Scientific Computing HW 4

Ryan Chen

September 24, 2024

1. e

2. Before answering each part, we cite the following

$$\boldsymbol{\nabla}\big(x^T Ax\big) = (A + A^T)x, \quad \boldsymbol{\nabla}\big(x^T x\big) = 2x$$

We compute

$$\boldsymbol{\nabla} Q(x) = \frac{(x^T x)(A + A^T)x - (x^T Ax)2x}{(x^T x)^2}$$

Thus

$$\boldsymbol{\nabla} Q(x) = 0 \iff (x^T x)(A + A^T)x = 2(x^T Ax)x \iff (A + A^T)x = \frac{2x^T Ax}{x^T x}x$$

(a) Under the condition A is symmetric, the above computation gives

$$\boldsymbol{\nabla} Q(x) = 0 \iff Ax = \frac{x^T Ax}{x^T x}x \iff \left(\frac{x^T Ax}{x^T x}, x\right) \text{ is an eigenpair of } A$$

(b) If A is asymmetric, the above calculation says $\boldsymbol{\nabla} Q(x) = 0$ iff $\left(\frac{2x^T Ax}{x^T x}, x\right)$ is an eigenpair of $A + A^T$.

3. (a) Lemma: If A is nonsingular and (λ, v) is an eigenpair of A then (λ^{-1}, v) is an eigenpair of A^{-1}.
Proof of lemma: From A being nonsingular, $\lambda \neq 0$. Then

$$Av = \lambda v \implies v = \lambda A^{-1} v \implies A^{-1} v = \lambda^{-1} v$$

Since μ is not an eigenvalue of A, we know $A - \mu I$ is nonsingular. Then

$$(A - \mu I)v = Av - \mu I v = \lambda v - \mu v = (\lambda - \mu)v$$

hence $((\lambda - \mu), v)$ is an eigenpair of $A - \mu I$. By the lemma, $((\lambda - \mu)^{-1}, v)$ is an eigenpair of $(A - \mu I)^{-1}$.

(b)

(c)