

Scientific Computing HW 8

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Problem 1. Code: <https://github.com/RokettoJanpu/Scientific-Computing-2/blob/main/hw8.ipynb>
Problem 2.

(a) **Pf.** Write

$$\begin{aligned} u(x) &= \int_0^1 G(x, y) f(y) dy = \int_0^x G(x, y) f(y) dy + \int_x^1 G(x, y) f(y) dy \\ &= (1-x) \int_0^x y f(y) dy + x \int_x^1 (1-y) f(y) dy = (1-x) \int_0^x y f(y) dy - x \int_1^x (1-y) f(y) dy \end{aligned}$$

Then compute

$$\begin{aligned} u'(x) &= - \int_0^x y f(y) dy + (1-x) x f(x) - \int_1^x (1-y) f(y) dy - x(1-x) f(x) \\ &= - \int_0^x y f(y) dy - \int_1^x f(y) dy + \int_1^x y f(y) dy = - \int_1^x f(y) dy \\ &\implies u''(x) = -f(x) \end{aligned}$$

(b) **Pf.** Compute

$$G_y(x, y) = \begin{cases} 1-x, & y < x \\ -x, & y > x \end{cases}$$

Then

$$\begin{aligned} \int_0^1 v'(y) G_y(x, y) dy &= \int_0^x v'(y) G_y(x, y) dy + \int_x^1 v'(y) G_y(x, y) dy \\ &= (1-x) \int_0^x v'(y) dy - x \int_x^1 v'(y) dy = \int_0^x v'(y) dy - x \int_0^x v'(y) dy - x \int_x^1 v'(y) dy \\ &= v(x) - v(0) - x[v(1) - v(0)] = v(x) - 0 - x[0 - 0] = v(x) \end{aligned}$$

Since $u \in H_0^1([0, 1])$, the identity in particular holds for u .

(c) The linear interpolant $I_h u$, as a linear combination of the basis functions φ_j , is

$$I_h u = \sum_{j=1}^n u(x_j) \varphi_j$$

(d) **Pf.** Note that the function ψ_j is not differentiable at x_j , but is differentiable at all other points.

Suppose the functions ψ_j are linearly dependent, i.e. there exist scalars c_j , not all 0, so that

$$\sum_{j=1}^n c_j \psi_j = 0$$

In particular the linear combination

$$\sum_{j=1}^n c_j \psi_j$$

is differentiable. From the assumption we can pick i so that $c_i \neq 0$. Since the functions ψ_j , for $j \neq i$, are differentiable at x_i , the linear combination

$$\sum_{\substack{j=1 \\ j \neq i}}^n c_j \psi_j$$

is differentiable at x_i . Also, since ψ_i is not differentiable at x_i and $c_i \neq 0$, the function $c_i \psi_i$ is not differentiable at x_i . This means the linear combination $\sum_{j=1}^n c_j \psi_j$ is not differentiable at x_i , a contradiction. Thus the functions ψ_j are linearly independent.

Lastly, since $\dim(S_h) = n$, the n linearly independent functions ψ_j form a basis of S_h .

(e) **Pf.** Recall that

$$G_y(x, y) = \begin{cases} 1 - x, & y < x \\ -x, & y > x \end{cases}, \quad \varphi'_i(x) = \begin{cases} 0, & x < x_{i-1} \text{ or } x > x_{i+1} \\ \frac{1}{x_i - x_{i-1}}, & x_{i-1} < x < x_i \\ -\frac{1}{x_{i+1} - x_i}, & x_i < x < x_{i+1} \end{cases}$$

First write

$$\begin{aligned} \int_0^1 [I_h u]'(y) G_y(x_j, y) dy &= \int_0^1 \left[\sum_{i=1}^n u(x_i) \varphi'_i(y) \right] G_y(x_j, y) dy \\ &= \sum_{i=1}^n u(x_i) \int_0^1 \varphi'_i(y) G_y(x_j, y) dy = \sum_{i=1}^n u(x_i) \underbrace{\int_{x_{i-1}}^{x_{i+1}} \varphi'_i(y) G_y(x_j, y) dy}_{=: a_{ij}} \end{aligned}$$

Now examine cases.

