

Scientific Computing HW 3

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Problem 1. Pf. From the Butcher array,

$$A = \begin{bmatrix} \gamma & 0 \\ 1-\gamma & \gamma \end{bmatrix}, \quad b = \begin{bmatrix} 1-\gamma \\ \gamma \end{bmatrix}, \quad c = \begin{bmatrix} \gamma \\ 1 \end{bmatrix}$$

Check the 1st order accuracy condition.

$$\sum_{l=1}^2 b_l c_l = (1-\gamma) + \gamma = 1$$

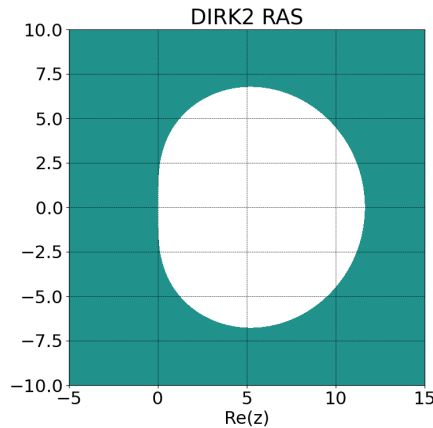
Check the 2nd order accuracy condition.

$$\sum_{l=1}^2 b_l c_l^2 = (1-\gamma)\gamma + \gamma \cdot 1 = \gamma - \gamma^2 + \gamma = 2\gamma - \gamma^2 = 2 - 2^{1/2} - 1 - 2^{-1} + 2^{1/2} = 1 - 2^{-1} = \frac{1}{2}$$

Thus the method is 2nd order accurate. To show it is A-stable, first find the stability function $R(z)$ and let $|z| \rightarrow \infty$.

$$\begin{aligned} I - zA &= \begin{bmatrix} 1-\gamma z & 0 \\ -(1-\gamma)z & 1-\gamma z \end{bmatrix} \implies D := \det(I - zA) = (1-\gamma z)^2 = \gamma^2 z^2 - 2\gamma z + 1 \\ \implies (I - zA)^{-1} &= \frac{1}{D} \begin{bmatrix} 1-\gamma z & 0 \\ (1-\gamma)z & 1-\gamma z \end{bmatrix} \implies (I - zA)^{-1} \mathbf{1}_{s \times 1} = \frac{1}{D} \begin{bmatrix} 1-\gamma z \\ (1-\gamma)z + 1 - \gamma z \end{bmatrix} = \frac{1}{D} \begin{bmatrix} 1-\gamma z \\ (1-2\gamma)z + 1 \end{bmatrix} \\ R(z) - 1 &= z b^T (I - zA)^{-1} \mathbf{1}_{s \times 1} = \frac{z}{D} [(1-\gamma)(1-\gamma z) + \gamma((1-2\gamma)z + 1)] = \frac{z}{D} [1 - \gamma z - \gamma + \gamma^2 z + (\gamma - 2\gamma^2)z + \gamma] \\ \implies R(z) - 1 &= \frac{z}{D} [1 - \gamma^2 z] = \frac{-\gamma^2 z^2 + z}{\gamma^2 z^2 - 2\gamma z + 1} \implies R(z) = \frac{-\gamma^2 z^2 + z}{\gamma^2 z^2 - 2\gamma z + 1} + 1 \xrightarrow{|z| \rightarrow \infty} -1 + 1 = 0 \end{aligned}$$

To finish showing A-stability, we plot the RAS and see that it contains the left half plane. Code in 2nd cell of: <https://github.com/RokettoJanpu/Scientific-Computing-2/blob/main/hw3%20RAS.ipynb>



Problem 2. From the Butcher array,

$$A = \begin{bmatrix} \gamma & 0 \\ 1-2\gamma & \gamma \end{bmatrix}, b = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}, c = \begin{bmatrix} \gamma \\ 1-\gamma \end{bmatrix}$$

