Scientific Computing HW 1

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P1. Pick $T < t^*$ where $t^* := t_0 + \frac{1}{y_0}$. Fix $r \in \mathbb{R}$. Then $f(t, y) := y^2$ is continuous on the cylinder $Q := \{t_0 \le t \le T,\ |y - y_0| \le r\}$. Now for all $(t, y) \in Q$,

$$|y - y_0| \le r \implies |y| = |y - y_0 + y_0| \le |y - y_0| + |y_0| \le r + |y_0| \implies |f(t, y)| = |y^2| = |y|^2 \le (r + |y_0|)^2$$

i.e. $|f| \le M$ on Q where $M := (r + |y_0|)^2$. Then by theorem 1, the IVP has a solution for $0 \le t - t_0 \le \min(\frac{r}{M}, T - t_0)$, i.e. $t_0 \le t \le t_0 + \min(\frac{r}{M}, T - t_0)$.

P2. Pick $0 \le T < \infty$. Fix $r \in \mathbb{R}$. Then $f(t, y) := 2y^{1/2}$ is continuous on the cylinder $Q := \{t_0 \le t \le T,\ |y| \le r\}$. Now for all $(t, y) \in Q$ with $y \ge 0$, using similar arguments as in P1,

$$|y| \le r \implies |f(t, y)| = |2y^{1/2}| = 2y^{1/2} \le 2r^{1/2}$$

i.e. $|f| \le M$ on Q where $M := 2r^{1/2}$. Then by theorem 1, the IVP has a solution for $0 \le t \le \min(\frac{r}{M}, T)$.

To see that $f(y) := 2y^{1/2}$ is not Lipschitz at $y = 0$, suppose there exists $C > 0$ such that $|f(x) - f(0)| \le C|x - 0|$ for all $x \ge 0$, i.e. $2x^{1/2} \le Cx$. But if we set $x = \frac{1}{C^2}$, then $2 \le Cx^{1/2} = C\frac{1}{C} = 1$, a contradiction.

P3. We compute the Picard iterates for $f(t, y) := y^2,\ t_0 = 0,\ y_0 = 1$, starting at $y_1(t) = 1$.

$$y_2(t) = 1 + \int_0^t f(s, y_1(s))ds = 1 + \int_0^t 1ds = 1 + t$$

$$y_3(t) = 1 + \int_0^t f(s, y_2(s))ds = 1 + \int_0^t (1 + s)^2 ds = 1 + \int_0^t (1 + 2s + s^2)ds = 1 + t + t^2 + \frac{1}{3}t^3$$

$$y_4(t) = 1 + \int_0^t \left(1 + 2s + 3s^2 + \frac{8}{3}s^3 + \frac{5}{3}s^4 + \frac{2}{3}s^5 + \frac{1}{9}s^6\right) ds = 1 + t + t^2 + t^3 + \frac{2}{3}t^4 + \frac{1}{3}t^5 + \frac{1}{9}t^6 + \frac{1}{63}t^7$$

The Picard iterates seem to converge to $\sum_{n \ge 0} t^n = \frac{1}{1-t}$ on $0 < t < 1$. Plugging values into the time interval in P1, we get

$$M = (r + |y_0|)^2 = (r + 1)^2$$

$$\frac{r}{M} = \frac{r}{(r+1)^2} \le 1$$

$$T - t_0 = T < t^* = t_0 + \frac{1}{y_0} = 1$$

$$0 \le t \le \min\left(\frac{r}{M}, T - t_0\right) < 1$$

which is a smaller interval.

