

Scientific Computing HW 11

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Problem 1.

(a) The form of solution is

$$u(x, t) = \frac{1}{2}[\varphi(x + at) + \varphi(x - at)] + \frac{1}{2a} \int_{x-at}^{x+at} \psi(s) ds$$

First find u_{tt} .

$$\begin{aligned} u_t &= \frac{1}{2}[\varphi'(x + at) \cdot a + \varphi'(x - at) \cdot (-a)] + \frac{1}{2a}[\psi(x + at) \cdot a - \psi(x - at) \cdot (-a)] \\ &= \frac{a}{2}[\varphi'(x + at) - \varphi'(x - at)] + \frac{1}{2}[\psi(x + at) + \psi(x - at)] \\ u_{tt} &= \frac{a^2}{2}[\varphi''(x + at) + \varphi''(x - at)] + \frac{a}{2}[\psi'(x + at) - \psi'(x - at)] \end{aligned}$$

Then find $a^2 u_{xx}$ and see that it equals u_{tt} , hence u solves the PDE.

$$\begin{aligned} u_x &= \frac{1}{2}[\varphi'(x + at) + \varphi'(x - at)] + \frac{1}{2a}[\psi(x + at) - \psi(x - at)] \\ u_{xx} &= \frac{1}{2}[\varphi''(x + at) + \varphi''(x - at)] + \frac{1}{2a}[\psi'(x + at) - \psi'(x - at)] \\ a^2 u_{xx} &= \frac{a^2}{2}[\varphi''(x + at) + \varphi''(x - at)] + \frac{a}{2}[\psi'(x + at) - \psi'(x - at)] = u_{tt} \end{aligned}$$

(b) The form of solution is

$$u(x, t) = \frac{1}{2}[\varphi(x + at) + \varphi(x - at)] + \frac{1}{2a} \int_{x-at}^{x+at} \psi(s) ds$$

From its terms we see that $u(x, t)$ depends precisely on the values of φ at $x \pm at$ and the values of ψ on $[x - at, x + at]$. Thus the domain of dependence of a point (x, t) is $\{(s, 0) : x - at \leq s \leq x + at\}$.

(c) We see that

$$w := \begin{bmatrix} u_t \\ u_x \end{bmatrix} \implies w_x = \begin{bmatrix} u_{tx} \\ u_{xx} \end{bmatrix}$$

so that

$$w_t = \begin{bmatrix} u_{tt} \\ u_{xt} \end{bmatrix} = \begin{bmatrix} 0u_{tx} + a^2u_{xx} \\ 1u_{tx} + 0u_{xx} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & a^2 \\ 1 & 0 \end{bmatrix}}_{=:A} \begin{bmatrix} u_{tx} \\ u_{xx} \end{bmatrix} = Aw_x$$

Then

$$u_x \Big|_{t=0} = \frac{1}{2}[\varphi'(x) + \varphi'(x)] + \frac{1}{2a}[\psi(x) - \psi(x)] = \varphi'(x)$$

so that the initial condition for w is

$$w \Big|_{t=0} = \begin{bmatrix} u_t|_{t=0} \\ u_x|_{t=0} \end{bmatrix} = \begin{bmatrix} \psi(x) \\ \varphi'(x) \end{bmatrix}$$

(d) The eigenvalues of A are

$$0 = \det(A - \lambda I) = \begin{vmatrix} -\lambda & a^2 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - a^2 = (\lambda - a)(\lambda + a) \implies \lambda_1 = a, \lambda_2 = -a$$

Eigenvectors v_1, v_2 of A are

$$A - \lambda_1 I = \begin{bmatrix} -a & a^2 \\ 1 & -a \end{bmatrix} \implies v_1 = \begin{bmatrix} a \\ 1 \end{bmatrix}$$

$$A - \lambda_2 I = \begin{bmatrix} a & a^2 \\ 1 & a \end{bmatrix} \implies v_2 = \begin{bmatrix} -a \\ 1 \end{bmatrix}$$

Diagonalizing A ,

$$A = C\Lambda C^{-1}, \quad C := [v_1, v_2], \quad \Lambda := \text{diag}(\lambda_1, \lambda_2)$$

Changing variable, we obtain independent PDEs.

$$y := C^{-1}w = \begin{bmatrix} \xi \\ \eta \end{bmatrix} \implies w = Cy \implies w_t = Cy_t, \quad w_x = Cy_x$$

$$\begin{aligned} \implies 0 &= w_t - Aw_x = Cy_t - C\Lambda C^{-1}Cy_x = C(y_t - \Lambda y_x) \implies y_t - \Lambda y_x = 0 \implies y_t = \Lambda y_x \\ &\implies \xi_t = \lambda_1 \xi_x = a \xi_x, \quad \eta_t = \lambda_2 \eta_x = -a \eta_x \end{aligned}$$

