# Scientific Computing HW 11

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1. (a) If $v = 0$ then $\langle u|v \rangle = 0$ and $\langle u|u \rangle \langle v|v \rangle = 0$, so the Cauchy–Schwarz inequality holds. Now assume $v \neq 0$. The function

$$p(t) := \langle u + tv|u + tv \rangle \,, \ t \in \mathbb{R}$$

is nonnegative due to positive–definiteness of the inner product. Expand $p(t)$ as

$$p(t) = \langle v|v \rangle t^2 + 2 \langle u|v \rangle t + \langle u, u|u, u \rangle$$

which is a quadratic in $t$ due to positive–definiteness ($\langle v|v \rangle > 0$). Its discriminant is

$$\Delta = 4 \langle u|v \rangle^2 - 4 \langle u|u \rangle \langle v|v \rangle = 4(\langle u|v \rangle^2 - \langle u|u \rangle \langle v|v \rangle)$$

Since $p$ is nonnegative, it does not change sign, hence $\Delta \leq 0$, from which the Cauchy–Schwarz inequality follows.

(b) Since $B$ is SPD, we can define the matrices $B^{1/2}$ and $B^{-1/2}$. Then

$$\begin{aligned}
(g^T Bg)(g^T B^{-1} g) &= (g^T B^{1/2} B^{1/2} g)(g^T B^{-1/2} B^{-1/2} g) \\
&= \left\langle B^{1/2} g \middle| B^{1/2} g \right\rangle \left\langle B^{-1/2} g \middle| B^{-1/2} g \right\rangle \\
&\geq \left\langle B^{1/2} g \middle| B^{-1/2} g \right\rangle^2 && \text{by Cauchy–Schwarz} \\
&= (g^T B^{1/2} B^{-1/2} g)^2 \\
&= (g^T g)^2
\end{aligned}$$

2. For convenience write $\lambda = \lambda^{(l)}$. To prove that the iterations are equivalent, we must show that

$$-\frac{\phi(\lambda)}{\phi'(\lambda)} = \frac{\|p_l\|^2}{\|z_l\|^2}\frac{\|p_l\| - \Delta}{\Delta}$$

First we establish some equalities. Observe that

$$\begin{aligned}\frac{q_j^T g}{\lambda_j + \lambda} &= [(B + \lambda I)^{-1} q_j]^T g \\ &= q_j^T (B + \lambda I)^{-1} g \\ &= -q_j^T p_l\end{aligned}$$

hence

$$\frac{(q_j^T g)^2}{(\lambda_j + \lambda)^2} = (q_j^T p_l)^2 \tag{1}$$

Also observe that

$$\begin{aligned}\frac{q_j^T g}{(\lambda_j + \lambda)^{1/2}} &= [(B + \lambda I)^{-1/2} q_j]^T g \\ &= q_j^T (B + \lambda I)^{-1/2} g \\ &= q_j^T L^{-1} p_l \\ &= q_j^T z_l\end{aligned}$$

This, along with (1), implies

$$\frac{(q_j^T g)^2}{(\lambda_j + \lambda)^3} = \frac{(q_j^T p_l)^2}{\lambda_j + \lambda} = (q_j^T p_l)^2 \tag{2}$$

Since $B$ is SPD, we may take the $q_j$'s to form an orthonormal basis of $\mathbb{R}^n$, so that

$$\|x\|^2 = \sum_j (q_j^T x)^2, \ x \in \mathbb{R}^n \tag{3}$$

Rewrite $\phi(\lambda)$.

$$\begin{aligned}\phi(\lambda) &= \Delta^{-1} - \left[\sum_j \frac{(q_j^T g)^2}{(\lambda_j + \lambda)^2}\right]^{-1/2} \\ &= \Delta^{-1} - \left[\sum_j (q_j^T p_l)\right]^{-1/2} && \text{by (1)} \\ &= \Delta^{-1} - \|p_l\|^{-1} && \text{by (3)}\end{aligned}$$

Obtain and rewrite $\phi'(\lambda)$.

$$\begin{aligned}\phi'(\lambda) &= -\left[\sum_j \frac{(q_j^T g)^2}{(\lambda_j + \lambda)^2}\right]^{-3/2} \sum_j \frac{(q_j^T g)^2}{(\lambda_j + \lambda)^3} \\ &= -\left[\sum_j (q_j^T p_l)^2\right]^{-3/2} \sum_j (q_j^T z_l)^2 && \text{by (1) and (2)} \\ &= -\|p_l\|^{-3} \|z_l\|^2 && \text{by (3)}\end{aligned}$$

Finally, we have

$$
\begin{aligned}
-\frac{\phi(\lambda)}{\phi'(\lambda)} &= \frac{\Delta^{-1} - \|p_l\|^{-1}}{\|p_l\|^{-3}\|z_l\|^2} \\
&= \frac{\Delta^{-1} - \|p_l\|^{-1}}{\|p_l\|^{-3}\|z_l\|^2} \cdot \frac{\Delta\|p_l\|}{\Delta\|p_l\|} \\
&= \frac{\|p_l\| - \Delta}{\|p_l\|^{-2}\|z_l\|^2\Delta} \\
&= \frac{\|p_l\|^2}{\|z_l\|^2}\frac{\|p_l\| - \Delta}{\Delta}
\end{aligned}
$$

3.

4.