Scientific Computing HW 11

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April 25, 2023

Problem 1.

(a) The form of solution is

$$u(x,t) = \frac{1}{2}[\varphi(x+at) + \varphi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(s)ds$$

First find u_{tt}.

$$\begin{aligned}
u_t &= \frac{1}{2}[\varphi'(x+at)\cdot a + \varphi'(x-at)\cdot(-a)] + \frac{1}{2a}[\psi(x+at)\cdot a - \psi(x-at)\cdot(-a)] \\
&= \frac{a}{2}[\varphi'(x+at) - \varphi'(x-at)] + \frac{1}{2}[\psi(x+at) + \psi(x-at)] \\
u_{tt} &= \frac{a^2}{2}[\varphi''(x+at) + \varphi''(x-at)] + \frac{a}{2}[\psi'(x+at) - \psi'(x-at)]
\end{aligned}$$

Then find $a^2 u_{xx}$ and see that it equals u_{tt}, hence u solves the PDE.

$$\begin{aligned}
u_x &= \frac{1}{2}[\varphi'(x+at) + \varphi'(x-at)] + \frac{1}{2a}[\psi(x+at) - \psi(x-at)] \\
u_{xx} &= \frac{1}{2}[\varphi''(x+at) + \varphi''(x-at)] + \frac{1}{2a}[\psi'(x+at) - \psi'(x-at)] \\
a^2 u_{xx} &= \frac{a^2}{2}[\varphi''(x+at) + \varphi''(x-at)] + \frac{a}{2}[\psi'(x+at) - \psi'(x-at)] = u_{tt}
\end{aligned}$$

(b) The form of solution is

$$u(x,t) = \frac{1}{2}[\varphi(x+at) + \varphi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(s)ds$$

From its terms we see that $u(x,t)$ depends precisely on the values of φ at $x \pm at$ and the values of ψ on $[x-at, x+at]$. Thus the domain of dependence of a point (x,t) is $\{(s,0) : x-at \le s \le x+at\}$.

(c) We see that

$$w := \begin{bmatrix} u_t \\ u_x \end{bmatrix} \implies w_x = \begin{bmatrix} u_{tx} \\ u_{xx} \end{bmatrix}$$

so that

$$w_t = \begin{bmatrix} u_{tt} \\ u_{xt} \end{bmatrix} = \begin{bmatrix} 0u_{tx} + a^2 u_{xx} \\ 1u_{tx} + 0u_{xx} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & a^2 \\ 1 & 0 \end{bmatrix}}_{=:A} \begin{bmatrix} u_{tx} \\ u_{xx} \end{bmatrix} = Aw_x$$

Then

$$u_x\Big|_{t=0} = \frac{1}{2}[\varphi'(x) + \varphi'(x)] + \frac{1}{2a}[\psi(x) - \psi(x)] = \varphi'(x)$$

so that the initial condition for w is

$$w\Big|_{t=0} = \begin{bmatrix} u_t\big|_{t=0} \\ u_x\big|_{t=0} \end{bmatrix} = \begin{bmatrix} \psi(x) \\ \varphi'(x) \end{bmatrix}$$

(d) The eigenvalues of A are

$$0 = \det(A - \lambda I) = \begin{vmatrix} -\lambda & a^2 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - a^2 = (\lambda - a)(\lambda + a) \implies \lambda_1 = a,\ \lambda_2 = -a$$

Eigenvectors v_1, v_2 of A are

$$A - \lambda_1 I = \begin{bmatrix} -a & a^2 \\ 1 & -a \end{bmatrix} \implies v_1 = \begin{bmatrix} a \\ 1 \end{bmatrix}$$

$$A - \lambda_2 I = \begin{bmatrix} a & a^2 \\ 1 & a \end{bmatrix} \implies v_2 = \begin{bmatrix} -a \\ 1 \end{bmatrix}$$

Diagonalizing A,

$$A = C\Lambda C^{-1}, \quad C := [v_1, v_2] = \begin{bmatrix} a & -a \\ 1 & 1 \end{bmatrix}, \quad \Lambda := \operatorname{diag}(\lambda_1, \lambda_2) = \operatorname{diag}(a, -a)$$

Changing variable, we obtain independent PDEs.

$$y := C^{-1}w = \begin{bmatrix} \xi \\ \eta \end{bmatrix} \implies w = Cy \implies w_t = Cy_t,\ w_x = Cy_x$$

$$\implies 0 = w_t - Aw_x = Cy_t - C\Lambda C^{-1}Cy_x = C(y_t - \Lambda y_x) \implies y_t - \Lambda y_x = 0 \implies y_t = \Lambda y_x$$
$$\implies \xi_t = a\xi_x,\ \eta_t = -a\eta_x$$

First find C^{-1}.

