

Scientific Computing HW 6

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March 10, 2023

Problem 1.

- (a) The system $Au = f$ is shown on the left, and the block structure of A is compactly written on the right.

$\begin{array}{ c c c c c c c c c c c c c c c c c c c } \hline & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ \hline 1 & -4 & 2 & & & & & & & & & & & & & \\ \hline 2 & 1 & -4 & 1 & & & & & & & & & & & & \\ \hline 3 & & 1 & -4 & 1 & & & & & & & & & & & \\ \hline 4 & & & 1 & -4 & 1 & & & & & & & & & & \\ \hline 5 & & & & 2 & -4 & & & & & & & & & & \\ \hline 6 & 1 & & & & -4 & 2 & & & & & & & & & \\ \hline 7 & & 1 & & & 1 & -4 & 1 & & & & & & & & \\ \hline 8 & & & 1 & & 1 & -4 & 1 & & & & & & & & \\ \hline 9 & & & & 1 & & 1 & -4 & 1 & & & & & & & \\ \hline 10 & & & & & 1 & & 1 & -4 & 1 & & & & & & \\ \hline 11 & & & & & & 1 & & -4 & 2 & & & & & & \\ \hline 12 & & & & & & & 1 & -4 & 1 & & & & & & \\ \hline 13 & & & & & & & & 1 & -4 & 1 & & & & & \\ \hline 14 & & & & & & & & & 1 & -4 & 1 & & & & \\ \hline 15 & & & & & & & & & & 2 & -4 & & & & \\ \hline \end{array}$ <p style="text-align: center;">A</p>	$u_1 = -1$ $u_2 = -1$ $u_3 = -1$ $u_4 = -1$ $u_5 = -1$ $u_6 = 0$ $u_7 = 0$ $u_8 = 0$ $u_9 = 0$ $u_{10} = 0$ $u_{11} = 0$ $u_{12} = 0$ $u_{13} = 0$ $u_{14} = 0$ $u_{15} = 0$ <p style="text-align: center;">$\bullet u_8 = 0$</p> <p style="text-align: center;">$A = \begin{bmatrix} (\mathcal{T}+1) \times (\mathcal{T}+1) \text{ blocks} & & & \\ \textcolor{red}{T} & \textcolor{blue}{I} & & \\ \textcolor{blue}{I} & \textcolor{red}{T} & \textcolor{blue}{I} & \\ & \ddots & \ddots & \ddots \\ & & \textcolor{blue}{T} & \textcolor{red}{I} \\ & & \textcolor{blue}{I} & \textcolor{red}{T} \end{bmatrix}$</p> <p style="text-align: center;">$T = \begin{bmatrix} -4 & 2 & & & \\ 1 & -4 & 1 & & \\ & 1 & -4 & 1 & \\ & & 1 & -4 & 1 \\ & & & 1 & -4 & 1 \\ & & & & 1 & -4 \\ & & & & & 2 & -4 \end{bmatrix}$</p>
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- (b) The system $Au = f$ is shown on the left, and the block structure of A is compactly written on the right.

$\begin{array}{ c c c c c c c c c c c c c c c c c c c } \hline & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ \hline 1 & -4 & 1 & 2 & & & & & & & & & & & & \\ \hline 2 & 1 & -4 & 1 & 2 & & & & & & & & & & & \\ \hline 3 & & 1 & -4 & & 2 & & & & & & & & & & \\ \hline 4 & 1 & & -4 & 1 & & 1 & & & & & & & & & \\ \hline 5 & & 1 & & 1 & -4 & 1 & & & & & & & & & \\ \hline 6 & & & 1 & & 1 & -4 & 1 & & & & & & & & \\ \hline 7 & & & & 1 & & -4 & 1 & 1 & & & & & & & \\ \hline 8 & & & & & 1 & & 1 & -4 & 1 & & & & & & \\ \hline 9 & & & & & & 1 & & 1 & -4 & 1 & & & & & \\ \hline 10 & & & & & & & 1 & & -4 & 1 & 1 & & & & \\ \hline 11 & & & & & & & & 1 & & -4 & 1 & & & & \\ \hline 12 & & & & & & & & & 1 & & -4 & 1 & & & \\ \hline 13 & & & & & & & & & & 2 & & -4 & 1 & & \\ \hline 14 & & & & & & & & & & & 2 & & 1 & -4 & 1 \\ \hline 15 & & & & & & & & & & & & 1 & & 1 & -4 & \\ \hline \end{array}$ <p style="text-align: center;">A</p>	$u_1 = -1$ $u_2 = 0$ $u_3 = 0$ $u_4 = -1$ $u_5 = 0$ $u_6 = 0$ $u_7 = -1$ $u_8 = 0$ $u_9 = 0$ $u_{10} = -1$ $u_{11} = 0$ $u_{12} = 0$ $u_{13} = -1$ $u_{14} = 0$ $u_{15} = 0$ <p style="text-align: center;">$\bullet u_8 = 0$</p> <p style="text-align: center;">$A = \begin{bmatrix} (\mathcal{T}-1) \times (\mathcal{T}-1) \text{ blocks} & & & \\ \textcolor{red}{T} & 2\textcolor{blue}{I} & & \\ \textcolor{blue}{I} & \textcolor{red}{T} & \textcolor{blue}{I} & \\ & \ddots & \ddots & \ddots \\ & & \textcolor{blue}{T} & \textcolor{red}{I} \\ & & \textcolor{blue}{I} & \textcolor{red}{T} \\ & & & 2\textcolor{red}{I} & \textcolor{blue}{T} \end{bmatrix}$</p> <p style="text-align: center;">$T = \begin{bmatrix} -4 & 1 & \\ 1 & -4 & 1 & \\ & 1 & -4 & 1 & \\ & & 1 & -4 & 1 & \\ & & & 1 & -4 & 1 & \\ & & & & 2 & & -4 & 1 & & & & & & & & & \\ & & & & & 2 & & 1 & -4 & 1 & & & & & & & & \\ & & & & & & 1 & & 1 & -4 & 1 & & & & & & & \\ & & & & & & & 1 & & -4 & 1 & 1 & & & & & & \\ & & & & & & & & 1 & & -4 & 1 & & & & & & \\ & & & & & & & & & 2 & & -4 & 1 & & & & & \\ & 2 & & 1 & -4 & 1 & & & \\ & 1 & & 1 & -4 & \\ \end{bmatrix}$</p>
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- (c) The system $Au = f$ is shown on the left, and the block structure of A is compactly written on the right.

