Scientific Computing HW 3

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Problem 1. Pf. From the Butcher array,

$$A = \begin{bmatrix} \gamma & 0 \\ 1-\gamma & \gamma \end{bmatrix}, \quad b = \begin{bmatrix} 1-\gamma \\ \gamma \end{bmatrix}, \quad c = \begin{bmatrix} \gamma \\ 1 \end{bmatrix}$$

Check the 1st order accuracy condition.

$$\sum_{l=1}^{2} b_l = (1-\gamma) + \gamma = 1$$

Check the 2nd order accuracy condition.

$$\sum_{l=1}^{2} b_l c_l = (1-\gamma)\gamma + \gamma \cdot 1 = \gamma - \gamma^2 + \gamma = 2\gamma - \gamma^2 = 2 - 2^{1/2} - 1 - 2^{-1} + 2^{1/2} = 1 - 2^{-1} = \frac{1}{2}$$

Thus the method is 2nd order accurate. To show it is A–stable, we first find the stability function $R(z)$ and let $|z| \to \infty$.

$$I - zA = \begin{bmatrix} 1-\gamma z & 0 \\ -(1-\gamma)z & 1-\gamma z \end{bmatrix} \implies D := \det(I - zA) = (1-\gamma z)^2 = \gamma^2 z^2 - 2\gamma z + 1$$

$$\implies (I - zA)^{-1} = \frac{1}{D} \begin{bmatrix} 1-\gamma z & 0 \\ (1-\gamma)z & 1-\gamma z \end{bmatrix} \implies (I - zA)^{-1} 1_{s\times 1} = \frac{1}{D} \begin{bmatrix} 1-\gamma z \\ (1-\gamma)z + 1 - \gamma z \end{bmatrix} = \frac{1}{D} \begin{bmatrix} 1-\gamma z \\ (1-2\gamma)z + 1 \end{bmatrix}$$

$$R(z) - 1 = zb^T (I - zA)^{-1} 1_{s\times 1} = \frac{z}{D} [(1-\gamma)(1-\gamma z) + \gamma((1-2\gamma)z + 1)] = \frac{z}{D} \left[1 - \gamma z - \gamma + \gamma^2 z + (\gamma - 2\gamma^2)z + \gamma\right]$$

$$\implies R(z) - 1 = \frac{z}{D}\left[1 - \gamma^2 z\right] = \frac{-\gamma^2 z^2 + z}{\gamma^2 z^2 - 2\gamma z + 1} \implies R(z) = \frac{-\gamma^2 z^2 + z}{\gamma^2 z^2 - 2\gamma z + 1} + 1 \xrightarrow{|z|\to\infty} -1 + 1 = 0$$

To finish showing A–stability, we plot $|R(z)| < 1$ and see that it contains the left half plane.