Scientific Computing HW 2

Ryan Chen

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1. We obtain $\hat{f}(x)$ by computing $f(x)$ with relative error ϵ,

$$\epsilon = \frac{\hat{f}(x) - f(x)}{f(x)} \implies \hat{f}(x) = f(x) + \epsilon f(x)$$

Write a first order Taylor expansion of $f(x+h)$ with remainder,

$$f(x+h) = f(x) = f'(x)h + \frac{1}{2}f''(\xi)h^2, \quad x < \xi < x+h$$

This gives

$$\frac{f(x+h) - f(x)}{h} - f'(x) = \frac{1}{2}f''(\xi)h$$

Using the above, along with $f, f \in O(1)$, we can estimate the forward difference error as

$$\begin{aligned}
\left|\frac{\hat{f}(x+h) - \hat{f}(x)}{h} - f'(x)\right| &= \left|\frac{f(x+h) - f(x)}{h} - f'(x) + \epsilon\frac{f(x+h) - f(x)}{h}\right| \\
&\leq \frac{1}{2}|f''(\xi)|h + \frac{\epsilon}{h}|f(x+h) - f(x)| \\
&\sim \frac{1}{2}h + \frac{2\epsilon}{h} =: g(h)
\end{aligned}$$

Now we find h^* that minimizes $g(h)$.

$$g'(h) = 0 \implies \frac{1}{2} - \frac{2\epsilon}{h^2} = 0 \implies \frac{2\epsilon}{h^2} = \frac{1}{2} \implies h^2 = 4\epsilon \implies h^* = 2\sqrt{\epsilon}$$

2. (a) The first two Chebyshev polynomials are

$$T_0(x) = \cos(0 \cdot \arccos x) = 1, \quad T_1(x) = \cos(1 \cdot \arccos x) = x$$

The rest of the recurrence is given by

$$\begin{aligned}
T_{k+1}(x) + T_{k-1}(x) &= \cos((k+1)\arccos x) + \cos((k-1)\arccos x) \\
&= 2\cos\frac{(k+1)\arccos x + (k-1)\arccos x}{2}\cos\frac{(k+1)\arccos x - (k-1)\arccos x}{2} \\
&= 2xT_k(x)
\end{aligned}$$

i.e., $T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x)$.

(b) Label the bases $\mathcal{B} = \{T_0, \ldots, T_n\}$ of $\mathcal{P}_n$ and $\mathcal{C} = \{T_0, \ldots, T_{n-1}\}$ of $\mathcal{P}_{n-1}$. For convenience, given $f \in \mathcal{P}_{n-1}$ and $v \in \mathbb{R}^n$, we write $f \doteq v$ to mean that v is the coordinate vector of f wrt $\mathcal{C}$.

$$T_0' = 0 \doteq (0, \ldots, 0)^T, \quad T_1' = 1 = 1T_0 \doteq (1, 0, \ldots, 0)^T$$

From the recurrence in the last part,

$$T_2(x) = 2xT_1(x) - T_0(x) = 2x^2 - 1 \implies T_2' = 4T_1 \doteq (0, 4, 0, \ldots, 0)^T$$

For the rest of the calculations, we cite Eq. (3.25) from Chapter 3 in "Numerical Methods for Special Functions",

$$T_k = \frac{1}{2}\left(\frac{T_{k+1}'}{k+1} - \frac{T_{k-1}'}{k-1}\right), \ k \geq 2$$

This gives us a recurrence for computing derivatives,

$$T_{k+1}' = 2(k+1)T_k + \frac{k+1}{k-1}T_{k-1}', \ k \geq 2$$

Resuming calculations,

$$T_3' = 6T_2 + 3T_1' = 6T_2 + 3T_0 \doteq (3, 0, 6, 0, 0, 0, 0)^T$$

$$T_4' = 8T_3 + 2T_2' = 8T_3 + 8T_1 \doteq (0, 8, 0, 8, 0, 0, 0)^T$$

$$T_5' = 10T_4 + \frac{5}{3}T_3' = 10T_4 + 10T_2 + 5T_0 \doteq (5, 0, 10, 0, 10, 0, 0)^T$$

$$T_6' = 12T_5 + \frac{3}{2}T_4' = 12T_5 + 12T_3 + 12T_1 \doteq (0, 12, 0, 12, 0, 12, 0)^T$$

$$T_7' = 14T_6 + \frac{7}{5}T_5' = 14T_6 + 14T_4 + 14T_2 + 7T_0 \doteq (7, 0, 14, 0, 14, 0, 14)^T$$