1. **Pf.** Check the 1st order accuracy condition.

$$\sum_{l=1}^2 b_l = \frac{1}{2} + \frac{1}{2} = 1$$

Check the 2nd order accuracy condition.

$$\sum_{l=1}^2 b_l c_l = \frac{1}{2}\gamma + \frac{1}{2}(1-\gamma) = \frac{1}{2}(\gamma + 1 - \gamma) = \frac{1}{2}$$

2. **Pf.** Check the 3rd order accuracy conditions.

$$\sum_{p,q,r} b_p a_{pq} a_{pr} = \frac{1}{2}[\gamma^2 + 2\gamma \cdot 0 + 0^2] + \frac{1}{2}[(1-2\gamma)^2 + 2(1-2\gamma)\gamma + \gamma^2]$$

The quantity in the second bracket is

$$(1-2\gamma)^2 + 2(1-2\gamma)\gamma + \gamma^2 = 1 + 4\gamma^2 - 4\gamma + 2\gamma - 4\gamma^2 + \gamma^2 = \gamma^2 - 2\gamma + 1 = (\gamma - 1)^2$$

giving

$$\sum_{p,q,r} b_p a_{pq} a_{pr} = \frac{1}{2}[\gamma^2 + \gamma^2 - 2\gamma + 1] = \gamma^2 - \gamma + \frac{1}{2}$$

We find

$$\gamma^2 = \frac{1}{2} + \frac{3}{36} + 2\frac{3^{1/2}}{12} = \frac{1}{12}[3 + 1 + 2 \cdot 3^{1/2}] = \frac{1}{12}[4 + 2 \cdot 3^{1/2}] = \frac{1}{6}[2 + 3^{1/2}]$$

so finally,

$$\sum_{p,q,r} b_p a_{pq} a_{pr} = \frac{1}{6}[2 + 3^{1/2} - 3 - 3^{1/2} + 3] = \frac{1}{3}$$

3. First find the stability function $R(z)$.

$$I - zA = \begin{bmatrix} 1 - \gamma z & 0 \\ -(1-2\gamma)z & 1 - \gamma z \end{bmatrix} \implies D := \det(I - zA) = (1 - \gamma z)^2 = \gamma^2 z^2 - 2\gamma z + 1$$

$$\implies (I - zA)^{-1} = \frac{1}{D} \begin{bmatrix} 1 - \gamma z & 0 \\ (1-2\gamma)z & 1 - \gamma z \end{bmatrix} \implies (I - zA)^{-1} \mathbf{1}_{s \times 1} = \frac{1}{D} \begin{bmatrix} 1 - \gamma z \\ (1-2\gamma)z + 1 - \gamma z \end{bmatrix} = \frac{1}{D} \begin{bmatrix} 1 - \gamma z \\ (1-3\gamma)z + 1 \end{bmatrix}$$

$$R(z) - 1 = z b^T (I - zA)^{-1} \mathbf{1}_{s \times 1} = \frac{z}{2D} [1 - \gamma z + (1-3\gamma)z + 1] = \frac{z}{2D} [(1-4\gamma)z + 2] = \frac{1}{2} \frac{(1-4\gamma)z^2 + 2z}{\gamma^2 z^2 - 2\gamma z + 1}$$

