Homework 2. Due Wednesday, Sept. 11.

1. (**5 pts**) Suppose you need to evaluate the derivative of a function $f(x)$ by forward difference, i.e.,
$$f'(x) \approx \frac{f(x+h) - f(x)}{h}. \tag{1}$$
The function is not available analytically but can be evaluated at any point x with a relative error ϵ such that $|\epsilon| \le 10^{-14}$. Suppose the function and its second derivative are of the order of 1. Give a rough estimate of the optimal value for h that minimizes the error in $f'(x)$.

2. (**6 pts**) Consider the polynomial space $\mathcal{P}_n(x)$, $x \in [-1,1]$. Let T_k, $k = 0, 1, \ldots, n$, be the Chebyshev basis in it. The Chebyshev polynomials are defined via
$$T_k = \cos(k \arccos x).$$
   (a) Use the trigonometric formula
   $$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
   to derive the three-term recurrence relationship for the Chebyshev polynomials
   $$T_0(x) = 1, \quad T_1(x) = x, \quad T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x), \quad k = 1, 2, \ldots. \tag{2}$$
   (b) Consider the differentiation map
   $$\frac{d}{dx} : \mathcal{P}_n \to \mathcal{P}_{n-1}.$$
   Write the matrix of the differentiation map with respect to the Chebyshev bases in $\mathcal{P}_n$ and $\mathcal{P}_{n-1}$ for $n = 7$. *Hint: you might find helpful properties of Chebyshev polynomials presented in Section 3.3.1 of Gil, Segure, Temme, "Numerical Methods For Special Functions".*

3. (**6 pts**) Let $A = (a_{ij})$ be an $m \times n$ matrix.
   (a) Prove that the l_1-norm of A is
   $$\|A\|_1 = \max_j \sum_i |a_{ij}|,$$
   i.e., the maximal column sum of absolute values. Find the maximizing vector.
   (b) Prove that the max-norm or l_∞-norm of A
   $$\|A\|_{\max} = \max_i \sum_j |a_{ij}|,$$
   i.e., the maximal row sum of absolute values. Find the maximizing vector.