AMSC661, Spring 2023 Maria Cameron

Homework 11. Due April 26

Reading:

- [1], Chapter 10, in particular, Section 10.9 (Modified equations) and Appendix E3 (Fourier analysis).
- My lecture notes `Hyperbolic.pdf`.

1. **(15 pts)** Consider the wave equation of an infinite line:

$$u_{tt} = a^2 u_{xx}, \tag{1}$$
$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x).$$

The parameter $a > 0$ is the speed of wave propagation.

(a) Verify that the solution is given by d'Alembert's formula

$$u(x,t) = \frac{1}{2}(\phi(x+at) + \phi(x-at)) + \frac{1}{2a}\int_{x-at}^{x+at} \psi(s)ds. \tag{2}$$

(b) The domain of influence of a point x_0 is the set of points on the (x,t)-plane such that $u(x,t)$ depends on the initial conditions at the point x_0. More precisely, a point (x,t) belongs to the domain of influence of x_0, if for any neighborhood of x_0 there exist perturbations of the initial conditions supported within this neighborhood such that the solution changes at the point (x,t) due to these perturbations.

Plot the domain of influence of a point x_0 on the (x,t)-plane.

The domain of dependence of a point (x,t) is the minimal set of points on the line $(x,0)$ such that the solution at the point (x,t) is completely determined by the initial conditions restricted to this set. (The word minimal in the clause "minimal set satisfying some conditions" means that no its proper subset satisfies these conditions.)

Plot the domain of dependence of a point (x,t).

(c) Rewrite Eq. (1) as a hyperbolic system. Proceed as follows. Introducing the new vector-function

$$w = \begin{bmatrix} u_t \\ u_x \end{bmatrix}.$$

Derive the equation for w: $w_t = Aw_x$, where

$$A = \begin{bmatrix} 0 & a^2 \\ 1 & 0 \end{bmatrix}.$$

Express the initial conditions for w in terms of ϕ and ψ.

(d) Diagonalize the matrix A, i.e., find its eigenvalues and eigenvectors (λ_1, v_1) and (λ_2, v_2) and rewrite it of the form $A = C\Lambda C^{-1}$, where $C = [v_1, v_2]$, $\Lambda = \text{diag}\{\lambda_1, \lambda_2\}$. Introduce the new variable

$$y = \begin{bmatrix} \xi \\ \eta \end{bmatrix} \equiv C^{-1}w.$$

Observe that the system for y is decoupled into two independent advection equations:

$$\xi_t = \lambda_1 \xi_x, \quad \eta_t = \lambda_2 \eta_x. \tag{3}$$

Obtain the initial conditions for ξ and η in terms of ϕ and ψ.

(e) Set $\phi = \max\{1 - |x|, 0\}$, $\psi = 0$, $a = \sqrt{2}$, $h = 0.05$, and a reasonable time step k. Pick the numerical domain $-6 \le x \le 6$ and periodic boundary conditions. Solve the equations for ξ and η numerically using Lax-Friedrichs, appropriate Upwind (left for one and right for the other depending on signs of λ), Lax-Wendroff, and appropriate Beam-Warming methods. Return to the variable w and then to the original variable u. Plot the numerical solutions u obtained using each of the methods at times $t = 1/(2a)$, $t = 1/a$, $t = 2/a$, and $t = 4/a$ as well as the exact solution at these times given by Eq. (2). Write a summary of your observations.

2. **(15 pts)**

Consider the advection equation

$$u_t + au_x = 0, \quad u(x, 0) = \phi(x), \quad -\infty < x < \infty, \quad t \ge 0. \tag{4}$$

Assume that $a > 0$.

(a) Derive the modified equations for the Lax-Friedrichs method (LaxFr), the Leapfrog (LF) and the Left Beam Warming (BWL) method (Eq. (10.26) in [1]). For convenience, you can introduce the parameter $\nu := ak/h$ where $k \equiv \Delta t$ is the step in t, h is the step in x and make coefficients the modified equations more compact using it.

(b) Consider the modified equations for the Upwind Left (UL), Lax-Wendroff (LW) (derived in class, see [1]) and the ones that you derived for LaxFr, LF, and BWL. UL and LaxFr are first order, while LW, LF, and BWL) are second order. Use their modified equations for determining which of the first-order methods is most accurate and which of the second-order methods is the most accurate. Comment on the choice of the time step k given a and space step h that would lead to the smallest numerical error in each of these five methods.

(c) Apply the Fourier transform (Eq. (E.18) in Appendix E in [1]) to the modified equations for UL, LaxFr, LW, LF, and BWL. Solve them in the Fourier space (i.e., sole the ODEs for $\hat{u}$. Comment on the behavior of the Fourier modes $\hat{u}(\xi, t)$ as $t \to \infty$. Obtain the dispersion relations (Eq. (E.48) in [1]) and find the phase and group velocities. What are the phase and the group velocity for the original advection equation? Run the code `advection.m` available on ELMS. The choice of method is controlled by the variable `method` in the code. Explain the appearance and the location of oscillations produced by these methods using their modified equations.

References

[1] R. J. LeVeque, Finite Difference Methods for Ordinary and Partial Differential Equations, SIAM 2007 (Chapter 10, Appendix E)