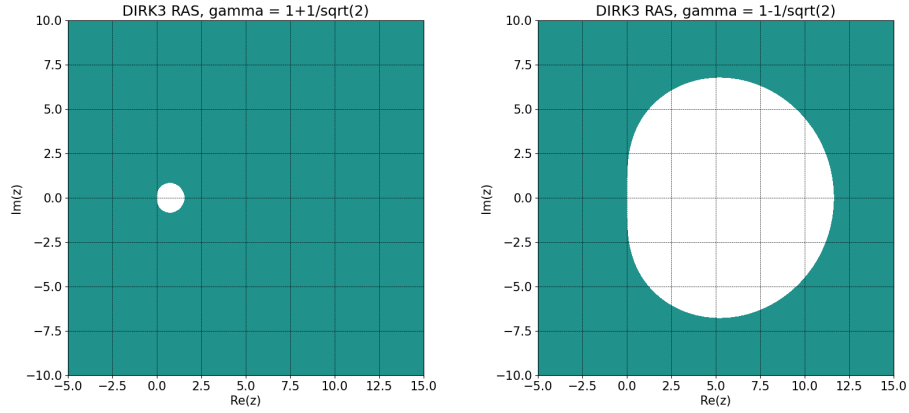
We find γ by imposing $\lim_{|z| \rightarrow \infty} R(z) = 0$.

$$\lim_{|z| \rightarrow \infty} R(z) = 0 \iff -1 = \frac{1}{2} \lim_{|z| \rightarrow \infty} \frac{(1-4\gamma)z^2 + 2z}{\gamma^2 z^2 - 2\gamma z + 1} \iff \lim_{|z| \rightarrow \infty} \frac{(1-4\gamma)z^2 + 2z}{\gamma^2 z^2 - 2\gamma z + 1} = -2 \iff \frac{1-4\gamma}{\gamma^2} = -2$$

$$\iff -2\gamma^2 = 1 - 4\gamma \iff 2\gamma^2 - 4\gamma + 1 = 0 \iff \gamma = \frac{4}{4} \pm \frac{(16-8)^{1/2}}{4} = 1 \pm \frac{2 \cdot 2^{1/2}}{4} = 1 \pm 2^{-1/2}$$

We check that the method for $\gamma = 1 \pm 2^{-1/2}$ is A-stable, hence L-stable, by plotting the RASes and seeing that they contain the left half plane. Code in 3rd cell of:

<https://github.com/RokettoJanpu/Scientific-Computing-2/blob/main/hw3%20RAS.ipynb>



Problem 3. Code for all parts of this problem:

<https://github.com/RokettoJanpu/Scientific-Computing-2/blob/main/hw3.ipynb>

- (a) Set $f(t, y) := -L(y - \phi(t)) + \phi'(t)$. To implement DIRK2, we first obtain explicit formulas for k_1, k_2, u_{n+1} .

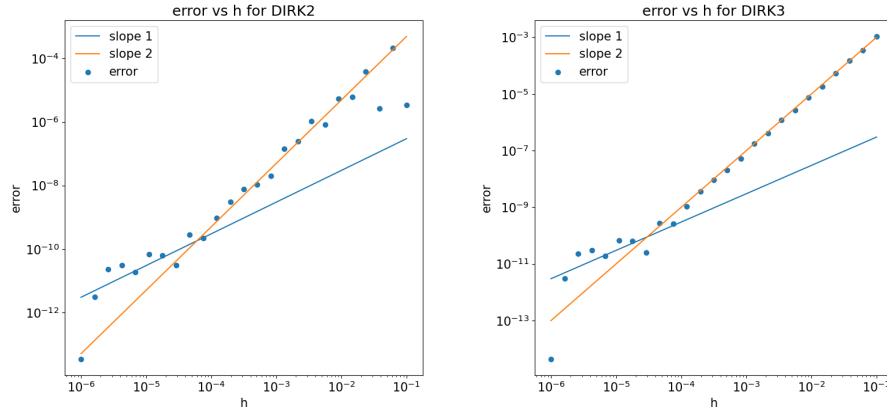
$$\begin{aligned}
 k_1 &= f(t_n + \gamma h, u_n + h\gamma k_1) = -L[u_n + h\gamma k_1 - \phi(t_n + \gamma h)] + \phi'(t_n + \gamma h) \\
 &\implies k_1 = -Lh\gamma k_1 - L[u_n - \phi(t_n + \gamma h)] + \phi'(t_n + \gamma h) \\
 \implies (1 + Lh\gamma)k_1 &= -L[u_n - \phi(t_n + \gamma h)] + \phi'(t_n + \gamma h) \implies k_1 = \frac{-L[u_n - \phi(t_n + \gamma h)] + \phi'(t_n + \gamma h)}{1 + Lh\gamma} \\
 k_2 &= f(t_n + h, u_n + h(1 - \gamma)k_1 + h\gamma k_2) = -L[u_n + h(1 - \gamma)k_1 + h\gamma k_2 - \phi(t_n + h)] + \phi'(t_n + h) \\
 &\implies k_2 = -Lh\gamma k_2 - L[u_n + h(1 - \gamma)k_1 - \phi(t_n + h)] + \phi'(t_n + h) \\
 &\implies (1 + Lh\gamma)k_2 = -L[u_n + h(1 - \gamma)k_1 - \phi(t_n + h)] + \phi'(t_n + h) \\
 &\implies k_2 = \frac{-L[u_n + h(1 - \gamma)k_1 - \phi(t_n + h)] + \phi'(t_n + h)}{1 + Lh\gamma} \\
 u_{n+1} &= u_n + h[(1 - \gamma)k_1 + \gamma k_2]
 \end{aligned}$$

