Scientific Computing HW 9

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Problem 2. Using separation of variables, write

$$u(x,t) = X(x)T(t)$$

Plug into the PDE.

$$X(x)T'(t) = X''(x)T(t) \implies \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda \implies X'' + \lambda X = 0,\ T' + \lambda T = 0$$

The eigenvalue problem for X,

$$X'' + \lambda X = 0,\ X(0) = X'(\pi) = 0$$

has eigenvalues and eigenfunctions

$$\lambda_n = (n+1/2)^2,\ X_n(x) = \sin[(n+1/2)x],\ n \geq 0$$

Solving the ODE for T,

$$T' + \lambda_n T = 0 \implies T(t) = e^{-\lambda_n t}$$

Thus the solution to the BVP is, for some coefficients a_n,

$$u(x,t) = \sum_{n\geq 0} a_n e^{-(n+1/2)^2 t} \sin[(n+1/2)x]$$

Applying the IC,

$$x = u\Big|_{t=0} = \sum_{n\geq 0} a_n \sin[(n+1/2)x]$$

Fix $m \geq 0$, multiply each side by X_m, and integrate on $[0,\pi]$, using the fact $\int_0^\pi X_n X_m = \frac{\pi}{2}\delta_{nm}$. The RHS is

$$\int_0^\pi \sum_{n\geq 0} a_n X_n X_m = \sum_{n\geq 0} a_n \int_0^\pi X_n X_m = \sum_{n\geq 0} a_n \frac{\pi}{2}\delta_{nm} = a_m \frac{\pi}{2}$$

Integrating by parts, the LHS is

$$\int_0^\pi x \sin[(m+1/2)x]dx = \left[-\frac{1}{m+1/2} x\cos[(m+1/2)x] + \frac{1}{(m+1/2)^2}\sin[(m+1/2)x]\right]\Bigg|_0^\pi$$

$$= -0 + 0 + \frac{1}{(m+1/2)^2}(-1)^m - 0 = \frac{(-1)^m}{(m+1/2)^2}$$

Thus

$$a_m \frac{\pi}{2} = \frac{(-1)^m}{(m+1/2)^2} \implies a_m = \frac{2(-1)^{m+1}}{\pi(m+1/2)^2}$$