Scientific Computing HW 13

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Problem 1.

(a) Using $f(u) = au$,

$$u_j^* = u_j^n - \frac{ak}{h}(u_{j+1}^n - u_j^n)$$

In turn,

$$\begin{aligned} u_j^{n+1} &= \frac{1}{2}\left[u_j^n + u_j^n - \frac{ak}{h}(u_{j+1}^n - u_j^n)\right] - \frac{ak}{2h}\left[u_j^n - \frac{ak}{h}(u_{j+1} - u_j^n) - u_{j-1}^n + \frac{ak}{h}(u_j^n - u_{j-1}^n)\right] \\ &= u_j^n - \frac{ak}{2h}(u_{j+1}^n - u_j^n + u_j^n - u_{j-1}^n) + \frac{a^2k^2}{2h^2}(u_{j+1}^n - u_j^n - u_j^n + u_{j-1}^n) \\ &= u_j^n - \frac{ak}{2h}(u_{j+1}^n - u_{j-1}^n) + \frac{a^2k^2}{2h^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) \end{aligned}$$

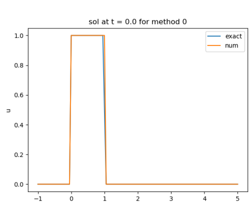
Which coincides with Lax–Wendroff.

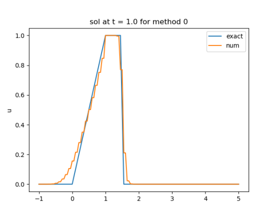
Problem 2.

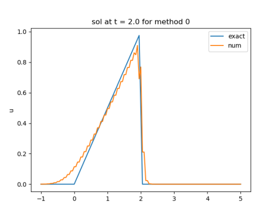
Pf. Recall $F(u_L, u_R) = f(u^*(u_L, u_R))$. From the fact $f'' > 0$, f' is increasing. Suppose $u_L \le u_R$. In case 1, from $f'(u_L) \ge 0$ we have $f' \ge 0$ hence f is increasing, so $F(u_L, u_R) = \min_{u_L \le u \le u_R} f(u) = f(u_L)$. In case 2, from $f'(u_R) \le 0$, we have $f' \le 0$ hence f is decreasing, so $F(u_L, u_R) = \min_{u_L \le u \le u_R} f(u) = f(u_R)$. In case 3, we have f is increasing, so $(f(u_L) - f(u_R))/(u_L - u_R) > 0$ hence $F(u_L, u_R) = f(u_L)$. Similar arguments can be made for when we suppose $u_L > u_R$.

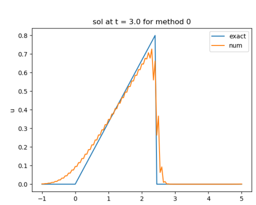
Problem 3.

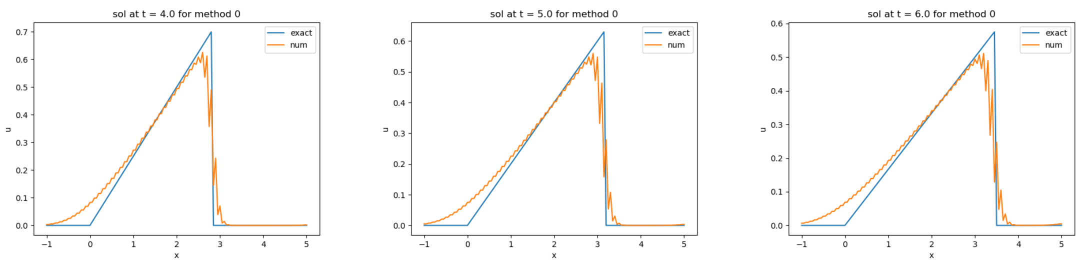
Code: `https://github.com/RokettoJanpu/Scientific-Computing-2/blob/main/hw13q3.ipynb`
Lax–Friedrichs:



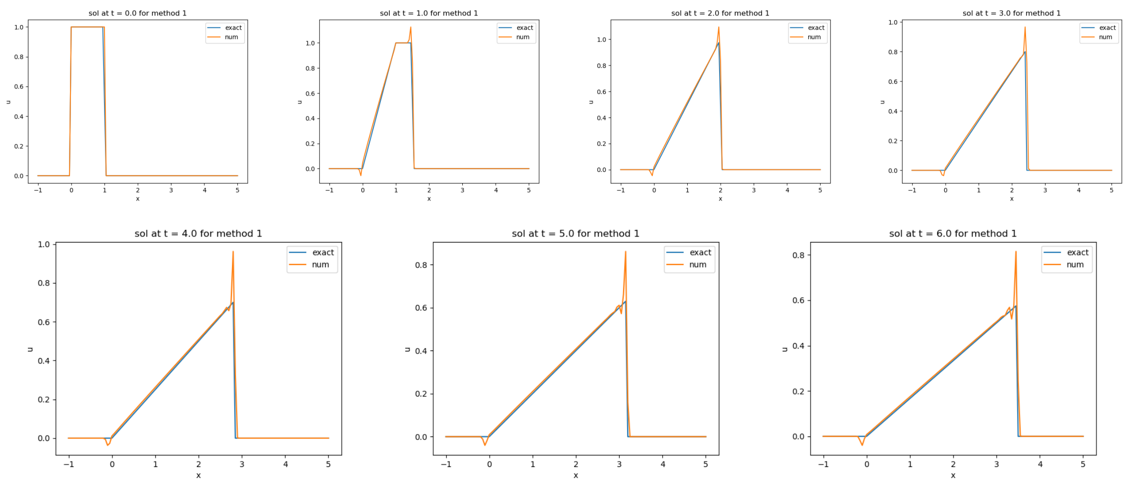




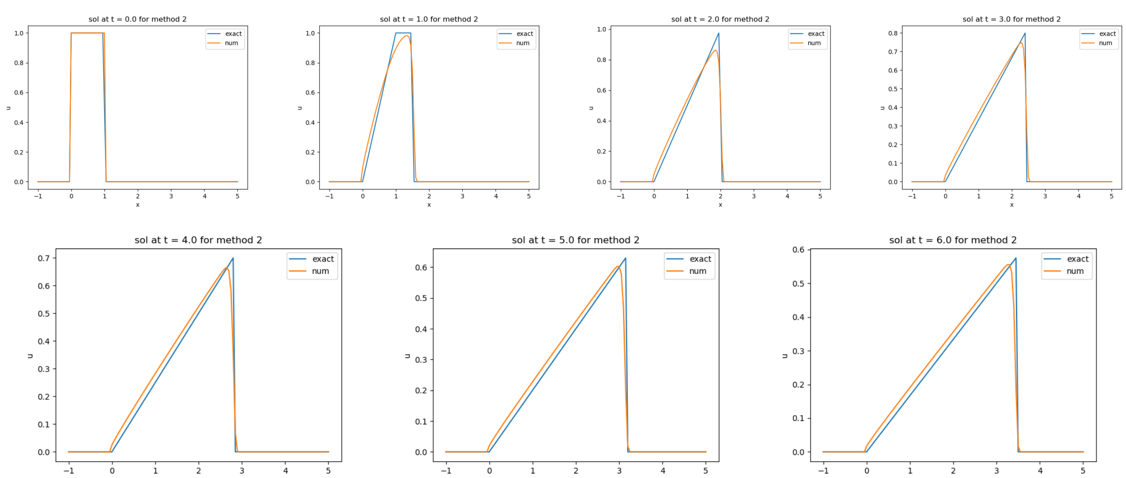




Richtmyer:



MacCormack:



Problem 4.

