AMSC661, Spring 2023 Maria Cameron

Homework 8. Due Wednesday, April 5

Please upload a single pdf file on ELMS. Link your codes to your pdf (i.e., put your codes to dropbox, Github, google drive, etc. and place links to them in your pdf file with your solutions.

1. **(10 pts)** *You can use Python or Matlab. If you choose Matlab, you can modify the code MyFEMcat.m. If you choose Python, you can modify FEM_TPT.py and any ipynb notebook available on my GitHub: Github.*

   Consider the problem of finding the density of the electric current in a thin square plate Ω made out of two metals with different conductivities $a = a_1$ and $a = a_2$ as shown in Fig. 1. The voltage $u = 0$ at the left side of the square, and $u = 1$ on the right side. Use the finite element method (FEM) to find the voltage and the density of the electric current inside the plate. Plot the voltage and the absolute value of the current density. Consider two cases: (a): $a_1 = 1.2$ and $a_2 = 1$; (b): $a_1 = 0.8$ and $a_2 = 1$. Comment on the distribution of the current in the plate. Link your codes to your pdf file.

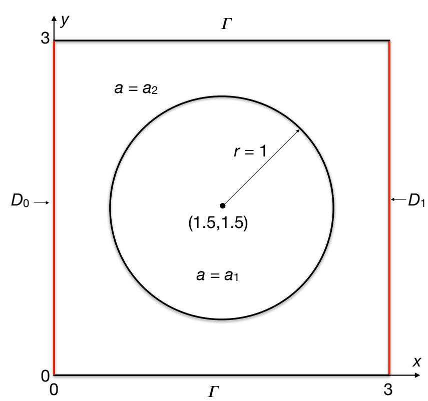


Figure 1: The conducting plate. An illustration to Problem 1.

The voltage is the solution to the following boundary value problem (BVP):

$$\begin{cases} -\nabla \cdot (a(x,y)\nabla u) = 0, & (x,y) \in \Omega := [0,3]^2, \\ u = 0, & (x,y) \in D_0, \\ u = 1, & (x,y) \in D_1, \\ \frac{\partial u}{\partial n} = 0, & (x,y) \in \Gamma. \end{cases} \tag{1}$$

Note that the solution must satisfy the flux continuity condition: at the boundary separating the regions where $a(x, y) = a_1$ and $a(x, y) = a_2$, the voltage u and the current density $a(x, y)\nabla u$ must be continuous. This condition is automatically satisfied if you generate the mesh so that $a(x, y)$ is continuous in each mesh triangle and then solve the BVP using FEM. This can be easily achieved using e.g. `distmesh2d.m` requiring the points on the boundary separating these two regions to be fixed.

Once you have computed the voltage u, you can find the current density:

$$j = -a(x, y)\nabla u.$$

Use the formula for the gradient found in HW7, Problem 4(b). As a result, you have the gradient evaluated at the centers of the mesh triangles. Then compute the absolute value of the density of the current. In order to visualize it using `trisurf` you need to recompute it at the vertices of the mesh triangles. This can be done by averaging over all triangles adjacent to a given vertex:

```
abs_current_verts = zeros(Npts,1);
count_tri = zeros(Npts,1);
for j = 1:Ntri
    abs_current_verts(tri(j,:)) = abs_current_verts(tri(j,:)) ...
         + abs_current_centers(j);
    count_tri(tri(j,:)) = count_tri(tri(j,:)) + 1;
end
abs_current_verts = abs_current_verts./count_tri;
```

2. **(8 pts)** The goal of this problem is to show that the FEM solution of the Poisson equation is the linear interpolant of the exact solution. We will examine only the 1D case. This problem is a composite of problems 2.2a, 2.4, and 5.4 from [2].

   Consider the BVP

$$-u'' = f(x), \quad 0 < x < 1, \quad u(0) = u(1) = 0, \tag{2}$$

   where $f(x)$ is a given function integrable on $[0, 1]$.

