Scientific Computing HW 7

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Problem 1.

(a) Multiplying the BVP by -1 and integrating it, we find $k(x)u' = M$ for some constant M, i.e. $u' = \frac{M}{k(x)}$. The solution is then

$$u(x) = u_a + \int_a^x \frac{M}{k(s)} ds$$

If $x \leq c$ then

$$u(x) = u_a + \int_a^x \frac{M}{k_1} ds = u_a + \frac{M}{k_1}(x-a)$$

If $x > c$ then

$$u(x) = u_a + \int_a^c \frac{M}{k(s)} ds + \int_c^x \frac{M}{k(s)} ds = u_a + \int_a^c \frac{M}{k_1} ds + \int_c^x \frac{M}{k_2} ds = u_a + \frac{M}{k_1}(c-a) + \frac{M}{k_2}(x-c)$$

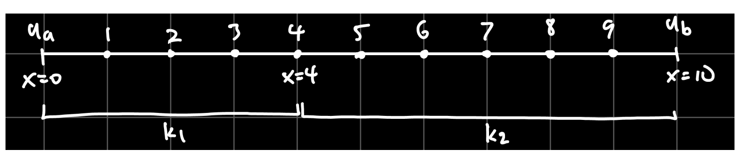
Apply BCs.

$$u_b = u(b) = u_a + \frac{M}{k_1}(c-a) + \frac{M}{k_2}(b-c) = u_a + M\left[\frac{c-a}{k_1} + \frac{b-c}{k_2}\right] \implies M = \frac{u_b - u_a}{\frac{c-a}{k_1} + \frac{b-c}{k_2}}$$

In summary, the solution is

$$u(x) = \begin{cases} u_a + \frac{M}{k_1}(x-a), & x \leq c \\ u_a + \frac{M}{k_1}(c-a) + \frac{M}{k_2}(x-c), & x > c \end{cases} \quad \text{where} \quad M = \frac{u_b - u_a}{\frac{c-a}{k_1} + \frac{b-c}{k_2}}$$

(b) Given the parameters, it is enough to solve for the values of the 9 mesh points shown below.



The finite difference scheme is

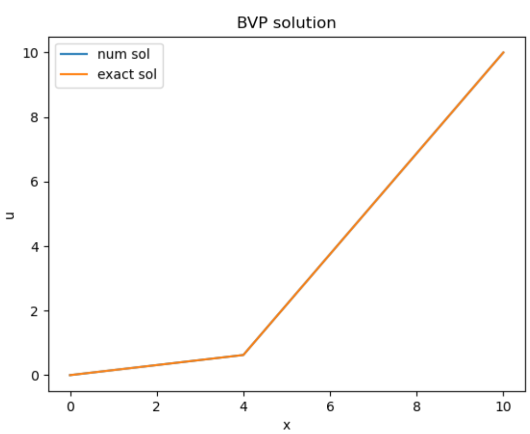
$$L_h u_P = -\frac{1}{h^2}\left[k_w u_W + k_e u_E - (k_e + k_w)u_P\right] = 0 \implies -(k_w + k_e)u_P + k_w u_W + k_e u_E = 0$$

Applying the scheme to each mesh point, we obtain a linear system.

$$
\begin{array}{c|ccccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
1 & -2k_1 & k_1 & & & & & & & \\
2 & k_1 & -2k_1 & k_1 & & & & & & \\
3 & & k_1 & -2k_1 & k_1 & & & & & \\
4 & & & k_1 & -k_1-k_2 & k_2 & & & & \\
5 & & & & k_2 & -2k_2 & k_2 & & & \\
6 & & & & & k_2 & -2k_2 & k_2 & & \\
7 & & & & & & k_2 & -2k_2 & k_2 & \\
8 & & & & & & & k_2 & -2k_2 & k_2 \\
9 & & & & & & & & k_2 & -2k_2
\end{array}
\cdot
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{bmatrix}
=
\begin{bmatrix} -k_1u_a \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -k_2u_b \end{bmatrix}
$$

We solve it and plot the numerical solution u along with the exact solution from part (a). In this case the solutions agree exactly.

Code: `https://github.com/RokettoJanpu/Scientific-Computing-2/blob/main/hw7q1.ipynb`



Problem 4.