We do the same for DIRK3 (abuse of notation for DIRK of order 3).

$$k_1 = f(t_n + \gamma h, u_n + h\gamma k_1)$$

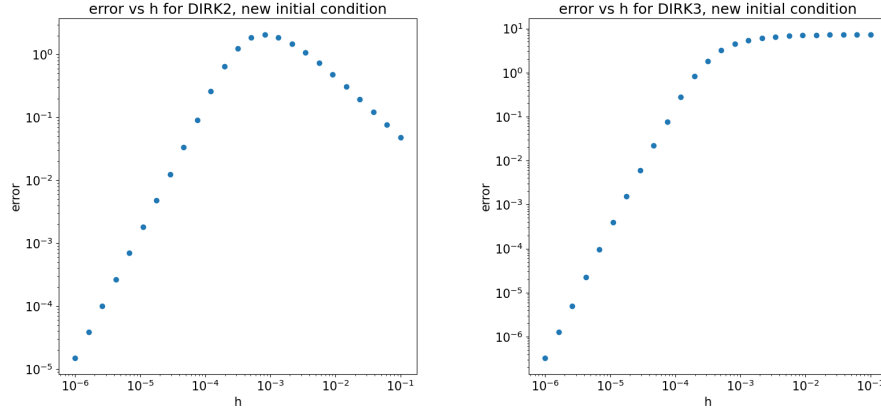
so the explicit formula for k_1 is the same as in DIRK2.

$$\begin{aligned}
 k_1 &= \frac{-L[u_n - \phi(t_n + \gamma h)] + \phi'(t_n + \gamma h)}{1 + Lh\gamma} \\
 k_2 &= f(t_n + (1 - \gamma)h, u_n + h(1 - 2\gamma)k_2 + h\gamma k_2) \\
 &= -L[u_n + h(1 - 2\gamma)k_1 + h\gamma k_2 - \phi(t_n + (1 - \gamma)h)] + \phi'(t_n + (1 - \gamma)h) \\
 &= -Lh\gamma k_2 - L[u_n + h(1 - 2\gamma)k_1 - \phi(t_n + (1 - \gamma)h)] + \phi'(t_n + (1 - \gamma)h) \\
 \implies (1 + Lh\gamma)k_2 &= -L[u_n + h(1 - 2\gamma)k_1 - \phi(t_n + (1 - \gamma)h)] + \phi'(t_n + (1 - \gamma)h) \\
 \implies k_2 &= \frac{-L[u_n + h(1 - 2\gamma)k_1 - \phi(t_n + (1 - \gamma)h)] + \phi'(t_n + (1 - \gamma)h)}{1 + Lh\gamma} \\
 u_{n+1} &= u_n + \frac{h}{2}[k_1 + k_2]
 \end{aligned}$$