Write the PDE as

$$u_t = \underbrace{-u_{xxxx} - u_{xx}}_{=:Lu} \underbrace{-\frac{1}{2}(u^2)_x}_{=:N(u)}$$

First solving $u_t = Lu$, write

$$u(x,t) = \sum_{k=-\infty}^{\infty} u_k(t)e^{ikx/16}$$

so that

$$u_t = \sum_{k=-\infty}^{\infty} u_k'(t)e^{ikx/16}, \quad u_{xx} = \sum_{k=-\infty}^{\infty} u_k(t)\left(-\left(\frac{k}{16}\right)^2\right)e^{ikx/16}, \quad u_{xxxx} = \sum_{k=-\infty}^{\infty} u_k(t)\left(\frac{k}{16}\right)^4 e^{ikx/16}$$

Plugging in these derivatives and using the fact that the basis functions $e^{ikx/16}$ are linearly independent,

$$u_k'(t) = \left[\left(\frac{k}{16}\right)^2 - \left(\frac{k}{16}\right)^4\right]u_k(t) \implies u_k(t) = u_k(0)e^{[(k/16)^2-(k/16)^4]t}$$

giving the solution

$$u(x,t) = \sum_{k=-\infty}^{\infty} u_k(0)e^{ikx/16}e^{[(k/16)^2-(k/16)^4]t}, \quad u_k(0) = \frac{1}{32\pi}\int_0^{32\pi} u(x,0)e^{-ikx/16}dx$$

Define the solution operator e^{tL} by specifying its action on the basis functions $e^{ikx/16}$.

$$e^{tL}(e^{ikx/16}) := e^{ikx/16}e^{[(k/16)^2-(k/16)^4]t}$$

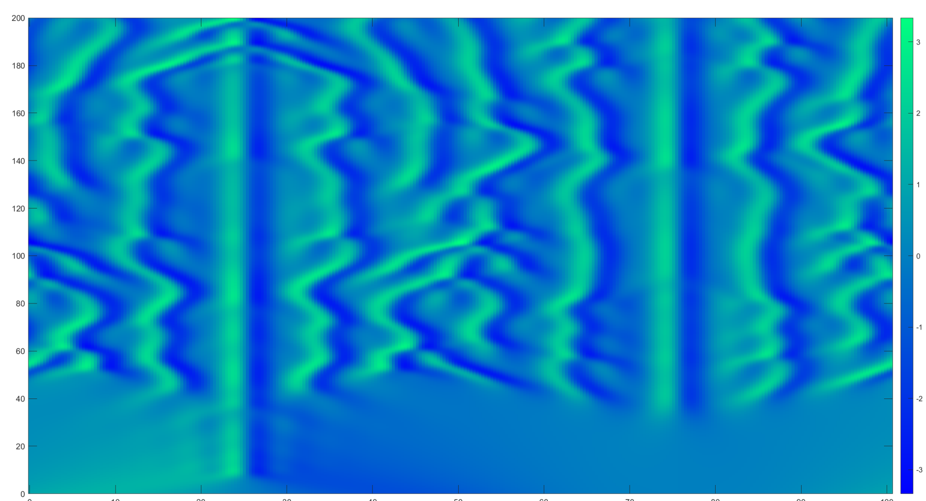
We check that we can rewrite the solution as

$$u(x,t) = \sum_{k=-\infty}^{\infty} u_k(0)e^{tL}(e^{ikx/16}) = e^{tL}\sum_{k=-\infty}^{\infty} u_k(0)e^{ikx/16} = e^{tL}u(x,0)$$

Let v satisfy $u = e^{tL}v$. Plugging into the equation $u_t = Lu + N(u)$, we obtain an equation for v.

$$Le^{tL}v + e^{tL}v_t = Le^{tL}v + N(e^{tL}v) \implies e^{tL}v_t = N(e^{tL}v) \implies v_t = e^{-tL}N(e^{tL}v)$$

The file KdVrkm.m was modified to solve this PDE:
`https://github.com/RokettoJanpu/Scientific-Computing-2/blob/main/KdVrkm.m`



The plot is very similar to the one in the linked article. The initial data does not vary significantly, but it splits into many high frequency waves, giving rise to a complex looking solution. On the other hand, characteristic lines appear, upon which solutions appear stationary, and moreover nearby solutions tend toward these lines.