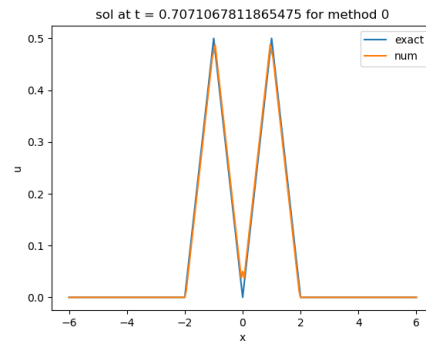
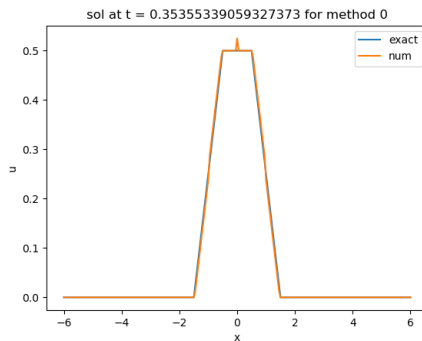
First find C^{-1} .

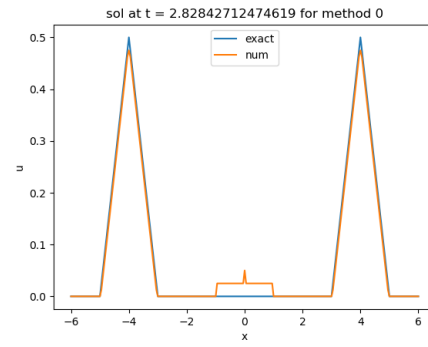
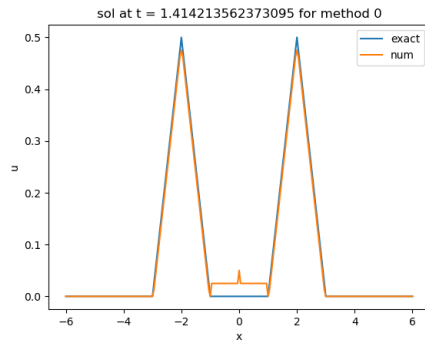
$$\det C = \begin{vmatrix} a & -a \\ 1 & 1 \end{vmatrix} = 2a \implies C^{-1} = \frac{1}{2a} \begin{bmatrix} 1 & a \\ -1 & a \end{bmatrix}$$

Then we find the initial condition for y , i.e. the initial conditions for ξ, η .

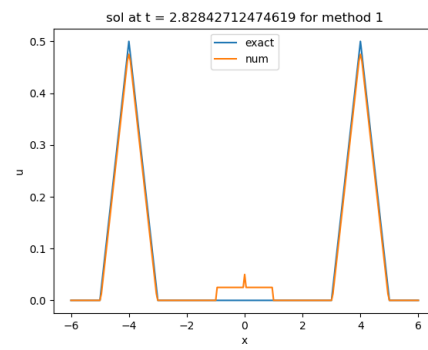
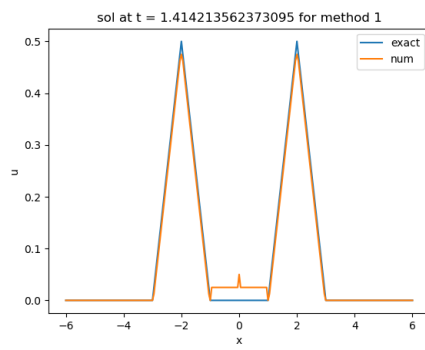
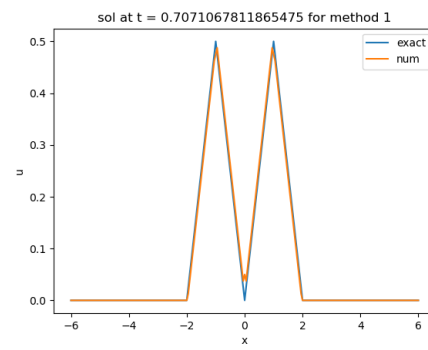
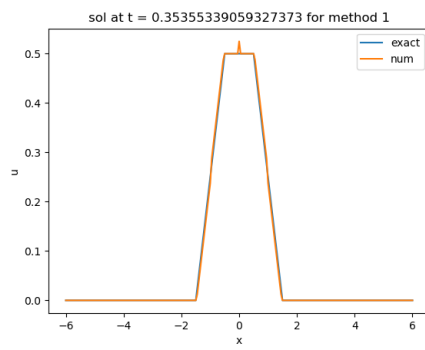
$$\begin{bmatrix} \psi(x) \\ \varphi'(x) \end{bmatrix} = w \Big|_{t=0} = Cy \Big|_{t=0} \implies y \Big|_{t=0} = C^{-1} \begin{bmatrix} \psi(x) \\ \varphi'(x) \end{bmatrix} = \frac{1}{2a} \begin{bmatrix} \psi(x) + a\varphi'(x) \\ -\psi(x) + a\varphi'(x) \end{bmatrix}$$

(e) Code: github.com/RokettoJanpu/Scientific-Computing-2/blob/main/hw11q1.ipynb
Lax–Friedrichs:





Upwind:



Beam-Warming:

