) - v(x, t-k)] = -\frac{a}{2h}[v(x+h, t) - v(x-h, t)]$$

Taylor expand at (x, t).

$$\begin{aligned}
v(x, t+k) &= v + kv_t + \frac{1}{2}k^2 v_{tt} + \frac{1}{6}k^3 v_{ttt} + O(k^4) \\
v(x, t-k) &= v - kv_t + \frac{1}{2}k^2 v_{tt} - \frac{1}{6}k^3 v_{ttt} + O(k^4) \\
v(x+h, t) &= v + hv_x + \frac{1}{2}h^2 v_{xx} + \frac{1}{6}h^3 v_{xxx} + O(h^4) \\
v(x-h, t) &= v - hv_x + \frac{1}{2}h^2 v_{xx} - \frac{1}{6}h^3 v_{xxx} + O(h^4)
\end{aligned}$$

Plug in the expansions.

$$\begin{aligned}
\frac{1}{2k}\left[2kv_t + \frac{2}{6}k^3 v_{ttt} + O(k^5)\right] &= -\frac{a}{2h}\left[2hv_x + \frac{2}{6}h^3 v_{xxx} + O(h^5)\right] \\
v_t + \frac{1}{6}k^2 v_{ttt} + O(k^4) &= -av_x - \frac{1}{6}ah^2 v_{xxx} + O(h^4) \\
v_t + av_x &= -\frac{1}{6}ah^2 v_{xxx} - \frac{1}{6}k^2 v_{ttt} + O(h^4) \qquad (4)
\end{aligned}$$

Now express v_{ttt} in terms of v_{xxx}. From (4),

$$\begin{aligned}
v_t + av_x &= O(h^2) \\
\partial_t + a\partial_x &= O(h^2) \\
\partial_t &= -a\partial_x + O(h^2) \\
\partial_t^3 &= -a^3\partial_x^3 + 3a^2\partial_x^2 O(h^2) - 3a\partial_x O(h^4) + O(h^6) \\
&= -a^3\partial_x^3 + O(h^2) \\
v_{ttt} &= -a^3 v_{xxx} + O(h^2)
\end{aligned}$$

Plug into (4).

$$\begin{aligned}
v_t + av_x &= -\frac{1}{6}ah^2 v_{xxx} - \frac{1}{6}k^2[-a^3 v_{xxx} + O(h^2)] + O(h^4) \\
&= \frac{1}{6}ah^2\left[-v_{xxx} + \frac{a^2k^2}{h^2}v_{xxx}\right] + O(h^4) \\
&= \frac{1}{6}ah^2[\nu^2 - 1]v_{xxx} + O(h^4)
\end{aligned}$$

Thus the modified equation for LF (second order) is

$$\boxed{v_t + av_x = \frac{1}{6}ah^2[\nu^2 - 1]v_{xxx}}$$

The BWL scheme is

$$u_j^{n+1} = u_j^n - \frac{1}{2}\nu[3u_j^n - 4u_{j-1}^n + u_{j-2}^n] + \frac{1}{2}\nu^2[u_j^n - 2u_{j-1}^n + u_{j-2}^n]$$

Let v satisfy $v(x_j, t_n) = u_j^n$.

$$v(x,t+k) = v(x,t) - \frac{1}{2}\nu[3v(x,t) - 4v(x-h,t) + v(x-2h,t)] + \frac{1}{2}\nu^2[v(x,t) - 2v(x-h,t) + v(x-2h,t)]$$

Taylor expand at (x,t).

$$\begin{aligned}
v(x,t+k) &= v + kv_t + \frac{1}{2}k^2 v_{tt} + \frac{1}{6}k^3 v_{ttt} + O(k^4) \\
v(x-h,t) &= v - hv_x + \frac{1}{2}h^2 v_{xx} - \frac{1}{6}h^3 v_{xxx} + O(h^4) \\
v(x-2h,t) &= v - 2hv_x + 2h^2 v_{xx} - \frac{4}{3}h^3 v_{xxx} + O(h^4)
\end{aligned}$$

Plug in the expansions. In computing $3v(x,t) - 4v(x-h,t) + v(x-2h,t)$, the coefficient of each term is:

$$\begin{aligned}
v : &\quad 3 - 4 + 1 = 0 \\
hv_x : &\quad 4 - 2 = 2 \\
h^2 v_{xx} : &\quad -2 + 2 = 0 \\
h^3 v_{xxx} : &\quad \frac{2}{3} - \frac{4}{3} = -\frac{2}{3}
\end{aligned}$$

so in total

$$3v(x,t) - 4v(x-h,t) + v(x-2h,t) = 2hv_x - \frac{2}{3}h^3 v_{xxx} + O(h^4)$$

In computing $v(x,t) - 2v(x-h,t) + v(x-2h,t)$, the coefficient of each term is:

$$\begin{aligned}
v : &\quad 1 - 2 + 1 = 0 \\
hv_x : &\quad 2 - 2 = 0 \\
h^2 v_{xx} : &\quad -1 + 2 = 1 \\
h^3 v_{xxx} : &\quad \frac{1}{3} - \frac{4}{3} = -1
\end{aligned}$$