   (a) Verify that the solution of Eq. (7) is given by

$$u(x) = \int_0^1 G(x, y)f(y)dy, \tag{3}$$

   where $G(x, y)$ is Green's function defined in $[0, 1]^2$ by

$$G(x, y) = \begin{cases} (1-x)y, & 0 \le y \le x \le 1, \\ x(1-y), & 0 \le x \le y \le 1. \end{cases} \tag{4}$$

(b) Verify that for any function $v(x) \in H_0^1([0,1])$, the following identity holds:

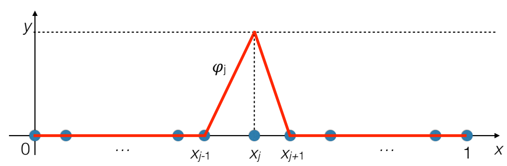
$$\int_0^1 v'(y) \frac{d}{dy} G(x,y) dy = v(x). \tag{5}$$

Conclude that since the exact solution u of Eq. (7) belongs to $H_0^1([0,1])$, Eq. (5) holds for u for all $x \in [0,1]$.

(c) The linear interpolant of the exact solution $u(x)$, denoted by $I_h u$, is a function that is equal to u at the nodes

$$0 \equiv x_0 < x_1 < \ldots < x_n < x_{n+1} \equiv 1,$$

and is linear in each interval $[x_j, x_{j+1}]$, $j = 0, 1, \ldots, n$. (We do not assume that the nodes are equispaced.) Write $I_h u$ as a linear combination of the basis functions shown in fig. below.



(d) Show that the set of functions $\psi_j(y) := G(x_j, y)$ is a basis in $S_h(x) = \text{span}\{\phi_1, \ldots, \phi_n\}$.

(e) Show that for the linear interpolant $I_h u$,

$$\int_0^1 [I_h u]'(y) \frac{d}{dy} G(x_j, y) dy = u(x_j). \tag{6}$$

(f) Write out the definition of the FEM solution. Prove, using all the facts above that the FEM solution

$$U^{FEM}(x) = \sum_{j=1}^{n} U_j^{FEM} \phi_j(x)$$

coincides with $I_h u$.

3. **(5 pts)** This problem suggests a way to modify a non-self-adjoint elliptic equation to make it amenable for solving using FEM.

(a) Let Ω be a compact domain in 2D. The boundary of Ω can be decomposed as $\partial\Omega = \Gamma_D \cup \Gamma_N$. Consider the following boundary-value problem:

$$\begin{cases} \beta^{-1}\Delta u + b(x) \cdot \nabla u = 0, & x \in \Omega \\ u = u_D & x \in \Gamma_D \\ \frac{\partial u}{\partial n} = g, & x \in \Gamma_N. \end{cases} \tag{7}$$

Let b be a smooth vector field and β be a constant. Suppose it can be decomposed into curl-free and divergence-free components:

$$b = -\nabla V(x) + f(x). \tag{8}$$

The curl-free component is ∇V and the divergence-free component is f: $\nabla \cdot f = 0$. Show that BVP (7) is equivalent to

$$\begin{cases} \nabla \cdot \left(e^{-\beta V(x)} \nabla u\right) + \beta e^{-\beta V(x)} f(x) \cdot \nabla u = 0, & x \in \Omega \\ u = u_D & x \in \Gamma_D \\ \frac{\partial u}{\partial n} = g, & x \in \Gamma_N. \end{cases} \tag{9}$$

(b) Consider the Maier-Stein drift field

$$b(x,y) = \begin{bmatrix} x - x^3 - 10xy^2 \\ -(1+x^2)y \end{bmatrix}. \tag{10}$$

Decompose $b(x,y)$ into divergence-free and curl-free components. Find the potential function $V(x,y)$.

References

[1] Jochen Alberty, Carsten Carstensen and Stefan A. Funken, Remarks around 50 lines of Matlab: short finite element implementation

[2] S. Larsson and V. Thomee, Partial Differential Equations with Numerical Methods, Springer-Verlag Berlin Heidelberg, 2003, 2009 (soft cover)