Scientific Computing HW 6

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1. (a) Consider the following.

$$\sum_i \sum_j \sum_k a_{ik} b_{kj} c_{ji} = \sum_i \sum_j \sum_k b_{kj} c_{ji} a_{ik} = \sum_i \sum_j \sum_k c_{ji} a_{ik} b_{kj}$$

The first, second, and third expressions are, respectively, $\mathrm{tr}(ABC)$, $\mathrm{tr}(BCA)$, and $\mathrm{tr}(CAB)$.

(b) Compute

$$\begin{aligned}
\|A\|_F^2 &= \sum_i \sum_j a_{ij}^2 \\
&= \mathrm{tr}\left(A^T A\right) \\
&= \mathrm{tr}\left(V \Sigma U^T U \Sigma V^T\right) && A = U \Sigma V^T \\
&= \mathrm{tr}\left(V \Sigma^2 V^T\right) \\
&= \mathrm{tr}\left(\Sigma^2 V^T V\right) && \text{cyclic property of trace} \\
&= \mathrm{tr}\left(\Sigma^2\right) \\
&= \sum_i \sigma_i^2
\end{aligned}$$

(c) Compute

$$\begin{aligned}
\|A + B\|_F^2 &= \sum_i \sum_j (a_{ij} + b_{ij})^2 \\
&= \sum_i \sum_j a_{ij}^2 + \sum_i \sum_j b_{ij}^2 + 2 \sum_i \sum_j a_{ij} b_{ij} \\
&= \|A\|_F^2 + \|B\|_F^2 + 2 \left\langle A, B \right\rangle_F
\end{aligned}$$

2. Let A have size $n \times d$ with $n \geq d$. Observe that

$$A_k = U_k \Sigma_k V_k^T = U \Sigma_k' V^T$$

where Σ_k' is obtained by adding zeros to Σ_k to make it the same size as Σ. Then

$$A - A_k = U(\Sigma - \Sigma_k')V^T$$

is an SVD of $A - A_k$ with the jth diagonal entry of $\Sigma - \Sigma_k'$ being 0 for $j \leq k$ and $\sigma_j(A)$ for $j > k$. Thus

$$\|A - A_k\|_{KF(p)}^p = \sum_{j=1}^{d} \sigma_j^p(A - A_k) = \sum_{j=k+1}^{d} \sigma_j^p(A)$$

Fix an $n \times d$ matrix M with $\operatorname{rank} M \leq k$. Using the above calculation,

$$\begin{aligned}
\|A - M\|_{KF(p)}^p &= \sum_{i=1}^{d} \sigma_i^p(A - M) \\
&\geq \sum_{i=1}^{d-k} \sigma_i^p(A - M) \\
&\geq \sum_{i=1}^{d-k} \sigma_{k+i}^p(A) && \text{by Lemma 1, } \sigma_{k+i}(A) \leq \sigma_i(A - M) + \sigma_{k+1}(M) = \sigma_i(A - M) \\
&= \sum_{j=k+1}^{d} \sigma_j^p(A) && j := k + i \\
&= \|A - A_k\|_{KF(p)}^p
\end{aligned}$$

Taking the pth root of both sides gives $\|A - M\|_{KF(p)}^p \geq \|A - A_k\|_{KF(p)}^p$.

3. (a)
   (b)
   (c)