so in total

$$v(x,t) - 2v(x-h,t) + v(x-2h,t) = h^2 v_{xx} - h^3 v_{xxx} + O(h^4)$$

Hence

$$v + kv_t + \frac{1}{2}k^2 v_{tt} + \frac{1}{6}k^3 v_{ttt} + O(k^4) = v - \frac{1}{2}\nu\left[2hv_x - \frac{2}{3}h^3 v_{xxx} + O(h^4)\right] + \frac{1}{2}\nu^2[h^2 v_{xx} - h^3 v_{xxx} + O(h^4)]$$

$$\begin{aligned}
kv_t + \frac{1}{2}k^2 v_{tt} + \frac{1}{6}k^3 v_{ttt} &= -\nu h v_x + \frac{1}{2}\nu^2 h^2 v_{xx} + \frac{\nu h^3}{k}\left[\frac{1}{3} - \frac{1}{2}\nu\right] v_{xxx} + O(h^4) \\
v_t + \frac{1}{2}kv_{tt} + \frac{1}{6}k^2 v_{ttt} &= -av_x + \frac{1}{2}\frac{\nu^2 h^2}{k} v_{xx} + \frac{\nu h^3}{k}\left[\frac{1}{3} - \frac{1}{2}\nu\right] v_{xxx} + O(h^3) \\
v_t + av_x &= \frac{1}{2}a^2 k v_{xx} + \frac{1}{6}ah^2\left[2 - 3\nu\right] v_{xxx} - \frac{1}{2}kv_{tt} - \frac{1}{6}k^2 v_{ttt} + O(h^3) \\
&= \frac{1}{2}k[a^2 v_{xx} - v_{tt}] + \frac{1}{6}ah^2\left[2 - 3\nu\right] v_{xxx} - \frac{1}{6}k^2 v_{ttt} + O(h^3)
\end{aligned}$$

Set $z := a^2 v_{xx} - v_{tt}$, so that

$$v_t + av_x = \frac{1}{2}kz + \frac{1}{6}ah^2\left[2 - 3\nu\right]v_{xxx} - \frac{1}{6}k^2 v_{ttt} + O(h^3)$$

We aim to show $z = O(k^2)$. Take ∂_t and ∂_x of the above separately.

$$v_{tt} + av_{xt} = \frac{1}{2}kz_t + \frac{1}{6}ah^2\left[2 - 3\nu\right]v_{xxxt} - \frac{1}{6}k^2 v_{tttt} + O(h^2) \tag{5}$$

$$v_{tx} + av_{xx} = \frac{1}{2}kz_x + \frac{1}{6}ah^2\left[2 - 3\nu\right]v_{xxxx} - \frac{1}{6}k^2 v_{tttx} + O(h^2) \tag{6}$$

Take (5) $-\, a$(6).

$$-z = \frac{1}{2}k[z_t - az_x] + \frac{1}{6}ah^2[2 - 3\nu][v_{xxxt} - av_{xxxx}] - \frac{1}{6}k^2[v_{tttt} - av_{tttx}] + O(h^2)$$

If z is $O(k)$ but not $O(k^2)$ then $z_t - az_x$ is $O(k)$ or smaller, so that the above gives $O(k) = O(k^2)$, a contradiction. Thus $z = O(k^2)$, and hence

$$v_t + av_x = \frac{1}{6}ah^2[2 - 3\nu]v_{xxx} - \frac{1}{6}k^2 v_{ttt} + O(h^3)$$

To express v_{ttt} in terms of v_{xxx}, use the above to write

$$\begin{aligned}
v_t + av_x &= O(h^2) \\
\partial_t + a\partial_x &= O(h^2) \\
\partial_t &= -a\partial_x + O(h^2) \\
\partial_t^3 &= -a^3\partial_x^3 + 3a^2\partial_x^2 O(h^2) - 3a\partial_x O(h^4) + O(h^6) \\
&= -a^3\partial_x^3 + O(h^2) \\
v_{ttt} &= -a^3 v_{xxx} + O(h^2)
\end{aligned}$$

Substituting for v_{ttt},

$$\begin{aligned}
v_t + av_x &= \frac{1}{6}ah^2[2 - 3\nu]v_{xxx} - \frac{1}{6}k^2[-a^3 v_{xxx} + O(h^2)] + O(h^3) \\
&= \frac{1}{6}ah^2\left[(2 - 3\nu)v_{xxx} + \frac{a^2k^2}{h^2}v_{xxx}\right] + O(h^3) \\
&= \frac{1}{6}ah^2[2 - 3\nu + \nu^2]v_{xxx} + O(h^3)
\end{aligned}$$

Thus the modified equation for BWL (second order) is

$$\boxed{v_t + av_x = \frac{1}{6}ah^2[2 - 3\nu + \nu^2]v_{xxx}}$$

(b) In the modified equations of UL and LaxFR, we compare the terms $f(\nu) = 1 - \nu$ and $g(\nu) = \frac{1-\nu^2}{\nu}$ to determine which scheme is more accurate. The equations are well–posed for $0 \leq \nu \leq 1$. For $0 < \nu < 1$ we have $f(\nu) < g(\nu)$, so UL is more accurate than LaxFR.