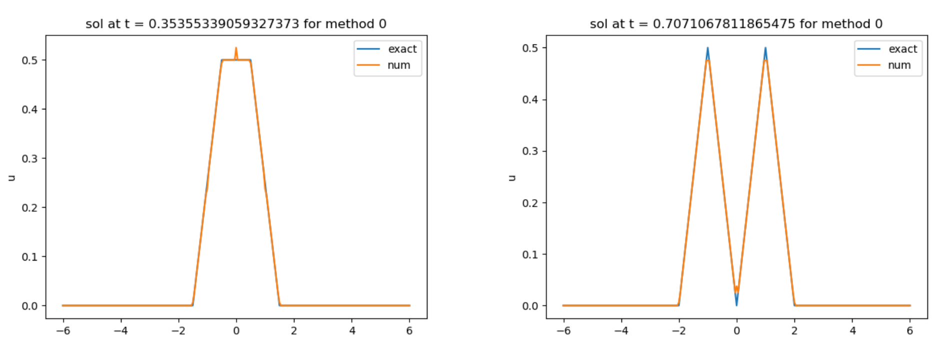
$$\det C = \begin{vmatrix} a & -a \\ 1 & 1 \end{vmatrix} = 2a \implies C^{-1} = \frac{1}{2a}\begin{bmatrix} 1 & a \\ -1 & a \end{bmatrix}$$

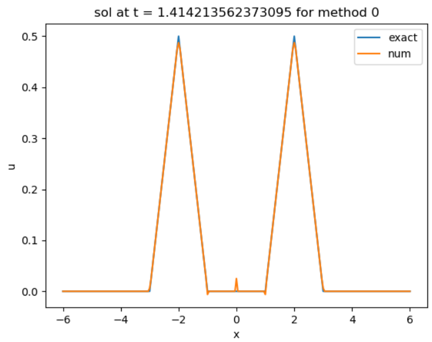
Then we find the initial condition for y, i.e. the initial conditions for ξ, η.

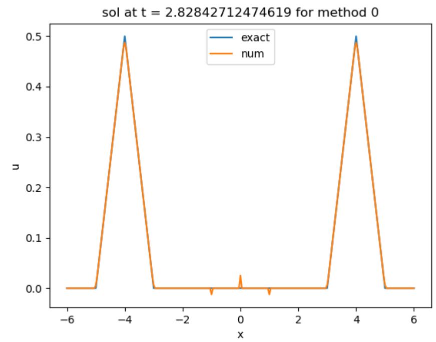
$$\begin{bmatrix} \psi(x) \\ \varphi'(x) \end{bmatrix} = w\Big|_{t=0} = Cy\Big|_{t=0} \implies y\Big|_{t=0} = C^{-1}\begin{bmatrix} \psi(x) \\ \varphi'(x) \end{bmatrix} = \frac{1}{2a}\begin{bmatrix} \psi(x) + a\varphi'(x) \\ -\psi(x) + a\varphi'(x) \end{bmatrix}$$

(e) Code: `https://github.com/RokettoJanpu/Scientific-Computing-2/blob/main/hw11%20q1.ipynb`

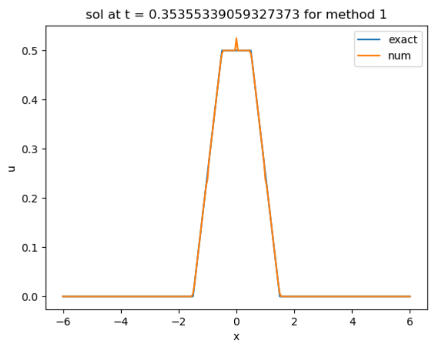
Lax–Friedrichs:

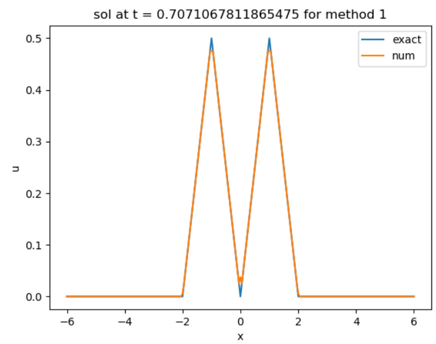


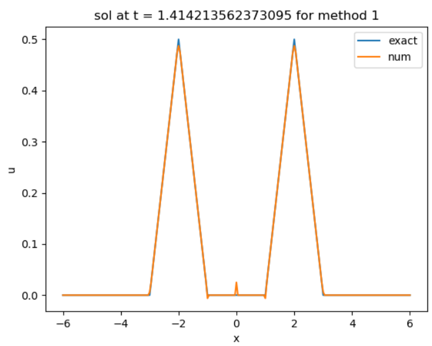


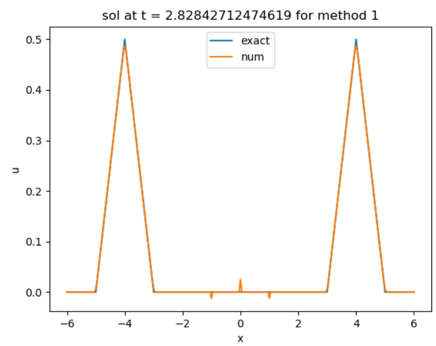


Upwind:

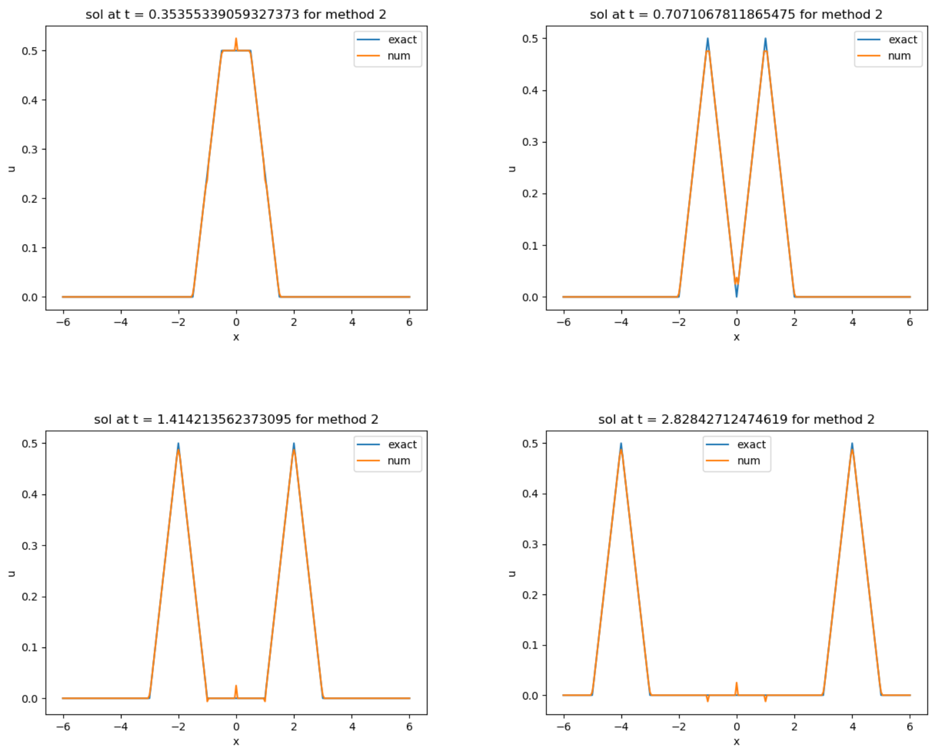




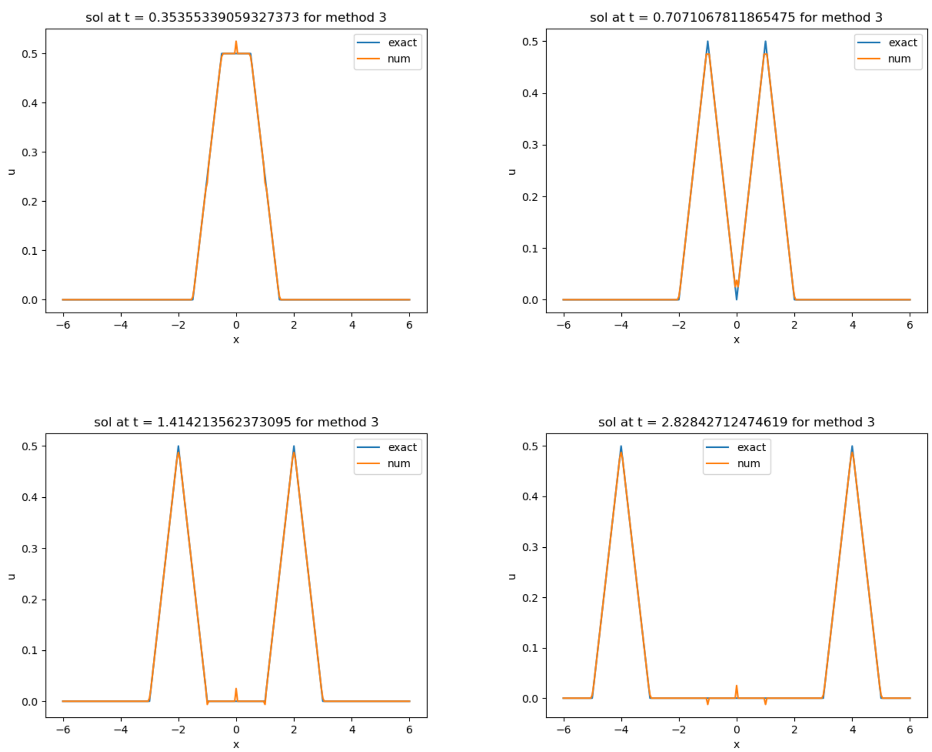




Lax–Wendroff:



Beam–Warming:



Each scheme is highly accurate in that it computes a solution that essentially coincides with the exact solution, with exceptions occurring at cusps. Some cusps are created and remain at points corresponding to cusps of the initial displacement $\varphi(x) = \max(1 - |x|, 0)$, i.e. at $x = 0, 1, -1$. From the exact solution

$$u(x,t) = \frac{1}{2}[\varphi(x + at) + \varphi(x - at)]$$

the other cusps occur at $x \pm at = 0, 1, -1$, i.e. $x = \mp at, 1 \mp at, -1 \mp at$, and this fact is visually shown by the fact that these cusps propagate with the solution.