Homework 9. Due Wednesday, Oct. 30.

The goal of this homework is to understand the Nested Dissection algorithm (George, 1973).

Reference: G. Martinsson, 10 lectures on fast direct solvers (2014). Lecture 6.

1. **(5 pts)** Suppose an invertible matrix A has a block form

$$A = \left[\begin{array}{c|c|c} A_{11} & & A_{13} \\ \hline & A_{22} & A_{23} \\ \hline A_{31} & A_{32} & A_{33} \end{array}\right]. \tag{1}$$

Assume that LU decompositions for A_{11} and A_{22} are available: $A_{11} = L_{11}U_{11}$, $A_{22} = L_{22}U_{22}$.

   (a) Show that A can be factored as

$$A = \left[\begin{array}{c|c|c} L_{11} & & \\ \hline & L_{22} & \\ \hline A_{31}U_{11}^{-1} & A_{32}U_{22}^{-1} & I \end{array}\right] \left[\begin{array}{c|c|c} I & & \\ \hline & I & \\ \hline & & S_{33} \end{array}\right] \left[\begin{array}{c|c|c} U_{11} & & L_{11}^{-1}A_{13} \\ \hline & U_{22} & L_{22}^{-1}A_{23} \\ \hline & & I \end{array}\right], \tag{2}$$

   where the matrix S_{33} is called the *Schur compliment*. Derive the formula for S_{33}.

   (b) Suppose that the LU decomposition of S_{33} is found: $S_{33} = L_{33}U_{33}$. Write out the LU decomposition of A.

2. **(5 pts)** Modify the provided Matlab or Python code implementing the nested dissection algorithm to replace the LU factorizations with Cholesky factorizations. This modification will be specifically designed for symmetric positive definite matrices A. You can use a built-in function that computes Cholesky factorization.

   Test it on the linear system from the problem with the maze from the previous homework. Save the symmetric positive definite linear matrix, the corresponding right-hand side, and the solution to it to a file and read this file in your new modified code. Paste your code to the pdf file with your homework. Report the norm of the difference between the solution computed by your code and the solution computed by a standard built-in linear solver.

3. **(5 pts)** Let the input matrix A be $N \times N$, symmetric positive definite. Estimate the number of flops in the resulting nested dissection with Cholesky factorizations. Do not count multiplications by permutation matrices as, if they were implemented in e.g. C, they would do only reindexing but involve no flops. Your answer should contain the exact coefficient next to the highest power of N. Terms with smaller powers of N can be incorporated in $O(\cdot)$.