In the modified equations of LW, LF, and BWL, we compare the terms $p(\nu) = \nu^2 - 1$ and $q(\nu) = 2 - 3\nu + \nu^2$ to determine which scheme is more accurate. The equations for LW and LF are well–posed for $\nu \geq 1$, and the equation for BWL is well–posed for $\nu = 1$ or $\nu > 2$. For $\nu > 2$ we have $q(\nu) < p(\nu)$, so BWL is more accurate than LW and LF.

Given a and h, pick $k = \frac{h}{a}$ so that $\nu = 1$, hence $f(\nu) = g(\nu) = p(\nu) = q(\nu) = 0$. This gives the smallest numerical error in all five schemes.

(c) For UL,

$$u_t + au_x = \frac{1}{2}ah(1-\nu)u_{xx}$$

Take the FT in x, using the fact $\widehat{\partial_x^n u}(z) = (iz)^n \hat{u}(z)$, and solve.

$$\begin{aligned}
\hat{u}_t + iaz\hat{u} &= \frac{1}{2}ah(1-\nu)(iz)^2\hat{u} \\
&= -\frac{1}{2}ah(1-\nu)z^2\hat{u} \\
\hat{u}_t + iaz\hat{u} + \frac{1}{2}ah(1-\nu)z^2\hat{u} = 0 \\
\hat{u}(z,t) &= \hat{u}_0(z)\exp(-iazt)\exp\left(-\frac{1}{2}ah(1-\nu)z^2 t\right)
\end{aligned}$$

For LaxFr,

$$u_t + au_x = \frac{1}{2}\frac{ah}{\nu}(1-\nu^2)u_{xx}$$

Take the FT in x and solve.

$$\begin{aligned}
\hat{u}_t + iaz\hat{u} &= -\frac{1}{2}\frac{ah}{\nu}(1-\nu^2)z^2\hat{u} \\
\hat{u}_t + iaz\hat{u} + \frac{1}{2}\frac{ah}{\nu}(1-\nu^2)z^2\hat{u} = 0 \\
\hat{u}(z,t) &= \hat{u}_0(z)\exp(-iazt)\exp\left(-\frac{1}{2}\frac{ah}{\nu}(1-\nu^2)z^2 t\right)
\end{aligned}$$

For LW and LF,

$$u_t + au_x = \frac{1}{6}ah^2(\nu^2-1)u_{xxx}$$

Take the FT in x and solve.

$$\begin{aligned}
\hat{u}_t + iaz\hat{u} &= -i\frac{1}{6}ah^2(\nu^2-1)z^3\hat{u} \\
\hat{u}_t + iaz\hat{u} + i\frac{1}{6}ah^2(\nu^2-1)z^3\hat{u} = 0 \\
\hat{u}(z,t) &= \hat{u}_0(z)\exp(-iazt)\exp\left(-i\frac{1}{6}ah^2(\nu^2-1)z^3 t\right)
\end{aligned}$$

The dispersion relation is

$$\omega(z) = az + \frac{1}{6}ah^2(\nu^2-1)z^3$$

The phase velocity is

$$c_p(z) = \frac{\omega(z)}{z} = a + \frac{1}{6}ah^2(\nu^2-1)z^2$$

The group velocity is

$$c_g(z) = \omega'(z) = a + \frac{1}{2}ah^2(\nu^2-1)z^2$$

For BWL,

$$u_t + au_x = \frac{1}{6}ah^2(2-3\nu+\nu^2)v_{xxx}$$

Take the FT in x and solve.

$$\hat{u}_t + iaz\hat{u} = -i\frac{1}{6}ah^2(2-3\nu+\nu^2)z^3\hat{u}$$

$$\hat{u}_t + iaz\hat{u} + i\frac{1}{6}ah^2(2-3\nu+\nu^2)z^3\hat{u}$$

$$\hat{u}(z,t) = \hat{u}_0(z)\exp(-iazt)\exp\left(-i\frac{1}{6}ah^2(2-3\nu+\nu^2)z^3 t\right)$$

The dispersion relation is

$$\omega(z) = az + \frac{1}{6}ah^2(2-3\nu+\nu^2)z^3$$

The phase velocity is

$$c_p(z) = \frac{\omega(z)}{z} = a + \frac{1}{6}ah^2(2-3\nu+\nu^2)z^2$$

The group velocity is

$$c_g(z) = \omega'(z) = a + \frac{1}{2}ah^2(2-3\nu+\nu^2)z^2$$

The original advection equation is

$$u_t + au_x = 0$$

Taking the FT in x gives

$$\hat{u}_t + iaz\hat{u} = 0$$

The dispersion relation is

$$\omega(z) = az$$

Hence the phase and group velocities are

$$c_p = c_g = a$$