(d) The system $Au = f$ is shown on the left, and the block structure of A is compactly written on the right.

(e) The system $Au = f$ is shown on the left. The block structure of A is highlighted.

Problem 2. The BVP on the domain $\Omega := [-\pi, \pi] \times [0, 2]$ is

$$u_{xx} + u_{yy} = g(x) := \begin{cases} -\cos x, & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ 0, & \text{else} \end{cases} \quad (2.1)$$

with BCs

$$u \Big|_{x=\pi} = u \Big|_{x=-\pi}, \quad u_x \Big|_{x=\pi} = u_x \Big|_{x=-\pi}, \quad u \Big|_{y=0} = 0, \quad u_y \Big|_{y=2} = 0$$

Fix $J \in \mathbb{N}$. Set mesh steps in the x and y directions,

$$h_x := \frac{2\pi}{J}, \quad h_y := \frac{2}{J}$$

Then

$$\begin{aligned} u_{xx}(x, y) &= \frac{1}{h_x^2} [u(x + h_x, y) - 2u(x, y) + u(x - h_x, y)] + O(h_x^2) \\ u_{yy}(x, y) &= \frac{1}{h_y^2} [u(x, y + h_y) - 2u(x, y) + u(x, y - h_y)] + O(h_y^2) \end{aligned}$$

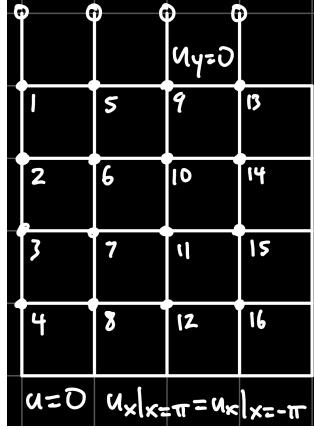
Plug these expressions into (2.1) and use the compass direction notation from lecture.

$$-2u_P \left[\frac{1}{h_x^2} + \frac{1}{h_y^2} \right] + \frac{1}{h_x^2}[u_E + u_W] + \frac{1}{h_y^2}[u_N + u_S] = g_P$$

Set $a := \frac{1}{h_x^2}$, $b := \frac{1}{h_y^2}$, $c := a + b$, so that

$$-2cu_P + a[u_E + u_W] + b[u_N + u_S] = g_P \quad (2.2)$$

To explore the appropriate numerical method, take $J = 4$. The mesh steps create a mesh from Ω with $(J+1)^2 = 25$ points. Using the BCs, it is enough to solve for the values of 16 points, labeled below.



Apply (2.2) to each point in the mesh to obtain a system $Au = f$, shown below on the left. The block structure of A is compactly written on the right.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2c	2b			a								a			
b	-2c	b			a							a			
b	-2c	b			a							a			
b	-2c			a								a			
a				-2c	2b		a								
a				b	-2c	b		a							
a				b	-2c	b			a						
a				b	-2c			a							
								-2c	2b						
								b	-2c	b					
								b	-2c	b					
								b	-2c	b					
								b	-2c						

A

U_1	$g(-\pi + \frac{2\pi}{n})$
U_2	$g(-\pi + \frac{\pi}{n})$
U_3	$g(-\pi + \frac{3\pi}{n})$
U_4	$g(-\pi + \frac{4\pi}{n})$
U_5	$g(-\pi + \frac{5\pi}{n})$
U_6	$g(-\pi + \frac{6\pi}{n})$
U_7	$g(-\pi + \frac{7\pi}{n})$
U_8	$g(-\pi + \frac{8\pi}{n})$
U_9	$g(-\pi + \frac{9\pi}{n})$
U_{10}	$g(-\pi + \frac{10\pi}{n})$
U_{11}	$g(-\pi + \frac{11\pi}{n})$
U_{12}	$g(-\pi + \frac{12\pi}{n})$
U_{13}	$g(-\pi + \frac{13\pi}{n})$
U_{14}	$g(-\pi + \frac{14\pi}{n})$
U_{15}	$g(-\pi + \frac{15\pi}{n})$
U_{16}	$g(-\pi + \frac{16\pi}{n})$

U

f

$\mathcal{T} \times \mathcal{T}$ blocks

$$A = \begin{bmatrix} \mathcal{T} & aI \\ aI & \mathcal{T} & aI \\ & \ddots & \ddots & \ddots \\ & & \mathcal{T} & aI \\ & & aI & \mathcal{T} \end{bmatrix} \quad \mathcal{T} = \begin{bmatrix} -2c & 2b \\ b & -2c & b \\ & b & -2c & b \\ & & b & -2c \end{bmatrix}$$