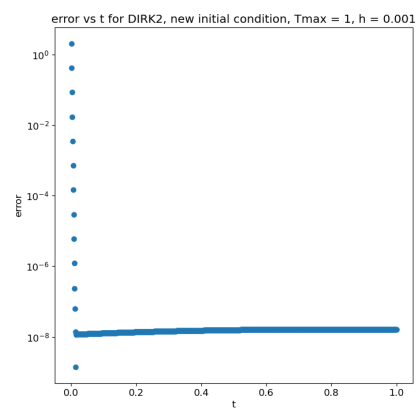
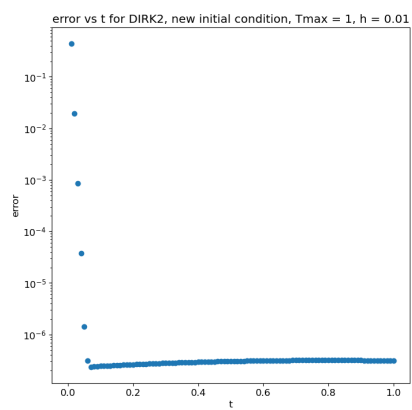
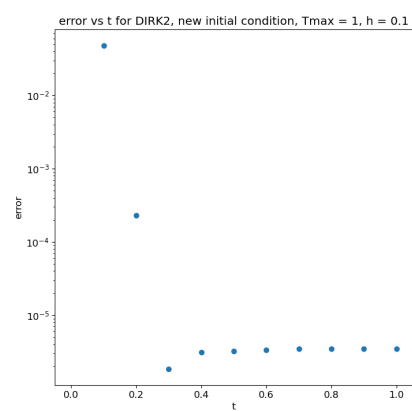
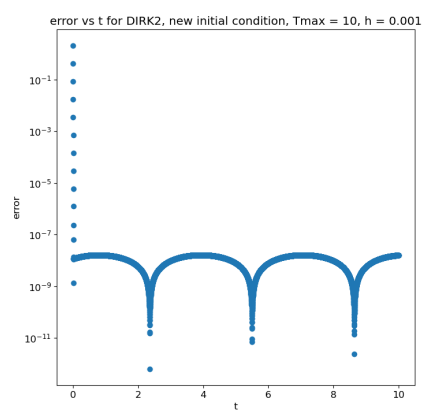
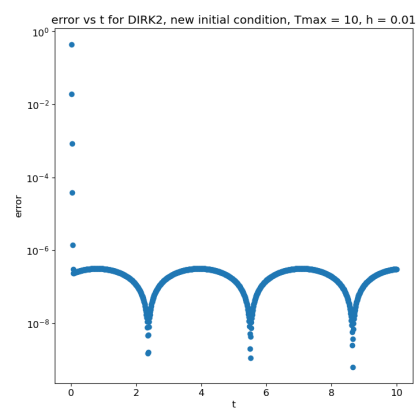
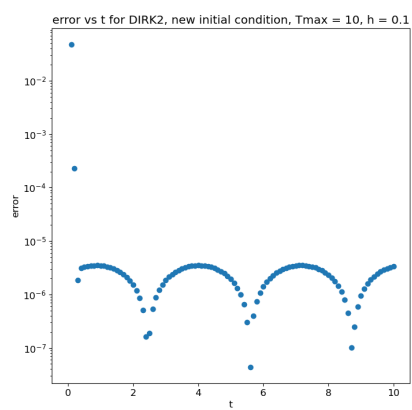


