

Scientific Computing HW 6

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Problem 1.

- (a) The system $Au = f$ is shown on the left, and the block structure of A is compactly written on the right.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	-4	2													
2	1	-4	1												
3		1	-4	1											
4			1	-4	1				1						
5				2	-4					1					
6	1				-4	2					1				
7	1			1	-4	1					1				
8		1			1	-4	1					1			
9	.	1			1	-4	1					1			
10	.		1			1	-4	1					1		
11				1			-4	2							
12					1		1	-4	1						
13						1		1	-4	1					
14							1		1	-4	1				
15								1		2	-4				

- (b) The system $Au = f$ is shown on the left, and the block structure of A is compactly written on the right.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	-4	1		2											
2	1	-4	1		2										
3	1	1	-4			2									
4	1		-4	1	1										
5	1		1	-4	1	1									
6	1		1	1	-4		1								
7		1		-4	1	1									
8		1		1	-4	1	1								
9		1		1	1	-4		1							
10			1		-4	1	1								
11			1		1	-4	1		1						
12			1		1	1	-4			1					
13					2		-4	1							
14					2		1	-4	1						
15					2		1	1	-4						

- (c) The system $Au = f$ is shown on the left, and the block structure of A is compactly written on the right.

(d) The system $Au = f$ is shown on the left, and the block structure of A is compactly written on the right.

(e) The system $Au = f$ is shown on the left. The block structure of A is highlighted.

Problem 2. The BVP on the domain $\Omega := [-\pi, \pi] \times [0, 2]$ is

$$u_{xx} + u_{yy} = g(x) := \begin{cases} -\cos x, & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ 0, & \text{else} \end{cases} \quad (2.1)$$

with BCs

$$u \Big|_{x=\pi} = u \Big|_{x=-\pi}, \quad u_x \Big|_{x=\pi} = u_x \Big|_{x=-\pi}, \quad u \Big|_{y=0} = 0, \quad u_y \Big|_{y=2} = 0$$

Fix $J \in \mathbb{N}$. Set mesh steps in the x and y directions,

$$h_x := \frac{2\pi}{J}, \quad h_y := \frac{2}{J}$$

Then

$$\begin{aligned} u_{xx}(x, y) &= \frac{1}{h_x^2} [u(x + h_x, y) - 2u(x, y) + u(x - h_x, y)] + O(h_x^2) \\ u_{yy}(x, y) &= \frac{1}{h_y^2} [u(x, y + h_y) - 2u(x, y) + u(x, y - h_y)] + O(h_y^2) \end{aligned}$$

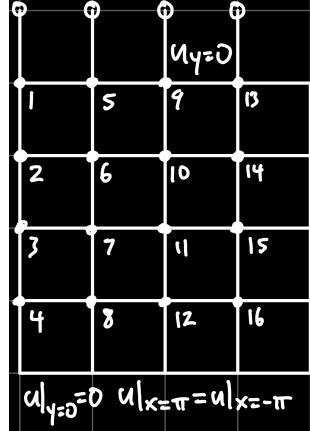
Plug these expressions into (2.1) and use the compass direction notation from lecture.

$$-2u_P \left[\frac{1}{h_x^2} + \frac{1}{h_y^2} \right] + \frac{1}{h_x^2}[u_E + u_W] + \frac{1}{h_y^2}[u_N + u_S] = g_P$$

Set $a := \frac{1}{h_x^2}$, $b := \frac{1}{h_y^2}$, $c := a + b$, so that

$$-2cu_P + a[u_E + u_W] + b[u_N + u_S] = g_P \quad (2.2)$$

To explore the appropriate numerical method, take $J = 4$. The mesh steps create a mesh from Ω with $(J+1)^2 = 25$ points. Using the BCs, it is enough to solve for the values of 16 points, labeled below.



Using the fact $u_y|_{y=2} = 0$, the values of the “ghost points” above the mesh from left to right are, respectively, u_2, u_6, u_{10}, u_{14} . Apply (2.2) to each point in the mesh to obtain a system $Au = f$, shown below on the left. The block structure of A is compactly written on the right.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	
1	-2c	2b		a					a				U ₁	g(-\pi + \frac{2\pi}{n})			
2	b	-2c	b		a					a			U ₂	g(-\pi - \frac{2\pi}{n})			
3	b	-2c	b			a					a		U ₃	g(\pi + \frac{2\pi}{n})			
4		b	-2c				a					a	U ₄	g(\pi - \frac{2\pi}{n})			
5	a			-2c	2b		a						U ₅	g(\pi - \frac{2\pi}{n})			
6		a		b	-2c	b		a					U ₆	g(\pi - \frac{4\pi}{n})			
7		a		b	-2c	b		a					U ₇	g(\pi - \frac{6\pi}{n})			
8		a		b	-2c			a					U ₈	g(\pi - \frac{8\pi}{n})	=		
9			a			-2c	2b		a				U ₉	g(\pi - \frac{10\pi}{n})			
10			a			b	-2c	b		a			U ₁₀	g(\pi - \frac{12\pi}{n})			
11				a		b	-2c	b		a			U ₁₁	g(\pi - \frac{14\pi}{n})			
12				a		b	-2c			a			U ₁₂	g(\pi - \frac{16\pi}{n})			
13	a				a			-2c	2b				U ₁₃	g(\pi - \frac{18\pi}{n})			
14		a				a			b	-2c	b		U ₁₄	g(\pi - \frac{20\pi}{n})			
15		a					a		b	-2c	b		U ₁₅	g(\pi - \frac{22\pi}{n})			
16			a				a		b	-2c	b		U ₁₆	g(\pi - \frac{24\pi}{n})			

$$A = \begin{array}{|ccc|} \hline & T & aI \\ aI & T & aI \\ & \ddots & \ddots \\ & \ddots & T & aI \\ aI & aI & T \\ \hline \end{array} \quad T = \begin{array}{|ccc|} \hline & -2c & 2b \\ b & -2c & b \\ b & -2c & b \\ b & \ddots & \ddots \\ \ddots & -2c & b \\ b & -2c & b \\ \hline \end{array}$$

The stationary heat distribution is solved for $J = 100$ and plotted below.

Code: <https://github.com/RokettoJanpu/Scientific-Computing-2/blob/main/hw6.ipynb>

