Homework 3. Due Feb. 22

Please upload a single pdf file on ELMS. Link your codes to your pdf (i.e., put your codes to dropbox, Github, google drive, etc. and place links to them in your pdf file with your solutions.

1. **(5 pts)** Solve exercise 13 in my lecture notes `ODEsolvers.pdf`.

2. **(5 pts)** Solve exercise 14 in my lecture notes `ODEsolvers.pdf`.

3. **(6 pts)** There is a curious phenomenon called *order reduction* observed when DIRK methods are applied to stiff problems with time steps h such that $h\lambda$ where $-\lambda$ is the largest negative eigenvalue of the linear part of the right-hand side of an ODE is not small – see papers by B. Seibold's group https://arxiv.org/pdf/1712.00897.pdf and https://arxiv.org/pdf/1811.01285.pdf. The goal of this exercise is to examine the performance of DIRK methods of orders 2 and 3 from the previous exercises on the Prothero-Robinson problem, plot the maximum absolute error as a function of the time step, and observe the two orders of convergence for each method, one for large hL, and one for small hL. You need to plot graphs similar to those in Fig. 2 in https://arxiv.org/pdf/1811.01285.pdf.

   Consider the Prothero-Robinson problem

   $$y' = -L(y - \phi(t)) + phi'(t), \quad y(0) = y_0 \tag{1}$$

   with $L = 10^4$ and $\phi(t) = \sin(t + \pi/4)$. Set the time interval $0 \le t \le T_{\max} = 10$. The exact solution to this problem is

   $$y = -e^{-Lt}(y_0 - \phi(0)) + \phi(t). \tag{2}$$

   (a) Pick the initial condition $y(0) = \sin(\pi/4)$. Compute the numerical solution using $\mathsf{DIRK2}$ on the interval $[0, T_{\max}]$ with time step h for each h from the following set:

   $$h = 10^{-p}, \quad \text{where} \quad p \in \{1, 1+d, 1+2d, \ldots, 6\},\ d = 5/24/. \tag{3}$$

   Plot the numerical error $e(h) = \max_{0 \le t_n \le T_{\max}} |u_n - y(t_n)|$ vs h. Use the log-log scale. Observe error decay $e = C_1 h$ and $e = C_2 h^2$ for large and small values of h, respectively. For reference, plot lines with slopes 1 and 2, i.e., $e = C_1 h$ and $e = C_2 h^2$ where you need to choose C_1 and C_2 so that the plot looks nice. Do the same for the DIRK of order 3. What orders of error decay do you observe? Also, plot reference lines.

(b) Repeat the task with $y(0) = \sin(\pi/4) + 10$. You will obtain a bit puzzling set of graphs. To understand what is going on, plot $|e(t)|$ for each method where $e(t)$ is the difference between the numerical and the exact solutions for three values of h: $h = 10^{-1}$, $h = 10^{-2}$, and $h = 10^{-3}$. Set the log scale in the y-axis. Do so for $T_{\max} = 10$ and $T_{\max} = 1$.

(c) Summarize what you have learned about the behavior of the error for DIRK2 and DIRK of order 3 from the numerical experiments in this problem.

4. **(5 pts)** Derive a three-step Adams-Moulton method with a variable timestep. Then compute its coefficients for the case where the timestep is constant.