P4. Pf. Using the Taylor expansion of y at t_n,

$$\begin{aligned}\tau_{n+1} &= y + hy' + \frac{1}{2}h^2y'' + \frac{1}{6}h^3y''' + O(h^4) - y \\ &\quad - h\left[\frac{23}{12}y' - \frac{4}{3}\left(y' - hy'' + \frac{1}{2}h^2y''' + O(h^3)\right) + \frac{5}{12}\left(y' - 2hy'' + 2h^2y''' + O(h^3)\right)\right]\end{aligned}$$

Collect coefficients of the following terms:

$$\begin{aligned} y: &\quad 1 - 1 = 0 \\ hy': &\quad 1 - \frac{23}{12} + \frac{4}{3} - \frac{5}{12} = \frac{1}{12}(12 - 23 + 16 - 5) = 0 \\ h^2y'': &\quad \frac{1}{2} - \frac{4}{3} + \frac{5}{6} = \frac{1}{6}(3 - 8 + 5) = 0 \\ h^3y''': &\quad \frac{1}{6} + \frac{2}{3} - \frac{5}{6} = \frac{1}{6}(1 + 4 - 5) = 0 \end{aligned}$$

Thus $\tau_{n+1} = O(h^4)$, i.e. the method is consistent of order 3.

P5. Pf. Substituting k into the recurrence,

$$u_{n+1} = u_n + hf\left(t_n + \frac{1}{2}h, u_n + \frac{1}{2}hk\right)$$

Using the Taylor expansion of y at $t_n + \frac{1}{2}h$,

$$\tau_{n+1} = y + hy' + \frac{1}{2}h^2y'' + O(h^3) - y - h\left[y' + \frac{1}{2}y'' + O(h^2)\right]$$

Collect coefficients of the following terms:

$$\begin{aligned} y: &\quad 1 - 1 = 0 \\ hy': &\quad 1 - 1 = 0 \\ h^2y'': &\quad \frac{1}{2} - \frac{1}{2} = 0 \end{aligned}$$

Thus $\tau_{n+1} = O(h^3)$, i.e. the method is consistent of order 2.

P6.

(a) Since we have four undetermined coefficients, we take a third order Taylor expansion. Any higher order expansion would give an overconstrained system.

$$\begin{aligned}\tau_{n+1} &= y + hy' + \frac{1}{2}h^2y'' + \frac{1}{6}h^3y''' + O(h^4) - a_0y - a_1\left[y' - hy' + \frac{1}{2}h^2y'' - \frac{1}{6}h^3y''' + O(h^4)\right] \\ &\quad - h\left[b_0y' + b_1\left(y' - hy'' + \frac{1}{2}h^2y''' + O(h^3)\right)\right]\end{aligned}$$

Collect coefficients of the following terms. To seek consistency of the highest possible order (in this case order 3), set each sum equal to 0:

$$\begin{aligned} y: &\quad 1 - a_0 - a_1 = 0 \\ hy': &\quad 1 + a_1 - b_0 - b_1 = 0 \\ h^2y'': &\quad \frac{1}{2} - \frac{1}{2}a_1 + b_1 = 0 \\ h^3y''': &\quad \frac{1}{6} + \frac{1}{6}a_1 - \frac{1}{2}b_1 = 0 \end{aligned}$$

The solution of this system (using WolframAlpha) is $a_0 = -4,\ a_1 = 5,\ b_0 = 4,\ b_1 = 2.$

(b) **Pf.** Applying the method (with coefficients from the last part) to $y' = 0 =: f(t, y)$ with initial condition $y(0) = a$,

$$u_{n+1} + 4u_n - 5u_{n-1} = 0$$

Using the ansatz solution r^n,

$$0 = r^2 + 4r - 5 = (r+5)(r-1) \implies r = 1, -5 \implies u_n = A + B(-5)^n$$

The exact solution of the IVP is $y(t) = a$. Perturb the values of the first two iterates, say $u_0 = a + \delta_0$ and $u_1 = a + \delta_1$. Then

$$a + \delta_0 = A + B,\ a + \delta_1 = A - 5B \implies \delta_0 - \delta_1 = 6B$$

If $\delta_0 \neq \delta_1$ then $B \neq 0$, in which case the solution blows up. Thus the method is unstable.

(c) `https://github.com/RokettoJanpu/Scientific-Computing-2/blob/main/hw1.ipynb`