- If $i = j$,

$$a_{ij} = \int_{x_{j-1}}^{x_j} \frac{1 - x_j}{x_j - x_{j-1}} dy + \int_{x_j}^{x_{j+1}} \frac{-x_j}{-(x_{j+1} - x_j)} dy = 1 - x_j + x_j = 1$$

- If $i \geq j + 1$,

$$a_{ij} = \int_{x_{i-1}}^{x_i} \frac{-x_j}{x_i - x_{i-1}} dy + \int_{x_i}^{x_{i+1}} \frac{-x_j}{-(x_{i+1} - x_i)} dy = -x_j + x_j = 0$$

- If $i \leq j - 1$,

$$a_{ij} = \int_{x_{i-1}}^{x_i} \frac{1 - x_j}{x_i - x_{i-1}} dy + \int_{x_i}^{x_{i+1}} \frac{1 - x_j}{-(x_{i+1} - x_i)} dy = 1 - x_j - (1 - x_j) = 0$$

This means $a_{ij} = \delta_{ij}$, thus

$$\int_0^1 [I_h u]'(y) G_y(x_j, y) dy = \sum_{i=1}^n u(x_i) \delta_{ij} = u(x_j)$$

(f) **Pf.** e

Problem 3.

- (a) **Pf.** Using the identity $\nabla \cdot (fv) = \nabla f \cdot v + f \nabla \cdot v$ for any scalar function f and vector field v ,

$$\begin{aligned} \nabla \cdot (e^{-\beta V(x)} \nabla u) + \beta e^{-\beta V(x)} f(x) \cdot \nabla u &= e^{-\beta V(x)} \Delta u - \beta e^{-\beta V(x)} \nabla V(x) \cdot \nabla u + \beta e^{-\beta V(x)} f(x) \cdot \nabla u \\ &= \beta e^{-\beta V(x)} [\beta^{-1} \Delta u - \nabla V(x) \cdot \nabla u + f(x) \cdot \nabla u] = \beta e^{-\beta V(x)} [\beta^{-1} \Delta u + b(x) \cdot \nabla u] \end{aligned}$$

and since $\beta e^{-\beta V(x)} \neq 0$ for all x ,

$$\beta^{-1} \Delta u + b(x) \cdot \nabla u = 0 \iff \nabla \cdot (e^{-\beta V(x)} \nabla u) + \beta e^{-\beta V(x)} f(x) \cdot \nabla u = 0$$

hence the two BVPs are equivalent.

- (b) Decompose

$$b(x, y) = \begin{bmatrix} x - x^3 - 10xy^2 \\ -y - x^2y \end{bmatrix} = \underbrace{\begin{bmatrix} x - x^3 - xy^2 \\ -y - 3y^3 - x^2y \end{bmatrix}}_{=: F(x, y)} + \underbrace{\begin{bmatrix} -9xy^2 \\ 3y^3 \end{bmatrix}}_{=: f(x, y)}$$

We check

$$\begin{aligned} \nabla \times F &= \partial_x(-y - 3y^3 - x^2y) + \partial_y(x - x^3 - xy^2) = -2xy + 2xy = 0 \\ \nabla \cdot f &= \partial_x(-9xy^2) + \partial_y(3y^3) = -9y^2 + 9y^2 = 0 \end{aligned}$$

so the decomposition is as desired. For the vector field

$$-F(x, y) = \begin{bmatrix} -x + x^3 + xy^2 \\ y + 3y^3 + x^2y \end{bmatrix}$$

a potential V is found by

$$V(x, y) = \int (-x + x^3 + xy^2) dx = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{2}x^2y^2 + g(y)$$

for some function g , and

$$V(x, y) = \int (y + 3y^3 + x^2y) dy = \frac{1}{2}y^2 + \frac{3}{4}y^4 + \frac{1}{2}x^2y^2 + h(x)$$

for some function h . Putting the calculations together,

$$V(x, y) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{2}x^2y^2 + \frac{1}{2}y^2 + \frac{3}{4}y^4$$