(a) The matrix G is given by

$$G = A^{-1}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A := \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

We find A^{-1} by its adjugate. By cofactor expansion over the first row,

$$D := \det(A) = \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = x_2y_3 - x_3y_2 - x_1y_3 + x_3y_1 + x_1y_2 - x_2y_1$$

The matrix of cofactors is

$$\mathrm{cof}(A) = \begin{bmatrix} x_2y_3 - x_3y_2 & -x_1y_3 + x_3y_1 & x_1y_2 - x_2y_1 \\ -y_3 + y_2 & y_3 - y_1 & -y_2 + y_1 \\ x_3 - x_2 & -x_3 + x_1 & x_2 - x_1 \end{bmatrix}$$

The adjugate of A is

$$\mathrm{adj}(A) = \mathrm{cof}(A)^T = \begin{bmatrix} x_2y_3 - x_3y_2 & -y_3 + y_2 & x_3 - x_2 \\ -x_1y_3 + x_3y_1 & y_3 - y_1 & -x_3 + x_1 \\ x_1y_2 - x_2y_1 & -y_2 + y_1 & x_2 - x_1 \end{bmatrix}$$

Thus

$$G = \frac{1}{D}\mathrm{adj}(A) \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{D} \begin{bmatrix} y_2 - y_3 & x_3 - x_2 \\ y_3 - y_1 & x_1 - x_3 \\ y_1 - y_2 & x_2 - x_1 \end{bmatrix}$$

Fix an even permutation (i, j, k) of $1, 2, 3$ (i.e. $(1, 2, 3)$, $(2, 3, 1)$, $(3, 1, 2)$). To find an expression for $\eta_i(x, y)$, we find an equation for the plane containing $(x_i, y_i, 1)$, $(x_j, y_j, 0)$, $(x_k, y_k, 0)$, which involves finding a normal vector n to the plane.

$$v := (x_k, y_k, 0) - (x_j, y_j, 0) = (x_k - x_j, y_k - y_j, 0), \quad w := (x_i, y_i, 1) - (x_j, y_j, 0) = (x_i - x_j, y_i - y_j, 1)$$

$$n := v \times w = (y_k - y_j, x_j - x_k, (x_k - x_j)(y_i - y_j) - (x_i - x_j)(x_k - y_j))$$

The last component of n is

$$x_ky_i - x_ky_j - x_jy_i + x_jy_j - x_iy_k + x_iy_j + x_jy_k - x_jy_j = x_ky_i - x_ky_j - x_jy_i - x_iy_k + x_iy_j + x_jy_k$$

$$= \begin{vmatrix} x_j & x_k \\ y_j & y_k \end{vmatrix} - \begin{vmatrix} x_i & x_k \\ y_i & y_k \end{vmatrix} + \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x_i & x_j & x_k \\ y_i & y_j & y_k \end{vmatrix}$$

Since the permutation (i, j, k) is even, this expression equals $\det(A) = D$. Using the point $(x_j, y_j, 0)$ and the components of $n = (y_k - y_j, x_j - x_k, D)$, we have an equation for the plane.

$$(y_k - y_j)(x - x_j) + (x_j - x_k)(y - y_j) + D\eta_i(x, y) = 0$$

$$\implies \eta_i(x, y) = -\frac{1}{D}\left[(y_k - y_j)(x - x_j) + (x_j - x_k)(y - y_j)\right] = \frac{1}{D}\left[(y_j - y_k)(x - x_j) + (x_k - x_j)(y - y_j)\right]$$

$$\implies \partial_x\eta_i(x, y) = \frac{1}{D}(y_j - y_k), \quad \partial_y\eta_i(x, y) = \frac{1}{D}(x_k - x_j)$$

Thus

$$\begin{bmatrix} \partial_x\eta_1(x, y) & \partial_y\eta_1(x, y) \\ \partial_x\eta_2(x, y) & \partial_y\eta_2(x, y) \\ \partial_x\eta_3(x, y) & \partial_y\eta_3(x, y) \end{bmatrix} = \frac{1}{D} \begin{bmatrix} y_2 - y_3 & x_3 - x_2 \\ y_3 - y_1 & x_1 - x_3 \\ y_1 - y_2 & x_2 - x_1 \end{bmatrix} = G$$