Thus the matrix of the derivative map $\frac{\mathrm{d}}{\mathrm{d}x} : \mathcal{P}_n \to \mathcal{P}_{n-1}$ wrt the bases $\mathcal{B}$ and $\mathcal{C}$ is

$$\left[\frac{\mathrm{d}}{\mathrm{d}x}\right]_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 & 3 & 0 & 5 & 0 & 7 \\ 0 & 0 & 4 & 0 & 8 & 0 & 12 & 0 \\ 0 & 0 & 0 & 6 & 0 & 10 & 0 & 14 \\ 0 & 0 & 0 & 0 & 8 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 & 0 & 14 \\ 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 14 \end{bmatrix}$$

3. The $p-$norm of A is

$$\|A\|_p := \max_{\|x\|_p = 1} \|Ax\|_p$$

(a) Let $A_1, \ldots, A_n$ denote the columns of A, so that

$$\sum_{i=1}^{m} |a_{ij} = \|A_j\|_1$$

For all x,

$$\begin{aligned}
\|Ax\|_1 &= \left\|\sum_{i=1}^n x_i A_i\right\|_1 \\
&\le \sum_{i=1}^n |x_i| \|A_i\|_1 \\
&\le \underbrace{\max_{1\le j\le n} \|A_j\|_1}_{=:M} \cdot \sum_{i=1}^n |x_i| \\
&= M\|x\|_1
\end{aligned}$$

This implies $\|Ax\|_1 \le M$ for $\|x\|_1 = 1$, thus $\|A\|_1 \le M$.
Pick j_0 that maximizes $\|A_j\|_1$ (as a function of j). Then the j_0'th standard basis vector e_{j_0} is the maximizing vector since $\|e_{j_0}\|_1 = 1$ and

$$\|A\|_1 \ge \|Ae_{j_0}\|_1 = \|A_{j_0}\|_1 = M$$

Moreover, this inequality establishes $\|A\|_1 = M$.

(b) Let us write $\|A\|_\infty$ instead of $\|A\|_{\max}$. For all x,

$$\begin{aligned}
\|Ax\|_\infty &= \max_{1\le i\le m} |(Ax)_i| \\
&\le \max_{1\le i\le m} \sum_{j=1}^n |a_{ij}||x_j| \\
&\le \underbrace{\max_{1\le i\le m} \sum_{i=1}^n |a_{ij}|}_{=:M} \cdot \max_{1\le j\le n} |x_j| \\
&= M\|x\|_\infty
\end{aligned}$$

This implies $\|Ax\|_\infty \le M$ for $\|x\|_\infty = 1$, thus $\|A\|_\infty \le M$.
Pick i_0 that maximizes $\sum_{j=1}^n |a_{ij}|$ (as a function of i). Then the vector x with components $x_j := \mathrm{sign}(a_{i_0 j})$ is the maximizing vector since $\|x\|_\infty = 1$ and

$$\begin{aligned}
\|A\|_\infty &\ge \|Ax\|_\infty \\
&= \max_{1\le i\le m} |(Ax)_i| \\
&= \max_{1\le i\le m} \left|\sum_{j=1}^n a_{ij}x_j\right| \\
&\ge \left|\sum_{j=1}^n a_{i_0 j}x_j\right| \\
&= \sum_{j=1}^n |a_{i_0 j}| && x_j = \mathrm{sign}(a_{i_0 j}) \\
&= M
\end{aligned}$$

Moreover, this inequality establishes $\|A\|_\infty = M$.