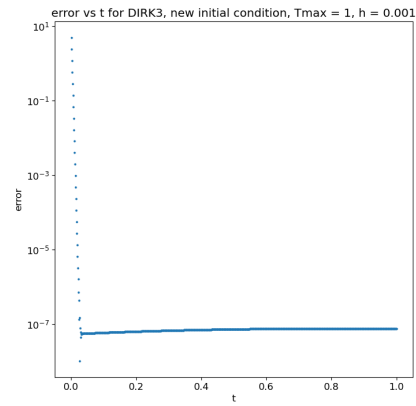
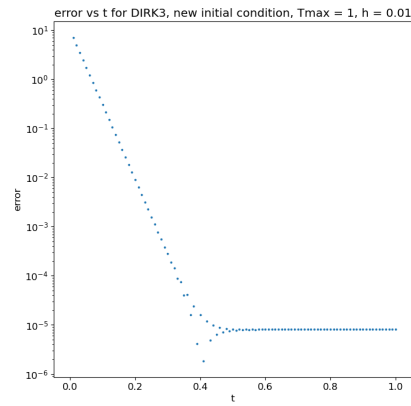
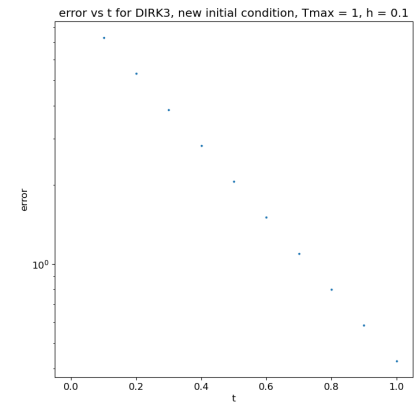
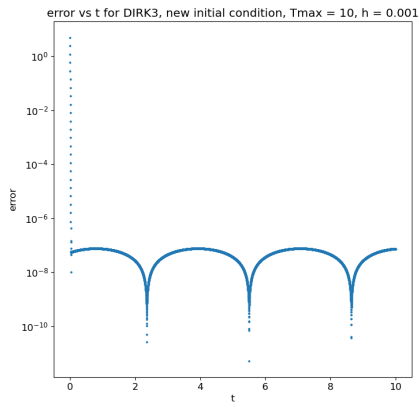
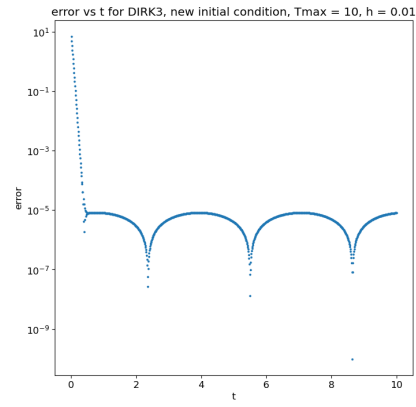
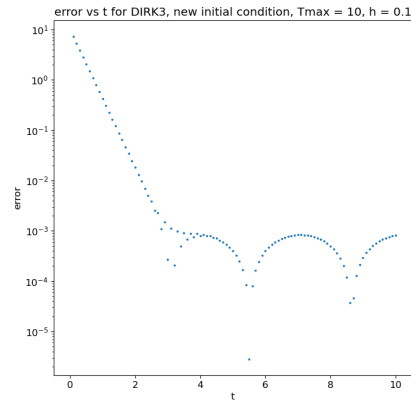
For $h > 10^{-4}$, the log-log graph roughly has a slope of 2, ie the error is on the order of h . For $10^{-5} < h < 10^{-4}$, the log-log graph roughly has a slope of 1, ie the error is on the order of h^2 . For $h < 10^{-5}$, the slope is considerably steeper, ie the error is on the order of a high power of h .

- (b) Repeating (a) with $y(0) = \sin(\frac{\pi}{4}) + 10$, we notably get a region for DIRK2 where error increases as h decreases, and a region for DIRK3 of very small slope.



For each method, we plot error vs time graphs for $h = 10^{-1}, 10^{-2}, 10^{-3}$ and $T_{\max} = 10, 1$.





(c) When the initial condition receives a