AMSC661, Spring 2023 Maria Cameron

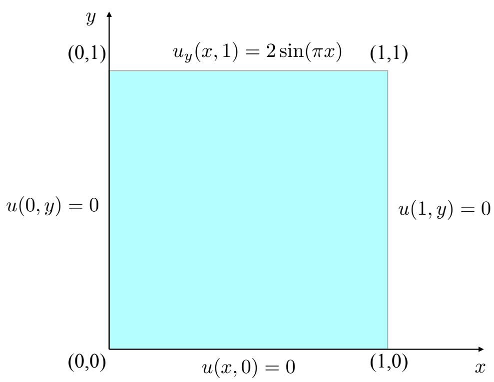
Homework 9. Due Wednesday, April 12

Please upload a single pdf file on ELMS. Link your codes to your pdf (i.e., put your codes to dropbox, Github, google drive, etc. and place links to them in your pdf file with your solutions.

1. **(10 pts)** Set up and solve Problem 6 from Lagaris, Likas, and Fotiadis (1998), a boundary-value problem for the Poisson equation

$$u_{xx} + u_{yy} = f(x, y) \quad \text{where} \quad f(x, y) = (2 - \pi^2 y^2)\sin(\pi x), \quad (x, y) \in [0, 1]^2, \tag{1}$$

with mixed boundary conditions:



The exact solution to this problem is given by

$$u_{ex}(x, y) = y^2 \sin(\pi x). \tag{2}$$

To solve this problem, you can mimic the provided codes for Problem 5 from Lagaris et al.:

- `Lagaris5.zip`, a Matlab package written by me from scratch, or
- `Lagaris5.ipynb`, a Python code written by me from scratch, or
- `Lagaris_Margot.ipynb`, a Python written by Margot Yuan (AMSC) with the use of automatic differentiation and built-in Adam optimizer.

Use a neural network with one hidden layer and $n = 10$ neurons of the form:

$$\mathcal{N}(x, y; w) = \sum_{j=0}^{n-1} w_{3j}\sigma\left(w_{0j}x + w_{1j}y + w_{2j}\right), \tag{3}$$

where $\sigma(z) = \tanh(z)$ acts entrywise. The total number of parameters to optimize is $4n$.

The proposed form of the solution $U(x, y; w)$ is given in Lagaris et al. in Eqs. (24)–(25). Take $N_{tr} = 49$ training points forming a uniform 7×7 `meshgrid` in $[0, 1]^2$ (set a 9×9 meshgrid and strip its boundaries). Set up the least squares loss function

$$\mathcal{L}(w) = \frac{1}{2} \sum_{i=0}^{N_{tr}-1} |U_{xx}(x_i, y_i; w) + U_{yy}(x_i, y_i; w) - f(x_i, y_i)|^2 . \tag{4}$$

Plot the computed solution and the numerical error using level sets or a heatmap. Plot the loss function versus the iteration number using a log scale for the y-axis.

2. **(5 pts)** Consider the following IBVP for the heat equation:

$$\begin{aligned}
u_t &= u_{xx}, \quad 0 \le x \le \pi, \quad t \ge 0, && (5)\\
u(x, 0) &= x, && (6)\\
u(0, t) &= 0, \quad \text{the left end is kept at temperature } 0, && (7)\\
u_x(\pi, t) &= 0, \quad \text{the right end is insulated.} && (8)
\end{aligned}$$

Solve Eq. (5)-(8) analytically using the method of separation of variables (if you are not familiar with it look it up e.g. in Wikipedia). You will obtain the solution of the form of infinite series involving sines $\sin[(n + 1/2)x]$, $n = 0, 1, 2, \ldots$. Calculate the coefficients for each harmonic of this series. Plot graphs of the sums of the first m terms of the series at time 0 and time 2 for $m = 1, 2, 5, 10, 100$. Find the maximum norm of the difference of the sum of the first 100 terms of the series at time 0 and the function $x = u(x, 0)$.