# Scientific Computing HW 9

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1. (a) Label the three factors in the claimed factorization as $L, S, U$. Compute

$$LS = \begin{bmatrix} L_{11} & & \\ & L_{22} & \\ A_{31}U_{11}^{-1} & A_{32}U_{22}^{-1} & S_{33} \end{bmatrix} \implies LSU = \begin{bmatrix} A_{11} & & A_{13} \\ & A_{22} & A_{23} \\ A_{31} & A_{32} & B \end{bmatrix}$$

where

$$B := A_{31}U_{11}^{-1}L_{11}^{-1}A_{13} + A_{32}U_{22}^{-1}L_{22}^{-1}A_{23} + S_{33}$$

Imposing $A = LSU$ gives a formula for the Schur complement $S_{33}$.

$$\begin{aligned} A = LSU &\iff B = A_{33} \\ &\iff A_{31}U_{11}^{-1}L_{11}^{-1}A_{13} + A_{32}U_{22}^{-1}L_{22}^{-1}A_{23} + S_{33} = A_{33} \\ &\iff S_{33} = A_{33} - A_{31}U_{11}^{-1}L_{11}^{-1}A_{13} - A_{32}U_{22}^{-1}L_{22}^{-1}A_{23} \end{aligned}$$

(b) Given $S_{33} = L_{33}U_{33}$, the LU decomposition of $A$ is

$$A = \begin{bmatrix} L_{11} & & \\ & L_{22} & \\ A_{31}U_{11}^{-1} & A_{32}U_{22}^{-1} & L_{33} \end{bmatrix} \begin{bmatrix} U_{11} & & L_{11}^{-1}A_{13} \\ & U_{22} & L_{22}^{-1}A_{23} \\ & & U_{33} \end{bmatrix}$$

2. Code: `https://github.com/RokettoJanpu/scientific-computing-1-redux/blob/main/hw9/hw9.ipynb`

   The norm of difference between methods is 1200. This seems too large; perhaps there is an error in how I set up the problem.

3. The algorithm does not compute the Cholesky decomposition of $A_{11}$ or $A_{22}$ until the grid size is small, so we only consider the costs of computing the Schur complement $S_{33}$ and its Cholesky decomposition on each level.

   We will focus on level 0. The block sizes of the tessellation of $A$ are

$$\begin{aligned} A_{11}, A_{22} &: \frac{N}{2} \times \frac{N}{2} \\ A_{13}, A_{23} &: \frac{N}{2} \times N^{1/2} \\ A_{31}, A_{32} &: N^{1/2} \times \frac{N}{2} \\ A_{33} &: N^{1/2} \times N^{1/2} \end{aligned}$$

   We compute $S_{33}$ via

$$S_{33} = A_{33} - A_{31}U_{11}^{-1}L_{11}^{-1}A_{13} - A_{32}U_{22}^{-1}L_{22}^{-1}A_{23}$$

   According to the block sizes, a product involving the inverse of a triangular matrix, e.g. $L_{11}^{-1}A_{13}$, may be considered as solving a sparse nonsingular triangular $\frac{N}{2} \times \frac{N}{2}$ system with $N^{1/2}$ RHS vectors, and there are 4 products to compute. Being sparse gives roughly $\frac{N}{2}$ nonzero entries, and being nonsingular triangular forces at least $\frac{N}{2}$ of the nonzero entries to lie on the diagonal, so the system is roughly diagonal. Thus the cost is $\frac{N}{2}N^{1/2}4 = 2N^{3/2}$.

   The cost of computing $AB$, where $A$ is $m \times k$ sparse and $B$ is $k \times n$ possibly nonsparse, is roughly $2nnz(A)ncols(B)$ with $nnz(A)$ the number of nonzero entries of $A$ and $ncols(B)$ the number of columns of $B$. There are two products involving a sparse matrix and a possibly nonsparse matrix, e.g. $A_{31}(U_{11}^{-1}L_{11}^{-1}A_{13})$. Considering the block sizes and $A_{31}$ sparse with roughly $\frac{N}{2}$ nonzero entries, estimate the cost as $2\frac{N}{2}N^{1/2}2 = 2N^{3/2}$.

   There are two sums of $N^{1/2} \times N^{1/2}$ possibly nonsparse matrices to compute, contributing $2N$ to the cost. Thus the total cost of computing $S_{33}$ is $4N^{3/2}$. Then the cost of computing the Cholesky decomposition of $S_{33}$, it being $N^{1/2} \times N^{1/2}$, is $\frac{1}{3}N^{3/2}$. In conclusion, the cost of level 0 is $aN^{3/2}$ where $a := \frac{13}{3}$.

   Now consider the cost on level $k$. The meshgrid is partitioned with $2^k$ "dividers" of length $\left(\frac{N}{2^k}\right)^{1/2}$ each. Applying the reasoning behind the cost of level 0, the cost of level $k$ is $a2^k\left(\frac{N}{2^k}\right)^{3/2} = aN^{3/2}(2^{-1/2})^k$. Thus the cost of the entire algorithm is estimated as

$$W \leq aN^{3/2}\sum_{k=0}^{\infty}(2^{-1/2})^k = \frac{13}{3}\frac{1}{1-2^{-1/2}}N^{3/2}$$